Information retrieval

Flexible querying systems and ranking systems

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Objectives of the course

- Acquire a culture in information retrieval
- Master the basics concepts allowing to understand:
  - what is at stake in novel IR methods
  - what are the technical limits

This will allow you to have the basics tools to analyze current limitations or lacks, and imagine novel solutions.

Today's outline

- Short summary of last lecture
- tf-idf
- Querying in the vector-space model
- Latent semantics
- Ranking

What to remember from last time?

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- Web IR is split in distinct steps:
  - Gathering and indexing data from the web (**crawling**)
  - Retrieving documents relevant to a query
  - Ranking the valid answers according to relevance
- The involved data is **big**

  Need efficient representation and algorithms

Drawbacks of the boolean querying systems

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The boolean querying does not rank

When querying using a boolean querying system, the output is binary.
→Unable to distinguish the relevant matches from non-relevant ones.

The vector space model and the latent semantics

Representing documents as vectors in $\mathbb{R}^T$

From binary presence/absence...

	tok 1	tok 2	tok 3	tok 4	tok 5	...
	election	president	crazy	united	United States	...
doc 1	1	1	0	0	1	...
doc 2	0	1	1	0	1	...
doc 3	1	1	1	0	1	...
...	...	...	...	...	...	...

Representing documents as vectors in $\mathbb{R}^T$

...to real vector space.

	tok 1	tok 2	tok 3	tok 4	tok 5	...
	election	president	crazy	united	United States	...
doc 1	0.01	0.02	0	0	0.006	...
doc 2	0	0.013	0.001	0	0.001	...
doc 3	0.0031	0.008	0.0043	0	0.0021	...
...	...	...	...	...	...	...

What numbers can be useful here ?

Not every term is informative

How do you quantify information according to Shannon theory?

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Example: which book are you talking about?

Piece of information	Probability	Information content
"the" is frequent	~ 1	Low
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- when e_1 and e_2 are independent, we would like that:

$$I(e_1 \& e_2) = I(e_1) + I(e_2)$$

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If we moreover ask for f to be continuous and non-zero, there is only one possible class of functions: $-log_b$

Information

The information of an event e is defined as $I(e) = -log(P(e))$

Definition

We can now compute the information of a token as:

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Exercise

I throw a die. What is the more informative:

- the outcome is even
- the outcome is ≥ 5

Vector representation of a document

A document can be represented by a vector of the fraction information associated to each of its token:

$$D_t = \frac{\text{\# t in D}}{\text{\# tokens in D}} \times I(t)$$

What does $||\vec{D}||_1$ represent?

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$||\vec{D}||_1$ carries the total information carried by a document:

- low if the document contains only common tokens
- average if the document contains few exceptional tokens
- high if the document contains only exceptional items

The tf-idf matrix

Definition

The matrix M which rows – corresponding to each document – are:

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is called the **tf-idf** (term frequency-inverse document frequency) representation.

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Question

What is the unit of elements of the tf-idf matrix?

Querying a set of vector

Represent the query the same way:

$$Q_t = \frac{\#\text{ t in Q}}{\#\text{ tokens in Q}} \times I(t)$$

How to retrieve documents related to the query?

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How to retrieve documents related to the query? Naïve approach: dot product.
Indeed, it makes sense: For each document, compute:

$$\vec{D} \cdot \vec{Q} = \sum_t D_t . Q_t$$

The higher the dot product, the more informative tokens $\vec{Q}$ and $\vec{D}$ share... and the more relevant should be the D with respect to the query Q.

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For querying purposes, one can select documents such that $\vec{D} \cdot \vec{Q} > \tau$, but it can directly be used for ranking documents.

Correcting for cheaters

Problem

Imagine a way of cheating with this approach.

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Content farms

$$
\begin{array}{rcl}
\vec{D} \cdot \vec{Q} & = & \sum_t D_t . Q_t \\
 & = & \sum_t \frac{\text{\# t in D}}{\text{\# tokens in D}} \times I(t) . \frac{\text{\# t in Q}}{\text{\# tokens in Q}} \times I(t) \\
 & \propto & \frac{1}{\text{\# tokens in D}} \sum_t \text{\#t in D} \times \text{\#t in Q} \times I(t)^2
\end{array}
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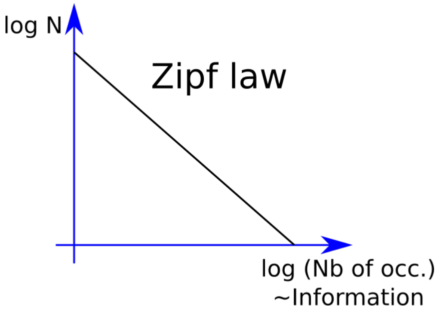
$$\begin{aligned}\vec{D}\cdot\vec{Q} &= \textstyle\sum_t D_t.Q_t \\ &= \textstyle\sum_t \frac{\text{\# t in D}}{\text{\# tokens in D}} \times I(t).\frac{\text{\# t in Q}}{\text{\# tokens in Q}} \times I(t) \\ &\propto \frac{1}{\text{\# tokens in D}} \textstyle\sum_t \#\text{t in D} \times \#\text{t in Q} \times I(t)^2\end{aligned}$$

Documents containing many informative words will be selected and ranked first.

Content farms: pull informative words together



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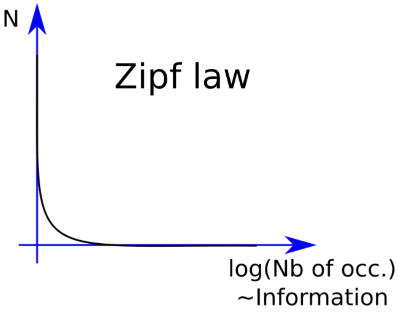

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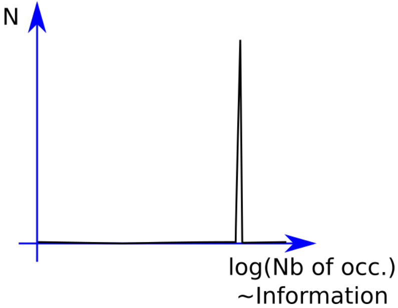

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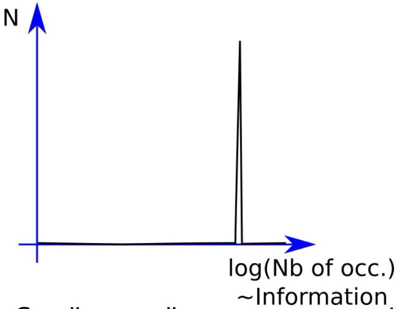

...Can "answer" many more queries

The cosine similarity

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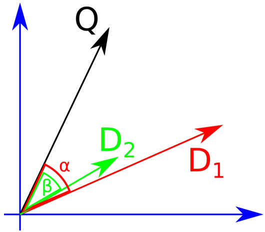

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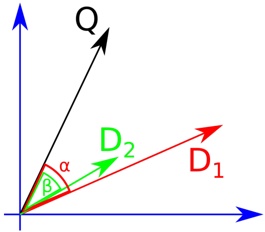

Correct by normalizing the similarity:

Consine similarity

$$\mathsf{cosim}(\vec{D},\vec{Q}) = \frac{\vec{D}\cdot\vec{Q}}{||\vec{D}||_2.||\vec{Q}||_2}$$

A flexible querying system?

With the vector space model, information of the tokens are now automatically taken into account.
Does it solve the synonymous problem?

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Query: `result elections United States`
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As already pointed out, we could use a semantic approach (ontologies), but need a fixed and manually curated work.
Can we work directly from the data?

Latent semantics

Special structure of the data: correlations

In practice a tf matrix looks like:

Interlude

Video

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How to recover automatically those blocks ?

Reminders from linear algebra

Exercise

If M is a tf matrix and Q a binary vector over tokens, what does MQ represent?

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What does it mean that $M^\top MQ = \lambda.Q$? What if λ is small? big?

Algebra theorem

Theorem

$M^\top M$ is symmetric and its eigenvectors $\vec{C}_i$ are orthogonal and form a basis of the token space.

$$\vec{D'} = \sum \alpha_i \vec{C}_i$$
$$\vec{Q'} = \sum \beta_i \vec{C}_i$$

We can compare search documents matching query Q using $\vec{D'}.\vec{Q'} = \sum \alpha_i.\beta_i$ or cosim$(\vec{D'}, \vec{Q'})$:)

Low rank approximation

Theorem

$M^\top M$ is symmetric and its eigenvectors are orthogonal and form a basis of the token space.

Theorem (Eckart–Young–Mirsky)

The best[a] r-rank approximation $\hat{M}$ of M is given by the projection on the subspace formed by the eigenvectors of $M^\top M$ corresponding to the r biggest eigen values.

[a] In the sense minimizing $||M - \hat{M}||_F = \sum_{i,j}(m_{i,j} - \hat{m}_{i,j})^2$

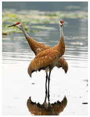
The projection to the low rank space (columns of $V^\top$ in SVD decomposition $M = U\Sigma V^\top$) collapse similar (i.e. *correlated*) tokens to the same component. This space is called the **Latent semantic space**.

Vector model: bright and dark side

The tf-idf vector model is good...

- Similarity based on information carried by tokens
- Flexible querying (latent semantics)
- Naturally rank documents
- Works well in practice

...but still not perfect:

- ignore polysemy  vs. 
- ignore the *truth* of the information 

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Recent techniques (well, mostly since 2013)

Machine learning techniques can be used to **learn better vector representation**[a] **of tokens**, and more generally of any data (document, sentence, word, image, etc.).

[a] aka embeddings

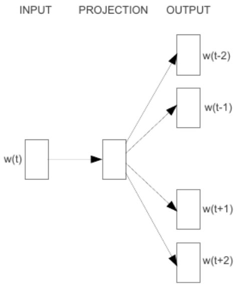
Embeddings: a general technique with many derivatives

Many models have been developed for representing various type of data. Here is a small list of freely available models:

Model	Data represented
`word2vec`	Tokens
`GloVe`	Tokens
`fastText`	Tokens
`doc2vec`	Documents
`dna2vec`	Genomic sequences

Word2vec: predict the context of a token

The core idea of word2vec is to learn a vector representation allows to predict the context of the token. Thereby, tokens appearing in similar context will be encoded closely in the vector space.



Skip-gram

[Mikolov, Tomas; et al. (2013)]

word2vec's latent semantics

The word2vec embeddings have interesting semantic features[1].

Table 8: *Examples of the word pair relationships, using the best word vectors from Table 4 (Skip-gram model trained on 783M words with 300 dimensionality).*

Relationship	Example 1	Example 2	Example 3
France - Paris	Italy: Rome	Japan: Tokyo	Florida: Tallahassee
big - bigger	small: larger	cold: colder	quick: quicker
Miami - Florida	Baltimore: Maryland	Dallas: Texas	Kona: Hawaii
Einstein - scientist	Messi: midfielder	Mozart: violinist	Picasso: painter
Sarkozy - France	Berlusconi: Italy	Merkel: Germany	Koizumi: Japan
copper - Cu	zinc: Zn	gold: Au	uranium: plutonium
Berlusconi - Silvio	Sarkozy: Nicolas	Putin: Medvedev	Obama: Barack
Microsoft - Windows	Google: Android	IBM: Linux	Apple: iPhone
Microsoft - Ballmer	Google: Yahoo	IBM: McNealy	Apple: Jobs
Japan - sushi	Germany: bratwurst	France: tapas	USA: pizza

[1] Note that `GloVe` is better at this

Dealing with the *truth*

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It is almost impossible to deal with truth judgment only from the document data.

However, we can assume that we trust information coming from *authorities* (well-known newspaper, official website, etc.).

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Idea

Rank the results of the querying system according to their authority.

How do we know who is the authority ?

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Idea

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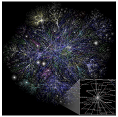
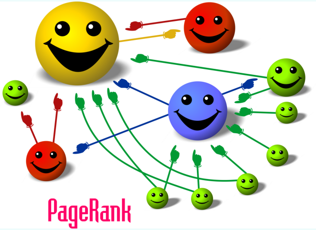
→ We extract it from the web structure  !

Authority and web structure

Who is the authority?

If you only represent the web by a graph where each node is a web page and each directed edge is an HTML link.



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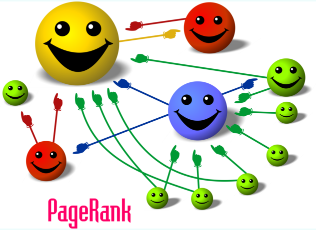

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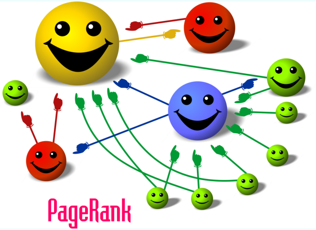

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Imagine an algorithm able to detect/rank authorities.

PageRank formalization (simple version)

Random surfer model

Imagine a user having the following behavior clicking on random links on the Internet.

The more links leading to a page, the more chance (and the more times) the user visits the page.

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The more links leading to a page, the more chance (and the more times) the user visits the page.
After a loooong time, we measure the average number of times the user visited a given page P, we denote R_P.

Definition of the rank according to PageRank

We define the authority/ranking of a page by the R_P value.

PageRank algorithm (simple version)

Data: $A :=$ graph of the WWW $\quad A_{ij} \quad = \quad \begin{cases} \frac{1}{N_j} & \text{if link from } j \text{ to } i \\ 0 & \text{else} \end{cases}$

Result: Ranking of web pages

$R_0 := S$;

repeat

$\quad R^{(i+1)} \leftarrow AR^{(i)}$

$\quad \delta \leftarrow ||R^{(i)} - R^{(i+1)}||_1$

until $\delta \leq \epsilon$;

Algorithm 1: simplified PageRank

Milestone of Google (algo designed by L. Page, Google co-founder), and drove the initial success of Google.

PageRank without sink effect

Sink effect

What if a page does not have any outgoing connection?

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The random eager surfer

Imagine the user having now the following behavior[a]

- click on a random link on the current web page with probability $p(t)$
- or jump to a random web page on the Internet with probability $1 - p(t)$

[a] In the original paper by Page, the balance between the two events is given by its trap feeling: the more trapped it gets, the more likely the user will jump somewhere else.

Full PageRank

To avoid a *sink* effect, we introduce random jumps to a set of pages encoded in E.

Data: Graph of the WWW

Result: Ranking of web pages

$R_0 := S$;

repeat

$R_{i+1} \leftarrow AR_i$

$d \leftarrow ||R_i||_1 - ||R_{i+1}||_1$

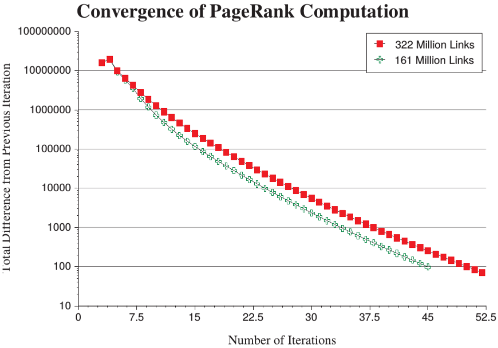
$R_{i+1} \leftarrow R_{i+1} + d.E$

$\delta \leftarrow ||R_i - R_{i+1}||_1$

until $\delta \leq \epsilon$;

Algorithm 2: PageRank

PageRank convergence



[L. Page, 98]

Full PageRank

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Note that the vector E encodes the distribution of pages where the user is willing to jump to.

6 Personalized PageRank

An important component of the PageRank calculation is E – a vector over the Web pages which is used as a source of rank to make up for the rank sinks such as cycles with no outedges (see Section 2.4). However, aside from solving the problem of rank sinks, E turns out to be a powerful parameter to adjust the page ranks. Intuitively the E vector corresponds to the distribution of web pages that a random surfer periodically jumps to. As we see below, it can be used to give broad general views of the Web or views which are focussed and personalized to a particular individual.

...

Such personalized page ranks may have a number of applications, including personal search engines. These search engines could save users a great deal of trouble by efficiently guessing a large part of their interests given simple input such as their bookmarks or home page. We show an example of this in Appendix A with the "Mitchell" query. In this example, we demonstrate that while there are many people on the web named Mitchell, the number one result is the home page of a colleague of John McCarthy named John Mitchell.

Summary

- Tf-Idf vector representation of a document
- Flexible vector queries (cosine similarity)
- Latent semantics (lower rank projection of the tf matrix)
- PageRank

Next lectures: can we make it?

- Machine learning in IR
- TP (Implemenation and experiments around IR systems)
  - Tokenizer
  - Tf-Idf matrix construction
  - Page Rank implementation
  - Mini-search engine

Information function is unique up to a $\times$ constant

Let $a \in \mathbb{R}_+$ and $p \in \mathbb{N}$.
$f(a) = f(a^{\frac{q}{q}}) = f((a^{\frac{1}{q}})^q) = q.f(a^{\frac{1}{q}})$.
So for any $p, q \in \mathbb{N}$,

$$f(a^{\frac{p}{q}}) = \frac{p}{q} f(a)$$

By density of $\mathbb{Q}$ in $\mathbb{R}$ and continuity of f, $f(a^x) = x.f(a)$.
If $f \neq 0$, there is a b such that $f(b) = 1$, so that $\forall x \in \mathbb{R}_+, f(b^x) = x$ so that $f = \log_b$

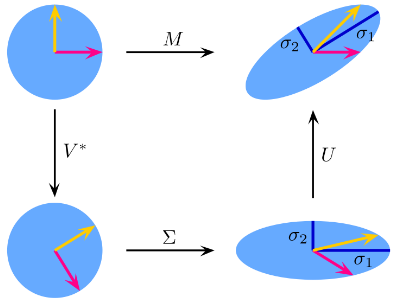
$||R_i||_1 < ||R_{i+1}||_1$ comes from sinks

If $\forall j$ there exists at least a page i and a link $j \rightarrow i$, then:

$$\begin{aligned} ||R_{i+1}||_1 &= ||A.R_i||_1 \\ &= \sum_i \sum_{j \rightarrow i} \frac{R_j}{N_j} \\ &= \quad ... \\ &= \quad 1 \end{aligned}$$

Reminders from linear algebra

We can decompose a matrix as a composition of orthogonal operation, scaling and again orthogonal operation.



$$M = U \cdot \Sigma \cdot V^*$$

This decomposition is coined the Singular Value Decomposition (SVD).

Low rank approximation of the tf-idf matrix

Eckart-Young-Mirsky Theorem

Let $M \in \mathbb{R}^{d \times t}, t < d$. If $M = U \Sigma V^\top$ is the SVD decomposition of M with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_t$, then the best[a] r-rank approximation of M is $(r < t)$:

$$\hat{M} := U_r \Sigma_{r,r} V_r^\top$$

where X_r is the restriction of X to the first r columns, and $\Sigma_{r,r}$ to the first r lines and columns.

[a] In the sense minimizing $||M - \hat{M}||_F = \sum_{i,j} (m_{i,j} - \hat{m}_{i,j})^2$