ENGR 123 Lab:

Induction Ceremony!

Purpose of the Lab
In this lab we will learn about inductive proofs and how they work.

For many students there are two stages to understanding inductive proofs. At first the method may seem simply wrong. It seems to involve the classic high schooler's logic error of assuming the statement you are trying to prove, e.g., assume a = b, and therefore a = b. When it seems like that his what is happening you are halfway there. When you understand that is not what is happening and the proof is actually rigorous, you have got it.

Outline
Core
In the core part of the lab you will follow an inductive proof, filling in steps and explaining the reasoning.

Completion
In the completion part of the lab you will use induction to prove the formula for a geometric series from ENGR 121 with some hints provided.

Challenge
In the challenge part of the lab you are asked to prove the arithmetic series formula from ENGR 121 without help.

CORE

CORE 1 (5 marks)
Based on your tutor's presentation, explain the overall logic of an inductive proof. In particular, make sure you explain the consequence of a conditional statement.

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Inductive proof is a regular proof that allows an arbitrary number (eg. n) to be used in order to prove if a statement is true.

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OK, now let's get to work.

We claim that,

$$\sum_{i=0}^{n} 2^{i} = 2^{n+1} - 1 \qquad \text{for all integers } n \geq 0$$

Or $1 + 2 + 4 + \ldots 2^{n} = 2^{n+1} - 1$

CORE 2 (5 marks)

First let's see if we believe this. Try a couple of examples: n = 1, n = 2. Does it work?
Also try for n = 0.

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N = 1:

$$\sum_{i=0}^{n} 2^{i} = 2^{n+1} - 1$$

LHS: =3 RHS = 3 so true

N = 2:

LHS = 7 RHS = 7 so true

N = 0:

LHS = 1 RHS = 1 so true

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So we now know the formula works for n = 0, 1, 2. That's a hint that it might work for all n greater than or equal to 0, but it is NOT a proof that it does.

Let's ***suppose*** the formula works for some value of n. We won't specify what this value is so long as it is at least 0. Let's call this the n formula. Next we ***suppose the n formula is correct.***

$$\sum_{i=0}^{n} 2^{i} = 2^{n+1} - 1$$

CORE 3 (5 marks)

Do we know the n formula is correct? Do we have proof?

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Without proof there is no definitive way to prove that the formula n is correct. So we need to find proof using induction

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Now we will **build the formula for n+1 from the n formula**. That's not as hard as it sounds in this particular problem, but in some problems it can be quite hard. Here what we need to do is add 2^{n+1} to each side of the n formula and do some algebra. So let's do that.

CORE 4 (5 marks)

Add 2^{n+1} to each side of the n formula. Note that we are assuming the n formula is correct, so adding the same thing to both sides is algebraically allowed.

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$$\sum_{i=0}^{n} 2^{i} + 2^{n+1} = 2^{n+1} + 2^{n+1} - 1$$

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CORE 5 (5 marks)

Simplify to get

$$\sum_{i=0}^{n+1} 2^{i} = 2^{n+1} + 2^{n+1} - 1$$

Just show the steps in the algebra

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Equivalence of (2^0 + 2^1 + 2^2 + 2^3) + 2^4

So by adding 2^n+1 to the formula you are just adding the next iteration of the sum to the value.

But increasing the range by one (n+1 rather than n) it is adding another iteration in the calculation.

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CORE 6 (5 marks)

Simplify further to get

$$\sum_{i=0}^{n+1} 2^{i} = 2^{n+2} - 1$$

Again just show the steps in the algebra to get there.

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Add the similar elements (2^n+1 + 2^2+1)

- 2*2^n+1
- 2^n+2

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CORE 7 (5 marks)

Now look carefully at your formula.

$$\sum_{i=0}^{n+1} 2^{i} = 2^{n+2} - 1$$

Suppose you just replace n with n+1 in the original formula. What do you get? Can you see that we have built up the n+1 formula from the n formula?

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$$\sum_{i=0}^{n+1} 2^{i} = 2^{n+1+1} - 1$$

Will prove true as it just does another iteration on both sides (treats n=2 as n=3 as n+1)

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Now let's understand what we have done.

Have we proven the n+1 formula works? Not quite yet. Let's summarize.

- We supposed the n formula is correct.
- We then built up the n+1 formula by just adding 2^{n+1} to each side.

So …

CORE 8 (5 marks)

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We now know that the formula is correct for ____n_____ ***if*** the formula is correct for _____n+1_____. But that's not worth much if we do not know that the n formula is correct.

Going back to our example from the presentation, we want to know whether the lab is on today. We use the timetable to prove the lab is on if it's Tuesday. Great, but we only know the lab is on today if we know it is Tuesday.

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But now the base case (n=0) helps us. We already showed the formula worlds for $n = 0$.

CORE 9 (5 marks)

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We know the formula works for n = ____0____ (base case) because we _______tested it________.

But if it works for n =0 then it works for n+1 = _____.

But if it works for _____ then it works for _____.

And so on. So now what have we proved?

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Congratulations. You have been inducted into the Hall of Provers.

COMPLETION

Recall from ENGR 121 the formula for a geometric series with $r \neq 1$ and integer $k \geqslant 1$

$$a + ar + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

We called this sum S_k.

$$S_k = a + ar + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

We will now follow the method we used in core.

COMPLETION 1 (10 marks)
Check a couple of values. Let's set r = 0.5 and k = 3, 4, 1 again. Does the formula seem to work? What is your base case?
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The base case is 1 as k => 1 so k cannot be less than 1.

S1 = (a1 (1 - 0.5^1) / (1 - 0.5)) = a1 (0.5)/(0.5) = **a1**. For the base case of n = 1.
S3 = (a3 (1 - 0.5^3) / (0.5)) = a3(0.125)/0.5) = a3(0.25)
S4 = (a4 (1 - 0.5^4) / (0.5)) = a4(0.0625/0.5) = a4(0.125)
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COMPLETION 2 (10 marks)
Now we **suppose** the k formula is correct and build up the k+1 formula. To do this add ar^k to both sides. Notice that the left side is the expression for S_{k+1}.
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Sk+1 = Sk + ak+1

$$= \frac{a_1(1-r^k)}{1-r} + a_1 r^{k+1-1}$$

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COMPLETION 3 (10 marks)
As in core, work on the right side. Get a common denominator and simplify. After a few steps you should get

$$S_{k+1}=a+ar+\ldots+ar^{k}=\frac{a\left(1-r^{k+1}\right)}{1-r}$$

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$$=\frac{a_1(1-r^k)}{1-r}+a_1r^{k+1-1}$$

$$=\frac{a_1(1-r^k)}{1-r}+\frac{a_1r^k(1-r)}{1-r}$$

$$=\frac{a_1(1-r^k)+a_1r^k(1-r)}{1-r}$$

$$=\frac{a_1\left[1-r^k+r^k(1-r)\right]}{1-r}$$

$$=\frac{a_1\left[1-r^k+r^k-r^{k+1}\right]}{1-r}$$

$$=\frac{a_1\left[1-r^{k+1}\right]}{1-r}$$

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COMPLETION 4 (10 marks)

Note your formula for S_{k+1} is exactly the same as the formula for S_k with k replaced by k+1. So...

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The formula works for ___Sk +1 ___ if it works for __Sk___.

But we know the formula works for k = __1___ from the _______base case_______.

So if it works for k = __2___ then it works for k = __3___ and then for k = ___4__ and so on. So it works for all k ___+1___ .

We're done.

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CHALLENGE (15 marks)

Recall the formula for the arithmetic series from ENGR 121

$$S_k=\sum_{i=0}^{k-1}a+id=\frac{k}{2}\left(2a+(k-1)d\right)$$

Prove this using induction.

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