CURS 9

Sisteme dinamice generate de sisteme planare de ecuații dif. autonome

$x = x(t)$
$y = y(t)$

$$(1)\quad \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$$

Fluxul generat de sistemul (1).

$$(2)\quad \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \qquad \eta = (\eta_1, \eta_2) \in \mathbb{R}^2.$$

<u>Teorema 1</u>. Dacă $f = (f_1, f_2) \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ atunci problema Cauchy (2) are o unică soluție saturată pt $\forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^2$.

unica sol. a probl. (2)

$$x(t, \eta) = x(t, \eta_1, \eta_2)$$

$$y(t, \eta) = y(t, \eta_1, \eta_2)$$

$$x(\cdot, \eta), y(\cdot, \eta) : I_\eta \to \mathbb{R}.$$

sol. saturată $\Rightarrow$ I_η interval maximal.

$$I_\eta = (\alpha_\eta, \beta_\eta)$$

$$0 \in I_\eta \Rightarrow \alpha_\eta < 0 < \beta_\eta$$

$$W = \{ I_\eta \times \{\eta\} \mid \eta \in \mathbb{R}^2 \}$$

$\varphi : W \to \mathbb{R}^2$

$$\varphi(t,\eta) = \varphi(t,\eta_1,\eta_2) = \left(x(t,\eta), y(t,\eta)\right).$$

φ – fluxul generat de sist. (1).

<u>Obs</u>. Dacă $I_\eta = \mathbb{R}$, $\forall \eta \in \mathbb{R}^2 \to W = \mathbb{R}\times\mathbb{R}^2 = \mathbb{R}^3$

operatorul : $\eta \mapsto \varphi(t,\eta)$ sist. dinamic generat de (1).

<u>Proprietățile fluxului:</u>

1. $\varphi(0,\eta) = \eta$, $\forall \eta \in \mathbb{R}^2$
2. $\varphi(t+s,\eta) = \varphi(t, \varphi(s,\eta))$ $\forall \eta \in \mathbb{R}^2, \forall t,s \in I_\eta$.
3. φ este <u>continuă</u>.

Def. $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$

$$\gamma^+(\eta) = \bigcup_{t \in [0, \beta_\eta)} \varphi(t, \eta)$$ orbita poz. a lui η.

$$\gamma^-(\eta) = \bigcup_{t \in (\alpha_\eta, 0]} \varphi(t, \eta)$$ orbita negativă a lui η.

$$\gamma(\eta) = \gamma^+(\eta) \cup \gamma^-(\eta)$$ orbita lui η.

Portret fazic: Reuniunea tuturor orbitelor împreună cu sensul de parcurgere al acestora.

Exemplu:
$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

$$x' = y \Rightarrow \begin{cases} x'' = y' \\ y' = -x \end{cases} \Rightarrow x'' = -x \Rightarrow \boxed{x'' + x = 0}$$

$r^2+1=0$ ec. caract

$r_{1,2}=\pm i$ $\begin{cases} \nearrow x_1(t)=e^{0.t}\cdot\cos(1.t)=\cos t \\ \searrow x_2(t)=e^{at}\cdot\sin(1.t)=\sin t \end{cases}$

$\alpha=0,\ \beta=1$

$\Rightarrow \boxed{x(t)=c_1\cos t+c_2\sin t,\ c_1,c_2\in\mathbb{R}}$

$y=x' \Rightarrow \boxed{y(t)=-c_1\sin t+c_2\cos t,\ c_1,c_2\in\mathbb{R}}$

$$\begin{cases} x'=y \\ y'=-x \\ x(0)=\eta_1 \\ y(0)=\eta_2 \end{cases}$$

$(\eta_1,\eta_2)\in\mathbb{R}^2$

$x(0)=\eta_1 \Rightarrow c_1=\eta_1$

$y(0)=\eta_2 \Rightarrow c_2=\eta_2$

$\Rightarrow$ $x(t,\eta_1,\eta_2)=\eta_1\cos t+\eta_2\sin t$

$y(t,\eta_1,\eta_2)=-\eta_1\sin t+\eta_2\cos t$

$\Rightarrow I_\eta=\mathbb{R},\ \forall\eta\in\mathbb{R}^2$ intervalul maximal.

$$\varphi(t,\underbrace{\eta_1,\eta_2}_{\eta}) = (x(t,\eta_1,\eta_2), y(t,\eta_1,\eta_2)) =$$

$$= (\eta_1\cos t + \eta_2 \sin t, -\eta_1 \sin t + \eta_2 \cos t)$$

$\varphi : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

fluxul generat de sistem.

<u>Orbite</u>:

1. $(\eta_1,\eta_2) = (0,0) \rightarrow$

$$\Rightarrow \varphi(t,0,0) = (0,0)$$

$$\gamma(0,0) = \bigcup_{t\in\mathbb{R}} \varphi(t,0,0) = \{(0,0)\}$$

2. $(\eta_1,\eta_2) \neq (0,0)$

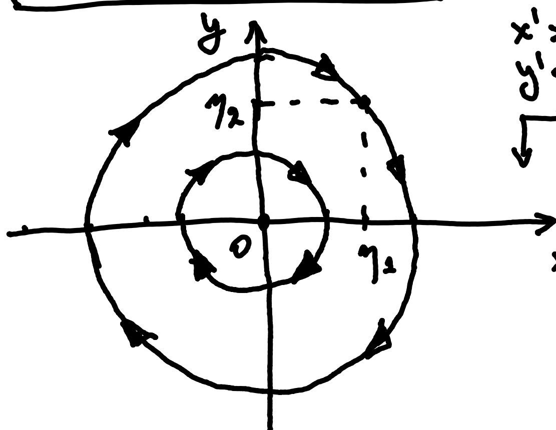
$$\gamma(\eta_1,\eta_2) = \bigcup_{t\in\mathbb{R}} \varphi(t,\eta_1,\eta_2) = \bigcup_{t\in\mathbb{R}} (\underbrace{\eta_1\cos t + \eta_2\sin t}_{x}, \underbrace{-\eta_1 \sin t + \eta_2 \cos t}_{y})$$

$$\begin{cases} x = \eta_1\cos t + \eta_2 \sin t \\ y = -\eta_1 \sin t + \eta_2\cos t \end{cases}, \quad t\in\mathbb{R}$$

$$x^2+y^2 = \eta_1^2\cos^2 t + \cancel{2\eta_1\eta_2\cos t\sin t} + \eta_2^2\sin^2 t + \eta_1^2 \sin^2 t - \cancel{2\eta_1\eta_2 \sin t\cos t} + \eta_2^2\cos^2 t =$$

$$= \eta_1^2(\underbrace{\cos^2 t+\sin^2 t}_{=1}) + \eta_2^2(\underbrace{\sin^2 t+\cos^2 t}_{=1})$$

$$\Rightarrow \boxed{x^2+y^2=\eta_1^2+\eta_2^2}$$



$x'>0$
$y'<0$

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

$$\begin{matrix} x>0 \\ y>0 \end{matrix} \Rightarrow \begin{matrix} x'>0 \\ y'<0 \end{matrix} \quad \begin{matrix} x \nearrow \\ y \searrow \end{matrix}$$

portretul fazic.

$$\begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad \begin{aligned} x' &= \frac{dx}{dt} = f_1(x,y) \\ y' &= \frac{dy}{dt} = f_2(x,y) \end{aligned} \Rightarrow$$

$$\Rightarrow \boxed{\frac{dx}{dy} = \frac{f_1(x,y)}{f_2(x,y)}}$$ ecuația diferențială a orbitelor

$$\downarrow$$

$$x'(y)$$

sau $$\boxed{\underbrace{\frac{dy}{dx}}_{\displaystyle\downarrow\; y'(x)} = \frac{f_2(x,y)}{f_1(x,y)}}$$

În cazul în care ecuația dif. a orbitelor este una rezolvabilă atunci se pot obține orbitele prin rezolvarea acesteia fără a rezolva sist. inițial.

In cazul exemplului

$f_1(x,y) = y$ $\qquad \frac{dx}{dy} = \frac{y}{-x} \Rightarrow y\,dy = -x\,dx \,/\cdot 2.$

$f_2(x,y) = -x$

$\Rightarrow \int 2y\,dy = \int -2x\,dx \quad \Rightarrow y^2 = -x^2 + c, \; c \in \mathbb{R}.$

$\Rightarrow \boxed{x^2 + y^2 = c, \; c \in \mathbb{R}}$

orbitele sunt cercuri centrate în $(0,0)$ și de rază $\sqrt{c}$ (c depinde de $x(0), y(0)$)

Soluții de echilibru

O soluție constantă a sist. (1) se numește sol. de echilibru

$\begin{cases} x(t) \equiv x^* \\ y(t) \equiv y^* \end{cases}$ sol. de echil. $\qquad \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases}$

punctul $X^*(x^*, y^*)$ — punct de echilibru.

punctele de echilibru $X^*(x^*,y^*)$ sunt soluții **reale** ale sist.

$$\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$$

$(X^*(x^*,y^*)\in\mathbb{R}^2)$

Def. Fie $X^*(x^*,y^*)\in\mathbb{R}^2$ un pct de echil. a sist. (1).

Spunem că $X^*(x^*,y^*)$ este:

a) local stabil $\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0$ a.î. dacă $\|\eta-X^*\|<\delta$ atunci $\|\varphi(t,\eta)-X^*\|<\varepsilon,\ \forall t\geq 0$, unde φ fluxul generat (1), $\|\cdot\|$ o normă din $\mathbb{R}^2$

b) local asimptotic stabil dacă este local stabil și în plus $\|\varphi(t,\eta)-X^*\|\xrightarrow[t\to+\infty]{}0$.

c) instabil dacă nu este local stabil.

Cazul sist. liniare

$$(3)\ \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \boxed{X' = A \cdot X.}$$

$X^*(0,0)$ este pct. de echil. pt sist. (3)

sol. sist. (3) $X = e^{t \cdot A} \cdot C$, $C \in \mathbb{R}^2$

$$\begin{cases} X' = AX \\ X(0) = \eta \end{cases}, \ \eta \in \mathbb{R}^2 \Rightarrow \underline{X(t,\eta) = e^{tA} \cdot \eta} \text{ sol. pr. Cauchy}$$

Teorema 2 (Criteriul de stabilitate pt. sist. liniare).

Fie sist. liniar (3)

Punctul de echilibru $(0,0)$ este:

a) local stabil $\Leftrightarrow$ $\operatorname{Re}\lambda \leq 0$, $\forall \lambda$ val. proprie a matricii A,
egalitatea cu 0 are loc pt. val. proprii simple

b) asimptotic stabil $\Leftrightarrow$ $\operatorname{Re}\lambda < 0$, $\forall \lambda$ val. proprie a matricii A.

c) instabil $\Leftrightarrow$ nu are loc (a).

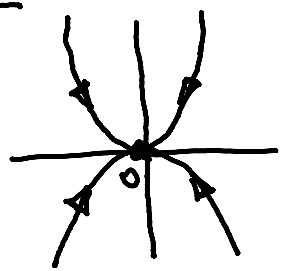
Ec. caract. $\underline{\det(\lambda I_2 - A) = 0} \Leftrightarrow$

$$\Leftrightarrow \boxed{\lambda^2 - tr(A)\cdot\lambda + \det A = 0}$$

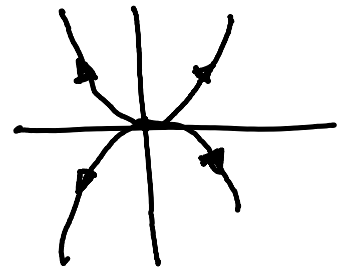
Clasificarea punctului de echilibru (0,0)

Spunem că (0,0) este:

- de tip nod dacă $\lambda_1, \lambda_2 \in \mathbb{R}$ cu $\lambda_1 \cdot \lambda_2 > 0$

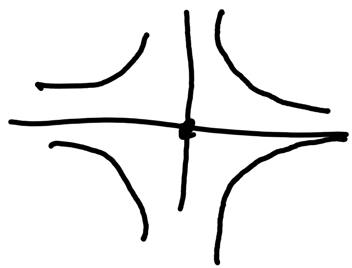


$\lambda_1 < 0$, $\lambda_2 < 0$
nod as. stab.



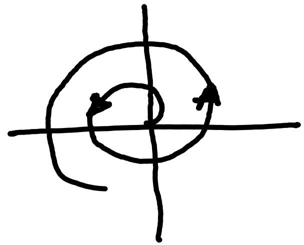
$\lambda_1 > 0$, $\lambda_2 > 0$
nod instab.

- de tip șa dacă $\lambda_1, \lambda_2 \in \mathbb{R}$ cu $\lambda_1 \cdot \lambda_2 < 0$



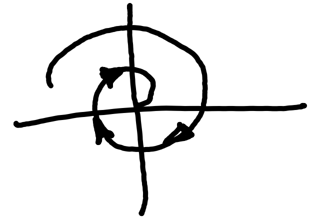
instabil.

– de tip focus (focal) dacă $\lambda_{1,2} = \alpha \pm i\beta \in \mathbb{C}$, $\alpha \neq 0$.



$\alpha > 0$

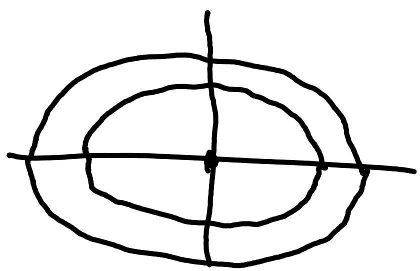
focus instabil.



$\alpha < 0$

focus stabil.

– de tip centru dacă $\lambda_{1,2} = \pm i\beta \in \mathbb{C}$



local stabil.

Exemple :

1) $\begin{cases} x' = y \\ y' = -x \end{cases}$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

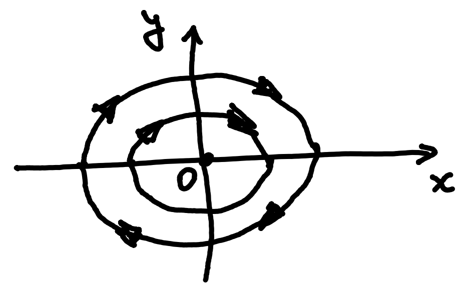
$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i$

$Re(\lambda_{1,2}) = 0$

$\lambda_{1,2}$ simple



portret. fazic

$\rightarrow (0,0)$ local stabil

de tip centru

2) $\begin{cases} x' = x \\ y' = 2y \end{cases}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 2 \end{vmatrix} = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda_1 = 1$

$\lambda_2 = 2$

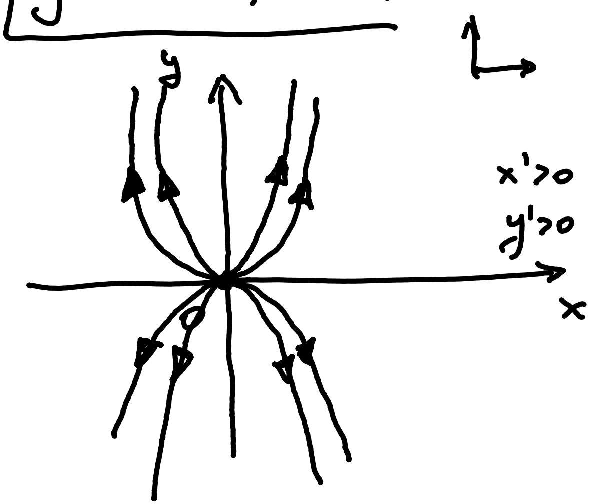
$\Rightarrow (0,0)$ este instabil de tip nod.

$\frac{dx}{dy} = \frac{x}{2y}$ ec. dif. a orbitelor.

$\int \frac{dy}{y} = \int \frac{2dx}{x}$

$\ln y = 2 \ln x + \ln c$

$\boxed{y = c \cdot x^2, c \in \mathbb{R}}$



$x' > 0$

$y' > 0$

3) $\begin{cases} x' = -x \\ y' = y \end{cases}$

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda+1)(\lambda-1) = 0$$

$$\lambda_1 = -1$$

$$\lambda_2 = 1$$

$$\operatorname{Re}\lambda_2 = \lambda_2 > 0$$

$\Rightarrow$ $(0,0)$ este instabil de tip şa

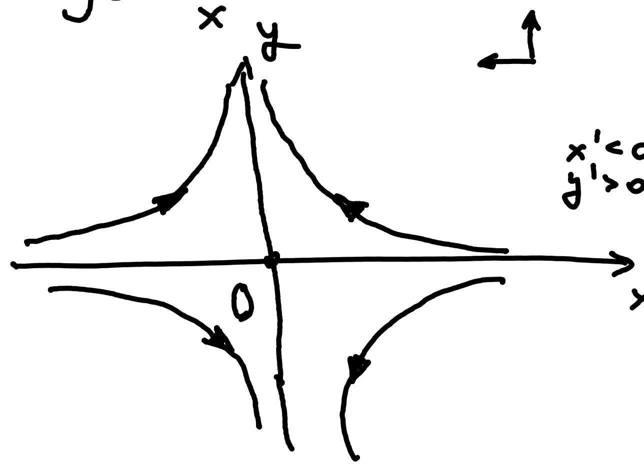
$$\frac{dx}{dy} = \frac{-x}{y}$$ ec. dif. a orbitelor

$$\int \frac{dy}{y} = \int -\frac{dx}{x}$$

$$\ln y = -\ln x + \ln c$$

$$y = c \cdot x^{-1}$$

$$y = \frac{c}{x}$$



Cazul neliniar

$$(1)\quad \begin{cases} x' = f_1(x,y) \\ y' = f_2(x,y) \end{cases} \qquad X^*(x^*, y^*) \text{ pct. de echil.}$$

$$f_1(x,y) \simeq \underbrace{f_1(x^*,y^*)}_{=0} + \frac{\partial f_1}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial f_1}{\partial y}(x^*,y^*)(y-y^*)$$

$$f_2(x,y) \simeq \underbrace{f_2(x^*,y^*)}_{=0} + \frac{\partial f_2}{\partial x}(x^*,y^*)(x-x^*) + \frac{\partial f_2}{\partial y}(x^*,y^*)(y-y^*)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \simeq \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x^*,y^*) & \frac{\partial f_1}{\partial y}(x^*,y^*) \\ \frac{\partial f_2}{\partial x}(x^*,y^*) & \frac{\partial f_2}{\partial y}(x^*,y^*) \end{pmatrix} \begin{pmatrix} x-x^* \\ y-y^* \end{pmatrix}$$

$$Y = X - X^* \iff \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x-x^* \\ y-y^* \end{pmatrix}$$

$$\Rightarrow \boxed{Y' = J_f(X^*)\cdot Y} \text{ sist. liniarizat în } X^*(x^*, y^*).$$

Teorema 3 (Criteriul de stabilitate în primă aproximație)

Fie sistemul (1) și $X^*(x^*, y^*)$ un punct de echilibru

a) Dacă $Re\,\lambda < 0$, $\forall \lambda$ val. proprie a $J_f(x^*, y^*) \Rightarrow$

$\Rightarrow X^*(x^*, y^*)$ este local asimptotic stabil.

b) Dacă $\exists \lambda$ cu $Re\,\lambda > 0$, λ val. proprie a matricii $J_f(x^*, y^*) \Rightarrow X^*(x^*, y^*)$ este instabil.