3 a) $\dim S + \dim T = \dim(S+T) + \dim(S\cap T)$

$S\cap T = \{0\} \Rightarrow \dim(S\cap T) = 0$ și are baza vidă

$\{y_1,\dots,y_n\}$ bază în S

$\{z_1,\dots,z_m\}$ bază în T

Arătăm că $\{y_1,\dots,y_n, z_1,\dots,z_m\}$ lin. ind.

Fie $\alpha_1,\dots,\alpha_n, \beta_1,\dots,\beta_m \in K$ a.î. $\alpha_1 y_1 + \dots + \alpha_n y_n + \beta_1 z_1 + \dots + \beta_m z_m = 0$

$$\left.\begin{array}{l}\underbrace{\alpha_1 y_1 + \dots + \alpha_n y_n}_{\in S} = -\underbrace{(\beta_1 z_1 + \dots + \beta_m z_m)}_{\in T} \in S\cap T \\ S\cap T = \{0\}\end{array}\right\} \Rightarrow$$

$\alpha_1 y_1 + \dots + \alpha_n y_n = 0 = +(\beta_1 z_1 + \dots + \beta_m z_m)$

$y_1,\dots,y_n$ l. ind. $\Downarrow$ $\qquad\qquad \Downarrow$ $z_1,\dots,z_m$ lin. ind.

$\alpha_1 = \dots = \alpha_n = 0 \qquad \beta_1 = \dots = \beta_m = 0$

Gen: $y_1,\dots,y_n, z_1,\dots,z_m \in S\cup T \subseteq \langle S\cup T\rangle = S+T$

Fie $x \in S+T \Rightarrow x = s+t$, $s\in S$, $t \in T$

$s \in S = \langle y_1,\dots,y_n\rangle \Rightarrow \exists\, \alpha_1,\dots,\alpha_n \in K : s = \alpha_1 y_1 + \dots + \alpha_n y_n$

$t \in T = \langle z_1,\dots,z_m\rangle \Rightarrow \exists\, \beta_1,\dots,\beta_m \in K : t = \beta_1 z_1 + \dots + \beta_m z_m$ $\quad +$

$x = s+t = \alpha_1 y_1 + \dots + \alpha_n y_n + \beta_1 z_1 + \dots + \beta_m z_m \in \langle y_1,\dots,y_n, z_1,\dots,z_m\rangle \Rightarrow S+T \subseteq \langle y_1,\dots,y_n, z_1,\dots,z_m\rangle$

$S+T = \langle y_1,\dots,y_n, z_1,\dots,z_m\rangle$ (generarea)

b) $S = \langle s_1, s_2, s_3, s_4\rangle \qquad T = \langle t_1, t_2, t_3\rangle$

$\dim S = \dim\langle s_1,s_2,s_3,s_4\rangle = \operatorname{rang}\begin{bmatrix} s_1\\ s_2\\ s_3\\ s_4\end{bmatrix} = \operatorname{rang}\begin{bmatrix} s_1\\ s_2\\ s_3\\ s_4\end{bmatrix}_e \qquad e =$ baza canonică din $\mathbb{R}^4$

$$\begin{bmatrix} s_1\\ s_2\\ s_3\\ s_4\end{bmatrix}_e = \begin{pmatrix} 1 & -1 & 3 & 1\\ 2 & 0 & 5 & 3\\ 1 & 1 & 2 & 2\\ 1 & 1 & 1 & 3\end{pmatrix} \xrightarrow[L_4\leftarrow L_4 - L_1]{\substack{L_2\leftarrow L_2-2L_1\\ L_3\leftarrow L_3-L_1}} \begin{pmatrix} 1 & -1 & 3 & 1\\ 0 & 2 & -1 & 1\\ 0 & 0 & -1 & 1\\ 0 & 2 & -2 & 2\end{pmatrix} \xrightarrow{L_4\leftarrow L_4-L_2} \begin{pmatrix} 1 & -1 & 3 & 1\\ 0 & 2 & -1 & 1\\ 0 & 0 & -1 & 1\\ 0 & 0 & -1 & 1\end{pmatrix} \xrightarrow{L_4\leftarrow L_4-L_3} \begin{pmatrix} 1 & -1 & 3 & 1\\ 0 & 2 & -1 & 1\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{bmatrix} s_1\\ s_2-2s_1\\ s_3-s_1\\ s_4-s_1\end{bmatrix}_e \qquad \begin{bmatrix} s_1\\ s_2-2s_1\\ s_3-s_1\\ (s_4-s_1)-(s_2-2s_1)\end{bmatrix}_e \qquad \begin{bmatrix} s_1\\ s_1-2s_2\\ s_3-s_1\\ s_1-s_2+s_4-s_3+s_1\end{bmatrix}_e$$

$(s_4-s_1)-(s_2-2s_1) = s_1 - s_2 + s_4$; $\quad s_1-s_2+s_4-s_3+s_1 = 0$

$2s_1 - s_2 - s_3 + s_4 = 0 \Rightarrow s_4 = -2s_1 + s_2 + s_3 \qquad S = \langle s_1, s_2, s_3, \cancel{s_4}\rangle$ lin. ind.

$\dim S = 3$

$\begin{Bmatrix} s_1\\ s_2\\ s_3\end{Bmatrix}$ bază în S.

$$\dim T = \dim \langle t_1, t_2, t_3 \rangle = \text{rang} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \text{rang} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}_e$$

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}_e = \begin{pmatrix} 1 & -3 & 3 & 0 \\ 1 & -1 & 4 & 1 \\ 1 & 1 & 5 & 2 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_3 - L_1]{L_2 \leftarrow L_2 - L_1} \begin{pmatrix} 1 & -3 & 3 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 2 \end{pmatrix} \xrightarrow{L_3 \leftarrow L_3 - 2L_2} \begin{pmatrix} \boxed{\begin{matrix} 1 & -3 \\ 0 & 2 \end{matrix}} & \begin{matrix} 3 & 0 \\ 1 & 1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix}$$

minor nenul!

$\dim T = 2$ şi $\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ bază în T

$$S+T = \langle S \cup T \rangle = \langle s_1, s_2, s_3, t_1, t_2 \rangle \Rightarrow \dim(S+T) = \text{rang} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ t_1 \\ t_2 \end{bmatrix} = \text{rang} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ t_1 \\ t_2 \end{bmatrix}_e$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ t_1 \\ t_2 \end{bmatrix}_e = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 2 & 0 & 5 & 3 \\ 1 & -1 & 2 & 2 \\ 1 & -3 & 3 & 0 \\ 1 & -1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} s_1 \\ s_2 - 2s_1 \\ s_3 - s_1 \\ t_1 - s_1 \\ t_2 - s_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} s_1 \\ s_2 - 2s_1 \\ s_3 - s_1 \\ t_1 + s_2 - 3s_1 \\ t_2 - s_1 \end{bmatrix} = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(t_1 - s_1) + (s_2 - 2s_1)$

$$\begin{matrix} s_1 \\ s_1 - 2s_2 \\ s_3 - s_1 \\ t_1 + s_2 - 3s_1 - (s_3 - s_1) \\ t_2 - s_1 + s_3 - s_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_1 - 2s_2 \\ s_3 - s_1 \\ t_1 + s_2 - s_3 + 2s_1 \\ (t_2 + s_3 - 2s_1) + t_1 + s_2 - s_3 + 2s_1 \end{bmatrix}$$

$\dim S+T = 4$; bază în $S+T$ este $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ t_1 \end{bmatrix}$

$$\underset{3}{\underset{\shortparallel}{\dim S}} + \underset{2}{\underset{\shortparallel}{\dim T}} = \underset{4}{\underset{\shortparallel}{\dim(S+T)}} + \dim(S \cap T) \Rightarrow \dim(S \cap T) = 1$$

Dar $(t_2 + \cancel{s_3} - 2s_1) + t_1 + s_2 - \cancel{s_3} + 2s_1 = 0 \Rightarrow \underset{T}{\underset{\cap}{t_2 + t_1}} = \underset{S}{\underset{\uparrow}{\cancel{2s_1} - s_2 - \cancel{2s_1}}} \in S \cap T \Rightarrow$

bază în $S \cap T$ este $[t_1 + t_2] = [(2, -4, 7, 1)]$.

<u>Variantă de rezolvare</u>: Lema substituţiei. Plecăm cu baza can. e

	s_1	s_2	s_3	s_4	t_1	t_2
e_1	1	2	1	1	1	1
e_2	-1	0	-1	1	-3	-1
e_3	3	5	2	1	3	4
e_4	1	3	2	3	0	1
s_1	1	2	1	1	1	1
e_2	0	2	0	2	-2	0
e_3	0	-1	-1	-2	0	1
e_4	0	1	1	2	-1	0
s_1	1	1	0	-1	1	2
e_2	0	2	0	2	-2	0
s_3	0	1	1	2	0	-1
e_4	0	0	0	0	-2	1

	t_1	t_2	t_3
e_1	1	1	1
e_2	-3	-1	1
e_3	3	4	5
e_4	0	1	2
t_1	1	1	1
e_2	0	2	4
e_3	0	1	2
e_4	0	1	2
t_1	1	0	-1
t_2	0	1	2
e_3	0	0	0
e_4	0	0	0

t_1, t_2 lin. ind

$t_3 = -t_1 + 2t_2$

$T = \langle t_1, t_2, \cancel{t_3} \rangle$

$\Downarrow$

$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ bază în T

$\dim T = 2$.

$$\begin{array}{c|cccccc}
 & s_1 & s_2 & s_3 & s_4 & t_1 & t_2 \\ \hline
s_1 & 1 & 0 & 0 & 2 & 2 & 2 \\
s_2 & 0 & 1 & 0 & 1 & -1 & 0 \\
s_3 & 0 & 0 & 1 & 1 & 1 & -1 \\
e_4 & 0 & 0 & 0 & 0 & -2 & 1 \\ \hline
s_1 & 1 & 0 & 0 & 2 & 6 & 0 \\
s_2 & 0 & 1 & 0 & 1 & -1 & 0 \\
s_3 & 0 & 0 & 1 & 1 & -1 & 0 \\
t_2 & 0 & 0 & 0 & 0 & -2 & 1
\end{array}$$

bază în $\mathbb{R}^4$

$\left.\begin{array}{l} s_1, s_2, s_3 \text{ lin. ind.} \\ s_4 = 2s_1 + s_2 + s_3 \end{array}\right\} \Rightarrow S = \langle s_1, s_2, s_3, \cancel{s_4} \rangle$

$\Rightarrow \begin{vmatrix} s_1 \\ s_2 \\ s_3 \end{vmatrix}$ bază în S

$\dim S = 3.$

$S + T = \langle s_1, s_2, s_3, \cancel{t_1}, t_2 \rangle$

s_1, s_2, s_3, t_2 lin. ind. $\quad t_1 = 6s_1 - s_2 - s_3 - 2t_2$

$\Rightarrow \dim(S+T) = 4$ și $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ t_2 \end{bmatrix}$ bază

$\underset{\substack{\parallel \\ 3}}{\dim S} + \underset{\substack{\parallel \\ 2}}{\dim T} = \underset{\substack{\parallel \\ 4}}{\dim (S+T)} + \dim(S \cap T) \Rightarrow \dim(S\cap T) = \underline{1}$

Bază din $(S\cap T)$ ~~se~~ obține din

$t_1 = 6s_1 - s_2 - s_3 - 2t_2 \Leftrightarrow \underset{\substack{\cap \\ T}}{t_1 + 2t_2} = \underset{\substack{\uparrow \\ S}}{6s_1 - s_2 - s_3} \in S \cap T$

OK.

formă bază

Obs Undeva am greșit la calcule!

4. a) $f: V \to W$ aplic. lin $\quad \operatorname{Im} f = \{ y \in W \mid \exists x \in V : f(x) = y \}$

Exemplu: $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (0,0)$ este lin. și $\operatorname{Im} f = \{(0,0)\} \neq \mathbb{R}^2$

b) $f: \mathbb{R}^3 \to \mathbb{R}^3 \quad f(x_1,x_2,x_3) = (x_1 - x_2 + 2x_3,\ 3x_2 + x_3,\ 2x_1 + x_2 + 5x_3) = (x_1, x_2, x_3) \cdot \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}}_{A}$

$f(x) = x \cdot A$

Atunci $\alpha, \beta \in \mathbb{R}^2$, $x, y \in \mathbb{R}^3$

$\left.\begin{array}{l} f(\alpha x + \beta y) = \ldots \\ \alpha f(x) + \beta(y) = \ldots \end{array}\right\}$ sunt egale?

$f(\alpha x + \beta y) = (\alpha x + \beta y) A = \alpha(x A) + \beta (y A) = \alpha f(x) + \beta f(y) \Rightarrow f$ liniară

b) $\operatorname{Im} f = f(\mathbb{R}^3) = f(\langle e_1, e_2, e_3 \rangle) = \langle f(e_1), f(e_2), f(e_3) \rangle$

$e = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ bază can. din $\mathbb{R}^3$

$f(e_1) = f(1,0,0) = (1 \quad 0 \quad 2)$

$f(e_2) = f(0,1,0) = (-1 \quad 3 \quad 1)$

$f(e_3) = f(0,0,1) = (2 \quad 1 \quad 5)$

$\dim \operatorname{Im} f = \operatorname{rang} \begin{bmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{bmatrix} = \operatorname{rang} \begin{bmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{bmatrix}_e = \operatorname{rang} \begin{vmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \operatorname{rang} A$

$\begin{vmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{vmatrix}_e = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \sim \begin{vmatrix} f(e_1) \\ f(e_1) + f(e_2) \\ f(e_3) - 2f(e_1) \end{vmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} f(e_1) \\ f(e_3) - 2f(e_1) \\ f(e_1) + f(e_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix}$

$$\sim \begin{bmatrix} f(e_1) \\ f(e_3)-2f(e_1) \\ f(e_1)+f(e_2)-3[f(e_3)-2f(e_1)] \end{bmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim \operatorname{Im} f = 2$$

$\begin{bmatrix} f(e_1) \\ f(e_3) \end{bmatrix}$ bază.

$$\underbrace{\dim \mathbb{R}^3}_{3} = \dim \operatorname{Ker} f + \underbrace{\dim \operatorname{Im} f}_{2} \Rightarrow \dim \operatorname{Ker} f = 1$$

Dar $f(e_1)+f(e_2)-3f(e_3)+6f(e_1)=0 \iff$

$7f(e_1)+f(e_2)-3f(e_3)=0$

$f(7e_1+e_2-3e_3)=0 \Rightarrow 7e_1+e_2-3e_3=(7,1,-3)\in \operatorname{Ker} f$

$[(7,1,-3)]$ bază.

<u>Variantă</u>

$$\operatorname{Im} f = f(\mathbb{R}^3) = f(\langle e_1,e_2,e_3\rangle) = \langle f(e_1), f(e_3), f(e_3)\rangle$$

$e=\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ baza can. din $\mathbb{R}^3$

$f(e_1)=f(1,0,0)=(1\ \ 0\ \ 2)$

$f(e_2)=f(0,1,0)=(-1\ \ 3\ \ 1)$

$f(e_3)=f(0,0,1)=(2\ \ 1\ \ 5)$

		$f(e_1)$	$f(e_2)$	$f(e_3)$
	e_1	1	-1	2
	e_2	0	3	1
	e_3	2	1	5
	$f(e_1)$	1	-1	2
	e_2	0	3	1
	e_3	0	3	1
bază $\mathbb{R}^3$	$f(e_1)$	1	-7	0
	$f(e_3)$	0	3	1
	e_3	0	0	0

$f(e_1), f(e_3)$ lin. ind.

$f(e_2)=-7f(e_1)+3f(e_3)$

$\operatorname{Im} f = \langle f(e_1), \cancel{f(e_2)}, f(e_3)\rangle$

$\Rightarrow \begin{bmatrix} f(e_1) \\ f(e_3) \end{bmatrix}$ bază în $\operatorname{Im} f$

$\dim \operatorname{Im} f = 2$

$$\underbrace{\dim \mathbb{R}^3}_{3} = \dim \operatorname{Ker} f + \underbrace{\dim \operatorname{Im} f}_{2} \Rightarrow \dim \operatorname{Ker} f = 1$$

$f(e_2)=-7f(e_1)+3f(e_3) \Rightarrow 7f(e_1)+f(e_2)-3f(e_3)=0$

$f(7e_1+e_2-3e_3)=0$

$[7e_1+e_2-3e_3]$ bază în <u>$\operatorname{Ker} f$</u>.

c) $[f]_{e,e}=?$ $\quad [f]_{b,b}=?$ $\quad$ b este baza de la 2 c.

$$[f]_{e,e} = \begin{bmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{bmatrix}_e = A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}; \qquad [f]_{b,b} = [b]_e \cdot [f]_{e,e} \cdot [b]_e^{-1}$$

$$[b]_e = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_e = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -2 & 5 \\ -2 & 1 & -1 \end{pmatrix}$$

$$[f]_{b,b} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -2 & 5 \\ -2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 1 & -2 & 5 \\ -2 & 1 & -1 \end{pmatrix}^{-1} = \dots$$

$$[f]_{b,e} = [b]_e \cdot [f]_{e,e} \underbrace{[e]_e}_{I_3} = [b]_e \cdot [f]_{e,e}$$

$$[f]_{e,b} = \underbrace{[e]_e}_{I_3} \cdot [f]_{e,e} \cdot [b]_e^{-1} = [f]_{e,e} \cdot [b]_e^{-1}.$$

1. a) V, W sunt sp. vect peste un corp K, $f: V \to W$ funcție

f se numește aplic. liniară dacă $f(x+y) = f(x) + f(y)$ & $f(\alpha x) = \alpha f(x)$, $\forall x, y \in V$, $\forall \alpha \in K$

Exemplu $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 0$

b) $U \subseteq V$.

$(i) \Rightarrow (ii)$ $a, b \in K$, $x, y \in U \overset{(i)}{\Rightarrow} ax, by \in U \overset{(i)}{\Rightarrow} ax + by \in U$.

$(ii) \Rightarrow (i)$. $x, y \in U \Rightarrow x + y = 1 \cdot x + 1 \cdot y \overset{(ii)}{\in} U$

$a \in K$, $x \in U \Rightarrow ax = ax + 0 \cdot y \overset{(ii)}{\in} U$.

c) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 + x_2 - 5x_3 = 0\}$

- $0 = (0,0,0) \in S$ pt. că $2 \cdot 0 + 0 - 5 \cdot 0 = 0$ (OK)
- $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in S \overset{?}{\Rightarrow} \alpha x + \beta y \in S$
  $\alpha, \beta \in \mathbb{R}$

Din $x, y \in S \Rightarrow 2x_1 + x_2 - 5x_3 = 0 = 2y_1 + y_2 - 5y_3$

$\alpha x + \beta y = \alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3) = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \in \mathbb{R}^3$

$2(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) - 5(\alpha x_3 + \beta y_3) = \alpha(\underbrace{2x_1 + x_2 - 5x_3}_{0}) + \beta(\underbrace{2y_1 + y_2 - 5y_3}_{0}) = 0 \Rightarrow$

$\alpha x + \beta y \in S$. Deci $S \leq \mathbb{R}^3$.

Rezolvăm sist. cu 3 nec. și o singură ec. $2x_1 + x_2 - 5x_3 = 0$

$$\begin{cases} x_1 = \alpha \\ x_2 = 5x_3 - 2x_1 = 5\beta - 2\alpha \\ x_3 = \beta \end{cases} \qquad \alpha, \beta \in \mathbb{R}.$$

$S = \{(\alpha, 5\beta - \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\} = \{\alpha(1, -1, 0) + \beta(0, 5, 1) \mid \alpha, \beta \in \mathbb{R}\} =$

$\langle \overset{\alpha_1}{(1, -1, 0)}, \overset{\alpha_2}{(0, 5, 1)} \rangle$

Dar $\text{rang}\begin{pmatrix}1 & -1 & 0\\ 0 & 5 & 1\end{pmatrix} = 2 \Rightarrow$ lin. ind. $\Rightarrow$ bază. ($\dim S = 2$).

$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$.

2) a) O listă de vectori $\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix}$ s.n. lin. ind. dacă $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$, $\alpha_1, \dots, \alpha_n \in K$ implică $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Ex. $\begin{bmatrix}e_1 = (1,0,0)\\ e_2 = (0,1,0)\end{bmatrix}$ lin. ind. în $\mathbb{R}^3$; sau $\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}$ de la 1 c.

b) $b = \begin{bmatrix}b_1\\ b_2\\ b_3\end{bmatrix}$ $b_1, b_2, b_3 \in V$.

"$\Rightarrow$" b bază $\Rightarrow$ lin. ind. și $V = \langle b \rangle$. Deci dacă $v \in V \Rightarrow \exists\, \alpha_1, \alpha_2, \alpha_3 \in K$ a.î.

$v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 \Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \underset{0}{\overset{\neq}{(-1)}} v = 0 \Rightarrow \begin{bmatrix}b_1\\ b_2\\ b_3\\ v\end{bmatrix}$ lin. dep.

"$\Leftarrow$" b lin. ind. Este clar că $\langle b_1, b_2, b_3 \rangle \leq V$.

Dacă $v \in V \Rightarrow \begin{bmatrix}b_1\\ b_2\\ b_3\\ v\end{bmatrix}$ lin. dep. $\Rightarrow \exists\, \alpha_1, \alpha_2, \alpha_3, \alpha \in K$ nu toți nuli a.î.

$\alpha^{*}/$ $\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \alpha v = 0$

Dacă presup. $\alpha = 0 \Rightarrow \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = 0$ cu $\alpha_1, \alpha_2, \alpha_3$ nu toți nuli $\Rightarrow b$ lin. dep. abs.

Deci $\alpha \neq 0 \Rightarrow \exists\, \alpha^{-1} \in K \Rightarrow v = 1 \cdot v = \alpha^{-1} \alpha v = \underset{K}{(-\alpha^{-1}\alpha_1)} b_1 + \underset{K}{(-\alpha^{-1}\alpha_2)} b_2 + \underset{K}{(-\alpha^{-1}\alpha_3)} b_3 \in \langle b_1, b_2, b_3 \rangle$

Deci $V \leq \langle b_1, b_2, b_3 \rangle$ de unde $V = \langle b_1, b_2, b_3 \rangle \Rightarrow$ gener. $\Rightarrow$ bază.

c) $[b]_e$, e – baza can.

b bază $\Leftrightarrow [b]_e$ invers. $\Leftrightarrow \det [b]_e \neq 0$

$[b]_e = \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}_e = \begin{pmatrix}1 & 3 & 2\\ 1 & -2 & 5\\ -2 & 1 & -1\end{pmatrix}$ se calc. $\det [b]_e$ ($\neq 0 \Rightarrow$ bază).

$[x]_b = (x_1, x_2, x_3) \Leftrightarrow x = x_1 b_1 + x_2 b_2 + x_3 b_3$

$(0,1,3) = x_1(1\ 3\ 2) + x_2(1\ -2\ 5) + x_3(-2\ 1\ -1) \Leftrightarrow$

$$\begin{cases} x_1 + x_2 - 2x_3 = 0\\ 3x_1 - 2x_2 + x_3 = 1 \quad | \cdot 2\\ 2x_1 + 5x_2 - x_3 = 3.\end{cases}$$

$$\begin{cases} 5x_1 + 3x_2 = 4\\ 7x_1 - 3x_2 = 2\end{cases} \quad +$$

$12x_1 = 6 \Rightarrow x_1 = \frac{1}{2}$

$3x_2 = 4 - 5x_1 = 4 - 5 \cdot \frac{1}{2} = \frac{8-5}{2} = \frac{3}{2}$

$x_2 = \frac{1}{2}$

$x_3 = 1 - 3x_1 + 2x_2 = 1 - \frac{3}{2} + \frac{2}{2} = \frac{4-3}{2} = \frac{1}{2}$

$\boxtimes$.

Variantă. Lema substituției

e = baza can. în $\mathbb{R}^3$

		b_1	b_2	b_3	x
$e =$	e_1	1	1	-2	0
	e_2	3	-2	1	1
	e_3	2	5	-1	3
	b_1	1	1	-2	0
	e_2	0	-5	7	1
	e_3	0	3	3	3
	b_1	1	3	0	2
	e_2	0	-12	0	-6
	b_3	0	1	1	1
bază	b_1	0	0	0	$1/2$
	b_2	0	1	0	$1/2$
	b_3	1	0	1	$1/2$

b

bază

$$\frac{(-12)\cdot 2 - 3\cdot(-6)}{-12} = \frac{-24+18}{-12} = \frac{(-6)(4-3)}{(-6)\cdot 2} = \frac{1}{2}$$

$$\frac{-12+6}{-12} =$$

$$x = \frac{1}{2}\cdot b_1 + \frac{1}{2}\cdot b_2 + \frac{1}{2} b_2 \Rightarrow [x]_b = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \boxtimes.$$