CS 3320 – Numerical Software
Module 4 Homework
Solution

1. Find the Taylor's Series of degree 2 centered at the point $x = 0$ for the following functions:
    a. (5 pts) $f(x) = \cos(5x)$
    b. (5 pts) $f(x) = \frac{1}{x+1}$

Solution:
Part a:
$f'(x) = -5\sin(5x)$
$f''(x) = -25\cos(5x)$
$f(0) = \cos(0) = 1; f'(0) = -5\sin(0) = 0; f''(0) = -25\cos(0) = -25$
Degree 2 Taylor's Series:
$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + 0 - \frac{25}{2}x^2$$
Or
$f(x) \approx 1 - 12.5x^2$

Part b.
$$f'(x) = -\frac{1}{(x+1)^2}, and\ f''(x) = \frac{2}{(x+1)^3}$$
$$f(0) = \frac{1}{0+1} = 1, f'(0) = -\frac{1}{(0+1)^2} = -1, f''(0) = \frac{2}{(0+1)^3} = 2$$
Degree 2 Taylor's Series:
$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 - x + \frac{2}{2}x^2$$
Or
$f(x) \approx 1 - x + x^2$

2. Do the following problems.
    a. (5 pts) Find the Taylor's Series of degree 4 for $f(x) = x^{-2}$ centered at the point $x = 1$.
    b. (5 pts) Use the result of (a) to approximate $f(0.9)$ and $f(1.1)$.
    c. (5 pts) Use the Taylor's Remainder to find an error formula for the Taylor's Series found in (a).
    d. (5 pts) Give error upper bounds for each of the two approximations made in part (b).

Solution:

a. The first five derivatives of $f(x) = x^{-2}$ are:
$f'(x) = -2x^{-3}, f''(x) = 6x^{-4}, f'''(x) = -24x^{-5}, f^{(4)}(x) = 120x^{-6},$
$and\ f^{(5)}(x) = -720x^{-7}$

Degree 4 Taylor's Series centered at $x = 1$:

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$f(x) \approx 1 - 2(x-1) + \frac{6}{2}(x-1)^2 - \frac{24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4$$

Or

$$f(x) \approx 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4$$

b. $f(0.9) = 1 - 2(0.9-1) + 3(0.9-1)^2 - 4(0.9-1)^3 + 5(0.9-1)^4$
$f(0.9) = 1 - 2 \cdot (-0.1) + 3 \cdot (0.01) - 4 \cdot (-0.001) + 5 \cdot (0.0001) = 1.2345$
$f(1.1) = 1 - 2(1.1-1) + 3(1.1-1)^2 - 4(1.1-1)^3 + 5(1.1-1)^4$
$f(1.1) = 1 - 2 \cdot (0.1) + 3 \cdot (0.01) - 4 \cdot (0.001) + 5 \cdot (0.0001) = 0.8265$

c. Taylor's Remainder:

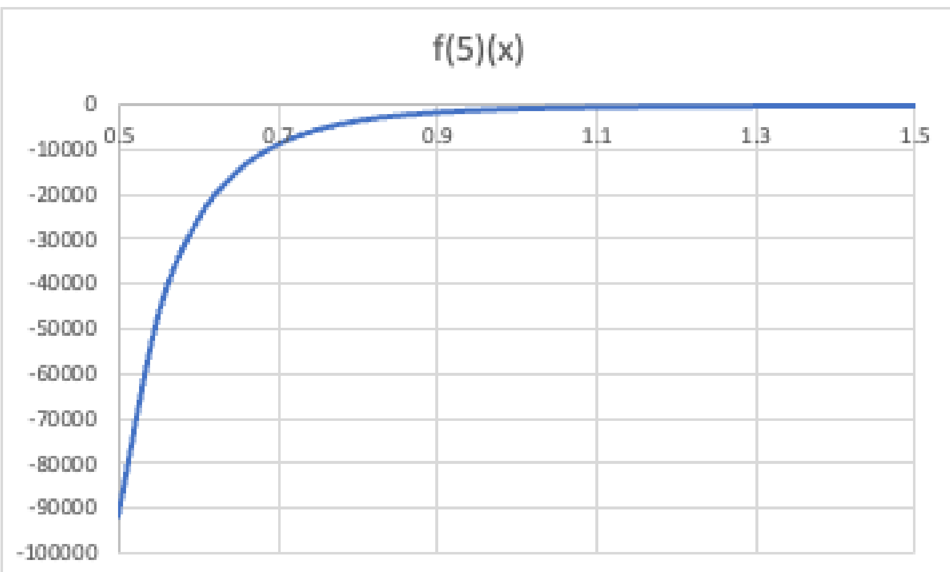
$$R_n = f^{(n+1)}(z)\frac{(x-a)^{n+1}}{(n+1)!} \; where \; a \le z \le x$$

For this problem,

$$R_4 = f^{(5)}(z)\frac{(x-1)^5}{5!} \; where \; 1 \le z \le x$$

$$f^5(x) = -720x^{-7}$$

This function is shown in the plot below from $x = 0.5$ to $x = 1.5$.



d. To find the upper bound of the error for each approximation, we need to find z in the appropriate interval the gives the largest absolute value of the fifth derivative.

Error bound for the estimate at $x = 0.9$:
In the interval from $x = 0.9$ to $x = 1$, the fifth derivative has the largest absolute value at $x = 0.9$. Therefore, the upper bound of error for this approximation is

evaluated with $z = 0.9$, where the largest magnitude occurs, and $x = 0.9$, where the function was evaluated.

$$\left|-720 \cdot 0.9^{-7} \cdot \frac{(0.9 - 1)^5}{5!}\right| = 0.0001254$$

Error bound for the estimate at $x = 1.1$:
In the interval from $x = 1.0$ to $x = 1.1$, the fifth derivative has the largest absolute value at $x = 1.0$. Therefore, the upper bound of error for this approximation is evaluated with $z = 1.0$, where the largest magnitude occurs, and $x = 1.1$, where the function was evaluated.

$$\left|-720 \cdot 1.0^{-7} \cdot \frac{(1.1 - 1)^5}{5!}\right| = 0.00006$$