CS 3320
Final
Review Sheet and Practice Problem

1. Use Lagrange interpolation to find a polynomial that passes through the following points: (1, 4), (2, 7), (3, 20).

   Lagrange Interpolation Polynomial:
   $$L(x) = 4\frac{(x-2)(x-3)}{(1-2)(1-3)} + 7\frac{(x-1)(x-3)}{(2-1)(2-3)} + 20\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

   $$L(x) = 2(x-2)(x-3) - 7(x-1)(x-3) + 10(x-1)(x-2)$$

2. Use Newton Divided Differences to find the Newton Interpolation Polynomial passes through (0, 1), (1, -5), (2, -9), and (3, -5)

   Newton Interpolation Polynomial:
   $$f(x) = a + bx + cx(x-1) + dx(x-1)(x-2)$$

   We will use divided differences to find $a, b, c, and\ d$.

Divided Differences				
x	y			
0	1			
		$\frac{-5-1}{1-0} = -6$		
1	-5		$\frac{-4-(-6)}{2-0} = 1$	
		$\frac{-9-(-5)}{2-1} = -4$		$\frac{4-1}{3-0} = 1$
2	-9		$\frac{4-(-4)}{3-1} = 4$	
		$\frac{-5-(-9)}{3-2} = 4$		
3	-5			

$$a = 1, b = -6, c = 1, and\ d = 1$$

$$f(x) = 1 - 6x + x(x-1) + x(x-1)(x-2)$$

3. Given four data points (0, 2), (1, 3), (2, 1), and (3, 12), the least squares second order polynomial fit is $y = 2.5x^2 - 4.7x + 2.8$. Find the SS_E of the solution.

   $SS_E = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

In this case, $y = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 12 \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2.5 \\ -4.7 \\ 2.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0.6 \\ 3.4 \\ 11.2 \end{bmatrix}$

$$SS_E = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = (2-2.8)^2 + (3-0.6)^2 + (1-3.4)^2 + (12-11.2)^2$$
$$SS_E = (-0.8)^2 + (2.4)^2 + (-2.4)^2 + (0.8)^2 = 12.8$$

4. Find the first three random numbers (to the fifth digits after the decimal point) using the Linear Congruent Generator with $a = 3, m = 17, b = 0$, and 11 as the seed.

   LCG: $x_i = (ax_{i-1} + b) \; mod \; m$ and $u_i = \frac{x_i}{m}$.

   In this problem, $a = 3, b = 0, m = 17, and \; x_0 = 11$.

   #1: $x_1 = (3 * 11 + 0) \; mod \; 17 = 33 \; mod \; 17 = 16$ and $u_1 = \frac{16}{17} = 0.94118$

   #2: $x_2 = (3 * 16 + 0) \; mod \; 17 = 48 \; mod \; 17 = 14$ and $u_2 = \frac{14}{17} = 0.82353$

   #3: $x_3 = (3 * 14 + 0) \; mod \; 17 = 42 \; mod \; 17 = 8$ and $u_3 = \frac{8}{17} = 0.47059$

5. Fit the data in the following table to the exponential model, $y = e^{(x-b)/a}$. In other words, find a and b to the fourth place after the decimal point.

x	1	2	3	4	5
y	0.5	2	2.9	3.5	4

Take the log on both sides of the model equation:
$$\ln(y) = \ln e^{(x-b)/a}$$
$$\ln(y) = \frac{x-b}{a} = \frac{1}{a}x - \frac{b}{a}$$

x	1	2	3	4	5
$\ln(y)$	-0.69315	0.69314	1.06471	1.27276	1.38629

Fit a straight line through x and $ln(y)$: $\ln(y) = cx + d$
$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} -0.69315 \\ 0.69314 \\ 1.06471 \\ 1.27276 \\ 1.38629 \end{bmatrix}$$
Solve the normal equation: $c = 0.4718 \; and \; d = -0.6748$.
Or $a = \frac{1}{c} = 2.1193 \; and \; b = -ad = 1.4301$.
$$y = e^{\frac{x-1.4301}{2.1193}}$$

6. Find the area under the function $f(x) = xe^{-x/2}$ from a=1 to b=4 using Simpson's 3/8 rule with $h = 0.5$. (Report your answer to the fourth place after the decimal point.)
With $h = 0.5$, we will be using the composite Simpson's 3/8 rule.

x	1	1.5	2	2.5	3	3.5	4
$f(x)$	0.6065	0.7085	0.7358	0.7163	0.6694	0.6082	0.5413
Multiplier	1	3	3	2	3	3	1
$m * f(x)$	0.6065	2.1255	2.2074	1.4326	2.0082	1.8246	0.5413

$$\int_1^4 xe^{-x/2}\,dx \approx \frac{3\cdot 0.5}{8}\sum m_i f(x_i) = \frac{1.5}{8}10.7461 = 2.0149$$

7. Given the following system of equations, what is the approximated solution after two iterations using Gauss-Seidel method with $x = 1, y = 0, and\ z = 2$ as the initial guess. (Make sure the algorithm with converge to the solution before you start the iterative process. Report your answer to the fourth place after the decimal point or in fraction.)
$x - 2y + 4z = -4$
$3x - y - z = 6$
$2x + 5y - 2z = 11$

We need to rearrange the rows to satisfy the diagonal dominance criteria.
$3x - y - z = 6$
$2x + 5y - 2z = 11$
$x - 2y + 4z = -4$

Iterative equations:
$x_{i+1} = \frac{1}{3}(6 + y_i + z_i)$
$y_{i+1} = \frac{1}{5}(11 - 2x_{i+1} + 2z_i)$
$z_{i+1} = \frac{1}{4}(-4 - x_{i+1} + 2y_{i+1})$

With $x_0 = 1, y_0 = 0, z_0 = 2$, first iteration:

$x_1 = \frac{1}{3}(6 + 0 + 2) = \frac{8}{3} = 2.6667$
$y_1 = \frac{1}{5}\left(11 - 2\cdot\frac{8}{3} + 2\cdot 2\right) = \frac{1}{5}\left(\frac{33-16+12}{3}\right) = \frac{29}{15} = 1.9333$
$z_1 = \frac{1}{4}\left(-4 - \frac{8}{3} + 2\cdot\frac{29}{15}\right) = \frac{1}{4}\left(\frac{-60-40+58}{15}\right) = \frac{1}{4}\cdot\frac{-42}{15} = -0.7$

Second iteration:
$x_2 = \frac{1}{3}\left(6 + \frac{29}{15} - \frac{7}{10}\right) = \frac{1}{3}\cdot\frac{180+58-21}{30} = \frac{217}{90} = 2.4111$
$y_2 = \frac{1}{5}\left(11 - 2\cdot\frac{217}{90} - 2\cdot\frac{7}{10}\right) = \frac{1}{5}\left(\frac{990-434-126}{90}\right) = \frac{430}{450} = 0.9556$
$z_2 = \frac{1}{4}\left(-4 - \frac{217}{90} + 2\cdot\frac{430}{450}\right) = \frac{1}{4}\left(\frac{-1800-1085+860}{450}\right) = \frac{1}{4}\cdot\frac{-2025}{450} = -\frac{1}{4}\cdot\frac{9}{2} = -1.125$

$x \approx 2.4111, y \approx 0.9556, z \approx -1.125$