CS 3320 – Numerical Software

Module 9 Homework

1. The following system of equations is designed to determine the concentrations (c's in g/m^3) in a series of coupled, well-mixed tanks as a function of mass input to each tank. The right-hand side of the equations below represent these inputs in g/day.
   $15c_1 - 3c_2 - c_3 = 4000$
   $-3c_1 + 18c_2 - 6c_3 = 1200$
   $-4c_1 - c_2 + 12c_3 = 2350$
   a. (5 pt) Determine the inverse of the coefficient matrix. (You can use Python to solve the inverse.)
   b. (10 pt) Use the inverse to determine the solution. (Do this by hand.)
   c. (10 pt) Determine how much the rate of mass input to tank 3 must increase to induce a 10 g/m^3 rise in the concentration in tank 1.

   Solution:
   a. The following Python code can be used to find the inverse:
```
import numpy as np
A=np.matrix('15, -3, -1;-3, 18, -6;-4,-1,12')

AInv = np.linalg.inv(A)
AInv
matrix([[0.07253886, 0.01278066, 0.01243523],
        [0.02072539, 0.06079447, 0.03212435],
        [0.02590674, 0.00932642, 0.09015544]])
```
   b. Solution
   $$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0.07253886 & 0.01278066 & 0.01243523 \\ 0.02072539 & 0.06079447 & 0.03212435 \\ 0.02590674 & 0.00932642 & 0.09015544 \end{bmatrix} \begin{bmatrix} 4000 \\ 1200 \\ 2350 \end{bmatrix}$$
   $$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 334.7150 \\ 231.3472 \\ 326.6839 \end{bmatrix}$$
   c. The relation between c_1 and the input to tank 3, I_3, is determined by AInv[1,3]=0.01243, or $\Delta c_1 = 0.012435 \cdot \Delta I_3$.
   $$\Delta I_3 = \frac{10}{0.012435} = 804.2 \frac{g}{day}$$

2. (10 pt) Determine $\|A\|_f$, $\|A\|_1$, and $\|A\|_\infty$ for
   $$A = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}.$$

   Solution: The textbook mentioned to scale each row by the largest number in magnitude of the row. If you do the scaling, then

$$A' = \begin{bmatrix} .8 & .2 & -1 \\ -1 & 1/9 & 1/3 \\ 1 & -1/15 & 2/5 \end{bmatrix}$$

$\|A\|_f = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}^2} = 1.9920$
$\|A\|_1 = 2.8$ (Maximum sum of absolute value for each column)
$\|A\|_\infty = 2$ (Maximum sum of absolute value for each row)

If you didn't scale the matrix:

$$\|A\|_f = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}^2} = 22.8254$$

$\|A\|_1 = 32$ (Maximum sum of absolute value for each column)
$\|A\|_\infty = 22$ (Maximum sum of absolute value for each row)

3. (10 pt) Solve the following system using three iterations of the Gauss-Seidel method. If necessary, rearrange the equations. Show all the steps in your solution. At the end of your computation, compute the true error of your final results. (Do this by hand.)
$3x_1 + 8x_2 = 11$
$7x_1 - x_2 = 5$
Solution:
Swap row to ensure diagonal dominance. The iterative equations are:

$$x_1 = \frac{5 + x_2}{7}$$

$$x_2 = \frac{11 - 3x_1}{8}$$

Use $x_2 = 0$ as the initial guess:

Iteration 1:
$x_1 = \frac{5}{7} = 0.7143$ and $x_2 = \frac{11-3\cdot 0.7143}{8} = 1.0714$

Iteration 2:
$x_1 = \frac{5+1.0714}{7} = 0.8725$ and $x_2 = \frac{11-3\cdot 0.8725}{8} = 1.0478$

Iteration 3:
$x_1 = \frac{5+1.0478}{7} = 0.8640$ and $x_2 = \frac{11-3\cdot 0.8640}{8} = 1.051$

Relative errors from iteration 2 to iteration 3:
$\left|\frac{0.8725-0.8640}{0.8640}\right| \approx 0.010 = 1.0\%$; $\left|\frac{1.0478-1.051}{1.051}\right| \approx 0.003 = 0.3\%$

True relative errors:

Exact solution: $x_1 \cong 0.86441$ and $x_2 \cong 1.05085$
$\left|\frac{0.8640-0.86441}{0.86441}\right| \approx 0.00047 = 0.047\%$; $\left|\frac{1.051-1.05085}{1.05085}\right| \approx 0.00014 = 0.014\%$

4. (10 pt) Use the Gauss-Seidel method (a) without relaxation and (b) with relaxation ($\lambda = 1.2$) to solve the following set of linear equation to meet an error tolerance of $\varepsilon_s = 5\%$. If necessary, rearrange the equations to achieve convergence.
$2x_1 - 6x_2 - x_3 = -38$
$-3x_1 - x_2 + 7x_3 = -34$
$-8x_1 + x_2 - 2x_3 = -20$

We need to swap the rows for diagonal dominance:
$-8x_1 + x_2 - 2x_3 = -20$
$2x_1 - 6x_2 - x_3 = -38$
$-3x_1 - x_2 + 7x_3 = -34$

The Gauss-Seidel form of the equations is:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8}$$
$$x_2 = \frac{-38 - 2x_1 + x_3}{-6}$$
$$x_3 = \frac{-34 + 3x_1 + x_2}{7}$$

a. Results are tabulated below.

Iteration	x1	x2	x3	ea		
0	0	0	0			
1	2.500	7.167	-2.762	100%	100%	100%
2	4.086	8.156	-1.941	39%	12%	42%
3	4.005	7.992	-1.999	2.0%	2.1%	2.9%
4	3.999	7.999	-2.001	0.1%	0.1%	0.1%

b. With relaxation:

lambda	1.2								
Iteration	x1	x1r	x2	x2r	x3	x3r	ea		
0	0	0	0	0	0	0			
1	2.500	3.000	7.333	8.800	-2.314	-2.777	100%	100%	100%
2	4.294	4.553	8.314	8.217	-1.732	-1.523	34%	7%	82%
3	3.908	3.779	7.847	7.773	-2.127	-2.248	20%	6%	32%
4	4.034	4.085	8.070	8.129	-1.945	-1.885	7%	4%	19%
5	3.987	3.968	7.970	7.938	-2.023	-2.050	2.9%	2.4%	8.1%
6	4.005	4.012	8.012	8.027	-1.991	-1.979	1.1%	1.1%	3.6%

Note: case a converges more quickly.