CS 3320 – Numerical Software
Module 11 Homework

1. (10 pt) Use a 5th order interpolating polynomial to estimate y at $x = 3.5$. In your answer, please show the interpolating polynomial. You can use Python library to find the interpolating polynomial.

x	0	1.8	5	6	8.2	9.2
y	2.6	16.415	5.375	3.5	2.015	2.54

Python script:

```
>>>import numpy as np
>>>x=np.array([0, 1.8, 5, 6, 8.2, 9.2])
>>>y=np.array([2.6, 16.415, 5.375, 3.5, 2.015, 2.54])
>>>pFit = np.polyfit(x,y,5)
>>>pFit
   array([ 5.74407918e-03, -1.73471191e-01,  1.99870979e+00, -
1.04592064e+01,
           2.09771368e+01,  2.60000000e+00])
>>>np.polyval(pFit, 3.5)
   10.574759742841737
```

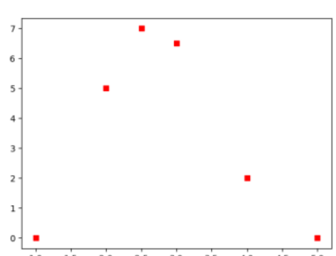
The 5th order interpolating polynomial is:
$\hat{y} = 5.744 \cdot 10^{-3} x^5 - 0.1735x^4 + 1.9987x^3 - 10.4592x^2 + 20.977x + 2.6$
$\hat{y}(3.5) = 10.5748$

2. (10 pt) Given the data:

x	1	2	2.5	3	4	5
y=f(x)	0	5	7	6.5	2	0

a. Plot the data and estimate $f(3.4)$ from the plot. (You can use excel to plot the data.)
b. Calculate $f(3.4)$ using Newton Interpolating polynomial of order 1. (Do this by hand. Also, you only need two data point for order 1 interpolation. Make sure you pick the right points)
c. Calculate $f(3.4)$ using Newton Interpolating polynomial of order 5. (Do this by hand. Use finite divided differences to find the coefficients for each Newton interpolating polynomial.)

a. See plot below. Base on the plot $f(3.4) \approx 4.0$



b. If a 1st order polynomial is used, the best choice will be using (3, 6.5) and (4,2). The Newton Interpolating polynomial will be

$$f_1(x) = 6.5 + \frac{2 - 6.5}{4 - 3}(x - 3) = 6.5 - 4.5(x - 3)$$

$$f_1(3.4) = 6.5 - 4.5(3.4 - 3) = 6.5 - 1.8 = 4.7$$

c. A 5th order polynomial will use all the points: The coefficients for the Newton Interpolating polynomial will be calculated as below:

x	y					
1	0					
		5				
2	5		-0.66667			
		4		-2.1666		
2.5	7		-5		1.16667	
		-1		1.3333		-0.2833
3	6.5		-2.3333		0.03333	
		-4.5		1.4333		
4	2		1.25			
		-2				
5	0					

The 5th order Newton Polynomial is

$$f_5(x) = 0 + 5(x - 1) - 0.6667(x - 1)(x - 2) - 2.1666(x - 1)(x - 2)(x - 2.5)$$
$$+1.1667(x - 1)(x - 2)(x - 2.5)(x - 3)$$
$$-0.2833(x - 1)(x - 2)(x - 2.5)(x - 3)(x - 4)$$

$$f_5(3.4) = 0 + 5(3.4 - 1) - 0.6667(3.4 - 1)(3.4 - 2)$$
$$-2.1666(3.4 - 1)(3.4 - 2)(3.4 - 2.5)$$
$$+1.1667(3.4 - 1)(3.4 - 2)(3.4 - 2.5)(3.4 - 3)$$
$$-0.2833(3.4 - 1)(3.4 - 2)(3.4 - 2.5)(3.4 - 3)(3.4 - 4) = 4.8248$$

3. (10 pt) Given the data:

x	1	2	3	5	6
y=f(x)	4.75	4	5.25	19.75	36

Estimate $f(4)$ using Lagrange polynomials of order one and three. (Do this by hand.)

Order 1 will be using (3, 5.25) and (5, 19.75)

$$f_1(x) = \frac{x - 5}{3 - 5}5.25 + \frac{x - 3}{5 - 3}19.75$$

$$f_1(4) = \frac{4 - 5}{3 - 5}5.25 + \frac{4 - 3}{5 - 3}19.75 = 0.5 * 5.25 + 0.5 * 19.75 = 12.5$$

Order 3 will be using: (2,4), (3, 5.25), (5, 19.75), (6, 36)

$$f_3(x) = 4\frac{(x - 3)(x - 5)(x - 6)}{(2 - 3)(2 - 5)(2 - 6)} + 5.25\frac{(x - 2)(x - 5)(x - 6)}{(3 - 2)(3 - 5)(3 - 6)}$$
$$+19.75\frac{(x - 2)(x - 3)(x - 6)}{(5 - 2)(5 - 3)(5 - 6)} + 36\frac{(x - 2)(x - 3)(x - 5)}{(6 - 2)(6 - 3)(6 - 5)}$$

$$f_3(4) = 4\frac{(4-3)(4-5)(4-6)}{(2-3)(2-5)(2-6)} + 5.25\frac{(4-2)(4-5)(4-6)}{(3-2)(3-5)(3-6)}$$
$$+19.75\frac{(4-2)(4-3)(4-6)}{(5-2)(5-3)(5-6)} + 36\frac{(4-2)(4-3)(4-5)}{(6-2)(6-3)(6-5)}$$

$$f_3(4) = -4\tfrac{2}{12} + 5.25\tfrac{4}{6} + 19.75\tfrac{2}{3} - 36\tfrac{1}{6}=10.0$$

4. (10 pt) Given the data.

x	1	2	2.5	3	4	5
y	1	5	7	8	2	1

Fit these data with (a) cubic splines with natural end conditions, (b) cubic splines with not-a-knot end conditions. Present comparative plots of 50 equally spaced interpolation points over the domain $1 \le x \le 5$. You may use Python for this problem. However, make sure you understand the definition of cubic spline and the various end conditions.

Set up for the splines:
$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \ \forall 1 \le i \le 6$

For example: $S_1(x) = a_i + b_i(x-1) + c_i(x-1)^2 + d_i(x-1)^3$
End point conditions:
$S_i(x_i) = a_i + b_i(x_i - x_i) + c_i(x_i - x_i)^2 + d_i(x_i - x_i)^3 = a_i = y_i$
$S_i(x_{i+1}) = a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 =$
$y_i + b_i\delta_i + c_i(\delta_i)^2 + d_i(\delta_i)^3 = y_{i+1}$
Or $b_i\delta_i + c_i(\delta_i)^2 + d_i(\delta_i)^3 = \Delta_i$

1st derivatives:
$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$
$b_i + 2c_i(x_{i+1} - x_i) + 3d_i(x_{i+1} - x_i)^2$
$= b_{i+1} + 2c_{i+1}(x_{i+1} - x_{i+1}) + 3d_{i+1}(x_{i+1} - x_{i+1})^2$
$b_i - b_{i+1} + 2c_i(\delta_i) + 3d_i(\delta_i)^2 = 0$

2nd derivatives:
$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$
$2c_i + 6d_i(x_{i+1} - x_i) = 2c_{i+1} + 6d_{i+1}(x_{i+1} - x_{i+1})$
$c_i - c_{i+1} + 3d_i(\delta_i) = 0$

Form:
$\delta_1 = x_2 - x_1 = 1, \delta_2 = 0.5, \delta_3 = 0.5, \delta_4 = 1, \delta_5 = 1.$
$\Delta_1 = y_2 - y_1 = 4, \Delta_2 = 2, \Delta_3 = 1, \Delta_4 = -6, \Delta_5 = -1.$

Part a.
Following the derivation in book, solve c first with natural end condition: $c_1 = c_6 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 2 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -24 \\ 15 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.08714 \\ -0.5228 \\ -9.9958 \\ 6.2490 \\ 0 \end{bmatrix}$$

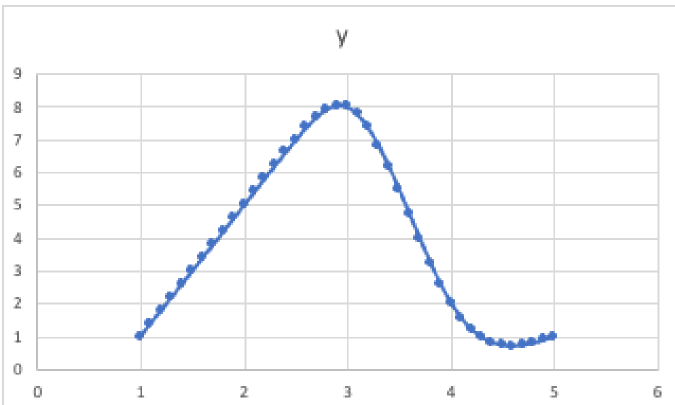
$a_1 = y_1 = 1, a_2 = 5, a_3 = 7, a_4 = 8, a_5 = 2.$

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i}$$

$d_1 = 0.02904, d_2 = -0.4066, d_3 = -6.3154, d_4 = 5.4149, d_5 = -2.08299.$

$$b_i = \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}(c_{i+1} + 2c_i)$$

$b_1 = 3.9710, b_2 = 4.0581, b_3 = 3.8402, b_4 = -1.4191, b_5 = -5.1660.$



Part b.
Following the derivation in book, solve c first with not-a-knot condition:

$\delta_2 c_1 - (\delta_1 + \delta_2)c_2 + \delta_1 c_3 = 0$ and

$\delta_{n-1} c_{n-2} - (\delta_{n-2} + \delta_{n-1})c_{n-1} + \delta_{n-2} c_n = 0$

$$\begin{bmatrix} 0.5 & -1.5 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 2 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -24 \\ 15 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0.4592 \\ -0.09184 \\ -0.3674 \\ -10.4388 \\ 7.5 \\ 0 \end{bmatrix}$$

$a_1 = y_1 = 1, a_2 = 5, a_3 = 7, a_4 = 8, a_5 = 2.$

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i}$$

$d_1 = -0.38776, d_2 = -0.38776, d_3 = -5.2857, d_4 = 3.7347, d_5 = 3.7347.$

$$b_i = \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}(c_{i+1} + 2c_i)$$

$b_1 = 3.4184, b_2 = 4.1939, b_3 = 3.7092, b_4 = -1.0306, b_5 = -7.2347.$