CS 3320 – Numerical Software
Module 12 Homework
Key

1. Evaluate the following integral.

$$\int_0^4 (1 - e^{-x})dx$$

   a. (5 pts) Analytically.
   b. (5 pts) Single application of the trapezoidal rule.
   c. (5 pts) Composite trapezoidal rule with $n = 2$ and 4.
   d. (5 pts) Single application of Simpson's 1/3 rule.
   e. (5 pts) Composite Simpson's 1/3 rule with $n = 4$.

   For each of the numerical estimates (b) through (e), determine the true percent relative error based on (a).

   Solution:

   a. $\int_0^4 (1 - e^{-x})dx = (x + e^{-x})|_0^4 = (4 + e^{-4}) - (0 + e^0) = 3.0183$
   b. Let $f(x) = 1 - e^{-x}$. $f(0) = 1 - 1 = 0$ and $f(4) = 1 - e^{-4} = 0.9817$
   $$\int_0^4 (1 - e^{-x})dx \approx \frac{4-0}{2}(0 + 0.9817) = 1.9634$$
   Relative Error $= \frac{abs(1.9634 - 3.0183)}{3.0183} = 0.3495 = 34.95\%$
   c. Composite trapezoidal with $n = 2$:
   $f(2) = 1 - e^{-2} = 0.8647$
   $$\int_0^4 (1 - e^{-x})dx \approx \frac{2}{2}(0 + 2 * 0.8647 + 0.9817) = 2.7111$$
   Relative Error $= \frac{abs(2.7111 - 3.0183)}{3.0183} = 0.1018 = \%$

   Composite trapezoidal with $n = 4$:
   $f(1) = 1 - e^{-1} = 0.6321$ and $f(3) = 1 - e^{-3} = 0.9502$
   $$\int_0^4 (1 - e^{-x})dx \approx \frac{1}{2}(0 + 2 * 0.6321 + 2 * 0.8647 + 2 * 0.9502 + 0.9817)$$
   $= 2.9379$
   Relative Error $= \frac{abs(2.9379 - 3.0183)}{3.0183} = 0.02663 = 2.663\%$

   d. Simpson's 1/3 rule:
   $$\int_0^4 (1 - e^{-x})dx \approx \frac{h}{3}\big(f(0) + 4f(2) + f(4)\big) = \frac{2}{3}(0 + 4 * 0.8647 + 0.9817)$$
   $= 2.9603$
   Relative Error $= \frac{abs(2.9603 - 3.0183)}{3.0183} = 0.01922 = 1.922\%$

   e. Composite Simpson's 1/3 rule:

$$\int_0^4 (1-e^{-x})dx \approx \frac{h}{3}\big(f(0)+4*f(1)+2*f(2)+4*f(3)+f(4)\big)$$
$$=\frac{1}{3}(0+4*0.6321+2*0.8647+4*0.9502+0.9817)=3.0134$$
Relative Error $=\frac{abs(3.0134-3.0183)}{3.0183}=0.001623=0.1623\%$

2. Determine the distance traveled from the following velocity data:

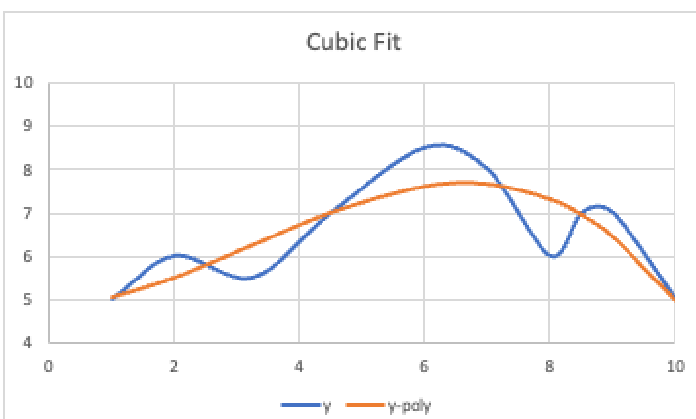
t	1	2	3.25	4.5	6	7	8	8.5	9	10
v(t)	5	6	5.5	7	8.5	8	6	7	7	5

a. (10 pts) Use the trapezoidal rule. In addition, determine the average velocity.
b. (10 pts) Fit the data with a cubic equation using polynomial regression. Integrate the polynomial to determine the distance.

(Hint: Distance traveled between t_i and t_f is $\int_{t_i}^{t_f} v(t)dt$.)

Solution:

a. Since the time intervals are not the same, we won't be able to use a single composite trapezoidal formula.
Distance traveled: $\frac{1}{2}(5+6)+\frac{1.25}{2}(6+5.5)+\frac{1.25}{2}(5.5+7)+\frac{1.5}{2}(7+8.5)$
$+\frac{1}{2}(8.5+2*8+6)+\frac{0.5}{2}(6+2*7+7)+\frac{1}{2}(7+5)=60.125$
$$v_{average}=\frac{60.125}{9}=6.6806\ m/s$$
b. Polynomial Fit: $v(t)\approx a_3t^3+a_2t^2+a_1t+a_0=y$
Normal Equation: $A^TAa=A^Ty$ where $a=\begin{bmatrix}a_3\\a_2\\a_1\\a_0\end{bmatrix}$. Solve the normal equation:
$v(t)=-0.0180x^3+0.1753x^2+0.0703x+4.8507$
Distance traveled: $\int_1^{10}(-0.0180x^3+0.1753x^2+0.0703x+4.8507)dx=$
$\left(\frac{-0.0180}{4}x^4+\frac{0.1753}{3}x^3+\frac{0.0703}{2}x^2+4.8507x)\right|_1^{10}=60.0359\ m$



3. (10 pt) The total mass of a variable density rod of variable cross-section is given by:

$$m = \int_0^L \rho(x)A(x)dx$$

where m = mass, ρ = density, A = cross-sectional area, x= the distance along the rod, and L = the total length of the rod. The following data have been measured for a 20-m rod. Determine the mass of the rod in grams using the Python functions `trapezoid` and `simpson` in `scipy.integrate`. What is the percentage difference between the two results? (Use the result from `simpson` as your base. Report your mass in kg.)

x*, *m	0	4	6	8	12	16	20
$\boldsymbol{\rho, {}^{g}/_{cm^3}}$	4.00	3.95	3.89	3.80	3.60	3.41	3.30
$\boldsymbol{A, cm^2}$	100	103	106	110	120	133	150

Python Code:
```
import numpy as np
from scipy import integrate

x=np.array([0, 4, 6, 8, 12, 16, 20])
rho=np.array([4.00, 3.95, 3.89, 3.80, 3.60, 3.41, 3.30])
area=np.array([100, 103, 106, 110, 120, 133, 150])
xCm=100*x
rhoAreaInKg = rho*area/1000
massTrap=integrate.trapezoid(rhoAreaInKg,xCm)
massSimp=integrate.simpson(rhoAreaInKg,xCm)
percentDiff = (abs(massTrap-massSimp)/massSimp)*100

print("Mass of the rod (in Kg):")
print("\tUsing trapezoidal
method:{0:12.4f}".format(massTrap))
print("\tUsing Simpson method:
{0:12.4f}".format(massSimp))
print("\nThe percent difference is
{0:8.4f}%".format(percentDiff))
```

Results:
```
Mass of the rod (in Kg):
	Using trapezoidal method:    863.1350
	Using Simpson method:        861.4652

The percent difference is   0.1938%
```