CS 3320 – Numerical Software

Module 8 Homework Key

1. Write the following set of equations in matrix form then use `solve` in `numpy.linalg` to solve for the unknowns.
   $50 = 5x_3 - 7x_2$
   $4x_2 + 7x_3 + 30 = 0$
   $x_1 - 7x_3 = 40 - 3x_2 + 5x_1$

   The given equations can be rewritten as
   $0x_1 - 7x_2 + 5x_3 = 50$
   $0x_1 + 4x_2 + 7x_3 = -30$
   $4x_1 - 3x_2 + 7x_3 = -40$

   These equations can be expressed in the matrix form as
   $$\begin{bmatrix} 0 & -7 & 5 \\ 0 & 4 & 7 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ -30 \\ -40 \end{bmatrix}$$

   The Python sequence is listed below:
```
>>> import numpy as np
>>> A=np.matrix('0, -7, 5; 0, 4, 7; 4, -3, 7')
>>> b=np.matrix('50; -30; -40')
>>> x=np.linalg.solve(A,b)
>>> x
    matrix([[-15.18115942],
            [ -7.24637681],
            [ -0.14492754]])
>>> result=np.matmul(A,x)
>>> result
    array([[ 50.],
           [-30.],
           [-40.]])
```

   $x_1 = -15.18115942, x_2 = -7.24637681, x_3 = -0.14492754$

2. Given the system equations
   $-3x_2 + 7x_3 = 4$
   $x_1 + 2x_2 - x_3 = 0$
   $5x_1 - 2x_2 = 3$

   a. Use `det` function in `numpy.linalg` to find the determinant.
   b. If the determinant is not zero, use Gaussian elimination with partial pivot to solve for x. (Do this manually. Show all the steps of computation.)
   c. Substitute your results into the original equations to verify your solution.

Key:

a. Python sequence is listed below:

```
>>> A=np.array([[0, -3, 7], [1, 2, -1], [5, -2, 0]])
>>> aDet = np.linalg.det(A)
>>> aDet
    -68.99999999999996
```

b. Gauss elimination is shown below:

Form augmented matrix: $\begin{bmatrix} 0 & -3 & 7 & 4 \\ 1 & 2 & -1 & 0 \\ 5 & -2 & 0 & 3 \end{bmatrix}$

Pivot: Exchange rows 1 and 3: $\begin{bmatrix} 5 & -2 & 0 & 3 \\ 1 & 2 & -1 & 0 \\ 0 & -3 & 7 & 4 \end{bmatrix}$

Eliminate the first column below the main diagonal: $\begin{bmatrix} 5 & -2 & 0 & 3 \\ 0 & 2.4 & -1 & -0.6 \\ 0 & -3 & 7 & 4 \end{bmatrix}$

Pivot: Exchange rows 2 and 3: $\begin{bmatrix} 5 & -2 & 0 & 3 \\ 0 & -3 & 7 & 4 \\ 0 & 2.4 & -1 & -0.6 \end{bmatrix}$

Eliminate the second column below the main diagonal: $\begin{bmatrix} 5 & -2 & 0 & 3 \\ 0 & -3 & 7 & 4 \\ 0 & 0 & 4.6 & 2.6 \end{bmatrix}$

Start backward substitution: $x_3 = \frac{2.6}{4.6} = 0.565217$

$x_2 = -\frac{1}{3}(4 - 7x_3) = -0.014493$

$x_1 = \frac{1}{5}(3 + 2x_2) = 0.594203$

c. Substitute the x's back into the equation resulted in the original right-hand side.

$-3x_2 + 7x_3 = -3 * (-0.014493) + 7 * 0.565217 = 0.043479 + 3.956521$
$= 4.0$

$x_1 + 2x_2 - x_3 = 0.594203 + 2 * (-0.014493) - 0.565217 = 0.0$
$5x_1 - 2x_2 = 5 * 0.594203 - 2 * (-0.014493) = 3.0$

3. Given the matrix $A = \begin{bmatrix} 2 & 1 & -2 \\ 4 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, find (by hand!) matrices P, L, and U which constitute the *LU-factorization* of A, such that LU = PA, where of course L is lower-triangular, U is upper triangular, and P is a permutation matrix (a row re-ordering of the identity matrix). Use these resultant matrices to solve the system $Ax = b$ for the following two right-hand sides:

$b = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$.

Do all this manually. Use a calculator if you must. Use fractions if you're manly.

Key:

Problem setup:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & -2\\4 & -1 & 2\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}$$

Pivot: Swap rows 1 and 2

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & -1 & 2\\2 & 1 & -2\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}$$

First row of U is the same as the permutated A. Also perform the elementary row operation of $E_{1,2}(-\frac{1}{2})$ and $E_{1,3}(-\frac{1}{2})$ to eliminate the first row under the main diagonal:

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & -1 & 2\\2 & 1 & -2\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\\frac{1}{2} & 1 & 0\\\frac{1}{2} & l_{32} & 1\end{bmatrix}\begin{bmatrix}4 & -1 & 2\\0 & \frac{3}{2} & -3\\0 & -\frac{1}{2} & 0\end{bmatrix}$$

On the second column, no pivot is required. The elementary row operation $E_{2,3}(\frac{1}{3})$ will eliminate the entry in U under the main diagonal on the second column:

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & -1 & 2\\2 & 1 & -2\\2 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\\frac{1}{2} & 1 & 0\\\frac{1}{2} & -\frac{1}{3} & 1\end{bmatrix}\begin{bmatrix}4 & -1 & 2\\0 & \frac{3}{2} & -3\\0 & 0 & -1\end{bmatrix}$$

Therefore:

$$P=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}, L=\begin{bmatrix}1 & 0 & 0\\\frac{1}{2} & 1 & 0\\\frac{1}{2} & -\frac{1}{3} & 1\end{bmatrix}, and\ U=\begin{bmatrix}4 & -1 & 2\\0 & \frac{3}{2} & -3\\0 & 0 & -1\end{bmatrix}.$$

Solve for $b=\begin{bmatrix}1\\5\\2\end{bmatrix}$.

$Ly=Pb$ or $\begin{bmatrix}1 & 0 & 0\\\frac{1}{2} & 1 & 0\\\frac{1}{2} & -\frac{1}{3} & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\5\\2\end{bmatrix}=\begin{bmatrix}5\\1\\2\end{bmatrix}$

Or

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}5\\(1-\frac{1}{2}\cdot 5)\\2-\frac{5}{2}+\frac{y_2}{3}\end{bmatrix}=\begin{bmatrix}5\\-\frac{3}{2}\\-1\end{bmatrix}$$

Then solve $Ux=y$.

$$\begin{bmatrix} 4 & -1 & 2 \\ 0 & \frac{3}{2} & -3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -\frac{3}{2} \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(5 + x_2 - 2) \\ \frac{2}{3}(-\frac{3}{2} + 3) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For the second b:

$$Ly = Pb \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ (0 - \frac{1}{2} \cdot 6) \\ 3 - \frac{6}{2} + \frac{y_2}{3} \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 2 \\ 0 & \frac{3}{2} & -3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(6 + x_2 - 2) \\ \frac{2}{3}(-3 + 3) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$