CS 3320 Homework Solution
Module 7 – Chapter 7
Optimization

1. (10 pts) Problem 7.2
   a. $f(x) = -x^2 + 8x - 12$
      $f'(x) = -2x + 8$
      Maximum occurs at $f'(x) = -2x + 8 = 0$ or $x = 4$.
   b. $x_1 = 0$, $x_2 = 2$, and $x_3 = 6$. Using Eq (7.10)

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2-x_1)^2[f(x_2)-f(x_3)]-(x_2-x_3)^2[f(x_2)-f(x_1)]}{(x_2-x_1)[f(x_2)-f(x_3)]-(x_2-x_3)[f(x_2)-f(x_1)]}$$

      $f(x_0) = -12$, $f(x_1) = 0$, and $f(x_2) = 0$.

$$x_4 = 2 - \frac{1}{2}\frac{(2-0)^2[0-0]-(2-6)^2[0-(-12)]}{(2-0)[0-0]-(2-6)[0-(-12)]} = 2 - \frac{1}{2}\frac{(-16\cdot 12)}{4\cdot 12} = 2 + 2 = 4$$

2. (10 pts) Problem 7.5 – Do it by hand for two iterations (not three).
   $f(x) = -1.5x^6 - 2x^4 + 12x$
   Since we are using the rules that are setup for finding a minimum, we need to change the function to $-f(x) = -(-1.5x^6 - 2x^4 + 12x) = 1.5x^6 + 2x^4 - 12x$
   Use Golden-Section search with $x_l = 0, x_u = 2$.
   $r = 0.61803$
   $d = r(x_u - x_l) = 2r = 1.23606$
   $f(x_l) = 0; f(x_u) = -104$

   Iteration 1:
   $x_1 = x_l + d = 1.23606$ and $x_2 = x_u - d = 0.76393$
   $f(x_1) = -4.81418; f(x_2) = -8.18788$
   Since $f(x_1) > f(x_2)$, we will keep the left bracket: $x_u = x_1, x_1 = x_2, x_2 = x_l + (x_u - x_1)$

   Iteration 2:
   $x_l = 0, x_u = 1.23606,\ x_1 = 0.76393, x_2 = 0 + (1.23606 - 0.76393) = 0.47214$
   $f(x_1) = -8.18788; f(x_2) = -5.54964$
   Since $f(x_1) < f(x_2)$, we will keep the right bracket for the next iteration: $x_l = x_2, x_2 = x_1, x_1 = x_u - (x_2 - x_l)$
   At the end of this iteration, the estimated minimum is -8.18788 at x=0.76393

   Tabulating the info above:

Iteration	x_l	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u
1	0	0.76393	−8.18788	1.23606	−4.81418	2
2	0	0.47214	−5.54964	0.76393	−8.18788	1.23606

3. (10 pts) Problem 7.6 – Do it by hand for two iterations (not three).
   $f(x) = -1.5x^6 - 2x^4 + 12x$
   Since the rules are designed for finding a minimum, we need to change the sign:
   $$-f(x) = -(-1.5x^6 - 2x^4 + 12x) = 1.5x^6 + 2x^4 - 12x$$
   Use Parabolic iteration with $x_1 = 0, x_2 = 1$, and $x_3 = 2$.
   Equation for the parabolic iteration:
   $$x_4 = x_2 - \frac{1}{2}\frac{(x_2-x_1)^2[f(x_2)-f(x_3)]-(x_2-x_3)^2[f(x_2)-f(x_1)]}{(x_2-x_1)[f(x_2)-f(x_3)]-(x_2-x_3)[f(x_2)-f(x_1)]}$$

   Initial evaluation:
   $$f(x_1) = 0, f(x_2) = -8.5, f(x_3) = 104$$

   Iteration 1:
   Using the formula, we have $x_4 = 0.57025, f(x_4) = -6.57991$
   x_4 falls between x_1 and x_2 and $f(x_4) \nless f(x_2)$, so $x_4 = x_1$ for the next iteration.

   Iteration 2:
   $x_1 = 0.57025, x_2 = 1$, and $x_3 = 2$.
   $x_4 = 0.81243$ and $f(x_4) = -8.44652$
   x_4 falls between x_1 and x_2 and $f(x_4) \nless f(x_2)$, so $x_4 = x_1$ for the next iteration.
   At the end of the second iteration, the estimated minimum is -8.44652 occurs at x=0.81243

   Summarize in a table form:

Ite.	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	x_4	$f(x_4)$
1	0	0	1	-8.5	2	104	0.57025	-6.57991
2	0.57025	-6.57991	1	-8.5	2	104	0.81243	-8.44652

4. (20 pts) Implement the Golden Search algorithm for finding a local minimum in a given interval. The basic algorithm is shown in Figure 7.7 in the book. However, it has some efficiencies. Python is the preferred language of this assignment. Your function should have the following signature:

```
golden(func, left, right, tol)
```

   `func`: function to find the minimum
   `left`: left end value of the interval
   `right`: right end value of the interval
   `tol`: Tolerance of the absolute relative error for ending the algorithm, i.e.,

   $$(1-r)\frac{x_{right}-x_{left}}{|x_{opt}|} < tol \text{ where } r = \frac{\sqrt{5}-1}{2}.$$

   The improvement that you will make is that you will only use r (or ϕ) once at the beginning of the algorithm to find the original shifting distance, d. Finding the subsequent distance, d, for a new interval is a matter of subtraction.

Test your algorithm on $\frac{x^2}{10} - 2\sin(x)$ with an initial interval of [0, 4] and a tolerance of machine epsilon. Report where the minimum is and the number of function evaluations in finding the minimum.

The minimum is -1.77572565314742 at x= 1.42755177280649.
Took 40 iterations with 42 function evaluation.