Certainly! Here's a step-by-step process for adaptive quadrature, along with an example:

Step 1: Choose a function to integrate.

Let's say we want to integrate the function f(x) = x^2 over the interval [0, 1].

Step 2: Choose an initial number of subintervals.

To begin, we'll choose an initial number of subintervals to use for our quadrature rule. Let's start with 4 subintervals.

Step 3: Compute the quadrature approximation for each subinterval.

Using a quadrature rule such as Simpson's rule or the midpoint rule, compute the quadrature approximation for each subinterval. In this example, we'll use Simpson's rule.

The quadrature approximation for the first subinterval, [0, 0.25], is:

(0.25 - 0)/6 * (f(0) + 4f(0.125) + f(0.25))

= 0.03125

The quadrature approximation for the second subinterval, [0.25, 0.5], is:

(0.5 - 0.25)/6 * (f(0.25) + 4f(0.375) + f(0.5))

= 0.09375

The quadrature approximation for the third subinterval, [0.5, 0.75], is:

(0.75 - 0.5)/6 * (f(0.5) + 4f(0.625) + f(0.75))

= 0.171875

The quadrature approximation for the fourth subinterval, [0.75, 1], is:

(1 - 0.75)/6 * (f(0.75) + 4f(0.875) + f(1))

= 0.2890625

Step 4: Compute the total quadrature approximation.

Add up the quadrature approximations for each subinterval to get the total quadrature approximation. In this example, the total quadrature approximation is:

0.03125 + 0.09375 + 0.171875 + 0.2890625 = 0.5859375

Step 5: Compute the error estimate.

Compute the error estimate for the quadrature approximation using Richardson extrapolation or a similar method. In this example, we'll use Richardson extrapolation with p = 4.

The error estimate is:

|(0.5859375 - 0.3333333)/15| = 0.0140625

Step 6: Check the error estimate against the desired tolerance.

Compare the error estimate to the desired tolerance. If the error estimate is smaller than the tolerance, stop and return the quadrature approximation. If not, proceed to step 7.

Step 7: Bisect the subintervals with the largest error estimate.

Identify the subintervals with the largest error estimate and bisect them. In this example, the third subinterval has the largest error estimate. We'll bisect it to create two new subintervals: [0.5, 0.625] and [0.625, 0.75].

Step 8: Repeat steps 3-7 on the new subintervals.

Compute the quadrature approximations and error estimates for the new subintervals, and compare the error estimates to the desired tolerance. If any error estimates are larger than the tolerance, bisect the corresponding subintervals and repeat the process until all error estimates are smaller than the tolerance.

That's it! By repeating steps 3-7 on smaller and smaller subintervals, adaptive quadrature is able to provide accurate