Exercice 1 :

1/ Les contraintes principales :

$$\sigma_{ij} = \begin{bmatrix} 120 & 0 & 0 \\ 0 & 40 & 10 \\ 0 & 10 & 40 \end{bmatrix}$$

$$\det(\sigma_{ij} - \lambda I)\det\begin{bmatrix} 120-\lambda & 0 & 0 \\ 0 & 40-\lambda & 10 \\ 0 & 10 & 40-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (120-\lambda)(30-\lambda)(50-\lambda) = 0$$

$$\begin{cases} \sigma_1 = \lambda_1 = 120 \\ \sigma_2 = \lambda_2 = 50 \\ \sigma_3 = \lambda_3 = 30 \end{cases}$$

2/ Déterminons la contrainte totale T :

$$T = \sigma_{ij}.n = \begin{bmatrix} 120 & 0 & 0 \\ 0 & 40 & 10 \\ 0 & 10 & 40 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} \frac{120}{\sqrt{3}} & \frac{40\sqrt{2}}{\sqrt{3}} & \frac{10\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \|T\| = \sqrt{\left(\frac{120}{\sqrt{3}}\right)^2 + \left(\frac{40\sqrt{2}}{\sqrt{3}}\right)^2 + \left(\frac{10\sqrt{2}}{\sqrt{3}}\right)^2} \approx 77\ MPa$$

La contrainte normale σ_n :

$$\sigma_n = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \end{bmatrix} . \begin{bmatrix} 120 & 0 & 0 \\ 0 & 40 & 10 \\ 0 & 10 & 40 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \\ 0 \end{bmatrix}$$

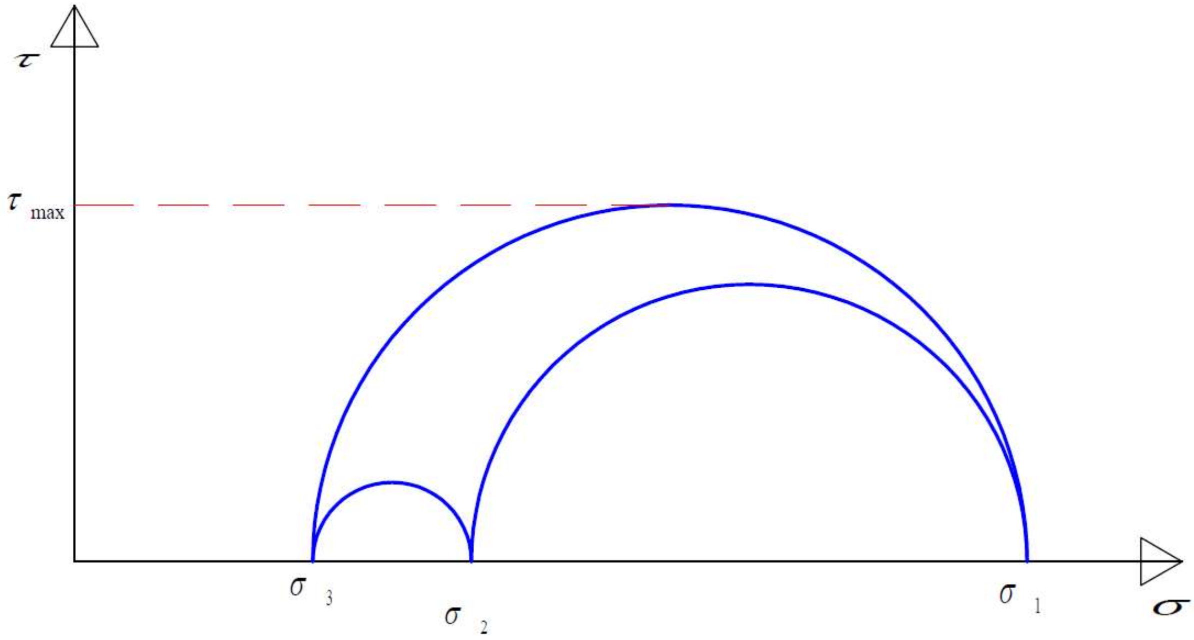
$$\Rightarrow \sigma_n = 66{,}6\ MPa$$

La contrainte tangentielle τ :

$$\vec{\tau} = \vec{T} - \overrightarrow{\sigma_n}$$

$$\Rightarrow \tau = \sqrt{T^2 - \sigma_n^2} \Rightarrow \tau = 38{,}5\ MPa$$

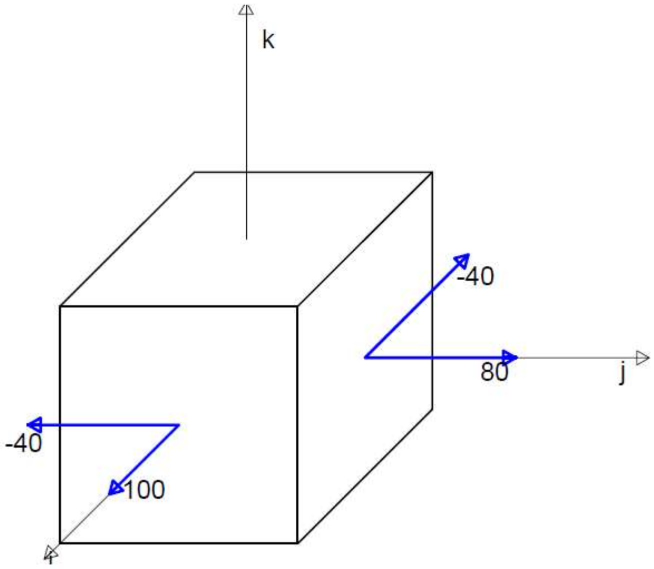
La contrainte tangentielle maximale :



$$\tau_{max} = \frac{\sigma_3 - \sigma_1}{2} = 45\ MPa$$

Exercice 2 :

1/



2/

a/ Calculons les composantes du vecteur contrainte :

$$\vec{T}(M,\vec{n}) = [\sigma(M)][\vec{n}]$$

$$\{n\} = \frac{1}{\sqrt{5}}\begin{Bmatrix}1\\2\\0\end{Bmatrix}$$

$$\Rightarrow \vec{T}(M,\vec{n}) = \begin{bmatrix}100 & -40 & 0\\ -40 & 80 & 0\\ 0 & 0 & 0\end{bmatrix}.\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}20\\120\\0\end{bmatrix}$$

b/ Calculons la contrainte normale :

$$\sigma_n = \vec{n}.\vec{T}(M,\vec{n}) = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2 & 0\end{pmatrix}.\frac{1}{\sqrt{5}}\begin{bmatrix}20\\120\\0\end{bmatrix} = 52\ MPa$$

c/ Les composantes du vecteur cisaillement :

$$\vec{\tau}_n = \vec{T}(M,\vec{n}) - \sigma_n\vec{n} = \frac{1}{\sqrt{5}}\begin{bmatrix}20\\120\\0\end{bmatrix} - \frac{52}{\sqrt{5}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-32/\sqrt{5}\\16/\sqrt{5}\\0\end{bmatrix} MPa$$

Exercice 3 :

$$\sigma = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & \sqrt{3} \\ 1 & \sqrt{3} & 2 \end{bmatrix} MPa$$

$$\det(\sigma - \lambda I)\det\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & \sqrt{3} \\ 1 & \sqrt{3} & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left((2-\lambda)^2 - 4\right) = 0$$

$$\begin{cases} \sigma_1 = \lambda_1 = 4\ MPa \\ \sigma_2 = \lambda_2 = 2\ MPa \\ \sigma_3 = \lambda_3 = 0 \end{cases}$$

$$Pour\ \sigma_1 = \lambda_1 = 4\ MPa$$

$$(\sigma - \sigma_1 I)\vec{V}_1 = \vec{0} \quad \Rightarrow \quad \vec{V}_1 = \begin{pmatrix} 1/\sqrt{8} \\ \sqrt{2}/8 \\ 2/\sqrt{8} \end{pmatrix}$$

$$Pour\ \sigma_2 = \lambda_2 = 2\ MPa$$

$$(\sigma - \sigma_2 I)\vec{V}_2 = \vec{0} \quad \Rightarrow \quad \vec{V}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \\ 0 \end{pmatrix}$$

$$Pour\ \sigma_3 = \lambda_3 = 0$$

$$(\sigma - \sigma_3 I)\vec{V}_3 = \vec{0} \quad \Rightarrow \quad \vec{V}_3 = \begin{pmatrix} 1/\sqrt{8} \\ \sqrt{3/8} \\ -1/8 \end{pmatrix}$$

Exercice 4 :

1/ Les contraintes principales :

$$\sigma = \begin{bmatrix} -4 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & -1 & 3 \\ -\sqrt{2} & 3 & -1 \end{bmatrix} (N/mm^2)$$

$$\det(\sigma - \lambda I) = \det \begin{bmatrix} -4-\lambda & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & -1-\lambda & 3 \\ -\sqrt{2} & 3 & -1-\lambda \end{bmatrix} = 0$$

$$\begin{cases} \sigma_1 = \lambda_1 = 2\ MPa \\ \sigma_2 = \lambda_2 = -2\ MPa \\ \sigma_3 = \lambda_3 = -6\ MPa \end{cases}$$

2/ Les directions principales normalisées :

$$Pour\ \sigma_1 = \lambda_1 = 2\ MPa$$

$$(\sigma - \sigma_1 I)\vec{V}_1 = \vec{0} \quad \Rightarrow \quad \vec{V}_1 = \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

$$Pour\ \sigma_2 = \lambda_2 = -2\ MPa$$

$$(\sigma - \sigma_2 I)\vec{V}_2 = \vec{0} \quad \Rightarrow \quad \vec{V}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$Pour\ \sigma_3 = \lambda_3 = -6\ MPa$$

$$(\sigma - \sigma_3 I)\vec{V}_3 = \vec{0} \quad \Rightarrow \quad \vec{V}_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/2 \\ 1/2 \end{pmatrix}$$

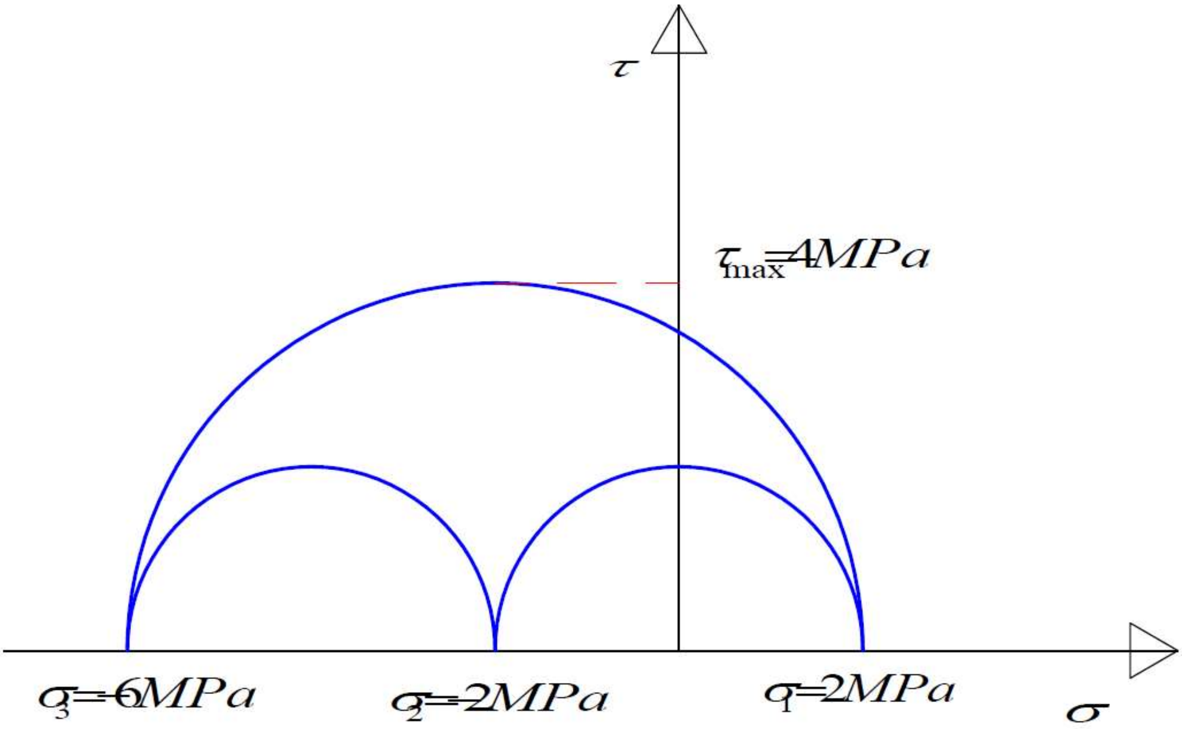
3/ La matrice C des cosinus directeurs des axes principaux :

$$C = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

4/ La contrainte normale moyenne :

$$\sigma_{moy} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -2$$

La contrainte tangentielle maximale :



5/ Vérifions les invariants des contraintes I_1 et I_3 :

$$I_1 = Trace(\sigma) = -6\ MPa$$
$$I_3 = \sigma_1 \sigma_2 \sigma_3 = 24$$

Exercice 5 :

1/ La dimension de T est : MPa

2/

$$\vec{T} = \sigma_{ij}.\vec{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} . \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}$$

$$on\ a \quad :(*)\ \vec{T}(\vec{e}_1) = \sigma_0(-\vec{e}_1 - \gamma\vec{e}_3)$$

$$\sigma_{11} = -\sigma_0 \quad et \quad \sigma_{21} = 0 \quad et \quad \sigma_{31} = -\gamma\sigma_0$$

$$(*) \quad \vec{T}(\vec{e}_3) \wedge \vec{e}_1 = -\sigma_0\vec{e}_2$$

$$\begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sigma_0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ \sigma_{33} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sigma_0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \sigma_{31} = 0 \\ \sigma_{33} = -\sigma_0 \\ \sigma_{32} = 0 \end{cases}$$

$$(*) \quad \vec{T}(\vec{e}_2).\vec{e}_2 = -\sigma_0 \quad \Rightarrow \begin{bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \sigma_{22} = -\sigma_0$$

$$\Rightarrow \overline{\overline{\sigma}} = \begin{bmatrix} -\sigma_0 & 0 & -\gamma\sigma_0 \\ 0 & -\sigma_0 & 0 \\ -\gamma\sigma_0 & 0 & -\sigma_0 \end{bmatrix}$$

3/ Les valeurs propres :

$$\det(\sigma - \lambda I) = \det \begin{bmatrix} -\sigma_0 - \lambda & 0 & -\gamma\sigma_0 \\ 0 & -\sigma_0 - \lambda & 0 \\ -\gamma\sigma_0 & 0 & -\sigma_0 - \lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \sigma_1 = \lambda_1 = -\sigma_0 \\ \sigma_2 = \lambda_2 = \sigma_0(\gamma - 1) \\ \sigma_3 = \lambda_3 = \sigma_0(-\gamma - 1) \end{cases}$$

$$Si \quad \gamma > 1 \quad : \quad \lambda_2 > \lambda_1 > \lambda_3$$

$$Si \quad \gamma < 1 \quad : \quad \lambda_3 > \lambda_1 > \lambda_2$$

Les vecteurs propres :

$$Pour\ \sigma_1 = \lambda_1 = -\sigma_0$$

$$(\sigma - \sigma_1 I)\vec{V}_1 = \vec{0} \quad \Rightarrow \quad \vec{V}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$Pour\ \sigma_2 = \lambda_2 = \sigma_0(\gamma - 1)$$

$$(\sigma - \sigma_2 I)\vec{V}_2 = \vec{0} \quad \Rightarrow \quad \vec{V}_2 = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}$$

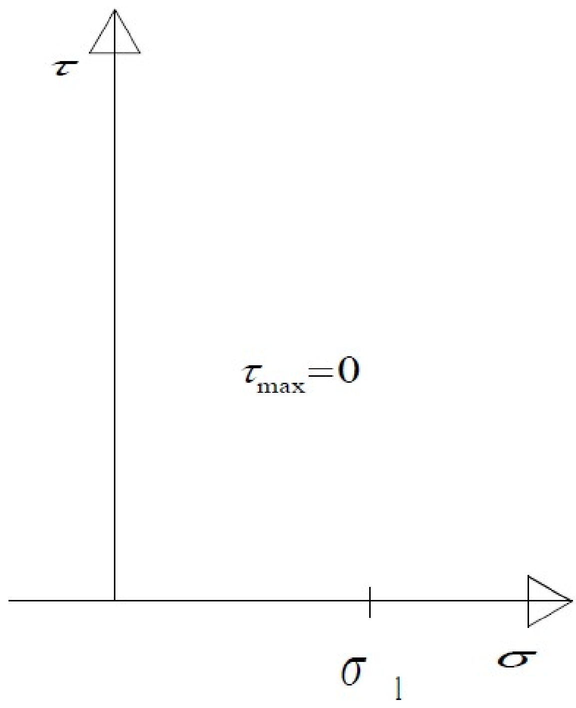
$$Pour\ \sigma_3 = \lambda_3 = \sigma_0(-\gamma - 1)$$

$$(\sigma - \sigma_3 I)\vec{V}_3 = \vec{0} \quad \Rightarrow \quad \vec{V}_3 = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{pmatrix}$$

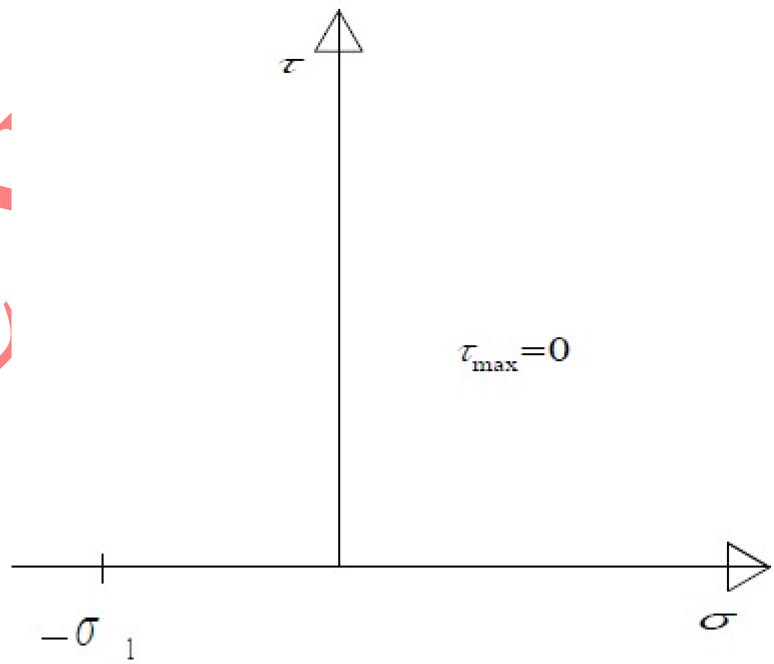
Exercice 6 :

1/

a/

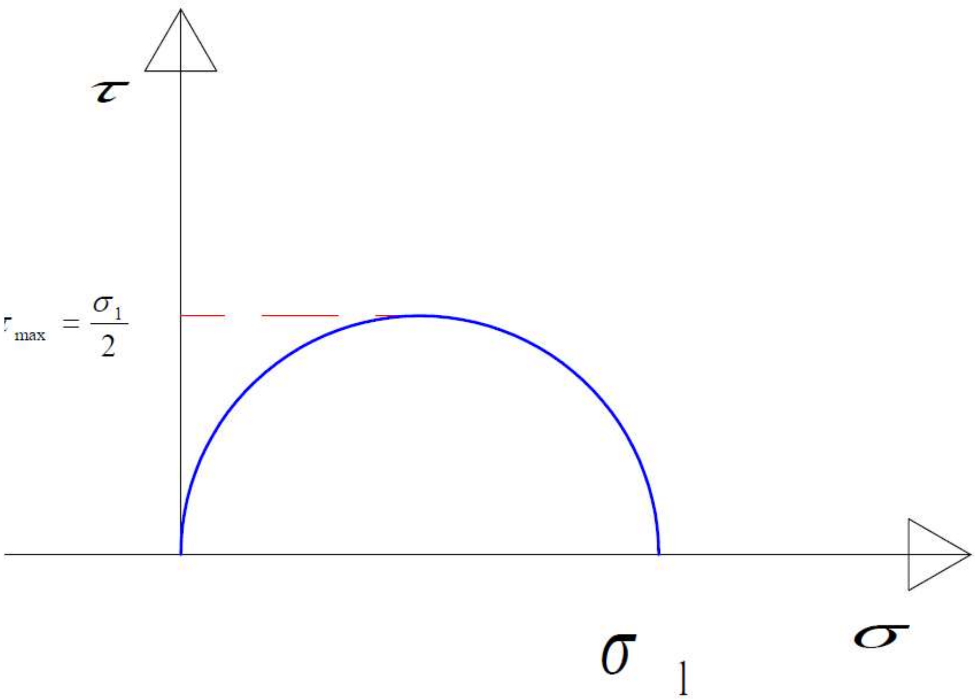


b/



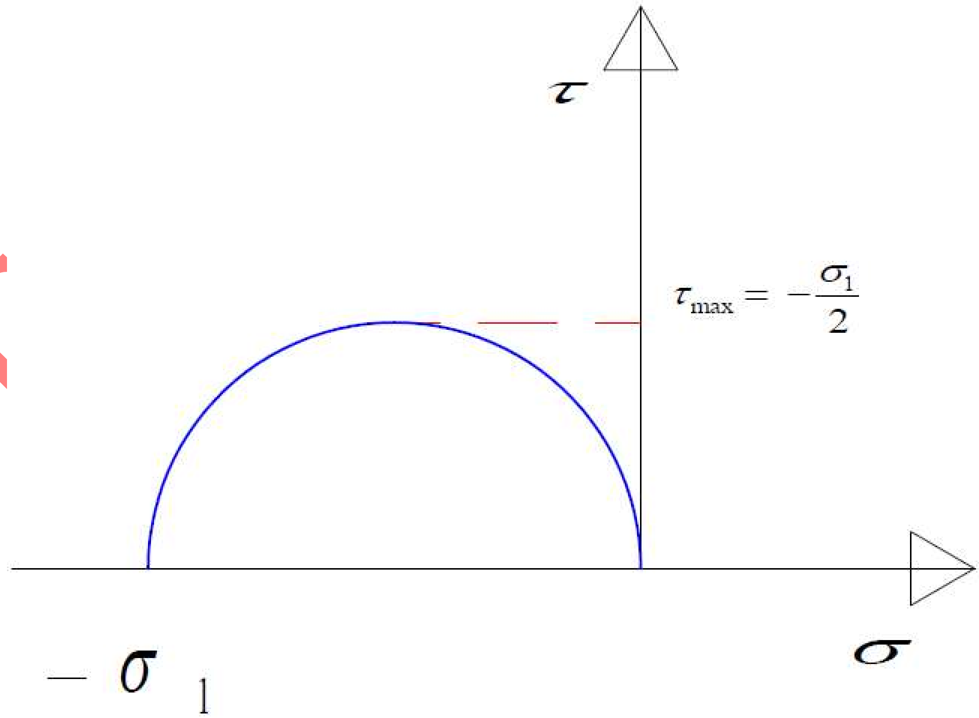
2/

a/



τ
$\tau_{max} = \frac{\sigma_1}{2}$
σ_1
σ

b/



τ
$\tau_{max} = -\frac{\sigma_1}{2}$
$-\sigma_1$
σ

3/ Cisaillement simple :

$$\det(\sigma - \lambda I) = \det\begin{bmatrix} -\lambda & \sigma_{12} & 0 \\ \sigma_{12} & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \sigma_1 = \lambda_1 = 0 \\ \sigma_2 = \lambda_2 = \sigma_{12} \\ \sigma_3 = \lambda_3 = -\sigma_{12} \end{cases}$$