THE LOCAL-GLOBAL CONJECTURE FOR APOLLONIAN CIRCLE PACKINGS IS FALSE

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Abstract. In a primitive integral Apollonian circle packing, curvatures that appear must fall into one of six or eight residue classes modulo 24. The Local-Global Conjecture states that every sufficiently large integer in one of these residue classes will appear as a curvature in the packing. We prove that this conjecture is false for many packings, by proving that certain quadratic and quartic families are missed. We then formulate a new conjecture, and give computational evidence in support of it.

Contents

1. Introduction

New methods for understanding arithmetic properties of thin groups have had a proving ground in the study of Apollonian circle packings (Figure 1), but also apply to other problems such as Zaremba's conjecture for continued fractions. The central conjecture is that "thin orbits" in $\mathbb{Z}$, such as orbits of a thin group like the Apollonian group (defined below), satisfy a *local-global* property, namely that they are subject to certain congruence restrictions (local), but otherwise should admit all sufficiently large integers (global). In the Apollonian case, the Local-Global Conjecture specifying the exact congruence obstructions is due to Graham-Lagarias-Mallows-Wilks-Yan [GLM+03] and Fuchs-Sanden [FS11]. Significant progress has been made using a variety of techniques, including analytic methods, the spectral theory of graphs, and others. This has culminated in the theorem of Bourgain and Kontorovich that amongst the admissible curvatures for an Apollonian circle packing (those values not obstructed by congruence conditions), a set of density one

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does appear [BK14]. In this paper, we demonstrate that, for infinitely many (and perhaps most, in a suitable sense) Apollonian circle packings, the Local-Global Conjecture is nevertheless false.

1.1. **Apollonian circle packings.** A theorem often attributed to Apollonius of Perga states that, given three mutually tangent circles in the plane, there are exactly two ways to draw a fourth circle tangent to the first three. By starting with three such circles, we can add in the two solutions of Apollonius (sometimes called Soddy circles after an ode by the famous chemist), obtaining five. New triples appear, and by continuing this process, one obtains a fractal called an *Apollonian circle packing* (Figure 1).

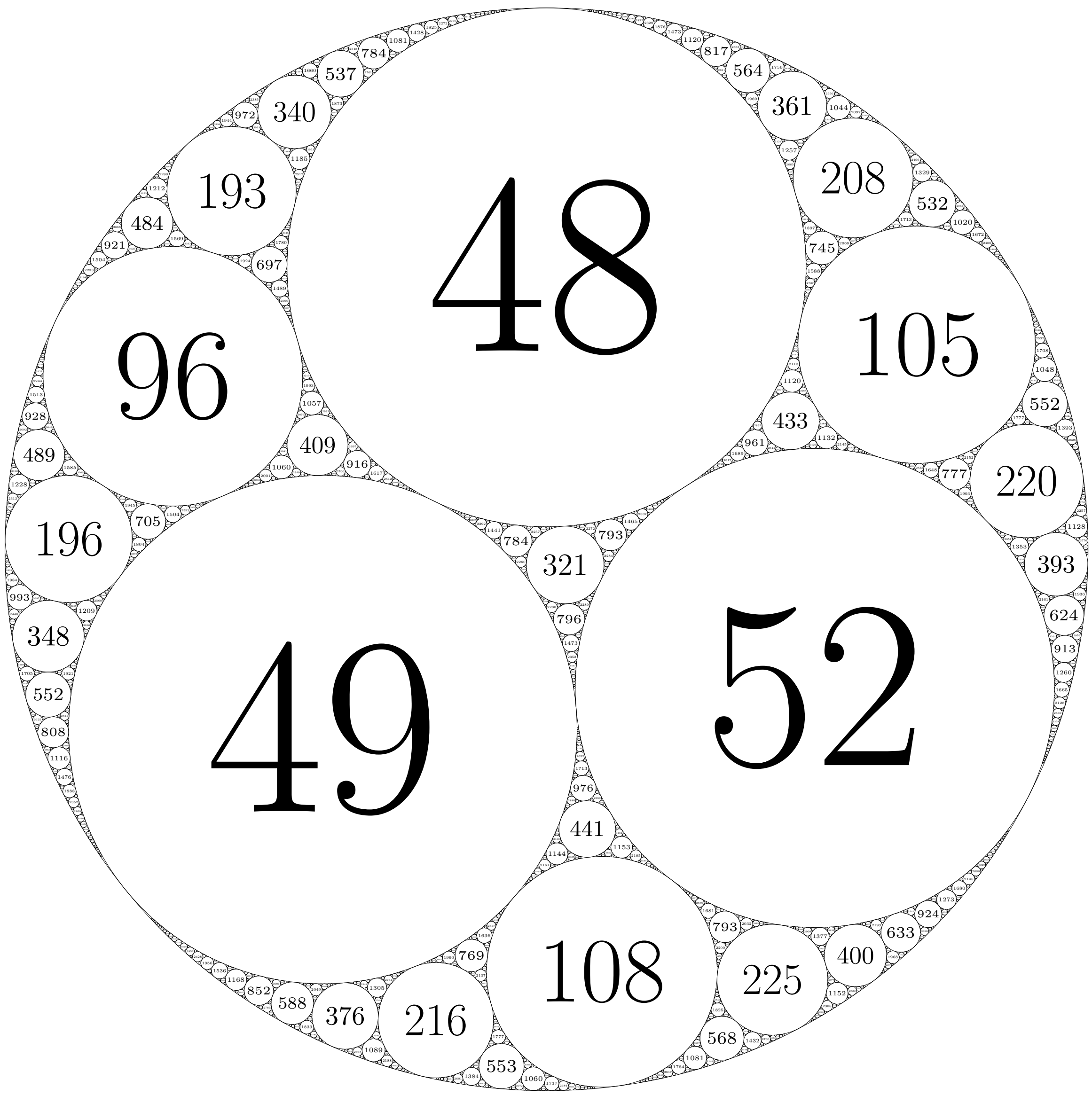

FIGURE 1. Circles of curvature ≤ 15000 in the Apollonian circle packing corresponding to $(-23, 48, 49, 52)$. The Local-Global Conjecture is false for every residue class modulo 24 in this packing.

Any solution to Apollonius' problem produces four mutually tangent circles. A *Descartes quadruple* is a quadruple of four real numbers (a, b, c, d) such that

- $(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2)$ (the Descartes equation);
- $a + b + c + d > 0$.

Given any Descartes quadruple, there exist four mutually tangent circles in the plane with those curvatures, with the following conventions:

- A negative curvature denotes that the interior is on the outside of the circle, i.e. contains the point at infinity. This ensures that the interiors of the four circles are disjoint.
- A curvature of zero corresponds to a straight line.

In particular, a Descartes quadruple generates a unique Apollonian circle packing (whereas an Apollonian circle packing corresponds to many Descartes quadruples). For this and general background on Apollonian circle packings, see [GLM+03].

A simple but remarkable consequence of the Descartes equation is that if $a, b, c, d \in \mathbb{Z}$, then all curvatures in the packing are integral. Call such a configuration, and the packing it generates, *integral*, and if we furthermore have $\gcd(a, b, c, d) = 1$, call both *primitive*. Figure 1 depicts part of the packing obtained from the Descartes quadruple $(-23, 48, 49, 52)$, with circles labelled by curvature (the outer circle being the unique one of negative curvature). Note that the four curvatures in the Descartes quadruple are the smallest four that appear in the packing; such a quadruple is unique, and is called the *root quadruple*. We tend to describe circle packings by their associated root quadruple.

1.2. **The set of curvatures of an Apollonian circle packing.** Given a primitive Apollonian circle packing $\mathcal{A}$, we study the set of curvatures which appear. This question was first addressed in the work of Graham, Lagarias, Mallows, Wilks, and Yan in [GLM+03] as the "Strong Density Conjecture", appearing near the end of Section 6. This was later revised by Fuchs and Sanden in [FS11], and has come to be known as the "Local-Global Conjecture" or "Local-Global Principle."

Conjecture 1.1 ([GLM+03, FS11])**.** *Let $\mathcal{A}$ be a primitive Apollonian circle packing containing curvatures equivalent to $r \pmod{24}$. The set of positive integers $x \equiv r \pmod{24}$ not occurring in $\mathcal{A}$ is finite.*

Call a positive curvature c *missing* in $\mathcal{A}$ if curvatures equivalent to $c \pmod{24}$ appear in $\mathcal{A}$ but c does not. By computing the frequency of curvatures that appear up to $5 \cdot 10^8$ in the packings corresponding to $(-1, 2, 2, 3)$ and $(-11, 21, 24, 28)$, Fuchs and Sanden observe a tendency toward increasing multiplicity for larger curvatures, which is evidence towards there being few missing curvatures.

The current best known result is due to Bourgain and Kontorovich.

Theorem 1.2 ([BK14])**.** *The number of missing curvatures up to N is at most $O(N^{1-\eta})$ for some effectively computable $\eta > 0$.*

In this paper, we prove that Conjecture 1.1 is false for many packings.

Theorem 1.3. *There exist infinitely many primitive Apollonian circle packings for which the number of missing curvatures up to N is $\Omega(\sqrt{N})$. In particular, the local-global conjecture is false for these packings.*

This theorem is proven by showing that certain quadratic and quartic families of curvatures (of the form ux^2 and ux^4 for a fixed integer u) are missing from some packings. A precise description of the obstructions is found in Theorem 2.5.

While Conjecture 1.1 is false in general, there still remain some families where it could be true, as we find no obstructions. For the other packings, we can also remove the quadratic and quartic obstructions, and ask if the remaining set of curvatures is now finite.

Definition 1.4. Let $\mathcal{A}$ be a primitive Apollonian circle packing, and define $S_{\mathcal{A}}$ to be the set of missing curvatures which do not lie in one of the quadratic or quartic obstruction classes as described in Theorem 2.5. Call this set the "sporadic set" for $\mathcal{A}$. For a positive integer N, define $S_{\mathcal{A}}(N)$ to be the sporadic curvatures in $\mathcal{A}$ that are at most N.

By writing an efficient program using C and PARI/GP [PAR23] to compute missing curvatures, we are able to find $S_{\mathcal{A}}(N)$ for various packings $\mathcal{A}$ and bounds N in the range $[10^{10}, 3\cdot 10^{11}]$. This code can be found in the GitHub repository [Ric23a], and the sets $S_{\mathcal{A}}(N)$ are found in the GitHub repository [Ric23b]. Some of this data is summarized in Tables 1 and 2, which support a revised conjecture. Note that this revised conjecture is not in contradiction to the analysis of Fuchs and Sanden, which indicates a sparsity of missing curvatures.

Conjecture 1.5. *Let $\mathcal{A}$ be a primitive Apollonian circle packing. Then $S_{\mathcal{A}}$ is finite.*

In other words, every sufficiently large curvature will appear in the given primitive Apollonian circle packing $\mathcal{A}$, except for curvatures lying in one of the linear, quadratic, or quartic families described by Theorem 2.5.

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2. Precise statement of the results

Curvatures modulo 24 in a primitive Apollonian circle packing fall into a set of six or eight possible residues, called the *admissible residues* for the packing. There are six possible sets of these residue classes, detailed in the following proposition.

Proposition 2.1. *Let $\mathcal{A}$ be a primitive Apollonian circle packing. Let $R(\mathcal{A})$ be the set of residues modulo 24 of the curvatures in $\mathcal{A}$. Then $R(\mathcal{A})$ is one of six possible sets, labelled by a* type *as follows:*

Type	$R(\mathcal{A})$
$(6,1)$	$0,1,4,9,12,16$
$(6,5)$	$0,5,8,12,20,21$
$(6,13)$	$0,4,12,13,16,21$
$(6,17)$	$0,8,9,12,17,20$
$(8,7)$	$3,6,7,10,15,18,19,22$
$(8,11)$	$2,3,6,11,14,15,18,23$

The set $R(\mathcal{A})$ is called the *admissible set* of the packing. Each admissible set is labelled by a *type* (x, k) where x is its cardinality and k is the smallest residue in R that is coprime to 24.

Definition 2.2. Let $\mathcal{A}$ be a primitive Apollonian circle packing, let u and d be positive integers, and let $S_{d,u} := \{un^d : n \in \mathbb{Z}\}$. We say that the set $S_{d,u}$ forms a *power obstruction* to $\mathcal{A}$ if

- Infinitely many elements of $S_{d,u}$ are admissible in $\mathcal{A}$ modulo 24;
- No element of $S_{d,u}$ appears as a curvature in $\mathcal{A}$.

If $d = 2$ we call it a *quadratic obstruction*, and if $d = 4$ it is a *quartic obstruction*.

Note that some power obstructions are stronger than others, as (for example) we have $S_{2,3} \supseteq S_{2,18}$ and $S_{2,9} \supseteq S_{4,9}$. We will tend to only associate "maximal" power obstruction classes to packings, i.e. where no stronger power obstructions exist.

It is clear that if there exists a power obstruction for $\mathcal{A}$, then the Local-Global Conjecture 1.1 cannot hold for $\mathcal{A}$, and more specifically, for any of the admissible residue classes intersecting $S_{d,u}$. For each of these classes, adding a single gcd condition to n will give us the intersection of $S_{d,u}$ with that class.

In order to describe the quadratic and quartic obstructions, it is necessary to subdivide the types further. There exists a function

$$\chi_2 : \{\text{circles in a primitive Apollonian circle packing}\} \to \{\pm 1\}$$

which relates to the possible curvatures of circles tangent to the input circle $\mathcal{C}$, and is constant across the packing containing $\mathcal{C}$. In particular, this gives a well defined value for $\chi_2(\mathcal{A})$ for primitive Apollonian circle packings $\mathcal{A}$. Furthermore, there exists a function

$$\chi_4 : \{\text{circles in a primitive Apollonian circle packing of type } (6, 1) \text{ or } (6, 17)\} \to \{1, i, -1, -i\}$$

that satisfies $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$, and is also constant across a packing (necessarily of type $(6, 1)$ or $(6, 17)$; it is left undefined for other types). This again gives a well defined value for $\chi_4(\mathcal{A})$.

The value of χ_2 determines which quadratic obstructions occur, and the value of χ_4 determines which quartic obstructions occur. We compute χ_2 in terms of the quadratic residuosity of the curvatures tangent to a circle, and χ_4 in terms of a quartic residue symbol. Full definitions come in Sections 4.1 and 5.2, but we can give a quick way to compute $\chi_2(\mathcal{A})$ here (which follows directly from the definition).

Proposition 2.3. *Let $\mathcal{A}$ be a primitive Apollonian circle packing, and let (a, b) be a pair of curvatures of circles tangent to each other in $\mathcal{A}$ that also satisfies:*

- *a is coprime to $24b$;*
- *if $\mathcal{A}$ is of type $(8, k)$, then $a \equiv 7 \pmod 8$.*

Then $\chi_2(\mathcal{A}) = \left(\frac{b}{a}\right)$.

The definition of χ_4 relies on a finer invariant using the quartic residue symbol for Gaussian integers.

The essential fact that the symbol is constant across a packing is a direct consequence of quadratic and quartic reciprocity.

Definition 2.4. The (extended) type of a primitive Apollonian circle packing $\mathcal{A}$ is either the triple (x, k, χ_2) or (x, k, χ_2, χ_4), where $\mathcal{A}$ has type (x, k) and corresponding values of χ_2 (and χ_4, if relevant). We will refer to the type as any of the three possible variants, as they are uniquely distinguished by the number of entries.

Theorem 2.5. *The type of a primitive Apollonian circle packing $\mathcal{A}$ implies the existence of certain quadratic and quartic obstructions, as described by the following table (which also includes the list of residues modulo* 24 *where Conjecture 1.1 is false, and those where it is still open):*

Type	Quadratic Obstructions	Quartic Obstructions	1.1 false	1.1 open
$(6,1,1,1)$				$0,1,4,9,12,16$
$(6,1,1,-1)$		$n^4,4n^4,9n^4,36n^4$	$0,1,4,9,12,16$	
$(6,1,-1)$	$n^2,2n^2,3n^2,6n^2$		$0,1,4,9,12,16$	
$(6,5,1)$	$2n^2,3n^2$		$0,8,12$	$5,20,21$
$(6,5,-1)$	$n^2,6n^2$		$0,12$	$5,8,20,21$
$(6,13,1)$	$2n^2,6n^2$		0	$4,12,13,16,21$
$(6,13,-1)$	$n^2,3n^2$		$0,4,12,16$	$13,21$
$(6,17,1,1)$	$3n^2,6n^2$	$9n^4,36n^4$	$0,9,12$	$8,17,20$
$(6,17,1,-1)$	$3n^2,6n^2$	$n^4,4n^4$	$0,9,12$	$8,17,20$
$(6,17,-1)$	$n^2,2n^2$		$0,8,9,12$	$17,20$
$(8,7,1)$	$3n^2,6n^2$		$3,6$	$7,10,15,18,19,22$
$(8,7,-1)$	$2n^2$		18	$3,6,7,10,15,19,22$
$(8,11,1)$				$2,3,6,11,14,15,17,23$
$(8,11,-1)$	$2n^2,3n^2,6n^2$		$2,3,6,18$	$11,14,15,23$

Remark 2.6. The intersection of quadratic and quartic obstructions with a residue class can be described by adding a condition on n. For example, the obstruction $2n^2$ in type $(6,17,-1)$ intersects the class $8 \pmod{24}$ as $2(6n\pm 2)^2$, and the class $0 \pmod{24}$ as $2(6n)^2$.

Remark 2.7. We could consider the χ_4 value for packings of types $(6,1,-1)$ and $(6,17,-1)$, which would be either i or $-i$. Both of these χ_4 values imply that the families $n^4,4n^4,9n^4,36n^4$ are quartic obstructions. However, we already have n^2 as a quadratic obstruction, which is strictly stronger. This is why they are not listed, as the quartic obstruction does not give anything the quadratic did not (see the discussion below Definition 2.2). This is why we did not attempt to extend the definition of χ_4 to other packing types: we did not find any further quartic obstructions in our computations (that were not already ruled out by quadratic obstructions).

A few direct corollaries from Theorem 2.5 are noted next.

Corollary 2.8. *The Local-Global Conjecture 1.1 is false for at least one residue class in all primitive Apollonian circle packings that are not of type* $(6,1,1,1)$ *or* $(8,11,1)$.

The exceptions where the Local-Global Conjecture may yet hold include the strip packing (generated from the root quadruple $(0,0,1,1)$), and the bug-eye packing (generated from $(-1,2,2,3)$).

The following corollary is the phenomenon which led to the discovery of square/quadratic obstructions.

Corollary 2.9. *Curvatures* $24m^2$ *(necessarily* $0 \bmod 24$*) and* $8n^2$ *with* $3\nmid n$ *(necessarily* $8 \bmod 24$*) cannot appear in the same primitive Apollonian circle packing, despite* $0 \pmod{24}$ *and* $8 \pmod{24}$ *being simultaneously admissible in packings of type* $(6,5)$ *or* $(6,17)$.

Remark 2.10. Apollonian circle packings have been generalized in a variety of ways. In [Sta18a], K-Apollonian packings were defined for each imaginary quadratic field K, where the $\mathbb{Q}(i)$-Apollonian case is the subject of this paper. It is quite possible that square obstructions occur in these packings, as they share many features with Apollonian packings, including the fact that quadratic forms control the family of tangent curvatures. The existence of quartic obstructions is less likely, since these depend on quartic reciprocity, which is defined only for $K = \mathbb{Q}(i)$. The family of packings studied in [FSZ19] are also governed by quadratic forms (this was the essential feature needed for the positive density results of that paper), and include the K-Apollonian packings; these are likely subject to quadratic obstructions as well. It would also be interesting to ask the same question about an even wider class of packings studied by Kapovich and Kontorovich [KK23].

In Section 3, we cover the background material required, including the connection between circle packings and values of binary quadratic forms. The proof of the quadratic obstructions comes in Section 4, and the quartic obstructions are found in Section 5. Computational evidence to support Conjecture 1.5 is found in Section 6.

3. Residue classes and quadratic forms

In this section we prove Proposition 2.1, and give the background material on the connection between quadratic forms and circle packings.

To begin, consider a Descartes quadruple (a, b, c, d) contained in an Apollonian circle packing. The act of swapping the i^{th} circle to obtain the other solution described by Apollonius is called a *move*, and is denoted S_i. In terms of the quadruples, S_1 corresponds to

$$S_1 : (a, b, c, d) \to (2b + 2c + 2d - a, b, c, d).$$

Analogous equations for S_2 to S_4 hold. It is possible to move between every pair of Descartes quadruples in a fixed circle packing via a finite sequence of these moves (up to a permutation of the entries of the quadruples).

3.1. **Residue classes.** In [GLM$^+$03], Graham-Lagarias-Mallows-Wilks-Yan proved that the residues modulo 12 of a primitive Apollonian circle packing $\mathcal{A}$ fall into one of the following four sets:

$$\{0, 1, 4, 9\}, \{2, 3, 6, 11\}, \{3, 6, 7, 10\}, \{0, 5, 8, 9\}.$$

They also proved that if m is coprime to 30, then every residue class modulo m occurs in $\mathcal{A}$. In her PhD thesis [Fuc10], Fuchs proved that there are in fact restrictions modulo 24, and that this is the "best possible modulus." However, to the best of our knowledge, the exact list of the possible admissible sets modulo 24 found in Proposition 2.1 has not appeared in the literature until now, and requires justification. We give a self-contained proof of Proposition 2.1. The key observation is the following result.

Proposition 3.1. *Let tangent circles in a primitive Apollonian circle packing have curvatures a, b. Then $a + b \not\equiv 3, 6, 7 \pmod 8$.*

Proof. Assume otherwise. Then the odd part of $a+b$ is equivalent to $3 \pmod 4$, so there exists an odd prime p with $p \equiv 3 \pmod 4$ and $v_p(a + b)$ is odd. Let (a, b, c, d) be a Descartes quadruple corresponding to our starting circles, and rearranging the Descartes equation gives

$$(a-b)^2 + (c-d)^2 = 2(a+b)(c+d).$$

Since the left hand side is a sum of two squares that is a multiple of $p \equiv 3 \pmod 4$, it follows that $p \mid a-b, c-d$, and $v_p(\text{LHS})$ is even. Therefore $v_p(c+d)$ is odd, hence $p \mid c+d$ as well. Thus $p \mid a, b, c, d$, so the quadruple is not primitive, contradiction. □

Rather than work modulo 24, we consider modulo 3 and 8 separately.

Lemma 3.2. *Let $\mathcal{A}$ be a primitive Apollonian circle packing. Then the set of Descartes quadruples in $\mathcal{A}$ taken modulo 3 is one of the following sets:*

(a) $\{(0,0,1,1),(0,1,1,1)\}$ *and all permutations;*
(b) $\{(0,0,2,2),(0,2,2,2)\}$ *and all permutations.*

Proof. By dividing into cases based on the number of curvatures that are multiples of 3, a straightforward computation shows the claimed sets are the only solutions to Descartes's equation modulo 3. By considering the moves S_1 to S_4, we see that they fall into the two classes. □

Lemma 3.3. *Let $\mathcal{A}$ be a primitive Apollonian circle packing. Then the set of Descartes quadruples in $\mathcal{A}$ taken modulo 8 is one of the following sets:*

(a) $\{(0,0,1,1),(0,4,1,1),(4,0,1,1),(4,4,1,1)\}$ *and all permutations;*
(b) $\{(0,0,5,5),(0,4,5,5),(4,0,5,5),(4,4,5,5)\}$ *and all permutations;*
(c) $\{(2,2,3,7),(2,6,3,7),(6,2,3,7),(6,6,3,7)\}$ *and all permutations.*

Proof. Let (a,b,c,d) be a primitive Descartes quadruple in $\mathcal{A}$. Considering the Descartes equation modulo 2, there is an even number of odd numbers amongst a,b,c,d. This cannot be zero (as the quadruple would not be primitive), and it cannot be 4, as otherwise $(a+b+c+d)^2 \equiv 8 \pmod{16}$. Therefore there are always two odd and two even, so without loss of generality assume that c,d are odd and a,b are even.

Assume $c \equiv 1 \pmod 4$. By Proposition 3.1, $a \equiv b \equiv 0 \pmod 4$ (else $a+c \equiv 3 \pmod 4$). Thus we also have $d \equiv 1 \pmod 4$, and in fact $d \equiv c \pmod 8$, as otherwise $c+d \equiv 6 \pmod 8$. This gives the 8 quadruples listed in (a) and (b), and by applying S_1 through S_4, we see them fall into two classes.

Finally, assume $c \equiv 3 \pmod 4$. In this case we must have $a \equiv b \equiv 2 \pmod 4$, which implies that $d \equiv 3 \pmod 4$ as well. Consequently we have $c \not\equiv d \pmod 8$, else $c+d \equiv 6 \pmod 8$. This gives the final four quadruples, and again, the moves S_1 to S_4 show that they form one class. □

A consequence of Lemmas 3.2 and 3.3 is that

- $R(\mathcal{A}) \pmod 3 = \{0,1\}$ or $\{0,2\}$;
- $R(\mathcal{A}) \pmod 8 = \{0,1,4\}$ or $\{0,4,5\}$ or $\{2,3,6,7\}$.

The Chinese remainder theorem gives six ways to combine these two congruences into a congruence set modulo 24, resulting in exactly the six sets described in Proposition 2.1. For each of the six sets, there do exist primitive packings with those admissible sets. The only remaining issue is justification that we do actually get curvatures corresponding to every pair of residues modulo 3 and 8, i.e. that they do not conspire against each other in the packing (which would lead to a subset of the claimed admissible set). A more general version of the following lemma exists in [Fuc10].

Lemma 3.4. *Let $\mathcal{A}$ be a primitive Apollonian circle packing containing a curvature equivalent to $r_1 \pmod 3$ and another curvature equivalent to $r_2 \pmod 8$. Then there exists a curvature in $\mathcal{A}$ that is simultaneously equivalent to $r_1 \pmod 3$ and $r_2 \pmod 8$.*

Proof. Let (a, b, c, d) be a Descartes quadruple in $\mathcal{A}$, and assume that $a \equiv r_2 \pmod 8$. If $(a, b, c, d) \pmod 3$ has two zeroes, then swapping either of them produces the non-zero residue. If it has a single zero, then swapping a non-zero residue gives a zero. In particular, we can always apply a sequence of moves to get (a', b', c', d') where $a' \equiv r_1 \pmod 3$. Furthermore, we can assume that if $r_1 \equiv 0$ then $b \equiv c \equiv d \not\equiv 0 \pmod 3$, and if $r_1 \not\equiv 0$, then $b \equiv r_1 \pmod 3$ and $c \equiv d \equiv 0 \pmod 3$. Note that the moves do not change the curvatures modulo 4, hence $a' \equiv r_2, r_2 + 4 \pmod 8$. If it is $r_2 \pmod 8$ we are done. Otherwise it is $r_2 + 4 \pmod 8$, and then the move S_1 will take it to $r_2 \pmod 8$ and will not change it modulo 3, completing the proof. □

In particular, Proposition 2.1 follows.

Proposition 3.1 is not just useful for eliminating possible admissible sets; it can also rule out behaviours in certain packings. A direct corollary that will be useful later is as follows.

Corollary 3.5. *Let $\mathcal{A}$ be a primitive Apollonian circle packing of type $(8, k)$, and assume two circles with odd curvatures a, b are tangent to each other. Then one of a, b is $3 \pmod 8$, and the other is $7 \pmod 8$.*

3.2. **Quadratic forms.** Generating integral Apollonian circle packings seems tricky at first glance, as it requires integral solutions to the Descartes equation. This step is solved by a direct connection to quadratic forms, as first described in [GLM+03].

Definition 3.6. A primitive integral positive definite binary quadratic form is a function of the form $f(x, y) = Ax^2 + Bxy + Cy^2$, where $A, B, C \in \mathbb{Z}$, $\gcd(A, B, C) = 1$, $A > 0$, and $D := B^2 - 4AC < 0$. Call D the discriminant of f. The group $\mathrm{PGL}(2, \mathbb{Z})$ acts on the set of forms as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ f := f(ax + by, cx + d).$$

This action preserves the discriminant, and divides the set of forms of a fixed discriminant into a finite number of equivalence classes.

See the books by Buell [Bue89] or Cohen [Coh93] for a longer exposition on quadratic forms. Note that while most references study the $\mathrm{PSL}(2, \mathbb{Z})$ action, we will need the extended $\mathrm{PGL}(2, \mathbb{Z})$ action, which no longer gives rise to a group.

Proposition 3.7 ([GLM+03, Theorem 4.2])**.** *Let (a, b, c, d) be a primitive Descartes quadruple. Then*

$$(a + b)x^2 + (a + b + c - d)xy + (a + c)y^2$$

is a primitive integral positive definite binary quadratic form of discriminant $-4a^2$.

Similarly, let $Ax^2 + Bxy + Cy^2$ be a primitive integral positive definite binary quadratic form of discriminant $-4a^2$. Then

$$(a, A - a, C - a, A + C - B - a)$$

is a primitive Descartes quadruple.

These two notions are inverse to each other, and induce a bijection between quadratic forms of discriminant $-4a^2$ and Descartes quadruples containing a as the first curvature.

Given a circle $\mathcal{C}$ of curvature a in a primitive Apollonian circle packing, we can associate a Descartes quadruple (a, b, c, d) to $\mathcal{C}$, and therefore a quadratic form. The ambiguity in choosing the Descartes quadruple exactly corresponds to taking a $\mathrm{PGL}(2, \mathbb{Z})-$equivalence class of quadratic forms (hence the need to go beyond the $\mathrm{PSL}(2, \mathbb{Z})$ action). This is implicit in [GLM+03]; see also Proposition 3.1.3 of [Ric23c].

Definition 3.8. Let $\mathcal{C}$ be a circle of curvature $n \neq 0$ in a primitive Apollonian circle packing, and define $f_{\mathcal{C}}$ to be a primitive integral positive definite binary quadratic form of discriminant $-4n^2$ that corresponds to $\mathcal{C}$ via Proposition 3.7.

Remark 3.9. For the rest of the paper we will assume that $n \neq 0$ for convenience. The results should still hold for $n = 0$, but as this only corresponds to the strip packing $(0, 0, 1, 1)$, it will be of no use here.

While $f_{\mathcal{C}}$ is only defined up to a $\mathrm{PGL}(2, \mathbb{Z})-$equivalence class, this will not matter, and it is more convenient to always refer to it as a bona fide quadratic form, and not just a representative of its equivalence class. Furthermore, if the circle packing $\mathcal{A}$ containing $\mathcal{C}$ has symmetries, multiple distinct circles may correspond to the same Descartes quadruple, and hence the same quadratic form. This fact is of no importance in this paper.

Given a quadratic form f, the values properly represented by f are the numbers of the form $f(x, y)$ where x, y are coprime integers. This set is constant across a $\mathrm{PGL}(2, \mathbb{Z})-$equivalence class of forms. In [Sar07], Sarnak made a crucial observation connecting curvatures of circles tangent to $\mathcal{C}$ and properly represented values of $f_{\mathcal{C}}$.

Proposition 3.10. *The set of curvatures of circles tangent to $\mathcal{C}$ in $\mathcal{A}$ is in bijection with the set of $f(x, y)-n$, where n is the curvature of $\mathcal{C}$ and (x, y) is a pair of coprime integers.*

This observation has been a key ingredient in nearly every result on Apollonian circle packings since, and this paper is no exception.

4. Quadratic obstructions

Let $\mathcal{A}$ be a primitive integral Apollonian circle packing, and fix a set $S_{2,u} = \{uw^2 : w \in \mathbb{Z}\}$ for some fixed positive integer u which has no prime divisors larger than 3. The strategy to prove that no element of $S_{2,u}$ appears as a curvature in $\mathcal{A}$ is:

(1) For each circle $\mathcal{C} \in \mathcal{A}$, define $\chi_2(\mathcal{C}) \in \{\pm 1\}$ which relates to the possible curvatures of circles tangent to $\mathcal{C}$, and prove that it is an invariant of $\mathcal{A}$:
   (a) Prove that the value of χ_2 is equal for tangent circles with coprime curvatures;
   (b) Prove that given any two circles in $\mathcal{A}$, there exists a path from one to the other via tangencies, where each consecutive pair of curvatures is coprime;

   Thus, we define a value $\chi_2(\mathcal{A}) \in \{\pm 1\}$ for each packing $\mathcal{A}$.
(2) Prove that for a fixed u, packings with a certain $\chi_2(\mathcal{A})$ value and type cannot accommodate curvatures in $S_{2,u}$ in circles tangent to a "large" subset of $\mathcal{A}$.
(3) Prove that every circle in $\mathcal{A}$ is tangent to an element of this large subset.

For circle packings with the certain $\chi_2(\mathcal{C})$ value and type, these steps imply that curvatures in $S_{2,u}$ cannot appear anywhere in the packing. For each admissible residue class contained in $S_{2,u}$, we get a quadratic obstruction.

4.1. **Definition of χ_2.** Let $f(x,y) = Ax^2 + Bxy + Cy^2$ be a primitive integral positive definite binary quadratic form of discriminant $-4n^2$, for an integer $n \neq 0$.

Proposition 4.1. *The set of properly represented invertible values of $f(x,y)$ modulo n taken modulo squares, i.e.,*

$$\{f(x,y) \pmod{n} : \gcd(x,y) = \gcd(f(x,y),n) = 1\} \subseteq (\mathbb{Z}/n\mathbb{Z})^\times/((\mathbb{Z}/n\mathbb{Z})^\times)^2,$$

is a singleton set. Denote any lift of this value to the positive integers by $\rho(f)$.

Proof. If A is coprime to n, observe that

$$f(x,y) = A\left(x + \frac{B}{2A}y\right)^2 + \frac{n^2}{A}y^2 \equiv A\left(x + \frac{B}{2A}y\right)^2 \pmod{n},$$

hence this lies in the coset containing A. If A is not coprime to n, then we can replace f by a $\mathrm{PSL}(2,\mathbb{Z})$ translation of f for which the first coefficient is coprime to n. $\square$

Note that the class of $\rho(f)$ is well-defined across a $\mathrm{PGL}(2,\mathbb{Z})-$equivalence class of forms, as these forms properly represent the same integers. By using the Kronecker symbol, this allows us to determine a condition for numbers that are not represented by f.

Proposition 4.2. *Let $n' = \frac{n}{2}$ if $n \equiv 2 \pmod{4}$ and $n' = n$ otherwise. Then the Kronecker symbol $\left(\frac{\rho(f)}{n'}\right)$ is independent of the choice of $\rho(f)$ and takes values in $\{\pm 1\}$. Furthermore, let uw^2 be coprime to n, where u and w are positive integers. If $\left(\frac{\rho(f)}{n'}\right) \neq \left(\frac{u}{n'}\right)$, then $f(x,y)$ does not properly represent $n + uw^2$.*

Proof. Let ρ_1 and ρ_2 be two choices for $\rho(f)$. Then there exists integers s,t such that $\gcd(s,n) = 1$ and $\rho_1 = s^2\rho_2 + tn$. Since n' is either odd or a multiple of 4, the Kronecker symbol is well defined modulo n'. Therefore

$$\left(\frac{\rho_1}{n'}\right) = \left(\frac{s^2\rho_2 + tn}{n'}\right) = \left(\frac{s^2\rho_2}{n'}\right) = \left(\frac{\rho_2}{n'}\right),$$

so the symbol $\left(\frac{\rho(f)}{n'}\right)$ is well-defined as a function of f (the assumption that $\rho_1, \rho_2 > 0$ is implicitly used if $n' < 0$). Since $\rho(f)$ is coprime to n, the symbol takes values in $\{\pm 1\}$.

Finally, assume that $f(x,y)$ properly represents $n + uw^2$. Since $\gcd(uw^2, n) = 1$, we can take $\rho(f) = u$, and the result follows from above. $\square$

Taking n' instead of n in Proposition 4.2 was crucial, as if $n \equiv 2 \pmod{4}$, the Kronecker symbol is *not* well-defined modulo n.

By using the correspondence between circles and quadratic forms, we can now assign a sign ± 1 to each circle in an Apollonian circle packing, which will dictate what quadratic obstructions can occur adjacent to it. By Proposition 4.2, the following is well-defined.

Definition 4.3. Let $\mathcal{C}$ be a circle in a primitive Apollonian circle packing of curvature $n \neq 0$, and let $\rho = \rho(f_{\mathcal{C}})$. Define

$$\chi_2(\mathcal{C}) := \begin{cases} \left(\frac{\rho}{n}\right) & \text{if } n \equiv 0,1 \pmod{4}; \\ \left(\frac{-\rho}{n/2}\right) & \text{if } n \equiv 2 \pmod{4}; \\ \left(\frac{2\rho}{n}\right) & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Assume that $\mathcal{C}$ is a circle having curvature n coprime to u and equivalent to either $1 \pmod 4$ or $7 \pmod 8$. Then $\chi_2(\mathcal{C}) = \left(\frac{\rho}{n}\right)$, so $\chi_2(\mathcal{C}) \neq \left(\frac{u}{n}\right)$ would rule out the existence of a circle tangent to $\mathcal{C}$ of curvature uw^2 with w coprime to n.

However, to rule out uw^2 in the entire packing, we need to show that all such circles in a packing have the same value of χ_2. This requires walking through a tangency chain of intermediate circles not necessarily coprime to u. The various cases in the definition are designed for this purpose.

4.2. **Propagation of χ_2.** For a circle packing of type $(6,x)$, all curvatures are $0, 1 \pmod 4$, which makes the propagation fairly easy. For the $(8,x)$ packings, all curvatures are $2, 3 \pmod 4$, and Corollary 3.5 will come into play.

Proposition 4.4. *Let $\mathcal{C}_1$, $\mathcal{C}_2$ be tangent circles with non-zero coprime curvatures in a primitive Apollonian circle packing. Then $\chi_2(\mathcal{C}_1) = \chi_2(\mathcal{C}_2)$.*

Proof. Let the curvatures of $\mathcal{C}_1, \mathcal{C}_2$ be a, b respectively. Since $\gcd(a+b,a) = \gcd(a+b,b) = 1$, by Proposition 3.7, we can take $\rho(f_{\mathcal{C}_1}) = a+b = \rho(f_{\mathcal{C}_2})$ (noting that $a+b>0$). We separate into cases based on the type of packing and $a, b \pmod 4$.

First, assume the packing has type $(6,k)$. Then $a, b \equiv 0, 1 \pmod 4$, with at least one being odd. Therefore

$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{a+b}{a}\right)\left(\frac{a+b}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1,$$

by quadratic reciprocity. This implies that $\chi_2(\mathcal{C}_1) = \chi_2(\mathcal{C}_2)$, as claimed.

Otherwise, the packing has type $(8,k)$, and we make two cases. If both a and b are odd, by Corollary 3.5, we can assume $a \equiv 3 \pmod 8$ and $b \equiv 7 \pmod 8$. Thus

$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{2a+2b}{a}\right)\left(\frac{2a+2b}{b}\right) = \left(\frac{2}{ab}\right)\left(\frac{b}{a}\right)\left(\frac{a}{b}\right) = (-1)(-1) = 1,$$

where we used the fact that $ab \equiv 5 \pmod 8$ and quadratic reciprocity.

Finally, assume that a is odd and b is even, necessarily $2 \pmod 4$. Write $b = 2b'$, and we compute

$$\chi_2(\mathcal{C}_1)\chi_2(\mathcal{C}_2) = \left(\frac{2a+2b}{a}\right)\left(\frac{-a-b}{b'}\right) = \left(\frac{4b'}{a}\right)\left(\frac{-a}{b'}\right) = \left(\frac{-1}{b'}\right)\left(\frac{b'}{a}\right)\left(\frac{a}{b'}\right) = (-1)^{(b'-1)/2}(-1)^{(b'-1)/2} = 1,$$

completing the proof. ☐

4.3. **Coprime curvatures.** Since Proposition 4.4 requires coprime curvatures, to extend χ_2 to the entire packing, we need some results on coprimality of tangent curvatures in the packing.

If p divides the curvatures of two tangent circles $\mathcal{C}_1$ and $\mathcal{C}_2$, then it is an immediate consequence of primitivity that there is a circle tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$ that is coprime to p. However, we need something slightly stronger: that for any $\mathcal{C}_1$ and $\mathcal{C}_2$, we can find a circle tangent and coprime to both. For this, we characterise the possible curvatures tangent to both by a polynomial, and apply the Chinese remainder theorem.

Lemma 4.5. *Let (a,b,c,d) be a Descartes quadruple, where curvatures a, b correspond to circles $\mathcal{C}_1, \mathcal{C}_2$ respectively. The curvatures of the family of circles tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$ are parameterized by*

$$f(x) = (a+b)x^2 - (a+b+c-d)x + c, \quad x \in \mathbb{Z}.$$

Proof. The relevant family of curvatures is given by applying powers of S_4S_3 to the initial quadruple. We have that $2(a+b+f(x+1))-f(x)=f(x+2)$ and $2(a+b+f(x-1))-f(x)=f(x-2)$. We show by two-tailed induction that

$$(S_4S_3)^k(a,b,f(0),f(1))=(a,b,f(2k),f(2k+1)), k\in\mathbb{Z}.$$

The forward inductive step is

$$\begin{aligned}(S_4S_3)^{k+1}(a,b,f(0),f(1))&=(S_4S_3)(a,b,f(2k),f(2k+1))\\&=S_4(a,b,f(2k+2),f(2k+1))\\&=(a,b,f(2(k+1)),f(2(k+1)+1)),\end{aligned}$$

and the other direction is similar. □

Lemma 4.6. *Let $\mathcal{C}_1,\mathcal{C}_2\in\mathcal{A}$ be tangent circles in a primitive Apollonian circle packing. Then there exists a circle $\mathcal{C}'$ that is tangent to both $\mathcal{C}_1$ and $\mathcal{C}_2$, and whose curvature is coprime to both $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. It suffices to show that the function $f(x)$ of Lemma 4.5 takes at least one value simultaneously coprime to a and b. By the Chinese remainder theorem, it suffices, for each prime p dividing a or b, to show that $f(x)$ does not vanish identically over $\mathbb{F}_p$. Suppose $p=2$, so at least one of a or b is even. Then, since any Descartes quadruple has two even curvatures and two odd curvatures, $f(x)$ is not always even. For $p>2$, a quadratic polynomial can only vanish on all of $\mathbb{F}_p$ if its coefficients vanish, in which case $p\mid a+b,a+b+c-d,c$. But then $p\mid a,b,c,d$, a contradiction in a primitive packing. □

Corollary 4.7. *Let $\mathcal{C},\mathcal{C}'\in\mathcal{A}$ be two circles in a primitive Apollonian circle packing. Then there exists a path of circles $\mathcal{C}_1,\mathcal{C}_2,\dots,\mathcal{C}_k$ such that*

- *$\mathcal{C}_1=\mathcal{C}$ and $\mathcal{C}_k=\mathcal{C}'$;*
- *$\mathcal{C}_i$ is tangent to $\mathcal{C}_{i+1}$ for all $1\le i\le k-1$;*
- *The curvatures of $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are non-zero and coprime for all $1\le i\le k-1$.*

Proof. There exists a path of circles $\mathcal{C}_1,\mathcal{C}_2,\dots,\mathcal{C}_k$ that satisfy the first two requirements. For any consecutive pair whose curvatures are not coprime, by Lemma 4.6 we can insert another circle between them which is tangent with coprime curvature to each, which yields a valid path. □

The following corollary now follows directly from Proposition 4.4 and Corollary 4.7.

Corollary 4.8. *The value of χ_2 is constant across all circles in a fixed primitive Apollonian circle packing $\mathcal{A}$. Denote this value by $\chi_2(\mathcal{A})$.*

Before proving the sets of quadratic obstructions, we have one final coprimality lemma.

Lemma 4.9. *Let $\mathcal{C}$ be a circle of curvature n in a primitive Apollonian circle packing. Then there exists a circle tangent to $\mathcal{C}$ with curvature coprime to $6n$.*

Proof. If there is a circle tangent to $\mathcal{C}$ of curvature divisible by $d=6/\gcd(6,n)$, then we can use Lemma 4.6 to finish.

Using Lemma 3.2, let a represent the unique non-zero residue modulo 3 attained by this packing. By the same lemma, quadruples modulo 3 are of the form $(0,a,a,a)$ or $(0,0,a,a)$ up to permutation, and furthermore, S_2 swaps between these two. Modulo 2, they are always of the form $(0,0,1,1)$ up to permutation. Therefore, if $d = 2$ or 3, taking any quadruple containing $\mathcal{C}$ will suffice to find a curvature which is divisible by d.

If $d = 6$ (in which case n is coprime to 6), take a quadruple containing $\mathcal{C}$ in the first position. If it contains a curvature divisible by 6, we are done. Otherwise, without loss of generality, it is simultaneously of the form $(a,0,a,a) \pmod 3$ and $(1,1,0,0) \pmod 2$; by applying the swap S_3 or S_4 we can obtain a curvature divisible by 6. □

4.4. **Quadratic obstructions.** For each type of packing, we can assemble the above results to determine which values of u and χ_2 produce quadratic obstructions.

Proposition 4.10. *Let $\mathcal{A}$ have type $(6,k)$. Then the following quadratic obstructions occur, as a function of type and $\chi_2(\mathcal{A})$:*

Type	$\chi_2(\mathcal{A})$	Quadratic obstructions
(6, 1)	1	
	-1	$n^2, 2n^2, 3n^2, 6n^2$
(6, 5)	1	$2n^2, 3n^2$
	-1	$n^2, 6n^2$
(6, 13)	1	$2n^2, 6n^2$
	-1	$n^2, 3n^2$
(6, 17)	1	$3n^2, 6n^2$
	-1	$n^2, 2n^2$

Proof. We work with the contrapositive: for each type, we assume a quadratic family appears, and compute $\chi_2(\mathcal{A})$. Assume that a circle of curvature uw^2 appears in a packing $\mathcal{A}$ of type $(6,k)$. By Lemma 4.9, it is tangent to a circle $\mathcal{C}$ with curvature n coprime to $24uw^2$, hence we can take $n \equiv k \pmod{24}$. By Proposition 4.2, the existence of the curvature uw^2 tangent to $\mathcal{C}$ means

$$\left(\frac{u}{n}\right) = \left(\frac{\rho(f_{\mathcal{C}})}{n}\right) = \chi_2(\mathcal{A}),$$

using Definition 4.3 and Corollary 4.8. By quadratic reciprocity, we have

$k \pmod{24}$	$\left(\frac{2}{n}\right)$	$\left(\frac{3}{n}\right)$
1	1	1
5	-1	-1
13	-1	1
17	1	-1

These values give the claimed table. □

Note that not listing a value of u as a quadratic obstruction in the table does not imply that it cannot be an obstruction, only that this proof method does not work for it. The completeness of these lists is discussed in Section 6.

In particular, the $(6,k)$ entries in the table of Theorem 2.5 are filled in by intersecting the quadratic obstructions with the possible residue classes. Next, we complete the computation for $(8,k)$, which is similar but a little trickier. The issue stems from there being two residue classes that are coprime to 24.

Proposition 4.11. *Let $\mathcal{A}$ have type $(8,k)$. Then the following quadratic obstructions occur, as a function of type and $\chi_2(\mathcal{A})$:*

Type	$\chi_2(\mathcal{A})$	Quadratic obstructions
(8, 7)	1	$3n^2, 6n^2$
	-1	$2n^2$
(8, 11)	1	
	-1	$2n^2, 3n^2, 6n^2$

Proof. Repeat the proof of Proposition 4.10: assume that a circle of curvature uw^2 appears in $\mathcal{A}$. Thus there exists a circle of curvature n coprime to $24uw^2$ that is tangent to our starting circle. Thus

$$\left(\frac{u}{n}\right) = \left(\frac{\rho(f_{\mathcal{C}})}{n}\right) = \left(\frac{2}{n}\right)\chi_2(\mathcal{A}),$$

i.e. $\chi_2(\mathcal{A}) = \left(\frac{2u}{n}\right)$. By quadratic reciprocity, we have

$k \pmod{24}$	$\left(\frac{2}{n}\right)$	$\left(\frac{3}{n}\right)$
7	1	-1
11	-1	1
19	-1	-1
23	1	1

First, assume $u = 2$. Then $\chi_2(\mathcal{A}) = 1$, giving the conditions.

Next, take $u = 3$. The only admissible residue class with elements of the form $3w^2$ is $3 \pmod{24}$, so we can assume that w is odd. If the packing has type $(8,7)$, we divide in two cases.

If $n \equiv 7 \pmod{24}$, then $\chi_2(\mathcal{A}) = \left(\frac{2u}{n}\right) = -1$. The other possibility is $n \equiv 19 \pmod{24}$. However, in this case the two circles under consideration have curvatures which are $3 \pmod 8$, in contradiction to Corollary 3.5.

For packing type $(8,11)$, a similar argument works. The case $n \equiv 11 \pmod{24}$ is ruled out by Corollary 3.5, and $n \equiv 23 \pmod{24}$ implies $\chi_2(\mathcal{A}) = \left(\frac{2u}{n}\right) = 1$.

The final case is $u = 6$. Then $\chi_2(\mathcal{A}) = \left(\frac{3}{n}\right)$, and the conclusion follows from the table of residues. $\square$

Remark 4.12. It is reasonable to ask if the results in this section can be extended to other values of u, as we focused on $u \mid 6$. The proof will work for larger values of u that have no prime divisors other than 2 or 3, but these obstructions are already contained in those with $u \mid 6$. If u has a prime divisor $p \geq 5$, then the Kronecker symbol $\left(\frac{p}{n}\right)$ cannot be determined from the residue class $n \pmod{24}$. It will rule out uw^2 from appearing tangent to a subset of the circles in $\mathcal{A}$, but this is not enough to cover the entire packing. Interestingly, this suggests that there may be "partial" obstructions: quadratic families whose members appear less frequently than other curvatures of the same general size.

5. Quartic obstructions

The proof strategy in this section is similar to the quadratic case, where we define an invariant on the circles in the packing, and show that this forbids certain curvatures. The main difference is the quartic restrictions will only apply to two types of packings, and the propagation of the invariant comes down to quartic reciprocity for $\mathbb{Z}[i]$.

5.1. **Quartic reciprocity.** We recall the main definitions and results of quartic reciprocity; see Chapter 6 of [Lem00] for a longer exposition.

The Gaussian integers $\mathbb{Z}[i]$ form a unique factorization domain, with units being $\{1, i, -1, -i\}$. For $\alpha = a + bi \in \mathbb{Z}[i]$, denote the norm of α by $N(\alpha) := a^2 + b^2$. We call α *odd* if $N(\alpha)$ is odd, and *even* otherwise. An even α is necessarily divisible by $1 + i$.

Definition 5.1. If $\alpha = a + bi \in \mathbb{Z}[i]$ is odd, call it *primary* if $\alpha \equiv 1 \pmod{2 + 2i}$. This is equivalent to $(a, b) \equiv (1, 0), (3, 2) \pmod 4$.

The *associates* of α are α, $i\alpha$, $-\alpha$, $-i\alpha$. If α is odd, then exactly one associate of α is primary.

Definition 5.2. The quartic residue symbol $\left[\frac{\alpha}{\beta}\right]$ takes in two coprime elements $\alpha, \beta \in \mathbb{Z}[i]$ with β odd, and outputs a power of i. Let π be an odd prime of $\mathbb{Z}[i]$. If α is coprime to π, define $\left[\frac{\alpha}{\pi}\right]$ to be the unique power of i that satisfies

$$\left[\frac{\alpha}{\pi}\right] \equiv \alpha^{\frac{N(\pi)-1}{4}} \pmod{\pi}.$$

Extend the quartic residue symbol multiplicatively in the denominator:

$$\left[\frac{\alpha}{u\pi_1\pi_2\cdots\pi_n}\right] = \left[\frac{\alpha}{u}\right]\left[\frac{\alpha}{\pi_1}\right]\left[\frac{\alpha}{\pi_2}\right]\cdots\left[\frac{\alpha}{\pi_n}\right],$$

where $\left[\frac{\alpha}{u}\right] = 1$ for any unit $u \in \mathbb{Z}[i]$.

The basic properties of this symbol, and the statement of quartic reciprocity are summarized next, following [Lem00].

Proposition 5.3. [Lem00, Propositions 4.1, 6.8] *The quartic residue symbol satisfies the following properties:*

a) *If $\alpha_1, \alpha_2 \in \mathbb{Z}[i]$ with $\alpha_1\alpha_2$ coprime to an odd $\beta \in \mathbb{Z}[i]$, then $\left[\frac{\alpha_1\alpha_2}{\beta}\right] = \left[\frac{\alpha_1}{\beta}\right]\left[\frac{\alpha_2}{\beta}\right]$.*

b) *If $\alpha_1, \alpha_2, \beta \in \mathbb{Z}[i]$ satisfy $\alpha_1 \equiv \alpha_2 \pmod{\beta}$, α_1 and β are coprime, and β is odd, then $\left[\frac{\alpha_1}{\beta}\right] = \left[\frac{\alpha_2}{\beta}\right]$.*

c) *If $a, b \in \mathbb{Z}$ are coprime integers with b odd, then $\left[\frac{a}{b}\right] = 1$.*

Proposition 5.4. [Lem00, Theorem 6.9] *Let $\alpha = a + bi$ be primary. Then*

$$\left[\frac{i}{\alpha}\right] = i^{\frac{1-a}{2}}, \quad \left[\frac{-1}{\alpha}\right] = i^{b}, \quad \left[\frac{1+i}{\alpha}\right] = i^{\frac{a-b-b^2-1}{4}}, \quad \left[\frac{2}{\alpha}\right] = i^{\frac{-b}{2}}.$$

If $\beta \in \mathbb{Z}[i]$ is relatively prime to α and primary, then

$$\left[\frac{\alpha}{\beta}\right] = (-1)^{\frac{N(\alpha)-1}{4}\frac{N(\beta)-1}{4}}\left[\frac{\beta}{\alpha}\right].$$

In particular, if either α or β is an odd primary integer, then $\left[\frac{\alpha}{\beta}\right] = \left[\frac{\beta}{\alpha}\right]$.

5.2. **Definition of χ_4.** Let $\mathcal{C}$ be a circle of curvature $n \neq 0$ in a primitive Apollonian circle packing. It is associated to $[f_{\mathcal{C}}]$, a $\mathrm{PGL}(2, \mathbb{Z})-$equivalence class of primitive integral positive definite binary quadratic forms of discriminant $-4n^2$. In order to define $\chi_2(\mathcal{C})$, we observed that the values of the quadratic form $f_{\mathcal{C}}$ were consistent in their residuosity modulo n: they were all either quadratic residues, or all non-residues (at least once we restricted to those coprime to n). By using the bijection between quadratic forms and fractional ideals, we obtain a unique homothety class of lattices $[\Lambda]$, for some $\Lambda \subseteq \mathbb{Q}[i]$. Recall that the values of $f_{\mathcal{C}}$ are exactly the norms of the elements of Λ (up to a global scaling factor). By considering the elements of this lattice, instead of their norms, we can recover finer information than just quadratic residuosity: we can access quartic residuosity. This is the key insight in defining χ_4.

Up to multiplication by a unit, there is a unique representative of $[\Lambda]$ that lies inside $\mathbb{Z}[i]$ with covolume n [Sta18b, Proposition 5.1] (the cited proposition is more general, but it is only in the case of class number one that this is true of every homothety class).

In general, any rank-2 lattice $\Lambda \subseteq \mathbb{Z}[i]$ has an order, $\mathrm{ord}(\Lambda) := \{\lambda \in \mathbb{Z}[i] : \lambda\Lambda \subseteq \Lambda\}$, which is an order of $\mathbb{Q}(i)$, not necessarily maximal. By the *conductor of* Λ, we will mean the conductor of this order, which is a positive integer. The following can be proven by observing that Λ is locally principal, or by choosing a Hermite basis for Λ, which will have one element in $\mathrm{ord}(\Lambda)$.

Proposition 5.5. [Sta18b, Proposition 6.3] *Let n be a positive integer. Let $\Lambda \subseteq \mathbb{Z}[i]$ be a lattice of covolume n and conductor n, and define*

$$S_\Lambda := \{\beta \in \Lambda : \beta \text{ is coprime to } n\}.$$

Then the image of S_Λ in $(\mathbb{Z}[i]/n\mathbb{Z}[i])^\times/(\mathbb{Z}/n\mathbb{Z})^\times$ consists of a single element.

The uniqueness property allows us to define the invariant χ_4.

Definition 5.6. Let $\mathcal{A}$ be a primitive Apollonian circle packing of type $(6,1)$ or $(6,17)$, and let $\mathcal{C}$ be a circle of non-zero curvature n in $\mathcal{A}$, necessarily satisfying $n \equiv 0, 1, 4 \pmod 8$. Suppose $\mathcal{C}$ corresponds to a lattice $\Lambda_{\mathcal{C}} \subseteq \mathbb{Z}[i]$ of covolume n. Let $\beta = a + bi \in S_{\Lambda_{\mathcal{C}}} \cup S_{i\Lambda_{\mathcal{C}}}$, where β is chosen to be primary if n is even. Write $n = 2^e n'$ where n' is odd. Define $\chi_4(\mathcal{C})$ as

$$\chi_4(\mathcal{C}) := \begin{cases} (-1)^{be/4}\left[\frac{\beta}{n'}\right] & \text{if } n \equiv 0 \pmod 8; \\ \left[\frac{\beta}{n}\right] & \text{if } n \equiv 1 \pmod 8; \\ \left(\frac{-1}{n'}\right)\left[\frac{\beta}{n'}\right] & \text{if } n \equiv 4 \pmod 8. \end{cases}$$

Proposition 5.7. *There exists a choice of β satisfying all requirements, and the definition of χ_4 is well-defined, independent of this choice, and lies in $\{1, i, -1, -i\}$.*

Proof. First, the set $S_{\Lambda_{\mathcal{C}}} \cup S_{i\Lambda_{\mathcal{C}}}$ is uniquely determined by $\mathcal{C}$. If n is odd and β, β' are two choices, then by Proposition 5.5 we have $\beta' = i^k(u\beta + \delta)$ for $k = 0, 1$, an integer u coprime to n, and $\delta \in n\mathbb{Z}[i]$. Using Propositions 5.3 and 5.4 (and recalling that $n \equiv 1 \pmod 8$), we compute

$$\left[\frac{\beta'}{n}\right] = \left[\frac{i}{n}\right]^k \left[\frac{u\beta+\delta}{n}\right] = \left[\frac{u\beta}{n}\right] = \left[\frac{u}{n}\right]\left[\frac{\beta}{n}\right] = \left[\frac{\beta}{n}\right], \tag{5.1}$$

so the symbol is well-defined.

Next, assume n is even, hence a multiple of 4. Pick an arbitrary $\beta \in S_{\Lambda_{\mathcal{C}}}$, which is necessarily odd, and by replacing it with an associate, we can assume it is primary, which proves that a choice of β is possible.

As before, assume that two valid choices are β, β', so that $\beta' = i^k(u\beta + \delta)$ for some integer $k = 0, 1$, integer u coprime to n, and $\delta \in n\mathbb{Z}[i]$. Also write $\beta = a + bi$ and $\beta' = a' + b'i$.

If $k = 1$, then b and b' have opposite parity, a contradiction to them being primary. Therefore $k = 0$, so $\beta' = u\beta + \delta$. In particular, as n' is an odd integer, the analogous computation to Equation 5.1 still holds, so $\left[\frac{\beta'}{n'}\right] = \left[\frac{\beta}{n'}\right]$. It remains to check that the extra factors in the definition of χ_4 in the even case are also independent.

If $n \equiv 4 \pmod 8$, the extra factor is $\left(\frac{-1}{n'}\right)$, which does not depend on β.

If $n \equiv 0 \pmod 8$, the extra factor is $(-1)^{be/4}$, which depends on b. We must verify that $b \equiv 0 \pmod 4$, so that the exponent of $be/4$ is integral. Assuming this is true, and using that $8 \mid \delta$ and u is odd, we have $b' \equiv ub \equiv b \pmod 8$, which completes the proof.

To prove that $b \equiv 0 \pmod 4$, note that $a^2 + b^2 = N(\beta)$ is a value properly represented by $f_{\mathcal{C}}(x, y)$. If $b \not\equiv 0 \pmod 4$, then $a^2 + b^2 \equiv 5 \pmod 8$, hence $f_{\mathcal{C}}(x, y) - n$ properly represents a number that is $5 - 0 \equiv 5 \pmod 8$, i.e. $\mathcal{A}$ contains a curvature of this form. However, we are in a packing of type either $(6, 1)$ or $(6, 17)$, where all odd curvatures are $1 \pmod 8$, contradiction. $\square$

5.3. **Propagation of χ_4.** Assume $\mathcal{A}$ is a primitive Apollonian circle packing of type $(6, 1)$ or $(6, 17)$. In order to relate the χ_4 value of tangent circles, we need a value of β that works for both.

Proposition 5.8. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be tangent circles of non-zero coprime curvatures n_1, n_2 in $\mathcal{A}$. Then there exists a $\beta \in \mathbb{Z}[i]$ such that*

- *$N(\beta) = n_1 + n_2$;*
- *β is a valid choice in Definition 5.6 for both $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. The β is described by [Sta18b, Proposition 4.6], and is defined up to unit multiple, which allows for choosing β to be primary if necessary. That β is in $\Lambda_{\mathcal{C}_1}$ and $\Lambda_{\mathcal{C}_2}$ is a consequence of [Sta18b, Theorem 1.4, Theorem 4.7]. That the norm is correct is a consequence of [Sta18b, Theorem 4.7].

Because this proof depends so heavily on the results from [Sta18b], we will provide an overview of the relevant ideas there. The orbit of the extended real line $\widehat{\mathbb{R}}$ under the Möbius transformations $\mathrm{PSL}(2, \mathbb{Z}[i])$ includes all primitive integral Apollonian circle packings (scaled by a factor of $1/2$); we call this the Schmidt arrangement. Thus, given the packing $\mathcal{A}$, we can place it within the extended complex plane $\widehat{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C})$ in a canonical way, so that the tangency points of any circle $\mathcal{C}$ are given by the projectivization of the $\mathbb{Z}$-span in $\mathbb{C}^2$ of two vectors $[\alpha, \beta]$, and $[\gamma, \delta]$, corresponding to tangency points α/β and γ/δ. Therefore the denominators of the tangency points in the packing form a lattice $\Lambda = \beta\mathbb{Z} + \delta\mathbb{Z}$. In [Sta18b], it is shown that these are the Λ described in the previous section. In particular, the lattices of tangent circles share a primitive vector corresponding to the tangency point. $\square$

Proposition 5.9. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be tangent circles of non-zero coprime curvature in $\mathcal{A}$. Then $\chi_4(\mathcal{C}_1) = \chi_4(\mathcal{C}_2)$.*

Proof. Let n_1 and n_2 be the curvatures of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Since we are in type $(6, 1)$ or $(6, 17)$, $n_1, n_2 \equiv 0, 1$, or $4 \pmod 8$. Take a β as promised by Proposition 5.8, and assume that n_1 is odd. If n_2 is also odd, then $N(\beta) = n_1 + n_2 \equiv 2 \pmod 8$, hence $\beta = (1 + i)\beta'$, with β' odd. By replacing β with an associate, we can assume that $\beta' = a + bi$ is primary. We compute

$$\chi_4(\mathcal{C}_1) = \left[\frac{\beta}{n_1}\right] = \left[\frac{1+i}{n_1}\right]\left[\frac{\beta'}{n_1}\right] = i^{\frac{n_1-1}{4}}\left[\frac{n_1}{\beta'}\right].$$

As $n_1 + n_2 = N(\beta) = \beta\overline{\beta}$, we have $n_1 \equiv -n_2 \pmod{\beta'}$. Thus

$$\chi_4(\mathcal{C}_1) = i^{\frac{n_1-1}{4}}\left[\frac{-n_2}{\beta'}\right] = i^{\frac{n_1-1}{4}}\left[\frac{-1}{\beta'}\right]\left[\frac{n_2}{\beta'}\right] = i^{\frac{n_1-1}{4}} i^b \left[\frac{\beta'}{n_2}\right] = i^{\frac{n_1-1}{4}} i^b i^{-\frac{n_2-1}{4}} \chi_4(\mathcal{C}_2).$$

In order to conclude that $\chi_4(\mathcal{C}_1) = \chi_4(\mathcal{C}_2)$, we must have $\frac{n_1-1}{4} + b - \frac{n_2-1}{4} \equiv 0 \pmod 4$, i.e.

$$n_1 - n_2 + 4b \equiv 0 \pmod{16}. \tag{5.2}$$

If $n_1 \equiv n_2 \pmod{16}$, then $2(a^2+b^2) = n_1+n_2 \equiv 2 \pmod{16}$, hence $a^2+b^2 \equiv 1 \pmod 8$. In particular, $4 \mid b$ (recall that β' is primary), and Equation 5.2 follows. Otherwise, $n_1 \equiv n_2 + 8 \pmod{16}$, so $2(a^2+b^2) \equiv 10 \pmod{16}$, and $a^2+b^2 \equiv 5 \pmod 8$. This implies that $b \equiv 2 \pmod 4$, and again Equation 5.2 is true.

If n_2 is even, then $\beta = a + bi$ is primary by assumption. We compute

$$\chi_4(\mathcal{C}_1) = \left[\frac{\beta}{n_1}\right] = \left[\frac{n_1}{\beta}\right] = \left[\frac{-n_2}{\beta}\right] = \left[\frac{n_2/4}{\beta}\right],$$

where the last equality follows from $\left[\frac{-4}{\beta}\right] = \left[\frac{1+i}{\beta}\right]^4 = 1$. We now separate into the cases $n_2 \equiv 0$ or $4 \pmod 8$.

If $n_2 \equiv 0 \mod 8$, write $n_2 = 2^e n_2'$ with n_2' odd, and

$$\chi_4(\mathcal{C}_2) = (-1)^{be/4}\left[\frac{\beta}{n_2'}\right] = (-1)^{be/4}\left[\frac{\pm n_2'}{\beta}\right] = (-1)^{be/4}\left[\frac{\mp 1}{\beta}\right]\left[\frac{-n_2'}{\beta}\right] = (-1)^{be/4}\left[\frac{\mp 1}{\beta}\right]\left[\frac{2^e}{\beta}\right]^{-1}\chi_4(\mathcal{C}_1),$$

where the sign of the $\pm$ depends on n_2' modulo 4. As in the proof of Proposition 5.7, we have $4 \mid b$, hence $\left[\frac{-1}{\beta}\right] = i^b = 1$, so the $\pm$ sign does not matter. We also compute

$$\left[\frac{2^e}{\beta}\right]^{-1} = \left(i^{-b/2}\right)^{-e} = (-1)^{be/4} = (-1)^{-be/4},$$

hence the terms cancel and $\chi_4(\mathcal{C}_2) = \chi_4(\mathcal{C}_1)$.

Finally, assume $n_2 \equiv 4 \pmod 8$, so $n_2' = n_2/4$. If $n_2' \equiv 1 \pmod 4$, then

$$\chi_4(\mathcal{C}_2) = \left(\frac{-1}{n_2'}\right)\left[\frac{\beta}{n_2'}\right] = \left[\frac{n_2'}{\beta}\right] = \chi_4(\mathcal{C}_1),$$

as desired. Otherwise, $n_2' \equiv 3 \pmod 4$ and

$$\chi_4(\mathcal{C}_2) = \left(\frac{-1}{n_2'}\right)\left[\frac{\beta}{n_2'}\right] = -\left[\frac{-n_2'}{\beta}\right] = -\left[\frac{-1}{\beta}\right]\chi_4(\mathcal{C}_1) = -i^{-b}\chi_4(\mathcal{C}_1).$$

Since $a^2+b^2 = n_1+n_2 \equiv 5 \pmod 8$, we have $b \equiv 2 \pmod 4$, so $-i^b = 1$, completing the proof. ☐

Corollary 4.7 and Proposition 5.9 combine to give the following corollary.

Corollary 5.10. *The value of χ_4 is constant across all circles in a fixed Apollonian circle packing $\mathcal{A}$ of type $(6,1)$ or $(6,17)$. Denote this value by $\chi_4(\mathcal{A})$.*

5.4. **Quartic obstructions.** Assume that $\mathcal{A}$ is a primitive Apollonian circle packing of type $(6,1)$ or $(6,17)$. As in the quadratic section, χ_4 determines the quartic obstructions.

Proposition 5.11. *Then the following quartic obstructions occur, as a function of type and $\chi_4(\mathcal{A})$:*

Type	$\chi_4(\mathcal{A})$	Quartic obstructions
(6, 1)	1	
	$-1, i, -i$	$n^4, 4n^4, 9n^4, 36n^4$
(6, 17)	1	$9n^4, 36n^4$
	-1	$n^4, 4n^4$
	$i, -i$	$n^4, 4n^4, 9n^4, 36n^4$

Proof. Let $u \in \{1, 4, 9, 36\}$, and assume that a circle $\mathcal{C}$ of curvature $n = uw^4$ appears in $\mathcal{A}$ for some positive integer w. The proof works in a "contrapositive" manner: we divide into cases based on $n \pmod 8$, and, for each u, restrict the values of $\chi_4(\mathcal{A})$ that may permit the curvature n to appear. Let n_2 be the curvature of a circle $\mathcal{C}'$ tangent to $\mathcal{C}$ that is coprime to n. Let β be chosen as in Proposition 5.8 for the circles $\mathcal{C}$ and $\mathcal{C}'$.

If $n \equiv 0 \pmod 8$, then $2 \mid w$, hence $n = 2^e n'$ with n' odd and $2 \mid e$. Thus

$$\chi_4(\mathcal{C}) = (-1)^{be/4}\left[\frac{\beta}{n'}\right] = \begin{cases} \left[\frac{\beta}{1}\right] & \text{if } u = 1, 4; \\ \left[\frac{\beta}{3}\right]^2 & \text{if } u = 9, 36. \end{cases}$$

Clearly $\left[\frac{\beta}{1}\right] = 1$, which gives the result for $u = 1, 4$ (specifically, if any curvature of the form n^4 or $4n^4$ appears, then $\chi_4(\mathcal{A}) = 1$). For $u = 9, 36$, we have $n \equiv 0 \pmod{24}$. Let $\beta = a + bi$, and as 3 is prime in $\mathbb{Z}[i]$,

$$\left[\frac{\beta}{3}\right]^2 \equiv \beta^4 \equiv (a + bi)^4 \equiv a^4 + b^4 + (a^3 b - ab^3)i \pmod 3.$$

Then $a^2 + b^2 = n + n_2 \equiv n_2 \pmod 3$. If the type of $\mathcal{A}$ is $(6, 1)$, then $a^2 + b^2 \equiv 1 \pmod 3$, so exactly one of (a, b) is $0 \pmod 3$. In either case, $a^4 + b^4 + (a^3 b - ab^3)i \equiv 1 \pmod 3$, so $\chi_4(\mathcal{C}) = 1$. If the type is $(6, 17)$, then $a^2 + b^2 \equiv 2 \pmod 3$, so $a^2 \equiv b^2 \equiv 1 \pmod 3$. Thus

$$a^4 + b^4 + (a^3 b - ab^3)i \equiv \left(a^2\right)^2 + \left(b^2\right)^2 + ab(a^2 - b^2)i \equiv 2 \equiv -1 \pmod 3,$$

so $\chi_4(\mathcal{C}) = -1$, again agreeing with the table.

Next, assume $n \equiv 1 \pmod 8$. If $u = 1$, we immediately have $\chi_4(\mathcal{C}) = \left[\frac{\beta}{w^4}\right] = 1$. Otherwise we have $u = 9$, and in fact $n \equiv 9 \pmod{24}$. Then $\chi_4(\mathcal{C}) = \left[\frac{\beta}{3^2 w^4}\right] = \left[\frac{\beta}{3}\right]^2$, from whence the analysis proceeds exactly as in the case of $n \equiv 0 \pmod 8$.

Finally, take $n \equiv 4 \pmod 8$, which allows $u = 4, 36$. If $u = 4$, then n' is an odd fourth power, so $\chi_4(\mathcal{C}) = \left(\frac{-1}{n'}\right)\left[\frac{\beta}{n'}\right] = 1$. If $u = 36$, then $n' = 9t^4$, so $\chi_4(\mathcal{C}) = \left[\frac{\beta}{3}\right]^2$, which is again identical to the case of $n \equiv 0 \pmod 8$. $\square$

There is a nice relationship between $\chi_4(\mathcal{A})$ and $\chi_2(\mathcal{A})$.

Proposition 5.12. $\chi_4(\mathcal{A})^2 = \chi_2(\mathcal{A})$.

Proof. Let $\mathcal{C}$ be a circle of odd curvature n in $\mathcal{A}$. Choose a circle $\mathcal{C}'$ tangent to $\mathcal{C}$ of coprime curvature n_2, and choose β as in Proposition 5.8 for circles $\mathcal{C}$ and $\mathcal{C}'$. Then $\chi_4(\mathcal{C})^2 = \left[\frac{\beta}{n}\right]^2$. Let $\left(\frac{\cdot}{\cdot}\right)$ also denote the quadratic residue symbol for $\mathbb{Z}[i]$, and it follows that $\chi_4(\mathcal{C})^2 = \left(\frac{\beta}{n}\right)$. By [Lem00, Proposition 4.2iii)], we have $\left(\frac{\beta}{n}\right) = \left(\frac{N(\beta)}{n}\right)$, with the second Kronecker symbol taken over $\mathbb{Z}$. Since $N(\beta) = n + n_2$, we can take $\rho(f_{\mathcal{C}}) = n + n_2$, proving that $\chi_4(\mathcal{C})^2 = \chi_2(\mathcal{C})$. As χ_2 and χ_4 are constant across $\mathcal{A}$, the result follows. $\square$

From this proposition, if $\chi_4(\mathcal{A}) = \pm i$, then $\chi_2(\mathcal{A}) = -1$, implying that there are no squares in the packing. Since all quartic obstructions are squares, these cases were already removed, and we get nothing new. This is why we only list quartic obstructions for types $(6, 1, 1)$ and $(6, 17, 1)$.

6. Computations

Table 1. $S_{\mathcal{A}}(N)$ for small packings: part 1.

Packing	Type	Quadratic	Quartic	N	$\lvert S_{\mathcal{A}}(N)\rvert$	$\max(S_{\mathcal{A}}(N))$	$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$
$(0,0,1,1)$	$(6,1,1,1)$			10^{10}	215	1199820	8334.58
$(-12,16,49,49)$				10^{11}	275276	5542869468	18.04
$(-20,36,49,49)$				10^{11}	2014815	55912619880	1.79
$(-8,12,25,25)$	$(6,1,1,-1)$		$n^4, 4n^4,$	10^{10}	47070	517280220	19.33
$(-12,25,25,28)$			$9n^4, 36n^4$	10^{11}	238268	5919707820	16.89
$(-15,24,40,49)$				$2 \cdot 10^{11}$	639149	12692531688	15.75
$(-15,28,33,40)$	$(6,1,-1)$	$n^2, 2n^2,$		10^{10}	80472	820523160	12.19
$(-20,33,52,57)$		$3n^2, 6n^2$		10^{11}	240230	4127189100	24.23
$(-23,40,57,60)$				10^{11}	392800	8689511520	11.51
$(-4,5,20,21)$	$(6,5,1)$	$2n^2, 3n^2$		10^{10}	3659	32084460	311.68
$(-16,29,36,45)$				10^{10}	80256	927211800	10.79
$(-19,36,44,45)$				10^{11}	177902	3603790320	27.75
$(-3,5,8,8)$	$(6,5,-1)$	$n^2, 6n^2$		10^{10}	676	3122880	3202.17
$(-12,21,29,32)$				10^{10}	30347	312225420	32.03
$(-19,32,48,53)$				$2.5 \cdot 10^{10}$	168264	2286209460	10.94
$(-3,4,12,13)$	$(6,13,1)$	$2n^2, 6n^2$		10^{10}	731	7354464	1359.72
$(-12,21,28,37)$				10^{11}	234386	3470731680	28.81
$(-11,16,36,37)$				10^{10}	20748	226988340	44.06
$(-8,13,21,24)$	$(6,13,-1)$	$n^2, 3n^2$		10^{10}	5273	45348900	220.51
$(-11,21,24,28)$				10^{10}	21003	176441136	56.68
$(-20,37,45,52)$				10^{11}	229356	4079861484	24.51
$(-16,32,33,41)$	$(6,17,1,1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	10^{10}	81777	841440840	11.88
$(-7,8,56,57)$				10^{10}	55057	595231740	16.80
$(-16,20,81,81)$				$3 \cdot 10^{11}$	1075024	26983035480	11.12
$(-4,8,9,9)$	$(6,17,1,-1)$	$3n^2, 6n^2$	$n^4, 4n^4$	10^{10}	2057	10742460	930.89
$(-7,9,32,32)$				10^{10}	34916	367956840	27.18
$(-15,32,32,33)$				10^{11}	585942	8505627180	11.76
$(-7,12,17,20)$	$(6,17,-1)$	$n^2, 2n^2$		10^{10}	3744	17141220	583.39
$(-12,17,41,44)$				10^{10}	31851	270186456	37.01
$(-15,24,41,44)$				10^{10}	80106	803343900	12.45

TABLE 2. $S_{\mathcal{A}}(N)$ for small packings: part 2.

Packing	Type	Quadratic	Quartic	N	$\lvert S_{\mathcal{A}}(N)\rvert$	$\max(S_{\mathcal{A}}(N))$	$\approx \frac{N}{\max(S_{\mathcal{A}}(N))}$
$(-5,7,18,18)$	$(8,7,1)$	$3n^2, 6n^2$		10^{10}	16417	86709570	115.33
$(-6,10,15,19)$				10^{10}	24305	133977255	74.64
$(-9,18,19,22)$				10^{10}	14866	82815750	120.75
$(-2,3,6,7)$	$(8,7,-1)$	$18n^2$		10^{10}	236	429039	23307.90
$(-5,6,30,31)$				10^{10}	19695	97583070	102.48
$(-14,27,31,34)$				$2\cdot 10^{10}$	99294	1643827935	12.17
$(-1,2,2,3)$	$(8,11,1)$			10^{10}	61	97287	102788.66
$(-9,14,26,27)$				10^{10}	17949	85926675	116.38
$(-10,18,23,27)$				10^{10}	25944	124625694	80.24
$(-6,11,14,15)$	$(8,11,-1)$	$2n^2, 3n^2,$		10^{10}	3381	20149335	496.29
$(-10,14,35,39)$		$6n^2$		$4\cdot 10^{10}$	256228	2934238515	13.63
$(-13,23,30,38)$				10^{10}	71341	598107510	16.72

In order to support Conjecture 1.5, code to compute the missing curvatures and remove the quadratic and quartic families was written with a combination of C and PARI/GP [PAR23]. This code (alongside other methods to compute with Apollonian circle packings) can be found in the GitHub repository [Ric23a].

Files containing the sporadic sets for many small root quadruples can be found in the GitHub repository [Ric23b]. We summarize the results for the smallest three quadruples (ordered by the sum of the root quadruple curvatures) of each type in Tables 1 and 2.

In every case, the sporadic set appears to thin out in a typical way as the curvature bound increases. This is illustrated in Figure 2, showing the decreasing proportion of sporadic curvatures, as curvature size increases, for some large sporadic sets. The last column of Table 1 and 2 shows the ratio of the largest curvature computed to the last known sporadic curvature. As this ratio increases, we can be increasingly confident that we have found the full sporadic set.

In Figure 3, we compare our predicted values of $\#S_{\mathcal{A}}$ to the "size" of the packing, and observe an upward trend.

Remark 6.1. There are only 5 pairs of residue class and packing for which it appears that *every single* positive residue in that class appears. They are:

- $(-3,5,8,8)$ and $5 \pmod{24}$;
- $(-3,4,12,13)$ and $13 \pmod{24}$;
- $(-1,2,2,3)$ and $11, 14, 23 \pmod{24}$.

Near misses are

- $(0,0,1,1)$ and $1 \pmod{24}$, which only misses the curvature 241 up to 10^{10};
- $(-1,2,2,3)$ and $2 \pmod{24}$, which only misses the curvature 13154 up to 10^{10}.

Remark 6.2. An intrepid observer of the raw sporadic sets may remark that, toward the tail end, the sporadic curvatures are disproportionately multiples of 5. In fact, they generally prefer prime divisors which are $1 \pmod 4$. We speculate that this is another local phenomenon: a result of certain symmetries of the

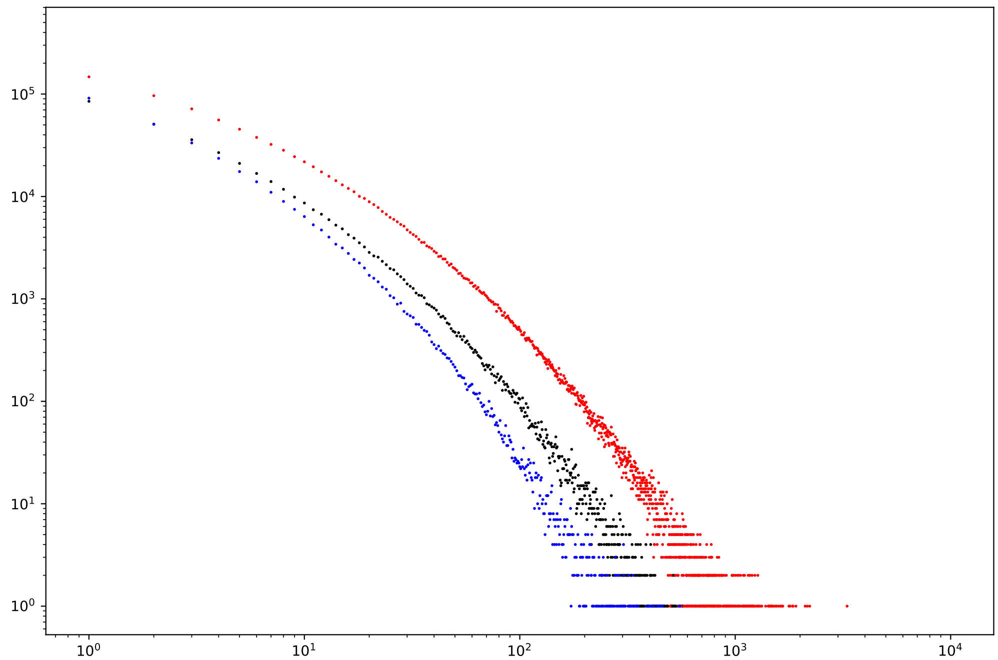

FIGURE 2. The decay of the sporadic sets for the packings $(-15, 17, 128, 128)$ (type $(6, 17, 1, 1)$, red, upper), $(-15, 24, 40, 49)$ (type $(6, 1, 1, -1)$, black, middle) and $(-21, 30, 71, 74)$ (type $(8, 11, 1)$, blue, lower). The x-th position on the x-axis represents bin $B_x = [x \cdot 10^4, (x + 1) \cdot 10^4) \cap \mathbb{Z}$ of size 10^4; we plot $y(x) = \#(S_{\mathcal{A}}(10^{11}) \cap B_x)$ on loglog axes. Loosely speaking, the resulting plot is a loglog plot of the probability that a curvature is sporadic, as curvature size increases.

distribution of curvatures in the orbit of quadruples modulo $p \equiv 1 \pmod 4$ (similar to [FS11, Figures 3 and 4]).

Remark 6.3. One particularly visually appealing way to observe the power obstructions is to plot the successive differences of the exceptional set. Since quadratic and quartic sequences have patternful successive differences, even a union of quadratic and quartic sequences reveals a prominent visual pattern once the sporadic set begins to thin out. See Figure 4.

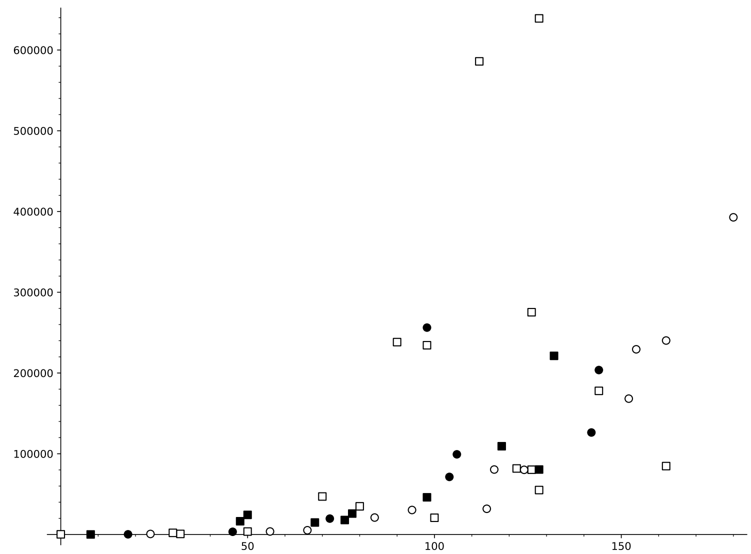

FIGURE 3. Scatterplot of $|a|+|b|+|c|+|d|$ on the x-axis vs. $\#S_{\mathcal{A}}(N)$ on the y-axis for small packings $\mathcal{A}$ of root quadruple (a, b, c, d) for which we are fairly confident in our prediction for $S_{\mathcal{A}}$ (those for which $N/\max(S_{\mathcal{A}}(N)) \geq 5$). Packings of type $(6, k)$ are in black, and type $(8, k)$ are in white. Packings with $\chi_2(\mathcal{A}) = -1$ appear as circles, and those with $\chi_2(\mathcal{A}) = 1$ are squares.

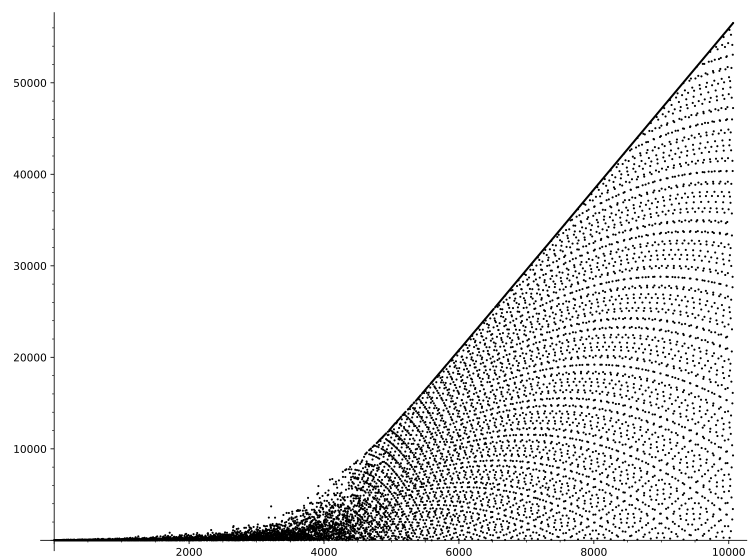

FIGURE 4. Successive differences of missing curvatures in the packing $(-4, 5, 20, 21)$ (type (6,5,1)). Around the 5000th missing curvature, the quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and they continue to occur even past the region of this graph, but very sparsely.).

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