

The Local-Global Conjecture is False

Clyde Kertzer

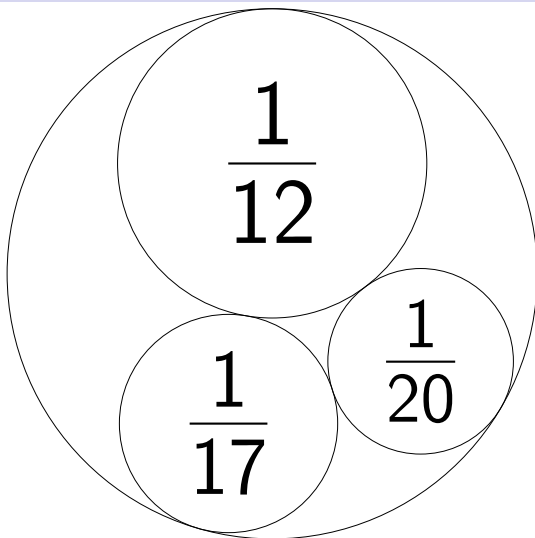
University of Colorado Boulder

Feb 21, 2026

Circle Packings

The Local-
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is False

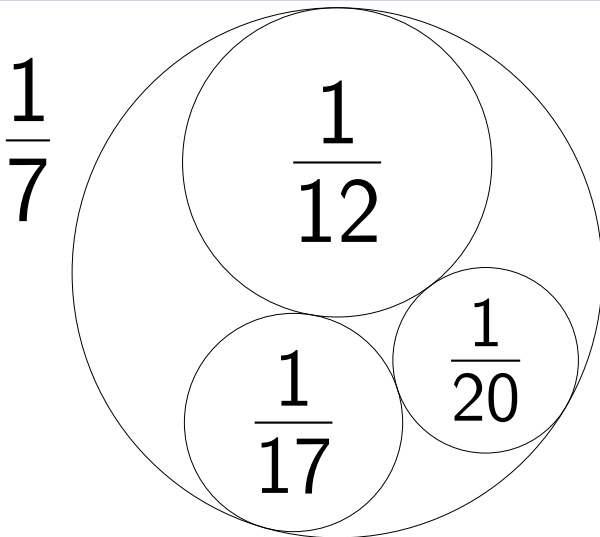
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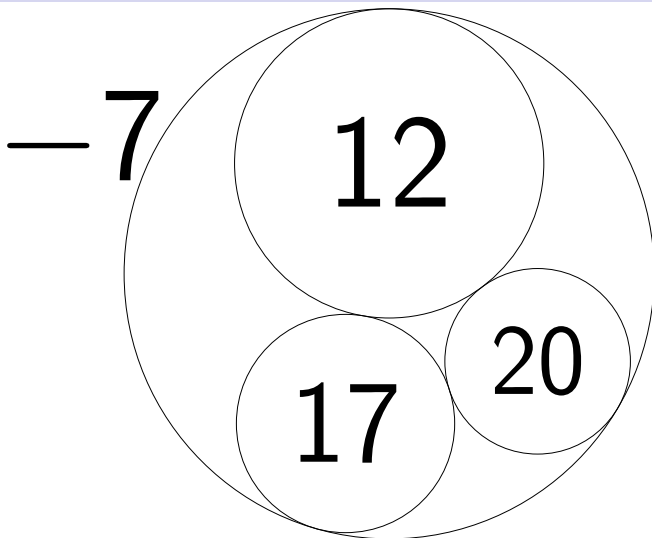
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Summary: The Descartes Equation

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Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

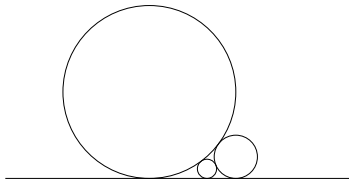
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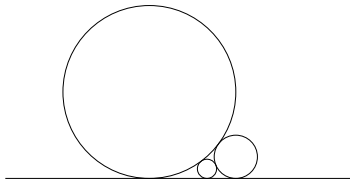
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Circle with infinite radius (Curvature 0)

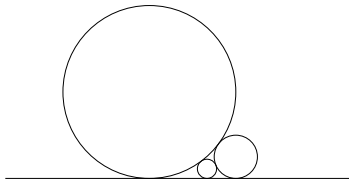
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Four mutually tangent circles are called a *Descartes Quadruple*.

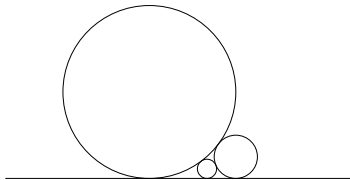
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Descartes Equation

If four mutually tangent circles have curvatures a , b , c , d then

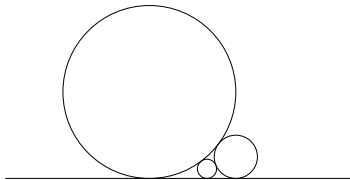
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Descartes Equation

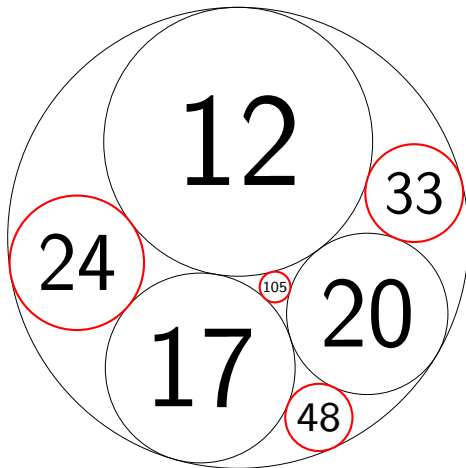
If four mutually tangent circles have curvatures a, b, c, d then

$$(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2).$$

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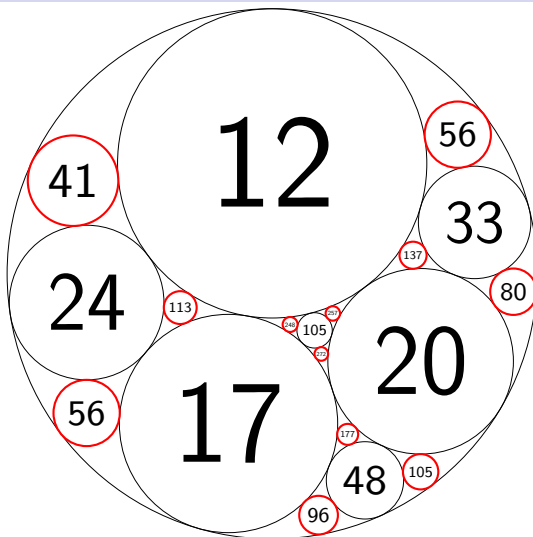
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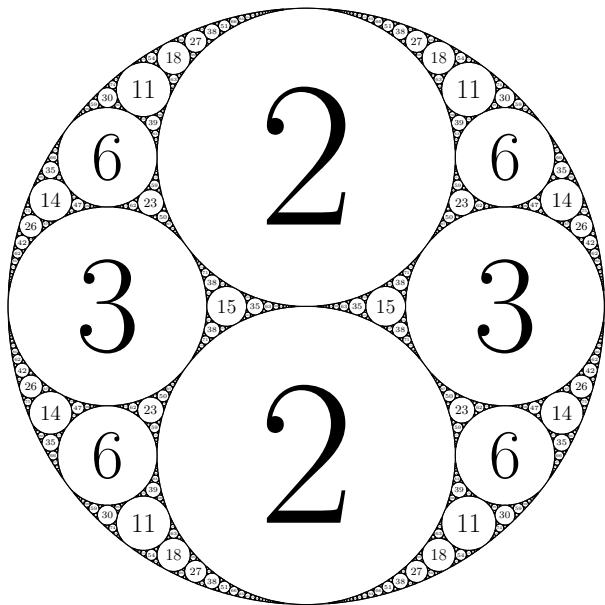
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A circular diagram illustrating a mathematical process. The outermost ring contains the number 12. Inside this ring, there are several smaller circles containing numbers: 17, 20, 24, 41, 56, 68, 89, 105, 132, 152, 185, 200, 240, 249, 264, 273, 277, 280, 283, 287, 290, 293, 296, 299, 302, 305, 308, 311, 314, 317, 320, 323, 326, 329, 332, 335, 338, 341, 344, 347, 350, 353, 356, 359, 362, 365, 368, 371, 374, 377, 380, 383, 386, 389, 392, 395, 398, 401, 404, 407, 410, 413, 416, 419, 422, 425, 428, 431, 434, 437, 440, 443, 446, 449, 452, 455, 458, 461, 464, 467, 470, 473, 476, 479, 482, 485, 488, 491, 494, 497, 500, 503, 506, 509, 512, 515, 518, 521, 524, 527, 530, 533, 536, 539, 542, 545, 548, 551, 554, 557, 560, 563, 566, 569, 572, 575, 578, 581, 584, 587, 590, 593, 596, 599, 602, 605, 608, 611, 614, 617, 620, 623, 626, 629, 632, 635, 638, 641, 644, 647, 650, 653, 656, 659, 662, 665, 668, 671, 674, 677, 680, 683, 686, 689, 692, 695, 698, 701, 704, 707, 710, 713, 716, 719, 722, 725, 728, 731, 734, 737, 740, 743, 746, 749, 752, 755, 758, 761, 764, 767, 770, 773, 776, 779, 782, 785, 788, 791, 794, 797, 800, 803, 806, 809, 812, 815, 818, 821, 824, 827, 830, 833, 836, 839, 842, 845, 848, 851, 854, 857, 860, 863, 866, 869, 872, 875, 878, 881, 884, 887, 890, 893, 896, 899, 902, 905, 908, 911, 914, 917, 920, 923, 926, 929, 932, 935, 938, 941, 944, 947, 950, 953, 956, 959, 962, 965, 968, 971, 974, 977, 980, 983, 986, 989, 992, 995, 998, 1000. The innermost ring contains the numbers 17, 20, 24, 41, 56, 68, 89, 105, 132, 152, 185, 200, 240, 249, 264, 273, 277, 280, 283, 287, 290, 293, 296, 299, 302, 305, 308, 311, 314, 317, 320, 323, 326, 329, 332, 335, 338, 341, 344, 347, 350, 353, 356, 359, 362, 365, 368, 371, 374, 377, 380, 383, 386, 389, 392, 395, 398, 401, 404, 407, 410, 413, 416, 419, 422, 425, 428, 431, 434, 437, 440, 443, 446, 449, 452, 455, 458, 461, 464, 467, 470, 473, 476, 479, 482, 485, 488, 491, 494, 497, 500, 503, 506, 509, 512, 515, 518, 521, 524, 527, 530, 533, 536, 539, 542, 545, 548, 551, 554, 557, 560, 563, 566, 569, 572, 575, 578, 581, 584, 587, 590, 593, 596, 599, 602, 605, 608, 611, 614, 617, 620, 623, 626, 629, 632, 635, 638, 641, 644, 647, 650, 653, 656, 659, 662, 665, 668, 671, 674, 677, 680, 683, 686, 689, 692, 695, 698, 701, 704, 707, 710, 713, 716, 719, 722, 725, 728, 731, 734, 737, 740, 743, 746, 749, 752, 755, 758, 761, 764, 767, 770, 773, 776, 779, 782, 785, 788, 791, 794, 797, 800, 803, 806, 809, 812, 815, 818, 821, 824, 827, 830, 833, 836, 839, 842, 845, 848, 851, 854, 857, 860, 863, 866, 869, 872, 875, 878, 881, 884, 887, 890, 893, 896, 899, 902, 905, 908, 911, 914, 917, 920, 923, 926, 929, 932, 935, 938, 941, 944, 947, 950, 953, 956, 959, 962, 965, 968, 971, 974, 977, 980, 983, 986, 989, 992, 995, 998, 1000.

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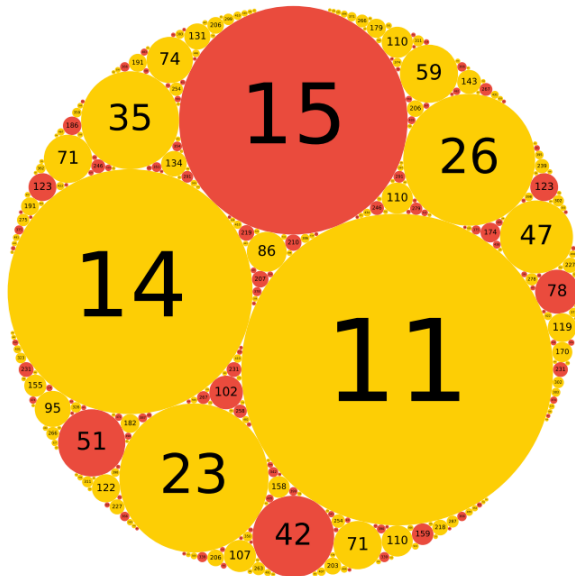
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Allowed Residues

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Theorem (Fuchs-Sanden (2010))

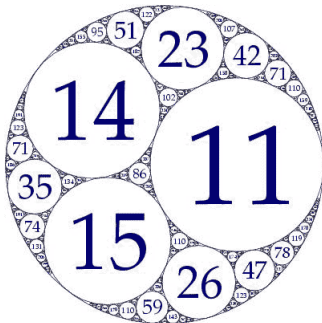
If a congruence obstruction appears, then it appears modulo 24.

Type	Allowed Residues
(6, 1)	0, 1, 4, 9, 12, 16
(6, 5)	0, 5, 8, 12, 20, 21
(6, 13)	0, 4, 12, 13, 16, 21
(6, 17)	0, 8, 9, 12, 17, 20
(8, 7)	3, 6, 7, 10, 15, 18, 19, 22
(8, 11)	2, 3, 6, 11, 14, 15, 18, 23

Allowed Residues

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$[-6, 11, 14, 15]$

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No room for 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 17, ...

Missing Curvatures?

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Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$

$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

and observed for $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \bmod 24$.

Local-to-global

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Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan 2003, Fuchs-Sanden, 2010)

Curvatures satisfy a congruence condition mod 24, and all sufficiently large integers satisfying this condition appear.

Theorem (Bourgain-Kontorovich, 2014)

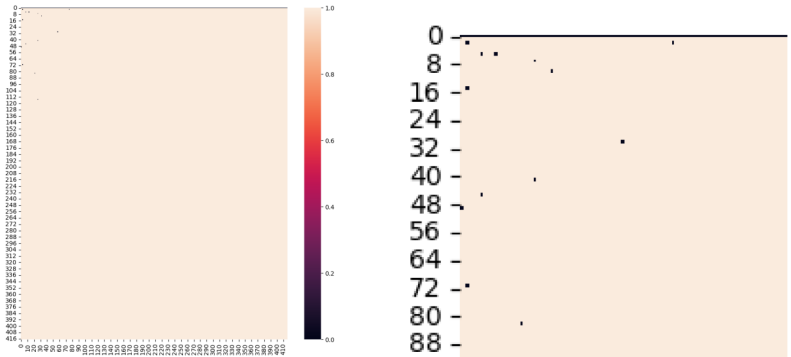
The number of missing curvatures up to N is at most $O(N^{1-\varepsilon})$ for some computable $\varepsilon > 0$.

1. Fix a pair of curvatures, and study what packings contain them.
2. Plot: for an admissible pair of residue classes modulo 24, black dot if no packing has that pair.
3. Local-global: finitely many black dots on any row or column.

Typical graph

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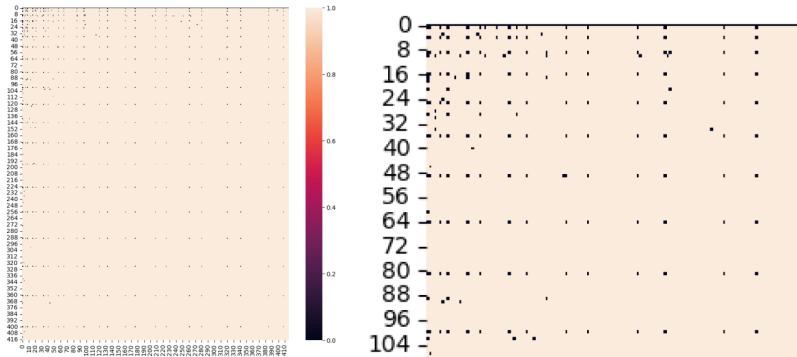


Residue classes 0 (mod 24) and 12 (mod 24) (Summer Haag)

One weird graph

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Residue classes 0 (mod 24) and 8 (mod 24) (Summer Haag)

No bug—The conjecture is false!

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Theorem (Haag-Kertzer-Rickards-Stange)

The packing $[-3, 5, 8, 8]$ has no square curvatures.

The New Conjecture

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The New Conjecture

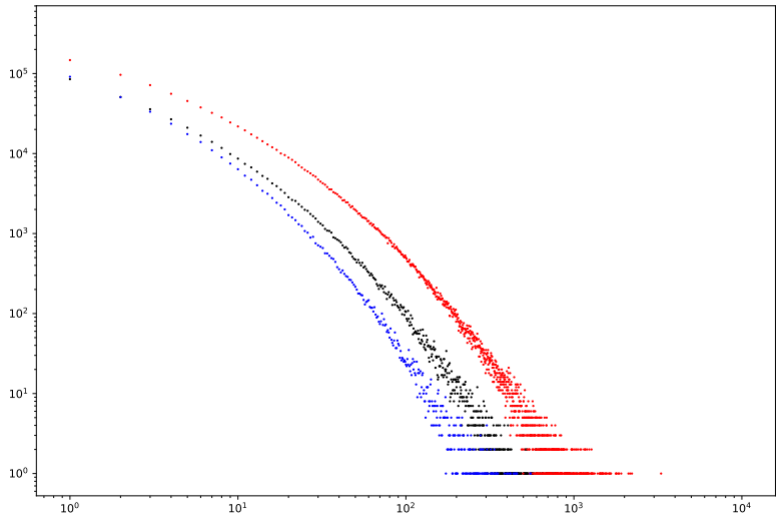
The type of a packing implies the existence of certain quadratic and quartic obstructions:

Type	n^2 Obstructions	n^4 Obstructions	L-G false	L-G open
(6, 1, 1, -1)		$n^4, 4n^4, 9n^4, 36n^4$	0, 1, 4, 9, 12, 16	
(6, 1, -1)	$n^2, 2n^2, 3n^2, 6n^2$		0, 1, 4, 9, 12, 16	
(6, 5, 1)	$2n^2, 3n^2$		0, 8, 12	5, 20, 21
(6, 5, -1)	$n^2, 6n^2$		0, 12	5, 8, 20, 21
(6, 13, 1)	$2n^2, 6n^2$		0	4, 12, 13, 16, 21
(6, 13, -1)	$n^2, 3n^2$		0, 4, 12, 16	13, 21
(6, 17, 1, 1)	$3n^2, 6n^2$	$9n^4, 36n^4$	0, 9, 12	8, 17, 20
(6, 17, 1, -1)	$3n^2, 6n^2$	$n^4, 4n^4$	0, 9, 12	8, 17, 20
(6, 17, -1)	$n^2, 2n^2$		0, 8, 9, 12	17, 20
(8, 7, 1)	$3n^2, 6n^2$		3, 6	7, 10, 15, 18, 19, 22
(8, 7, -1)	$2n^2$		18	3, 6, 7, 10, 15, 19, 22
(8, 11, -1)	$2n^2, 3n^2, 6n^2$		2, 3, 6, 18	11, 14, 15, 23

Missing curvatures dropping off

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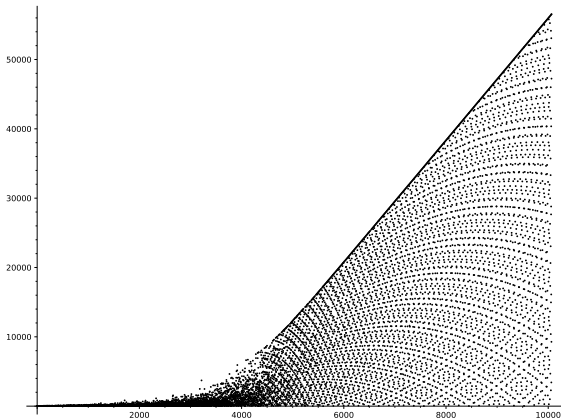


A loglog plot of the probability a curvature is missing, as curvature size increases.

Successive differences

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Successive differences of missing curvatures in the packing $[-4, 5, 20, 21]$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the missing set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

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