

Packing Problems & Number Theory

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December 12, 2024

A classic game

Packing
Problems &
Number
Theory

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A classic game

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A guessing jar:

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A guessing jar: How many marbles?

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A guessing jar: How many marbles? 223!

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A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?

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A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

A classic game

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A guessing jar: How many marbles? 223!

1. What's the largest number of marbles that can fit in the container?
2. Given 223 marbles, what's the smallest container holding them all?

We need to simplify the problem...

Square Packing

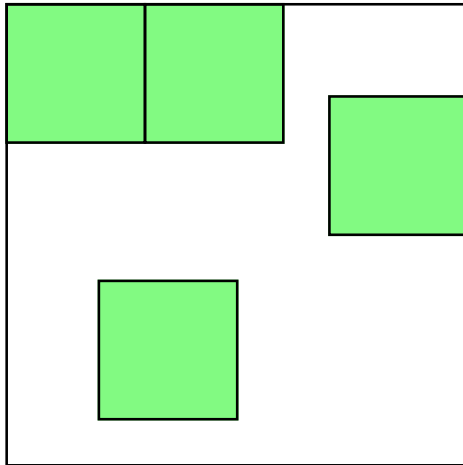
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Square Packing

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Square Packing

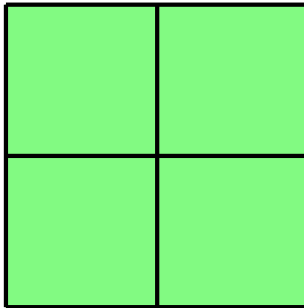
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What's the smallest square we can fit 4 squares inside of?

Square Packing

What's the smallest square we can fit 4 squares inside of?

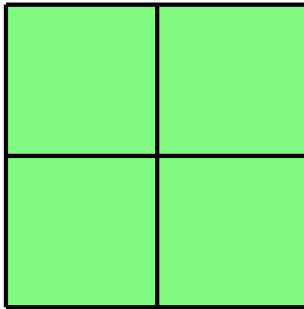


Square Packing

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What's the smallest square we can fit 4 squares inside of?



Side length: 2

Packing Problems: Square Packing

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Packing Problems: Square Packing

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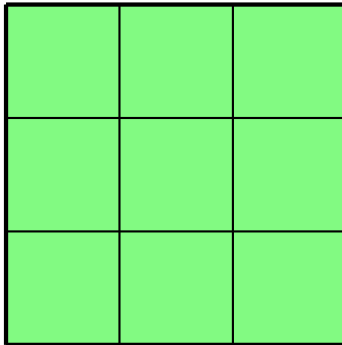
What's the smallest square we can fit 9 squares inside of?

Packing Problems: Square Packing

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What's the smallest square we can fit 9 squares inside of?

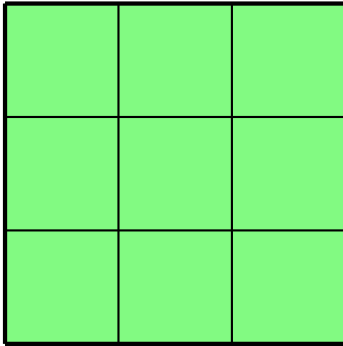


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What's the smallest square we can fit 9 squares inside of?



Side length: 3

Square Packing

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Square Packing

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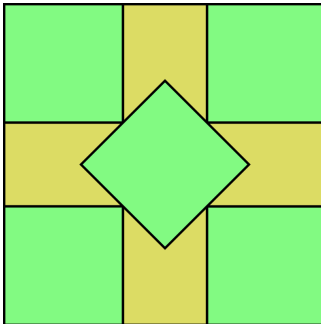
What about 5 squares?

Square Packing

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What about 5 squares?

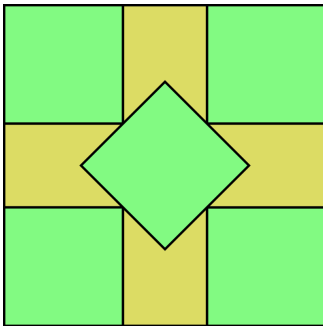


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What about 5 squares?



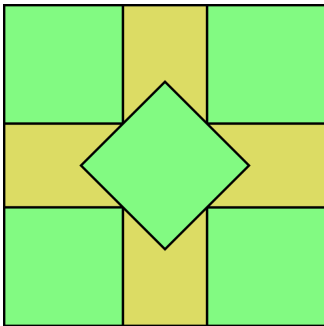
Side length:

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What about 5 squares?



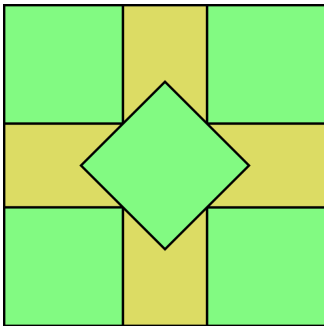
Side length: $\approx 2.707 \dots$

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What about 5 squares?



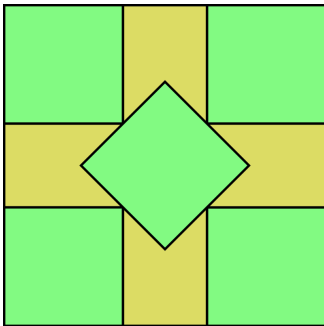
Side length: $\approx 2.707 \dots = 2 + \sqrt{2}/2$

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What about 5 squares?



Side length: $\approx 2.707 \dots = 2 + \sqrt{2}/2$

Can we use this packing to find the optimal packing of 10 squares?

Square Packing

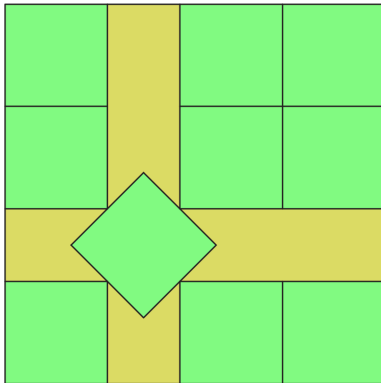
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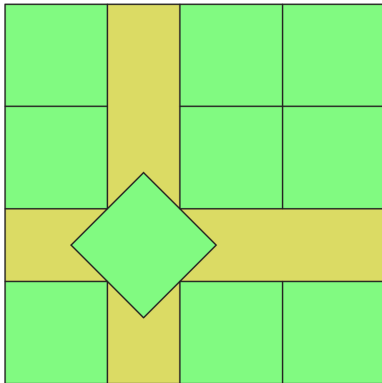
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Side length: $\approx 3.707 \dots = 3 + \frac{\sqrt{2}}{2}$

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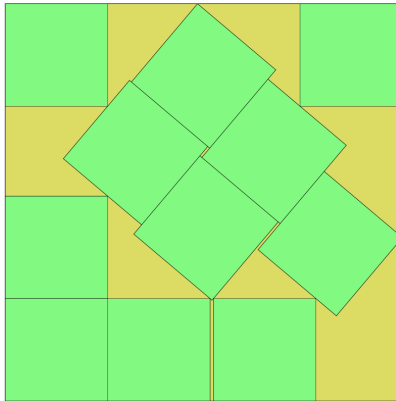
What about 11 squares?

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What about 11 squares?

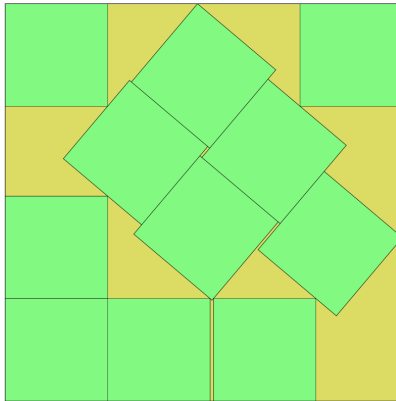


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What about 11 squares?



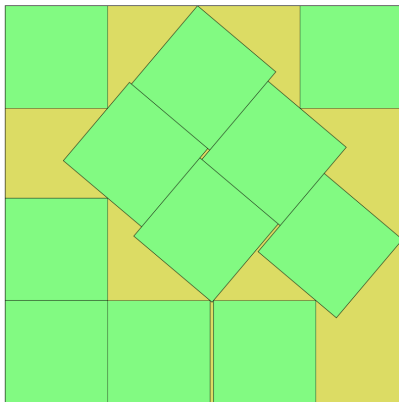
Side length: $\approx 3.877 \dots$

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What about 11 squares?



Side length: $\approx 3.877 \dots$

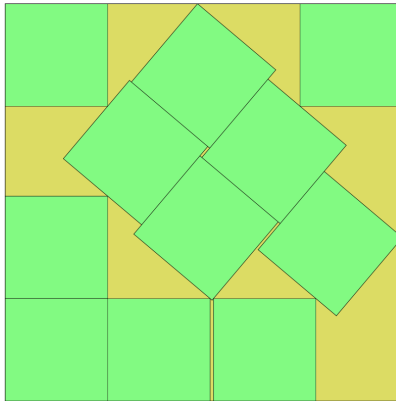
Is this the best possible packing?

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What about 11 squares?



Side length: $\approx 3.877 \dots$

Is this the best possible packing? Mathematicians still don't know...

Circle Packing

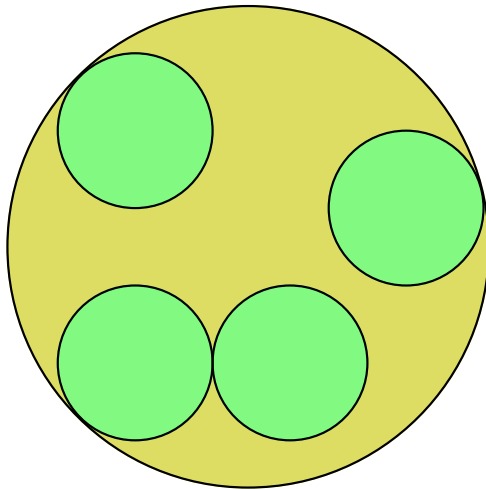
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Circle Packing

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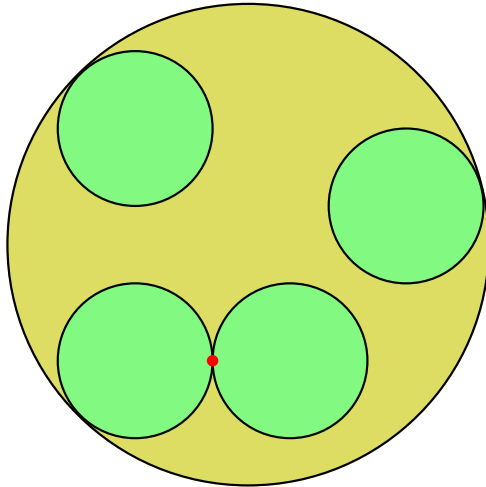
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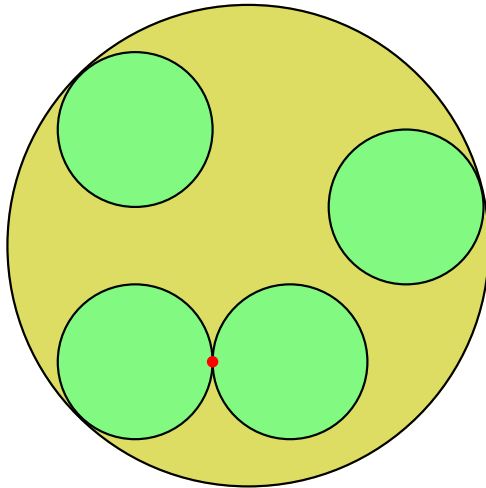
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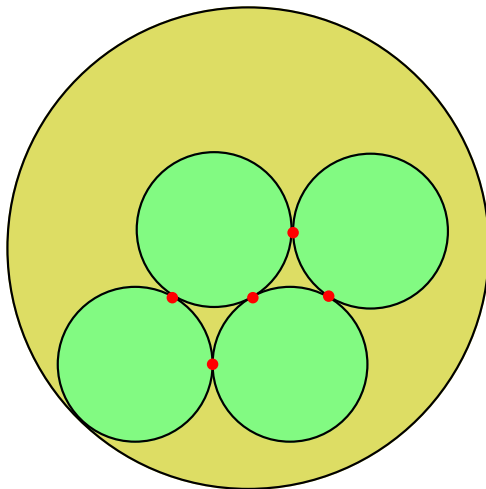


Try it yourself: Can you find an arrangement of 4 circles such that each one is tangent to the other?

Circle Packing

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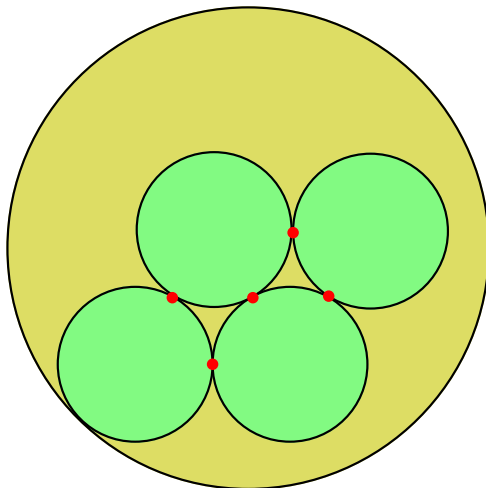
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What if the circles aren't all the same size...

Circle Packing

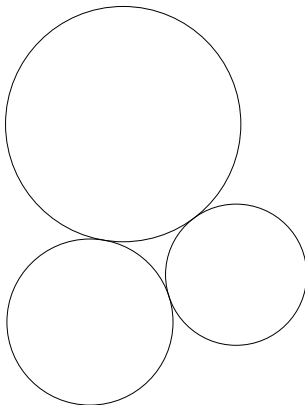
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Circle Packing

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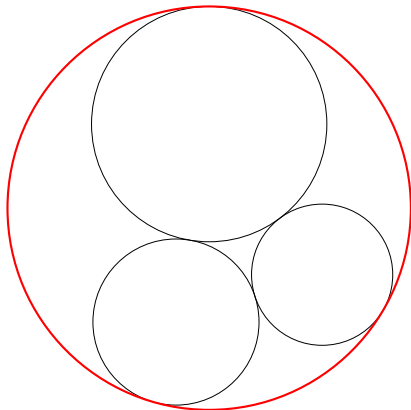
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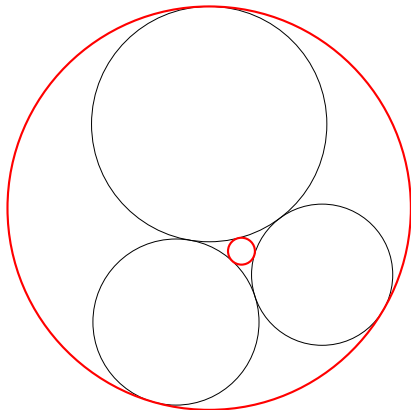
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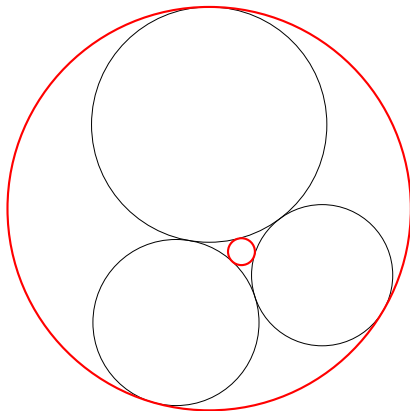
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Definition

A set of four mutually tangent circles is called a *Descartes Quadruple*

Circle Packing: Curvature

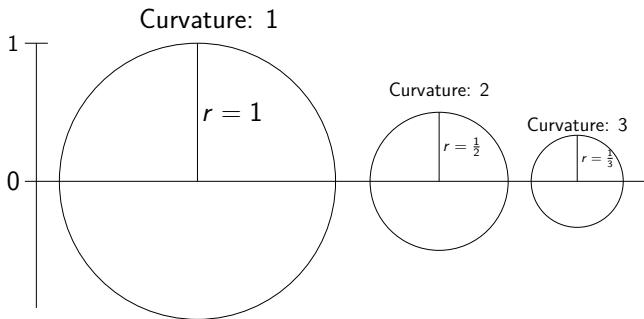
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Circle Packing: Curvature

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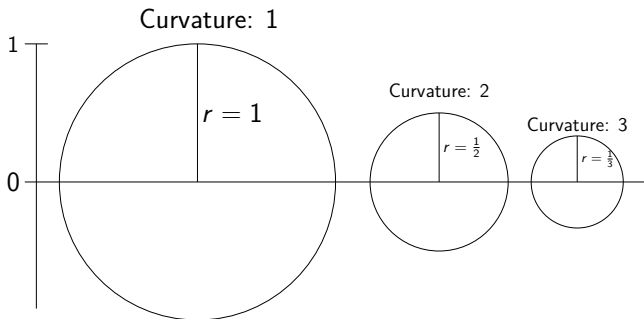
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Circle Packing: Curvature

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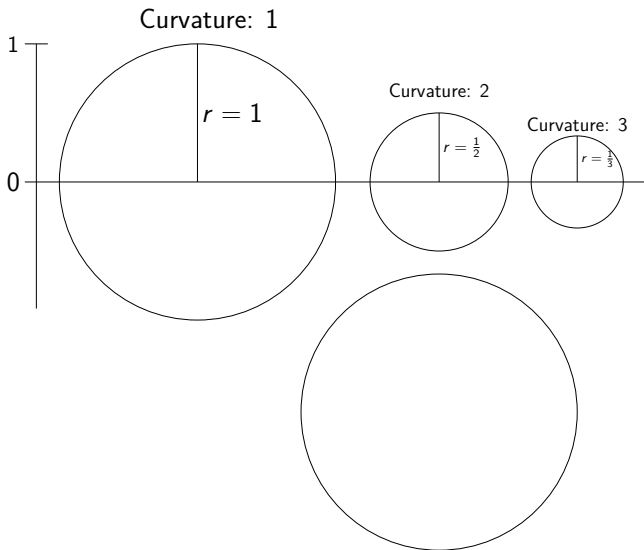
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Circle Packing: Curvature

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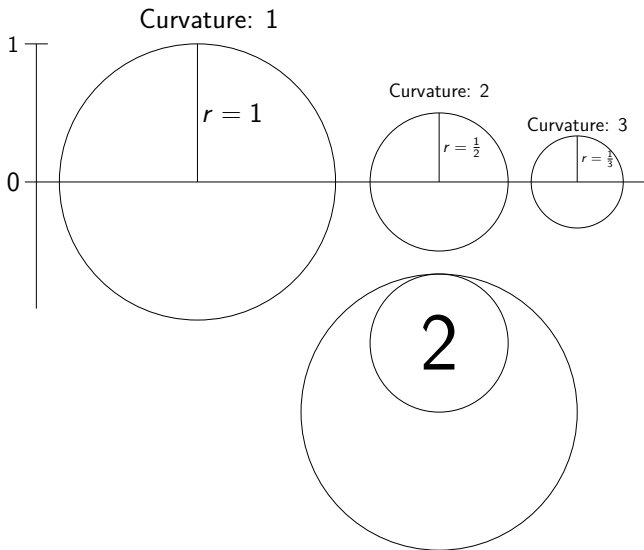
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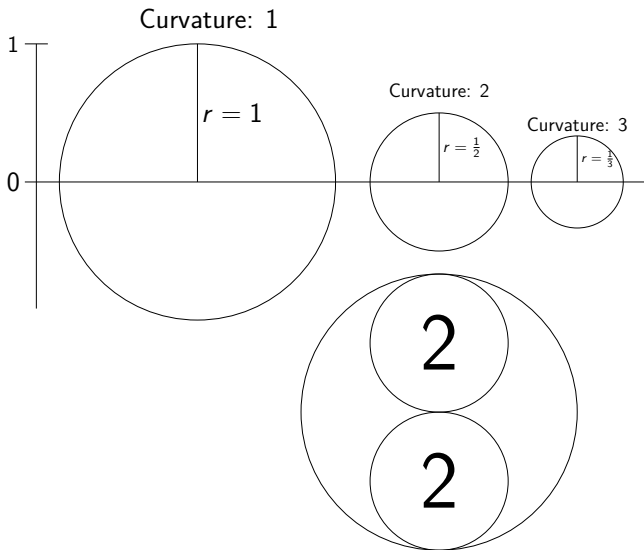
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Circle Packing: Curvature

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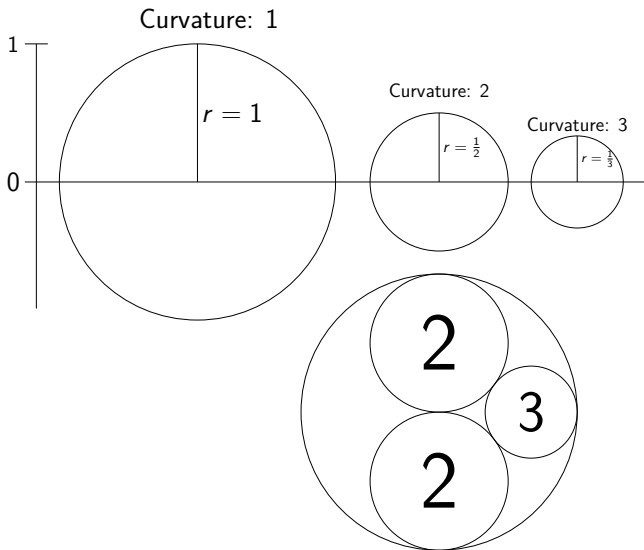
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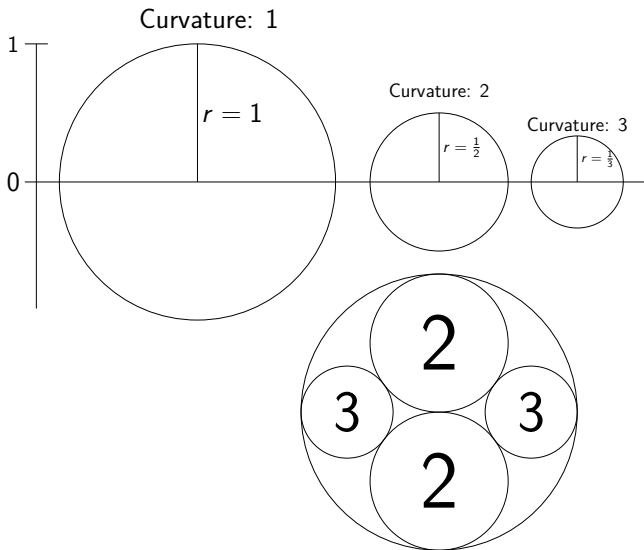
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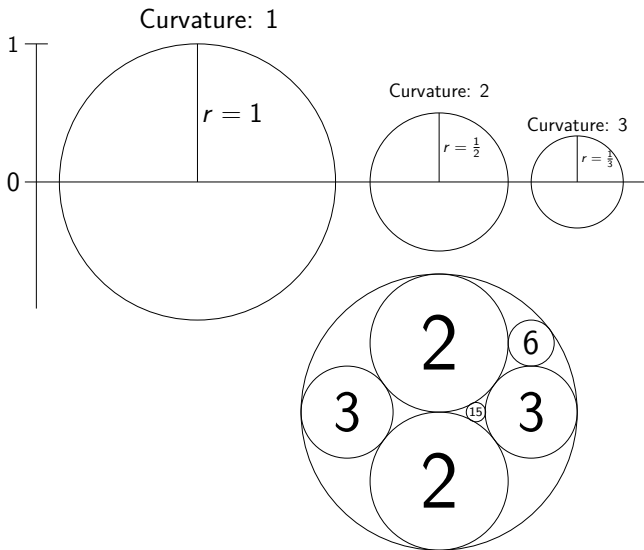
Circle Packing: Curvature



Circle Packing: Curvature

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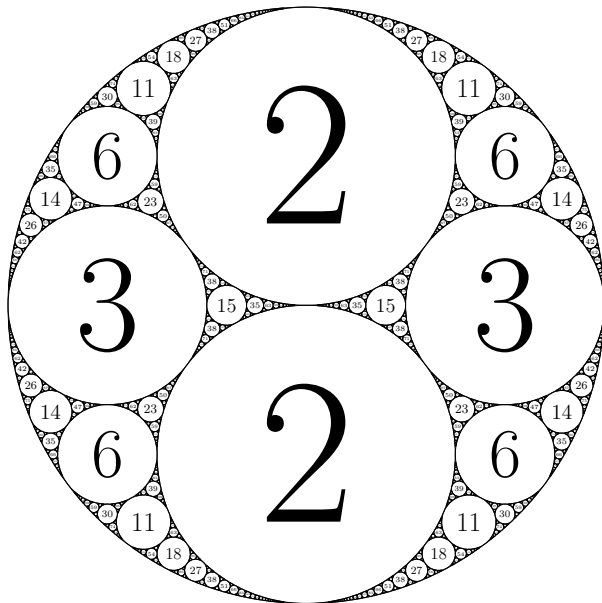
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Circle Packing: Curvature

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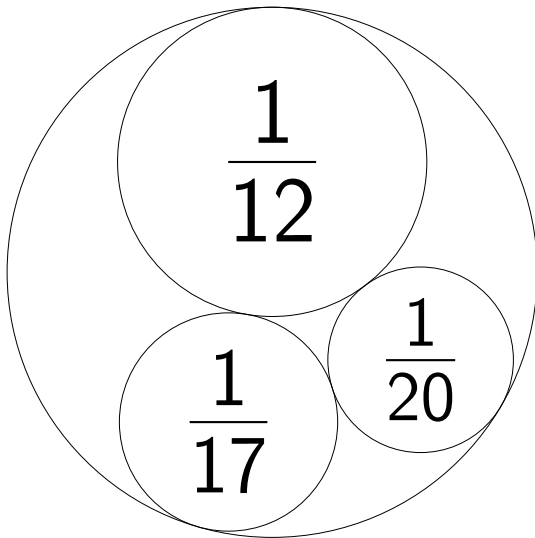
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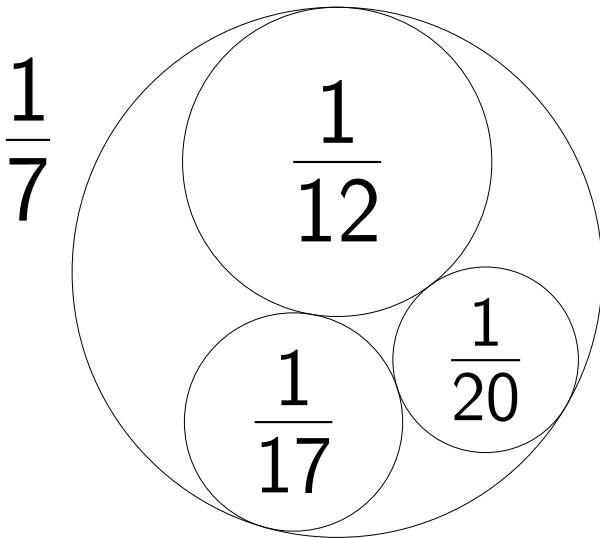
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Circle Packing

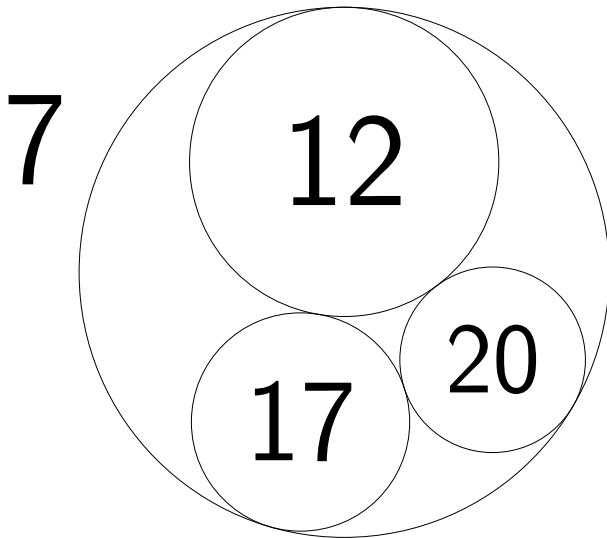
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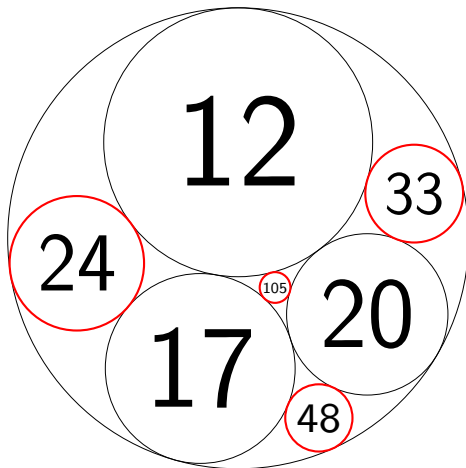
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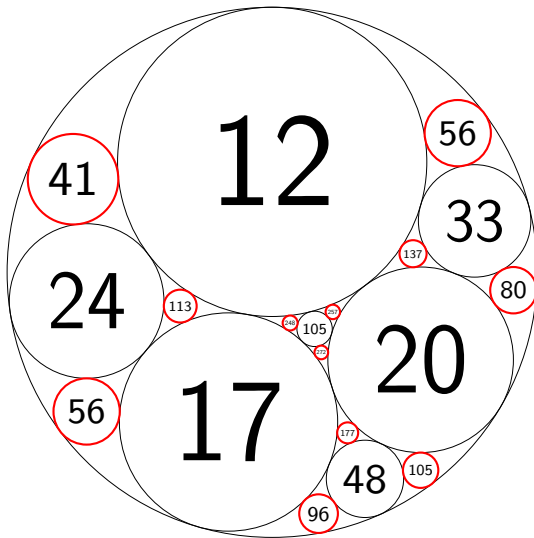
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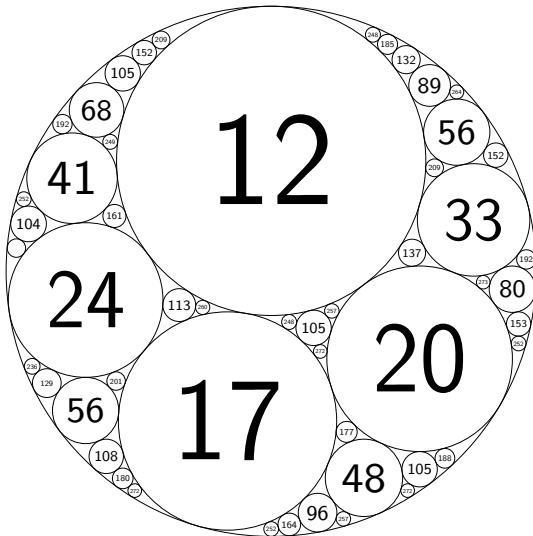
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The Descartes Equation

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The Descartes Equation

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Definition

The Descartes Equation

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Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

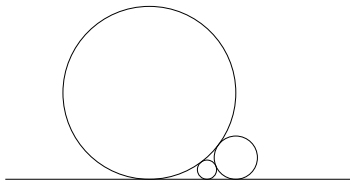
The Descartes Equation

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Definition

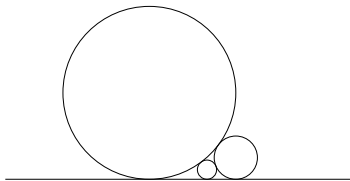
The *curvature* of a circle with radius r is defined to be $1/r$.



The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

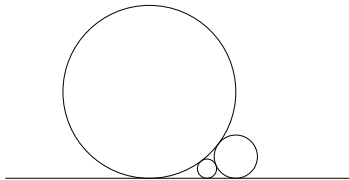


Circle with infinite radius (Curvature 0)

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



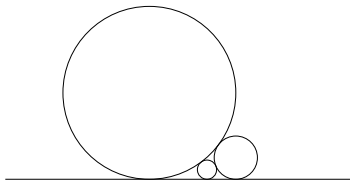
Circle with infinite radius (Curvature 0)

Descartes Equation

The Descartes Equation

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Circle with infinite radius (Curvature 0)

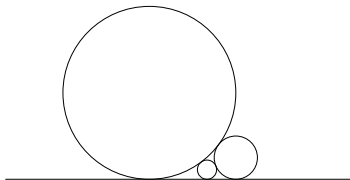
Descartes Equation

If four mutually tangent circles have curvatures a , b , c , d then

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius (Curvature 0)

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2).$$

The Descartes Equation

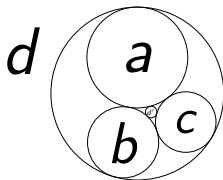
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The Descartes Equation

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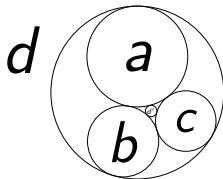
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The Descartes Equation

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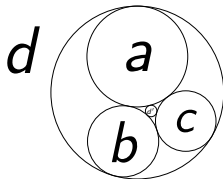
Corollary

If three mutually tangent circles have curvatures a , b , and c , then the two circles of Apollonius, d and d' have curvatures

The Descartes Equation

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Corollary

If three mutually tangent circles have curvatures a , b , and c , then the two circles of Apollonius, d and d' have curvatures

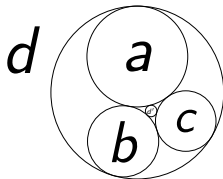
$$d = a + b + c + 2\sqrt{ab + ac + bc}$$

$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Descartes Equation

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Corollary

If three mutually tangent circles have curvatures a , b , and c , then the two circles of Apollonius, d and d' have curvatures

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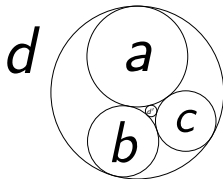
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Key Relation

The Descartes Equation

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Corollary

If three mutually tangent circles have curvatures a , b , and c , then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$

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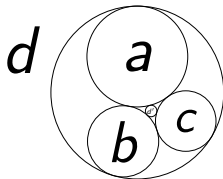
The Key Relation

$$d + d' = 2(a + b + c)$$

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Corollary

If three mutually tangent circles have curvatures a , b , and c , then the two circles of Apollonius, d and d' have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$

$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

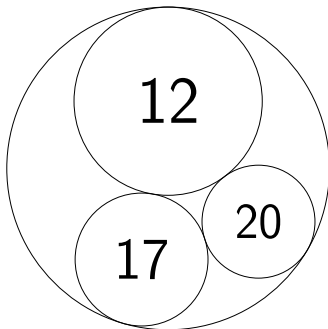
The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

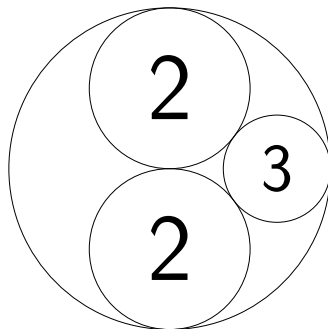
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$[-7, 12, 17, 20]$

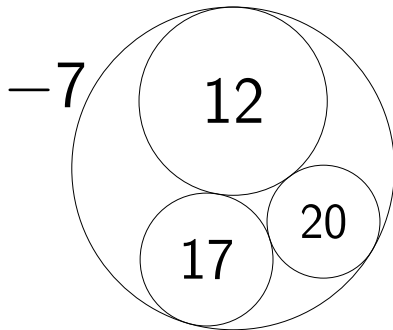


$[-1, 2, 2, 3]$

The Descartes Equation

Packing
Problems &
Number
Theory

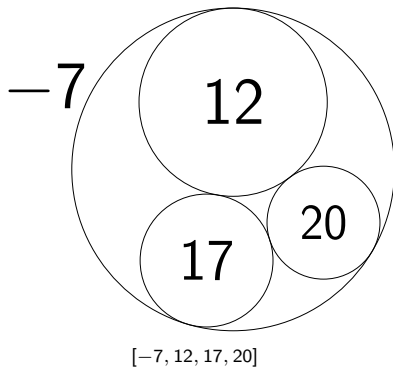
Clyde
Kertzer



The Descartes Equation

Packing
Problems &
Number
Theory

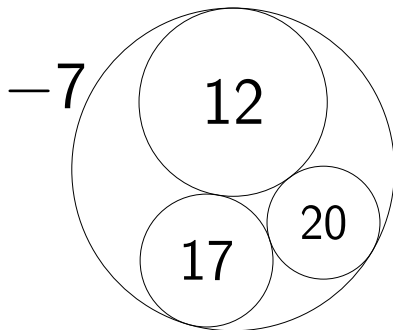
Clyde
Kertzer



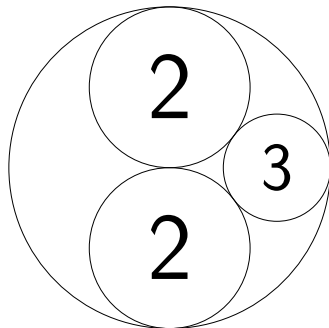
The Descartes Equation

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Problems &
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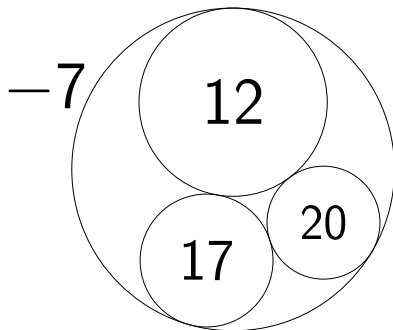
$[-7, 12, 17, 20]$



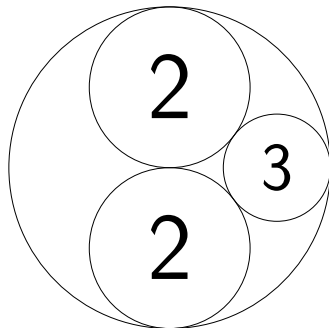
The Descartes Equation

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Kertzer



$[-7, 12, 17, 20]$



$[-1, 2, 2, 3]$

The Descartes Equation

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Number
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Kertzer

The Descartes Equation

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Kertzer

Proof.

The Descartes Equation

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Number
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Proof.

First, we solve for d from the Descartes Equation to find that

The Descartes Equation

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Proof.

First, we solve for d from the Descartes Equation to find that

$$\begin{aligned}2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 &= 0 \\ d^2 - 2d(a + b + c) + (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac) &= 0.\end{aligned}$$

The Descartes Equation

Proof.

First, we solve for d from the Descartes Equation to find that

$$2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = 0$$
$$d^2 - 2d(a + b + c) + (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac) = 0.$$

The quadratic formula gives

The Descartes Equation

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The quadratic formula gives

$$\begin{aligned}d &= (a + b + c) \\&\quad \pm \frac{\sqrt{4(a + b + c)^2 - 4(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)}}{2} \\&= a + b + c \pm 2\sqrt{ab + bc + ca}.\end{aligned}$$

The Descartes Equation

Proof.

First, we solve for d from the Descartes Equation to find that

$$\begin{aligned}2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 &= 0 \\d^2 - 2d(a + b + c) + (a^2 + b^2 + c^2 - 2ab - 2bc - 2ac) &= 0.\end{aligned}$$

The quadratic formula gives

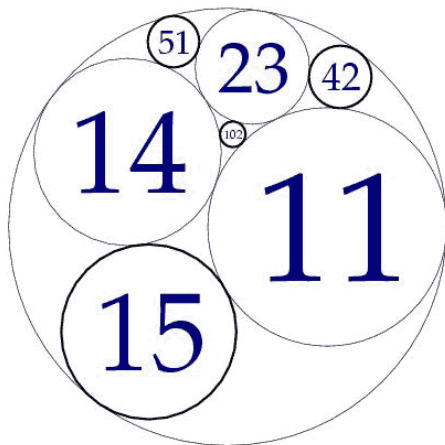
$$\begin{aligned}d &= (a + b + c) \\&\quad \pm \frac{\sqrt{4(a + b + c)^2 - 4(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)}}{2} \\&= a + b + c \pm 2\sqrt{ab + bc + ca}.\end{aligned}$$

Thus, there are two options for d . Their sum is $2(a + b + c)$. □

Apollonian Circle Packings

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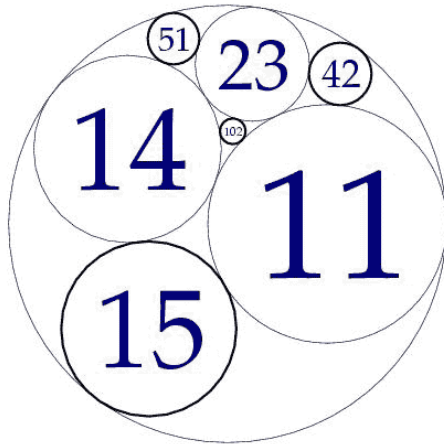


$[-6, 11, 14, 23]$

Apollonian Circle Packings

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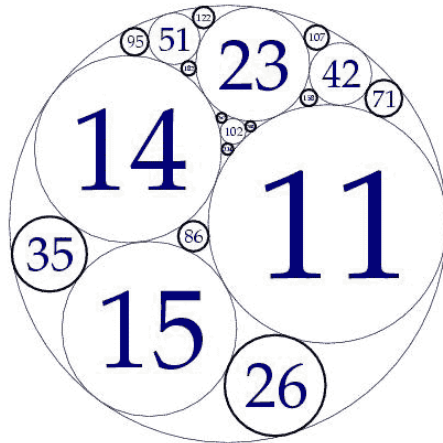


$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings

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Kertzer

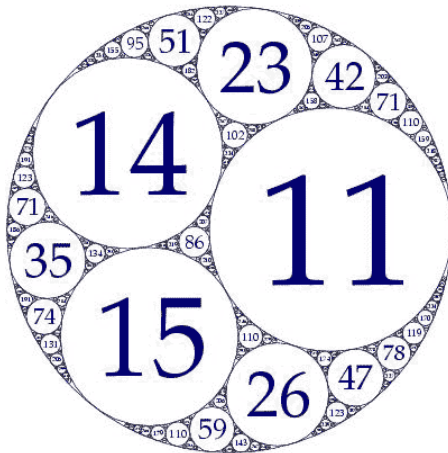


$[-6, 11, 14, 15]$

Apollonian Circle Packings

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$[-6, 11, 14, 15]$

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Apollonian Circle Packings

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Definition

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Definition

A positive integer a *has a packing*

Apollonian Circle Packings

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Definition

A positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

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Definition

A positive integer a *has a packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Example: $a = 7$

Apollonian Circle Packings

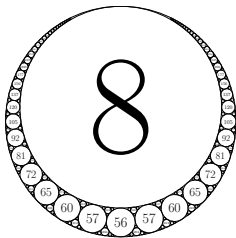
Packing
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Definition

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Apollonian Circle Packings

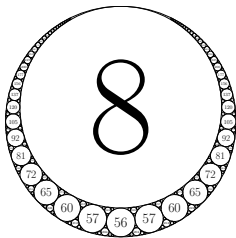
Packing
Problems &
Number
Theory

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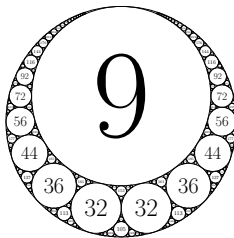
Definition

A positive integer a has a *packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

Example: $a = 7$



$[-7, 8, 56, 57],$



$[-7, 9, 32, 32],$

Apollonian Circle Packings

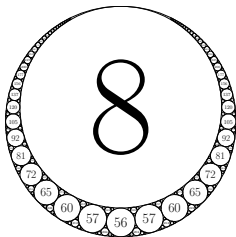
Packing
Problems &
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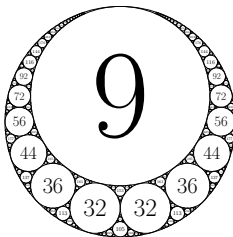
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A positive integer a has a *packing* if there exists a primitive reduced Descartes quadruple $[-a, b, c, d]$.

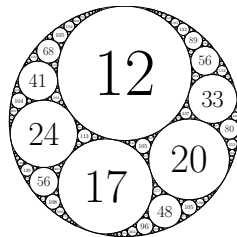
Example: $a = 7$



$[-7, 8, 56, 57],$



$[-7, 9, 32, 32],$



$[-7, 12, 17, 20].$

Symmetric Packings

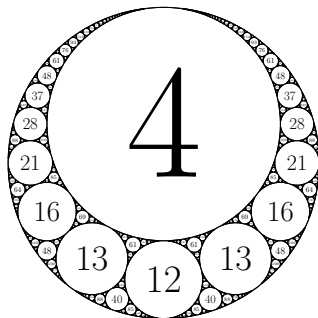
Packing
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Symmetric Packings

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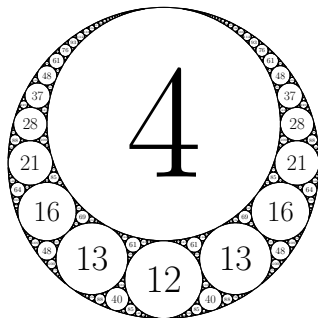


$[-3, 4, 12, 13]$

Symmetric Packings

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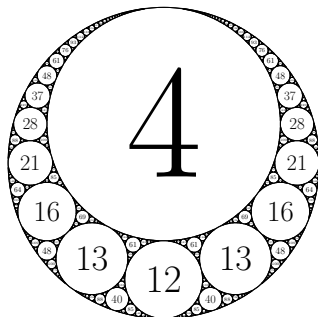
Clyde
Kertzer



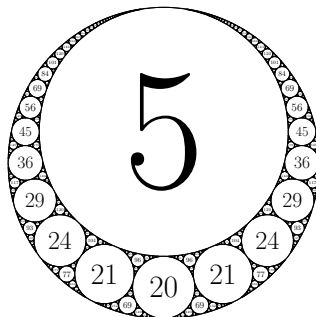
Symmetric Packings

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$[-3, 4, 12, 13]$



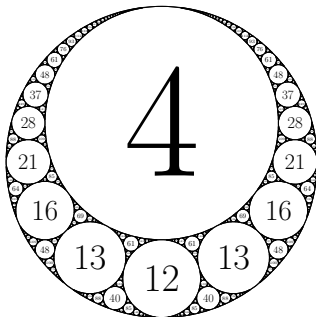
$[-4, 5, 20, 21]$

Definition

Symmetric Packings

Packing
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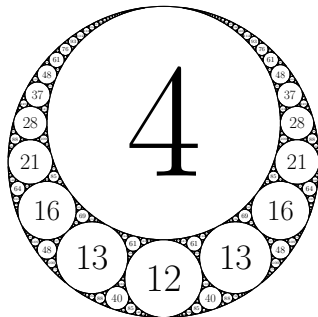
Clyde
Kertzer



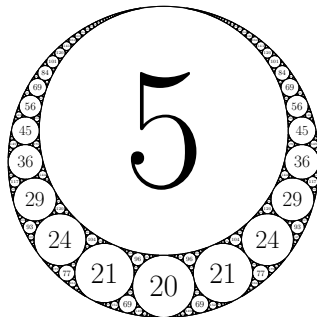
Symmetric Packings

Packing
Problems &
Number
Theory

Clyde
Kertzer



$[-3, 4, 12, 13]$



$[-4, 5, 20, 21]$

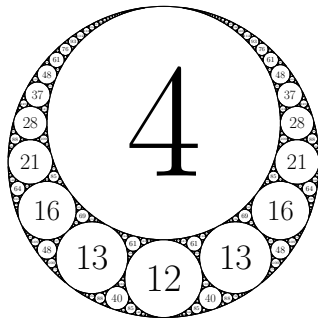
Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

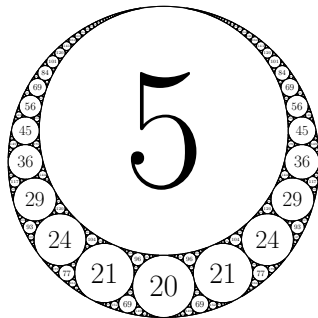
Symmetric Packings

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$[-3, 4, 12, 13]$



$[-4, 5, 20, 21]$

Definition

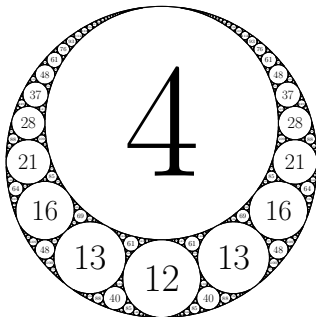
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$$2(a + b + c) - d = d$$

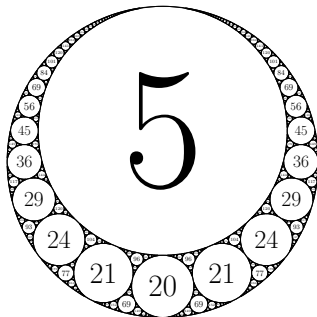
Symmetric Packings

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$[-3, 4, 12, 13]$



$[-4, 5, 20, 21]$

Definition

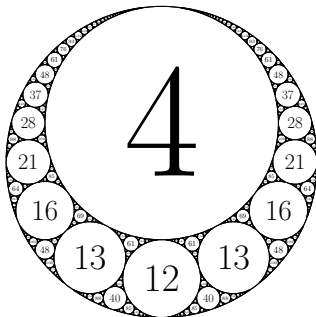
A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

$$2(a + b + c) - d = d \implies 2(a + b + c) = 2d$$

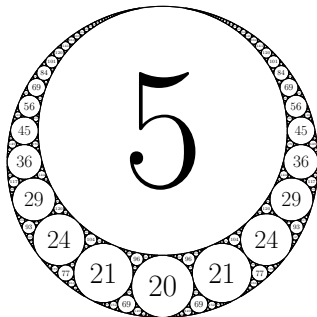
Symmetric Packings

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$[-3, 4, 12, 13]$



$[-4, 5, 20, 21]$

Definition

A *sum-symmetric* quadruple is a primitive reduced Descartes quadruple satisfying $2(a + b + c) - d = d$.

$$2(a + b + c) - d = d \implies 2(a + b + c) = 2d \implies a + b + c = d$$

Symmetric Packings

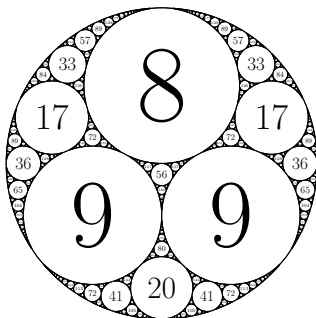
Packing
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Symmetric Packings

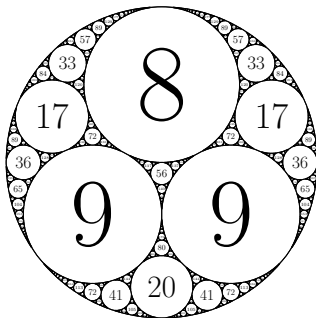
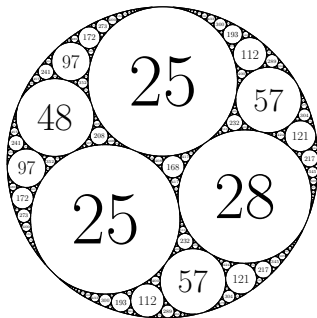
Packing
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$[-4, 8, 9, 9]$

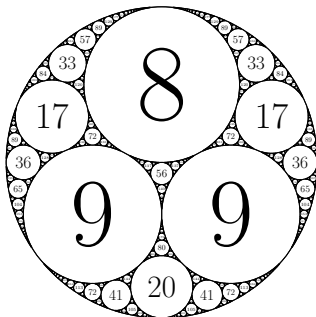
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Kertzer

 $[-4, 8, 9, 9]$  $[-12, 25, 25, 28]$

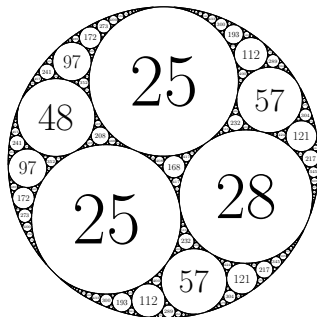
Symmetric Packings

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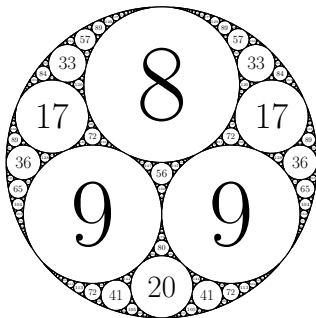
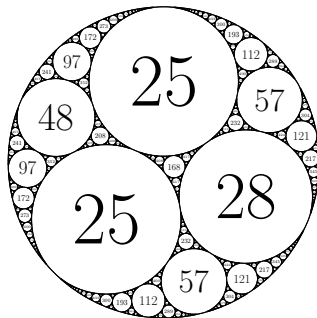
$[-4, 8, 9, 9]$



Symmetric Packings

Packing Problems & Number Theory

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 $[-4, 8, 9, 9]$ 
$$[-12, 25, 25, 28]$$

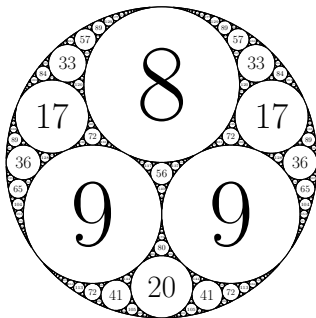
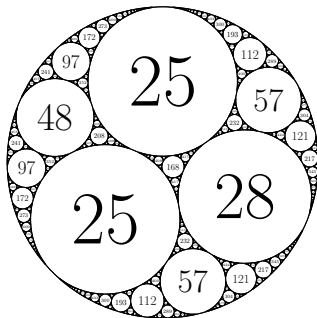
Definition

A *twin-symmetric* quadruple

Symmetric Packings

Packing Problems & Number Theory

Clyde
Kertzer

 $[-4, 8, 9, 9]$ 
$$[-12, 25, 25, 28]$$

Definition

A *twin-symmetric* quadruple is a primitive reduced Descartes quadruple with $c = d$ or $c = b$.

The Two Unusual Symmetric Packings

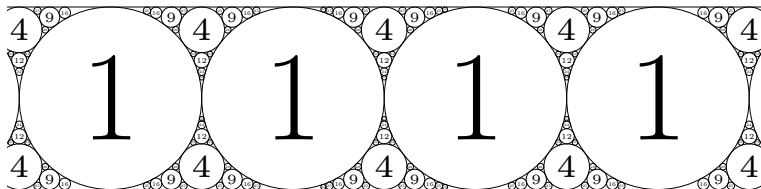
Packing
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The Two Unusual Symmetric Packings

Packing
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Number
Theory

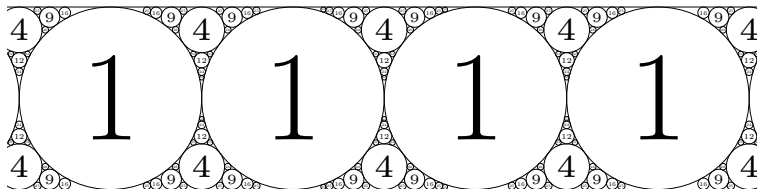
Clyde
Kertzer



The Two Unusual Symmetric Packings

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Kertzer



The strip packing: $[0, 0, 1, 1]$

The Two Unusual Symmetric Packings

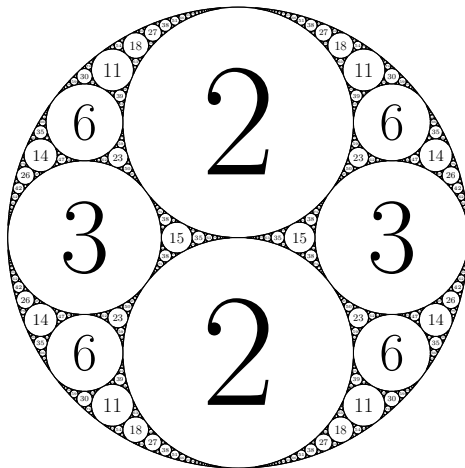
Packing
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Theory

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Kertzer

The Two Unusual Symmetric Packings

Packing
Problems &
Number
Theory

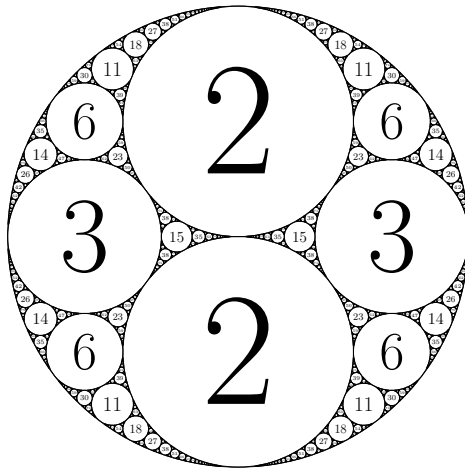
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The Two Unusual Symmetric Packings

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The bug-eye packing: $[-1, 2, 2, 3]$

Symmetric Packings

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Theory

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Kertzer

Proposition

Symmetric Packings

Packing
Problems &
Number
Theory

Clyde
Kertzer

Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Symmetric Packings

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Proposition

Symmetric Packings

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Proposition

A symmetric packing is either sum-symmetric or twin-symmetric.

Proposition

Only the strip and bug-eye packing are both sum-symmetric and twin-symmetric.

Sum-Symmetric Packings

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Sum-Symmetric Packings

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$$\underline{\underline{[-a, b, c, d] \quad | \quad d - c \quad | \quad d - b \quad | \quad d + a}}$$

Sum-Symmetric Packings

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Kertzer

$$\begin{array}{c|ccc} [-a, b, c, d] & d - c & d - b & d + a \\ \hline [-6, 10, 15, 19] & & & \end{array}$$

Sum-Symmetric Packings

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$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25

Sum-Symmetric Packings

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Kertzer

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49

Sum-Symmetric Packings

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Kertzer

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	4	9	25
$[-12, 21, 28, 37]$	9	16	49
$[-18, 22, 99, 103]$	4	81	121
$[-20, 36, 45, 61]$	16	25	81
$[-21, 30, 70, 79]$	9	49	100

Sum-Symmetric Packings

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Kertzer

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2		3^2		5^2
$[-12, 21, 28, 37]$	3^2		4^2		7^2
$[-18, 22, 99, 103]$	2^2		9^2		11^2
$[-20, 36, 45, 61]$	4^2		5^2		9^2
$[-21, 30, 70, 79]$	3^2		7^2		10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
$[-20, 36, 45, 61]$	4^2	4^2	5^2	5^2	9^2
$[-21, 30, 70, 79]$	3^2	3^2	7^2	7^2	10^2

Given the factorization of a , we can find the entire quadruple!

Sum-Symmetric Packings

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$[-a, b, c, d]$	$d - c$	$b + a$	$d - b$	$c + a$	$d + a$
$[-6, 10, 15, 19]$	2^2	2^2	3^2	3^2	5^2
$[-12, 21, 28, 37]$	3^2	3^2	4^2	4^2	7^2
$[-18, 22, 99, 103]$	2^2	2^2	9^2	9^2	11^2
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Given the factorization of a , we can find the entire quadruple!

$$\left[\underbrace{-(2 \cdot 3)}_{-6}, \underbrace{2^2 + 2 \cdot 3}_{10}, \underbrace{3^2 + 2 \cdot 3}_{15}, \underbrace{(2 + 3)^2 - 2 \cdot 3}_{19} \right]$$

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$$[-xy, x^2 + xy, y^2 + xy, (x + y)^2 - xy]$$

Sum-Symmetric Packings

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Kertzer

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Given the factorization of a , we can find the entire quadruple!

$$\left[\underbrace{-(2 \cdot 3)}_{-6}, \underbrace{2^2 + 2 \cdot 3}_{10}, \underbrace{3^2 + 2 \cdot 3}_{15}, \underbrace{(2 + 3)^2 - 2 \cdot 3}_{19} \right]$$

$$[-xy, x^2 + xy, y^2 + xy, (x + y)^2 - xy]$$

$$[-xy, x(x + y), y(x + y), (x + y)^2 - xy]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
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Sum-Symmetric Packings

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Try with $12 = 6 \cdot 2$:

Sum-Symmetric Packings

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Try with $12 = 6 \cdot 2$:

$$[-xy, x(x+y), y(x+y), (x+y)^2 - xy] =$$

Sum-Symmetric Packings

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Try with $12 = 6 \cdot 2$:

$$[-xy, x(x+y), y(x+y), (x+y)^2 - xy] =$$

$$[-2 \cdot 6, 2(2+6), 6(2+6), (2+6)^2 - 2 \cdot 6] =$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
$[-6, 10, 15, 19]$	2^2	3^2	5^2
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$$[-12, 16, 48, 52]$$

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Try with $12 = 6 \cdot 2$:

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$$[-12, 16, 48, 52] = [-3, 4, 12, 13]$$

Sum-Symmetric Packings

$[-a, b, c, d]$	$d - c$	$d - b$	$d + a$
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$[-21, 30, 70, 79]$	3^2	7^2	10^2

Try with $12 = 6 \cdot 2$:

$$[-xy, x(x+y), y(x+y), (x+y)^2 - xy] =$$

$$[-2 \cdot 6, 2(2+6), 6(2+6), (2+6)^2 - 2 \cdot 6] =$$

$$[-12, 16, 48, 52] = [-3, 4, 12, 13] \quad (x=3, y=1)$$

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Theorem

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Theorem

A sum-symmetric quadruple $[a, b, c, d]$ is of the form

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Sum-Symmetric Packings

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Theorem

A sum-symmetric quadruple $[a, b, c, d]$ is of the form

$$[-xy, x(x+y), y(x+y), (x+y)^2 - xy]$$

with $\gcd(x, y) = 1$, and $x, y \geq 0$.

The Number of Sum-Symmetric Packings

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Corollary

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Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n .

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A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n .

Proof.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n .

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y , write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$.

The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n .

Proof.

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The Number of Sum-Symmetric Packings

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The Number of Sum-Symmetric Packings

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The Number of Sum-Symmetric Packings

Corollary

A natural number n has $2^{\omega(n)-1}$ sum-symmetric packings, where $\omega(n)$ is the number of distinct prime divisors of n .

Proof.

Because $n = -xy$ determines the sum-symmetric packing for coprime x and y , write $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, so $\omega(n) = k$. For each prime power we can choose to put it as a factor of x or y , so there 2^k total factor pairs xy but we divide by two to account for symmetry. Thus, n has $2^k/2 = 2^{k-1} = 2^{\omega(n)-1}$ sum-symmetric packings. □

Sum-Symmetric packings of 60

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Sum-Symmetric packings of 60

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Write $60 = 2^2 \cdot 3 \cdot 5$,

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Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.
These correspond to the coprime factor pairs

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Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.
These correspond to the coprime factor pairs $(1, 60)$,

Sum-Symmetric packings of 60

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Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.
These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$,

Sum-Symmetric packings of 60

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Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.
These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$,

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$.

Sum-Symmetric packings of 60

Write $60 = 2^2 \cdot 3 \cdot 5$, so 60 has $2^{3-1} = 2^2 = 4$ sum-symmetric packings.

These correspond to the coprime factor pairs $(1, 60)$, $(4, 15)$, $(3, 20)$, $(5, 12)$. They are

$$(1, 60) \implies [-60, 61, 3660, 3661]$$

$$(4, 15) \implies [-60, 76, 285, 301]$$

$$(3, 20) \implies [-60, 69, 460, 469]$$

$$(5, 12) \implies [-60, 85, 204, 229]$$

Twin-Symmetric Packings

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Twin-Symmetric Packings

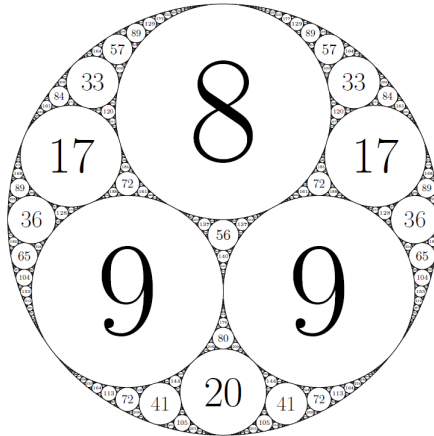
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Packings where one of the numbers is the same:

Twin-Symmetric Packings

Packings where one of the numbers is the same:

 $[-4, 8, 9, 9]$

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-2 |

none

Twin-Symmetric Packings

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-2	none
-3	$[-3, 5, 8, 8]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$

Twin-Symmetric Packings

-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$

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-2	none
-3	$[-3, 5, 8, 8]$
-4	$[-4, 8, 9, 9]$
-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

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-2	none
-3	$[-3, 5, 8, 8]$
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-5	$[-5, 7, 18, 18]$
-6	none
-7	$[-7, 9, 32, 32]$
-8	$[-8, 12, 25, 25]$
-9	$[-9, 11, 50, 50]$
-10	none
-11	$[-11, 13, 72, 72]$
-12	$[-12, 16, 49, 49], [-12, 25, 25, 28]$
-13	$[-13, 15, 98, 98]$
-14	none
-15	$[-15, 17, 128, 128], [-15, 32, 32, 33]$
-16	$[-16, 20, 81, 81]$

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Over the summer:

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Over the summer:

Theorem

Twin-Symmetric Packings

Over the summer:

Theorem

All primitive ACPs with $c = d$ are given by

$$\left[-x, x + y^2, \left(\frac{2x + y^2}{2y} \right)^2, \left(\frac{2x + y^2}{2y} \right)^2 \right] \quad y \text{ even}$$
$$\left[-x, x + 2y^2, 2 \left(\frac{x + y^2}{2y} \right)^2, 2 \left(\frac{x + y^2}{2y} \right)^2 \right] \quad y \text{ odd}$$

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All primitive ACPs with $c = d$ are given by

$$\left[-x, x + y^2, \left(\frac{2x + y^2}{2y} \right)^2, \left(\frac{2x + y^2}{2y} \right)^2 \right] \quad y \text{ even}$$
$$\left[-x, x + 2y^2, 2 \left(\frac{x + y^2}{2y} \right)^2, 2 \left(\frac{x + y^2}{2y} \right)^2 \right] \quad y \text{ odd}$$

Not ideal, not in terms of factorization.

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Improved to:

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Theorem

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Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\left\{ \left[-xy, xy + 2y^2, \frac{(x+y)^2}{2}, \frac{(x+y)^2}{2} \right] \quad x \text{ odd}, y \text{ odd} \quad x > y \right.$$

Twin-Symmetric Packings

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Improved to:

Theorem

A twin-symmetric quadruple is of the form

$$\left\{ \begin{array}{ll} \left[-xy, xy + 2y^2, \frac{(x+y)^2}{2}, \frac{(x+y)^2}{2} \right] & x \text{ odd}, y \text{ odd } x > y \\ \left[-xy, xy + 4y^2, \left(\frac{x}{2} + y \right)^2, \left(\frac{x}{2} + y \right)^2 \right] & 4 \mid x, \quad x > 2y \end{array} \right.$$

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Improved to:

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with $\gcd(x, y) = 1$.

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Further improved to:

Twin-Symmetric Packings

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A twin-symmetric quadruple is one of following two forms

$$\begin{cases} \left[-xy, xy + 2y^2, \frac{1}{2}(x+y)^2, \frac{1}{2}(x+y)^2 \right] & x, y \text{ odd} & x > y \\ \left[-2xy, 2xy + 4y^2, (x+y)^2, (x+y)^2 \right] & xy \text{ even} & x > y \end{cases}$$

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Twin-Symmetric Packings

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$$[-3, 3 + 2(3)^2, 5^2, 5^2] \implies [-3, 48, 25, 25]$$

Twin-symmetric Packings

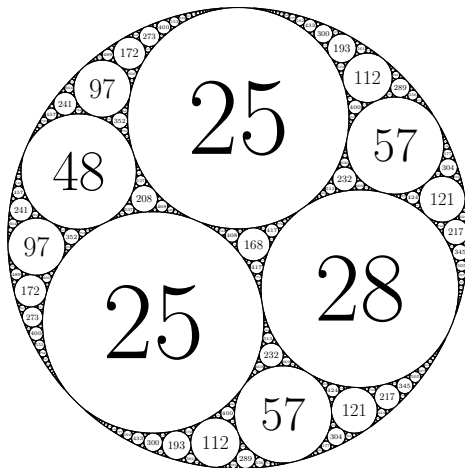
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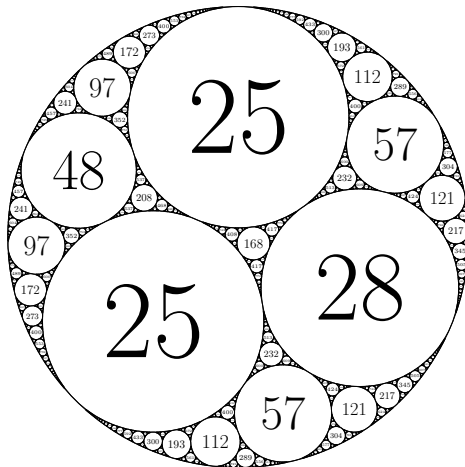


$[-12, 48, 25, 25]$

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$$[-12, 48, 25, 25] \implies [-12, 28, 25, 25]$$

The Number of Twin-symmetric Packings

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We define δ_n as

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We define δ_n as

$$\delta_n = \begin{cases} 1 & n \equiv 2 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary

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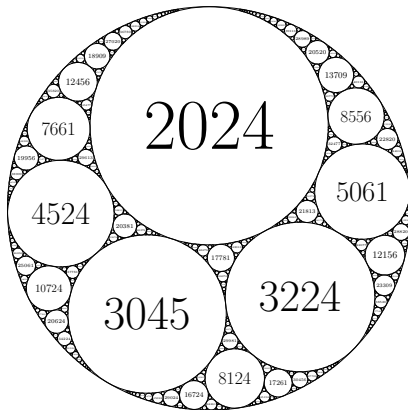
Corollary

A natural number n has $(1 - \delta_n) \cdot 2^{\omega(n)-1}$ twin-symmetric packings where $\omega(n)$ is the number of distinct prime divisors of n .

Thank You!

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Images generated using James Rickards' Code.

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