Numerical Solver for NH2018 S3

Clare S. Y. Huang

October 22, 2019

1 Discretization of Equation (S3)

Equation (S3) in NH2018 reads

$$\frac{\partial}{\partial \mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}(u_{\text{REF}}\cos\phi)\right] + \frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde\theta}{\partial z}}\frac{\partial}{\partial z}u_{\text{REF}}\cos\phi\right] = -a\frac{\partial}{\partial\mu}\left(\frac{q_{\text{REF}}}{2\Omega\mu}\right) \tag{1}$$

where $\mu \equiv \sin\phi$. With the substitutions $\tilde u \equiv u_{\text{REF}}\cos\phi$, $\tilde q \equiv \frac{q_{\text{REF}}}{2\Omega\mu}$ and static stability $S \equiv \frac{\partial\tilde\theta}{\partial z}$, equation (1) becomes

$$\frac{\partial}{\partial\mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}\tilde u\right] + \frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{S}\frac{\partial\tilde u}{\partial z}\right] = -a\frac{\partial\tilde q}{\partial\mu} \tag{2}$$

Discretizing each term in equation (2) on the uniform ϕ (index j) and z (index k) grids yields:

$$\frac{\partial}{\partial\mu}\left[\frac{1}{2\Omega\mu}\frac{\partial}{\partial\mu}\tilde u\right]$$
$$= \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{1}{2\Omega\sin\phi\cos\phi}\frac{\partial}{\partial\phi}\tilde u\right]$$
$$\approx \frac{1}{2\Omega(\Delta\phi)^2\cos\phi_j}\left[\frac{\tilde u_{j+1,k}}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}} + \tilde u_{j,k}\left(\frac{1}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}} + \frac{1}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}}\right) + \frac{\tilde u_{j-1,k}}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}}\right] \tag{3}$$

$$\frac{2\Omega a^2 H\mu}{R(1-\mu^2)}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde\theta}{\partial z}}\frac{\partial}{\partial z}u_{\text{REF}}\cos\phi\right]$$
$$= \frac{2\Omega a^2 H\sin\phi}{R\cos^2\phi}e^{z/H}\frac{\partial}{\partial z}\left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial\tilde\theta}{\partial z}}\frac{\partial}{\partial z}u_{\text{REF}}\cos\phi\right]$$
$$\approx \frac{2\Omega^2 a^2 H\sin\phi_j}{R\cos^2\phi_j}\frac{e^{z_k/H}}{(\Delta z)^2}\left[\tilde u_{j,k+1}\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} - \tilde u_{j,k}\left(\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} + \frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}}\right) + \tilde u_{j,k-1}\frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}}\right] \tag{4}$$

$$-a\frac{\partial\tilde q}{\partial\mu}$$
$$= -\frac{a}{\cos\phi}\frac{\partial\tilde q}{\partial\phi}$$
$$\approx -\frac{a}{2\Delta\phi\cos\phi_j}(\tilde q_{j+1} - \tilde q_{j-1}) \tag{5}$$

Substituting (3), (4) and (5) into (2) gives

$$a_{j,k}\tilde u_{j+1,k} + b_{j,k}\tilde u_{j-1,k} + c_{j,k}\tilde u_{j,k+1} + d_{j,k}\tilde u_{j,k-1} + e_{j,k}\tilde u_{j,k} = f_{j,k} \tag{6}$$

where

$$a_{j,k} = \frac{1}{\sin\phi_{j+1/2}\cos\phi_{j+1/2}} \tag{7}$$

$$b_{j,k} = \frac{1}{\sin\phi_{j-1/2}\cos\phi_{j-1/2}} \tag{8}$$

$$c_{j,k} = \frac{2\Omega^2 a^2 H \sin\phi_j}{R\cos\phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} \frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} \tag{9}$$

$$d_{j,k} = \frac{2\Omega^2 a^2 H \sin\phi_j}{R\cos\phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} \frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}} \tag{10}$$

$$e_{j,k} = -(a_{j,k} + b_{j,k} + c_{j,k} + d_{j,k}) \tag{11}$$

$$f_{j,k} = -\Omega a(\Delta\phi)(\tilde{q}_{j+1} - \tilde{q}_{j-1}) \tag{12}$$

Equation (6) is then solved using the successive over-relaxation algorithm.

2 The Successive Over-relaxation (SOR) Solver

The iterative procedure to solve equation (6) is

$$\tilde{u}_{j,k}^{new} = \tilde{u}_{j,k}^{old} - \omega \frac{\zeta_{j,k}}{e_{j,k}} \tag{13}$$

where $\eta_{j,k}$ is the residual

$$\zeta_{j,k} \equiv a_{j,k}\tilde{u}_{j+1,k} + b_{j,k}\tilde{u}_{j-1,k} + c_{j,k}\tilde{u}_{j,k+1} + d_{j,k}\tilde{u}_{j,k-1} + e_{j,k}\tilde{u}_{j,k} - f_{j,k} \tag{14}$$

and the over-relaxation parameter at the n-th step is given by

$$\omega_n = \begin{cases} \frac{1}{1-\rho_{\text{jacobi}}^2/2} & \text{when } n = 1 \\ \frac{1}{1-\rho_{\text{jacobi}}^2\omega_{n-1}/4} & \text{when } n > 1 \end{cases}$$

In NH2018, ρ_{jacobi}^2 used is 0.95. The condition to stop the iteration is when

$$\sum_1^{j_{\max}-1}\sum_1^{k_{\max}-1} \|\zeta_{j,k}\| < \epsilon \sum_1^{j_{\max}-1}\sum_1^{k_{\max}-1} \|f_{j,k}\| \tag{15}$$

where $\epsilon = 10^{-5}$ in NH2018.