Applications of Burger's Equation and the Lighthill-Whitham-Richards model on Traffic Flow

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Abstract

In the field of traffic flow, mathematicians use multiple different models in order to predict the behaviour of drivers. In this paper, we seek to analyze two prominent methods of prediction in the Lighthill-Whitham-Richards model, and the Burger's equation. We discuss the similarity of the LWR model to the transport equation, and illustrate a solution using the method of characteristics. Using the viscous model of the nonlinear Burger's equation, we describe the utility of the Cole-Hopf transformation and use it to solve an example of a one-lane traffic scenario. We discuss much of the current research centering around the existence of chaos and uncertainty into real-world scenarios, and how both models can adapt to better predict those changes.

1 Introduction

Traffic flow represents a complex phenomenon that shapes the dynamics of modern urban landscapes. The management and understanding of traffic patterns are important for both the functionality of transportation systems and the quality of life in urban environments. To address the nature of traffic flows, mathematical models have emerged as tools, providing insights into the dynamics and helping in the development of effective traffic management strategies. We'll be focusing on the application of two key equations: the Burgers equation and the Lighthill-Whitham-Richards (LWR) equation. These equations serve as a fundamental framework for capturing the dynamics of vehicle movement within our roads.

2 Lighthill-Whitham-Richards Model

2.1 History

The LWR model's origins can be traced back to 1955, when mathematicians Michael James Lighthill and G. B. Whitham first formulated the equation. They treated it as a one-dimensional flow model, similar to that of the wave equation or a transport equation. To them modeling the flow of water had plenty of resemblance to that of moving on a roadway. [7] With their model in particular, they stressed the relationship between kinematic waves, which model more free-flowing movement such as in flash floods or ocean waves, and dynamic waves, which rely more on continuity and Newton's second law. In the context of traffic, potential crashes and ripple effects found in breaking were described as "kinematic shock waves." [11]

This terminology would be used again the following year, when Paul I. Richards published his own take on the fluid model for traffic flow. Similar to Lighthill and Whitham, Richards used the model to show how one minor change in traffic was able to create large variations, again describing the motion as a type of "shock wave." [14] Because all three mathematicians' findings occurred so close together, this fluid model for traffic has been since referred to as the LWR-model.

Many people would find new facets of LWR as time went by. One key contributor was Richard Payne, whose contributions have been noteworthy enough to where his extensions to LWR have been specified as the LWRP model, which is considered a separate model from LWR.

Other extensions and variations of this model have been found over time; some have focused on multiple lanes and/or shifting between them, while some have taken a discrete approach in describing traffic movement.[7] Further case studies and applications have also been found using the LWR Model within a physical context, some of which will be discussed further below.

2.2 The Equation

The Lighthill-Whitham-Richards (LWR) equation serves as a fundamental model in traffic flow dynamics, extensively applied and modified over time. The most common formulation for traffic flow is expressed as:

$$\frac{\partial p}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

Here, x represents the spatial coordinate along the road, and t is time. In this equation, q denotes the traffic flux, and p represents traffic density. Specifically, $\frac{\partial p}{\partial t}$ signifies the time rate of change of traffic density, indicating how the number of vehicles per unit length changes over time. Additionally, $\frac{\partial q}{\partial x}$ represents the partial derivative of q with respect to x, portraying how traffic flow changes as one moves along the road.

This partial derivative serves as a crucial indicator: if q is constant with respect to x, there is no change in flow along the road. A positive value may represent areas of traffic congestion, while a negative value may indicate areas where vehicles are starting to disperse or where traffic gaps are increasing.

The equation can also be expressed in an alternative form:

$$\frac{\partial p}{\partial t} + \frac{\partial F(p)}{\partial x} = 0 \tag{2}$$

Here, $F(p)$ is the flux function, which simplifies the mathematical representation. The flux function $F(p)$ encapsulates the relationship between traffic density and traffic flux. In this context, $F(p)$ essentially represents the rate of flow of vehicles (q) as a function of traffic density (p). The flux function characterizes how the traffic flux changes concerning variations in traffic density[5].

In addition to the LWR equation, we were introduced to the transport equation, a variation of the LWR model:

Transport Equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{3}$$

This equation models the transport of a substance, such as a pollutant, in a uniform fluid flow moving with velocity c[12]. The connection between Equation (1) and Equation (3) lies in their analogous structures. In the LWR equation, traffic density (p) and traffic flux (q) are interconnected through their partial derivatives, reflecting the dynamic nature of traffic flow. Similarly, in the transport equation, the concentration of the substance being transported are interconnected, mirroring the transport dynamics.

2.3 Solving and Applications to Traffic Flow

To gain insight into the Lighthill-Whitham-Richards (LWR) equation and its application, we will explore a specific example employing the Greenshields traffic flow model. The Greenshields model provides a flux function ($F(p)$) that describes the relationship between traffic density (p) and traffic flux ($F(p)$):

$$F(p) = V_m p(1 - \frac{p}{p_m}) \tag{4}$$

The Greenshields model introduces a nonlinear relationship where traffic flux (F(p)) depends on both the current traffic density (p) and the difference between the maximum speed (V_m) and the current speed ($\frac{p}{p_m}$). This model reflects real-world scenarios where traffic flow is influenced by the interplay between vehicle density and available space for traffic movement.[15]

2.3.1 Solution Using the Method of Characteristics

Various methods, such as the method of characteristics or numerical techniques like finite difference methods, can be employed to solve the LWR equation. Here, we'll illustrate the solution using the method of characteristics.

Consider an initial condition: $p(x, 0) = 10$ vehicles/m.

The characteristic equations associated with the LWR equation are given by:

$\frac{dt}{ds} = 1$, $\frac{dx}{ds} = V(p)$, $\frac{dp}{ds} = 0$

Here, s represents a parameter along the characteristics, and V(p) is a function describing how speed varies with traffic density.

With parameter values $V_m = 1$ m/s, and $p_m = 20$ vehicles/m, integrate these characteristic equations to obtain:

$t = s + t_0$, $x = s + x_0$,$p = p_0$

To derive a specific solution, we apply the initial condition $p(x, 0) = 10$ vehicles/m. This leads to the following solution:

$p(x,t) = 10(1 - \frac{x-t}{20})$

This solution provides the traffic density p as a function of space x and time t based on the Greenshields traffic flow model and the specified initial condition.

The solution demonstrates that the traffic density decreases over time and space. The rate of decrease is influenced by the initial density and the Greenshields model parameters. The parameters V_m and p_m in the Greenshields model influence the overall shape and dynamics of the traffic density evolution. Understanding these parameters provides insights into the behavior of the traffic flow.

By examining this detailed solution, we gain a comprehensive understanding of how the Greenshields traffic flow model, combined with the method of characteristics, can be applied to solve and interpret the LWR equation in the context of traffic dynamics.

2.4 Graph

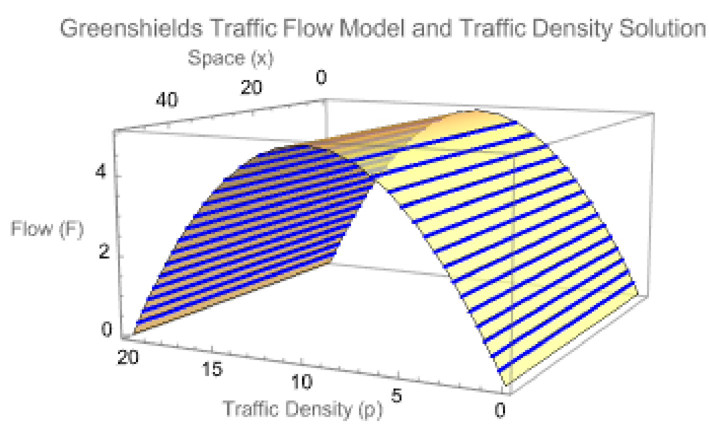

Figure 1: Visualization of traffic density over space and time.

In the presented Mathematica visualization, the graph portrays the Greenshields traffic flow model and the evolution of traffic density over both space and time. The blue surface corresponds to the Greenshields traffic flow model F(p), representing the relationship between traffic density (p) and flow. Simultaneously, the evolving traffic density is depicted by the transparent blue surface, calculated using the Greenshields model and the specified parameter values. On the graph, Traffic Density (p), the vertical axis is labeled Flow (F), and the depth axis is labeled Space (x). The constant appearance of the graph in the spatial direction (x) can be intuitively understood by considering the dynamics of the Greenshields traffic flow model and the chosen initial conditions. In the Greenshields traffic flow model, the traffic flux (F) is influenced by both the current traffic density (p) and the difference between the maximum speed (V_m) and the current speed ($\frac{p}{p_m}$). The model captures the idea that traffic flow is constrained by the available space on the road and the density of vehicles.

3 Burger's Equation

3.1 History

Burger's equation was once derived and used in a physical context back in 1915, as it was used by English mathematician Henry Bateman for researching fluid motion [1]. However, the equation's namesake was cemented through Dutch physicist Johannes Martinus Burgers work in 1948. He was able to find solutions to the equation that would relate it to the Theory of Turbulence, which claims that an abundance in kinetic energy in a fluid can overcome the damping effect found in its viscosity [3]. It was through this notoriety that Burger's name would be connected to this model. At times, this has also been called the Bateman-Burger's Equation, emphasizing Bateman's initial contributions as well.

Not so long after Burger's findings, both Julian D. Cole and Eberhard Hopf had discovered (independently from each other) that Burger's equation could be transformed into the linear heat equation. This transformation has since become known as the Cole-Hopf transformation. This transformation is useful as it allowed for Burger's equation to be more easily solved, especially when related to viscosity. [4]

Cole had been using Burger's in relation to aerodynamics, just one of many applications that Burger's equation had begun to be used for. For instance, Daniel Hayes used it to find shock waves for Navier-Stokes fluids. Lighthill himself used it in relation to how sound waves are affected by a surface's viscocity. Furthermore, Whitham would further apply Burger's to traffic flow in 1979 [6]. Each of these studies show how Burger's can be useful in more when modeling different shock waves or shock-like situations, such as with traffic flow.

3.2 The Equation

Burger's equation has been used in many different applications, and thus has been transformed into many different versions. The generic Burger's equation, however, is represented in the form

$$\frac{\partial u}{\partial t} + u * \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{5}$$

where the solutions are some nonlinear $u(t, x)$, and ν is the diffusivity of the model[12]. This diffusivity has similar mechanics to the heat equation, where in that instance, ν models the heat diffusion across the surface. In this case, the rate of diffusivity is how fast a traffic model changes. With specific solutions to the Burger's equation, it will model how fast the traffic model goes to a normal, unimpeded flow of traffic[17].

Next, we describe the application to traffic (how cars are modeled), and the concept of density ρ and the equation changes to:

$$\frac{\partial \rho}{\partial t} + c(\rho) * \frac{\partial \rho}{\partial x} = \nu \frac{\partial^2 \rho}{\partial x^2} \tag{6}$$

with c as some characteristic speed, and ν as the kinematic viscosity. In the case of traffic, kinematic viscosity would describe the flow of cars through an environment and the properties of how quickly the traffic moves through the studied environment.

One key initial condition to use for a one-lane traffic simulation is $u(x, 0) = f(x)$, which describes some initial position of the first car in the lane [2]. Later, we will look at a one-lane case where the initial conditions are more directly defined.

3.3 Solving and Applications to Traffic Flow

One of the easiest ways to solve specific solutions for the Burger's equation is the Col-Hopf transformation. Pioneered by Eberhard Hopf and Julian Cole, the transformation uses nonlinear separation of variables to convert the nonlinear diffusion equation into an initial value problem for the heat equation, a much more familiar model[12]. This process is called linearization. We will convey the method of transforming Burger's equation as discussed in Peter Olver's textbook "Introduction to Partial Differential equations".

In order to illustrate the derivation, we will reverse the process and go from the heat equation to Burger's equation. Beginning with the linear heat equation, where ν is a constant:

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} \tag{7}$$

We want to define $v(t, x)$ as some nonlinear solution. We will guess an exponential solution, and thus define $v(t, x)$ as:

$$v(t, x) = e^{\alpha \varphi(t,x)} \tag{8}$$

α is some constant, and φ is a real function (provided that $v(t, x)$ is positive). Substituting the various derivatives of $v(t, x)$ into the heat equation gives us:

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2} + \nu \alpha \frac{\partial \varphi^2}{\partial x} \tag{9}$$

The next step is to integrate the whole equation with respect to x, and then set

$$\frac{\partial \varphi}{\partial x} = u(t, x). \tag{10}$$

Given this substitution, the equation can now be represented as:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + 2\nu\alpha u \frac{\partial u}{\partial x}, \tag{11}$$

which turns into Burger's equation with a substitution of $\alpha = -\frac{1}{2\nu}$. Since we have defined a derivative of φ for u, we can use the earlier definition of $v(t,x)$ to find u. Given

$$\varphi(t,x) = \frac{1}{\alpha}\log[v(t,x)], \tag{12}$$

we find that

$$u(t,x) = \frac{\partial}{\partial x}(-2\nu\log[v(t,x)]) = -2\nu\frac{1}{v}\frac{\partial v}{\partial x}. \tag{13}$$

These solutions $u(t,x)$ fulfill the viscous Burger's equation for both infinite and finite domains, and its resulting heat equation can be solved for any homogeneous boundary conditions[2].

3.3.1 One-lane traffic flow example

Bitari (2019) illustrates a simple model for one lane of traffic that has no exits or entrances, and retains the same number of cars. The model uses the variables ρ for traffic density, and $f(\rho)$ for the flux of cars through a given point on the road. The equations and boundary conditions defined were

$$\begin{array}{rc} \rho_t + (f(\rho)_x = 0, & x \in [x_1, x_2], t > 0 \\ \rho(x,0) = f(x), & x \in [x_1, x_2] \\ \rho(x_1,t) = \rho_1(t), & t > 0 \\ \rho(x_2,t) = \rho_2(t), & t > 0 \end{array} \tag{14}$$

The equation for the flux $f(\rho)$ is given by the Greenshield model, which assumes that the velocity is linearly decreasing with traffic density:

$$f(\rho) = v_f[\rho - \frac{\rho^2}{\rho_{max}}] - D\frac{\partial \rho}{\partial x}. \tag{15}$$

By this relation, if the density is zero, then the car's velocity corrects to the maximum velocity v_f. Inversely, if the traffic is bumper-to-bumper, the car would stop. By scaling ρ using $u = 1 - \frac{2\rho}{\rho_{max}}$, we get the system

$$\begin{array}{rc} u_t + u * u_x = \nu u_{xx}, & x \in [x_1, x_2], t > 0, \nu > 0 \\ u(x,0) = h(x), & x \in [x_1, x_2] \\ u(x_1,t) = u_1(t), & t > 0 \\ u(x_2,t) = u_2(t), & t > 0 \end{array} \tag{16}$$

Once a system like this is acquired with a recognizable Burger's equation and a set of initial and boundary conditions, researchers then apply the Cole-Hopf transformation, and solve using the given initial conditions. One graphical example of this is shown below.

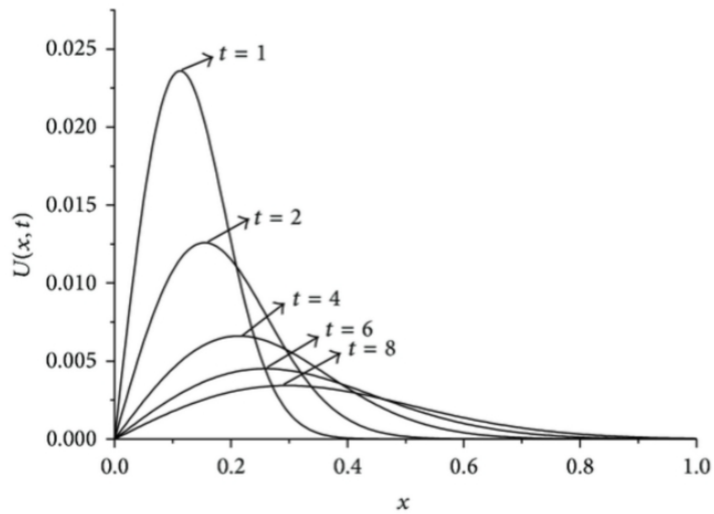

Figure 2: Solutions for traffic density vs the spatial coordinate. $t = 0.01, \nu = 0.01$

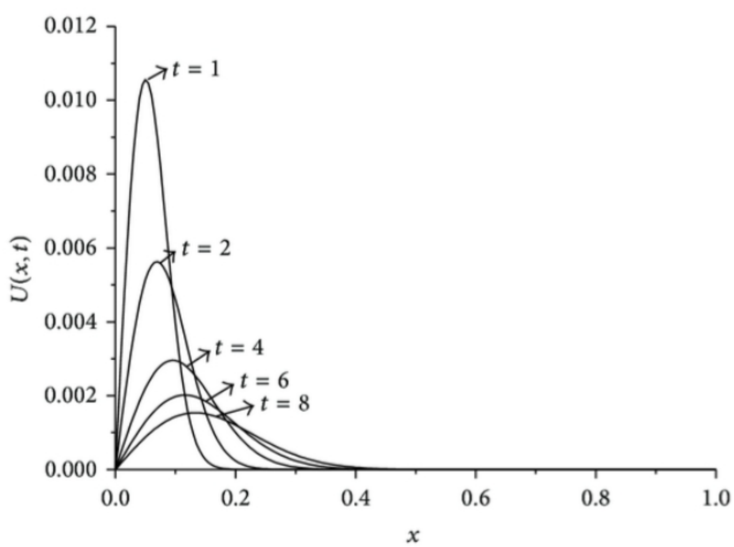

Figure 3: Solutions for traffic density vs the spatial coordinate. $t = 0.01, \nu = 0.001$

These plots show the solutions graphed over 8 time steps, with the only change being the constant ν [16]. Each line shows the traffic density at a different point of time, with the solution eventually smoothing out to a density of zero (where all cars are moving freely). Karakoc et al. showed that solutions to the 1D Burger's Equation appear as shown above, starting as a parabolic function and eventually decreasing to zero. Boundary conditions would change whether or not the parabola starts at the origin, while the initial conditions and diffusivity ν affect the speed at which it decreases.

4 Comparison and current research

Both LWR and Burger's equation are considered fairly important when attempting to model different forms of traffic flow. However, it seems Burger's is a more well-rounded approach as it accounts for more of the physical aspects of traffic. Since LWR is a more general approach to the model, it has less room for extra detail and nuance that would come with everyday traffic. Factors like inertia, visibility, and reaction time were not implemented in the basic model, though the authors have since developed a second-order ODE to address several of these factors. [9]

Since then, LWR has received much attention from several researchers to understand just how accurate it can be with patterns in traffic movement. For instance, in 2015, four professors from the Technical University of Valencia had used LWR in different situational models, such as when dealing with drivers who think and react at different speeds. They found the existence of chaos in each situation, showing that it occurs with LWR on a macroscopic level [9]. Considering how rudimentary the original model was, it is understandable that more researchers would point out its flaws. In 2011, four American researchers from different universities found that when using the LWR model along with a flux function for random free-flow speed, the amount of uncertainty in the location of traffic disturbance would increase over time, though the disturbance itself would still be accurate. [10] Thus, in its current state, LWR could not necessarily be used as a long-term model, as it would be difficult to find each instance of a crash as time went on.

In a similar fashion, Burger's equation seems to falter when dealing with similarly random input. When a polynomial chaos expansion (in which a solution is represented with a truncated series of orthogonal polynomials) was used for a stochastic equation for Burger's, meaning the initial and boundary values are left uncertain, it was found that when the series was truncated, Burger's model had several discontinuities. A smooth, continuous model could only be viewed when the series was infinite. They had surmised the need for time-dependent boundaries when using Burger's, as otherwise, one could not get a full picture using only a finite amount of terms. [13]. With that said, Burger's has still been given credit for many different studies focusing on the context of traffic. In 1998, When analyzing a kinetic clustering of cars, the clustering itself could be described in a hydrodynamic equation, which could be derived into the Burger's equation, showing a link between the model and the idea of cars moving closer together [17]. Furthermore, Burger's was involved in a study investigating how electric bikes could better fit into the flow of traffic, particularly at intersections.[8]

5 Conclusion

Through our studies into both of these equations, it is clear how great of an impact each has had on how we model traffic. As cars continue to play a prominent role in our daily life, having a model to better predict the chaos of traffic is a necessity. Having these models that stem from the transport equation allows us to better root our understanding of traffic in the realm of differential equations. Though they may not be devoid of uncertainty, they have still allowed us to have a more concrete, mathematical understanding of how a series of cars move with each other. Furthermore, as more people study these equations, more people have added greater nuance to the discussion, making it clear what parts of traffic can be implemented to make a traffic model more distinct.

We have found that the original LWR model has much depth in its simplicity, though it may not hold up as well in long-term situations. Between the two, it would seem Burger's model has a greater amount of use in real-world applications, along with the Cole-Hopf transformation. Even though it also has moments of uncertainty when used with certain random inputs, it still has seen some use in real-world traffic interactions in the modern day. It would seem Burger's model has allowed for more accurate depictions of traffic, despite its imperfections. While these models may not be perfect, often challenged in the presence of chaos, they still have made an immense impact in representing traffic flow, allowing researchers to better understand and predict a variety of interactions found on the road.

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