PGL_2 Skein Algebras on the Torus, the Punctured Torus, and the T-Shirt

Cole Ellison[1], Mark Sandey[2], and Andrew Sylvester[1]

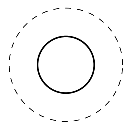
[1]University of California, Santa Barbara
[2]University of California, Riverside

Abstract

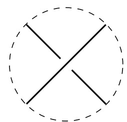
The Kauffman bracket skein algebra (KBSA) of an oriented surface Σ is defined as the free module spanned by diagrams of unoriented framed links in Σ modulo the Kauffman bracket relations with multiplication defined by stacking diagrams using over crossings. One particular modification of this construction is obtained by instead considering the free module spanned by trivalent graphs embedded in Σ modulo the Yamada relations and the same multiplication. This algebra is called the PGL_2 skein algebra (otherwise known as the chromatic skein algebra or graph skein algebra) of Σ. We compute finite presentations of the PGL_2 skein algebras for the punctured torus and the T-shirt, and a finitely generated presentation and multiplication table of the PGL_2 skein algebra of the torus. Additonally, we show that the PGL_2 skein algebras of the punctured torus and T-shirt embed into their respective KBSAs. We show that this fails for the PGL_2 skein algebra of the torus, which is the first instance of this failing in the literature.

1 Introduction

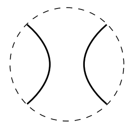
Let $q \in \mathbb{C}^\times$ and let Σ be an oriented surface. We define the Kauffman bracket skein algebra (KBSA) of Σ, denoted $S_q(\Sigma)$, as the free $\mathbb{C}(q)$-module spanned by diagrams of unoriented framed links in Σ, up to planar isotopy, modulo the local relations



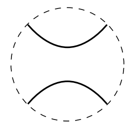
$$= -[2]\varnothing \tag{1}$$



$$= q^{\frac{1}{2}}$$



$$+ q^{-\frac{1}{2}}$$

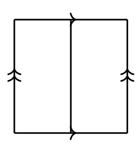


$$\tag{2}$$

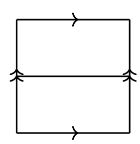
where we define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

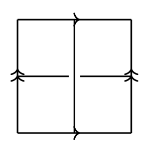
for all positive integers n, and $\varnothing$ is the empty diagram. We call the relations (1) and (2) the Kauffman bracket skein relations. Multiplication in $S_q(\Sigma)$ is defined on link diagrams D_1 and D_2 by stacking D_1 on top of D_2 using over crossings and extended linearly for all elements of $S_q(\Sigma)$. For example, in $S_q(T^2)$, the product of a meridian and a longitude is computed as



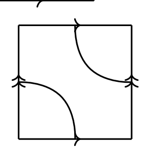
$$*$$



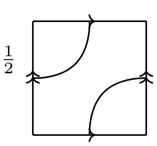
$$=$$



$$= q^{\frac{1}{2}}$$



$$+ q^{-\frac{1}{2}}$$

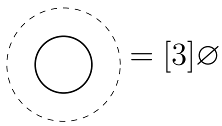


where we represent the torus T^2 as a square with opposite edges identified.

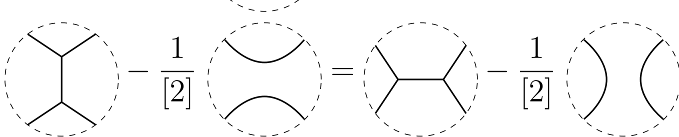
One particularly notable property of $S_q(\mathbb{R}^2)$ is that it is related to the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$. More specifically, with some modifications, $S_q(\mathbb{R}^2)$ coincides with the graphical calculus of $U_q(\mathfrak{sl}_2)$-equivariant maps $\varphi : V_1^{\otimes n} \to V_1^{\otimes m}$ where V_1 and $V_0 = V^{\otimes 0} = \mathbb{C}$ are the fundamental and trivial representations of $U_q(\mathfrak{sl}_2)$. In a more down-to-earth fashion, this means that the relations (1) and (2) are short hand for equations related to evaluation map $V_1 \otimes V_1 \to V_0$, the coevaluation map $V_0 \to V_1 \otimes V_1$, and the braiding map $V_1 \otimes V_1 \to V_1 \otimes V_1$ defined by $x \otimes y \mapsto y \otimes x$.

The construction of $S_q(\Sigma)$ for other surfaces Σ, then can be seen as being the answer to the question "What happens if we draw the same kinds of pictures but on other surfaces?" Of course, nothing here is special to the Lie group SL_2 and its Lie algebra $\mathfrak{sl}_2$ — we can play this game with other Lie groups, such as $\mathrm{PGL}_2 = \mathrm{GL}_2/\sim$ where $S \sim T$ if and only if they differ up to scaling.

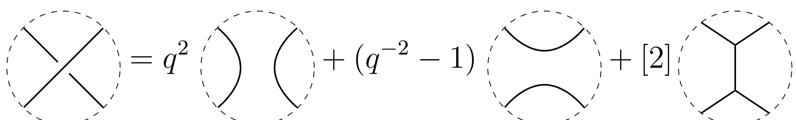
Now, with the representation theory of $U_q(\mathfrak{pgl}_2)$ in mind, one defines the PGL_2 skein algebra (otherwise known as the chromatic skein algebra, or graph skein algebra) of an oriented surface Σ, denoted $S_q^Y(\Sigma)$, as the free $\mathbb{C}(q)$-module spanned by diagrams of trivalent spatial ribbon graphs embedded in Σ, up to planar isotopy, modulo the following modified Yamada relations:



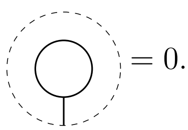
(3)



(4)



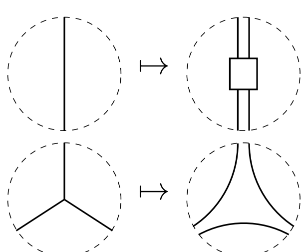
(5)



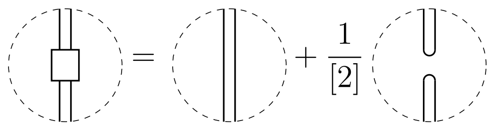
(6)

Multiplication in $S_q^Y(\Sigma)$ is defined in the same way as in $S_q(\Sigma)$.

Since the representation category of $U_q(\mathfrak{pgl}_2)$ is a subcategory of the representation category of $U_q(\mathfrak{sl}_2)$, we ought to be able to include $S_q^Y(\mathbb{R}^2)$ into $S_q(\mathbb{R}^2)$. Indeed, this this inclusion is given by the map $JW : S_q^Y(\mathbb{R}^2) \to S_q(\mathbb{R}^2)$ defined by



where



is the Jones-Wenzl indempotent of the Temperley-Leib algebra on two strands. In particular, this map is injective. A reasonable question to ask then, is "For what surfaces Σ is the map $JW : S_q^Y(\Sigma) \to S_q(\Sigma)$ (defined in the same fashion) injective?" The following is a conjecture from the folklore:

Conjecture (Folklore)**.** *For all $q \in \mathbb{C}^\times$ and oriented Σ, the map JW : $S_q^Y(\Sigma) \to S_q(\Sigma)$ is injective.*

In this paper, we prove that this is true for all surfaces with boundary. In so doing, we construct a finite presentations for for the PGL_2 skein algebra of the punctured torus and the T-shirt. We also show that this conjecture is *not* true in general by showing that the kernel of the JW map for the torus is one-dimensional. Moreover, we show that the PGL_2 skein algebra of the torus is finitely generated and compute its multiplication table.