Abstract Algebra I

Set

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Notion of a set

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1. A set S is made up of elements. If a is one of the elements, we shall denote this fact by $a \in S$.
2. There is exactly one set with no elements. It is the empty set and is denoted by $\emptyset$.
3. We will describe a set either by giving a characterizing property of the elements, such as "the set of all even integers", or by listing the elements. If a set is described by a characterizing property $P(x)$ of its elements x, the notation $\{x : P(x)\}$ or $\{x|P(x)\}$ is used, and is read "the set of all x such that the statement $P(x)$ about x holds.". Thus,

$$\{2, 4, 6, 8\} = \{x : x \text{ is an even positive integer } \leq 8\}.$$

Examples

Let $\mathbb{Z}$ be the set of integers.

$$\begin{aligned}
&\text{the set of odd integers}\\
=&\{x : x \text{ is an odd integer}\}\\
=&\{x \in \mathbb{Z} : x \equiv 1 \mod 2\}\\
=&\{2x + 1 : x \in \mathbb{Z}\}\\
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&\text{the set of nature numbers}\\
=&\{x|\ x \text{ is a positve integer}\}\\
=&\{x \in \mathbb{Z} | x > 0\}\\
=&\mathbb{Z}_{>0}.
\end{aligned}
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Cartesian product

Definition (0.4)

Let A and B be two sets. The set $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ is the Cartesian product of A and B.

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Example

Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Then $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$.

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$\mathbb{N}$= the set of natural numbers,

$\mathbb{Z}$= the set of integers,

$\mathbb{Q}$= the set of rational numbers,

$\mathbb{R}$= the set of real numbers,

$\mathbb{C}$= the set of complex numbers.

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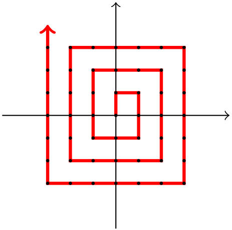
Schröder–Bernstein theorem

If $|A| \geq |B|$ and $|A| \leq |B|$, then $|A| = |B|$.

Examples

- The map $f(x) = 2x$ is a bijection from the set of integers to the set of even integers. Therefore, these two sets have the same cardinality.
- Irreducible fraction expression defines an injection from $\mathbb{Q}$ to $\mathbb{Z}^2$, so $|\mathbb{Q}| \leq |\mathbb{Z}^2|$.



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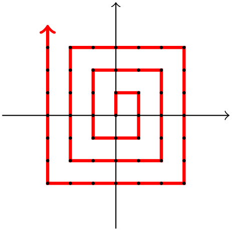

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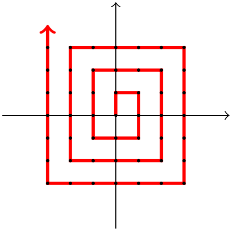

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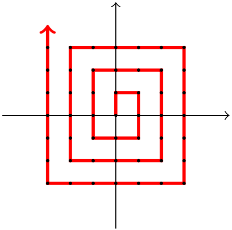

- Since $\mathbb{N}$ is a subset of $\mathbb{Q}$, $|\mathbb{N}| \leq |\mathbb{Q}|$.
- We conclude that $|\mathbb{N}| = |\mathbb{Q}|$ (and hence $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$).

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Examples

If $\mathrm{A} = \{1, 2\}$, then $2^{\mathrm{A}} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Especially, the empty set is an element of 2^{A}. To avoid the ambiguity, let us denote this element by $\emptyset_{\mathrm{A}}$.

Cantor's diagonal argument

Theorem

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$$S = \{a \in A | a \notin f(a)\}.$$

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$$S = \{a \in A | a \notin f(a)\}.$$

Note that S is a subset of A, which is also an element of 2^A (even if S is the empty set). Since f is surjective, there is some $b \in A$, such that $f(b) = S$. If $b \in S$, then by the definition of S, $b \notin f(b) = S$, which is a contradiction.

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Continuum hypothesis

Let $\aleph_{\mathrm{k}}$ be the cardinality of the k-th power set of $\mathbb{N}$. The above theorem implies the following result.

Corollary

$\aleph_0 < \aleph_1 < \aleph_2 < \cdots$.

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Since

$$|\mathbb{C}| = |\mathbb{R}| = |2^{\mathbb{N}}| > |\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Z}|,$$

it is natural to ask the following.

$$\text{Is there a set A satisfying } \aleph_0 < |A| < \aleph_1?$$

This is the first problem of the well-known Hilbert's 23 problems, also called Continuum hypothesis.

Russell's paradox

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In 1963, Cohen showed that the CH (continuum hypothesis) is independent of ZFC. In other words, CH can not be prove or disprove under ZFC. Cohen won the Fields Medal in 1966 for this important work.