

Notes on Topological and Differentiable Manifolds

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1 Elementary Topology

Let us begin with the definition of a topology:

Definition 1. A **topology** on a set X is a collection \mathcal{T} of subsets of X , called **open sets**, satisfying the following properties:

1. X and \emptyset are elements of \mathcal{T} .
2. \mathcal{T} is closed under finite intersections: If $U_1 \dots U_n \in \mathcal{T}$, then their intersection $U_1 \cap \dots \cap U_n$ is in \mathcal{T} .
3. \mathcal{T} is closed under arbitrary unions: If $U_1 \dots U_n \dots$ is any (finite or infinite) collection of elements of \mathcal{T} , then their union $\cup_{\alpha} U_{\alpha}$ is in \mathcal{T} .

A pair (X, \mathcal{T}) consisting of a set X and a topology \mathcal{T} on X is called a **topological space**. The elements of a topological space are usually called its **points**.

Definition 2. If X is a topological space and $q \in X$, a **neighborhood** of q is just an open set containing q . More generally, a neighborhood of a subset $K \subset X$ is an open set containing K .

Definition 3. If X is a topological space and $\{q_i\}$ is any sequence of points in X , we say that the sequence **converges** to $q \in X$, and q is the **limit** of the sequence, if for every neighborhood U of q there exists N such that $q_i \in U$ for all $i \geq N$. We denote this as $q_i \rightarrow q$ or $\lim_{i \rightarrow \infty} q_i = q$.

Example 1. Let Y be a trivial topological space (i.e. the only open sets are X and \emptyset). Each point has only 1 neighborhood: X itself. Thus, any sequence can be entirely contained in the neighborhood X , and consequently, any sequence converges to any point in X .

Example 2. Let X be a discrete topological space (i.e. all every subset of X is open). Take any sequence of points $\{q_i\}$. If the sequence converges to q , every open set containing q must contain all but a finite elements of the sequence. By virtue of the discrete topology, there exists an open set that contains only q . Obviously, then, there must exist an N such that $q_i = q$ for all $i \geq N$. Consequently, the only convergent sequences in X are the ones that are “eventually constant.”

Definition 4. If X and Y are topological spaces, a map $f : X \rightarrow Y$ is said to be **continuous** if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X .

Lemma 1. *Let X, Y, Z be topological spaces.*

1. *Any constant map $f : X \rightarrow Y$ is continuous.*
2. *The identity map $\text{Id} : X \rightarrow X$ is continuous.*
3. *If $f : X \rightarrow Y$ is continuous, so is the restriction of f to any open subset of X .*
4. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, so is their composition $g \circ f : X \rightarrow Z$.*

Proof. Let us begin with the constant map. Suppose f maps X to the constant $\lambda \in Y$. We wish to show that the preimage of f of every open set U in Y is open. There are two cases: U either does or does not contain λ . If it does, $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \emptyset$. As both X and \emptyset are open sets, f is continuous.

The continuity of the identity map follows trivially from the fact that Id maps any open set back to the same open set.

To prove the third statement, take any open set U in Y . U can be written as a union of points in and outside $f(V) \subset Y$: $U = U_i \cup U_o$. We want to show that $g^{-1}(U)$ is open in V . Since $g^{-1}(U_o) = \emptyset$, which is open, and $g^{-1}(U_i) \subset V$ and is open in X by the continuity of f , $g^{-1}(U_o \cup U_i) = g^{-1}(U_o) \cup g^{-1}(U_i)$ is open in V .

To prove the fourth statement, it suffices to show that $(g \circ f)^{-1}(U)$, with $U \subset Z$ open, is open in X . First note that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U)$ is an open set in Y . Similarly, f^{-1} of an open set in Y is open in X as f is continuous, and we are done. \square

Lemma 2 (Local Criterion for Continuity). *A map $f : X \rightarrow Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous.*

Proof. If f is continuous, each point of X has a neighborhood on which f is continuous; namely, X itself.

To prove the converse, suppose that each point of X has a neighborhood on which f is continuous - we wish to show that for any open set $U \subset Y$, $f^{-1}(U)$ is open in X . By continuity at each point, we know that any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous. In other words, $(f|_{V_x})^{-1}(U) = f^{-1}(U) \cap V_x$ is open in X and is contained in $f^{-1}(U)$. As $f^{-1}(U)$ is the union of such sets for all V_x , and these sets are open, it follows that $f^{-1}(U)$ is open, and we are done. \square

Definition 5. If X and Y are topological spaces, a **homeomorphism** from X to Y is defined to be a continuous bijective map $\phi : X \rightarrow Y$ with continuous inverse. If there exists a homeomorphism between X and Y , we say that X and Y are **homeomorphic** or **topologically equivalent**. Sometimes this is abbreviated $X \approx Y$.

Exercise 1. Show that homeomorphisms are an equivalence relation.

Proof. To show that homeomorphisms are an equivalence relation, we show

- $X \approx X$: The identity map Id is a homeomorphism from X to X .
- $X \approx Y \implies Y \approx X$: There exists a homeomorphism from X to Y . Its inverse is clearly a homeomorphism from Y to X .
- $X \approx Y$ and $Y \approx Z \implies X \approx Z$: As the composition of two homeomorphisms is also a homeomorphism (from elementary set theory), the homeomorphism from X to Z is simply the composition of the homeomorphisms from Y to Z and from X to Y , respectively.

\square

Example 3. Any open ball in \mathbb{R}^n is homeomorphic to any other open ball. The homeomorphism can be constructed simply by composition translations $x \mapsto x + x_0$ and dilations $x \mapsto cx$. This shows that size is not a topological property.

Example 4. If \mathbb{B}^n is the open unit ball, we can define $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$ by

$$y = F(x) = \frac{x}{1 - |x|^2}.$$

The inverse is given by

$$x = F^{-1}(y) = \frac{2y}{1 + \sqrt{1 + 4|y|^2}}.$$

As both are continuous and bijective, F is a homeomorphism, so \mathbb{R}^n is homeomorphic to \mathbb{B}^n . This shows that boundedness is not a topological property.

Example 5. Take the surface of the unit sphere in \mathbb{R}^3 and the surface of the cube of side 2, centered at the origin. There exists a homeomorphism between these two surfaces,

$$\phi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

whose inverse is given by

$$\phi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Thus, corners are not a topological property either.

Example 6. Now for an example of a continuous bijection that is not a homeomorphism by failure of its inverse to be continuous. Let X be the interval $[0, 1) \subset \mathbb{R}$, and let \mathbb{S}^1 denote the unit circle in \mathbb{R}^2 (both with the Euclidean metric topologies). Define a map $a : X \rightarrow \mathbb{S}^1$ by $a(t) = (\cos 2\pi t, \sin 2\pi t)$. It is clear that the map is continuous and bijective. The inverse, however, is not continuous. To see why, take any neighborhood of the point $(1, 0)$ - the inverse of a on this neighborhood will inevitably contain the point $0 \in [0, 1)$. Thus the preimage of open sets is not necessarily open, and thus a^{-1} is not continuous.

Definition 6. A map $f : X \rightarrow Y$ is said to be an **open map** if for any open set $U \subset X$, the image set $f(U)$ is open in Y . A map can be open but not continuous, continuous but not open, both, or neither.

Definition 7. We say that a continuous map $f : X \rightarrow Y$ between topological spaces is a **local homeomorphism** if every point $x \in X$ has a neighborhood $U \subset X$ such that $f(U)$ is an open subset of Y and $f|_U : U \rightarrow f(U)$ is a homeomorphism.

Exercise 2. *Show that:*

1. *every local homeomorphism is an open map.*
2. *every homeomorphism is a local homeomorphism.*
3. *a bijective continuous open map is a homeomorphism.*

4. *a bijective local homeomorphism is a homeomorphism.*

Proof.

1. Take any open set $U \in X$. For every $x \in U$, there exists some neighborhood $V = U_x \cap U$ of x for which the local homeomorphism $f(V)$ is open. Note that U is the union of all such V (for various x) and thus $f(U)$ is the union of all $f(V)$, and as the union of open sets must be an open set, $f(U)$ must be open. Consequently, f is an open map.
2. Take f to be a homeomorphism from X to Y . f is trivially a local homeomorphism; the neighborhood needed by the definition is simply X itself.
3. Let f be a bijective, continuous, open map. To show that f is a homeomorphism, we must show that f^{-1} is a continuous map. In other words, we must show that the preimage of f on open sets of Y are open sets in X . This is true as f is open and bijective: open sets in X are taken to open sets in Y (open), and all open sets in Y are images of open sets in X (bijective).
4. Let f be a bijective local homeomorphism.

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