Notes on Differentiable Manifolds

Nilay Kumar

Last updated: January 31, 2013

0 Administrativia

Recommended textbooks:

- John Lee: *Introduction to Smooth Manifolds* (2012)
- Spivak: *Differential Geometry: A Comprehensive Introduction*
- L. Tu: *An Introduction to Manifolds* (2008 E-book available)

Problem sets will be assigned every one/two weeks through email. Some homework problems will be taken from Lee (second edition). There will most likely be two midterms and a final, all in-class.

Office hours are on Fridays from 11am to 12pm.

1 Introduction

Definition 1. A function f defined on $\mathbb{R}^n$ is C^k for a positive integer k if $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_l}}$ exists and is continuous for any positive integer $l \leq k$, where $1 \leq i_1 \cdots i_l \leq n$. f is C^∞ if it is C^k for any positive integer k.

Example 1. $f(x) = x^{1/3}$ for $x \in \mathbb{R}$ is C^0 but not C^1.

Example 2. $f(x) = x^{1/3} + k$ for $x \in \mathbb{R}$ is C^k but not C^{k+1}, for $k \geq 1$.

Definition 2. A **coordinate chart** (U, ϕ) on a topological space X is an open set $U \subset X$ together with a map $\phi : U \to \mathbb{R}^n$ such that ϕ is a homeomorphism onto $\phi(U)$, an open set in $\mathbb{R}^n$. In other words, (U, ϕ) gives each $p \in U$ a coordinate.

Example 3. Let $S^2 = \left\{(x, y, z)|x^2 + y^2 + z^2 = 1\right\} \subset \mathbb{R}^3$. Let $U = \{z > 0\} \cap S^2$ be the upper hemisphere. x, y, z are not good coordinates in that they are not free - they are constrained to the surface. Note that if we define $\phi : U \to \mathbb{R}^2$ such that it takes $(x, y, z) \to (x, y)$, we have a projection map and we now have free coordinates (z can be computed). This is a **graphical coordinate chart**. We can work similarly with the lower hemisphere. But what about the equator? We can build similar charts for the equator by projecting onto different planes. It is clear, however, that to cover every point, we will need a total of 6 graphical charts to cover S^2.

This is nice because we can now do calculus on the open sets that ϕ maps us to in Euclidean space.

Example 4 (Stereographic projection of S^2)**.** Use a different model for S^2. Consider $\left\{(x, y, z)|x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}\right\} \subset \mathbb{R}^3$, the sphere of radius 1/2 centered at $(0, 0, \frac{1}{2})$. Note that the south pole's coordinates are $(0, 0, 0)$ and the north pole's are $(0, 0, 1)$. Imagine that there is a light source at the north pole, which projects through the sphere onto the xy-plane. If the line hits the point (x, y, z) on the sphere, we can solve for the point at which it hits the xy-plane. The line is given by $(0, 0, 1) + t(x, y, z - 1)$ for $t \in \mathbb{R}$. Solving this for where $z = 0$ yields $t = \frac{1}{1-z}$. The point is then $(0, 0, 1) + \frac{1}{1-z}(x, y, z-1) = (\frac{x}{1-z}, \frac{y}{1-z}, 0)$. This gives a coordinate chart (U, ϕ) with $U = S - \{(0, 0, 1)\}$ (as the chart is undefined there) and $\phi : U \to \mathbb{R}^2$ that maps $(x, y, z) \to (\frac{x}{1-z}, \frac{y}{1-z})$.

In order to cover the south pole as well, we can perform stereographic projection from the south pole onto $z = 1$ plane. The corresponding line is now given by $(0, 0, 0) + t(x, y, z)$ which has $z = 1$ for $t = \frac{1}{z}$, and the corresponding point is $(\frac{x}{z}, \frac{y}{z}, 1)$. This gives us a (V, ψ) where $V = S - \{(0, 0, 0)\}$ with $\psi : V \to \mathbb{R}^2$ mapping $(x, y, z) \to (\frac{x}{z}, \frac{y}{z})$.

Example 5. Let X be the set of all lines on $\mathbb{R}^2$. X is a topological space (check this!). Take the set $U = \{$ lines of the form $y = mx + c\}$, which is the collection of all non-vertical lines. To cover the vertical lines, we can have $V = \{$ lines of the form $x = \bar{m}y + \bar{c}\}$, the collection of all non-horizontal lines. We now define $\phi : U \to \mathbb{R}^2$ that maps $y = mx + c \to (m, c)$ and $\psi : V \to \mathbb{R}^2$ that maps $x = \bar{m}y + \bar{c} \to (\bar{m}, \bar{c})$.

Notice that for this example and the stereographic projection example, there are instances where the charts overlap. For the sake of consistency, we want the coordinate charts to be compatible with one another. For example, a function that is differentiable in one chart should be differentiable in the other as well.

Definition 3. Given a coordinate chart (U, ϕ) and a function f defined on U, we can consider $f \circ \phi^{-1}$ as a function on $\phi(U)$ and differentiate $f \circ \phi^{-1}$. Suppose (V, ψ) is another coordinate chart and that $U \cap V \neq \varnothing$. Now we can consider $f \circ \psi^{-1}$ as a function to do calculus with. Now the question is: is $f \circ \phi^{-1}$ differentiable the same as $f \circ \psi^{-1}$ differentiable? This should be the case! We are considering a function on an abstract space, and the coordinates should respect properties such as differentiablity. So let us write $f \circ \psi^{-1} \circ (\psi \circ \phi^{-1})$, which is differentiable if the term in the parentheses is differentiable. We can do the same, but with ϕ and ψ switched. Thus, we want to make sure that both $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are differentiable. These are called **transition maps** of these coordinate charts. Two coordinate charts are **smoothly compatible** if their transition maps are diffeomorphisms (or if, trivially, they don't intersect). These diffeomorphisms are from $\phi(U \cap V)$ to $\psi(U \cap V)$ or vice versa.

Returning to the previous manifold of lines, suppose we have a line labeled by (m, c) with $m \neq 0$. This can be expressed in the other chart as $(m^{-1}, -m^{-1}c)$. It turns out, that for $m \neq 0$, these transition maps are differentiable. For the stereographic projection of S^2, it is the same, just a little trickier.

Definition 4. An **atlas** $\mathcal{A}$ for a topological space X is a collection of coordinate charts that covers X such that any two charts in $\mathcal{A}$ are smoothly compatible.

Example 6. Take the set of all lines through the origin in $\mathbb{R}^3$. Since only the direction matters, each line might be represented by a non-zero vector. This is the same as the quotient space $\mathbb{R}^3 \setminus \{(0,0,0)\} / \sim$ with $(x_1, x_2, x_3) \sim \lambda(x_1, x_2, x_3)$. Thus it is equipped with the quotient topology; i.e.$\Pi : \mathbb{R}^3 \ \{(0,0,0)\} \to \mathbb{RP}^2$ is continuous. We use $[x_1 x_2 x_3]$ to denote the equivalence class of (x_1, x_2, x_3). On the open set (check this) $U_1 = \{x_1 \neq 0\}$, we can use the coordinate chart $\phi_1 : U_1 \to \mathbb{R}^2$ that takes $(x_1, x_2, x_3) \to (x_2/x_1, x_3/x_1)$. However, we have not covered the whole set, so we repeat this process for $x_2 \neq 0$ and $x_3 \neq 0$. We claim that this set of charts is an atlas. It is obvious that these charts covers the space. We must now check that the transition maps are smoothly compatible. For example, we must check that $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism. For $[x_1, x_2, x_3] \in U_1 \cap V_2$, it's clear that $\phi_2 \circ \phi_1^{-1}(x_2/x_1, x_3/x_1) = (x_1/x_2, x_3/x_2)$. To show that this is a diffeomorphism, we choose coordinates (u, v) on $\mathbb{R}^2$ and write the function in terms of these coordinates: $u = x_2/x_1$ and $v = x_3/x_1$. We know that $x_1 \neq 0, x_2 \neq 0$ because of our domain. Thus, $\phi_2 \circ \phi_1^{-1} = (1/u, v/u)$. Again, by our domains, $u \neq 0$, and thus this map is differentiable. The inverse is found by writing $(1/u, v/u)$ as (p, q) which yields $(u, v) = (1/p, q/p)$ which is differentiable as well. One can check that this map is also one-to-one and onto, and we are done.

In general, it turns ou that $\mathbb{RP}^n$ requires $n + 1$ coordinate charts.

Definition 5. Two atlases are compatible (or equivalent) if their union is another atlas.

Definition 6. A **differentiable (smooth) structure** on a topological space X is an equivalence class of atlases (a maximal atlas).

Example 7. Take the curve $\mathcal{C} : y = x^{\frac{1}{3}}$ in $\mathbb{R}^2$. Recall that this curve has a vertical tangent at $x = 0$. This curve has the subspace topology. We can consider the graphical coordinate charts $\phi_1 : (x, y) \to x$ and take $U_1 = \mathcal{C}$. We can define an atlas $\mathcal{A}_1 = \{(U_1, \phi_1)\}$. We can also take $\phi_2 : (x, y) \to y$ and $\mathcal{A}_1 = \{(U_2, \phi_2)\}$. Are these equivalent? Consider the height function h on $\mathcal{C}$ (the y coordinate). Consider $h \circ \phi_1^{-1}(x) = x^{\frac{1}{3}}$. Consider instead $h \circ \phi_2^{-1}(y) = y$. But h is only differentiable on the second chart! Thus these atlases are not equivalent.

Example 8. Take the real line. $f(x) = x^{1/3}$ is not differentiable. Suppose we make a new coordinate system that assigns each point its cubic root. The points, of course go home all happy, but now f is differentiable!

In order to define what a "smooth manifold" we must impose global topological conditions to rid ourselves of certain pathological examples.

Definition 7. A topological space X is **Hausdorff** if any two points can be separated by disjoint open sets; i.e. $\forall p, q \in X$, there exist U, v open such that $p \in U, q \in V$ and $U \cap V = \varnothing$.

Definition 8. A topological space X is **second countable** if it has a countable basis of open sets.

Recall that a basis $\mathcal{B}$ is a subset of the collection of all open sets such that any open set can be written as a union of members of $\mathcal{B}$.

Example 9. Note that $\mathbb{R}^n$ is second countable because we can take a basis $\mathcal{B}$ that is the collection of all open balls with rational centers and rational radii. Additionally, it is clear that $\mathbb{R}^n$ is Hausdorff.

Note that a subspace of a Hausdorff, second-countable topological space is itself Hausdorff and second countable. Thus any subset of $\mathbb{R}^n$ is Hausdorff and second countable. Pathological spaces:

- the disjoint union of uncountably many copies of $\mathbb{R}$ is not second countable.
- a real line with two origins is not Hausdorff.

Definition 9. A **differentiable/smooth manifold** is a Hausdorff, second countable topological space with a smooth (differentiable) structure.

Example 10. Suppose $U \subset \mathbb{R}^n$ is open with $F : U \to \mathbb{R}^m$ that is C^∞. The graph of F is $\Gamma = \{(x, y)|x \in U, y = F(x)\} \subset \mathbb{R}^n \times \mathbb{R}^m$ is a differentiable manifold. Γ is Hausdorff and second countable because $\Gamma \subset \mathbb{R}^{n+m}$. We can take $\mathcal{A} = \{(\Gamma, \Pi)\}$ where $\Pi : \Gamma \to U$ maps $(x, F(x)) \to x$ (check that this is a homeomorphism).

Example 11. Take $S^2 = \left\{(x, y, z)|x^2 + y^2 + z^2 = 1\right\} \subset \mathbb{R}^3$. This is Hausdorff and second countable. We can take the 6 graphical coordinate charts $\{x > 0\}, \{x < 0\}, \{y > 0\}, \{y < 0\}, \{z > 0\}, \{z < 0\}$, with the appropriate homeomorphisms (projections, in this case) to map from hemispheres to the planes "below" them. Take for example the maps ϕ_1, ϕ_2 that map from the upper and lower hemispheres to the x-y plane respectively. Also, take ϕ_3 to be the map for $\{y > 0\}$ to the x-z plane. Then, on $U_1 \cap U_1$ we have that $\phi_3 \circ \phi_1^{-1} : (x, y) \to (x, z) = \sqrt{1 - x^2 - y^2}$. Clearly this is differentiable as long as $z \neq 0$. Indeed, all the charts turn out to be smoothly compatible, and we have a manifold. This manifold is what we call a **level set** of $F(x, y, z) = x^2 + y^2 + z^2$.

In general, though, what about $\{x|F(x) = c\} \subset \mathbb{R}^n$? Is it a manifold? (Indeed, it turns out that a set of equations $F_i(x_1 \cdots x_m) = c_n$ has solutions that often form a manifold.)

Note: take $U \subset \mathbb{R}^n$ open. Take $F : U \to \mathbb{R}$ continuous for any $c \in \mathbb{R}$. Then, $F^{-1}(c)$ is a closed subset of U. In fact, F^{-1} is always Hausdorff and second countable as a subset of $\mathbb{R}^n$. When is $F^[- 1](c)$ a differentiable manifold? Let us first ask: when can we solve one variable in terms of the others? In other words, given $F(x, y, z) = c$, can we write, for example, $z = f(x, y)$?

Example 12. Suppose F is linear: $F(x, y, z) = ax + by + dz$. On $F(x, y, z)$, we can solve x in terms of y, z as long as $a = \frac{\partial F}{\partial x} \neq 0$.

Theorem 1 (Implicit function theorem)**.** *Suppose U is open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}$. Suppose $F(a) = c$, with $a \in U$, and $\frac{\partial F}{\partial x^n}(a) \neq 0$. Then, there exists a neighborhood V of a and a unique function $f(x_1 \cdots x_{n-1})$ such that $\frac{\partial F}{\partial x_n} \neq 0$ on V and*

1. $V \cap F^{-1}(c) = V \cap \{(x_1 \cdots x_{n-1}) | x_n = f(x_1 \cdots x_n\}$
2. $\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} / \frac{\partial F}{\partial x_n}$ *for* $i = 1 \cdots n-1$

Proof. Go over this in your own time/past notes. □

Theorem 2. *Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞. Take $c \in \mathbb{R}$ and $F^{-1} \neq \varnothing$ and suppose $\nabla F(x) \neq 0$ for all $x \in F^{-1}(c)$. c is called a **regular value** of f. Then, $F^{-1}(c)$ is an $(n-1)$ dimensional smooth manifold.*

Proof. $F^{-1}(c)$ is Hausdorff and second countable. Now we wish to produce a differentiable structure. Consider $\tilde{U}_i = \left\{ \frac{\partial F}{\partial x_i} \neq 0 \right\} \cap F^{-1}(c)$. U_i is open in $F^{-1}(c)$ and $F^{-1}(c) \subset \cup_{i=1}^n U_i$ by the assumption that the gradient is nonzero on $F^{-1}(c)$. For each point $a \in U_i$, by the implicit function theorem, there exists a neighborhood U_a of a (may assume $U_a \subset U_i$ by taking an intersection) such that $U_a \cap F^{-1}(c)$ is the graph of a function $\{x_i = f_i(x_1 \cdots x_{i+1} \cdots x_n)\}$ and this f_i is unique.

We take the collection of all these U_a $a \in F^{-1}(c)$ as coordinate charts. When two coordinate charts overlap, if they belong to the same U_i, by uniqueness of f_i the transition map must be the identity map. But when two coordinate charts overlap but belong to different U_i, the transition maps consist going from one projection to another (or unprojecting), which is simply a matter of dropping (or adding back) certain x_i. Since, by the implicit function theorem, we may write x_i smoothly in terms of the other coordinates, the transition maps are smooth. □

For $F(x, y, z) = x^2 + y^2 + z^2$, the gradient is $\nabla F = (2x, 2y, 2z)$. Thus, the sphere is a differentiable manifold as the choice of $c = 1$ yields a gradient that is not the zero vector.

Remark (invariance of dimension): If a smooth manifold is connected, then any coordinate chart is homeomorphic to an open subset of $\mathbb{R}^n$ for a fixed n, this n is called the dimension of the manifold.

Theorem 3. *Take U open in $\mathbb{R}^{n+m}$ and $F : U \to \mathbb{R}^m$ smooth. Let $c \in \mathbb{R}^m$ and $F^{-1}(c) \neq \varnothing$. Suppose that for all $a \in F^{-1}(c)$, $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is of rank m (full rank). Then, $F^{-1}(c)$ is a n dimensional smooth manifold and c is called a **regular value** of F. Recall that DF_a is simply the $m \times (n+m)$*

matrix of partial derivatives of F_i. The corresponding implicity function theorem says that if the determinant of the $m \times m$ submatrix of DF_a is not zero, then $x_1 \cdots x_m$ can be locally solved in terms of $x_{m+1} \cdots x_{m+n}$.

Example 13. Let's look at the simple case $m = 3$, $n = 2$:

$$F_1(x_1, x_2, x_3) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 F_2(x_1, x_2, x_3) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

It's clear that if the 2×2 left submatrix has non-zero determinant, and we set $F_1 = c_1, F_2 = c_2$, we will be able to solve for x_3 in terms of x_1, x_2.