Introduction to Algebraic Topology PSET 5

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Proposition 1. *Hatcher exercise 1.2.6*

Proof. We follow the proof of Hatcher proposition 1.26. Suppose Y is a space obtained from a path-connected subspace X by attaching n-cells for a fixed $n \geq 3$. Note that each n-cell e_α^n is attached to X via attaching maps $\phi_\alpha : S^{n-1} \to X$. If we choose in each e_α^n a point y_α, and let $A = Y - \cup_\alpha \{y_\alpha\}$ and let $B = Y - X$. Note that $\pi_1(B) = 0$, and hence by van Kampen's theorem applied to the cover $\{A, B\}$, we find that $\pi_1(Y)$ is isomorphic to the quotient of $\pi_1(A)$ by the normal subgroup generated by the image of the map $\pi_1(A \cap B) \to \pi_1(A)$. Note, however, that $\pi_1(A \cap B)$ can be computed by van Kampen's theorem to be the free product of $\pi_1(A_\alpha)$ where $A_\alpha = A \cap B - \cup_{\beta \neq \alpha} e_\beta^n$. But A_α is simply a punctured n-cell, which is homotopy equivalent to S^{n-1}. Hence $\pi_1(A \cap B) = 0$, and we see that the map $\pi_1(A \cap B) \to \pi_1(A)$ is the trivial map, and thus $\pi_1(Y) = \pi_1(A) = \pi_1(X)$, since A deformation retracts to X (in particular, the n-cells retract to the S^{n-1} by which they are attached to X). This proves the claim.

Consider now a discrete subspace $X \subset \mathbb{R}^n$. We claim that $Y = \mathbb{R}^n \setminus X$ is simply-connected. It is clear that Y is path-connected; to show that the fundamental group is trivial, we construct open n-balls B_i around each of the points of X (containing only one point of X). Then Y deformation retracts onto $Y - \cup_i B_i$ and since we can attach to $Y - \cup_i B_i$ n-balls to produce $\mathbb{R}^n$, the theorem above implies that for $n \geq 3$, $\pi_1(Y) = \pi_1(\mathbb{R}^n) = 0$. □

Proposition 2. *Problem 2*

Proof.

(a) Recall that the stereographic projection gives a homeomorphism between the punctured sphere and $\mathbb{R}^2$. Hence, if we take an open cover of S^2 to be two punctured spheres with holes at the north and south poles, van Kampen's theorem implies a surjection $\pi_1(\mathbb{R}^2) * \pi_1(\mathbb{R}^2) \twoheadrightarrow \pi_1(S^2)$. But $\mathbb{R}^2$ is contractible, and hence $\pi_1(\mathbb{R}^2) = 0$. Such a surjection is then only possible if $\pi_1(S^2) = 0$ as well. This generalizes to higher dimensions in exactly the same way, as the stereographic projection holds for n-spheres and n-planes, and $\pi_1(\mathbb{R}^n) = 0$ for all $n \in \mathbb{N}$.

(b) The cell decomposition for S^n is a 0-cell with an attached n-cell. As the 0-cell is a point, it is clearly path-connected, and hence by the previous exercise, we find that $\pi_1(S^n) = 0$ for $n \geq 3$.

□

Proposition 3. *Hatcher exercise 1.3.2*

Proof. This is clear. Take $\{U_\alpha\}$ and $\{V_\beta\}$ to be the chosen open covers of X_1 and X_2, respectively. Then $\{U_\alpha \times V_\beta\}$ furnishes an open cover of $X_1 \times X_2$. If we let $\rho : \tilde{X}_1 \times \tilde{X}_2 \to X_1 \times X_2$ be the natural projection, then $\rho^{-1}(U_\alpha \times V_\beta) = p_1^{-1}(U_\alpha) \times p_2^{-1}(V_\beta)$. This is, of course a disjoint union of subspaces that map homeomorphically down to $U_\alpha \times V_\beta$, by the fact that $\tilde{X}_1 \to X_1$ and $\tilde{X}_2 \to X_2$ are covering spaces. □

Proposition 4. *Hatcher exercise 1.3.4*

Proof. Consider first $X = S^2 \cup D$, where D is a diameter. It is not hard to see that $X \simeq S^2 \vee S^1$. Hence the construction of the universal cover proceeds almost exactly as in the case of S^1: we take $\mathbb{R}$ and attach to every integer point an S^2 (by identifying say, the point with the north pole of the sphere). This is simply-connected as it path-connected and is homotopy equivalent to the infinite wedge sum of spheres (which by van Kampen has trivial fundamental group). It is quite clear that this is a covering space for X.

If we now have a circle intersected the sphere at two points denoted by X, it is clear that X is homotopy equivalent to $S^2 \vee S^1 \vee S^1$. Hence the covering space is constructed in analogy to that of $S^1 \vee S^1$, where one can take the Cayley complex discussed in class (and shown on p. 59 in Hatcher) and attach a sphere to every intersection point. This is of course simply connected, as it is again the wedge sum of spheres, and clearly projects down to $S^2 \vee S^1 \vee S^1$ as a covering map. □

Proposition 5. *Hatcher exercise 1.3.9*

Proof. Let X be a path-connected, locally path-connected space with finite fundamental group. Consider a map $f : X \to S^1$. Note that the induced map $f_* : \pi_1(X) \to \mathbb{Z}$ is a morphism from a finite group to $\mathbb{Z}$ and hence must be trivial (otherwise the image would form a finite subgroup, of which there are none in $\mathbb{Z}$), showing that $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\mathbb{R}, 0)) = 0$. By Hatcher proposition 1.33, we find that f lifts to a map $\tilde{f} : (X, x_0) \to (\mathbb{R}, 0)$, but by contractibility of $\mathbb{R}$, this map can be homotoped to a constant map. Projecting this homotopy down to S^1 via p, we find that f is in fact homotopic to a constant map on S^1, as desired. □