Vector Bundles and Line Bundles over $\mathbb{CP}^1$

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0   Introduction

Before we discuss vector bundles in full generality, let us look at what motivates the definition. Recall that given a smooth manifold M, we can associate to each point on M a vector space called the tangent space to M at p, T_pM. These tangent spaces can be constructed in a variety of different ways, ranging from geometrically intuitive (equivalence classes of curves) to rather abstract (space of derivations at p) [Lee12]. However the tangent space is defined, one problem arises: how do we "compare" tangent vectors that live in different tangent spaces? To be able to define vector fields, for example, one must deal with all the tangent spaces of a manifold at the same time!

It is this problem that motivates the introduction of the **tangent bundle**

$$TM = \coprod_{p \in M} T_pM,$$

the disjoint union of all the tangent spaces to M. Elements of the tangent bundle are typically written as an ordered pair (p, v) with $p \in M$ and $v \in T_pM$. Furthermore, every tangent bundle over M comes equipped with a projection map $\pi : TM \to M$ which sends each vector in T_pM to the point p at which the tangent space is attached: $\pi(p, v) = p$. Note, however, that the tangent bundle is much more than just a disjoint union of vector spaces – it has a smooth structure that makes it into a $2n$-dimensional smooth manifold.

Theorem 1. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection map $\pi : TM \to M$ is smooth.*

Proof. See [Lee12], chapter 3. □

Now that we recognize TM as a manifold, we can ask what coordinates we can use to locally describe its points. The most useful coordinates are the natural coordinates that are induced by M and the basis of the tangent vector spaces; they look like (x^i, v^i) where $x = \phi(p)$ and $v = v^i \partial/\partial x^i$.

Let us look at two simple examples of tangent bundles. First consider the tangent bundle over $\mathbb{R}^n$. Since the tangent spaces to $\mathbb{R}^n$ are each isomorphic to $\mathbb{R}^n$, we see that

$$T\mathbb{R}^n = \coprod_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

In this case, the tangent bundle can be written simply as a direct product of two smooth manifolds; this is not in general the case, as we will discuss in the next section. As another example, take $M = S^1$, the unit circle. The space tangent to the circle at every point is simply $\mathbb{R}^1$. To visualize
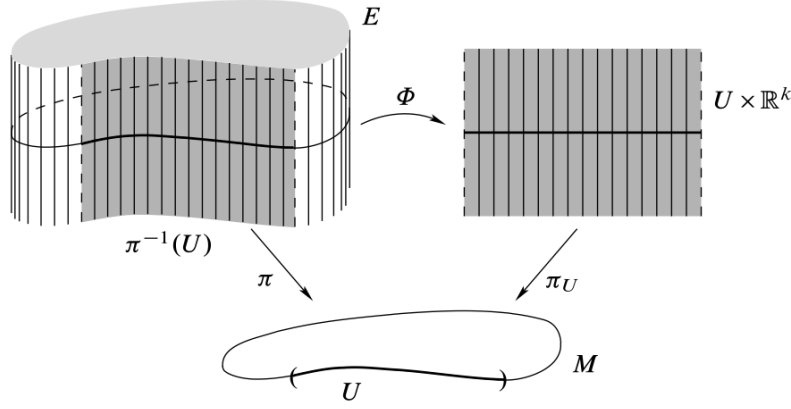
Figure 1: A schematic view of a real vector bundle [Lee12].

the tangent bundle TS^1, then, imagine orienting the circle horizontally and attaching a vertical line to every point of the circle to obtain an infinite cylinder, $S^1 \times \mathbb{R}^1$.

Now that we can talk about tangent vectors at different points on the manifold through the structure of the tangent bundle, we can define: a **smooth vector field** on M is a smooth section of the map $\pi : TM \to M$. In other words, a vector field is a smooth map $X : M \to TM$ (written $p \mapsto X_p$) with the property that $\pi \circ X = \mathrm{Id}_M$, i.e. $X_p \in T_pM$ for each $p \in M$. In words, a vector field is simply a smooth map that assigns each point p on a manifold to a tangent vector living in the tangent space at p.

1 Vector bundles

Now that we have an intuitive grasp of tangent bundles, let us do what mathematicians do best – generalize. We will fix a field k either $\mathbb{R}$ (with real manifolds) or $\mathbb{C}$ (with complex manifolds) to work in, and by smooth we will mean either C^∞ or holomorphic, and by diffeomorphic we will mean either diffeomorphic or biholomorphic.

Definition 1. A **vector bundle of rank** r over a smooth manifold M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a r-dimensional k-vector space.

(ii) For each $p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times k^r$, called a **local trivialization** of E over U, satisfying the following conditions:

- $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times k^r \to U$ is the projection;
- for each $q \in U$, the restriction of Φ to E_q is a k-vector space isomorphism from E_q to $\{q\} \times k^r \cong k^r$.

In the important special case that $r = 1$, we call the vector bundle a **line bundle**. E is typically called the **total space** and M is called the **base space**.

This definition is quite a mouthful, so let's dissect it, keeping Fig. (1) in mind. Informally, the vector bundle E is formed simply by attaching an r-dimensional k-vector space to each point of the

manifold M. This is precisely part (i) of the definition. It is important to note, however, that these fibers must be "put together" in a way that makes the total space E a manifold, i.e. E locally has a nice description, much like manifolds are required to be locally affine. This is precisely what the local trivializations in (ii) are meant to enforce: a vector bundle must locally look like the direct product of the base manifold M with the vector space k^r. Globally, however, the structure of the vector bundle might be "twisted" in some fashion. A good example to keep in mind is the (infinite) Mobius strip: locally, for some open set $U \in S^1$ one can describe it as $U \times \mathbb{R}^1$; globally, though, it becomes clear that the Mobius strip is twisted and cannot be written as a direct product. In this vein, we say that a vector bundle is **trivial** if there exists a smooth local trivialization of E over all of M, i.e. a diffeomorphism from E to the product $M \times k^r$.

It may be useful to think of local trivializations for vector bundles as analogous to charts for manifolds. Any bundle that is not trivial will clearly require more than one local trivialization. Just as in the case for transition maps between charts of a manifold, the transitions between smooth local trivializations are well-behaved, as shown in the following lemma.

Lemma 2. *Let $\pi : E \to M$ be a smooth vector bundle of rank r over M. Suppose $\Phi : \pi^{-1}(U) \to U \times k^r$ and $\Psi : \pi^{-1}(V) \to V \times k^r$ are two smooth local trivializations of E with $U \cap V \neq \varnothing$. Then there exists a smooth map $\tau : U \cap V \to GL_r(k)$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times k^r \to (U \cap V) \times k^r$ has the form*

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$$

where $\tau(p)v$ denotes the usual action of the $r \times r$ matrix $\tau(p)$ on the vector $v \in k^r$.

Proof. See [Lee12], chapter 10. □

This simply means that the transition function between two local trivializations takes fiber to fiber as a linear isomorphism. Consider again the example of the Möbius strip; seen as a line bundle over the circle, we can take an open cover for S^1 to be the usual two north/south-pole stereographic charts. On the circle, these charts overlap about the east/west-poles, and thus there are two transition functions: on one of the regions $\tau(p) = -1$, and on the other, $\tau(p) = +1$. It is precisely this -1 that "flips" one of the attached $\mathbb{R}^1$ fibers to give the characteristic Möbius twist.

The transition functions associated with the vector bundle carry quite a bit of information, as the following theorem shows.

Theorem 3. *Suppose M is a manifold with open cover $\{U_\alpha\}_{\alpha \in A}$. Suppose for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, k)$ such that the **cocycle condition***

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma$$

is satisfied for all $\alpha, \beta, \gamma \in A$. Then there exists a smooth rank-r vector bundle $E \to M$ with smooth local trivializations $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times k^r$ whose transition functions are the given maps $\tau_{\alpha\beta}$.

Proof. Left as an exercise to the reader. Hint: consider the disjoint union $\coprod_{\alpha \in A} (U_\alpha \times k^r)$ and define an appropriate equivalence relation. Quotienting by this equivalence relation should yield a vector bundle. □

This fact is quite useful, as we will see in the next section.

It should be clear that the tangent bundle that we discussed in the previous section is a prime example of a vector bundle. The vector spaces comprising the bundle are simply the tangent spaces to the manifold; the open cover and the transition functions follow in a straightforward manner from those for the base manifold. Incidentally, we can take smooth vector fields as a motivating

example and talk about smooth sections over a general vector bundle, which turns out to be quite useful. The study of Riemannian geometry, for example, is the study of Riemannian metrics on manifolds, which are smooth symmetric positive-definite two-tensor fields, i.e. smooth sections of a "tensor" (vector) bundle.

It turns out that we can actually use given vector bundles to produce new bundles! Indeed, most operations valid for vector spaces can be extended to vector bundles by performing the vector space operations fiberwise. Given vector bundles $E \to M, F \to M$, for example, we can define the dual bundle E^*, which has fibers $(k^r)^*$ at each point, or the direct sum bundle, which is a vector bundle $E \oplus F$ whose fibers at a point p are $E_p \oplus F_p$, or the tensor product bundle $E \otimes F$ given by the fiberwise tensor product, etc. [Hat03] We will see some of these constructions in action in the next section.

2 Holomorphic line bundles over $\mathbb{CP}^1$

Let us now turn to line bundles over $\mathbb{CP}^1$. This case is especially interesting because the line bundles over $\mathbb{CP}^1$ form a group isomorphic to $(\mathbb{Z}, +)$! We will not prove this here, but let us see why this is at least plausible.

The complex projective line is defined as $\mathbb{C}^2 - (0,0)/\sim$ where $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for $\lambda \in \mathbb{C}^\times$. In other words, $\mathbb{CP}^1$ is the space of one-dimensional subspaces of $\mathbb{C}^2$. Thus a natural vector bundle over $\mathbb{CP}^1$ is the **tautological bundle**: to each point p of $\mathbb{CP}^1$ attach the one-dimensional subspace associated to p as a fiber. This yields a line bundle that is typically denoted $\mathcal{O}(-1)$. Let us denote the usual charts for $\mathbb{CP}^1$ as U_1, U_2, where points in $\mathbb{CP}^1$ are written $[z_1 : z_2]$ with $z_1 \neq 0$ in U_1 and $z_2 \neq 0$ in U_2. We can now compute the transition function $\tau_{12} : U_1 \cap U_2 \to \mathrm{GL}(\mathbb{C})$. First note that we can construct the local trivializations Φ_1 as mapping $([1 : z], (\lambda, \lambda z)) \in \pi^{-1}(U_1)$ to $([1 : z], \lambda) \in U_1 \times \mathbb{C}$ and Φ_2 as mapping $([w : 1], (\lambda w, \lambda)) \in \pi^{-1}(U_2)$ to $([w : 1], \lambda) \in U_2 \times \mathbb{C}$. Hence, we have that
$$\Phi_1 \circ \Phi_2^{-1}([w : 1], \lambda) = \Phi_1\left([1 : w^{-1}], (\lambda w, \lambda)\right) = ([1 : w^{-1}], \lambda w).$$
From this we can read off that $\tau_{12}([1 : z]) = w = z^{-1}$.

Now, given the tautological line bundle $\mathcal{O}(-1)$, how can we produce more line bundles? The simplest construction is the dual bundle, $\mathcal{O}(1)$, whose fibers are linear maps $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$ instead of $\mathbb{C}$. We could simply take the dual (transpose) of our transition function from above – this however, would give us the transition function from U_1 to U_2, as dualizing maps flips the domain and codomain. Hence, we also have to throw in an inverse to get $\tau_{12}^*([1 : z]) = w^{-1} = z$. This now sheds some light on the notation $\mathcal{O}(\pm 1)$ that we've been using, and in the context of having an integers' worth of line bundles over $\mathbb{CP}^1$, it is natural to ask whether we can construct bundles with transition function z^n for $n \in \mathbb{Z}$ and if so, whether we can describe them nicely.

It should be clear that the trivial bundle $\mathcal{O}(0)$ has transition function $\tau_{12} = z^0 = \mathrm{Id}$. The next question to address is: how do we construct $\mathcal{O}(-2)$? The answer, it turns out, is through tensor products. Given two copies of $\mathcal{O}(-1)$, consider the tensor product $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$ as a vector bundle over $\mathbb{CP}^1$ whose fibers are $\mathbb{C} \otimes \mathbb{C}$. The transition function is now given by the tensor product of the transition functions of the individual bundles: $\tau_{12}^{\otimes 2}([1 : z]) = w \otimes w = w^2 = z^{-2}$. Thus we see that $\mathcal{O}(-2) = \mathcal{O}(-1)^{\otimes 2}$. More generally, it is easy to see that $\mathcal{O}(n)$ can be constructed by repeated tensor products of $\mathcal{O}(\pm 1)$.

If we take on faith that these are the only line bundles over $\mathbb{CP}^1$ (upto bundle isomorphism), we see that line bundles over $\mathbb{CP}^1$ form an abelian group under the tensor product that is isomorphic to $(\mathbb{Z}, +)$.

Now that we know what the line bundles over $\mathbb{CP}^1$ look like, we might inquire as to what the holomorphic sections of the bundle look like. In this case, a local section of $\mathcal{O}(n)$ simply associates each point p in $U_i \subset \mathbb{CP}^1$ with a complex number in the fiber of $\mathcal{O}(n)$ above p. Explicitly constructing local holomorphic sections is not particularly difficult: U_i is homeomorphic to a subset of $\mathbb{C}$ and thus the problem reduces to constructing a holomorphic functions on a subset of $\mathbb{C}$. More subtle is the problem of constructing a global holomorphic section, i.e. a section that is defined and holomorphic over the whole manifold $\mathbb{CP}^1$. If we construct our line bundles over the usual charts U_i, we have one overlap region $V = U_1 \cap U_2$. Using the magic of complex numbers we can construct a (local) holomorphic section over U_2, whose restriction to V can be extended uniquely to obtain a (hopefully) holomorphic section over U_1.

Consider the case for $\mathcal{O}(1)$. Take, for example, the monomial $f_2(w) = w$ on U_2. On U_1, then, we get (via extension) the function

$$f_1(z) = \tau_{12}(z) f_2(z^{-1}) = z \cdot z^{-1} = 1.$$

It's clear that w is holomorphic on U_2 and 1 is holomorphic on U_1 (and that they agree on V), and thus that we have a global holomorphic section on $\mathbb{CP}^1$. We can do the same if we define the function $f_2(w) = 1$, which extends to

$$f_1(z) = \tau_{12}(z) f_2(z^{-1}) = z \cdot 1 = z.$$

We again obtain a global holomorphic section! Any other monomial, however, does not induce a global holomorphic section. Since any holomorphic function can be written as a power series, we see that the only global holomorphic sections on $\mathcal{O}(1)$ are polynomials $a_0 + a_1 z$ (or $a_0 w + a_1$).

Similarly, for $\mathcal{O}(2)$, we can take $f_2(w)$ to be defined by $1, w, w^2$, which will extend to $f_1(z)$ being $z^2, z, 1$ resp. In general, then, the space of global holomorphic sections on $\mathcal{O}(n)$ for $n > 0$ is simply the space of polynomials in z (or w) of degree upto n. But what about if $n \leq 0$? Well the trivial bundle, $\mathcal{O}(0) = \mathbb{CP}^1 \times \mathbb{C}$, is easy. Global holomorphic sections simply correspond to holomorphic functions $f : \mathbb{CP}^1 \to \mathbb{C}$, which are constants (Liouville's theorem). Next consider $n = -1$, i.e. $\mathcal{O}(-1)$, which has transition function $\tau_{12}(z) = z^{-1}$. Does $f_2(w) = 1$ extend holomorphically to a global section? The usual calculation shows that

$$f_1(z) = \tau_{12}(z) f_2(z^{-1}) = z^{-1},$$

which blows up at $[1 : 0]$, and is thus not holomorphic. Furthermore, if $f_2(w) = w$ we get

$$f_1(z) = \tau_{12}(z) f_2(z^{-1}) = z^{-2},$$

which is clearly no good either. Indeed, it is easy to see that there is no holomorphic monomial in w that extends to a holomorphic monomial in z, and hence there are no non-trivial global sections of $\mathcal{O}(-1)$. The same holds for all $\mathcal{O}(n)$ with $n \leq -1$.

In conclusion, the space of global holomorphic sections of $\mathcal{O}(n)$ is trivial for negative n, and isomorphic to $\mathbb{C}^{n+1}$ for non-negative n. That we can explicitly determine these spaces of sections is a good measure of our understanding of the bundles $\mathcal{O}(n)$. This understanding for line bundles in general, in fact, turns out to be very useful in representation theory (the Borel-Weil-Bott theorem) and algebraic geometry (the line bundles over a variety yield information about the codimension 1 subvarieties).

References

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[Lee12] John M Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, 2012.