Introduction to Algebraic Topology PSET 3

Nilay Kumar

Last updated: February 9, 2014

Proposition 1. *Let $f_0 \diamond g_0 \simeq f_1 \diamond g_1$ and $g_0 \simeq g_1$. Then $f_0 \simeq f_1$.*

Proof. Consider the path $\bar{g}_1$ that traverses g_1 backwards. Then

$$\begin{aligned} f_0 \diamond g_0 &\simeq g_1 \diamond g_1 \\ (f_0 \diamond g_0) \diamond \bar{g}_1 &\simeq (f_1 \diamond g_1) \diamond \bar{g}_1 \\ f_0 \diamond (g_0 \diamond \bar{g}_1) &\simeq f_1 \diamond (g_1 \diamond \bar{g}_1) \\ f_0 \diamond (g_0 \diamond \bar{g}_1) &\simeq f_1 \diamond c_{g_1(0)} \simeq f_1, \end{aligned}$$

where $c_{g_1(0)}$ is the constant path at $c_{g_1(0)}$. Using the fact that $g_0 \simeq g_1$, we find that $g_0 \diamond \bar{g}_1 \simeq g_1 \diamond \bar{g}_1 \simeq c_{g_1(0)}$ and thus that $f_0 \simeq f_1$. □

Proposition 2. *The change-of-basepoint homomorphism β_h depends only on the homotopy class of h.*

Proof. Recall that the homomorphism is defined as $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $[f] \mapsto [h \diamond f \diamond \bar{h}]$, where $[f]$ is a loop based at x_1. Suppose instead of β_h we consider the map β_g given by $[f] \mapsto [g \diamond f \diamond \bar{g}]$, where $g \simeq h$. Of course, since $g \simeq h$, we have that $g \diamond \bar{h} \simeq c$ and $h \diamond \bar{g} \simeq c$. Then $[h \diamond f \diamond \bar{h}] = [g \diamond \bar{h} \diamond h \diamond f \diamond \bar{h} \diamond h \diamond \bar{g}] = [g \diamond f \diamond \bar{g}]$. as we can always attach constant maps and reparametrize upto homotopy. □

Proposition 3. *Let X be a path-connected space. Then $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.*

Proof. Suppose $\pi_1(X)$ is abelian. Then, for any $[f], [g] \in \pi_1(X)$, $[f \diamond g] = [g \diamond f]$. Given two distinct paths h_0, h_1 from x_0 to x_1 we obtain a loop $\bar{h}_1 \diamond h_0$ at x_1 with the property that

$$\bar{h}_1 \diamond h_0 \diamond f \simeq f \diamond \bar{h}_1 \diamond h_0.$$

Concatenating by h_1 on the right and $\bar{h}_0$ on the right, we find that

$$h_0 \diamond f \diamond \bar{h}_0 \simeq h_1 \diamond f \diamond \bar{h}_1,$$

and hence that the homomorphism is independent of the path chosen between the two endpoints.

Conversely, suppose that the basepoint-change homomorphisms only depend on the endpoints of the path h. Then, given a loop f at x_1, we can consider the basepoint homomorphism between $\pi_1(X, x_1)$ and itself, with two different paths: h_0 constant at x_1 and h_1 a loop at x_1. Then we find that

$$h_0 \diamond f \diamond \bar{h}_0 \simeq f \simeq h_1 \diamond f \diamond \bar{h}_1.$$

This shows that $f \diamond h_1 \simeq h_1 \diamond f$, proving that $\pi_1(X, x_1)$ is abelian. □

Proposition 4. *Given a space X, the following three conditions are equivalent.*

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point;

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$;

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Furthermore, a space X is simply connected if and only if all maps $S^1 \to X$ are homotopic (without regard to basepoints).

Proof. The implication $(c) \implies (a)$ is trivial. All loops are homotopic, and in particular, all loops are homotopic to the constant loop. Since the image of any map $S^1 \to X$ is a loop that can be contracted to a point, such maps must be homotopic to constant maps (with image being said point).

The implication $(a) \implies (b)$ is simple as well. The condition (a) furnishes a homotopy $F : S^1 \times I \to X$ between a loop in X and the constant loop at $x \in X$. Visualizing this as a shrinking circle that sweeps out a disk, we define the map $G : D^2 \times I \to X$ that takes $(r, \theta) \mapsto F(\theta, 1 - r)$. This map is continuous by virtue of continuity of the homotopy F, and is well-defined at $r = 0$, as $(0, \theta) \mapsto F(\theta, 1) = x$, which is independent of θ.

Let us now prove $(b) \implies (c)$. Consider a loop $f_0 : S^1 \to X$ thought of as an element of $\pi_1(X, x)$ for $x = f_0(0) = f_0(1)$. The map f_0 extends to a map $G : D^2 \to X$ by hypothesis and hence we define a family of maps $F = f_t : S^1 \times I \to X$ given by $f_t(\theta) = G(1 - t, \theta)$. The map F is continuous by continuity of G, but is not a homotopy, as its endpoints vary with t. To fix this, let $H = h_t : I \times I \to X$ be a family of maps connecting $f_t(0)$ to x (inside the disk). Then, defining $\bar{h}_t$ in the usual way, we find that $h_t \diamond f_t \diamond \bar{h}_t$ yields a homotopy between the constant loop at x and f_0. This shows that any loop in $\pi_1(X, x)$ is homotopic to the constant loop at x, implying that $\pi_1(X, x)$ consists of one element, and is thus the trivial group.

Finally, note that if X is simply connected, it is path-connected and its fundamental group is trivial, which implies that any loop $S^1 \to X$ is homotopic to a constant loop. Since any two constant loops are homotopic (by path-connectedness), it follows by the transitivity of homotopy that any two loops are homotopic. Conversely, suppose all maps $S^1 \to X$ are homotopic. This implies that all loops are homotopic to a constant loop, and hence by the statements above, the fundamental group must be trivial (without regards to basepoint). □

Proposition 5. *Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Then f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$]*

Proof. The homotopy connecting the identity to f is, of course, given by $f_t(\theta, s) = (\theta + 2\pi st, s)$, which clearly does not leave the $s = 1$ boundary circle stationary. Now suppose instead that we have a homotopy $g_t : S^1 \times I \times I \to S^1 \times I$ connecting the identity to f that is stationary on both boundary circles. Consider a fixed $\theta_0 \in S^1$. For any t we know by hypothesis that $g_t(\theta_0, 0) = g_t(\theta_0, 1)$. The key step is now to note that for any fixed t, $g_t(\theta_0, s)$ gives a loop in S^1 by simply projecting to the first factor $\rho : S^1 \times I \to S^1$. At $t = 0$ the loop is, of course, the constant loop $\omega_0 \in \pi_1(S^1, \theta_0)$. At $t = 1$, however, projecting the path determined by θ_0 yields the loop ω_1. But by the hypothesis (and the fact that everything in sight is continuous), we find that $\rho \circ g_t(\theta_0, s)$ is a homotopy in S^1 between ω_0 and ω_1. Of course, this contradicts what we know about the fundamental group of the circle (in particular that $\omega_i \simeq \omega_j$ if and only if $i = j$), and hence no such homotopy g_t can exist. □