Modern Geometry: Lecture Notes

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1 Covering Spaces

Theorem 1. *Any connected covering space of a topological manifold M is a topological manifold, $\tilde{M}$, and any smooth structure on M induces one on $\tilde{M}$ such that the projection $\pi : \tilde{M} \to M$ is a local diffeomorphism.*

Proof. What it means to be a covering space is that for all $x \in M$ there exists an open neighborhood $V_x \ni x$ such that $\pi^{-1}(V_x) \cong V_x \times \Lambda$ (homemorphism) where Λ is a discrete topological space. Additionally, the homemorphism must commute with the obvious projections to V_x. Clearly if M is Hausdorff, then $\tilde{M}$ must be Hausdorff (draw a picture and check this!). Recall from undergraduate topology that the classification of connected covering spaces is done via subgroups Γ of the fundamental group. This implies that Λ is bijective to cosets of Γ, and hence Λ is countable. If U_i is a countable basis for M, those $U_i \subset V_x$ evenly covered for some V_x are still a countable basis; then $U_i \times \{\lambda\}$ are a countable basis for $\tilde{M}$. Finally, the fact that the covering space is locally Euclidean is obvious since it's locally homemorphic to M.

There exists an atlass $\phi_i : U_i \to \mathbb{R}^n$ for M such that each $U_i \subset V_x$ for some evenly covered V_x. Then $\tilde{\phi}_{i,\lambda} : U_i \times \{\lambda\} \to \mathbb{R}^n$ constitues an atlas for $\tilde{M}$; compatibility of charts and smoothness of π are obvious (due to the overlaps being exactly the same as usual, and locally π being an identity). $\square$

Example 1. Consider $M = S^1$, with fundamental group $\pi(M) \cong \mathbb{Z}$. If we let $\Gamma = n\mathbb{Z} \subset \mathbb{Z}$, then we get n-fold covers $S^1 \to S^1$ given by $z \mapsto z^n \in \mathbb{C}$. For $\Gamma = \{0\} \subset \mathbb{Z}$, we get the universal cover $\mathbb{R} \to S^1$ given by $t \mapsto e^{it}$.

2 Lie Groups

Definition 1. A **Lie group** is a group G that is also a smooth manifold such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps of manifolds. In other words, Lie groups are the group objects in the category of smooth manifolds.

Example 2. There are many many obvious examples:

- $(\mathbb{R}, +), (\mathbb{R}^n, +)$
- $(\mathbb{Z}, +)$
- $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \cdot)$
- $\mathbb{R}^\times = (\mathbb{C} \setminus \{0\}, \cdot)$

- the **general linear group**, $GL(n, \mathbb{R})$, the set of invertible $n \times n$ matrices, an open submanifold of $\mathbb{R}^{n^2}$

- $GL(n, \mathbb{C}), GL(n, \mathbb{H})$

Remark. Given G, H Lie groups, the product $G \times H$ is easily seen to be a Lie group as well.

Recall from last time: if M, N are smooth manifolds, $f : M \to N$ smooth, and if $M' \subset M, N' \subset N$ are regular submanifolds, and if $f(M') \subset N'$, then $f\Big|_{M'} : M' \to N'$ is also smooth. Note that we could make N' an immersed submanifold without harm.

Example 3. Consider $S^1 \subset \mathbb{C}^\times$. Multiplication, $\mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$ is smooth. One can check that $S^1 \times S^1$ is a regular submanifold, and hence multiplication $S^1 \times S^1 \to S^1$ is smooth as well. A similar observation holds for inversion. This tells us that the circle is a Lie group. Hence $T^n = \prod_{i=1}^n S^1$ is also a Lie group. These are all abelian Lie groups.

Note that we can define the **special linear group** $SL(n, \mathbb{R})$ as $\det^{-1}(1)$. We claim that $SL(n, \mathbb{R})$ is in fact a regular submanifold. This follows from last time, if 1 is a regular value of the determinant map $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$. In other words, for all $A \in SL(n, \mathbb{R})$ we need $D_A \det : T_A GL(n, \mathbb{R}) \to T_{\det A} \mathbb{R}^\times$ is surjective. Let us show this. If $A = I$, we can compute the directional derivative of det along $B \in M_{n \times n}(\mathbb{R}) = \mathbb{R}^{n^2}$. This is the curve $1 + tB$. We must apply the determinant:

$$\det(I + tB) = 1 + t \operatorname{tr} B + O(t^2).$$

Taking a derivative at $t = 0$ gives us $\operatorname{tr} B$. Hence the derivative at the identity of the determinant is a trace, which is plainly surjective. Consider now a general $A \in SL(n, \mathbb{R})$. Multiplication by A is a diffeomorphism $GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$. The derivatives satisfy another commutative diagram simply by the chain rule. Hence $D_I \det$ is surjective and hence $D_A \det$ is surjective. Consequently, $SL(n, \mathbb{R})$ is a regular submanifold of $GL(n, \mathbb{R})$ and hence a Lie group. We could have done exactly the same thing for $SL(n, \mathbb{C})$ or $SL(n, \mathbb{H})$.

One can work similarly with the **orthogonal group**, $O(n)$. In fact, not only is $O(n)$ a Lie group, it is compact as well (closed and bounded). Likewise for the **unitary group**, $U(n)$ and the **symplectic group**, $Sp(n)$. Note carefully that this is the compact $Sp(n)$, with a close relative, the non-compact group appearing in symplectic geometry.

Observe that even the case $n = 1$ is not trivial. For example, $O(1) = \{\pm 1\} = S^0$, $U(1) = S^1 \subset \mathbb{C}^\times$, $Sp(1) = S^3 \subset \mathbb{H}^\times$. These are in fact the only spheres with Lie group structures.

Definition 2. If G, H are Lie groups, then a **Lie group homomorphism**, $f : G \to H$, is a smooth map of manifolds that is also a group homomorphism.

Example 4. There are "zillions" of examples:

- $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$

- the inclusions we were mentioning earlier

- the multiplcation map for any abelian Lie group

- $\det : O(n) \to \{\pm 1\}$

- consider $i : \mathbb{R} \to T^2$ where $i(t) = (e^{it}, e^{i\alpha t}$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (the dense torus). This is a Lie group homomorphism and an injective immersion, but not an embedding.

Theorem 2. *Given a Lie group homomorphism $f : G \to H$ with closed image, then the image is a regular submanifold, and hence a Lie group.*

Proof. We will prove this later. $\square$

Proposition 1. *If G is a Lie group and G_0 is the (path-)component containing the identity element $e \in G$, then G_0 is an embedded Lie subgroup of G, and all (path-)components are diffeomorphic to G_0.*

Proof. G_0 is clearly a regular submanifold, so for the first statement it suffices to show that G_0 is a subgroup. If γ is a path from e to g, and δ is a path from e to h, then $(\gamma\delta)(t) = \gamma(t)\delta(t)$ is a path from e to gh. Hence G_0 is closed under multiplication (and similarly for inversion). The second statement is easy since multiplication by any $g \in G$ is a diffeomorphism from G_0 to the componenent containing g. $\square$

Definition 3. The **special orthogonal group** $SO(n) = \det^{-1}(1)$ where $\det : O(n) \to \{\pm 1\}$.

Remark. Note that $SO(n) = O(n)_0$, i.e. the identity component. Similarly, $SO(2) \cong S^1$. More generally, any closed and open subgroup of a Lie group is a Lie group. Indeed, if $f : G \to H$ is a group homomorphism and a diffeomorphism, then the inverse is also a group homomorphism.

Theorem 3. *If G is a connected Lie group, then its fundamental group is abelian.*

The basic idea of this proof is quite simple. Take two loops γ, δ based at the identity. Then one can show $[\gamma] \cdot [\delta] = [\gamma\delta] = [\delta] \cdot [\gamma]$.

Proof. Let $\gamma, \delta : [0, 1] \to G$ be two loops in G, based at $e \in G$. We can multiply loops in two different ways: pointwise, which we denote $\gamma\delta(t) = \gamma(t)\delta(t)$, or the usual concatenation, which we denote $\gamma \cdot \gamma(t)$. Note that $\gamma\delta$ is also a loop based at e. Let us define some homotopies (fig 1):

$$\Gamma(s,t) = \begin{cases} \gamma((2-s)t) & \text{if } (2-s)t \leq 1 \\ e & \text{if not} \end{cases}$$

and

$$\Delta(s,t) = \begin{cases} \delta((2-s)t - (1-s)) & \text{if } (2-s)t - (1-s) \geq 0 \\ e & \text{if not.} \end{cases}$$

Concatenating these homotopies shows that $\delta \cdot \gamma \overset{\Gamma\Delta}{\sim} \delta\gamma \overset{\Delta\Gamma}{\sim} \gamma \cdot \delta$, and we are done. $\square$

Remark. Note that in this argument we showed that the product in the fundamental group is the same as the pointwise product,

$$[\gamma] \cdot [\delta] = [\gamma\delta].$$

Theorem 4. *Any connected covering space $\tilde{G}$ of a connected Lie group G is a Lie group so that $\pi : \tilde{G} \to G$ is a Lie group homomorphism.*

Proof. For some $\Gamma \leq \pi_1(G)$, we can write

$$\tilde{G} = \{\text{paths } \gamma : [0,1] \to G | \gamma(0) = e\} / \sim$$

where $\gamma \sim \delta$ if and only if $[\bar{\gamma} \cdot \delta] \in \Gamma$, where $\bar{\gamma}(t) = \gamma(1-t)$ is the **retrograde** of γ (fig 2). By the remark above, we see that pointwise multiplication of paths determines a group operation $\tilde{G} \times \tilde{G} \to \tilde{G}$, as one can check (indeed note that if $\mathcal{G} = \{\text{paths } \gamma : [0,1] \to G | \gamma(0) = e\}$ is a group under pointwise multiplcation; the set of $\gamma \in \mathcal{G}$ such that $[\gamma] \in \Gamma$ is normal since any path δ based at the identity satisfies $\delta \sim e$ via paths based at e so $\delta \cdot \gamma \cdot \delta \sim e\gamma e = \gamma$). Hence pointwise multiplication of paths endows $\tilde{G}$ with a group structure.

Smoothness of the group operations is established by looking over evenly covered sets in G. If $\tilde{g}_1, \tilde{g}_2 = \tilde{g} \in \tilde{G}$, and $\pi(\tilde{g}_1) = g_1 \in G$ etc., let B be a contractible neighborhood of $g \in G$. Now let B_1, B_2 be contractible neighborhoods of g_1, g_2 such that $B_1 B_2 \subset B$ (existence by homework 2). Hence there exist contractible $\tilde{B}, \tilde{B}_1, \tilde{B}_2$ that contain $\tilde{g}, \tilde{g}_1, \tilde{g}_2$ respectively such that $\pi|_{\tilde{B}} \tilde{B} \to B$ is a diffeomorphism and similarly for $\tilde{B}_1, \tilde{B}_2$. Then $\tilde{B}_1 \tilde{B}_2 \subset \tilde{B}$ and the following diagram commutes:

$$\begin{tikzcd} \tilde{B}_1 \times \tilde{B}_2 \arrow[r, "\text{mult in G}"] \arrow[d, "\text{diffeo.}"] & \tilde{B} \arrow[d, "\pi"] \\ B_1 \times B_2 \arrow[r, "\text{mult in G}"] & B \end{tikzcd}$$

Since multiplcation in G is smooth, multiplcation in $\tilde{G}$ is smooth. The case for inversion holds similarly. $\square$

3 Group actions on manifolds

Definition 4. A (smooth) action of a Lie group G on a (smooth) manifold M is simply an action $G \times M \to M$ given by $g, m \mapsto g \cdot m$ (satisfying the usual properties) which is smooth as a map of manifolds.

Note that this definition implies that for all $g \in G$, the map $m \mapsto g \cdot m$ defines a diffeomorphism $M \to M$. Indeed, it is a restriction of a smooth map and hence smooth, with inverse $m \mapsto g^{-1} \cdot m$.

Example 5.
- $GL(n, \mathbb{R}) \curvearrowright \mathbb{R}^n$ as $A \cdot v = AV$
- $O(n) \curvearrowright \mathbb{R}^n$ using the restriction property for regular submanifolds
- $O(n) \curvearrowright S^{n-1}$ for the same reason
- $G \curvearrowright G$ given by left multiplcation: $g \cdot h = gh$
- $G \curvearrowright G$ given by the adjoint action: $g \cdot h = ghg^{-1}$.

Theorem 5 (Rank theorem)**.** *If $f : M \to N$ has $\mathrm{rank} D_x f = k$ for all x in some neighborhood of $p \in M$, then there exist charts $\phi : U \to V$ on M, $\psi : U' \to V'$ on N, such that $\phi(p) = 0, \psi(f(p)) = 0$ and $\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_k, 0 \ldots, 0)$.*

Proof. Without loss of generality, assume that $M = \mathbb{R}^m, N = \mathbb{R}^n, p = 0 \in \mathbb{R}^m, f(p) = 0 \in \mathbb{R}^n$. Furthermore, we can permute the coordinates such that the upper left hand $k \times k$ minor of $D_0 f$ is nonsingular, i.e. for $\vec{u} \in \mathbb{R}^k, \vec{v} \in \mathbb{R}^{m-k}$, $f(\vec{u}, \vec{v}) = (g(\vec{u}, \vec{v}), h(\vec{u}, \vec{v}))$. $\partial g / \partial u$ nonsingular. Define $\phi : \mathbb{R}^m \to \mathbb{R}^m$ by $\phi(\vec{u}, \vec{v}) = (g(\vec{u}, \vec{v}), \vec{v})$ and now the derivative $D_0 \phi$ has $\partial g / \partial u$ on the top left, $\partial g / \partial v$ on the top right, zero on the bottom left, and the identity in the bottom right. By the inverse function theorem, there exists a local inverse $\phi^{-1}(\vec{u}, \vec{v}) = (q(\vec{u}, \vec{v}), \vec{v})$ defined on a neighborhood of $0 \in \mathbb{R}^m$. Now, by the chain rule, since we're assuming $D_x f$ has rank k near 0, $D_y(f \circ \phi^{-1})$ must have rank k near 0 but $f \circ \phi^{-1}(\vec{u}, \vec{v}) = (\vec{u}, h(q(\vec{u}, \vec{v}), \vec{v}))$. Hence $D_y(f \circ \phi^{-1})$ is the identity on the upper left block, 0 on the upper right block, something on the bottom left, and it must have zeroes on the bottom right to keep the rank k. But this means that the derivative of the second component with respect to $\vec{v}$ is zero, and hence $h(q(\vec{u},\vec{v}),\vec{v})$ is independent of $\vec{v}$. Let us call this function $r(\vec{u})$. Now $f\circ\phi^{-1}(\vec{u},\vec{v}) = (\vec{u}, r(\vec{u}))$. Now if we let $\psi(\vec{u},\vec{v}) = (\vec{u},\vec{v}-r(\vec{u}))$, then $D_0\psi$ will be invertible, and hence a local diffeomorphism by the inverse function theorem. Now precomposing, $\psi\circ f\circ\phi^{-1}(\vec{u},\vec{v}) = (\vec{u},\vec{0})$. □

Corollary 6. *If $f : M \to N$ is injective and constant rank, then f is an immersion, i.e. $\mathrm{rank} D_x f = \dim M$.*

Proof. If not, say with rank $k < m$, it locally looks like $(x_1,\dots,x_m) \mapsto (x_1,\dots,x_k,0,\dots,0)$, which is not even locally injective. □

Definition 5. For X, Y topological spaces, a continuous $f : X \to Y$ is said to be **proper** if, for all compact $C \subset Y$, $f^{-1}(C)$ is compact.

Example 6. Here are a few examples:

- if C is closed in Y, then the inclusion $i : C \to Y$ is proper.
- if X is compact and Y is Hausdorff, then any continuous f is proper
- compositions of proper maps are proper
- any homeomorphism is proper
- projection onto the first factor $X \times C \to X$ is proper iff C is compact
- the restriction of any proper map to a closed subset is proper

We leave it as an exercise to show that f is proper iff it extends to a continuous map of the one-point compactifications. Additionally, if X, Y are topological manifolds, $f : X \to Y$ proper, then f is closed. Hence, if f is both proper and injective, it is a homemorphism onto its image. A corollary is that a proper injective immersion of smooth manifolds is an embedding.

Definition 6. A group action $G \times M \to M$ is **proper** if $\mu : G \times M \to M \times M$ given by $(g,x) \mapsto (g\cdot x, x)$ is proper.

Example 7.

- Any Lie group acts properly on itself by left multiplcation. In this case the map takes $(g,h) \mapsto (gh^{-1}, h)$, which is a diffeomorphism.
- A closed subgroup $H \subset G$ acts properly on G: $H \times G$ is closed in $G \times G$ and apply the examples above
- $O(n)$ acts properly on $\mathbb{R}^n$ since if $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is compact, then $\mu^{-1}(C)$ is closed and is a subset of $O(n) \times \pi_2(C)$, which is a product of two compact sets. Here we really only used that G is compact and that M is Hausdorff. Hence any compact Lie group acts properly on any manifold.
- But, $GL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$ is not proper because we have a closed subgroup $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = G$ for $t \in \mathbb{R}$ and the inverse image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{R}^2 = G \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} \subset G \times \mathbb{R}^2$. This contradicts one of the last examples from above.

Theorem 7. *If a Lie group G acts smoothly, freely, and properly, on a smooth manifold M, then the quotient M/G is a smooth manifold of dimension $\dim M - \dim G$ so that the natural projection $M \to M/G$ is smooth.*