Riemann Surfaces: Lecture Notes

Nilay Kumar

Last updated: February 25, 2014

Class 10

Recall that we are studying function theory on the torus $\mathbb{C}/\Lambda$, where $\Lambda = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$. We had produced a candidate

$$\sigma(z) = z \prod_{\omega \in \Lambda^\times} \left(1 + \frac{z}{w}\right) e^{-\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}.$$

Let us check that this product converges by examining its logarithm:

$$\begin{aligned} \log\{\cdots\} &= \log(1 + \frac{z}{\omega}) - \frac{z}{\omega} + \frac{z^2}{2\omega^2} \\ &= \left(\frac{z}{\omega} - \frac{1}{2}\frac{z^2}{\omega^2} + \dots\right) - \frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}, \end{aligned}$$

which clearly converges. Hence $\sigma(z)$ is holomorphic for $z \in \mathbb{C}$. Recall that $\sigma'(z)/\sigma(z) = \zeta(z)$ and so $\partial_z \log \sigma(z + \omega_a) = \zeta(z + \omega_a)$. Thus we have (from before) that

$$\eta_a = \zeta(z + \omega_a) - \zeta(z) = \partial_z \log \sigma(z + \omega_a) - \partial_z \log \sigma(z),$$

which gives us periodicity information. Integrating and exponentiating, we see that

$$\sigma(z + \omega_a) = \sigma(z) e^{\eta_a z + c_a}$$

where c_a is the constant of integration, and taking $z = -\omega_a/2$, we find that

$$\sigma(\omega_a/2) = \sigma(-\omega_a/2) e^{-\eta_a \frac{\omega_a}{2} + c_a}.$$

It is easy to check, however, that σ is odd, and hence we find that

$$\sigma(z + \omega_a) = -\sigma(z) e^{\eta_a (z + \frac{\omega_a}{2})}.$$

So we have found that $\sigma(z)$ is holomorphic on $\mathbb{C}$ and that $\sigma(z) = 0$ if and only if $z = 0 \mod \Lambda$. Now that we have constructed such a σ, let us give another proof of Abel's theorem. First recall our previous statement of Abel's theorem.

Theorem 1 (Abel's theorem)**.** *Let $P_1, \dots, P_M, Q_1, \dots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M A(P_i) = \sum_{i=1}^N A(Q_i)$.*

Recall that the Abel map takes $\mathbb{C}/\Lambda \ni p \mapsto A(p) = \int_{p_0}^{p} \omega$ where the value of the integral is taken modulo the lattice generated by $\oint_A \omega, \oint_B \omega$. Take $p_0 = 0$ and $\omega = dz$, which is a well-defined form, and if we take A to align with ω_2 and B to align with ω_1, we see that $\oint_A \omega = \oint_A dz = \omega_1$ and similarly $\oint_B \omega = \omega_2$. Hence the map simply takes p to $\int_0^p dz \mod \Lambda = p$ where p is viewed as a complex number.

Let us now restate Abel's theorem.

Theorem 2 (Abel's theorem, v.2)**.** *Let $P_1, \ldots, P_M, Q_1, \ldots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^{M} P_i = \sum_{i=1}^{N} Q_i$* mod Λ.

Proof. Consider the function

$$f(z) = \frac{\prod_{i=1}^{M} \sigma(z - P_i)}{\prod_{i=1}^{N} \sigma(z - Q_i)}.$$

We should be a little careful to note that σ is a function not on the torus $\mathbb{C}/\Lambda$, but a function on $\mathbb{C}$ (it transforms under a lattice translation!). Hence we must be cognizant of the fact that P_i, Q_i here are some chosen representatives in $\mathbb{C}$ of the equivalence classes of the points P_i, Q_i. It should be clear that $f(z)$ is meromorphic with zeroes at every representative of each P_is and poles at every representative of each Q_i. The natural question, now, is whether this function extends to a function on the torus. To check this, let us see whether it is doubly periodic using what we know about σ:

$$\begin{aligned} f(z + \omega_a) &= f(z) \frac{\prod_{i=1}^{M} e^{\eta_a (z - P_i)}}{\prod_{i=1}^{N} e^{\eta_a (z - Q_i)}} \\ &= f(z) e^{-\eta_a \left(\sum_{i=1}^{M} P_i - \sum_{i=1}^{N} Q_i \right)}. \end{aligned}$$

Hence we wish to choose P_i, Q_i representatives such that the exponential becomes unity. By hypothesis, this can be done (by shifting one, if necessary). $\square$

Let us now return to Weierstrass theory. Given $\omega = dz$, we defined $\omega_0 = \mathcal{P}(z) dz$ which has a double pole at 0 and $\partial_z \log \sigma(z) = \zeta(z)$ and $\zeta'(z) = -\mathcal{P}(z)$. Now we can construct a form ω_{PQ} with residues $1, -1$ at P, Q respectively, by assigning $\omega_{PQ}(z) = (\zeta(z - P) - \zeta(z - Q))\,\omega = \partial_z \log \frac{\sigma(z-P)}{\sigma(z-Q)} dz$. What Weierstrass theory tells us that we can write everything in terms of σ, our analog of z.

Jacobi theory: θ-functions

Consider again the torus $\mathbb{C}/\Lambda$, where we now normalize the lattice as $\Lambda = \{m + n\tau; m, n \in \mathbb{Z}\}$ with $\operatorname{Im} \tau > 0$ (by linear independence, it cannot be real). This simply corresponds to picking $\omega_1 = 1, \omega_2/\omega_1 = \tau$. Next define the **theta-function**

$$\theta(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

in which the structure of the lattice is explicitly clear (unlike in the Weierstrass theory). Let us examine its main properties.

First, note that $\theta(z|\tau)$ is holomorphic in $z \in \mathbb{C}$ because the series converges for all z; this is due to the term

$$|e^{\pi i n^2 (\tau_1 + i\tau_2)}| = |e^{\pi i n^2 \tau_1} e^{-\pi^2 n^2 \tau_2}| = e^{-\pi n^2 \tau_2}$$

for $\tau = \tau_1 + i\tau_2$, whose decay dominates due to the n^2. Next, notice that

$$\begin{aligned}\theta(z+1|\tau) &= \theta(z|\tau) \\ \theta(z+\tau|\tau) &= e^{-\pi i\tau - 2\pi i z}\theta(z|\tau),\end{aligned}$$

where the second is obtained by completing the square. Though θ is not invariant, its zeroes are.

Furthermore, we claim that $\theta(z|\tau)$ vanishes at exactly one point modulo lattice translates. It suffices to compute the integral $\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz$, as it yields $2\pi i$ times the difference in the number of zeroes and poles in a given region. We shall integrate over the curve C where C traverses the circumference of one lattice segment (i.e. the whole torus):

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = \oint_B \left(-\frac{\theta'(z|\tau)}{\theta(z|\tau)} + \frac{\theta'(z+1|\tau)}{\theta(z+1|\tau}\right) + \oint_A \left(\frac{\theta'(z|\tau)}{\theta(z|\tau)} - \frac{\theta'(z+\tau|\tau)}{\theta(z+\tau|\tau)}\right).$$

But these are just the shifts in the logarithmic derivative, and since $\partial_z \log\theta(z+\tau|\tau) = -2\pi i + \partial_z \log\theta(z|\tau)$ using the transformation rules above, we see that our integral simplifies to

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = 2\pi i \oint_A dz = 2\pi i.$$

Of course, since θ is holomorphic, it has no poles, and hence we see that we have one zero. The zero, in fact, occurs in the center: $\theta\left((1+\tau)/2|\tau\right) = 0$. To see this, consider the following function:

$$\begin{aligned}\theta\left(z + \frac{1+\tau}{2}|\tau\right) &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i n\left(z + \frac{1+\tau}{2}\right)\right) \\ &= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\sum_{n\in\mathbb{Z}}\exp\left(\pi i(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})\right) \\ &= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\theta_1(z|\tau)\end{aligned}$$

where we have completed the square and defined the function θ_1. We claim that θ_1 is an odd function, which would imply that θ_1 vanishes at zero, which would prove the claim about the location of the zero. Hence let us verify that θ_1 is odd; switching $z \mapsto -z$ yields in the exponent

$$\log\theta_1(z|\tau) = \pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right).$$

If we switch the indices $n \mapsto m$ such that $n + \frac{1}{2} = -\left(m+\frac{1}{2}\right)$, we find that the exponent is now

$$\log\theta_1(z|\tau) = \pi i\left(m+\frac{1}{2}\right)^2\tau + 2\pi i\left(m+\frac{1}{2}\right)\left(\left(z+\frac{1}{2}\right) - 2\pi i\left(m+\frac{1}{2}\right)\right),$$

and hence θ_1 is odd. Now we see that the function we want is in fact $\theta_1(z|\tau)$ as it is odd, holomorphic, and has one zero.

We leave it as an exercise to show that

$$\sigma(z) = \omega_1\exp\left(\eta_1\frac{z^2}{\omega_1}\right)\frac{\theta_1\left(\frac{z}{\omega_1}|\tau\right)}{\theta_1'(0|\tau)}$$

Class 11

We claim that the theta-function theory is more powerful than what we have been using so far - to see this, let us prove Abel's theorem. Recall that the theorem states that there exists a meromorphic f with zeroes at P_i and poles and Q_j if and only if $N = M$ and $\sum_i A(P_i) = \sum_j A(Q_j)$. The idea is to express

$$f(z) = \frac{\prod_{i=1}^N \theta_1(x - P_i)}{\prod_{i=1}^N \theta_1(z - Q_i)}$$

and check double-periodicity. It is an exercise to check that

$$\begin{aligned}
&\theta_1(z+1|\tau) = -\theta_1(z|\tau) \\
&\theta_1(z+\tau|\tau) = \exp\left(-\pi i\tau - 2\pi i(z+1/2)\right)\theta_1(z|\tau),
\end{aligned}$$

from which periodicity follows easily. Of course, we must be careful to note that the P_i, Q_i used here are in fact chosen representatives.

Next let us define a meromorphic form

$$\omega_{PQ} = \partial_z \log\frac{\theta_1(z-P)}{\theta_1(z-Q)}dz,$$

which, it is easy to check, has poles at P, Q with opposite residues. Additionally, one can check that this expression is well-defined on the lattice, i.e. invariant under a shift. We leave it as a simple exercise to show that

$$\omega_P(z) = \partial_z^2 \log\frac{\theta_1(z-P|\tau)}{\theta_1'(0|\tau)}dz$$

is a meromorphic form with a double pole at P and is well-defined on the lattice.

But in fact, we can go even farther with this theta-function. Indeed, one attractive feature is that there exists a product expansion for $\theta(z|\tau)$.

Theorem 3. *We can expand*

$$\theta(z|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz})$$

where $q \equiv e^{\pi i\tau}$.

Proof. Define

$$T(z|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz}).$$

We claim that $T(z|\tau)$ is equal to zero exactly when z is $(1+\tau)/2 \mod \Lambda$ and that the zeros are simple. This can be checked by some simple algebra. It's also easy to show that $T(z|\tau)$ is holomorphic in $\mathbb{C}$ and that $\tau(z+1|\tau) = T(z|\tau)$. Moreover

$$\begin{aligned}
T(z+\tau|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})\prod\left(1+q^{2n+1}e^{2\pi iz}\right)\left(1+q^{2n-3}e^{-2\pi iz}q^{-2}\right) \\
&= \prod_{n=1}^{\infty}(1-q^{2n})\frac{\prod_{n=1}^{\infty}\left(1+q^{2n-1}e^{2\pi iz}\right)}{1+qe^{2\pi iz}}\prod_{n=1}^{\infty}\left(1+q^{2n-1}e^{-2\pi iz}\right) \quad = \frac{1-q^{-1}e^{-2\pi iz}}{1-qe^{2\pi iz}}T(z|\tau) = q^{-1}e^{-2\pi iz}.
\end{aligned}$$

Recall that θ followd a similar condition. This shows that $\theta(z|\tau)/T(z|\tau) = c$, where c is a constant independent of z that can depend on τ. Next we claim that $c(\tau) = 1$. For this we show that there exists a c such that $c(\tau) = c(4\tau) = c(4^k\tau)$ and $c(\tau) = \lim_{k\to\infty} c(4^k\tau) = 1$, which shows the proof. Hence let us prove that $c(\tau) = c(4\tau)$ using $\theta(z|\tau) = C(\tau)T(z|\tau)$.

Take $z = 1/2$. Then $e^{2\pi i z = e^{\pi i}} \geq 1-$ and $\theta(1/2|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}(-1)^n$ but $T(1/2|\tau) = \prod_{n=1}^{\infty}(1-q^{2n)(1-q^{2n-1}}(1-q^{2n-1})$. and hence $c(\tau) = \sum e^{\pi i n\tau}(-1)^n / \prod(1-q^n)(1-q^{2n-1})$. Next take $z = 1/4$ $\square$

1 Semester 2

This semester we will start by describing the L^2 estimates of Hormander. Later we will delve into its applications, including the Kodaira embedding theorem, the lower bounds for the Bergman kernel, and the ideas of canonical metrics and stability.

Remark. Suppose we have a holomorphic line bundle $L \to X$. We may ask the following questions.

(a) $H^0(X, L) = 0$?

(b) Take some $s \in H^0(X, L)$ with $||s||_{L^2} = 1$. How big can $s(z_0)$ be at a given point z_0?

We will be building machinery to address these questions, which depend sensitively on the geometry.

1.1 Review

Let us recall some techniques from last semester. Let $X = \cup_\mu X_\mu$ be a complex n-manifold with X_μ coordinate charts. Hence each X_μ is homeomorphic (and thus biholomorphic) to $\mathbb{C}^n$ and $\Phi_\mu \circ \Phi_\nu^{-1}$ is holomorphic with invertible differentials. Let $E \to X$ be a holomorphic vector bundle. Recall what? that a rank-r vector bundle is completely characterized by its transition functions $t^\alpha_{\mu\nu\beta}(z)$ (matrix valued in general) defined on $X_\mu \cap X_\nu$ with $1 \leq \alpha, \beta \leq r$. Note that the transition functions satisfy the cocycle condition. We denote by $\Gamma(X, E)$ the space of sections of E (recall that this means that $\phi^\alpha_\mu(z_\mu) = t^\alpha_{\mu\nu\beta}(z)\phi^\beta_\nu(z_\nu)$. For E to be holomorphic, it must have holomorphic transition functions.

Given a section $\phi \in \Gamma(X, E)$, we obtain a section $\bar\partial\phi \in \Gamma(X, E \otimes \Lambda^{0,1})$ via **covariant differentiation**. More explicitly, on X_μ, we write naively

$$\bar\partial\phi^\alpha \equiv \left(\frac{\partial}{\partial \bar z^j_\mu}\phi^\alpha_\mu\right)(z_\mu).$$

Fortunately, this is indeed a section. To see this, we note that $\phi^\alpha_\mu(z_\mu) = t^\alpha_{\mu\nu\beta}(z)\phi^\beta_\nu(z_\nu)$ and hence

$$\begin{aligned}\frac{\partial}{\partial \bar z^j_\mu}\phi^\alpha_\mu(z_\mu) &= t^\alpha_{\mu\nu\beta}(z)\frac{\partial}{\partial \bar z^j_\mu}\left(\phi^\beta_\nu(z_\nu)\right) \\ &= t^\alpha_{\mu\nu\beta}(z)\overline{\frac{\partial z^k_\nu}{\partial z^j_\mu}}\frac{\partial \phi^\beta_\nu}{\partial \bar z^k_\nu}(z).\end{aligned}$$

Thus we define $\Lambda^{0,1}$ to be the (antiholomorphic) vector bundle with transition functions $\overline{\partial z^k_\nu/\partial z^j_\mu}$.

Next, recall the definition of a **Hermitian metric** $H = H_{\bar\alpha\beta}(z)$ on a vector bundle E: we have $(H_\mu)_{\bar\alpha\beta}(z_\mu)$ on X_μ such that it is a positive-definite matrix for each z_μ satisfying

$$|\phi|^2_H \equiv (H_\mu)_{\bar\alpha\beta}\overline{\phi^\alpha_\mu}\phi^\beta_\mu = (H_\nu)_{\bar\gamma\delta}\overline{\phi^\gamma_\nu}\phi^\delta_\nu$$

on $X_\mu \cap X_\nu$. This quantity can be thought of as the length of the vector ϕ with respect to the metric H, which is by construction invariant of coordinate chart. Using metrics, we can introduce covariant derivatives of sections on a holomorphic vector bundle E with respect to a metric $H_{\bar{\alpha}\beta}$. Take $\phi \in \Gamma(X, E)$ and define on X_μ

$$(\nabla_j \phi)^\alpha \equiv H^{\alpha\bar{\gamma}} \partial_j (H_{\bar{\gamma}\beta} \phi^\beta_\mu),$$

where $H^{\alpha\bar{\gamma}} H_{\bar{\gamma}\beta} = \delta^\alpha_\beta$. It is easy to see that $\nabla\phi \in \Gamma(X, E \otimes \Lambda^{1,0})$, whose transition functions are $\partial z^k_\nu / \partial z^j_\mu$.

In summary, we write

$$\begin{aligned}
\nabla_{\bar{j}} \phi^\alpha &= \partial_{\bar{j}} \phi^\alpha \\
\nabla_j \phi^\alpha &= H^{\alpha\bar{\gamma}} \partial_j (H_{\bar{\gamma}\beta} \phi^\beta_\mu) \\
&= \partial_j \phi^\alpha + (H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta}) \phi^\beta \\
&= \partial_j \phi^\alpha + A^\alpha_{j\beta} \phi^\beta,
\end{aligned}$$

where, in matrix notation, we have the **connection** $A_j = H^{-1} \partial_j H$. It is a priori not obvious that these two derivatives must commute. Indeed, we define the **curvature** F of the metric $H_{\bar{\alpha}\beta}$ on $E \to X$ to be

$$[\nabla_{\bar{j}}, \nabla_k] \phi^\alpha = -F^\alpha_{\bar{j}k\beta} \phi^\beta.$$

We leave it as an exercise that $[\nabla_{\bar{j}}, \nabla_{\bar{k}}] = 0$ and $[\nabla_j, \nabla_k] = 0$. More explicitly, we can write

$$\begin{aligned}
[\nabla_{\bar{j}}, \nabla_k] \phi^\alpha &= \nabla_{\bar{j}} (\nabla_k \phi^\alpha) - \nabla_k \left(\nabla_{\bar{j}} \phi^\alpha\right) \\
&= (\partial_{\bar{j}} A^\alpha_{k\beta}) \phi^\beta,
\end{aligned}$$

and hence $F^\alpha_{\bar{j}k\beta} = -\partial_{\bar{j}} A^\alpha_{k\beta}$. In matrix notation, we can simply write

$$F_{\bar{j}k} = -\partial_{\bar{j}} A_k = -\partial_{\bar{j}} \left(H^{-1} \partial_k H\right).$$

We define the corresponding **curvature form** to be

$$F = \frac{i}{2\pi} F^\alpha_{\bar{j}k\beta}(z) dz^k \wedge d\bar{z}^j \in \Gamma(X, E \otimes E^* \otimes \Lambda^{1,1}) = \Gamma(X, \mathrm{End}(E) \otimes \Lambda^{1,1}),$$

i.e. a $\mathrm{End}(E)$-valued $(1,1)$-form.

1.2 Bochner-Kodaira Formulas

Let X be a complex manifold, compact without boundary. Take $E \to X$ to be a holomorphic vector bundle of rank r on X. We consider the $\bar{\partial}$ **complex**:

$$\cdots \xrightarrow{\bar{\partial}} \Gamma(X, E \otimes \Lambda^{p,q}) \xrightarrow{\bar{\partial}} \Gamma(X, E \otimes \Lambda^{p,q+1}) \xrightarrow{\bar{\partial}} \cdots.$$

Let us be more precise. Consider some $\phi \in \Gamma(X, E \otimes \Lambda^{p,q})$. We can write explicitly:

$$\phi^\alpha = \frac{1}{p!q!} \sum \phi^\alpha_{\bar{j}_1, \ldots \bar{j}_q, i_1, \ldots, i_p}(z) dz^{i_p} \wedge \cdots \wedge dz^{i_1} \wedge d\bar{z}^{j_q} \wedge \cdots \wedge d\bar{z}^{j_i}.$$

Now what exactly do we mean by $\bar{\partial}$? We define

$$\begin{aligned}
\bar{\partial}\phi &\equiv \frac{1}{p!q!}\sum\left(\bar{\partial}\phi_{\bar{j}_1,\ldots\bar{j}_q,i_1,\ldots,i_p}\right)\wedge dz^{i_p}\wedge\cdots\wedge dz^{i_1}\wedge d\bar{z}^{j_q}\wedge\cdots\wedge d\bar{z}^{j_i}\\
&= \frac{1}{p!q!}\sum\left(\partial_{\bar{k}}\phi_{\bar{j}_1,\ldots\bar{j}_q,i_1,\ldots,i_p}d\bar{z}^k\right)\wedge dz^{i_p}\wedge\cdots\wedge dz^{i_1}\wedge d\bar{z}^{j_q}\wedge\cdots\wedge d\bar{z}^{j_i}.
\end{aligned}$$

We leave it as an exercise for the reader to check that this is well-defined (follows as per the usual de Rham exterior derivative).

Example 1. What is $\bar{\partial}$ on $\Gamma(X, E\otimes\Lambda^{0,0}) = \Gamma(X,E)$? By definition, $\bar{\partial}\phi = \partial_{\bar{k}}\phi^\alpha d\bar{z}^k$.

Example 2. What is $\bar{\partial}$ on $\Gamma(X, E\otimes\Lambda^{0,1})$? Given a section, we can write $\phi = \sum\phi^\alpha_{\bar{j}}d\bar{z}^j$. In this case,

$$\begin{aligned}
\bar{\partial}\phi^\alpha &= \sum\left(\bar{\partial}\phi^\alpha_{\bar{j}}\right)\wedge d\bar{z}^j\\
&= \sum\left(\partial_{\bar{k}}\phi^\alpha_{\bar{j}}d\bar{z}^k\right)\wedge d\bar{z}^j\\
&= \frac{1}{2}\sum\left(\partial_{\bar{k}}\phi^\alpha_{\bar{j}} - \partial_{\bar{j}}\phi^\alpha_{\bar{k}}\right)d\bar{z}^k\wedge d\bar{z}^j.
\end{aligned}$$

Hence one finds the coefficient $(\bar{\partial}\phi)_{\bar{j}\bar{k}} = \left(\partial_{\bar{k}}\partial_{\bar{j}} - \partial_{\bar{j}}\partial_{\bar{k}}\right)$ with no factor of 1/2 out front, because we now have a two-form.

Let us now introduce a metric $H_{\bar{\alpha}\beta}$ on E and a metric $g_{\bar{k}j}$ on $T^{1,0}(X)$. This allows us to compute scalar norms of sections as $|\phi|^2_H = H_{\bar{\alpha}\beta}\overline{\phi^\alpha}\phi^\beta$ for $\phi\in\Gamma(X,E)$. We will work with the metric and hope that in the end, our results will be independent of the metric (where lengths are not involved). There is an induced L^2 metric on $\Gamma(X,E\otimes\Lambda^{p,q})$: given $\phi,\psi\in\Gamma(X,E\otimes\Lambda^{p,q})$, we define, using multi-index notation,

$$\langle\phi,\psi\rangle = \frac{1}{p!q!}\sum\int\phi^\alpha_{\bar{J}I}\overline{\psi^\beta_{\bar{K}L}}H_{\bar{\beta}\alpha}g^{K\bar{J}}g^{I\bar{L}}\frac{\omega^n}{n!},$$

where, by definition,

$$g^{K\bar{J}} = g^{k_1\bar{j}_1}\cdots g^{k_q\bar{j}_q}$$

if $K = (k_1,\ldots,k_q)$ and $J = (j_1,\ldots,j_q)$, and

$$\omega \equiv \frac{i}{2}g_{\bar{k}j}dz^j\wedge d\bar{z}^k.$$

We now define the **formal adjoint of** $\bar{\partial}$ by

$$\langle\bar{\partial}\phi,\psi\rangle = \langle\phi,\bar{\partial}^\dagger\psi\rangle$$

for all $\phi\in C^\infty(X,E\otimes\Gamma^{p,q})$ and $\psi\in C^\infty(X,E\otimes\Gamma^{p,q+1})$. Hence we can draw:

$$\begin{tikzcd}
\Gamma(X,E\otimes\Lambda^{p,q}) \arrow[r, bend left, "\bar{\partial}"] & \Gamma(X,E\otimes\Lambda^{p,q+1}) \arrow[l, bend left, "\bar{\partial}^\dagger"]
\end{tikzcd}$$

Next define the **Laplacian** $\Box : \Gamma(X, E \otimes \Lambda^{p,q}) \to \Gamma(X, E \otimes \Lambda^{p,q})$ by $\Box = \bar{\partial}^\dagger \bar{\partial} + \bar{\partial}\bar{\partial}^\dagger$. This raises the question: when do we have that

$$\dim \ker \Box \Big|_{\Gamma(X, E \otimes \Lambda^{p,q})} = 0?$$

It will turn out that for $q = 1$, if this is true, we will indeed be able to find sections on this bundle.

To approach this question, we use the Bochner-Kodaira formulas. We claim that

$$(\Box \phi)^{\alpha}_{\bar{J}I} = -g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} \phi^{\alpha}_{\bar{J}I} + \text{t.t} + \text{c.t.}$$

where t.t. and c.t. stand for torsion and curvature forms respectively. This relates a kind of geometric laplacian (lhs) to a more analytic laplacian (rhs) modulo certain correction terms. In practice, we will work with Kähler metrics, in which the torsion terms disappear. In this sense, we can schematically write (after integrating by parts)

$$\langle \Box \phi, \phi \rangle = ||\nabla_{\bar{l}} \phi^{\alpha}_{\bar{J}I}||^2 + \langle \text{c.t.}\phi, \phi \rangle.$$

This immediately implies that if the curvature terms are positive (loosely speaking), then $\ker \Box = 0$.

We begin by deriving the Bochner-Kodaira formula. We will simplify life by working with $(0,1)$-forms, but the results will extend fairly easily. Let us first compute $\bar{\partial}^\dagger$ more explicitly. In this case, we look at

$$\begin{tikzcd}
\Gamma(X, E \otimes \Lambda^{0,0}) \arrow[r, bend left, "\bar{\partial}"] & \Gamma(X, E \otimes \Lambda^{0,1}) \arrow[l, bend left, "\bar{\partial}^\dagger"] \arrow[r, bend left, "\bar{\partial}"] & \Gamma(X, E \otimes \Lambda^{0,2}). \arrow[l, bend left, "\bar{\partial}^\dagger"]
\end{tikzcd}$$

Let us compute the second formal adjoint that appears in the diagram. Pick some $\phi \in \Gamma(X, E \otimes \Lambda^{0,1})$ such that $\phi = \sum \phi^{\alpha}_{\bar{j}} d\bar{z}^j$. Then $\bar{\partial}\phi^{\alpha} = \sum \partial_{\bar{k}} \phi^{\alpha}_{\bar{j}} d\bar{z}^k \wedge d\bar{z}^j$. Also, pick $\psi = \frac{1}{2} \sum \psi_{\bar{j}\bar{k}} d\bar{z}^k \wedge d\bar{z}^j$ in $\Gamma(X, E \otimes \Lambda^{0,2})$. We impose that

$$\langle \bar{\partial}\phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle$$

The left hand side appears to be

$$\frac{1}{2} \left(\int (\partial_{\bar{p}} \phi^{\alpha}_{\bar{m}} - \partial_{\bar{m}} \phi^{\alpha}_{\bar{p}}) \overline{\psi^{\beta}_{\bar{j}\bar{k}}} H_{\bar{\beta}\alpha} g^{j\bar{m}} g^{k\bar{p}} \frac{\omega^n}{n!} \right)$$

But recall that

$$\nabla_{\bar{p}} \phi^{\alpha}_{\bar{m}} = \partial_{\bar{p}} \phi^{\alpha}_{\bar{m}} - \Gamma^{\bar{l}}_{\bar{p}\bar{m}} \phi^{\alpha}_{\bar{l}}$$

where $\Gamma^{\bar{l}}_{\bar{p}\bar{m}} = g^{k\bar{l}}(\partial_{\bar{p}} g_{\bar{m}k})$. Hence we may write

$$\partial_{\bar{p}} \phi^{\alpha}_{\bar{m}} - \partial_{\bar{m}} \phi^{\alpha}_{\bar{p}} = \nabla_{\bar{p}} \phi^{\alpha}_{\bar{m}} - \nabla_{\bar{m}} \phi^{\alpha}_{\bar{p}} + (\Gamma^{\bar{l}}_{\bar{p}\bar{m}} - \Gamma^{\bar{l}}_{\bar{m}\bar{p}}) \phi^{\alpha}_{\bar{l}}.$$

We denote the term in the parentheses by $T^{\bar{l}}_{\bar{p}\bar{m}}$ and call it the **torsion of the covariant derivative** $\nabla_{\bar{p}}$. Let us now move to the Kähler case. The metric $g_{\bar{k}j}$ is said to be **Kähler** if $\Gamma^{\bar{l}}_{\bar{p}\bar{m}} = \Gamma^{\bar{l}}_{\bar{m}\bar{p}}$. Note that this condition is equivalent to $\partial_{\bar{p}} g_{\bar{m}k} = \partial_{\bar{m}} g_{\bar{p}k}$ or $\partial_p g_{\bar{k}m} = \partial_m g_{\bar{k}p}$. Henceforth we assume that $g_{\bar{k}j}$ is Kähler.

Now in the computation of the formal adjoint above, we can write

$$\begin{aligned}\langle\bar{\partial}\phi,\psi\rangle &= \int (\nabla_{\bar{p}}\phi^{\alpha}_{\bar{m}})\overline{\psi^{\beta}_{\bar{j}\bar{k}}}H_{\bar{\beta}\alpha}g^{j\bar{p}}g^{k\bar{m}}\frac{\omega^n}{n!}\\ &= \int \phi^{\alpha}_{\bar{m}}\overline{(-g^{p\bar{j}}\nabla_p\psi^{\beta}_{\bar{j}\bar{k}})}H_{\bar{\beta}\alpha}g^{k\bar{m}}\frac{\omega^n}{n!},\end{aligned}$$

where we have de-antisymmetrized the covariant derivatives and then integrated by parts. This yields immediately the expression for the formal adjoint:

$$(\bar{\partial}^{\dagger}\psi)^{\beta}_{\bar{k}} = -g^{p\bar{j}}\nabla_p\psi^{\beta}_{\bar{j}\bar{k}}.$$

To see this in more detail, see the next paragraph, where we will perform the computation explicitly for the other formal adjoint. So much for the computation of the second formal adjoint in the diagram above. check this!

Let us now compute the first formal adjoint. Of course,

$$\langle\bar{\partial}\phi,\psi\rangle = \langle\phi,\bar{\partial}^{\dagger}\psi\rangle$$

for $\phi\in C^{\infty}(X,E)$ and $\psi\in C^{\infty}(X,E\otimes\Lambda^{0,1})$. We can write $\bar{\partial}\phi = \partial_{\bar{j}}\phi^{\alpha}d\bar{z}^{j}$ and $\psi=\psi^{\alpha}_{\bar{k}}d\bar{z}^{k}$. The above equation requires

$$\int \partial_{\bar{j}}\phi^{\alpha}\overline{\psi^{\beta}_{\bar{k}}}H_{\bar{\beta}\alpha}g^{k\bar{j}}\frac{\omega^n}{n!} = \int \phi^{\alpha}\overline{(\bar{\partial}^{\dagger}\psi)^{\beta}}H_{\bar{\beta}\alpha}\frac{\omega^n}{n!}.$$

Let us for now define $W^{\bar{j}}_{\alpha}\equiv\overline{\psi^{\beta}_{\bar{k}}H_{\bar{\alpha}\beta}g^{j\bar{k}}}$. Observe now that

$$\begin{aligned}\left(\partial_{\bar{j}}\phi^{\alpha}\right)W^{\bar{j}}_{\alpha} &\equiv \left(\nabla_{\bar{j}}\phi^{\alpha}\right)W^{\bar{j}}_{\alpha}\\ &= \nabla_{\bar{j}}\left(\phi^{\alpha}W^{\bar{j}}_{\alpha}\right) - \phi^{\alpha}\left(\nabla_{\bar{j}}W^{\bar{j}}_{\alpha}\right).\end{aligned}$$

This will be useful when integrating by parts. Now note that the nth wedge power of ω simplifies to yield

$$\int\left(\partial_{\bar{j}}\phi^{\alpha}\right)W^{\bar{j}}_{\alpha}\left(\det g_{\bar{q}p}\right) = \int\nabla_{\bar{j}}\left(\phi^{\alpha}W^{\bar{j}}_{\alpha}\right)\det g_{\bar{q}p} - \int\phi^{\alpha}\left(\nabla_{\bar{j}}W^{\bar{j}}_{\alpha}\right).$$

We claim that if the metric $g_{\bar{k}j}$ is Kähler, then $\int\nabla_{\bar{j}}(\phi^{\alpha}W^{\bar{j}}_{\alpha})\det g_{\bar{q}p}=0$. To see this, first define $V^{\bar{j}}=\phi^{\alpha}W^{\bar{j}}_{\alpha}$. Consider

$$\begin{aligned}\left(\nabla_{\bar{j}}V^{\bar{j}}\right)\det g_{\bar{q}p} &= \left(\partial_{\bar{j}}V^{\bar{j}}+\Gamma^{\bar{j}}_{\bar{j}\bar{k}}V^{\bar{k}}\right)\det g_{\bar{q}p}\\ &= \partial_{\bar{j}}\left(V^{\bar{j}}\det g_{\bar{q}p}\right) - V^{\bar{j}}\left(\partial_{\bar{j}}\det g_{\bar{q}p}\right)+\Gamma^{\bar{j}}_{\bar{j}\bar{k}}V^{\bar{k}}\det g_{\bar{q}p}.\end{aligned}$$

Now note that

$$\partial_{\bar{j}}\det g_{\bar{q}p} = \left(\det g_{\bar{q}p}\right)g^{l\bar{n}}\partial_{\bar{j}}g_{\bar{n}l},$$

which comes from the fact that

$$\begin{aligned}\delta\log\left(\det A\right) &= \sum\frac{\delta\lambda_j}{\lambda_j}\\ &= \operatorname{tr}\left(A^{-1}\delta A\right).\end{aligned}$$

But now recall that for $g_{\bar{k}j}$ is Kähler if and only if $\Gamma^{\bar{j}}_{\bar{k}\bar{m}} = \Gamma^{\bar{j}}_{\bar{m}\bar{k}}$, and hence the we see the last two terms in the expression above cancel. Hence the term picked up by integration by parts vanishes. Hence we are left with the equality

$$-\int \phi^\alpha \nabla_{\bar{j}} \overline{\left(\psi^\beta_{\bar{k}} H_{\bar{\alpha}\beta} g^{j\bar{k}}\right)} \frac{\omega^n}{n!} = \int \phi^\alpha \overline{\left(-g^{j\bar{k}} \nabla_j \psi^\beta_{\bar{k}}\right)} H_{\bar{\beta}\alpha} \frac{\omega^n}{n!}.$$

This yields the desired formula:

$$(\bar{\partial}^\dagger \psi)^\beta = -g^{j\bar{k}} \nabla_j \psi^\beta_{\bar{k}}.$$

Now let us compute the Laplacian $\Box = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. Set $\phi = \sum \phi^\alpha_{\bar{j}} d\bar{z}^j \in \Gamma(X, E \otimes \Lambda^{0,1})$. First note that

$$\begin{aligned}\left(\bar{\partial}\bar{\partial}^\dagger \phi\right) &= \bar{\partial}\left(-g^{j\bar{k}} \nabla_j \phi^\alpha_{\bar{k}}\right) \\ &= \partial_{\bar{l}}\left(-g^{j\bar{k}} \nabla_j \phi^\alpha_{\bar{k}}\right) d\bar{z}^l.\end{aligned}$$

Hence, noting that the expression in parentheses is a section of a holomorphic bundle, the $\partial_{\bar{j}}$ is simply a covariant derviative, which commutes with the metric, and we can write

$$\left(\bar{\partial}\bar{\partial}^\dagger \phi\right)^\alpha_{\bar{l}} = -g^{j\bar{k}} \nabla_{\bar{l}} \nabla_j \phi^\alpha_{\bar{k}}.$$

Next note that

$$\bar{\partial}\phi^\alpha = \frac{1}{2} \sum \left(\nabla_{\bar{k}} \phi^\alpha_{\bar{j}} - \nabla_{\bar{j}} \phi^\alpha_{\bar{k}}\right) d\bar{z}^k \wedge d\bar{z}^j$$

for $g_{\bar{k}j}$ Kähler. Hence we can write

$$\begin{aligned}\left(\bar{\partial}^\dagger \bar{\partial}\phi\right)^\beta_{\bar{l}} &= -g^{k\bar{m}} \nabla_k (\bar{\partial}\phi)^\beta_{\bar{l}\bar{m}} \\ &= -g^{k\bar{m}} \nabla_k \left(\nabla_{\bar{m}} \phi^\alpha_{\bar{l}} - \nabla_{\bar{l}} \phi^\alpha_{\bar{m}}\right) \\ &= -g^{k\bar{m}} \nabla_k \nabla_{\bar{m}} \phi^\alpha_{\bar{l}} + g^{k\bar{m}} \nabla_k \nabla_{\bar{l}} \phi^\alpha_{\bar{m}}.\end{aligned}$$

Summing the two terms of the Laplacian and switching appropriate dummy indices, we find that

$$\begin{aligned}(\Box\phi)^\alpha_{\bar{l}} &= -g^{k\bar{m}} \nabla_k \nabla_{\bar{m}} \phi^\alpha_{\bar{l}} + g^{k\bar{m}} \nabla_k \nabla_{\bar{l}} \phi^\alpha_{\bar{m}} - g^{k\bar{m}} \nabla_{\bar{l}} \nabla_{\bar{k}} \phi^\alpha_{\bar{m}} \\ &= -g^{k\bar{m}} \nabla_k \nabla_{\bar{m}} \phi^\alpha_{\bar{l}} + g^{k\bar{m}} [\nabla_k, \nabla_{\bar{l}}] \phi^\alpha_{\bar{m}} \\ &= -g^{k\bar{m}} \nabla_k \nabla_{\bar{m}} \phi^\alpha_{\bar{l}} + g^{k\bar{m}} \left(F^\alpha_{\bar{l}k\beta} \phi^\beta_{\bar{m}} + R^{\bar{p}}_{\bar{l}k\bar{m}} \phi^\alpha_{\bar{p}}\right),\end{aligned}$$

where as usual $F^\alpha_{\bar{l}k\beta} = -\partial_{\bar{l}}(J^{\alpha\bar{\gamma}} \partial_k H_{\bar{\gamma}\beta})$ and $R^{\bar{p}}_{\bar{l}k\bar{m}} = g^{\bar{p}q} R^r_{\bar{l}kq} g_{\bar{m}r}$ where $R^r_{\bar{l}kq} = -\partial_{\bar{l}}(g^{r\bar{s}} \partial_k g_{\bar{s}q})$. We can simplify this a little bit more, obtaining

$$(\Box\phi)^\alpha_{\bar{l}} = -g^{k\bar{m}} \nabla_k \nabla_{\bar{m}} \phi^\alpha_{\bar{l}} + F^{\bar{m}\alpha}_{\bar{l}\beta} \phi^\beta_{\bar{m}} + R^{\bar{p}}_{\bar{l}} \phi^\alpha_{\bar{p}}, \tag{1}$$

where $R^{\bar{p}}_{\bar{l}} \equiv g^{k\bar{m}} R^{\bar{p}}_{\bar{l}k\bar{m}}$ is the **Ricci curvature**.

Consider now the inner product with ϕ:

$$\begin{aligned}\int (\Box\phi)^\alpha_{\bar{l}} \overline{\phi^\beta_{\bar{m}}} H_{\bar{\beta}\alpha} g^{m\bar{l}} \frac{\omega^n}{n!} &= -\int g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} \phi^\alpha_{\bar{l}} \overline{\phi^\beta_{\bar{m}}} H_{\bar{\beta}\alpha} g^{m\bar{l}} \frac{\omega^n}{n!} + \int (F^{\bar{p}\alpha}_{\bar{l}\gamma} \phi^\gamma_{\bar{p}} + R^{\bar{p}}_{\bar{l}} \phi^\alpha_{\bar{p}}) \overline{\phi^\beta_{\bar{m}}} H_{\bar{\beta}\alpha} g^{m\bar{l}} \frac{\omega^n}{n!} \\ \langle \Box\phi, \phi \rangle &= \int \nabla_{\bar{k}} \phi^\alpha_{\bar{l}} \overline{\nabla_{\bar{j}} \phi^\beta_{\bar{m}}} g^{j\bar{k}} H_{\bar{\beta}\alpha} g^{m\bar{l}} \frac{\omega^n}{n!} + \int F^{m\bar{p}}_{\bar{\beta}\alpha} \phi^\alpha_{\bar{p}} \overline{\phi^\beta_{\bar{m}}} + (R^{m\bar{p}} H_{\bar{\beta}\alpha}) \phi^\alpha_{\bar{p}} \overline{\phi^\beta_{\bar{m}}} \frac{\omega^n}{n!} \\ \langle \Box\phi, \phi \rangle &= ||\bar{\nabla}\phi||^2 + \int (F^{m\bar{p}}_{\bar{\beta}\alpha} + R^{m\bar{p}} H_{\bar{\beta}\alpha}) \phi^\alpha_{\bar{p}} \overline{\phi^\beta_{\bar{m}}} \frac{\omega^n}{n!}.\end{aligned}$$

This yields the following simple corollary.

Corollary 4. *If* $F^{m\bar{p}}_{\bar{\beta}\alpha} + R^{m\bar{p}}_{\bar{\beta}\alpha} > 0$*, then* $\ker \Box\Big|_{\Gamma(X, E\otimes\Lambda^{0,1})} = 0.$

Further, we can prove the following result.

Corollary 5. *Let* $L \to X$ *be a* ***positive*** *holomorphic line bundle over a compact manifold* X*, i.e. there exists a metric* h *on* L *with* $-\partial_j\partial_{\bar{k}} \log h > 0$*. Set* $\omega = -\frac{i}{2}\partial\bar{\partial}\log h$ *(as coefficients). Since* L *is positive,* ω *is a metric, which is automatically Kähler:*

$$\partial_l g_{\bar{k}j} = -\partial_l\partial_j\partial_{\bar{k}} \log h = \partial_j g_{\bar{k}l}.$$

Equip X *with the Kähler metric* ω *and consider* $\Box$ *on* $L^M \otimes \Lambda^{0,1}$*. Then for* $M \geq 1$*,*

$$\ker \Box\Big|_{\Gamma(X, L^m\otimes\Lambda^{0,1})} = 0.$$

Proof. When $E = L$, we can write

$$F_{\bar{k}j} = -\partial_j\partial_{\bar{k}} \log h$$

and hence in the case of line bundles, we have that

$$F^{m\bar{p}} \equiv F^{m\bar{p}}_{\bar{\beta}\alpha} = g^{m\bar{k}} g^{j\bar{p}} F_{\bar{k}j} h.$$

Hence the Bochner-Kodaira formula simplifies to

$$\langle \Box\phi, \phi\rangle = ||\bar{\nabla}\phi||^2 + \int (F^{m\bar{p}} + R^{m\bar{p}})\phi_{\bar{p}}\overline{\phi_{\bar{m}}} h \frac{\omega^n}{n!}.$$

Now if we take $L \mapsto L^M$, R does not change, as it is the Ricci curvature of the metric on the base manifold. On the other hand, the curvature F of L is now multiplied by M, as the curvature is given by the logarithm above (and hence the power M becomes multiplicative). For $L^M \to X$, then, the Bochner-Kodaira formula reads:

$$\langle \Box\phi, \phi\rangle = ||\bar{\nabla}\phi||^2 + \int (MF^{m\bar{p}} + R^{m\bar{p}})\phi_{\bar{p}}\overline{\phi_{\bar{m}}} h \frac{\omega^n}{n!}.$$

In the case of positive line bundle L, we have a Kähler metric on X, which is precisely the curvature of the metric on L, ω, which yields the fact that

$$F^{m\bar{p}} = g^{m\bar{p}}.$$

On $L^M \to X$, then, we can further simplify the Bochner-Kodaira formula to

$$\langle \Box\phi, \phi\rangle = ||\bar{\nabla}\phi||^2 + \int (Mg^{m\bar{p}} + R^{m\bar{p}})\phi_{\bar{p}}\overline{\phi_{\bar{m}}} h \frac{\omega^n}{n!}.$$

Since $g^{m\bar{p}} > 0$, we can choose M large enough such that

$$Mg^{m\bar{p}} + R^{m\bar{p}} > 0,$$

which concludes the proof. $\Box$

Exercise 1. *Derive the Bochner-Kodaira formula for the case of* $\Lambda^{0,2}$*. It will be good for your soul.*

1.3 Other Bochner-Kodaira Formulae

Definition 1. Define the **Hodge** operator Λ as follows. Let $\Phi \in \Gamma(X, E \otimes \Lambda^{p+1,q+1})$ for some bundle $E \to X$. Then $(\Lambda\Phi)_{\bar{K}J} = g^{l\bar{p}}\Phi^{\alpha}_{\bar{p}l\bar{K}J} \in \Gamma(X, E \otimes \Lambda^{p,q})$.

Exercise 2. *Show that* $[\partial, \Lambda] = \bar{\partial}^\dagger$ *and* $[\bar{\partial}, \Lambda] = -\partial^\dagger$.

Now consider $\bar{\Box} = \partial\partial^\dagger + \partial^\dagger\partial$ on $\Gamma(X, E \otimes \Lambda^{p,q})$.

Theorem 6 (Kodaira-Akizuki-Nakano)**.** *Let* $L \to X$ *be a line bundle. Then*

$$\Box = \bar{\Box} + [F, \Lambda].$$

Proof. We sketch the proof:

$$\begin{aligned}
\Box - \bar{\Box} &= [\partial, \Lambda]\bar{\partial} + \bar{\partial}[\partial, \Lambda] + [\bar{\partial}, \Lambda]\partial + \partial[\bar{\partial}, \Lambda] \\
&= (\bar{\partial}\partial + \partial\bar{\partial})\Lambda - \Lambda(\bar{\partial}\partial + \partial\bar{\partial}) \\
&= [\bar{\partial}\partial + \partial\bar{\partial}, \Lambda] \\
&= [F, \Lambda]
\end{aligned}$$

□

In practice, we use the following lemma when we apply the KAN theorem, which gives us a handle on positivity.

Lemma 7. *Let* ω *and* F *be simultaneously diagonalized, i.e. if* $\zeta^a = \zeta^a_j dz^j$ *is a basis of forms, then* $\omega = \frac{i}{2}\sum_a \zeta^a \wedge \overline{\zeta^a}$ *and* $F = \frac{i}{2}\sum_a \lambda_a \zeta^a \wedge \overline{\zeta^a}$*. Then*

$$\langle [F, \Lambda]\Psi, \Psi\rangle = \frac{1}{2}\sum_{KJ}\left(\sum_{a\in J}\lambda_a + \sum_{b\in K}\lambda_b - \sum_{c=1}^{n}\lambda_c\right)|\Psi_{\bar{K}J}|^2.$$

Positivity can be tested directly from the term in parentheses. In particular, on positive line bundles, the sums are particularly simple: $p + q - n$. Hence we find via this lemma that for a positive line bundle L,

$$\ker \Box\Big|_{L\otimes\Lambda^{p,q}} = 0$$

when $p + q - n > 0$.

This concludes the easy part of this theory. Now we will turn to the cohomology of the $\bar{\partial}$ complexes.

Cohomology

Let $E \to X$ be a holomorphic line bundle, with $H_{\bar{\alpha}\beta}$ a metric on E and $g_{\bar{k}j}$ a (Kähler) metric on X. Recall the $\bar{\partial}$ complex from above. We define the **cohomology** of the $\bar{\partial}$ complex as

$$H^{p,q}_{\bar{\partial}}(X, E \otimes \Lambda^{p,q}) = \ker \bar{\partial}|_{\Gamma(X,E\otimes\Lambda^{p,q})}/\mathrm{Im}\ \bar{\partial}|_{\Gamma(X,E\otimes\Lambda^{p,q-1})}.$$

This makes sense because the complex is exact by construction. This is the analogue of de Rham cohomology, where the **de Rham cohomology**

$$H^p_{dR}(X) = \ker d|_{\Gamma(x,\Lambda^p)}/\mathrm{Im}\ d|_{\Gamma(X,\Lambda^{p-1})}$$

is defined for smooth compact manifolds X, and carries some sort of topological data of X. In our case, we are dealing with complex manifolds and hence the cohomology of the $\bar{\partial}$ complex depends sensitively on the complex structure of the manifold. There are two approaches to computing $H^p(X)$ (or at least, to determine when it is 0), namely Hodge theory and the L^2 estimates of Hörmander. In Hodge theory, the key statment is that $H^p_{dR}(X)$ is isomorphic to $\ker\Delta$, where Δ is the Laplacian $\Delta = dd^\dagger + d^\dagger d$ (having introduced a metric). What is of course remarkable is that the left-hand side of the equality is metric-independent. Then one can use certain vanishing theorems to show that this kernel, and hence the cohomology group, is zero. On the other hand, the L^2 estimates approach gives conditions under which the equation $d = v$ for $v \in L^2(X, \Lambda^{p+1})$ and $dv = 0$, admits solutions. This approach has intrinsic interest also, however, because this equation involves only one derivative.

The key step in the Hodge theory approach is the construction of an operator G such that

$$G\Delta = \Delta G = I - \Pi$$

where $\Pi : L^2(X, \Lambda^p) \to \ker\Delta$ is the orthogonal projection. Furthermore, $dG = Gd$ and $d^\dagger G = Gd^\dagger$. Assume the existence of G. Then

$$u = (\Delta g)u + \Pi u = d(d^\dagger G u) + d^\dagger(dGu) + \Pi u$$

for any smooth u. If $u \in \ker d$ then $du = 0$ and we are left with

$$\begin{aligned} u &= d(d^\dagger G u) + \Pi u \\ [u] &= [\Pi u]. \end{aligned}$$

But by definition, $\Pi u \in \ker\Delta$ and we are done. Moreover, suppose we want to solve the d equation $du = v$. Just take $u = d^\dagger G v$. Then

$$du = dd^\dagger G v = (\Delta - d^\dagger d)Gv = \Delta G v = v - \Pi v.$$

Hence if v is of class zero, then v is given by this equation. Otherwise, the equation cannot be solved.

Now how does one prove the existence of G? This follows (with more work, of course) from the following estimate

$$||u||_{W^{k+2,2}} \leq c(||\Delta u||^2_{W^{k,2}} + ||u||^2_{W^{k,2}})$$

for all $u \in C^\infty(X, \Lambda^p)$ where we have the Sobolev norm

$$||u||^2_{W^{k,p}} \equiv \sum_{|\alpha|\leq k} ||D^\alpha u||^2_{L^p}.$$

for $p > 1$. These kinds of estimates hold for elliptic operators (in this case the Laplacian).

The L^2 estimates approach

Let $L \to X$ be a holomorphic line bundle over X a compact complex manifold and let h be a metric on L and $g_{\bar{k}j}$ to be Kähler. We will in fact work with distributions instead of functions. Consider $L^1_{loc}(\mathbb{R}^n)$. Let $f \in L^1_{loc}(\mathbb{R}^n)$. The derivative $\partial f/\partial x_1$ in the sense of distributions is the functional

$$C_0^\infty(\mathbb{R}^n) \ni \phi \mapsto -\int f \frac{\partial\phi}{\partial x_1}$$

If $f \in C^1_{loc}(\mathbb{R}^n)$ we can integrate by parts and the functional becomes

$$C_0^\infty(\mathbb{R}^n) \ni \phi \mapsto \int \frac{\partial f}{\partial x_1}\phi.$$

This gives us a generalized way of thinking about the derivative. Hence we define the domain of $\bar{\partial}$ to be

$$\text{Dom } \bar{\partial}_{p,q} = \left\{\phi \in L^2(X, L \otimes \Lambda^{p,q}) \mid \exists \psi \in L^2(X, L \otimes \Lambda^{p,q+1}) \text{ with } \bar{\partial}\phi = \psi\right\},$$

where the equality is taken in the sense of distributions. Similarly, we must define the domain of $\bar{\partial}^\dagger$:

$$\text{Dom } \bar{\partial}^\dagger = \left\{\psi \in L^2(X, L \otimes \Lambda^{p,q+1}) \mid \exists \phi \in L^2(X, L \otimes \Lambda^{p,q}) \text{ with } \bar{\partial}^\dagger\psi = \phi\right\}$$

again where the equality is taken in the sense of distributions as well as $\langle \bar{\partial}\lambda, \psi\rangle = \langle \lambda, \phi\rangle$ for all $\lambda \in \text{Dom } \bar{\partial}$. Restricting this extra equation to $\lambda \in C_0^\infty$, we obtain the weaker condition that $\bar{\partial}^\dagger\psi = \phi$ in the sense of distributions.

Our main goal is to solve $\bar{\partial}u = f$, where $f \in L^2(X, L \otimes \Lambda^{0,1})$. The key lemma is as follows.

Lemma 8. *Let $g_{\bar{k}j}$ be a Kähler metric on X and h a metric on L. Assume that the following inequality holds:*

$$||\bar{\partial}u||^2_{L^2} + ||\bar{\partial}^\dagger u||^2_{L^2} \geq \int_X \langle Au, u\rangle$$

for all $u \in$ Dom $\bar{\partial}_1 \cap$ Dom $\bar{\partial}_0^\dagger$ and A is a positive definite matrix. Then, for all $f \in L^2(X, L\otimes\Lambda^{0,1})$, with $\bar{\partial}f = 0$ in the sense of distributions, and there exists $v \in L^2(X, L)$ satisfying $\bar{\partial}v = f$ and $\int_x |v|^2 \leq \int_X \langle A^{-1}f, f\rangle$.

Here we are distinguishing the two different $\bar{\partial}$ operators that are present. More explicitly, we write $u = \sum u_{\bar{j}} d\bar{z}^j$ with $u_{\bar{j}}$ a section of L and

$$\langle Au, u\rangle = A^{l\bar{j}} u_{\bar{j}} \overline{u_{\bar{l}}} h.$$

Then

$$\int_X \langle Au, u\rangle = \int_X A^{l\bar{j}} u_{\bar{j}} \overline{u_{\bar{l}}} h \frac{\omega^n}{n!}$$

Furthermore, $v \in L^2(X, L \otimes \Lambda^{0,0})$ and hence

$$\int |v|^2 = \int |v|^2 h \frac{\omega^n}{n!}$$

and

$$\int \langle A^{-1}f, f\rangle = \int (A^{-1})^{l\bar{j}} f_{\bar{j}} \overline{f_{\bar{l}}} h \frac{\omega^n}{n!}.$$

Hence we can write the restriction on v more explicitly as

$$\int |v|^2 h \frac{\omega^n}{n!} \leq \int (A^{-1})^{l\bar{j}} f_{\bar{j}} \overline{f_{\bar{l}}} h \frac{\omega^n}{n!}.$$

We use the following easy fact from functional analysis.

Lemma 9. *Let $\phi \in$ Dom $\bar{\partial}_0^\dagger$, and let $\phi = \phi_1 + \phi_2$ where $\phi_1 \in \ker \bar{\partial}_1$ and $\phi_2 \in (\ker \bar{\partial}_1)^\perp$. Then $\phi_1, \phi_2 \in$ Dom $\bar{\partial}_0^\dagger$.*

Note that the decomposition in the lemma makes sense because $\ker \bar{\partial}_1$ is a closed subspace of L^2.

Proof. Since $\bar{\partial}_1 \bar{\partial}_0 = 0$, it follows that the range of $\bar{\partial}_0$ is contained in $\ker \bar{\partial}_1$. This implies that if $\phi_2 \perp \ker \bar{\partial}_1$ then $\phi_2 \perp (\text{Range } \bar{\partial}_0)^{\perp}$, i.e. perpendicular to the space $0 = \langle \bar{\partial}_0 \psi, \phi \rangle$ for $\psi \in \text{Dom } \bar{\partial}_0$. This in turn means that $\langle \bar{\partial}_0 \psi, \phi_2 \rangle = \langle \psi, 0 \rangle$ for all $\psi \in \text{Dom } \bar{\partial}$, and thus that $\phi_2 \in \text{Dom } \bar{\partial}_0^{\dagger}$ and $\bar{\partial}_0^{\dagger} \phi_2 = 0$. ☐

Proof of earlier lemma. Consider the functional T defined by: $\bar{\partial}_0^{\dagger} \phi \mapsto \langle \phi, f \rangle$ for $\phi \in \text{Dom } \bar{\partial}_0^{\dagger}$. Note this is defined only on a subspace. It is not *a priori* obvious that this is well defined - there might be multiple such ϕ. Decompose $\phi = \phi_1 + \phi_2$ with $\phi_2 \in \ker \bar{\partial}_1, \phi_2 \perp \ker \bar{\partial}_1$. We can write

$$\begin{aligned}\langle \phi, f \rangle &= \langle \phi_1, f \rangle + \langle \phi_2, f \rangle \\ &= \langle \phi_1, f \rangle.\end{aligned}$$

By the lemma just proved, we see that $\phi_1 \in \text{Dom } \bar{\partial}_0^{\dagger} \cap \text{Dom } \bar{\partial}_1$. Using Cauchy-Schwarz with respect to the L^2 norm determined by A (it is positive definite), we find

$$\begin{aligned}|\langle \phi_1, f \rangle|^2 &\leq \left(\int \langle A\phi_1, \phi_1 \rangle \right) \left(\int \langle A^{-1} f, f \rangle \right) \\ &\leq \left(||\bar{\partial}\phi_1||^2 + ||\bar{\partial}^{\dagger}\phi_1||^2 \right) \left(\int \langle A^{-1} f, f \rangle \right) \\ &= ||\bar{\partial}^{\dagger}\phi_1||^2 \int \langle A^{-1} f, f \rangle.\end{aligned}$$

This shows that the functional defined above does indeed make sense, once in the context of the estimates of the hypothesis. Now, by the Hahn-Banach theorem stated below, applied to this functional, we obtain a functional $\tilde{T} : L^2 \to \mathbb{C}$ extending T such that $||\tilde{T}|| = ||T||$. This implies that there exists a $u \in L^2$ such that $\tilde{T}(\psi) = \langle \psi, u \rangle$ and $||u|| = ||\tilde{T}||$ for any $\psi \in L^2$. In particular, take $\psi = \bar{\partial}_0^{\dagger} \phi$ with $\phi \in C_0^{\infty}$. Then $\langle \bar{\partial}_0^{\dagger} \phi_1, u \rangle = T(\bar{\partial}_0^{\dagger} \phi) = \langle \phi, f \rangle$, which is equivalent to the fact that $\bar{\partial} u = f$ in the sense of distributions. ☐

Theorem 10 (Hahn-Banach)**.** *Let $V \subset \mathcal{B}$ a Banach space and T be a linear functional $V \ni v \mapsto T(v)$ with $|T(v)| \leq A||v||$. Then there exists an extension $\tilde{T}$ of T to the whole of $\mathcal{B}$, such that $\tilde{T}$ satisfies $|T(v)| \leq A||v||$ for all $v \in \mathcal{B}$.*

Proof. Omitted. ☐

When do the estimates in the hypothesis of the lemma hold? Recall the Bochner-Kodaira formula on $C^{\infty}(X, L \otimes \Lambda^{0,1})$,

$$\langle \Box \phi, \phi \rangle = ||\bar{\partial}\phi||^2 + ||\bar{\partial}^{\dagger}\phi||^2$$

where $\Box = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}$. Acting on smooth forms, we found

$$\langle \Box \phi, \phi \rangle = ||\nabla_{\bar{k}} \phi_{\bar{j}}||_{L^2}^2 + \int (F_{\bar{k}j} + R_{\bar{k}j}) \phi^j \overline{\phi^k} h \frac{\omega^n}{n!}.$$

Recall that $F_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log h$ and $R_{\bar{k}j} = R_{\bar{k}j} P_p = R_{\bar{k}p} P_j$. Note that the above lemma will hold if $F_{\bar{k}j} + R_{\bar{k}j} > 0$ and the inequality required by the lemma extends from C_0^{∞} to $\text{Dom } \bar{\partial}_0 \cap \text{Dom } \bar{\partial}_1^{\dagger}$.

Note first that on a compact manifold C_0^{∞} is dense in $\text{Dom } \bar{\partial}_0 \cap \text{Dom } \bar{\partial}_1^{\dagger}$ with respect to the norm $||u||_{L^2} + ||\bar{\partial}u||_{L^2} + ||\bar{\partial}^{\dagger}u||_{L^2}$.

Theorem 11. *Let $L \to X, h, g_{\bar{k}j}$ be as above. Writing $h = e^{-\phi}$, assume that*

$$-\partial_j \partial_k \phi + R_{\bar{k}j} \geq \epsilon g_{\bar{k}j}$$

for some $\epsilon > 0$. Then for any $f \in L^2(X, L \otimes \Lambda^{0,1})$ with $\bar{\partial} f = 0$, there exists $u \in L^2(X, L)$ solving $\bar{\partial} u = f$ and

$$\int |u|^2 e^{-\phi} \frac{\omega^n}{n!} \leq \frac{1}{\epsilon} \int g^{k\bar{j}} f_{\bar{j}} \overline{f_k} e^{-\phi} \frac{\omega^n}{n!}.$$

Note that a useful variation of this setup is to solve the equation $\bar{\partial} u = f$ for $f \in L^2(X, L \otimes \Lambda^{n,1})$. Why? Observe that if $u \in L^2(X, L \otimes \Lambda^{n,0})$, then $\int |u|_h^2 \equiv \int u \bar{u} e^{-\phi}$. Note that here there is no volume form needed, as the integrand is an n, n form. Hence there is no need for a metric, and the estimates tend to be better in this case. Note also that $L \otimes \Lambda^{n,1} = (L \otimes \Lambda^{n,0}) \otimes \Lambda^{0,1}$. But $L' = L \otimes \Lambda^{n,0}$ is simply another holomorphic line bundle. Hence we can apply our previous theorem with L replaced by L'. Thus we impose $\varepsilon g_{\bar{k}j} \leq F'_{\bar{k}} j + R_{\bar{k}j} = (F_{\bar{k}j} - R_{\bar{k}j}) + R_{\bar{k}j} = F_{\bar{k}j}$ (the curvature of $\Lambda^{n,0}$ is negative the Ricci curvature). Hence we obtain the theorem above but for this case of $f \in L^2(X, L \otimes \Lambda^{n,1})$.

Exercise 3. *Write this theorem down carefully and supply the missing steps.*

Observe that this in fact extends to X not compact, but complete as a metric space, i.e. the estimate in the hypothesis still extends from C_0^∞ to Dom $\bar{\partial}_0^\dagger \cap$ Dom $\bar{\partial}_1$. Even further, if $(X, g_{\bar{k}j})$ is not necessarily complete, but instead there exists a metric $g'_{\bar{k}j}$ that is Kähler and complete, then one applies the theorem to $(X, g^\delta_{\bar{k}j} = g_{\bar{k}j} + \delta g'_{\bar{k}j})$. Applying the theorem now, we obtain a sequence u_δ, uniformly bounded, which weakly converges to a solution u (c.f. J.P. Demailly's online book).

fill in missing lectures

1.4 Kodaira embedding

Theorem 12 (Kodaira embedding)**.** *Let $L \to X$ be a positive, holomorphic line bundle, over a compact, Kähler manifold (X, ω). Let $H^0(X, L^k)$ denote the space of holomorphic sections of L^k, with $N_{k+1} \equiv \dim H^0(X, L^k)$. Let $\{s_\alpha(z)\}_{\alpha=0}^{N_{k+1}}$ be a basis for $H^0(X, L^k)$. The Kodaira map is defined*

$$\iota_k : X \ni z \mapsto [s_0(z) : \cdots : s_{N_k}(z)] \in \mathbb{CP}^{N_k}.$$

Then there exists a k_0 such that for $k \geq k_0$, the map ι_k is an embedding of X into $\mathbb{CP}^{N_k}$.

Let us first sketch the idea of the proof. We claim that this theorem can be reformulated in terms of sheaves and Čech cohomology, which we will introduce shortly. Indeed, given N points $z_1, \ldots, z_N \in X$, with N integers $k_1, \ldots, k_N$, we can define a sheaf $\mathcal{I}_{k_1 z_1 + \cdots + k_n z_N}$ of functions vanishing at z_i of order k_i, respectively. Kodaira embedding follows if we can prove that for all k_i, there exists a k_0 such that the maps

$$\check{H}^0(X, L^k) \to \check{H}^0(X, L^k \otimes \mathcal{O}/\mathcal{I}_{k_1 z_1 + \cdots + k_N z_N})$$

are surjective, where $\mathcal{O}$ is the local ring of holomorphic germs. To do this, we will use the fact that this map sits inside the exact sequence induced by

$$0 \longrightarrow L^k \otimes \mathcal{I} \longrightarrow L^k \otimes \mathcal{O} \longrightarrow L^k \otimes \mathcal{O}/\mathcal{I} \longrightarrow 0,$$

which is a long exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \check{H}^0(X, L^k \otimes \mathcal{I}) & \longrightarrow & \check{H}^0(X, L^k \otimes \mathcal{O}) & \longrightarrow & \check{H}^0(X, L^k \otimes \mathcal{O}/\mathcal{I}) & & \\ & & & & \downarrow & & \\ & & & & \check{H}^1(X, L^k \otimes \mathcal{I}) & \longrightarrow & \cdots \end{array}$$

Hence it suffices to show, by exactness, that

$$\check{H}^1(X, L^k \otimes \mathcal{I}_{k_1 z_1 + \cdots + k_N z_N}) = 0.$$

We shall show that there exists a ϕ plurisubharmonic such that for any $k_1, \ldots, k_N$, there exists a k depending only on $k_1, \ldots, k_N$, such that $\mathcal{J}_\phi \subset \mathcal{I}_{k_1 z_1 + \cdots + k_N z_N}$ and $e^{-\phi}$ is a singular metric on $L^k \otimes \Lambda^{n,0}$ with $i\partial\bar{\partial}\phi \geq \varepsilon\omega$. Note that $\mathcal{J}_\phi$ is the sheaf of holomorphic functions f near z such that $\int_{U_z} |f|^2 e^{-\phi} < \infty$. This implies that

$$H^1_{\bar{\partial}}(X, L^k \otimes \Lambda^{n,0} \otimes \mathcal{J}_\phi) = 0,$$

where this cohomology group is defined to be $\ker \bar{\partial}|_{L^k \otimes \Lambda^{n,1} \otimes \mathcal{J}_\phi} / \mathrm{im}\ \bar{\partial}|_{L^k \otimes \Lambda^{n,0} \otimes \mathcal{J}_\phi}$. But this is essentially just Hörmander's theorem. To complete the proof, we will use the fact that

$$H^1_{\bar{\partial}}(X, L^k \otimes \Lambda^{n,0} \otimes \mathcal{J}_\phi) = \check{H}^1(X, L^k \otimes \Lambda^{n,0} \otimes \mathcal{J}_\phi),$$

which one might think of as a version of the de Rham theorem.

Let us now consider the basics of Čech cohomology. Let X be a topological space with an open cover $\underline{U} = \{U_\alpha\}$. A sheaf $\mathcal{E}$ on X is an assignment of a group $\Gamma(U_\alpha, \mathcal{E})$ to each U_α, with restriction maps $\rho_{VU} : \Gamma(V, \mathcal{E}) \to \Gamma(U, \mathcal{E})$ for $U \subset V$, with the obvious consistency conditions, as well as the extension property that for $s_1 \in \Gamma(U_1, \xi), s_2 \in \Gamma(U_2, \xi)$ with $\rho_{U_1, U_1 \cap U_2} s_1 = \rho_{U_2, U_1 \cap U_2} s_2$ then there exists a unique $t \in \Gamma(U_1 \cap U_2, \xi)$ with $\rho_{(U_1 \cup U_2), U_1} s_1, \rho_{U_1 \cup U_2, U_2} s_2$.

Let us look at some examples of sheaves. The simplest is the constant sheaf, say for example $\mathcal{S} = \underline{\mathbb{Z}}$, where $\Gamma(U, \mathcal{S})$ is the group of integer-valued continuous (constant) functions on U, or $\mathcal{S} = \underline{R}$ where $\Gamma(U, \mathcal{R})$ is the group of smooth real-valued functions on U. A more sophisticated example is as follows. Take a subvariety $V \subset X$. One can define a sheaf $\mathcal{V}$ where $\Gamma(U, \mathcal{V})$ is the group of holomorphic functions on U vanishing on V. Another example is $\mathcal{J}_\phi$, where $\Gamma(U, \mathcal{J}_\phi)$ is the group of holomorphic functions f on U such that $\int_U |f|^2 e^{-\phi} < \infty$.

Let us now turn to cohomology groups. Given a sheaf $\mathcal{E}$, we define the space of p-cycles as

$$\mathcal{C}^p = \big\{\sigma \mid U_{\alpha_0}, \ldots, U_{\alpha_p} \to \sigma_{\alpha_0, \ldots, \alpha_p} \in \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{E})\big\}$$

and the δ operator $\delta : \mathcal{C}^p \to \mathcal{C}^{p+1}$ such that

$$(\delta\sigma)_{\alpha_0, \ldots, \alpha_{p+1}} = \sum (-1)^j \sigma_{\alpha_0, \ldots, \hat{\alpha}_j, \alpha_{p+1}} \in \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{E})$$

It is easy to check that $\delta^2 = 0$. Hence we define

$$\check{H}^p(\underline{U}, \mathcal{E}) \equiv \ker \delta|_p / \mathrm{im}\ \delta|_{\mathcal{C}^{p+1}}.$$

Now let us see how these quantities depend on the choice of open covering. Let $\underline{U} = \{U_\alpha\}$ and $\underline{U'} = \{U'_\beta\}$ be two coverings of X. We say that $\underline{U}$ is a refinement of $\underline{U'}$ if there exists a $\phi : \{\alpha\} \to \{\beta\}$ with $U_\alpha \subset U'_{\phi(\alpha)}$. Then we have a map on cocycles

$$(\phi\sigma)_{\alpha_0, \ldots, \alpha_p} \equiv \sigma_{\phi(\alpha_0), \ldots, \phi(\alpha_p)} \in \Gamma(\underline{U'}, \mathcal{E}),$$

and thus a map $\Gamma(\underline{U}', \mathcal{E}) \ni \sigma \to \phi(\sigma) \in \Gamma(\underline{U}, \mathcal{E})$. This map in turn induces a map on cohomology groups $\check{H}^p(\underline{U}', \mathcal{E}) \to \check{H}^p(\underline{U}, \mathcal{E})$. We can now define

$$\check{H}^p(X, \mathcal{E}) = \lim_{j \to \infty} \check{H}^p(\underline{U}_j, \mathcal{E}) = \cup_j \check{H}^p(\underline{U}_j, \mathcal{E}) / \sim$$

where $\underline{U}_{j+1} \subset \underline{U}_j$ is a sequence of refinements, and the equivalence relation is given as $\sigma \in \check{H}^p(\underline{U}_j, \mathcal{E})$ and $\tau \in \check{H}^p(\underline{U}_k, \mathcal{E})$ are equivalent if there exists an $m > j, k$ with $\phi_{jm}\sigma = \phi_{km}\tau$.

Lemma 13 (Leray). *If $\underline{U}$ is a covering with $\check{H}^p(U_\alpha \cap \cdots \cap U_{\alpha_p}, \mathcal{E}) = 0$ for all $p \geq 1$, then $\check{H}^p(X, \mathcal{E}) = \check{H}^p(\underline{U}, \mathcal{E})$.*

This practical lemma allows us to bypass working with the complicated inductive limit defined above, and instead work with such coverings $\underline{U}$.

Let us now turn to some examples of cohomology groups.

Example 3. By definition, $\check{H}^0(X, \mathcal{E}) = \{\sigma \in \mathcal{C}^0, \delta\sigma = 0\}$. What does this condition mean? Well clearly $\sigma : U_\alpha \to \Gamma(U_\alpha, \mathcal{E})$. Meanwhile,

$$\mathcal{C}^1 \ni (\delta\sigma)_{\alpha\beta} = \sigma_\beta - \sigma_\alpha,$$

and hence for this to be zero means that $\sigma_\beta = \sigma_\alpha$ on $U_\alpha \cap U_\beta$. Of course, by the sheaf axiom, this implies that there exists a $\sigma \in \Gamma(U_\alpha \cup U_\beta, \mathcal{E})$ that restricts to the $\sigma_\alpha, \sigma_\beta$ appropriately. Hence we find that

$$\check{H}^0(X, \mathcal{E}) = \Gamma(X, \mathcal{E})$$

.

As another example, consider the Cousin problem. Fix a domain Σ in some Riemann surface X. Given a sequence of points $\{p_j\} \subset \Omega$, does there exist a meromorphic function with a pole of order n_j at p_j? This is certainly doable on U_j, where U_j are small neigbhorhoods of the p_j, by the charts that transport U_j to a disk in the complex plane. Hence this problem is trivially locally, but the problem of matching these U_j is not so easy. It is not surprising that this problem can be rephrased as a question of cohomology. Consider $\sigma \in \mathcal{C}^1(X, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of holomorphic functions. Indeed $\sigma_{jk} \equiv f_j - f_k$ on $U_j \cap U_k$ is now holomorphic as the poles have cancelled. Observe now that if $\check{H}^1(X, \mathcal{O}) = 0$, then a global function exists with the given properties because this implies that $\sigma_{jk} = (\delta\tau)$ for $\tau \in \mathcal{C}^0(X, \mathcal{O})$. But this means that $\sigma_{jk} = \tau_j - \tau_k$, i.e. $f_j - f_k = \tau_j - \tau_k$ and hence $f_j - \tau_j = f_k - \tau_k$, which gives us the global meromorphic function, as these match.

Consider now the second Cousin problem. Given the same setup as before, does there exist a holomorphic function with the given orders of vanishing at the p_j? We leave it as an exercise to show that this is possible if $\check{H}^1(X, \mathcal{O}^*) = 0$, where $\Gamma(U, \mathcal{O}^*)$ are the non-vanishing holomorphic functions on U. As another exercise, we claim that an element $\sigma \in \mathcal{C}^1(X, \mathcal{O}^*)$ gives rise to a line bundle $L \to X$.[1]

[1] The way to see this is to note that σ associates to $U_\alpha \cap U_\beta$ a section $\sigma_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*)$, which is multiplicative and hence $0 = \delta\sigma \iff \sigma_{\alpha\beta} = \sigma_{\alpha\gamma}\sigma_{\gamma\beta}$: the cocycle condition. This implies that $\check{H}^1(X, \mathcal{O}^*)$ is the space of line bundles L with transition functions $\sigma_{\alpha\beta}$ modulo $\sigma_{\alpha\beta} \sim \sigma'_{\alpha\beta}$ if $\sigma_{\alpha\beta = \sigma'_{\alpha\beta}} f_\alpha f_\beta^- 1$. From the point of view of line bundles, this makes sense because on one line bundle L, $\phi_\alpha = \sigma_{\alpha\beta}\phi_\beta$, while on the other $\phi'_\alpha = \sigma'_{\alpha\beta}\phi'_\beta$, between which there is a natural correspondence because we can write $(f_\alpha^{-1}\phi_\alpha) = \sigma'_{\alpha\beta}(f_\beta^{-1}\phi_\beta)$, which gives us a mapping $\Gamma(X, L) \ni \phi \mapsto \phi' \equiv f_\alpha^{-1}\phi_\alpha \in \Gamma(X, L')$. This one-to-one mapping allows us to think of the line bundles as equivalent, as f_α are holomorphic and non-vanishing.

The moral of these examples is that global problems can be rephrased in terms of cohomology.

Now let us turn to short exact sequences of sheaves. Suppose we have sheaves $\mathcal{E}, \mathcal{F}, \mathcal{G}$ and maps

$$0 \longrightarrow \mathcal{E} \xrightarrow{\Phi} \mathcal{F} \xrightarrow{\Psi} \mathcal{G} \longrightarrow 0$$

which are exact as maps of sheaves. By this we mean that for each z, the sequence

$$0 \longrightarrow \mathcal{E}_z \longrightarrow \mathcal{F}_z \longrightarrow \mathcal{G}_z \longrightarrow 0$$

is exact as a sequence of groups. Then we obtain the following long exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X,\mathcal{E}) \arrow[r] & \check{H}^0(X,\mathcal{F}) \arrow[r] & \check{H}^0(X,\mathcal{G}) \arrow[d, "\delta*"] & \\ & & & \check{H}^1(X,\mathcal{E}) \arrow[r] & \cdots \end{tikzcd}$$

Let us now try to construct δ^*, the **coboundary operator**. Consider first some $\sigma \in \mathcal{C}^p(X,\mathcal{G})$, i.e. $\delta\sigma = 0$. There exists, for each stalk, a $\tau \in \mathcal{C}^p(X,\mathcal{F})$ with $\Psi(\tau) = \sigma$, by surjectivity of Ψ. Note now that $\delta\tau = \Phi(\lambda)$ with $\lambda \in \mathcal{C}^{p+1}(X,\mathcal{E})$, because $\Psi(\delta\tau) = \delta\Psi(\tau) = \delta\sigma = 0$, and hence belongs to the kernel of Ψ and hence lies in the image of Φ. Indeed, λ is closed, as $\Phi(\delta\lambda) = \delta\Phi(\lambda) = \delta^2\tau = 0$, which implies that $\delta\lambda = 0$, as Φ is injective. Hence we obtain a map

$$\mathcal{C}^p(X,\mathcal{G}) \ni \sigma \mapsto \lambda \in \mathcal{C}^{p+1}(X,\mathcal{E}),$$

which yields the δ^* operator.

We want to equate the de Rham/Dolbeaut cohomology groups with Čech cohomology groups, as the former can be computed with methods discussed in previous lectures. Note the following important fact: if $\mathcal{E}$ is a **fine** sheaf, then $\check{H}^p(X,\mathcal{E}) = 0$ for $p \geq 1$. In particular, a fine sheaf is a sheaf such that if $\sum \chi_\alpha = 1$ is a partition of unity (with $\operatorname{supp}\chi_\alpha \subset U_\alpha$), then there is a map

$$\Gamma(U_\alpha,\mathcal{E}) \ni s \mapsto \chi_\alpha s \in \Gamma(U_\alpha,\mathcal{E}).$$

In other words, fineness means that there is a sensical way of using partitions of unity, compatible with the sheaf. As an example, consider the sheaf of smooth p-forms – this is clearly fine as we can multiply by smooth bump functions. As a non-example, simply consider something valued in the integers, for which one can obviously not do the same. Similarly for sheafs of holomorphic functions, as bump functions will destroy holomorphicity (recall bump functions are smooth but not analytic). The above observation can be proved fairly simply: let $\sigma \in \mathcal{C}^p(X,\mathcal{E})$, i.e. $\delta\sigma = 0$. Then $\sigma = \delta\tau$, and $\tau_{\alpha_0,\ldots,\alpha_{p-1}} = \sum_\gamma \chi_\gamma \sigma_{\gamma_0,\ldots,\gamma_p-1}$ for $\tau \in \mathcal{C}^{p-1}$. We leave the explicit details as an exercise. As it happens, the sheaves that we will be interested in will not be fine, but we will place them in exact sequences with other sheaves that are fine.