Commutative Algebra: Problem Set 11

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Problem 1

Consider the polynomial $F = X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2$ and the curve $D = V(F)$ in $\mathbb{P}^2$. Let us determine the singular points of F. Note first that $\nabla F = \langle 2X_0X_1^2 + 2X_0X_2^2, 2X_1X_0^2 + 2X_1X_2^2, 2X_2X_1^2 + 2X_2X_0^2 \rangle$. In the open where $X_0 \neq 0$, for $\nabla F = 0$, we see that $X_1 = X_2 = 0$ and hence $[1 : 0 : 0]$ is a singular point, as it clearly lies on D. Similarly, it is easy to see that $[0 : 1 : 0]$ and $[0 : 0 : 1]$ are singular as well.

Problem 2

Let us determine the genus of the curve $D = V(F)$ above. Recall from class that we have the genus-degree formula $g = (d-1)(d-2)/2 = 3$ if D were nonsingular. However, since D has 3 singularities, the genus must fall by at least 3 and hence $g = 0$.

Problem 3

Consider the curve $D = V(X_0^2 + X_1^2 + X_2^2 + X_3^2, X_0^3 + X_1^3 + X_2^3 + X_3^3)$.

(a) A sharp upper bound for the number of intersection points of a plane in $\mathbb{P}^3$ with D is 6. This bound is achieved by the plane $X_3 = 0$, for which we find that $(X_0^3 + X_1^3)^2 + (X_0^2 + X_1^2)^3 = 0$, and hence we obtain the equation:

$$2X_0^6 + 3X_0^4X_1^2 + 2X_0^3X_1^3 + 3X_0^2X_1^4 + 2X_1^6 = 0,$$

which has 6 distinct roots (which can be checked either numerically or by taking derivatives).

(b) We wish to find an irreducible curve D' in $\mathbb{P}^2$ that is birational to D by projection. In particular, we consider the map $\pi(X_0, X_1, X_2, X_3) = (X_0, X_1, X_2)$. Note that we can eliminate X_3 from the two polynomials defining D:

$$\begin{aligned} 0 &= X_0^2 + X_1^2 + X_2^2 + X_3^2 \\ X_3^2 &= -(X_0^2 + X_1^2 + X_2^2) \\ 0 &= X_0^3 + X_1^3 + X_2^3 + X_3^3 \\ X_3^3 &= -(X_0^3 + X_1^3 + X_2^3) \end{aligned}$$

to obtain

$$F(X_0, X_1, X_2) = (X_0^3 + X_1^3 + X_2^3)^2 + (X_0^2 + X_1^2 + X_2^2)^3 = 0.$$

This gives us the equation for D' in $\mathbb{P}^2$. It is now easy to see that the preimage is given by

$$\pi^{-1}(X_0, X_1, X_2) = \left(X_0, X_1, X_2, \frac{X_0^3 + X_1^3 + X_2^3}{X_0^2 + X_1^2 + X_2^2}\right).$$

Of course, for this to be well-defined, we must have $X_0^2 + X_1^2 + X_2^2 \neq 0$ or equivalently, $X_3 \neq 0$. Hence we see that for $X_3 \neq 0$, points in D' have a single preimage in D, but if $X_3 = 0$, there are a number of solutions, as seen in (a).

(c) D' has degree 6.

(d) To compute the singularities of D', we compute:

$$\begin{aligned}\frac{\partial F}{\partial X_i} &= 0 = 6X_i^2(X_0^3 + X_1^3 + X_2^3) + 6X_i(X_0^2 + X_1^2 + X_2^2)\\ 0 &= 6(X_0^2 + X_1^2 + X_2^2)(X_0^3 + X_1^3 + X_2^3 + (X_0^2 + X_1^2 + X_2^2)(X_0 + X_1 + X_2))\end{aligned}$$

Note that if $X_0^2 + X_1^2 + X_2^2 = 0$ we recover the points mentioned in part (a), i.e. the points intersecting the plane $X_3 = 0$ are singular. It is fairly easy to see that the second term has no solution in $\mathbb{P}^2$ and hence D' has only these 6 singularities.

(e) One would guess the genus to be the (possible) genus of D': $5 \cdot 4/2 - 6 = 4$, as genus is a birational invariant.