Riemann Surfaces: Lecture Notes

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Class 10

Recall that we are studying function theory on the torus $\mathbb{C}/\Lambda$, where $\Lambda = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$. We had produced a candidate

$$\sigma(z) = z \prod_{\omega \in \Lambda^\times} \left(1 + \frac{z}{w}\right) e^{-\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}.$$

Let us check that this product converges by examining its logarithm:

$$\begin{aligned} \log\{\cdots\} &= \log(1 + \frac{z}{\omega}) - \frac{z}{\omega} + \frac{z^2}{2\omega^2} \\ &= \left(\frac{z}{\omega} - \frac{1}{2}\frac{z^2}{\omega^2} + \dots\right) - \frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}, \end{aligned}$$

which clearly converges. Hence $\sigma(z)$ is holomorphic for $z \in \mathbb{C}$. Recall that $\sigma'(z)/\sigma(z) = \zeta(z)$ and so $\partial_z \log \sigma(z + \omega_a) = \zeta(z + \omega_a)$. Thus we have (from before) that

$$\eta_a = \zeta(z + \omega_a) - \zeta(z) = \partial_z \log \sigma(z + \omega_a) - \partial_z \log \sigma(z),$$

which gives us periodicity information. Integrating and exponentiating, we see that

$$\sigma(z + \omega_a) = \sigma(z) e^{\eta_a z + c_a}$$

where c_a is the constant of integration, and taking $z = -\omega_a/2$, we find that

$$\sigma(\omega_a/2) = \sigma(-\omega_a/2) e^{-\eta_a \frac{\omega_a}{2} + c_a}.$$

It is easy to check, however, that σ is odd, and hence we find that

$$\sigma(z + \omega_a) = -\sigma(z) e^{\eta_a (z + \frac{\omega_a}{2})}.$$

So we have found that $\sigma(z)$ is holomorphic on $\mathbb{C}$ and that $\sigma(z) = 0$ if and only if $z = 0 \mod \Lambda$. Now that we have constructed such a σ, let us give another proof of Abel's theorem. First recall our previous statement of Abel's theorem.

Theorem 1 (Abel's theorem)**.** *Let $P_1, \dots, P_M, Q_1, \dots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M A(P_i) = \sum_{i=1}^N A(Q_i)$.*

Recall that the Abel map takes $\mathbb{C}/\Lambda \ni p \mapsto A(p) = \int_{p_0}^{p} \omega$ where the value of the integral is taken modulo the lattice generated by $\oint_A \omega, \oint_B \omega$. Take $p_0 = 0$ and $\omega = dz$, which is a well-defined form, and if we take A to align with ω_2 and B to align with ω_1, we see that $\oint_A \omega = \oint_A dz = \omega_1$ and similarly $\oint_B \omega = \omega_2$. Hence the map simply takes p to $\int_0^p dz \mod \Lambda = p$ where p is viewed as a complex number.

Let us now restate Abel's theorem.

Theorem 2 (Abel's theorem, v.2). *Let $P_1, \ldots, P_M, Q_1, \ldots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M P_i = \sum_{i=1}^N Q_i$* mod Λ.

Proof. Consider the function

$$f(z) = \frac{\prod_{i=1}^M \sigma(z - P_i)}{\prod_{i=1}^N \sigma(z - Q_i)}.$$

We should be a little careful to note that σ is a function not on the torus $\mathbb{C}/\Lambda$, but a function on $\mathbb{C}$ (it transforms under a lattice translation!). Hence we must be cognizant of the fact that P_i, Q_i here are some chosen representatives in $\mathbb{C}$ of the equivalence classes of the points P_i, Q_i. It should be clear that $f(z)$ is meromorphic with zeroes at every representative of each P_is and poles at every representative of each Q_i. The natural question, now, is whether this function extends to a function on the torus. To check this, let us see whether it is doubly periodic using what we know about σ:

$$\begin{aligned} f(z + \omega_a) &= f(z) \frac{\prod_{i=1}^M e^{\eta_a (z - P_i)}}{\prod_{i=1}^N e^{\eta_a (z - Q_i)}} \\ &= f(z) e^{-\eta_a \left(\sum_{i=1}^M P_i - \sum_{i=1}^N Q_i \right)}. \end{aligned}$$

Hence we wish to choose P_i, Q_i representatives such that the exponential becomes unity. By hypothesis, this can be done (by shifting one, if necessary). $\square$

Let us now return to Weierstrass theory. Given $\omega = dz$, we defined $\omega_0 = \mathcal{P}(z) dz$ which has a double pole at 0 and $\partial_z \log \sigma(z) = \zeta(z)$ and $\zeta'(z) = -\mathcal{P}(z)$. Now we can construct a form ω_{PQ} with residues $1, -1$ at P, Q respectively, by assigning $\omega_{PQ}(z) = (\zeta(z - P) - \zeta(z - Q)) \omega = \partial_z \log \frac{\sigma(z-P)}{\sigma(z-Q)} dz$. What Weierstrass theory tells us that we can write everything in terms of σ, our analog of z.

Jacobi theory: θ-functions

Consider again the torus $\mathbb{C}/\Lambda$, where we now normalize the lattice as $\Lambda = \{m + n\tau; m, n \in \mathbb{Z}\}$ with $\operatorname{Im} \tau > 0$ (by linear independence, it cannot be real). This simply corresponds to picking $\omega_1 = 1, \omega_2/\omega_1 = \tau$. Next define the **theta-function**

$$\theta(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

in which the structure of the lattice is explicitly clear (unlike in the Weierstrass theory). Let us examine its main properties.

First, note that $\theta(z|\tau)$ is holomorphic in $z \in \mathbb{C}$ because the series converges for all z; this is due to the term

$$|e^{\pi i n^2 (\tau_1 + i\tau_2)}| = |e^{\pi i n^2 \tau_1} e^{-\pi^2 n^2 \tau_2}| = e^{-\pi n^2 \tau_2}$$

for $\tau = \tau_1 + i\tau_2$, whose decay dominates due to the n^2. Next, notice that

$$\begin{aligned}
\theta(z+1|\tau) &= \theta(z|\tau) \\
\theta(z+\tau|\tau) &= e^{-\pi i\tau - 2\pi i z}\theta(z|\tau),
\end{aligned}$$

where the second is obtained by completing the square. Though θ is not invariant, its zeroes are.

Furthermore, we claim that $\theta(z|\tau)$ vanishes at exactly one point modulo lattice translates. It suffices to compute the integral $\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz$, as it yields $2\pi i$ times the difference in the number of zeroes and poles in a given region. We shall integrate over the curve C where C traverses the circumference of one lattice segment (i.e. the whole torus):

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = \oint_B \left(-\frac{\theta'(z|\tau)}{\theta(z|\tau)} + \frac{\theta'(z+1|\tau)}{\theta(z+1|\tau}\right) + \oint_A \left(\frac{\theta'(z|\tau)}{\theta(z|\tau)} - \frac{\theta'(z+\tau|\tau)}{\theta(z+\tau|\tau)}\right).$$

But these are just the shifts in the logarithmic derivative, and since $\partial_z \log\theta(z+\tau|\tau) = -2\pi i + \partial_z \log\theta(z|\tau)$ using the transformation rules above, we see that our integral simplifies to

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = 2\pi i \oint_A dz = 2\pi i.$$

Of course, since θ is holomorphic, it has no poles, and hence we see that we have one zero. The zero, in fact, occurs in the center: $\theta\left((1+\tau)/2|\tau\right) = 0$. To see this, consider the following function:

$$\begin{aligned}
\theta\left(z + \frac{1+\tau}{2}|\tau\right) &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i n\left(z + \frac{1+\tau}{2}\right)\right) \\
&= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})\right) \\
&= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\theta_1(z|\tau)
\end{aligned}$$

where we have completed the square and defined the function θ_1. We claim that θ_1 is an odd function, which would imply that θ_1 vanishes at zero, which would prove the claim about the location of the zero. Hence let us verify that θ_1 is odd; switching $z \mapsto -z$ yields in the exponent

$$\log\theta_1(z|\tau) = \pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right).$$

If we switch the indices $n \mapsto m$ such that $n + \frac{1}{2} = -\left(m+\frac{1}{2}\right)$, we find that the exponent is now

$$\log\theta_1(z|\tau) = \pi i\left(m+\frac{1}{2}\right)^2\tau + 2\pi i\left(m+\frac{1}{2}\right)\left(\left(z+\frac{1}{2}\right) - 2\pi i\left(m+\frac{1}{2}\right)\right),$$

and hence θ_1 is odd. Now we see that the function we want is in fact $\theta_1(z|\tau)$ as it is odd, holomorphic, and has one zero.

We leave it as an exercise to show that

$$\sigma(z) = \omega_1 \exp\left(\eta_1\frac{z^2}{\omega_1}\right)\frac{\theta_1\left(\frac{z}{\omega_1}|\tau\right)}{\theta_1'(0|\tau)}$$

Class 11

We claim that the theta-function theory is more powerful than what we have been using so far - to see this, let us prove Abel's theorem. Recall that the theorem states that there exists a meromorphic f with zeroes at P_i and poles and Q_j if and only if $N = M$ and $\sum_i A(P_i) = \sum_j A(Q_j)$. The idea is to express

$$f(z) = \frac{\prod_{i=1}^N \theta_1(x - P_i)}{\prod_{i=1}^N \theta_1(z - Q_i)}$$

and check double-periodicity. It is an exercise to check that

$$\begin{aligned} \theta_1(z+1|\tau) &= -\theta_1(z|\tau) \\ \theta_1(z+\tau|\tau) &= \exp\left(-\pi i \tau - 2\pi i(z+1/2)\right)\theta_1(z|\tau), \end{aligned}$$

from which periodicity follows easily. Of course, we must be careful to note that the P_i, Q_i used here are in fact chosen representatives.

Next let us define a meromorphic form

$$\omega_{PQ} = \partial_z \log \frac{\theta_1(z-P)}{\theta_1(z-Q)} dz,$$

which, it is easy to check, has poles at P, Q with opposite residues. Additionally, one can check that this expression is well-defined on the lattice, i.e. invariant under a shift. We leave it as a simple exercise to show that

$$\omega_P(z) = \partial_z^2 \log \frac{\theta_1(z-P|\tau)}{\theta_1'(0|\tau)} dz$$

is a meromorphic form with a double pole at P and is well-defined on the lattice.

But in fact, we can go even farther with this theta-function. Indeed, one attractive feature is that there exists a product expansion for $\theta(z|\tau)$.

Theorem 3. *We can expand*

$$\theta(z|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z})$$

where $q \equiv e^{\pi i \tau}$.

Proof. Define

$$T(z|\tau) = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}e^{2\pi i z})(1+q^{2n-1}e^{-2\pi i z}).$$

We claim that $T(z|\tau)$ is equal to zero exactly when z is $(1+\tau)/2 \mod \Lambda$ and that the zeros are simple. This can be checked by some simple algebra. It's also easy to show that $T(z|\tau)$ is holomorphic in $\mathbb{C}$ and that $\tau(z+1|\tau) = T(z|\tau)$. Moreover

$$\begin{aligned} T(z+\tau|\tau) &= \prod_{n=1}^{\infty}(1-q^{2n})\prod\left(1+q^{2n+1}e^{2\pi i z}\right)\left(1+q^{2n-3}e^{-2\pi i z}q^{-2}\right) \\ &= \prod_{n=1}^{\infty}(1-q^{2n})\frac{\prod_{n=1}^{\infty}\left(1+q^{2n-1}e^{2\pi i z}\right)}{1+qe^{2\pi i z}}\prod_{n=1}^{\infty}\left(1+q^{2n-1}e^{-2\pi i z}\right) \quad = \frac{1-q^{-1}e^{-2\pi i z}}{1-qe^{2\pi i z}}T(z|\tau) = q^{-1}e^{-2\pi i z}. \end{aligned}$$

Recall that θ followd a similar condition. This shows that $\theta(z|\tau)/T(z|\tau) = c$, where c is a constant independent of z that can depend on τ. Next we claim that $c(\tau) = 1$. For this we show that there exists a c such that $c(\tau) = c(4\tau) = c(4^k\tau)$ and $c(\tau) = \lim_{k\to\infty} c(4^k\tau) = 1$, which shows the proof. Hence let us prove that $c(\tau) = c(4\tau)$ using $\theta(z|\tau) = C(\tau)T(z|\tau)$.

Take $z = 1/2$. Then $e^{2\pi i z = e^{\pi i}} \geq 1-$ and $\theta(1/2|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}(-1)^n$ but $T(1/2|\tau) = \prod_{n=1}^{\infty}(1-q^{2n)(1-q^{2n-1}}(1-q^{2n-1})$. and hence $c(\tau) = e^{\pi i n\tau}(-1)^n/\prod(1-q^n)(1-q^{2n-1})$. Next take $z = 1/4$ $\square$