Introduction to Algebraic Topology PSET 9

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Proposition 1. *Problem 1*

Proof.

(a) Consider the Δ-complex structure on X as follows:

(b) The chain complex associated to X is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^4 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0 \tag{1}$$

with $\partial_2 U = a + d - c, \partial_2 L = a + b - d, \partial_1 a = \partial_1 c = -\partial_1 b = v_0 - v_1$, and $\partial_1 d = 0$. Clearly $H_0^\Delta(X) = \mathbb{Z}$ and $H_2^\Delta(X) = \ker \partial_2 = 0$. Finally,

$$\begin{aligned} H_1^\Delta(X) &= \frac{\ker \partial_1}{\text{im } \partial_2} = \frac{\langle d, a - c, a + b \rangle}{\langle a + d - c, a + b - d \rangle} \\ &= \frac{\langle d, a - c, a + b - d \rangle}{a + d - c, a + b - d \rangle} = \frac{\langle a - c, d \rangle}{\langle a + d - c \rangle} \\ &= \mathbb{Z}. \end{aligned}$$

Next, if we consider $A = \partial X$ with the inherited Δ-complex structure, we find the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0 \tag{2}$$

with $H_0^\Delta(A) = \mathbb{Z}$ and $H_1^\Delta(A) = \ker \partial_1 = \mathbb{Z}$.

(c) Let $i : A \to X$ be the usual inclusion. Then $i_* : H_0^\Delta(A) \to H_0^\Delta(X)$ is an isomorphism, taking $v_0 \in H_0^\Delta(A)$ to $v_0 \in H_0^\Delta(X)$ respectively. Similarly, $i_* : H_1^\Delta(A) \to H_1^\Delta(X)$ takes $b + c \mapsto 2d$, as d is the generator for $H_1^\Delta(X)$ and the relations $a + b - d = 0, a + d - c = 0$ yield $2d = b + c$.

(d) We obtain a long exact sequence

$$0 \longrightarrow H_2(X, A) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow H_1(X, A) \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow H_0(X, A) \longrightarrow 0 \tag{3}$$

By exactness and part (c), it follows that all $H_n(X, A) = 0$ except for $n = 1$. In that case, since the first i_* is multiplication by two (taking the generator $b + c$ to twice the generator, d), we find that $H_1(X, A) = \mathbb{Z}_2$.

$\square$

Proposition 2. *Hatcher 2.2.1*

Proof. Suppose $f : D^n \to D^n$ has no fixed points. Then, define a map $g : S^n \to S^n$ that takes the northern hemisphere to the southern via reflection and then applies f, while on the southern hemisphere simply applies f. Note that the southern hemisphere is homeomorphic to D^n, and hence g has degree $(-1)^{n+1}$ as f has no fixed points. This contradicts that g has degree zero (as it is not surjective). Hence f must have at least one fixed point. $\square$

Proposition 3. *Hatcher 2.2.2*

Proof. Consider $f : S^{2n} \to S^{2n}$. If f has a fixed point, we are done. Otherwise, we find that $\deg f = (-1)^{2n+1} = -1$. Suppose now that $f(x) \neq -x$ for all x. Then the line from $f(x)$ to x does not pass through the origin. The map $g_t(x) = ((1-t)f(x) + tx) / |(1-t)f(x) + tx|$ thus furnishes a homotopy from f to the identity. This is a contradiction, as the identity has degree 1. Hence there must exist an x such that $f(x) = -x$.

An easy corollary of this fact is that every map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ must have a fixed point, because $\mathbb{RP}^{2n}$ can be seen as S^{2n} with antipodal points identified: hence there will always exist an $[x]$ such that $\tilde{f}([x]) = [x]$ or $\tilde{f}([x]) = [-x] = [x]$. We can construct a map $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$ without fixed point by finding a linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no eigenvectors. This can be done by consider the transformation

$$T = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. The characteristic equation is given by $\lambda^{2n} + 1 = 0$, which has no real roots, as desired. $\square$

Proposition 4. *Hatcher 2.2.4*

Proof. Consider the map $p : S^n \to D^n$ given by projection, and the quotient map $q : D^n \to D^n/\partial D^n = S^n$. The composition $qp : S^n \to S^n$ is clearly surjective for $n \geq 1$, but has degree zero, as the induced map q_* on homology is zero by contractibility of D^n. $\square$

Proposition 5. *Hatcher 2.2.6*

Proof. If $f : S^n \to S^n$ has fixed points, we are done. Otherwise, if f has no fixed points, $\deg f = (-1)^{n+1}$. If n is even, $\deg f = -1$, and we can homotope f to a reflection, which has fixed points. If n is odd, $\deg f = 1$, and we can homotope f to the identity, which fixes every point. $\square$