Representation Theory PSET 1

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Proposition 1. *Consider the category of topological abelian groups with continuous homomorphisms between them. Show this category is additive but not abelian.*

Proof. We first show that this category is additive, which is straightforward. Let A, B be topological abelian groups, and note that $\mathrm{Hom}(A, B)$ is an abelian group, as any two morphisms $f, g : A \to B$ can be added (or subtracted) pointwise to obtain another continuous homomorphism. This operation obviously forms a group (with identity the zero morphism). Note that composition of two morphisms is bilinear, as

$$f(g + h)(x) = f(g(x) + h(x)) = f(g(x)) + f(h(x))$$

and

$$(f + g)(h)(x) = f(h(x)) + g(h(x)).$$

The zero object in this category is the trivial group 0 (topologically a point); clearly $\mathrm{Hom}(0, 0) = \{0\}$, the zero morphism. Finally we wish to show the existence of a product. Given topological abelian groups A and B, we define the product as usual to be $A \oplus B = \{(a, b) \mid a \in A, b \in B\}$, where the group operation is performed componentwise. It is clear that $A \oplus B$ is an abelian group; it can be given the usual product topology induced by the topologies of A and B. The new group operation is continuous simply by the componentwise continuity, and similarly for inversion. It is easy to check that $A \oplus B$ satisfies the universal property of products. Similar statements hold for $A \oplus B$ as a coproduct. Hence the category of topological abelian groups is additive.

This category is not abelian, as the following counterexample shows. Consider the group $(\mathbb{R}, +)$ with the usual topology and the group $(\mathbb{Q}, +)$ with the subspace topology. It is clear that $\mathbb{R}$ and $\mathbb{Q}$, so defined, form topological abelian groups. The inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is a homomorphism because $\iota(q_1 + q_2) = q_1 + q_2 = \iota(q_1) + \iota(q_2)$ and continuous by construction of the subspace topology. Suppose the cokernel of ι exists. By definition, the cokernel is a choice of $\operatorname{coker} \iota$ and ψ making the diagram

$$\begin{tikzcd} \mathbb{Q} \arrow[rr, "\iota"] \arrow[dr, "0"'] & & \mathbb{R} \arrow[dl, "\psi"] \\ & \operatorname{coker} \iota & \end{tikzcd}$$

commute. But this means that $\psi(q) = 0$ for all $q \in \mathbb{Q}$. As any real number x can be written as $x = \lim_{i\to\infty} x_i$ for some sequence of rationals x_i, continuity implies that $\psi(x) = \psi(\lim_{i\to\infty} x_i) = \lim_{i\to\infty} \psi(x_i) = 0$. Hence ψ is, in fact, the zero map. Notice that if we consider $\operatorname{coker} \iota$ to be the trivial group 0, it satisfies the universality condition: for any map $\tilde{\psi} : \mathbb{R} \to A$ for some topological abelian group A, we know that $\tilde{\psi}$ is the zero map using the same reasoning as above, and hence $\tilde{\psi}$

clearly factors uniquely through $\operatorname{coker}\iota$ via $0_{0\to A}\circ\psi$. By the universal property, then, $\operatorname{coker}\iota = 0$. Now note that

$$\text{Image } \iota = \ker\psi = \mathbb{R},$$

which contradicts the fact that the image of the inclusion of $\mathbb{Q}$ in $\mathbb{R}$ is just $\mathbb{Q}$. Hence the cokernel of ι cannot exist, and the category of topological abelian groups is not abelian. □

Proposition 2. *Let $S \to R$ be a homomorphism of rings. It induces the restriction and induction functors between the corresponding categories of modules, Res : R-Mod → S-Mod and Ind : S-Mod → R-Mod, where Ind $A = R\otimes_S A$ and Res B is the module B viewed as an S-module. Show that these functors are adjoint in the sense that* $\operatorname{Hom}_R(\textit{Ind } A, B) = \operatorname{Hom}_S(A, \textit{Res } B)$.

Proof. Recall that for the two functors Res and Ind to be adjoint, there must be a natural bijection for all $A\in$ S-Mod and $B \in$ R-Mod

$$\tau_{AB} : \operatorname{Hom}_R(\text{Ind } A', B) = \operatorname{Hom}_S(A, \text{Res } B).$$

The bijection can be constructed as follows. Given any $\psi \in \operatorname{Hom}_R(\text{Ind } A, B)$ we define a map $\tilde{\psi} : A \to \text{Res } B$ that takes $a \mapsto \psi(1\otimes a)$. This association is injective, as it depends only on ψ: different ψ yield distinct $\tilde{\psi}$. Conversely, given any map $\tilde{\psi} : A \to$ Res B, we define the map $\psi : \text{Ind } A \to B$ to take $r\otimes_S a \mapsto r\cdot\tilde{\psi}(a)$. This association is also injective, and hence we have a bijection, call it $\tau_{AB} : \operatorname{Hom}_R(\text{Ind } A', B) = \operatorname{Hom}_S(A, \text{Res } B)$.

For this bijection to be natural, for any $f : A \to A', g : B \to B'$ the diagrams

$$\begin{array}{ccc}
\operatorname{Hom}_R(\text{Ind } A', B) & \xrightarrow{\text{Ind } f^*} & \operatorname{Hom}_R(\text{Ind } A, B) \\
\downarrow{\scriptstyle \tau_{A'B}} & & \downarrow{\scriptstyle \tau_{AB}} \\
\operatorname{Hom}_S(A', \text{Res } B) & \xrightarrow{f^*} & \operatorname{Hom}_S(A, \text{Res } B)
\end{array}$$

and

$$\begin{array}{ccc}
\operatorname{Hom}_S(A, \text{Res } B) & \xrightarrow{\text{Res } g^*} & \operatorname{Hom}_S(A, \text{Res } B') \\
\downarrow{\scriptstyle \tau^{-1}_{AB}} & & \downarrow{\scriptstyle \tau^{-1}_{AB'}} \\
\operatorname{Hom}_R(\text{Ind } A, B) & \xrightarrow{g^*} & \operatorname{Hom}_R(\text{Ind } A, B')
\end{array}$$

must commute. Let us start with the first diagram, on the top left. Given a $\psi : \text{Ind } A' \to B$, we want that

$$f^*(\tau_{A'B}(\psi)) = \text{Ind } f^*(\tau_{AB}(\psi)).$$

It is straighforward to see that both of these are equal to a map that takes $a \mapsto \psi(1\otimes f(a))$. Similarly, given a $\tilde{\psi} : A \to \text{Res } B$, we want that

$$\tau^{-1}_{AB'}(\text{Res } g^*(\psi)) = g^*(\tau^{-1}_{AB}(\psi)).$$

Both of these yield a map that takes $r\otimes a \mapsto r\cdot g(\tilde{\psi}(a))$. Hence the bijection τ_{AB} is natural, and we find that Res and Ind are adjoint functors. □

Proposition 3. *Show that every module in category $\mathcal{O}$ is finitely generated as a $\mathcal{U}\mathfrak{n}_-$-module.*

Proof. Take some M in $\mathcal{O}$. We know that M is a finitely-generated $\mathcal{U}\mathfrak{g}$-module, say with generators v_i. Note that $\mathcal{U}\mathfrak{g}\cdot v_i = \mathcal{U}\mathfrak{n}_- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}\cdot v_i$ generates M. Furthermore, $\mathcal{U}\mathfrak{n}\cdot v_i$ is a finite dimensional vector space for each i. Acting by $\mathcal{U}\mathfrak{h}$ simply scales, and thus acting by $\mathcal{U}\mathfrak{n}_-$ must generate M from these finite dimensional vector spaces. Hence if e_{j_i} is a basis for $\mathcal{U}\mathfrak{n}\cdot v_i$, the e_{j_i} generate a finite-dimensional $\mathcal{U}\mathfrak{b}$-submodule of M that generates M as a $\mathcal{U}\mathfrak{n}_-$-module. □

Proposition 4. *For $V_1, V_2 \in$ Obj $\mathcal{O}$, consider the tensor product $V_1 \otimes V_2$. Show that it is in $\mathcal{O}$ is one of the factors is finite-dimensional, but not in general.*

Proof. Let $\mathfrak{g}$ semisimple have the root decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Recall that the BGG category $\mathcal{O}$ is defined to be the full subcategory of $\mathcal{U}\mathfrak{g}$-Mod whose objects M satisfy the conditions

1. M is a finitely generated $\mathcal{U}\mathfrak{g}$-module;
2. M is $\mathfrak{h}$-semisimple, that is, M is a weight module: $M = \oplus_{\lambda\in\mathfrak{h}^*} M_\lambda$;
3. M is locally $\mathfrak{n}$-finite: for each $v \in M$, the subspace $\mathcal{U}\mathfrak{n}\cdot v$ of M is finite dimensional.

Now let $V, W \in$ Obj $\mathcal{O}$. Consider the tensor product $V \otimes W$. We claim that $V \otimes W$ satisfies properties 2 and 3, but will satisfy property 1 only if one of V, W is finite-dimensional. Hence $V \otimes W$ is in category $\mathcal{O}$ if and only if one of V, W is finite-dimensional.

It is fairly obvious that condition 2 is preserved: given $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ and $W = \oplus_{\mu\in\mathfrak{h}^*} W_\mu$. Then we can write:

$$V \otimes W = \bigoplus_{\lambda,\mu\in\mathfrak{h}^*} V_\lambda \otimes W_\mu.$$

Since the Lie algebra acts on tensor products as derivations, $V_\lambda \otimes W_\mu$ is a space with weight $\lambda+\mu$. We can regroup the direct sum to be over all possible $\nu = \lambda + \mu$, which gives us a weight decomposition for $V \otimes W$. Now let us check condition 3. Take some $v \otimes w \in V \otimes W$. The action of some $n \in \mathfrak{n}$ on $v \otimes w$ is as

$$n \cdot (v \otimes w) = n \cdot v \otimes w + v \otimes n \cdot w.$$

Repeated application via the action of $\mathcal{U}\mathfrak{n}$ will annihilate $v \otimes w$, as repeated application of $\mathcal{U}\mathfrak{n}$ annihilates by hypothesis v and w.

Next we show that condition 1 is preserved if one of V or W is finite-dimensional; Let V be finite dimensional. Denote by $\{v_i\}, \{w_i\}$ the basis of of V and the generators of W respectively. Let us show that $v_i \otimes w_j$ form a set of generators of $V \otimes W$. Denote by M the submodule of $V \otimes W$ generated by $v_i \otimes w_j$. Clearly $v \otimes w_j \in M$ for any $v \in V$. If we act by some $X \in \mathfrak{g}$, we find that

$$X \cdot (v \otimes w_j) = X \cdot v \otimes w_j + v \otimes X \cdot w_j.$$

The left hand side and the first term on the right are contained in M so $v \otimes X \cdot w_j \in M$. Repeated application of $\mathfrak{g}$ shows that $v \otimes p \cdot w_j \in M$ where p is a PBW monomial, but since the w_j generate W under the application of such monomials, we find that M is in fact all of $V \otimes W$.

Finally, let us show that the tensor product of two infinite-dimensional modules in $\mathcal{O}$ does not lie in $\mathcal{O}$. Take $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$ and consider the two Verma modules M_λ, M_μ for $\lambda \neq \mu$ both not even. The weights of M_λ are $\lambda - 2i$ for $i \in \mathbb{Z}$ and $\mu - 2j$ for $j \in \mathbb{Z}$ because there exist vectors $v_\lambda \in M_\lambda$ and $v_\mu \in M_\mu$ that generate the respective modules. Consider now $M_\lambda \otimes M_\mu$. Suppose that $m_i \in M_\lambda \otimes M_\mu$ are a finite set of generators. The m_i can be written as finite sums of fundamental tensors $v_a \otimes v_b$ (with weight, say, $a + b$). Now consider the vector $v_x \otimes v_y + \sum_i m_i$, where $x + y$ is not a weight that already appears in any of the m_i and whose difference from such weights is not a factor of two. This is possible since the Verma modules have infinitely negative weights and λ, μ are not both even. For a concrete example, consider $M_2 \otimes M_3$. Take the vector $v_3 \otimes v_2 + v_1 \otimes v_2$, for example: it which has monomials of weights of 5 and 3. The claim is simply that any finite set of generators m_i will not be able to generate $v_x \otimes v_y + \sum_i m_i$ if x and y are suitably chosen (e.g. if our generating set is $\{v_3 \otimes v_2, v_1 \otimes v_2\}$ then $v_3 \otimes v_2 + v_1 \otimes v_2 + v_1 \otimes v_0$ is not spanned by the set). Hence there is no finite $\mathcal{U}\mathfrak{g}$-generating set for $M_\lambda \otimes M_\mu$. $\square$

Proposition 5. *Consider the action of $GL(3)$ on polynomials of degree d in x_1, x_2, x_3. Resolve this representation by Verma modules.*