Introduction to Algebraic Topology: Class Notes

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Class 1

Remark. The goal of algebraic topology is to develop algebraic tools to study topological spaces. To this end, all maps are assumed to be continuous unless otherwise stated.

0.1 Homotopy

Definition 1. A **deformation retract** of a space X onto a subspace A is a family of maps $f_t : X \to X$ for $t \in I = [0,1]$ such that

1. $f_0 = \mathrm{Id}_X$
2. $f_1(X) = A$
3. $f_t|_A = \mathrm{Id}_A$ for all t
4. f_t is continuous as a map $X \times I \to X$ sending $(x,t) \mapsto f_t(x)$.

Example 1. Consider the annulus $X = \{x \in \mathbb{C} \mid 1/2 \leq |x| \leq 3/2\}$ with the subspace $A = \{x \in \mathbb{C} \mid |x| = 1\}$. We can construct a deformation retract of X onto A as

$$f_t(x) = \frac{x}{1 - t + t|x|}.$$

Definition 2. Given $f : X \to Y$, we define the **mapping cylinder** M_f to be the quotient space

$$(X \times I) \sqcup Y / \sim$$

where $(x,1) \sim f(x)$. Note that M_f deformation retracts to Y by sliding each point (x,t) along $\{x\} \times I \subset M_f$ to $\{x\} \times \{1\} = f(x) \in Y$.

Example 2. If $f : X \to Y$ is an inclusion, say of the circle into the plane, then the mapping circle is simply a cylinder attached to the plane.

Definition 3. A **homotopy** is a family of maps $f_t : X \to Y$ for $t \in I$ such that $F : X \times I \to Y$ given by $F(x,y) = f_t(x)$ is continuous. We say that two maps f_0 and f_1 from $X \to Y$ are **homotopic** if there exists a homotopy f_t connecting them. We write this as $f_0 \simeq f_1$.

Definition 4. A **retraction** of X onto A is a map $r : X \to X$ such that

1. $r(X) = A$
2. $r|_A = \mathrm{Id}_A$

In this case, A is called a **retract** of X. Note that $r^2 = r$, as r is the identity on its image. does this character-ize r?

Example 3. A deformation retract is a homotopy between the identity map to a retraction.

Example 4. Let X be any space with $x_0 \in X$. Consider the map $f : X \to X$ given by $x \mapsto x_0$. This is clearly a retract.

One might ask when there exists a deformation retract $X \to \{x_0\}$. For example, given an annulus and some x_0 inside, does there exist a deformation retract? The answer is in fact no, and later we develop some tools to show this. Similarly for the disjoint union of two disks, with x_0 contained in one of them.

Definition 5. Let $A \subset X$. A homotopy $f_t : X \to Y$ such that $f_t|_A$ is independent of t. Then we say that f_t is a **homotopy relative to** A.

Example 5. For example, a deformation retract of X onto A is a homotopy rel A.

Example 6. Consider the unit disk $X = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with $f_t(z) = ze^{2\pi it}$. In this case, f_t is a homotopy rel O. Note that this is a homotopy from the identity to itself.

Remark. Let $f_t : X \to X$ be a deformation retract of X onto A. Let $r = f_1$ be the resulting retract and let $i : A \to X$ be the inclusion. Then $r \circ i = \mathrm{Id}_A$ and $i \circ r \simeq \mathrm{Id}_X$ (via the homotopy f_t). In this sense r and i are inverses up to homotopy.

Definition 6. A map $f : X \to Y$ is a **homotopy equivalence** if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. Then we say that X and Y are **homotopy equivalent** and that they have the same **homotopy type** written $X \simeq Y$.

Definition 7. A space that is homotopy equivalent to a point is called **contractible**. This is equivalent to asking that the identity map be homotopic to a constant map, i.e. **nullhomotopic**.

Example 7. The spaces $\mathbb{R}^n$ and D^n are contractible, while S^n is not.

Remark. It will be useful to think of the sphere as $S^n = D^n/\partial D^n$.

0.2 Operations on spaces

We are of course already familiar with products and quotients.

Definition 8. Given a space X, we define the **cone of** X CX to be $X \times I/X \times \{0\}$. The prototypical example is given by $S^1 \times I/S^1 \times \{0\}$.

Definition 9. Given a space X, we define the **suspension of** X SX to be $X \times \{0\}$ collapsed to a point and $X \times \{1\}$ collapsed to another point. Note that suspension is a functor, and we can suspend maps as well.

Class 2

Definition 10. We define the **wedge sum** of two topological spaces as follows. Pick $x_0 \in X, y_0 \in Y$, and define

$$X \vee Y = X \sqcup Y/\sim$$

where $x_0 \sim y_0$.

Definition 11. We define the **join** of two spaces to be

$$X * Y = X \times Y \times I / \sim$$

where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. In other words, at $t = 0$, the product collapses to X while at $t = 1$, it collapses to Y. As an example, consider $X = Y = I$. In this case, the product is a cube and the join becomes a tetrahedron. Note that one can also consider the join $X * Y$ as formal convex linear combinations of points of X and Y.

Definition 12. We define the **smash product** of two spaces with $x_0 \in X, y_0 \in Y$ as

$$X \wedge Y = X \times Y / X \vee Y,$$

where the wedge sum is taken at x_0, y_0 and we are consider $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ as sitting inside $X \times Y$. One can it is true that $S^n \wedge S^m = S^{n+m}$.

0.3 CW-complexes

The motivation of CW-complexes is to build higher dimensional surfaces from polygons. The analogy is to the torus, which can be represented by a square in the plane with opposite edges oriented. This, in fact, can be extended to higher-genus surfaces. Note that the interior of such a polygon is a 2-cell, i.e. open 2-disk. The edges, on the other hand, are 1-cells, i.e. open 1-disks/intervals. Finally, we also have 0-cells as points. To build a torus, then, we take a 0-cell and add two 1-cells as the characteristic circles of the torus, and we attach a 2-cell to the 1-cells in the appropriate manner.

Definition 13. We define a **cell complex** (or **CW-complex**), built inductively as follows:

1. Start with X^0, as discrete set of points;
2. Form the n**-skeleton** X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi : S^{n-1} = \partial D^n \to X^{n-1}$. Hence $X^n = X^{n-1} \sqcup D^n_\alpha / x \sim \phi_\alpha(x)$, where $x \in \partial D^n$. Hence, setwise $X^n = X^{n-1} \sqcup e^n_\alpha$ where each e^n_α is an open n-disk.
3. Stop at some finite n to obtain a **finitely-generated cell complex** $X = X^n$ or continue indefinitely to obtain a cell complex $X = \cap_{n=0}^{\infty} X^n$. In the infinite case, we consider the **weak topology**, where $A \subset X$ is open if and only if $A \cap X^n$ is open for all n.

Example 8. Let us consider some examples of cell complexes. The most obvious such example is a graph, which is a 1-dimensional complex, with vertices as 0-cells and edges as 1-cells. We can construct a sphere S^1 by simply taking a 0-cell x_0 and attaching a 1-cell (via the constant attaching map sending boundaries to x_0). The same holds for S^n, where we take an n-cell and attach its boundary to a 0-cell (the attaching map again constant).

Next, we can construct the torus via a 0-cell e^0, 2 1-cells e^1_1 and e^1_2, and a 2-cell e^2. The attaching maps for the 1-cells are obvious. The boundary S^1 of e^2 is attached by going first positively along e^1_1, then positively along e^1_2, then negatively along e^1_1, and finally, negatively along e^1_2.

As a final example, recall the definition of the real projective space $\mathbb{R}P^n$. We can treat it as a quotient of D^n by identifying antipodal points. But this is simply D^n with antipodal boundary points identified. But the boundary is simply S^{n-1}, and hence identifying its boundary's antipodal points yields $\mathbb{R}P^{n-1}$. In this sense, we can consider $\mathbb{R}P^n$ as $\mathbb{R}P^{n-1} \sqcup D^n / x \sim \phi(x)$ where $\phi : S^{n-1} \to \mathbb{R}P^{n-1}$ is the quotient map. Hence we can start with a 0-cell and then attach a 1-cell by the antipodal quotient map to get $\mathbb{R}P^1$. More generally, we can write $\mathbb{R}P^n$ as the complex consisting of a 0-cell, 1-cell, 2-cell, ..., n-cell, where the attaching maps are all antipodal quotient maps.

Definition 14. Given a CW-complex X, each cell e^n_α has a **characteristic map** $\Phi_\alpha : D^n \to X$ which

1. extends the attaching map $\phi_\alpha : S^{n-1} \to X^{n-1}$;
2. is a homeomorphism from the interior of D^n_α to the cell e^n_α.

Note that Φ_α is the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$.

Example 9. For the sphere, the characteristic map is $\Phi : D^n \to S^n$, which collapses ∂D^n to a point. For $\mathbb{R}P^n$, on the other hand, the characteristic map is $\Phi : D^i \hookrightarrow \mathbb{R}P^i \subset \mathbb{R}P^n$ which identifies antipodal points on ∂D^i.

Definition 15. A **subcomplex** of a CW-complex X is a closed subspace $A \subset X$ that is a union of cells of X. We call (X, A) a **CW pair**.

Remark. Note that A is itself a CW-complex.

Example 10. Consider one of the 1-cells in the cell decomposition for the torus. This 1-cell, together with the 0-cell of the torus, yields a subcomplex. As another example, note that $\mathbb{R}P^k \subset \mathbb{R}P^n$ for $k \leq n$ is a subcomplex.

As a non-example, note that S^1 is not a subcomplex of S^2 with the usual cell structure, though it is a subspace. However, one can choose a different cell structure for S^2 in which we create the equator from two 0-cells and two 1-cells and two 2-cells (the northern and southern hemispheres), and in this structure, S^1 is indeed a subcomplex of S^2. More generally, one can obtain a CW-complex for S^n by attaching two n-cells to S^{n-1}, though this becomes difficult to visualize.

Definition 16. If X, Y are CW complexes, their **product** $X \times Y$ has the structure of a CW complex with cells $e^m_\alpha \times e^n_\beta$ where e^m_α ranges over cells in X and e^n_β ranges over cells in Y.

Example 11. The cell decomposition of $S^1 \times S^1$ will be $\{e^0, e^1\} \times \{e^0, e^1\} = \{e^0, e^1_1, e^1_2, e^2\}$, which is the usual decomposition we have for the torus.

Definition 17. If (X, A) is a CW pair, then the **quotient** X/A inherits a natural CW structure which is the usual cells structure away from A but has e^0 replacing A.

Example 12. If we take a 0-cell and a 1-cell of the torus as a subcomplex and quotient by it, we obtain a 2-sphere, as we are left with $\{e^0, e^2\}$.

0.4 Homotopy equivalence

Let us now present two criteria for determining homotopy equivalence.

Lemma 1. *If (X, A) is a CW pair consisting of a CW complex X and a contractible subspace A, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Example 13. For example, one might use this to determine homotopy equivalence for graphs. Indeed, if an edge has distinct endpoints, one can simply collapses the edge to a point. After repeatedly applying the above, all edge are loops, and each component of X is either a single vertex or a wedge of circles $\vee_m S^1$. Hence the question becomes: how can we prove that $\vee_m S^1 \simeq \vee_n S^1$ if and only if $m = n$? We will soon develop the tools needed to answer this question.

Example 14. Consider the space X which is a union of a torus with a disk embedded inside it. Then X/A is homotopy equivalent to X, and we get a croissant shape.

Lemma 2. *Given spaces X_0 and X_1, we can attach X_0 to X_1 by identifying points in a subspace $A \subset X_1$ with points in X_0. That is, given a map from $A \to X_0$, we form a quotient space $X_0 \sqcup_f X_1 \equiv X_0 \coprod X_1/a \sim f(a)$ for all $a \in A$. If (X_1, A) is a CW pair and the two attaching maps $f, g : A \to X_0$ are homotopic then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$.*

Example 15. Recall the mapping cylinder of two spaces X and Y with $f : X \to Y$. We consider $X_1 = X \times I, A = X \times \{1\}, X_0 = Y$.

Another example is given by $X_0 = X_1 = S^1 \times D^2$ and $A = \partial X_1$. Taking $f : \partial X_1 \to \partial X_0 \subset X_0$ by $(x, y) \mapsto (y, x)$. Then $X_0 \sqcup_f X_1 = S^3$.

Consider the 2-sphere with a subcomplex A be an arc from the north to the south pole. Take this arc and send its points to S^1 in a way that wraps around all of S^1. The attachment via this map yields another croissant. Of course, this map is homotopic to a constant map (since A is contractible). Attaching via this constant map yields the wedge $S^2 \vee S^1$. This shows that the croissant is homotopy equivalent to $S^2 \vee S^1$.

0.5 The fundamental group

We will be associating topological spaces to algebraic structures (such as groups or rings) and continuous maps get associated to group homomorphisms. In the case of the fundamental group, for example we will obtain a covariant functor. The fundamental group will be useful as something that can tell various spaces apart.

Class 3

Definition 18. A **path** in a space X is a continuous map $f : I \to X$.

Definition 19. A **homotopy of paths** in X is a family $f_t : I \to X$, $t \in I$ such that

1. the endpoints $f_t(0) = X_0$ and $f_t(1) = x_1$ are independent of t;

2. the associated map $F : I \times I \to X$ is continuous.

If these conditions are satisfied, then f_0 and f_1 are **homotopic**, denoted $f_0 \simeq f_1$.

Example 16. Let f_0, f_1 be two paths in $\mathbb{R}^n$ with the same endpoints x_0 and x_1. Then $f_0 \simeq f_1$ via the linear homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$. Here we have used only the convexity of $\mathbb{R}^n$; more generally, then, this is true in any convex subset of $\mathbb{R}^n$.

Proposition 1. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation. We will denote the homotopy class of a path f by $[f]$.*

Proof. Reflexivity is clear via the constant homotopy, and symmetric by taking f_{1-t} in place of f_t. Finally, for transitivity, suppose that $f_0 \simeq f_1$ via f_t and $f_1 = g_0$ and $g_0 \simeq g_1$ via g_t. Then $f_0 \simeq g_1$ by the homotopy that traverses f_{2t} and then g_{2t-1}. Note that this is well-defined at $t = 1/2$ and that the associated map $H(s, t)$ is continuous because a function on the union of two closed sets is continuous if the restriction to each of the closed sets is continuous. □

Definition 20. Let f and g be paths such that $f(1) = g(0)$. The **composition** or **product path** $f \diamond g$ is $f(2s)$ for $0 \le s \le 1/2$ and $g(2s-1)$ for $1/2 \le s \le 1$.

Remark. Note that the product operation on paths respects homotopy classes. In particular, if $f_0 \simeq f_1$ and $g_0 \simeq g_1$, via f_t and g_t respectively, and $f_0(1) = g_0(0)$, then $f_t \diamond g_t$ provides a homotopy between $f_0 \diamond g_0$ and $f_1 \diamond g_1$.

Definition 21. A **reparametrization** of a path f is a composition $f \circ \phi$ where $\phi : I \times I$ is a continuous map such that $\phi(0) = 0$ and $\phi(1) = 1$.

Remark. Note that $f \circ \phi \simeq f$ via $f \circ \phi_t$ where $\phi_t(s) = (1-t)\phi(s) + ts$.

Definition 22. A **loop** is a path $f : I \to X$ with $f(0) = f(1) = x_0$. The point x_0 is the **basepoint**.

Definition 23. The set of all homotopy classes $[f]$ of loops $f : I \to X$ (or equivalently $f : S^1 \to X$) at the basepoint x_0 is $\pi_1(X, x_0)$, the **fundamental group of X at basepoint** x_0.

Proposition 2. *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \diamond g]$.*

Proof. Since we are restricted to loops with a fixed basepoint x_0, we can always concatenate them. The product $[f][g] = [f \diamond g]$ is well-defined since we showed that the product respects homotopy classes. Let us now check the group axioms. The product of two loops at x_0 is again a loop at x_0, and hence the set is closed under the operation. Let c be the constant path at x_0, that is $c(s) = x_0$ for all $s \in I$. Let ϕ be the reparametrization that traverses f in the first half, and stays at the endpoint $f(1)$ for the second half. Then $f \circ \phi \simeq f \diamond c$ so $f \diamond c \simeq f$. Similarly, if ϕ stays at $f(0)$ for the first half and then traverse f in the second half, we find that $f \circ \phi \simeq c \diamond f$ and hence $c \diamond f \simeq f$. Hence c is the identity of $\pi_1(X, x_0)$.

Next let us show that the group operation is associative. Given loops f, g, h, we want to show that $f \diamond (g \diamond h) \simeq (f \diamond g) \diamond h$. But note that $f \diamond (g \diamond h) \simeq (f \diamond g) \diamond h \circ \phi$ where ϕ sends $1/2$ to $1/4$ and $3/4$ to $1/2$ and is linear everywhere else, and hence associativity holds.

Next let us determine inverses. Given a path f, define the inverse path to be $\bar{f}(s) = f(1-s)$. Let f be a loop. First note that $f \diamond \bar{f} \simeq c$ via $h_t = f_t \diamond g_t$ where f_t is f on $s \in [0, 1-t]$ and $f(1-t)$ on $s \in [1-t, 1]$ and $g_t = \bar{f}_t$. This gives us the desired homotopy between $f \diamond \bar{f}$ to $c \diamond \bar{c} = c$. Similarly for the other direction. Hence $\bar{f}$ is an inverse for f. This concludes the proof. ☐

Example 17. Suppose $X \subset \mathbb{R}^n$, X convex with $x_0 \in X$. Then $\pi_1(X, x_0) = 0$ since any two loops f and g are homotopic via the linear homotopy $h_t(s) = (1-t)f(s) + tg(s)$.

It is natural to ask how $\pi_1(X, x_0)$ depends on the basepoint x_0; of course, $\pi_1(X, x_0)$ only detects the path-component of X that contains x_0. Hence let x_1 lie in the same path component of X as x_0 and let $h : I \to X$ be a path from x_0 to x_1 with inverse $\bar{h}(s) = h(1-s)$. Then if f is a loop at x_1, then $h \diamond f \diamond \bar{h}$ is a loop at x_0.

Proposition 3. *The map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ given by $[f] \mapsto [h \diamond f \diamond \bar{h}]$ is an isomorphism.*

Proof. It f_t is a homotopy of a loops based at x_1, then $h \diamond f_t \diamond \bar{h}$ is a homotopy of loops at x_0, so the map is well-defined. Next, note that β_h is indeed a homomorphism, as $\beta_h[f \diamond g] = [h \diamond f \diamond \bar{h} \diamond h \diamond g \diamond \bar{h}] = \beta_h[f]\beta_h[g]$. It now suffices to find an inverse to β_h. Let us try $\beta_{\bar{h}}$. Note that $\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \diamond f \diamond h] = [h \diamond \bar{h} \diamond f \diamond h \diamond \bar{h}] = [f]$. Similarly to show that $\beta_{\bar{h}}$ is a right-inverse. This shows that β_h is an isomorphism. ☐

Class 4

Theorem 3. *The map $\Phi : \mathbb{Z} \to \pi_1(S^1)$ given by $n \mapsto [\omega_n]$ is an isomorphism, where $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$.*

Proof. See Hatcher p.29-31. ☐

Class 5

Theorem 4 (Brouwer's fixed point theorem)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point, i.e. there exists an $x \in D^2$ such taht $h(x) = x$.*

Proof. Suppose not, i.e. that $h(x) \neq x$ for all $x \in D^2$. Then, define a map $r : D^2 \to S^1$ where $r(x)$ is the point where the ray from $h(x)$ to x hits $S^1 = \partial D^2$. Notice that $r(x)$ is continuous, and that if $x \in S^1$ then $r(x) = x$. In particular, r is a retraction of D^2 onto S^1. Let f_0 be any loop in S^1 based at x_0. In D^2, $f_0 \simeq c$, where c is the constant loop at x_0, via the linear homotopy f_t, for example. Since $r|_{S^1} = \mathrm{Id}|_{S^1}$, $r \circ f_t$ is a homotopy in S^1 from $r \circ f_0 = f_0$ to c. This is a contradiction, as $\pi_1(S^1) \neq 0$. □

Theorem 5 (Borsuk-Ulam)**.** *Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists a pair of antipodal points – say x and $-x$ in S^2 – such that $f(x) = f(-x)$.*

Proof. For sake of contradiction, assume that no such x exists. Then define $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

This is well-defined, as the denominator is never zero and furthermore, $g(-x) = -g(x)$. Now define $\eta : I \to S^2 \subset \mathbb{R}^3$ given by $s \mapsto (\cos 2\pi s, \sin 2\pi s, 0)$, whose image sits on the equator, and let $h : I \to S^1$ be $g \circ \eta$, which yields a loop in S^1. Since $g(-x) = -g(x)$, we can write that $h(s + 1/2) = -h(s)$ for all $s \in [0, 1/2]$. If we lift the loop h to a path $\tilde{h} : I \to \mathbb{R}$, we can write $\tilde{h}(s + 1/2) = \tilde{h}(s) + q/2$ for some odd $q \in \mathbb{Z}$. *A priori*, q might depend (continuously) on s, but integrality forces it to be constant. Then we find that

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2, \tilde{h}(1/2) = \tilde{h}(0) + q/2,$$

which implies that $\tilde{h}(1) = \tilde{h}(0) + q$. Hence h is q times a generator for $\pi_1(S^1)$. As q is odd, h cannot be nullhomotopic. However, $h = g \circ \eta : I \to S^2 \to S^1$ and η is nullhomotopic in S^2 via, say f_t,then $g \circ \eta$ is nullhomotopic via $g \circ f_t$, which is a contradiction. □

Proposition 4. *Suppose X and Y are path-connected. Then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Proof. Recall that if $X \times Y$ is endowed with the product topology, then $f : Z \to X \times Y$ is continuous if and only if $g : Z \to X$ and $h : Z \to Y$ are continuous, where $f(z) = (g(z), h(z))$. Hence a loop in $X \times Y$ based at (x_0, y_0) is equivalent to a pair of loops (under the two projections) based at x_0 and y_0 respectively. Denote these loops by $g : I \to X$ and $h : I \to X$. Furthermore, a homotopy $f_t : I \to X \times Y$ is equivalent to a pair of homotopies g_t, h_t. Hence there is a bijection between $\pi_1(X \times Y, (x_0, y_0))$ and $\pi_1(X, x_0) \times \pi_1(Y, y_0)$. This is easily checked to be a homomorphism. □

Induced homomorphisms

Suppose $\phi : (X, x_0) \to (Y, y_0)$, i.e. $\phi : X \to Y$ with $\phi(x_0) = y_0$. Then ϕ induces a homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, given by $[f] \mapsto [\phi \circ f]$. Note that ϕ is indeed well-defined since a homotopy f_t of loops in X at x_0 gets sent by ϕ to a homotopy of loops in Y at y_0. Hence

$$\phi_*[f_0] = [\phi \circ f_0] = [\phi \circ f_1] = \phi_*[f_1].$$

Note also that it is a homomorphism, as $\phi(f \diamond g) = (\phi \circ f) \diamond (\phi \circ g)$. Two important properties of these induced homomorphisms is that

$$(\phi \circ \psi)_* = \phi_* \circ \psi_*,$$

i.e. it is covariant, and that

$$(\mathrm{Id}_{(X,x_0)})_* = \mathrm{Id}_{\pi_1(X,x_0)}\,.$$

These two properties show that π_1 is a covariant functor from pointed topological space to groups. One simple consequence of this is that if ψ is the inverse of ϕ then ψ_* is the inverse of ϕ_*, since $\phi_* \circ \psi_* = (\phi \circ \psi)_* = \mathrm{Id}_* = \mathrm{Id}$ (and similarly for the other order of composition).

Proposition 5. *If X retracts onto a subspace A, then the homomorphism $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $\iota : A \to X$ is injective.*

Proof. If $v : X \to A$ is a retraction than $r \circ i = \mathrm{Id}_A$, so $r_* \circ \iota_* = \mathrm{Id}$, so ι_* must be injective. □

Proposition 6. *If $\phi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism for all $x_0 \in X$.*

Lemma 6. *If $\phi_t : X \to Y$ is a homotopy and h is a path $\phi_t(x_0)$ formed by images of the basepoint $x_0 \in X$ then the following diagram commutes.*

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[rr, "(\phi_1)_*"] \arrow[dr, "(\phi_0)_*"'] & & \pi_1(Y,\phi_1(x_0)) \arrow[dl, "\beta_h"] \\
& \pi_1(Y,\phi_0(x_0)) &
\end{tikzcd}$$

Proof. Let $h_t(s) = h(ts)$. This is the restriction of h to $[0, t]$, reparametrized so the domain is still $[0, 1]$. If f is a loop in X at x_0 then $h_t \diamond \phi_t \circ f \diamond \bar{h}_t$ is a homotopy of loops at $\phi_0(x_0)$ (h is a path from $\phi_0(x_0)$ to $\phi_1(x_0)$). Restricting to $t = 0$ and $t = 1$ we have $(\phi_0)_*([f]) = \beta_h\left((\phi_1)_*[f]\right)$, homotopic. □

Proof of proposition. Let $\psi : Y \to X$ be the homotopy inverse of ϕ, i.e. $\phi\circ\psi \simeq \mathrm{Id}_Y$ and $\psi\circ\phi \simeq \mathrm{Id}_X$. Consider the sequence of maps

$$\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, \phi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi \circ \phi(x_0)) \longrightarrow \cdots$$

Note that $\psi_* \circ \phi_*$ is an isomorphism since $\psi \circ \phi \simeq \mathrm{Id}_X$ so $\psi_* \circ \phi_* = \beta_h$ for some h, by the lemma above. Thus ϕ_* is injective; similarly $\phi_* \circ \psi_*$ is an isomorphism for ϕ_* is surjective. □

Class 6

Followed Hatcher's proof of the Seifert-van Kampen theorem word-for-word.

Remark. We will have our first midterm on Feb. 24.

Covering spaces

Definition 24. A **covering space** of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

Remark. Note that we do not require $p^{-1}(U_\alpha)$ to be nonempty, so p need not be surjective.

The distinctive feature of covering spaces is their behavior with respect to lifting of maps. Recall that a **lift** of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$.

Let us cite some useful lemmas.

Lemma 7 (Homotopy lifting property)**.** *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.*

Proof. We omit this proof, as the version proved for $\pi_1(S^1) = 0$ holds for covering spaces in general. $\square$