An analogy and an example

Let F be a field and let $f(x) = c_d x^d + \cdots + c_0 \in F[x]$ be a polynomial of degree $d > 0$ (so that $c_d \neq 0$). We want to compare the quotient ring $F[x]/(f(x))$ with the more familiar ring $\mathbb{Z}/n\mathbb{Z}$ to see that they are similar in many ways.

The ring $\mathbb{Z}/n\mathbb{Z}$	The ring $F[x]/(f(x))$
Elements are cosets $a + n\mathbb{Z}$	Elements are cosets $g(x) + (f(x))$
Elements are usually written as $0, 1, \ldots, n-1$	Elements are uniquely described by $a_0 + a_1\alpha + \cdots + a_{d-1}\alpha^{d-1}$, $a_i \in F$ where $\alpha = x + (f(x))$
We compute (both addition and multiplication) by setting multiples of n to be 0	We compute by adding coefficients and multiplying according to the rule $\alpha^d = -c_d^{-1}(c_{d-1}\alpha^{d-1} + \cdots + c_0)$
$\mathbb{Z}/n\mathbb{Z}$ is a field $\iff$ n is a prime	$F[x]/(f(x))$ is a field $\iff$ $f(x)$ is irreducible in $F[x]$
Chinese Remainder Theorem n, m relatively prime $\implies$ $\mathbb{Z}/nm\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$	Chinese Remainder Theorem $f(x), g(x)$ relatively prime $\implies$ $F[x]/(f(x)g(x)) \cong$ $(F[x]/(f(x))) \times (F[x]/(g(x)))$

An example: let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ be the field with two elements, and let $f(x) = x^2 + x + 1 \in \mathbb{F}_2[x]$. Then $f(0) = f(1) = 1$ so that $f(x)$ has no roots in $\mathbb{F}_2$. Since $\deg f(x) = 2$, $f(x)$ is irreducible. The quotient ring $E = \mathbb{F}_2[x]/(f(x))$ is therefore a field. It has four elements $0, 1, \alpha, 1+\alpha$, so E is a **new** finite field! As an additive group, $(E, +) \cong ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}), +)$. The element α satisfies $\alpha^2 = -\alpha - 1 = \alpha + 1 = 1 + \alpha$, since, in characteristic 2, $-r = r$. As a consequence, $\alpha + \alpha^2 = 1$, and hence $\alpha(1 + \alpha) = 1$. Thus we see directly that each of the three nonzero elements of $E^* = \{1, \alpha, 1+\alpha\}$ has a multiplicative inverse, giving a direct argument that E is a field. Also note that E^* is a cyclic group of order 3 under multiplication, with generators α and hence $\alpha^{-1} = 1 + \alpha$. (Note that $(1 + \alpha)^2 = 1^2 + \alpha^2 = 1 + \alpha + 1 = \alpha$, because, as the characteristic is 2, $(1 + \alpha)^2 = 1^2 + \alpha^2$.)