Introduction to Algebraic Topology PSET 8

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Proposition 1. Hatcher exercise 2.1.11

Proof. Let $\iota: A \to X$ be the inclusion of A into X, and $r: X \to X$ be the retract of X onto A. The composition $r \circ \iota: A \to A$ yields the identity $\mathrm{Id}_A: A \to A$. The induced maps on the homology are $(r \circ \iota)_* = r_* \circ \iota_* = \mathrm{Id}: H_n(A) \to H_n(A)$. This map is of course injective, which implies that $\iota_*: H_n(A) \to H_n(X)$ must be injective as well.

Proposition 2. Hatcher exercise 2.1.12

Proof. Let us show that the relation of chain homotopy between chain maps is an equivalence relation. Consider $f_{\#}, g_{\#}, h_{\#}: C_n(A) \to C_{n+1}(B)$. The relation is clearly reflexive, as $f_{\#} \sim f_{\#}$ by the zero morphism $0: C_n(A) \to C_{n+1}(B)$. Symmetry holds as follows: if $f_{\#} \sim g_{\#}$ via a chain homotopy h, then $g_{\#} \sim f_{\#}$ via the chain homotopy -h, because then

$$f_{\#} - g_{\#} = \partial h + h\partial$$

$$g_{\#} - f_{\#} = -(\partial h + h\partial)$$

$$= \partial (-h) + (-h)\partial.$$

Finally, the relation is transitive, because given $f_{\#} \sim g_{\#}$ via H_1 and $g_{\#} \sim h_{\#}$ via H_2 , we can add the two commutation relations to obtain that

$$f_{\#} - h_{\#} = \partial H_1 + H_1 \partial + \partial H_2 + H_2 \partial$$

= $\partial (H_1 + H_2) + (H_1 + H_2) \partial$,

as desired. \Box

Proposition 3. Hatcher exercise 2.1.14

Proof. Consider the sequence

$$0 \longrightarrow \mathbb{Z}_4 \stackrel{\phi}{\longrightarrow} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \stackrel{\psi}{\longrightarrow} \mathbb{Z}_4 \longrightarrow 0$$

with ϕ taking the generator of \mathbb{Z}_4 to $(2,1) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$, which is clearly a well-defined injective map. If we now quotient $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ by $H = \text{im } \phi$, we obtain four cosets: (0,0)H, (0,1)H, (1,0)H, and (1,1)H, which is clearly isomorphic to \mathbb{Z}_4 (with (0,1)H as the generator). This yields a short exact sequence.

Consider now more generally the sequence of groups

$$0 \longrightarrow \mathbb{Z}_{p^m} \stackrel{\phi}{\longrightarrow} A \stackrel{\psi}{\longrightarrow} \mathbb{Z}_{p^n} \longrightarrow 0$$

Note first that $A = \mathbb{Z}_{p^{m+n-k}} \oplus \mathbb{Z}_{p^k}$, for $k \leq \min(m,n)$, fits into the sequence. Indeed, we let $\phi(1) = (p^{n-k},1)$, which is injective because the image along the first factor is injective. We now need to choose ψ such that $\ker \psi = (jp^{n-k},j)$ for $j \in \mathbb{Z}_{p^m}$. We claim that choosing $\psi(1,0) = 1$ fixes the map because $\psi(i,j) = i\psi(1,0) + j\psi(0,1)$ but since $\psi(jp^{n-k},p) = 0$, we find that $\psi(0,1) = -p^{n-k}$. Hence, with this choice of $\psi(1,0)$, we find that $\psi(i,j) = i-p^{n-k}j$. It is easy to see that $\ker \psi = \operatorname{im} \phi$. The image of ψ is cyclic, generated by $\psi(1,0)$ (as $\psi(i,j) = (i-jp^{n-k})\psi(1,0)$), and has order p^{m+n}/p^m , and hence isomorphic to \mathbb{Z}_{p^n} . Note, however, that it is unclear that these are the *only* groups that fit into this sequence (though it might be possible to invoke the fundamental theorem of finitely generated abelian groups).

Finally, consider the sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{\phi}{\longrightarrow} A \stackrel{\psi}{\longrightarrow} \mathbb{Z}_n \longrightarrow 0$$

We claim that $A = \mathbb{Z} \oplus \mathbb{Z}_d$ fits into this sequence for d|n. Indeed, we define $\phi(1) = (1, n/d)$ and then $\psi(0,1) = 1$, which forces $\psi(i,j) = (j-in/d)\psi(0,1) = j-in/d$. The sequence is clearly exact, and im ψ is cyclic. It suffices to compute the order of im ψ . This is done by counting the number of lattice points of \mathbb{Z}^2 contained in the parallelogram spanned by (0,d) and (1,n/d), which is simply $d \cdot n/d = n$. Hence we obtain \mathbb{Z}_n , as desired. Again, it is not clear that these are the *only* groups that fit into this sequence.

Proposition 4. Hatcher exercise 2.1.15

Proof. Consider the exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

Exactness at B requires $\ker \beta = \operatorname{im} \alpha$, and hence α is surjective if and only if $\ker \beta = B$. Exactness at D requires $\ker \delta = \operatorname{im} \gamma$, and hence δ is injective if and only if $\operatorname{im} \gamma = 0$. Hence if (and only if) α is surjective and δ is injective then $\gamma = 0$ and $\beta = 0$ and the exactness at C (requiring that $\ker \gamma = \operatorname{im} \beta$) forces C = 0.

Hence for a good pair (X, A), we find that $H_n(X, A) = 0$ if and only if the inclusion $A \to X$ induces isomorphisms on all homology groups, as the long exact sequence of theorem 2.13 splits into sequences

$$0 \longrightarrow \tilde{H}_n(A) \stackrel{\iota_*}{\longrightarrow} \tilde{H}_n(X) \longrightarrow 0$$

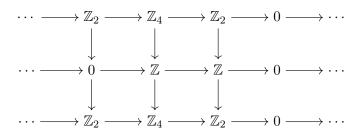
for all n.

Proposition 5. Let A and B be chain complexes. A chain map $f: A \to B$ is a chain homotopy equivalence if there exists a chain map $g: B \to A$ such that $f \circ g \sim \operatorname{Id}_B$ and $g \circ f \sim \operatorname{Id}_A$ in the sense of chain homotopies.

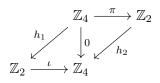
- (a) Prove that if $f: A \to B$ is a chain homotopy equivalence, then f induces an isomorphism on homology.
- (b) Give an example of chain complexes A and B with isomorphic homology but no chain homotopy equivalence between them. (Hint: let A be \mathbb{Z} in two consecutive gradings and zero everywhere else.)

Proof.

- (a) Recall that chain-homotopic maps induce the same homorphism on homology. Hence $(f \circ g)_* = f_* \circ g_* = (\mathrm{Id}_B)_* = \mathrm{Id}_{H_n(B)}$ and $(g \circ f)_* = g_* \circ f_* = (\mathrm{Id}_A)_* = \mathrm{Id}_{H_n(A)}$. As $\mathrm{Id}_{H_n(B)}$ is injective, $f_* : H_n(A) \to H_n(B)$ must be as well, and as $\mathrm{Id}_{H_n(A)}$ is surjective, f_* must be as well. Hence f_* is an isomorphism.
- (b) Consider the map of chain complexes $f: A_{\bullet} \to B_{\bullet}$ given by the first two rows of



where each square clearly commutes. The homology groups of the two sequences are $H_{\bullet}(A) = 0$ and $H_{\bullet}(B) = 0$. However, there does not exist a chain homotopy equivalence between A_{\bullet} and B_{\bullet} , as we now show. If there did exist one, there would exist a chain map $g: B_{\bullet} \to A_{\bullet}$ (between the second two rows) such that the appropriate compositions of f and g would be chain homotopic to Id_B and Id_A via some chain homotopy h. Of course, the only possible compositions are zero, and hence



there must exist h_1, h_2 such that $\mathrm{Id}_{\mathbb{Z}_4} = \iota \circ h_1 + h_2 \circ \pi$, but h_1 can only be the zero map or the quotient map and h_2 can only be the zero map or the inclusion map. It is easy to see that none of these combinations recover $\mathrm{Id}_{\mathbb{Z}_4}$.