Introduction to Algebraic Topology PSET 7

Nilay Kumar

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Proposition 1. *Hatcher exercise 2.1.1*

Proof. The Δ-complex obtained from the 2-simplex $[v_0, v_1, v_2]$ with edges $[v_0, v_1]$ and $[v_1, v_2]$ identified with order preserved is just a cone $C(S^1)$. $\square$

Proposition 2. *Hatcher exercise 2.1.5*

Proof. Let us compute the simplicial homology groups of the Klein bottle using the Δ-complex structure K described on page 102 of Hatcher. It's clear that there is one 0-simplex (v), three 1-simplices (a, b, c), and two 2-simplices (U, L). We obtain the chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

with $\partial_0 v = 0$, $\partial_1 a = \partial_1 b = \partial_1 c = 0$, and $\partial_2 U = a + b - c, \partial_2 L = a - b + c$. The Klein bottle is path-connected and hence $H_0^\Delta(K) = \mathbb{Z}$. Next note that $\ker \partial_2 = 0$ as no linear combination of U and L will map to 0 under ∂_2, and thus $H_2^\Delta(K) = 0$. Computing $H_1^\Delta(K)$ is a bit more tedious: $H_1^\Delta(K) = \ker \partial_1 / \mathrm{im}\ \partial_2 = \langle a, b, c\rangle / \langle a + b - c, a - b + c\rangle$. We can simplify this as follows:

$$\begin{aligned}
\frac{\langle a, b, c\rangle}{\langle a + b - c, a - b + c\rangle} &= \frac{\langle a - b + c, b, c\rangle}{\langle a + b - c, a - b + c\rangle} \\
&= \frac{\langle b, c\rangle}{\langle 2b - 2c\rangle} = \frac{\langle b - c, c\rangle}{\langle 2(b - c)\rangle} = \frac{\langle d, c\rangle}{\langle 2d\rangle} \\
&= \mathbb{Z} \oplus \mathbb{Z}_2.
\end{aligned}$$

Hence the non-trivial simplicial homology groups are $H_0^\Delta(K) = \mathbb{Z}$ and $H_1^\Delta(K) = \mathbb{Z} \oplus \mathbb{Z}_2$. $\square$

Proposition 3. *Find a Δ-complex structure for the orientable surface of genus two and compute the simplicial homology groups of this Δ-complex.*

Proof. Consider the Δ-complex Σ_2 obtained by taking an octagon (with sides identified appropriately) and drawing segments from a fixed vertex to the other vertices (see figure). This yields one 0-simplex (v), nine 1-simplices $(a, \ldots, i)$, and six 2-simplices $(\alpha, \ldots, \zeta)$ and the chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z}^6 \xrightarrow{\partial_2} \mathbb{Z}^9 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0.$$

Clearly $H_0^\Delta(\Sigma_2)=\mathbb{Z}$. More tedious is $H_1^\Delta(\Sigma_2)$:

$$\begin{aligned}
H_1^\Delta(\Sigma_2) &= \frac{\ker\partial_1}{\text{im }\partial_2}\\
&= \frac{\langle a,b,c,d,e,f,g,h,i\rangle}{\langle d+c-e, f+d-e, g+c-f, g+b-h, h+a-i, a+b-i\rangle}\\
&= \frac{\langle b,c,d,e,f,g,h,i\rangle}{\langle c+d-e, d-e+f, c-f+g, b+g-h, h-b\rangle}\\
&= \frac{\langle b,c,d,e,f,g,i\rangle}{\langle c+d-e, d-e+f, c-f+g, g\rangle} = \frac{\langle b,c,d,e,f,i\rangle}{\langle c+d-e, d-e+f, c-f\rangle}\\
&= \frac{\langle b,c,d,e,i\rangle}{\langle c+d-e, c+d-e\rangle} = \frac{\langle b,c,d,e,i\rangle}{\langle c+d-e\rangle}\\
&= \mathbb{Z}^4.
\end{aligned}$$

Finally, $H_2^\Delta(\Sigma_2)=\ker\partial_2/\text{im }\partial_3=\ker\partial_2$. It is straightforward to check that $\ker\partial_2=\langle\alpha-\beta-\gamma+\delta+\varepsilon-\zeta\rangle$ (one can read this off of the figure via the the orientations of the 2-simplicies). Hence the non-trivial simplicial homology groups are $H_0^\Delta(\Sigma_2)=\mathbb{Z}, H_1^\Delta(\Sigma_2)=\mathbb{Z}^4, H_2^\Delta(\Sigma_2)=\mathbb{Z}$. □

Proposition 4. *Find a Δ-complex structure for S^3 and compute the simplicial homology groups of this Δ-complex.*

Proof. Consider the Δ-complex of S^3 given by identifying (in the obvious way) all the faces of two 3-simplices T_1 and T_2. We obtain the chain complex

$$0 \xrightarrow{\partial_4} \mathbb{Z}^2 \xrightarrow{\partial_3} \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^4 \xrightarrow{\partial_0} 0.$$

Clearly $H_0^\Delta(S^3)=H_1^\Delta(S^3)=\mathbb{Z}$. Next, note that

$$\partial_1 a = y-x, \partial_1 b = z-y, \partial_1 c = z-x, \partial_1 d = w-x, \partial_1 e = w-y, \partial_1 f = w-z,$$

and

$$\partial_2\alpha = a-d+e, \partial_2\beta = b-e+f, \partial_2\gamma = c-d+f, \partial_2\delta = a+b-c,$$

and

$$\partial_3 T_1 = \partial_3 T_2 = \alpha+\beta-\gamma-\delta.$$

We can check that $\ker\partial_1$ is generated by $-c+d-f, -b+e-f, -a+d-e, a+b-c$, and that these are precisely generate im ∂_2. Hence $H_1^\Delta(S^3)=0$. Next, note that $\ker\partial_2$ is generated by $\alpha+\beta-\gamma-\delta$, which is precisely im ∂_3, and thus $H_2^\Delta(S^3)=0$ as well. □

Proposition 5. *Hatcher exercise 2.1.8*

Proof. Denote by L the given Δ-complex. Clearly $H_0^\Delta(L)=\mathbb{Z}$. To find the first homology group, note that there are two 1-simplices in L that are always present, independent of n, call them b and c. There are n other 1-simplices, denote them by $a_1,\ldots,a_n$. It is easy to see that since there are only two unique vertices, we must have

$$\ker\partial_1 = \langle a_i - a_{i+1}, b, c\rangle$$

for all i except for n (as otherwise we would obtain a linear dependence). Then, applying ∂_2 to the 2-simplicies we obtain

$$\text{im }\partial_2 = \langle a_i - a_{i+1} + b, a_{i+1} - a_i + c, nb\rangle = \langle a_i - a_{i+1} + b, b+c, nb\rangle$$

where again i excludes n due to the dependence (which we have explicitly written here as $nb = 0$). If we now change basis of our free $\mathbb{Z}$-module $\ker \partial_1$ appropriately, we find that

$$H_1^\Delta(L) = \frac{\langle b, c\rangle}{\langle b + c, nb\rangle} = \mathbb{Z}_n.$$

Now, $H_2^\Delta = \ker \partial_2 / \mathrm{im}\ \partial_3$, and it is easy to see that both numerator and denominator are precisely $V_{i+1} - V_i + H_i - H_{i+1}$ where the V's are the vertical 2-simplices, and the H's are the the slanted 2-simplices. Hence $H_2^\Delta = 0$. Finally, note that $H_3^\Delta(L) = \ker \partial_3$, which is of course just the appropriate alternating sum of the 3-simplices, and hence $H_3^\Delta(L) = \mathbb{Z}$. $\square$