# Commutative Algebra: Problem Set 9

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## Problem 2

Let $C$ be a hyperelliptic curve with function field $K/k$. We wish to show that $C$ is birational to a curve $D$ of the form $y^2 = f(x)$ with $f \in k[x]$ some monic squarefree polynomial. We may write, without loss of generality,

$$K = \operatorname{frac}\left(k[x,y]/(f_0(x)y^2 + f_1(x)y + f_2(x)\right)$$

for some $f_i(x) \in k[x]$ with no common factor. Let us first find a rational map $C \to D$, i.e. a map that takes points on $C$ and gives us points satisfying $y^2 - f(x) = 0$. Completing the square yields

$$\begin{aligned} 0 &= f_0(x)\left(y^2 + \frac{f_1(x)}{f_0(x)}y + \frac{f_2(x)}{f_0(x)}\right) \\ &= f_0(x)\left(y + \frac{f_1(x)}{2f_0(x)}\right)^2 + f_2(x) - \frac{f_1^2(x)}{4f_0(x)} \\ &= (2f_0(x)y + f_1(x))^2 + 4f_0(x)f_2(x) + f_1^2(x) \end{aligned}$$

Hence we see that the map $(x,y) \overset{\phi}{\mapsto} (x, 2f_0(x)y + f_1(x))$ for points $(x,y)$ on $C$ lands in $D$. This is clearly a rational map, and we see that $f(x) = -4f_0(x)f_2(x) - f_1^2(x)$, which is squarefree as the $f_i$ have no common factor. We now wish to show that $\phi$ has a rational inverse. But from above, we can simply take $(x', y') \overset{\phi}{\mapsto} (x', (y' - f_1(x))/2f_0(x))$. for some point $(x', y') \in D$. This is rational as well, and hence $\phi$ is birational.

## Problem 3

This is obvious from previous problem. Given any squarefree $f \in k[x]$, we may simply consider the function field $K$ as

$$K = \operatorname{frac}\left(k[x,y]/(y^2 - f(x))\right).$$

## Problem 4

Consider the two monic squarefree polynomials $x$ and $x - 1$. The hyperelliptic curves defined by $y^2 = x$ and $y^2 = x - 1$ are clearly birational to each other via the map $(x,y) \mapsto (x - 1, y)$.

## Problem 5

Let $C$ be a hyperelliptic curve with $D$ the zero divisor of $x$ on $C$. The degree of $D$ is 2. Let us show that $\ell(D) = 2$ if $g > 0$. By Lemma 78 of our class notes, we see that $\ell(D) \leq \deg(D) + 1 = 3$. It's clear that $\ell(D) > 1$, as $L(D)$ contains both 1 and $1/x$. Let us now suppose $\ell(D) = 3$ and reach a contradiction. Since $D = P + Q$ for some $P, Q$ on $C$, we see that $\ell(P) \geq 3$, i.e. we have contained in $L(P)$ some $f \neq 1$. Hence we can consider, at least, the set $\{1, f, \ldots, f^n\} \subset L(nP)$. By Riemann-Roch applied to $nP$ we see that

$$\ell(nP) = n - g + 1$$

if we take $n$ arbitrarily large (as $\ell(K - nP)$ will go to zero). But this gives us that

$$n + 1 \leq \ell(nP) = N - g + 1,$$

which implies that $g = 0$, a contradiction. Hence $\ell(D) = 2$.

## Problem 6

Consider the map $C \to \mathbb{P}^{\ell(D)-1} = \mathbb{P}^1$. As $C$ has a degree 2 divisor, the map has degree 2, and we see that the function field of $K$ is a degree two extension of $k(x)$, and hence we get a hyperelliptic curve.

## Problem 7

Let $C : y^2 = f(x)$ as above. Consider the differential form $\omega = dx$, which is clearly $dx = 2y dy / f'(x)$.