Introduction to Algebraic Topology PSET 1

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Proposition 1. *If a space X is contractible then X is path-connected.*

Proof. Take any $x, y \in X$. It suffices to show that there exists a continuous path $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since X is contractible, the identity map Id_X is homotopic to a constant map. In other words, there exists a point $z \in X$ and a homotopy $f_t : X \times I \to X$ such that $f_0 = \mathrm{Id}_X$ and $f_1(p) = z$ for all $p \in X$. Now consider the path $f_t(x) : I \to X$. Clearly $f_0(x) = x$ and $f_1(x) = z$. Similarly, for $f_t(y) : I \to X$, we see that $f_0(y) = y$ and $f_1(y) = z$. Both of these paths $f_t(x), f_t(y)$ are continuous as they are simply restrictions of the full homotopy $f_t : X \times I \to X$. But given continuous paths from x to z and y to z, we can concatenate to obtain a path from x to y. This proves path-connectedness. $\square$

Proposition 2. *Let X, Y, Z be topological spaces. Consider the maps*

$$\begin{tikzcd} X \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] & Y \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] & Z \end{tikzcd}$$

and suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $g_0 \circ f_0 : X \to Z$ and $g_1 \circ f_1 : X \to Z$ are homotopic.

Proof. There must exist a homotopy $F : X \times I \to Y$ such that $F_0 = f_0$ and $F_1 = f_1$, as well as a homotopy $G : Y \times I \to Z$ such that $G_0 = g_0$ and $G_1 = g_1$. Now consider the composition $H : X \times I \to Z$ defined by $(x, t) \mapsto G_t(F_t(x))$. The composition H is continuous by virtue of the continuity of the homotopies F and G. Moreover, H satisfies $H_0 = g_0 \circ f_0$ and $H_1 = g_1 \circ f_1$, and hence H is a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$. $\square$

Proposition 3. *Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.*

Proof. Consider the function $f : (\mathbb{R}^n - \{0\}) \times I \to \mathbb{R}^n - \{0\}$ given by

$$f_t(x) = x - t\left(x - \frac{x}{|x|}\right).$$

It is clear that f is continuous. Moreover, at $t = 0$, we have $f_0(x) = x$ and hence $f_0 = \mathrm{Id}_{\mathbb{R}^n - \{0\}}$. At $t = 1$, on the other hand, we see that $f_1(x) = x/|x|$, which is on S^{n-1} and hence $f_1(\mathbb{R}^n - \{0\}) = S^{n-1}$ (f_1 obviously surjects onto S^{n-1}). Finally, note that $f_t|_{S^{n-1}} = \mathrm{Id}_{S^{n-1}}$ for all t because $f_t(x) = x - tx + tx = x$ if $|x| = 1$. This shows that f is indeed the desired deformation retract. $\square$

Proposition 4.

(a) *The composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Furthermore, homotopy equivalence is an equivalence relation.*

(b) The relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) A map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof.

(a) Let $X \simeq Y$ via $f : X \to Y$ and $g : Y \to X$ and $Y \simeq Z$ via $f' : Y \to Z$ and $g' : Z \to Y$. In other words, $g \circ f \simeq \mathrm{Id}_X, f \circ g \simeq \mathrm{Id}_Y$ and $g' \circ f' \simeq \mathrm{Id}_Y, f' \circ g' \simeq \mathrm{Id}_Z$. Note that

$$\begin{aligned} f' \circ f \circ g \circ g' &\simeq f' \circ \mathrm{Id}_Y \circ g' \\ &\simeq f' \circ g' \\ &\simeq \mathrm{Id}_Z \end{aligned}$$

and

$$\begin{aligned} g \circ g' \circ f' \circ f &\simeq g \circ \mathrm{Id}_Y \circ f \\ &\simeq g \circ f \\ &\simeq \mathrm{Id}_X \, . \end{aligned}$$

But this implies that $X \simeq Z$ via $f' \circ f : X \to Z$ and $g \circ g' : Z \to X$, as desired. This shows that homotopy equivalence is a transitive relation. In fact, homotopy equivalence is an equivalence relation because $X \simeq X$ by the identity map, and if $X \simeq Y$ via f, g, then $Y \simeq X$ simply by switching f, g.

(b) Note first that any map $\phi : X \to Y$ is homotopic to itself by the constant homotopy $f : X \times I \to Y$ given by $(x, t) \mapsto \phi(x)$. Furthermore, if f_t is a homotopy from f_0 to f_1, we can define a homotopy from f_1 to f_0 by simply taking f_{1-t}. Finally, let f_t be a homotopy from f_0 to f_1 and g_t be a homotopy from g_0 to g_1, where $f_1 = g_0$. We can define a homotopy $h : X \times I \to Y$ from f_0 to g_1 by traversing the homotopy f_{2t} followed by g_{2t}. Hence the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Let X, Y be topological spaces, with $X \simeq Y$ via $h : X \to Y$ and $h' : Y \to X$. Let $f_0 : X \to Y$ be a map homotopic to h, i.e. there exists a homotopy $f_t : X \times Y$ between f_0 and $f_1 = h$. Then $f_0 \circ h' \simeq h \circ h' \simeq \mathrm{Id}_Y$ and $h' \circ f_0 \simeq h' \circ h \simeq \mathrm{Id}_X$ and hence f_0 provides a homotopy equivalence $X \simeq Y$.

□

Proposition 5. *A retract of a contractible space is contractible.*

Proof. Let X be a topological space and $r : X \to X$ be a retraction of X onto a subspace A. It suffices to show that A is contractible, i.e. that the identity map Id_A is homotopic to a constant map (on A). Using the contractibility of X, we find a homotopy $f : X \times I \to X$ such that $f_0 = \mathrm{Id}_X$ and $f_1(p) = z$ for some $z \in X$. Consider the composition $r \circ f : X \times I \to X$ restricted to A. It is clear that $r \circ f_0|_A = \mathrm{Id}_A$ and that $r \circ f_1|_A = r(z)$ is a constant map on A. As the composition (and restriction) $r \circ f|_A$ is continuous, this gives us a homotopy between Id_A and a constant map on A, proving that A is contractible. □