Riemann Surfaces PSET 1

Nilay Kumar

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Problem 1

As D is a compact regular domain in $\mathbb{R}^2$, we can apply Stoke's theorem (equivalently, Green's theorem), $\int_{\partial D} \omega = \int_D d\omega$. We have $\omega = f(z)dz$ and thus

$$d\omega = d(f(z)dz) = d(f(z)) \wedge dz = \left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}\right) \wedge dz = \frac{\partial f}{\partial \bar{z}}d\bar{z} \wedge dz,$$

which yields

$$\int_{\partial D} f(z)dz = \int_D \frac{\partial f}{\partial \bar{z}}d\bar{z} \wedge dz,$$

as desired.

Problem 2

Define $g(z) : \Omega \to \mathbb{C}$ as 0 at $z = 0$ and $z^2 f(z)$ on $\Omega \setminus 0$. As $f(z)$ is holomorphic on $\Omega \setminus 0$, it is clear that $g(z)$ is holomorphic on $\Omega \setminus 0$ as well. It is easy to see that $g(z)$ is in fact holomorphic at $z = 0$ as well:

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} hf(h) = 0$$

where in the last step we have used the boundedness of $f(z)$ in the neighborhood of $z = 0$. We can now write out a power series for $g(z)$ on Ω (about $z = 0$), $g(z) = a_0 + a_1 z + a_2 z^2 + \ldots$. Note, however, that $g(0) = 0$ and hence $a_0 = 0$. In fact, $a_1 = 0$ as well; suppose it were not: then $f(z) = g(z)/z^2 = a_1/z + a_2 + \ldots$ on $\Omega \setminus 0$, which contradicts the boundedness of f at 0. Hence we see that

$$\begin{aligned} g(z) &= a_2 z^2 + a_3 z^3 + \ldots \\ f(z) &= a_2 + a_3 z + \ldots \text{ on } \Omega \setminus 0, \end{aligned}$$

which implies that $f(z)$ can be holomorphically extended to Ω by simply defining it to take the value a_2 at $z = 0$.

Problem 3

Define $I(z) = \int_0^1 f(x)x^{z-1}dx$.