MODERN ALGEBRA II SPRING 2013: THIRD PROBLEM SET

1. Let F be a field of characteristic zero and let $\phi\colon F \to F$ be an automorphism, i.e. a ring isomorphism from F to itself. Show that $\phi(a) = a$ for all a in the prime subfield of F (the subfield isomorphic to $\mathbb{Q}$).

2. (i) Recall that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Let $\phi\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ be an automorphism, i.e. a ring isomorphism from $\mathbb{Q}(\sqrt{2})$ to itself. What is $[\phi(\sqrt{2})]^2$? (Use Problem 1.) Show that either $\phi(\sqrt{2}) = \sqrt{2}$ and $\phi = \mathrm{Id}$ or that $\phi(\sqrt{2}) = -\sqrt{2}$ and hence that $\phi(a+b\sqrt{2}) = a - b\sqrt{2}$ for all $a, b \in \mathbb{Q}$ (Problem 1 again). Finally, if $\phi\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is defined by the formula $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$ for all $a, b \in \mathbb{Q}$, show that ϕ is an isomorphism.

   (ii) Recall that $\mathbb{Q}(\sqrt[3]{2}) = \{a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 : a, b, c \in \mathbb{Q}\}$. Let $\phi\colon \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{Q}(\sqrt[3]{2})$ be an automorphism. What is $[\phi(\sqrt[3]{2})]^3$? Is it possible for $\phi(\sqrt[3]{2}) = -\sqrt[3]{2}$? For $\phi(\sqrt[3]{2}) = (\sqrt[3]{2})^2$? Using the fact that $\sqrt[3]{2}$ is the unique real number t such that $t^3 = 2$, show that $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$ and then that $\phi = \mathrm{Id}$.

3. Which of the following are ideals in the given ring? Why or why not? If the subset is an ideal, is it a principal ideal?

   (a) The subset $\mathbb{Q}$ of the ring $\mathbb{R}$.

   (b) The subset $\mathbb{Z}$ of the polynomial ring $\mathbb{Z}[x]$.

   (c) The subset $\mathbb{Z}$ of the ring $\mathbb{Z}[i]$.

   (d) The set of all multiples of $3 - 2i$ in the ring $\mathbb{Z}[i]$, i.e. the set
   $$\{(n + mi)(3 - 2i) : n, m \in \mathbb{Z}\}.$$

   (e) The subset of $\mathbb{Q}[x]$ consisting of all polynomials whose first five terms are 0 (i.e. the set of all polynomials of the form $\sum_{i=0}^n a_i x^i$ with $a_0 = \cdots = a_4 = 0$). (Note: for this definition to makes sense, we assume that if $\deg f \leq 4$, then we add the missing terms $0 \cdot x^i$ through $i = 5$.)

   (f) The subset of $\mathbb{Q}[x]$ consisting of all polynomials whose linear term is 0 (i.e. the set of all polynomials of the form $\sum_{i=0}^n a_i x^i$ with $a_1 = 0$).

   (g) The subset of $\mathbb{Z}[x]$ consisting of all polynomials whose leading coefficient is divisible by 2, together with the constant polynomial 0.

4. In this problem, R denotes a ring and I and J are two ideals in R. Show that $I \cap J$ is an ideal of R contained in both I and J, and in fact every ideal of R contained in both I and J must be a subset of $I \cap J$.

   Is $I \cup J$ always an ideal of R? Give a proof or counterexample.

5. Show that, if R is a ring, then the set of all nilpotent elements in R is an ideal, called the *radical* of R and often denoted by the symbol $\sqrt{0}$. (Use Problem 3 from the last problem set.) What is $\sqrt{0}$ for $R = \mathbb{Z}$? $R = \mathbb{Z}/6\mathbb{Z}$? $R = \mathbb{Z}/27\mathbb{Z}$? $R = \mathbb{Z}/18\mathbb{Z}$?

6. (i) Let $\varphi \colon R \to S$ be a ring homomorphism, and let J be an ideal in S. Show that $\varphi^{-1}(J)$ is an ideal in R. In particular, if R is a subring of S and J is an ideal in S, then $R \cap J$ is an ideal in R.

   (ii) Let $\varphi \colon R \to S$ be a **surjective** homomorphism of rings. Suppose that I is an ideal in R. Show that $\varphi(I)$ is an ideal of S. Give an example to show that, if φ is not surjective, then $\varphi(I)$ need not be an ideal of S.