Commutative Algebra: Problem Set 4

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Problem 5

Let $A = k[x,y,z]/(x^2y^2z^2, x^3y^2z)$. We wish to compute the dimension of A at the maximal ideal (x,y,z). Since $x^2y^2z^2$ is a nonzerodivisor in $k[x,y,z]_{(x,y,z)}$, quotienting out by it yields a ring with dimension two (by lemma 46). Next note that after quotienting out by x^3y^2z we still have a chain of primes $(x) \subset (x,y) \subset (x,y,z)$ where the inclusions are still proper (we can't get a z in (x,y) for example, because all we can do with $x^2y^2z^2$ is dock powers) and hence the dimension is two.

Problem 6

Let $A = k[x,y,z]/(x^3 - y^2, x^5 - z^2, y^5 - z^3)$. We wish to compute the dimension of A at (x,y,z). Consider first quotienting by $x^3 - y^2$, which is clearly prime (and a nonzerodivisor), and hence yields a domain, docking the dimension from 3 to 2. Next consider quotienting by $x^5 - z^2$, which is no longer prime, but is still a nonzerodivisor and hence yields a ring of dimension 1. Furthermore, if we now quotient out the last generator, we will obtain a ring of dimension either 1 or 0. We claim that $\dim A$ is in fact 1. It suffices to show that there exists an integral map $A \to k[t]$, as this will imply that $\dim A \geq \dim k[t] = 1$, thus ruling out the possibility $\dim A = 0$. We can construct such a map, call it ϕ, by sending $x \mapsto t^2, y \mapsto t^3$, and $z \mapsto t^5$. It's clear that ϕ is a morphism of rings. Moreover, since using t^3 and t^5 we can write any t^n for $n \geq 2$ as $\phi(x^iy^j)$ for some $i,j \geq 0$. This shows that everything in $k[t]$ is integral over A except for t (take $t^n - \phi(x^iy^j) = 0$). But t is clearly integral as well, because $t^2 - \phi(x) = 0$. Hence ϕ is integral, and we are done.

Problem 7

Let k be a field. Let $f \in k[x,y]$ be a polynomial and $a,b \in k$ be elements such that $f(a,b) = 0$. Let $\mathfrak{m} = (x-a, y-b)$ be the corresponding maximal ideal in the ring $A = k[x,y]/(f)$. By construction, $A_\mathfrak{m}$ is a local ring. We wish to check that it is regular, i.e. that the maximal ideal has exactly $\dim A_\mathfrak{m} = 1$ generators. The number of generators is given by $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim_k \mathfrak{m}/\mathfrak{m}^2$. In this case it's clear that $\mathfrak{m}$ is the ideal of all polynomials with root at (a,b), i.e. every element of $\mathfrak{m}$ has a finite Taylor series about (a,b) given by $\sum_{i+j\geq 1} c_{ij}(x-a)^i(y-b)^j$ for $c_{ij} \in k$. The finiteness is obvious since we are simply dealing with polynomials. It follows, then, that elements of $\mathfrak{m}^2$ look like $\sum_{i+j\geq 2} d_{ij}(x-a)^i(y-b)^j$ for $d_{ij} \in k$, as we have squared away linear terms. The quotient $\mathfrak{m}/\mathfrak{m}^2$ is then composed of elements of the form $c_{10}(x-a) + c_{01}(y-b)$, which is 2-dimensional as a vector space and isomorphic to k^2; we must be careful to note, however that (f) is zero in our original ring and hence its image in $\mathfrak{m}/\mathfrak{m}^2$, $(\partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b))$ must also be zero, which is a dimension one subspace. Hence we see that the dimension of $\mathfrak{m}/\mathfrak{m}^2$ must be $2 - 1 = 1$, as desired.

Problem 9

Consider $f = xy^2 + x^2y = xy(y + x)$, which has zeros at $x = y = 0$ and at $x = y$. We compute $\partial_x f = y^2 + 2xy$ and $\partial_y f = 2yx + x^2$; the only singular point, then, is $(0, 0)$. Next consider $f = x^2 - 2x + y^3 - 3y^2 + 3y$; we compute $\partial_x f = 2x - 2$ and $\partial y f = 3y^2 - 6y + 3 = 3(y - 1)^2$. The singular point is then $(1, 1)$, as $f(1, 1) = 0$. Finally, consider $f = x^n + y^n + 1$, which has $\partial_x f = nx^{n-1}, \partial_y f = ny^{n-1}$. These derivatives are never zero except at $(0, 0)$, which is not a root of f, and hence f has no singular points.

Problem 10

Let k an algebraically closed field. Let $f \in k[x, y]$ be a squarefree polynomial of degree 1. In other words, $f = ax + by + c$. Clearly f can only have singular points if it is a constant, which contradicts the degree being 1, and hence has no singular points. Next consider degree 2: $f = ax^2 + bxy + cy^2 + dx + ey + f$. In this case, solving for the singular points involves simultaneously solving a linear system, which yields a single point. We can assume that the system is non-degenerate because of the squarefree condition; if it were, it is straightforward but tedious to show that f can then be written as $(\sqrt{a}x \pm \sqrt{b}y + d/2)^2$, a square. Finally, we want our singular point to land on the curve - this can always be done by adjusting f.

For degree 3, we solve two quadratics, and hence we expect 4 singular points, and again, not an infinite number due to the squarefree condition. In this way, we might guess that in general for degree d we have, at maximum, $(d - 1)^2$ singular points.