Hartshorne Solutions

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Problem I.1.1

(a) Let Y be the plane curve $y - x^2 = 0$. The coordinate ring $A(Y) = k[x, y]/(y - x^2)$ is isomorphic to $k[x]$, as any power of y is simply replaced by x^2.

(b) Let Z be the plane curve $xy - 1 = 0$. The corrdinate ring $A(Z) = k[x, y]/(xy - 1)$ is clearly not isomorphic to a polynomial ring of one variable, as there are non-constant elements of $A(Z)$ (namely, powers of y) that are invertible.

(c) We assume that the char $k \neq 2$. Let f be an irreducible quadratic polynomial in $k[x, y]$, i.e.

$$f(x, y) = x^2 + axy + by^2 + cx + dy + e,$$

and let W be the conic defined by f (we can choose f to be monic without loss of generality).

Suppose the degree two terms are a perfect square. Then we may change variables to obtain $z^2 + bz + dy + e$ (with new coefficients). Completing the square (here we use char $k \neq 2$) in z and changing variables again, we obtain $w^2 - y = 0$, which yields the curve Y from part (a) above.

If the degree two terms do not factor perfectly, finish

Problem I.1.2

Let $Y \subset \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Consider $\mathfrak{a} = (y - x^2, z - x^3) \subset k[x, y, z]$. It is clear that $Y = I(\mathfrak{a})$. The coordinate ring is thus given $A(Y) = k[x, y, z]/(y - x^2, z - x^3)$. Note that $\dim A(Y) = 1$ by dimension theory (as $z - x^3$ is not a zero divisor in $k[x, y, z]$ and $y - x^2$ is not a zero divisor in $k[x, y, z]/(z - x^3)$), and hence Y has dimension one. Indeed, it is easy to see that $A(Y) \cong k[x]$ as y is replaced by x^2 and z is replaced by x^3. why? division

Problem I.3.1

(a) By the results of problem 1.1, we know that any conic in $\mathbb{A}^2$ can be written as either a variety Y defined by $y - x^2 = 0$ or a variety Z defined by $xy - 1 = 0$. We know that $A(Y) = k[x]$ and $A(Z) = k[x, x^{-1}]$. Note that $A(Y) \cong A(\mathbb{A}^1)$, and hence by Corollary 3.7, $Y \cong \mathbb{A}^1$ as affine varieties. It remains to show that Z is isomorphic to $\mathbb{A}^1 - \{0\}$. Note first that $xy - 1 = 0$ can be parametrized as (t, t^{-1}), which suggests the map $\phi : Z \to \mathbb{A}^1 - \{0\}$ given by $\phi(t, t^{-1}) = t$ as well as the reverse $\psi : \mathbb{A}^1 - \{0\} \to Z$ given by $\psi(x) = (x, x^{-1})$. It is easy to check ϕ and ψ are morphisms with $\psi \circ \phi = \mathrm{Id}_Z$ and $\phi \circ \psi = \mathrm{Id}_{\mathbb{A}^1 - \{0\}}$.

(b) Let B be a proper open subset of $\mathbb{A}^1$. By definition of the Zariski topology, we can write $B = \mathbb{A}^1 \setminus \{p_1, \dots, p_n\}$ where p_i are a finite set of points in $\mathbb{A}^1$. The ring of regular functions of $\mathbb{A}^1$ is $\mathcal{O}(\mathbb{A}^1) = k[x]$. In B, however, polynomials that vanish only at any of the p_i are globally invertible, and hence $\mathcal{O}(B) = k[x, (x-p_1)^{-1}, \dots, (x-p_n)^{-1}]$. These two rings are clearly not isomorphic, which completes the proof.

(c) In the projective plane, we can write a conic as $F(x, y, z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$, which can be rewritten under an appropriate change of variables as $x^2 + y^2 + z^2$(assuming char $k \neq 2$). Hence every conic in the projective plane is isomorphic, and it will suffice to show that there exists a conic that is isomorphic to $\mathbb{P}^1$. This follows From the result of exercise I.3.4: the 2-uple embedding $\rho_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism onto its image do this

$$\rho_2(a_0, a_1) = (a_0^2, a_0 a_1, a_1^2),$$

which clearly traces out a conic $xz - y^2$.

(d) This is hard!! finish

(e) If an affine variety X is isomorphic to a projective variety Y, then we must have that $\mathcal{O}(X) = \mathcal{O}(Y) = k$. But for $k[x_1, \dots, x_n]/I(X) = k$, $I(X)$ must be maximal. Hence $I(X) = (x_1 - a_1, \dots, x_n - a_n)$, i.e. X is just a point.

Problem I.3.14

(a) Note first that ϕ is continuous, as the preimage of any closed subset $V \subset \mathbb{P}^n$ is the projective cone $\overline{C(V)}$, which is closed in $\mathbb{P}^{n+1}$. Furthermore, the point at which the line connecting any Q and P to the hypersurface (choose $x_0 = 0$ without loss of generality) is given by

$$\phi(Q) = [Q_1 - \frac{Q_0 P_1}{P_0} : \dots : Q_{n+1} - \frac{Q_0 P_{n+1}}{P_0}],$$

where P_i and Q_i, are the ith components of P and Q, respectively (the coordinates are written as for a point in $\mathbb{P}^n$). It is easy to see that ϕ pulls back regular functions to regular functions: given $g/h : \mathbb{P}^n \to k$, $g(\phi(Q))/h(\phi(Q))$ is regular as well, since inserting $\phi(Q)$ (as above) will retain homogeneity as well as keep the denominator non-zero (as h has no zeroes).

(b) The twisted cubic is given parametrically by $[x : y : z : w] = [t^3 : t^2 u : t u^2 : u^3]$. We wish to project from $P = [0 : 0 : 1 : 0]$ onto the hyperplane $z = 0$. This yields the points $[t^3 : t^2 u : u^3] \in \mathbb{P}^2$. Note that these points satisfy the equation $x_0^2 x_2 - x_1^3 = 0$. But this is precisely the projective closure of the cuspidal cubic $y^3 = x^2$.

Problem I.3.15

(a) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Consider the product $X \times Y \subset \mathbb{A}^{n+m}$ with the induced Zariski topology. Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subset Z_i\}$ for $i = 1, 2$. The irreducibility of Y guarantees that $X_1 \cup X_2 = X$: if there were an x for which $x \times Y$ were not contained in a Z_i, this would yield a covering of Y by closed sets $Z_1 \cap Y$ and $Z_2 \cap Y$. Now consider the inclusion map $\iota : X \to X \times Y$; if ι is continuous then X_i must be closed, as $\iota^{-1}(Z_i) = X_i$. But the inclusion is obviously continuous, as any closed set in $X \times Y$ is defined by the vanishing of polynomials $f_\alpha(x_1, \dots, x_n, y_1, \dots, y_m)$, whose pullback to X is $f_\alpha(x_1, \dots, x_n, 0, \dots, 0)$, which is by definition a closed set of X. But if X_i are closed and cover X, either $X_1 = X$ or $X_2 = X$ and thus $Z_1 = X \times Y$ or $Z_2 = X \times Y$, i.e. $X \times Y$ is irreducible.

(b) Consider the homomorphism $\psi : A(X) \otimes_k A(Y) \to A(X \times Y)$ given by taking $(f \otimes g)(x, y)$ to $f(x)g(y)$. This map is clearly onto, as it produces the coordinate functions $x_1, \ldots, x_n, y_1, \cdots, y_m$.[1] Injectivity is more tricky: we start from the fact that every element of $A(X) \otimes_k A(Y)$ is a linear combination of elements of the form $f \otimes g$, where f and g are monomials. Moreover, all such $f \otimes g$ are linearly independent. We claim that none of the $f \otimes g$ are in the kernel of ψ, since $\psi(f \otimes g)(x, y) = f(x)g(y) = 0$ for all x, y iff $f(x) = 0$ for all x or $g(y) = 0$ for all y. This would imply $f = 0$ or $g = 0$, and therefore $f \otimes g = 0$. Now we are in a position to compute the kernel of ψ:

$$\psi(\sum c_{ij} f_i \otimes g_j) = \sum c_{ij} f_i g_j = 0$$

Since monomials of the form $f_i g_j$ are linearly independent, this implies $c_{ij} = 0$ for all i, j, hence the kernel is trivial and ψ is injective.

For those more versed in category theory, there is a simpler but more indirect proof. The functor that associates to varieties their coordinate rings is contravariant, and therefore takes products to coproducts. But the coproduct in the category of rings is $\otimes$, which proves the statement.

(c) Let π_X, π_Y denote the projections on X and Y respectively. Using lemma 3.6, these are morphisms as long as $x_i \circ \pi_X$ and $y_i \circ \pi_Y$ are regular functions on $X \times Y$. But $x_i \circ \pi_X(x, y) = x_i(x, 0) = x_i$, which is a polynomial function, thus regular. This works analogously for π_Y. Then we just need to check the universal property of products. Assume there exist morphisms $\phi_X : Z \to X$, $\phi_Y : Z \to Y$; then we construct the morphism $\phi_{XY} : Z \to X \times Y$ defined by $\phi_{XY}(z) = (\phi_X(z), \phi_Y(z))$. We have $\pi_X \circ \phi_{XY} = \phi_X$, $\pi_Y \circ \phi_{XY} = \phi_Y$, and ϕ_{XY} is clearly unique with these properties.

(d) It suffices to show that $\dim A(X) \otimes_k A(Y) = \dim A(X) + \dim A(Y)$. By Noether normalization, $A(X)$ is module-finite over the polynomial ring $k[t_1, \ldots, t_{d_1}]$ and $A(Y)$ is module-finite over the polynomial ring $k[s_1, \ldots, s_{d_2}]$ with $d_1 = \dim A(X)$ and $d_2 = \dim A(Y)$. In other words, every element of $A(X)$ or $A(Y)$ is the solution to some polynomial over the above polynomial rings, respectively. Next note that $R = k[t_1, \ldots, t_{d_1}, s_1, \ldots, s_{d_2}]$ must inject into $A(X) \otimes_k A(Y)$ via a map ϕ. Recall that every element in the tensor product can be written as a sum of elementary tensors $x \otimes y$ with $x \in A(X), y \in A(Y)$. Hence every element in the tensor product must also solve some polynomial over the ring R, i.e. $A(X) \otimes_k A(Y)$ is module-finite over R and $\dim A(X) \otimes_k A(Y) = d_1 + d_2$, as desired.

Problem I.3.17

(a) Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ by problem I.3.1, and hence it suffices to show that $\mathbb{P}^1 \subset \mathbb{P}^2$ is normal. If we parametrize the projective line by $x = 0$, its coordinate ring becomes the graded ring $k[y, z]$. Then $A(Y)$ is a UFD, and it is straightforward to check that as is the degree-zero subring of any localization of $A(Y)$. Hence, by theorem I.3.4b, any conic in $\mathbb{P}^1$ is normal.

(b)

(c) Consider the cuspidal cubic C in $\mathbb{A}^2$ given by $y^2 - x^3 = 0$. The ring of regular functions on C is isomorphic to the coordinate ring $k[x, y]/(y^2 - x^3)$. The local ring at $(0, 0)$ is given by the localization

$$\mathcal{O}_{(0,0),C} = \big(k[x, y]/(y^2 - x^3)\big)_{(x,y)} = k[x, y]_{(x,y)}/(y^2 - x^3).$$

[1] One might worry that generators may be missing from $A(X)$ or $A(Y)$ and hence that ψ may not produce all the generators of $A(X \times Y)$. This is actually not a problem: if $x_i \in I(X)$ then $x_i \in I(X \times Y)$ as well.

Now notice that the quotient y/x sitting in the fraction field of $\mathcal{O}_{(0,0),C}$ solves the monic polynomial $t^2 - x \in \mathcal{O}_{(0,0),C}[t]$, but $y/x \notin \mathcal{O}_{(0,0),C}$. Hence C is not normal.

(d) Let Y be an affine variety with coordinate ring $A(Y) \cong \mathcal{O}(Y)$. Suppose first that $\mathcal{O}(Y)$ is normal (integrally closed in its field of fractions). Let us show that Y is normal. It suffices to show that, more generally, the localization $S^{-1}A$ of a normal domain A is normal. If x is an element of the fraction field of A integral over $S^{-1}A$, it must solve a polynomial

$$x^n + \frac{a_{n+1}}{s_{n-1}}x^{n-1} + \cdots + \frac{a_0}{s_0}$$

for $a_i \in A, s_0 \in S$. If we denote by $r = s_0 \cdots s_{n-1}$ the product of the s_i, and multiply by r^n, we obtain

$$(rx)^n + \frac{a_{n-1}r}{s_{n-1}}(r^{n-1}x^{n-1}) + \cdots + \frac{a_0 r^n}{s_0}.$$

This implies that rx is integral over A, and so $rx \in A$ and $x \in S^{-1}A$. Thus, as $\mathcal{O}_{P,Y}$ is a localization of $\mathcal{O}(Y)$, which is normal, we find that Y is normal.

Now suppose instead that Y is normal (i.e. $\mathcal{O}_{P,Y}$ is normal for every $P \in Y$); we wish to show that $\mathcal{O}(Y)$ is normal. finish

(e) Let Y be an affine variety with $A(Y) \cong \mathcal{O}(Y)$ its coordinate ring. The integral closure N of $\mathcal{O}(Y)$ in its field of fractions is of course a normal domain. By theorem I.3.9A, N is a finitely generated k-algebra, and hence by the equivalence of categories of corollary I.3.8, we obtain a a variety $\tilde{Y}$ such that $\mathcal{O}(\tilde{Y}) = N$. Consider now a normal variety Z with a dominant morphism $\phi : Z \to Y$. We claim that the induced map $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(Z)$ is injective. To see this, note that $f \in \mathcal{O}(Y)$ is pulled back to regular function $f \circ \phi \in \mathcal{O}(Z)$. Note that if $f(\phi(X)) = 0$ then since $0 \in \mathbb{A}^1$ is Zariski-closed, so is $f^{-1}(0) \supset \phi(X)$, by continuity of f. The density of $\phi(X)$ then implies that $f^{-1}(0) = Y$, and hence $f(Y) = 0$, i.e. $f = 0$. This shows that ϕ^* is injective. Now, as Z is normal, $\mathcal{O}(Z)$ is normal as well (by the previous part). Since N is the "smallest" normal ring containing $\mathcal{O}(Y)$ (universal property of the normalization as rings), there is a unique injection $N \hookrightarrow \mathcal{O}(Z)$, which induces a (unique) morphism $Z \to Y$, just as desired.

Problem I.3.21

(a) It suffices to show that the addition and inversion maps are morphisms of varieties. But this follows from Lemma 3.6, as $\mu(a,b) = a + b$ and $\iota(a) = -a$ clearly define regular functions.

(b) Note that $\mathbb{G}_m$ is, as a variety, simply $\mathbb{A}^1 - \{0\}$, which in turn is isomorphic to an affine variety (c.f. problem I.3.1). Hence $\mathbb{G}_m$ is an affine variety, and the multiplication and inversion maps are morphisms again by Lemma 3.6.

(c) We define the group operation $\cdot$ on $\mathrm{Hom}(X, G)$ as

$$(f \cdot g)(x) = \mu(f(x), g(x)),$$

where $f, g \in \mathrm{Hom}(X, G)$ and μ is the operation on G and inversion as

$$f^{-1}(x) = \iota(f(x)),$$

where ι is the inversion on G. Thus defined, $\mathrm{Hom}(X, G)$ becomes a group by virtue of the group structure on G.

(d) By part (c), $\mathrm{Hom}(X, \mathbb{G}_a)$ inherits a group structure from $\mathbb{G}_a$, while the group structure on $\mathcal{O}(X)$ is the usual one. Any $f \in \mathrm{Hom}(X, \mathbb{G}_a)$ defines a regular function on X, and hence $f \in \mathcal{O}(X)$. Conversely, any regular function $\tilde{f} \in \mathcal{O}(X)$ is a morphism from X to $\mathbb{G}_a = \mathbb{A}^1$ (by Lemma 3.1) and hence contained in $\mathrm{Hom}(X, \mathbb{G}_a)$. The set equality $\mathrm{Hom}(X, \mathbb{G}_a) = \mathcal{O}(X)$ clearly extends to a group isomorphism, as the additive structure is clearly preserved.

(e) By part (c), $\mathrm{Hom}(X, \mathbb{G}_m)$ inherits a group structure from $\mathbb{G}_m$, while the group of units H in $\mathcal{O}(X)$ is the group of invertible, globally regular functions on X. Just as in part (d), we have the setwise equality $\mathrm{Hom}(X, \mathbb{G}_m) = H$, which extends to a group isomorphism, as the multiplicative structure is preserved.