Commutative Algebra: Problem Set 2

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Problem 5

Let k be a field. Then it's clear that $k[[t]]$, the formal power series ring over k, is local, as we can write it as a (disjoint) union $(x) \sqcup$ (units). To see this, note that any power series with a constant term can be inverted by a geometric series trick; given $x = c + \sum_{i=1}^{\infty} a_i t^i, c \in k$,

$$\frac{1}{x} = \frac{1}{c} \cdot \frac{1}{1 + c^{-1}(\sum_{i=1}^{\infty} a_i t^i)} = 1 + \left(c^{-1} \sum_{i=1}^{\infty} a_i t^i \right) + \left(c^{-1} \sum_{i=1}^{\infty} a_i t^i \right)^2 + \ldots,$$

which clearly yields another power series. Now, if a power series does not have a constant term, it must belong to the ideal (t). This ideal is maximal, as adding an additional element would mean adding a unit. Then, by the theorem proved in class, we see that $(k[[t]], (t))$ is a local ring.

Problem 6

Consider again the formal power series ring $A[[t]]$, where A is an domain. Hence $A[[t]]$ is an domain as well, and thus has (0) as a prime ideal. Additionally, since (t) is clearly a minimal (and maximal, by locality) prime of the ring, we see that $A[[t]]$ has exactly two prime ideals.

Finding a local ring with three primes is a little more difficult; consider the ring $\mathbb{C}[x, y]/(xy)$. In the last problem set, we computed the primes to be $(x), (y), (x, y), (x - \lambda, y), (x, y - \mu)$ for $\lambda, \mu\mathbb{C}^{\times}$. Localizing this ring about the prime (x, y) eliminates the last two (as they are not contained in (x, y)). Hence we are left with the three primes $(x), (y), (x, y)$.

Problem 7

Let $R = k[[t]]$ where k is a field. We wish to find an example of a module M over R such that $M = tM$. This does not contradict Nakayama's lemma, as the lemma applies only to M a finite R-module. Consider the module $M = R[[s]]/(ts - 1) \cong k[[t, t^{-1}]]$. Elements of this ring are of the form $\sum_{i=-\infty}^{\infty} a_i t^i$, and clearly multiplying by t gives us back an element of M. Furthermore, $M \subset tM$, as every element $x = \sum_{i=-\infty}^{\infty} a_i t^i$ of M can be written as an element of tM, i.e. $t \sum_{i=-\infty}^{\infty} a_{i-1} t^i$. Hence, $M = tM$ and $M \neq 0$, as desired.

Problem 8

Let $R = \mathbb{C}[x]$ be the polynomial ring over the complex numbers. Let $\mathfrak{m}_n$ for $n = 1, 2, 3, \ldots$ be an infinite sequence of pairwise distinct maximal ideals of R. Consider the product space $S = \prod_i^{\infty} R/\mathfrak{m}_i$. Suppose there exists a surjection $R \overset{\phi}{\twoheadrightarrow} S$. It's clear that the ϕ sends $\mathbb{C} \subset \mathbb{C}[x]$ to $\mathbb{C} \cdot (1, \ldots) \subset S$. Note that there must exist an $a \in \mathbb{C}[x]$ such that $\phi(a) = (1, 0, 0, \ldots)$; since the maximal ideals under consideration are of the form $(x - \lambda)$ for $\lambda \in \mathbb{C}$, this implies that a is in $\mathfrak{m}_i$ for $i > 1$, i.e. a must have an infinite number of roots. Of course, this implies that $a = 0$, which is a contradiction, as $\phi(0) = (0, \ldots)$. Hence there exists no such surjection.

Problem 9

Let k be a field. We wish to find the minimal prime ideals of $A = k[x, y, z]/(xy, xz, yz)$. The minimal primes of A are in correspondence to the smallest primes of $k[x, y, z]$ containing (xy, xz, yz). Note first of all that whether or not k is algebraically closed is irrelevant here as we are looking only for minimal primes. None of the ideals $(x), (y), (z)$ contain (xy, xz, yz) as (x) cannot generate yz, for example. Furthermore, consider principal ideals generated by polynomials of higher order (or of the form $x-\lambda$) will not work either as they will not generate the needed elements (or will not be prime). Next one considers ideals generated by two elements. Right away we see that $(x, y), (x, z), (y, z)$ are prime and that they contain (xy, xz, yz). Let us check that these are minimal. First off, it's clear that any ideal generated by more than 2 elements will not necessarily be contained in these, and hence will not be minimal. Furthermore, placing any higher-order polynomials in the place of x, y, or z will either not be prime or will not generate xy, xz, or yz. Hence we conclude that $(x, y), (x, z)$, and (y, z) are indeed the minimal prime ideals of $k[x, y, z]/(xy, xz, yz)$. (Moreover we can see this geometrically by visualizing the coordinate axes, as discussed in class.)

Problem 10