Lie Groups PSET 1

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Proposition 1. *The matrix groups $SO(n)$ and $SU(n)$ are compact and connected.*

Proof. We will use throughout this problem that path-connectedness is equivalent to connectedness on topological manifolds.

Let us start by showing that $SU(n)$ is connected. We can use the fact that every unitary matrix has an orthonormal basis of eigenvectors to write any $U \in SU(n)$ as

$$U = U_1 \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} U_1^{-1}$$

where U_1 is unitary and $\theta_i \in \mathbb{R}$. Note that since $U \in SU(n)$, we must have that $\sum_i \theta_i$ is an integer multiple of 2π. Of course, we can simply add or subtract multiples of 2π from any of the θ_i to force the sum to zero. Hence if we consider the matrix

$$U(t) = U_1 \begin{pmatrix} e^{i(1-t)\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i(1-t)\theta_n} \end{pmatrix} U_1^{-1}$$

for $t \in [0, 1]$, we obtain a continuous path in $SU(n)$ from U to the identity; the path is wholly contained in $SU(n)$ as the determinant is of the form $\exp(i(1-t)\sum_i \theta_i)$, which must be 1 for all t if the sum is zero.

Let us now turn to $SO(n)$. The case $n = 1$ is trivial, so let $n \geq 2$. We wish to path-connect an arbitrary $T \in SO(n)$ to the identity. Let $\{e_j\}$ be the standard basis of $\mathbb{R}^n$ and $v_j = Te_j$ be the resulting orthonormal basis. Now suppose we have a function u that spits out orthonormal bases in a continuous fashion that at time 0 gives us $\{e_j\}$ and at time t gives us $\{v_j\}$ (we shall construct u shortly). Then the linear maps $T_t : V \to V$ definied by $T_t(e_j) = u_j(t)$ are orthogonal, and since $T_0 = \text{Id}$, they in fact have determinant one. This shows path-connectedness. Let us now construct u by inducting on n. Given the two bases $\{e_i\}, \{v_i\}$, let W be the span of e_1, v_1 if they are independent and any subspace containing the line spanned by e_1, v_1 if not. We can orthogonally decompose $\mathbb{R}^n$ as $W \oplus W^\perp$. It's clear that we can construct a rotation matrix R that rotates e_1 into v_1 in W (and leaves $W^\perp$ unchanged). It's clear now that if we define T_t to rotate in this manner we get a continuous path $t \mapsto T \circ T_t$ in $SO(n)$.

Let us now show that $SO(n)$ is compact. First let us show that $O(n)$ is closed as a subset of $\mathbb{R}^{n^2}$. Let A_n be a sequence of orthogonal matrices that converges to A; hence we have that $A_n^T A_n = 1$. We wish to show that A is orthogonal. We can do this by noting that A_n converges to A and A_n^T converges to A^T and hence $A_n^T A_n$ converges to $A^T A$. But since $A_n^T A_n = 1$ it's clear that $A^T A$ must converge to one as well. Now consider $SO(n)$ - it is clearly a closed subset of $O(n)$ as it is an inverse image of a closed set ($\{1\}$). Additionally, the matrix elements of $SO(n)$ are obviously bounded (in $\mathbb{R}^{n^2}$)and hence $SO(n)$ is compact. The proof that $SU(2)$ is compact proceeds in almost exactly the same fashion, as it is closed in $U(n)$, which can shown to be compact. □

Proposition 2. *$SO(n)/SO(n-1) = S^{n-1}$ and $SU(n)/SU(n-1) = S^{2n-1}$ (for $n \geq 2$).*

Proof. Consider the action of $SO(n)$ on S^{n-1} given by the usual rotation. This is clearly a smooth action, as it is a restriction of the action of $GL(n)$ on $\mathbb{R}^n$ to the regular submanifold S^{n-1}. It is easy to see that this action is transitive; it suffices to show that there exists a $g \in SO(n)$ such that $g \cdot e_1 = v$ for any $v \in S^{n-1}$, where $e_1 = (1, 0, \ldots, 0)$. But this is obvious, since one can take the identity matrix and replace the first column by the (unit) vector v - this yields an orthogonal matrix (that can be made special orthogonal as necessary by multiplying one of the other columns by -1) satisfying the required condition. Hence the orbit of, say, the north pole $(0, \cdots, 0, 1)$ is all of S^{n-1}. Note additionally that the stabilizer of this point is the subgroup of $SO(n)$ that keeps the last component fixed, i.e. the subgroup $SO(n-1)$, which rotates the first $n-1$ components and leaves the nth component fixed. Hence, by Kirilov's Corollary 2.21, since the orbit S^{n-1} is trivially a submanifold of S^{n-1}, the quotient $SO(n)/SO(n-1) \cong S^{n-1}$.

Consider S^{2n-1} as embedded in $\mathbb{C}^n$ and consider the action of $SU(n)$ on it. We may argue almost exactly as above. The action is smooth, as it is a restriction of $GL(n)$ (over $\mathbb{C}$) on $\mathbb{C}^n$ to the submanifold S^{2n-1}. The action is transitive using exactly the same argument as above in $\mathbb{C}^n$. Hence the orbit of the north pole is again the whole sphere S^{2n-1}. Of course, the stabilizer of this point is the subgroup of $SU(n)$ that acts only on the first $2n-2$ (real) coordinates, i.e. $SU(n-1)$. Hence we see that $SU(n)/SU(n-1) \cong S^{2n-1}$. □

Proposition 3. *Right-invariant vector fields, etc.*

Proof. Let G be a Lie group and let $\mathcal{R}$ be the set of right-invariant vector fields on G. It should be clear that $\mathcal{R}$ is a real vector space. Let us define the Lie bracket operation as usual to be $[X, Y] = XY - YX$ for $X, Y \in \mathcal{R}$. It is straightforward but tedious to check that the bracket operation satisfies the Lie algebra bracket conditions. Hence it suffices to show that $\mathcal{R}$ is closed under this bracket:

$$(R_g)_*[X, Y] = [(R_g)_*X, (R_g)_*Y] = [X, Y].$$

Here we have used the naturality of the Lie bracket (since R_g is a diffeomorphism) in the first step and the fact that X and Y are right-invariant in the second step. Hence $\mathcal{R}$ is a Lie algebra in a very natural way. Let us show that it is isomorphic to the Lie algebra of G, $\mathfrak{g} = T_1G$. Define the map $\phi : \mathfrak{g} \to \mathcal{R}$ as taking the vector $X \in \mathfrak{g}$ to the vector field defined as $v|_g = (R_g)_*X$. Let us first check that $v = \phi(X)$ is indeed a smooth vector field, i.e. that for any $f \in C^\infty(G)$, vf is smooth. Pick a smooth curve $\gamma : (-\delta, \delta) \to G$ such that $\gamma(0) = 1$ and $\gamma'(0) = X$. Then for all $g \in G$,

$$vf|_g = v|_g f = (R_g)_*Xf = \gamma'(0)\,(f \circ R_g) = \left.\frac{d}{dt}\right|_{t=0} (f \circ R_g \circ \gamma)\,(t),$$

which is clearly smooth. Next, let us check that v is right-invariant, i.e. that $(R_h)_*v|_g = v|_{gh}$:

$$(R_h)_*v|_g = (R_h)_*(R_g)_*X = (R_h \circ R_g)_*X = (R_{gh})_*X = v|_{gh},$$

as desired.

Note that ϕ is indeed a morphism of Lie algebras, as (evaluating the vector fields at g)

$$\phi([X, Y])|_g = (R_g)_*[X, Y] = [(R_g)_*X, (R_g)_*Y] = [\phi(X)|_g, \phi(Y)|_g]$$

again by the naturality of the Lie bracket. Furthermore, ϕ is injective:

$$\begin{aligned}\phi(X)|_g &= \phi(Y)|_g\\(R_{g^{-1}})_*(R_g)_*X &= (R_{g^{-1}})_*(R_g)_*Y\\(R_{g^{-1}}\circ R_g)_*X &= (R_{g^{-1}}\circ R_g)_*Y\\X &= Y.\end{aligned}$$

Surjectivity is also fairly clear. Given a right-invariant vector field v, let $X = v|_1$. Right-invariance tells us that $(R_h)_*v|_g = v|_{gh}$ and applying this at $g = 1$ gives us the condition that $(R_h)_*v_1 = (R_h)_*X = v|_h$. But this is precisely the statement that $\phi(X) = v$, and thus ϕ is surjective. Consequently we see that $\mathcal{R} \cong \mathfrak{g}$ as Lie algebras.

Now consider the diffeomorphism $\psi : g \in G \mapsto \psi(g) = g^{-1} \in G$. It's clear that, in general, $\phi \circ L_{g^{-1}} = R_g \circ \phi$ (consider the action on h - both sides yield $h^{-1}g$. Differentiating both sides, we find that upon acting on a left-invariant vector field we must have

$$d(\phi \circ L_{g^{-1}})|_h = d\phi_{L_{g^{-1}h}} \circ L_{g^{-1}} = d(R_g \circ \phi)|_h = dR_{g\phi(h)} \circ d\phi.$$

But this means that $dR_{gh^{-1}} \circ d\phi = d\phi$, i.e. our vector field must be right-invariant as well. Note additionally that if X is left-invariant then $d\phi(X)$ is a vector field whose value at the identity is $-X$ since

$$\left.\frac{d}{dt}\right|_{t=0} \phi(e^{tX}) = \left.\frac{d}{dt}\right|_{t=0} e^{-tX} = -X.$$

Indeed, the map $X \mapsto d\phi(X)$ is an isomorphism of the Lie algebras of left and right-invariant vector fields on G as ϕ is a diffeomorphism and hence it's derivative is an isomorphism; all that remains to check is the compatibility of the Lie bracket. But this holds just as above using naturality of the Lie bracket:

$$d\phi([X,Y]) = [d\phi(X), d\phi(Y)].$$

□

Proposition 4. *The Grassmanian $Gr(k,n)$ of n-dimensional subspaces of $\mathbb{R}^n$ is a $O(n,\mathbb{R}^n)$-space and can be identified as the quotient $O(n)/\left(O(k) \times O(n-k)\right)$.*

Proof. Take two distinct subspaces $V, W \subset \mathbb{R}^n$. Let $\{v_i\}, \{w_i\}$ be their orthonormal bases respectively. Since each of the v_i, w_i are normal, they live on S^{n-1}. Because S^{n-1} is a $O(n)$-space, it's clear that we can find an orthogonal transformation that rotates $\{v_i\}$ to $\{w_j\}$. Of course, points in $Gr(k,n)$ are k-dimensional subspaces and hence determined by bases such as these. Consequently, $Gr(k,n)$ is a homogeneous $O(n)$-space. Given any k-dimensional subspace $V \subset \mathbb{R}^n$, we can split $\mathbb{R}^n$ as $V \oplus V^\perp$. Note that we can rotate the summands independently by elements of $O(k)$ and $O(n-k)$ respectively. Since rotations that take V to V stabilize the point $V \in Gr(k,n)$ both $O(k)$ and $O(n-k)$ stabilize any point in $Gr(k,n)$. Because these rotations can be performed in a completely disjoint manner, the subgroup $O(k) \times O(n-k) \leqslant O(n)$ stabilizes any point of $Gr(k,n)$, and thus by Kirillov 2.21 we see that

$$Gr(k,n) = O(n)/\left(O(k) \times O(n-k)\right).$$

Since the dimension of the $O(n)$ is $n(n-1)/2$, the dimension of the Grassmanian can be found by subtracting appropriately to get $k(n-k)$. □

Proposition 5. *Kirillov 2.8, 2.9, 2.10*

Proof. Let $\phi : SU(2) \to GL(3, \mathbb{R})$ be the map that takes g to the matrix of $Ad(g)$ (in the basis of i times the Pauli matrices). In other words, we have a map $G \xrightarrow{\phi} GL(\mathfrak{su}_2)$ such that $Ad(g)X = gXg^{-1}$ for some $g \in G$ and $X \in \mathfrak{su}_2$. It is easy to see that $Ad(g)$ is a linear map, as $Ad(g)(aX_1 + bX_2) = g(aX_1+bX_2)g^{-1} = aAd(g)X_1+bAd(g)X_2$. Additionally, $Ad(g)$ is an element of $SO(\mathfrak{su}_2) \cong SO(3)$ as it preserves the standard inner product. We can see this by first writing out an element (x_1, x_2, x_3) of $\mathfrak{su}_2$ as

$$X = \begin{pmatrix} ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & -ix_3 \end{pmatrix},$$

and then noting via a simple computation that the determinant gives us the inner product $\det X = x_1^2 + x_2^2 + x_3^2$. Of course, the determinant is preserved: $\det Ad(g)X = \det gXg^{-1} = \det X$. Hence $Ad(g)$ also preserves the inner product and is orthogonal. Note that ϕ is a morphism of Lie groups:

$$Ad(gh)X = ghXh^{-1}g^{-1} = gAd(h)Xg^{-1}$$

and hence $Ad(gh) = Ad(g) \circ Ad(h)$.

Let us now compute explicitly the map of tangent spaces $\phi_* : \mathfrak{su}_2 \to \mathfrak{so}_3$. Consider an integral curve $\gamma(t)$ of some $X \in \mathfrak{su}_2$ about the identity of $SU(2)$ (i.e. $\gamma'(0) = X$). We wish to compute the derivative at $t = 0$ of $Ad(\gamma(t))Y = \gamma(t)Y\gamma(t)^{-1}$:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} Ad(\gamma(t))Y &= \gamma'(0)Y\gamma(0) + \gamma(0)Y\left(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}\right) \\ &= XY - YX \\ &= [X, Y]. \end{aligned}$$

The derivative is simply the Lie bracket operation. Since $\mathfrak{su}_2$ and $\mathfrak{so}_3$ are both three-dimensional Lie algebras, it's clear that they are isomorphic as vector spaces. All that remains is to show that the isomorphism is in fact a Lie algebra isomorphism, i.e. that $\phi_*([X, Y]) = [\phi_*(X), \phi_*(Y)]$. Let us explicitly compute the action of ϕ_* on the basis $i\sigma_j$. Let us first look at

$$\begin{aligned} \phi_*(i\sigma_1)(ai\sigma_1 + bi\sigma_2 + ci\sigma_3) &= [i\sigma_1, ai\sigma_1 + bi\sigma_2 + ci\sigma_3] \\ &= -b[\sigma_1, \sigma_2] - c[\sigma_1, \sigma_3] \\ &= -2bi\sigma_3 + 2ci\sigma_2 \end{aligned}$$

We can perform similar computations and find that ϕ_* sends

$$\begin{aligned} i\sigma_1 &\to \begin{pmatrix} & & \\ & & 2 \\ & -2 & \end{pmatrix} \equiv 2\ell_1 \\ i\sigma_2 &\to \begin{pmatrix} & & -2 \\ & & \\ 2 & & \end{pmatrix} \equiv 2\ell_2 \\ i\sigma_3 &\to \begin{pmatrix} & 2 & \\ -2 & & \\ & & \end{pmatrix} \equiv 2\ell_3. \end{aligned}$$

Now let us verify that ϕ_* is a Lie algebra morphism; we do this for one case as the rest are simple computations:

$$\begin{aligned} \phi_*([i\sigma_1, i\sigma_2]) &= \phi_*(-2i\sigma_3) \\ &= -4\ell_3 = [2\ell_1, 2\ell_2] \\ &= [\phi(i\sigma_1, i\sigma_2)]. \end{aligned}$$

Next consider the kernel of ϕ - as the kernel of a Lie group homomorphism, it must be a normal subgroup. Since the derivative ϕ_* is an isomorphism we see by the inverse function theorem that there exists an open set U about any element $g \in \ker \phi$ such that $U \to \phi(U)$ is a diffeomorphism. This of course means that no other element in U can be in $\ker \phi$, as otherwise this would violate injectivity. Hence the elements of $\ker \phi$ must not accumulate, and thus $\ker \phi$ is a discrete subgroup of $SU(2)$.

Moreover, since ϕ_* is surjective, ϕ is a (smooth) submersion. Hence the subgroup Im ϕ of $SO(3)$ is in fact open, simply because submersions are open maps (and $SU(2)$ is by definition open in its own topology).

Finally, note that ϕ is in fact a covering map as the fiber over any point is discrete; by the classification of covering spaces we know that $\ker \phi$ must be $\mathbb{Z}_2$ as $\pi_1(SO(3)) \cong \mathbb{Z}_2$. Another way of showing that $\ker \phi \cong \mathbb{Z}_2$ is to show that the only solution to the linear system of equations $gXg^{-1} = X$ is for $g = \pm\text{Id}$. This is straightforward but tedious, which is why we present the topological proof. Next, note that the map ϕ is in fact surjective (it covers all of $SO(3)$) because $SO(3)$ is connected and ϕ_* is surjective (see Kirillov 2.10). By the first isomorphism theorem, then, we have that $SU(2)/\mathbb{Z}_2 \cong SO(3)$, as desired. $\square$