Lie Groups PSET 2

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Proposition 1. *Kirillov 2.15*

Proof. Let us first show that $\mathrm{End}_{\mathbb{H}} \mathbb{H}^n$ is naturally identified with the algebra of $n \times n$ quaternionic matrices. Let $\{e_\alpha\}$ be an orthonormal basis for $\mathbb{H}^n$ over $\mathbb{H}$ (the orthonormality condition will be useful later). Then we may write any $\vec{h} = \sum_\alpha e_\alpha h_\alpha$, as we are considering $\mathbb{H}^n$ as a right $\mathbb{H}$-module. Applying an endomorphism A, we find that $A(\vec{h}) = A\left(\sum_\alpha e_\alpha h_\alpha\right) = \sum_\alpha A(e_\alpha h_\alpha) = \sum_\alpha A(e_\alpha) h_\alpha$. Denoting the β component of $A(e_\alpha)$ by the quaternion $A_{\beta\alpha}$, we can rewrite

$$A(\vec{h}) = \sum_{\alpha,\beta} e_\beta A_{\beta\alpha} h_\alpha.$$

From this, it is clear that the endomorphism is characterized precisely by this $n \times n$ matrix $A_{\beta\alpha}$ of quaternions.

Now let us define an $\mathbb{H}$-valued form on $\mathbb{H}^n$ given by

$$(\vec{h}, \vec{h}') = \sum_i \bar{h}_i h_i'$$

and let $U(n, \mathbb{H})$ be the group of unitary quaternionic transformations:

$$U(n, \mathbb{H}) = \left\{ A \in \mathrm{End}_{\mathbb{H}} \mathbb{H}^n | (A\vec{h}, A\vec{h}') = (\vec{h}, \vec{h}') \right\}.$$

It's clear that $U(n, \mathbb{H})$ is indeed a group - composing two transformations yields another such transformation, and the identity is contained in the group. The inverses are given by just the matrix inverses (that clearly exist because no $A \in U(n, \mathbb{H})$ has determinant 0):

$$(\vec{h}, \vec{h}') = (AA^{-1}\vec{h}, AA^{-1}\vec{h}') = (A^{-1}\vec{h}, A^{-1}\vec{h}').$$

Using the identification of endomorphisms with matrices that we established in the previous paragraph, let us find a matrix characterization for elements of $U(n, \mathbb{H})$:

$$\begin{aligned}
(A\vec{h}, A\vec{h}') &= \left(\sum_{\alpha,\beta} e_\beta A_{\beta\alpha} h_\alpha, \sum_{\gamma,\delta} e_\gamma A_{\gamma\delta} h_\delta' \right) \\
&= \sum_{\beta,\gamma} (e_\beta, e_\gamma) \left(\overline{\sum_\alpha A_{\beta\alpha} h_\alpha} \right) \left(\sum_\delta A_{\gamma\delta} h_\delta' \right) \\
&= \sum_{\alpha,\beta,\delta} \overline{A_{\beta\alpha} h_\alpha} A_{\beta\delta} h_\delta' = \sum_{\alpha,\beta,\delta} \overline{h_\alpha} \cdot \overline{A_{\beta\alpha}} A_{\beta\delta} h_\delta'.
\end{aligned}$$

But this is precisely $(\vec{h}, \vec{h}')$ if and only if the middle terms sum to the identity, i.e. $A^*A = 1$.

Define now a map $\phi : \mathbb{C}^{2n} \to \mathbb{H}^n$ given by $(z_1, \dots, z_{2n}) \mapsto (z_1 + jz_{n+1}, \dots, z_n + jz_{2n})$. If we treat $\mathbb{H}^n$ as a complex vector space via the scalar multiplication $z(h_1, \dots, h_n) = (h_1 z, \dots, h_n z)$, ϕ is in fact an isomorphism of complex vector spaces. To see this, it suffices to show that ϕ is injective, but this is obvious because $z_i + jz_{n+1} = a_i + ib_i + ja_{n+i} - kb_{n+i} = 0$ implies that the $a_i = a_{n+i} = b_i = b_{n+i} = 0$, and hence that $z_i = z_{n+i} = 0$, i.e. $\ker \phi = \{\vec{0}\}$. Next let us show that this isomorphism identifies $\operatorname{End}_{\mathbb{H}} \mathbb{H}^n$ with $\{A \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^{2n} | \bar{A} = J^{-1}AJ\}$ where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

First note that the map $\vec{h} \mapsto \vec{h}j$ is identified with $\vec{z} \mapsto J\bar{\vec{z}}$, because quaternionic multiplcation by j is equivalent to multiplying $(z_i + jz_{n+i})j = z_i j + jz_{n+i}j = -\bar{z}_{n+i} + j\bar{z}_i$.

Under the identification $\mathbb{C}^{2n} \cong \mathbb{H}^n$ above, the quaternionic form simplifies as follows. Let $h_i = z_i + jz_{n+i}$ and $h'_i = z'_i + jz'_{n+1}$. Then, by definition,

$$\begin{aligned}(\vec{h}, \vec{h}') &= \sum_l (\bar{z}_l + \overline{jz_{n+l}})(z'_l + jz'_{n+l}) \\ &= \sum_l \left(\bar{z}_l z'_l + \bar{z}_{n+l} z'_{n+l} - \bar{z}_{n+l} j z'_l + \bar{z}_l j z'_{n+l}\right).\end{aligned}$$

Anti-commuting the j and factoring it out of the last two terms, we see that the first term is the usual Hermitian inner product and the last terms are the standard bilinear skew-symmetric form in $\mathbb{C}^{2n}$ (multiplied by j). For the whole form to be preserved, each form must be preserved, and hence we see that $Sp(n) = Sp(n, \mathbb{C}) \cap SU(2n)$. □

Proposition 2. *Kirillov 3.16*

Proof. We wish to show that $\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$. Note that (from Kirillov) $\mathfrak{sp}(n, \mathbb{C})$ is the set of $2n \times 2n$ complex matrices x such that $x + J^{-1}x^t J = 0$. As $Sp(n)$ is the intersection of $Sp(n, \mathbb{C})$ and $SU(2n)$, it's clear that $\mathfrak{sp}(n)$ is the set of matrices that satsify this condition in addition to the condition that $x^\dagger = -x$. If we now complexify and consider $\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n)$, the first condition $x + J^{-1}x^t J = 0$ still holds (by linearity) but the second no longer holds, as $(ix)^\dagger = -ix^\dagger = ix$ instead of $-ix$. This is precisely what we wanted to show. □

Proposition 3. *Kirillov 3.1*

Proof. Let $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Suppose $x \in \mathfrak{sl}(2, \mathbb{R})$ such that $\exp(x) = X$. It's clear that the eigenvalues of x must exponentiate to the eigenvalues of X, as eigenvalues of powers are simply powers of eigenvalues. Hence we see that if $\lambda_1, \lambda_2 \in \mathbb{C}$ are the eigenvalues of x then $e^{\lambda_1} = e^{\lambda_2} = -1$. This yields

$$\begin{aligned}\lambda_1 &= (2n+1)\pi i \\ \lambda_2 &= (2m+1)\pi i\end{aligned}$$

for some $m, n \in \mathbb{Z}$. The condition that $x \in \mathfrak{sl}(2, \mathbb{R})$ requires that $\lambda_1 + \lambda_2 = 0$, i.e. that $m = -(n+1)$. Hence the λ_1, λ_2 must be distinct, i.e. diagonalizable over the complex numbers. In other words, we have that $P^{-1}xP = \lambda$ where λ is diagonal with λ_1, λ_2. Exponentiating, we find that

$$P^{-1} \exp(x) P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moving the P's to the other side, we find that $\exp(x) = -I$, which is clearly not equal to X. Hence we obtain a contradiction - there can exist no such x. $\square$

Proposition 4. *Kirillov 3.5*

Proof. Consider $\mathbb{R}^3$ as a vector space with a Lie bracket given by the cross-product. It's clear that this is indeed a Lie bracket, because for $\vec{v}, \vec{w}, \vec{u} \in \mathbb{R}^3$, the cross product is clearly bilinear, and satisfies $\vec{v} \times \vec{w} = -\vec{v} \times \vec{w}$. The Jacobi identity follows by first noting that for $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, the triple vector product rule gives us $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - c(\vec{a} \cdot \vec{b})$ where the product is the usual inner product. Then, permuting the orders, we find that

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

where the cancellations are acheieved by the commutativity of the inner product. Hence $\mathbb{R}^3$ with the cross product forms a Lie algebra.

Now consider the usual basis for $\mathfrak{so}(3, \mathbb{R})$ denoted by J_x, J_y, J_z and the basis for $\mathbb{R}^3$ denoted by x, y, z. Let ϕ be the vector isomorphism that takes J_x to x, J_y to y, and J_z to z. Moreover, ϕ is a Lie algebra homomorphism as one can check (it is straightforward but tedious), as the basis of $\mathfrak{so}(3, \mathbb{R})$ follows precisely the same commutation relations as does the basis for $\mathbb{R}^3$.

Consider the standard action of $\mathfrak{so}(3, \mathbb{R})$ on $\mathbb{R}^3$ given by matrix multiplication $a \cdot \vec{v} = \phi(a) \times \vec{v}$. We wish to show that this action coincides with the action of $\mathbb{R}^3$ on itself via the cross product: $a \cdot \vec{v} = \phi(a) \times \vec{v}$. To do this, first note that:

$$\begin{aligned}
J_x \cdot x &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \\
J_x \cdot y &= z \\
J_x \cdot z &= -y \\
J_y \cdot x &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -z \\
J_y \cdot y &= \vec{0} \\
J_y \cdot z &= x \\
J_z \cdot x &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = y \\
J_z \cdot y &= -x \\
J_z \cdot z &= \vec{0}
\end{aligned}$$

which we can use to simplify

$$\begin{aligned}
a \cdot v &= (\alpha J_x + \beta J_y + \gamma J_z) \cdot (v_x x + v_y y + v_z z) \\
&= \alpha \left(v_y z - v_z y \right) + \beta \left(-v_x z + v_z x \right) + \gamma \left(v_x y - v_y x \right) \\
&= \left(\beta v_z - \gamma v_y \right) x + \left(\gamma v_x - \alpha v_z \right) y + \left(\alpha v_y - \beta v_z \right) z.
\end{aligned}$$

But this is precisely the formula for the cross product of the two vectors $\phi(a)$ and v, as desired. $\square$

Proposition 5. *Kirillov 3.8*

Proof. Consider the open cell $U = \mathbb{C} \subset \mathbb{CP}^1$ with coordinates given by $t = x/y$. The action of $SL(2,\mathbb{C})$ on a point in U is given by

$$A(x:y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x:y) = (ax+by : cx+dy) = (at+b : ct+d)$$

which we will denote by $[\frac{at+b}{ct+d}]$. Denoting the representation as π, we can consider the usual action of $SL(2,\mathbb{C})$ on functions defined on U:

$$\pi(A)(f)([t]) = f(\pi(A^{-1})[t]).$$

The induced action of $\mathfrak{g}$ by vector fields on functions defined on $\mathbb{CP}^1$ is then given:

$$\pi'(X)(f)([t]) = \frac{d}{ds}\bigg|_{s=0} f(\pi(e^{-sX})[t]) = \frac{d}{ds}\bigg|_{t=0} \left(\pi(e^{-sX})([t])\right) \frac{df}{dt}([t]),$$

which we can evaluate for the basis $\{H, X, Y\} \subset \mathfrak{sl}_2\mathbb{C}$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

whose exponentials yield:

$$\exp(-sH) = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix},\ \exp(-sX) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix},\ \exp(-sY) = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}.$$

Let us now compute the coefficient in front of df/dt in the expression for π' above for the various basis elements:

$$\begin{aligned} \frac{d}{ds}\bigg|_{t=0} \pi \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{pmatrix}([t]) &= \frac{d}{ds}\bigg|_{t=0} \left[e^{-2s}t\right] = [-2t] \\ \frac{d}{ds}\bigg|_{t=0} \pi \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}([t]) &= \frac{d}{ds}\bigg|_{t=0} [t-s] = [-1] \\ \frac{d}{ds}\bigg|_{t=0} \pi \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}([t]) &= \frac{d}{ds}\bigg|_{t=0} \left[\frac{t}{-st+1}\right] = [t^2]. \end{aligned}$$

Hence we see, in local coordinates t for U, that the vector fields corresponding to H, X, Y are precisely

$$\begin{aligned} H &\longleftrightarrow -2t\frac{d}{dt} \\ X &\longleftrightarrow -\frac{d}{dt} \\ Y &\longleftrightarrow t^2\frac{d}{dt}. \end{aligned}$$

$\square$