QM for Mathematicians: PSET 7

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Problem 1

We wish to show that elementary antisymmetric matrices L_{jk} satisfy the same commutation relations as the quadratic elements $\gamma_j\gamma_k/2$ of the Clifford algebra $\mathrm{Cliff}(n, \mathbb{R})$. First note that $(L_{jk})_{\alpha\beta} = -\delta_{j\alpha}\delta_{k\beta} + \delta_{j\beta}\delta_{k\alpha}$, and so we have

$$\begin{aligned}
[L_{jk}, L_{mn}]_{\alpha\beta} =& (L_{jk}L_{mn})_{\alpha\beta} - (L_{mn}L_{jk})_{\alpha\beta} \\
=& (L_{jk})_{\alpha\gamma}(L_{mn})_{\gamma\beta} - (L_{mn})_{\alpha\gamma}(L_{jk})_{\gamma\beta} \\
=& \sum_\gamma \left(-\delta_{j\alpha}\delta_{k\gamma} + \delta_{j\gamma}\delta_{k\alpha}\right)\left(-\delta_{m\gamma}\delta_{n\beta} + \delta_{m\beta}\delta_{n\gamma}\right) \\
& - \sum_\gamma \left(-\delta_{m\alpha}\delta_{n\gamma} + \delta_{m\gamma}\delta_{n\alpha}\right)\left(-\delta_{j\gamma}\delta_{k\beta} + \delta_{j\beta}\delta_{k\gamma}\right) \\
=& \sum_\gamma \delta_{m\gamma}\delta_{j\gamma}L_{kn} + \delta_{n\gamma}\delta_{j\gamma}L_{mk} + \delta_{k\gamma}\delta_{n\gamma}L_{jm} + \delta_{k\gamma}\delta_{m\gamma}L_{nj} \\
=& \delta_{mj}L_{kn} + \delta_{nj}L_{mk} + \delta_{kn}L_{jm} + \delta_{km}L_{nj},
\end{aligned}$$

where we have expanded out the matrix multiplication and then, in the last step, collapsed the sum over γ via the Kronecker deltas.

Now, for the Clifford algebra, using the anticommutation relation $[\gamma_j, \gamma_k]_+ = 2\delta_{jk}$,

$$\begin{aligned}
\frac{1}{4}[\gamma_j\gamma_k, \gamma_m\gamma_n] &= \frac{1}{4}\left(\gamma_j\gamma_k\gamma_m\gamma_n - \gamma_m\gamma_n\gamma_j\gamma_k\right) \\
&= \frac{1}{4}\left(\gamma_j\gamma_k\gamma_m\gamma_n + \gamma_m\gamma_j\gamma_n\gamma_k + 2\delta_{jn}\gamma_m\gamma_k\right) \\
&= \frac{1}{4}\left(\gamma_j\gamma_k\gamma_m\gamma_n - \gamma_j\gamma_m\gamma_n\gamma_k + 2\delta_{jn}\gamma_m\gamma_k - 2\delta_{jm}\gamma_n\gamma_k\right) \\
&= \cdots \\
&= \frac{1}{2}\delta_{jm}\gamma_k\gamma_n + \frac{1}{2}\delta_{nj}\gamma_m\gamma_k + \frac{1}{2}\delta_{kn}\gamma_j\gamma_m + \frac{1}{2}\delta_{km}\gamma_n\gamma_j,
\end{aligned}$$

recovering the same commutation relations as above. Thus, the Lie algebras of Spin(n) and SO(n) are the same.

Problem 2

To show that conjugation by an exponential of the quadratic Clifford algebra $\frac{1}{2}\gamma_j\gamma_k$ yields a rotation, we first note that the square of this quadratic element is simply $-\frac{1}{4}$. This allows us to write,

$$e^{\frac{\theta}{2}\gamma_j\gamma_k} = \cos\left(\frac{\theta}{2}\right) + \gamma_j\gamma_k \sin\left(\frac{\theta}{2}\right).$$

We then have, using the anti-commutation relation $[\gamma_j, \gamma_k]_+ = 2\delta_{jk}$,

$$\begin{aligned}
e^{-\frac{\theta}{2}\gamma_j\gamma_k}(v_j\gamma_j + v_k\gamma_k)e^{\frac{\theta}{2}\gamma_j\gamma_k} &= \left(\cos\frac{\theta}{2} - \gamma_j\gamma_k\sin\frac{\theta}{2}\right)(v_j\gamma_j + v_k\gamma_k)\left(\cos\frac{\theta}{2} + \gamma_j\gamma_k\sin\frac{\theta}{2}\right) \\
&= \cos^2\frac{\theta}{2}(v_j\gamma_j + v_k\gamma_k) + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(v_j\gamma_j + v_k\gamma_k)\gamma_j\gamma_k \\
&\quad - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\gamma_j\gamma_k(v_j\gamma_j + v_k\gamma_k) - \sin^2\frac{\theta}{2}\gamma_j\gamma_k(v_j\gamma_j + v_k\gamma_k)\gamma_j\gamma_k \\
&= \cos^2\frac{\theta}{2}(v_j\gamma_j + v_k\gamma_k) - \sin\theta\gamma_j\gamma_k(v_j\gamma_j + v_k\gamma_k) + \delta_{jk}\sin\theta(v_j\gamma_j + v_k\gamma_k) \\
&\quad - \sin^2\frac{\theta}{2}(v_j\gamma_j + v_k\gamma_k + 2v_j\gamma_j\gamma_k\gamma_j\delta_{jk} + 2v_k\gamma_j\gamma_k\gamma_k\delta_{jk})
\end{aligned}$$

If we assume that $j \neq k$, then this simplifies via a double angle identity to:

$$\begin{aligned}
&\cos\theta(v_j\gamma_j + v_k\gamma_k) - \gamma_j\gamma_k\sin\theta(v_j\gamma_j + v_k\gamma_k) \\
=&\cos\theta(v_j\gamma_j + v_k\gamma_k) - \sin\theta(-v_j\gamma_k + v_k\gamma_j) \\
=&(v_j\cos\theta - v_k\sin\theta)\gamma_j + (v_j\sin\theta + v_k\cos\theta)\gamma_k,
\end{aligned}$$

and so we are left with a rotation in the $j - k$ plane, as desired.

Problem 3