Complex Analysis and Riemann Surfaces: Final

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Problem 1

Let $\tau \in \mathbb{C}$ with Im $\tau > 0$ and let $\Lambda = \{m + n\tau; m, n \in \mathbb{Z}\}$ be the lattice generated by 1 and τ.

(a) Consider the functions defined by

$$\begin{aligned}
\theta(z|\tau) &= \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n \tau z\right) \\
\theta_1(z|\tau) &= \exp\left(\frac{\pi i \tau}{4} + \pi i\left(z + \frac{1}{2}\right)\right) \theta\left(z + \frac{1+\tau}{2}|\tau\right) \\
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(n + \frac{1}{2}\right)^2 \tau + 2\pi i \left(n + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right).
\end{aligned}$$

θ_1 is an odd holomorphic function, and in fact, its only zero is at $0 \mod \Lambda$. Furthermore, it transforms in the following way

$$\begin{aligned}
\theta_1(z+1|\tau) &= -\theta_1(z|\tau) \\
\theta_1(z+\tau|\tau) &= -e^{-\pi i\tau - 2\pi i z}\theta_1(z|\tau).
\end{aligned}$$

(b) Let $P \neq Q$ be arbitrary points on the torus $\mathbb{C}/\Lambda$. Using θ_1, we can construct a meromorphic form $\omega_{PQ}(z)$ on $\mathbb{C}/\Lambda$ with simple poles at P and Q with residues ± 1 respectively. Consider the form

$$\omega_{PQ}(z) = \partial_z \left(\log \frac{\theta_1(z-P)}{\theta_1(z-Q)}\right) dz$$

on $\mathbb{C}$. Note first that, using the above transformation rules, $\omega_{PQ}(z)$ is doubly-periodic:

$$\begin{aligned}
\omega_{PQ}(z+1) &= \omega_{PQ}(z) \\
\omega_{PQ}(z+\tau) &= \partial_z \left(\log \frac{\exp\left(\pi i \tau - 2\pi i(z-P)\right)\theta_1(z-P)}{\exp\left(-\pi i\tau - 2\pi i(z-Q)\right)\theta_1(z-Q)}\right) \\
&= \partial_z \left(\log e^{2\pi i(P+Q)} + \log \frac{\theta_1(z-P)}{\theta_1(z-Q)}\right) \\
&= \omega_{PQ}(z)
\end{aligned}$$

and hence well-defined on $\mathbb{C}/\Lambda$. Rewriting $\omega_{PQ}(z)$ as

$$\omega_{PQ}(z) = \frac{\theta_1'(z-P)}{\theta_1(z-P)} - \frac{\theta_1'(z-Q)}{\theta_1(z-Q)},$$

we see that $\omega_{PQ}(z)$ has poles at P and Q (as θ_1 vanishes to order one at 0 and θ_1' does not because if it did, θ_1 would have a double zero).

(c) Let P be an arbitrary point on $\mathbb{C}/\Lambda$. Using θ_1 we can construct a meromorphic form $\omega_P(z)$ on $\mathbb{C}/\Lambda$ with a double pole at P. In particular, consider

$$\omega_P(z) = \partial_z^2 \left(\log \frac{\theta_1(z-P)}{\theta_1'(0)} \right).$$

First note that $\omega_P(z)$ is doubly-periodic:

$$\begin{aligned} \omega_P(z+1) &= \omega_P(z) \\ \omega_P(z+\tau) &= \partial_z^2 \left(\log \frac{\theta_1(z+\tau-P)}{\theta_1'(0)} \right) \\ &= \partial_z^2 \left(\log \frac{-e^{-\pi i \tau - 2\pi i (z-P)} \theta_1(z-P)}{\theta_1'(0)} \right) \\ &= \omega_P(z) \end{aligned}$$

as the derivatives annihilate the constant terms that pop out, and hence ω_P is well-defined on $\mathbb{C}/\Lambda$. Rewriting, we find that

$$\omega_P(z) = \partial_z \frac{\theta_1'(z-P)}{\theta(z-P)} = \frac{\theta_1''(z-P)\theta_1(z-P) - (\theta_1'(z-P))^2}{\theta_1^2(z-P)}.$$

When evaluated at $z = P$, since $\theta_1(0) = 0$, we are left with $\theta_1'(0)^2/\theta_1^2(0)$ and hence $\omega_P(z)$ has a double pole at $z = P$ (as $\theta'(0) \neq 0$, as before).

Problem 2

Let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X.

(a) Let $h(z)$ be a metric on L. Recall that a metric is a strictly positive section of $L^{-1} \otimes \bar{L}^{-1}$. Using h, we can differentiate sections of our line bundle L as follows. Recall that we defined the derivative

$$\begin{aligned} \nabla_{\bar{z}} = \bar{\partial} : \Gamma(X, L) &\to \Gamma(X, L \otimes \Lambda^{0,1}) \\ \phi = \{\phi_\alpha\} &\mapsto \nabla_{\bar{z}}\phi = \{\partial \phi_\alpha / \partial \bar{z}_\alpha\} \end{aligned}$$

as well as the derivative

$$\begin{aligned} \nabla_z : \Gamma(X, L) &\to \Gamma(X, L \otimes \Lambda^{0,1}) \\ \phi = \{\phi_\alpha\} &\mapsto h_\alpha^{-1} \frac{\partial}{\partial z_\alpha}(h_\alpha \phi_\alpha). \end{aligned}$$

We computed the failure of commutativity of these two derivatives, i.e. the curvature $F_{\bar{z}z} \in \Gamma(X, \Lambda^{1,1})$:

$$[\nabla_z, \nabla_{\bar{z}}]\phi = F_{\bar{z}z}\phi = -(\partial_{\bar{z}}\Gamma)\phi = -(\partial_{\bar{z}}\partial_z \log h)\phi.$$

We then define the first Chern class $c_l(L)$ of the bundle to be

$$c_1(L) = \frac{i}{2\pi} \int_X F_{\bar{z}z} dz \wedge d\bar{z}.$$

Recall that we showed in class that $c_1(L)$ is independent of the metric, as is suggested by the notation.

(b) We wish to show that if $c_1(L) < 0$ then the line bundle L does not admit any non-trivial holomorphic sections. To do this, we first show that given any section $\phi \in \Gamma(X, L)$ the number its zeros minus the number of its poles is precisely $c_1(L)$. Then it follows immediately that if $c_1(L) < 0$, every section must have at least one pole. To show this, we work locally. Locally we may write:

$$F_{\bar{z}z} dz \wedge d\bar{z} = -\partial_z \partial_{\bar{z}} \log h \, dz \wedge d\bar{z} = -d\left((\partial_{\bar{z}} \log h) d\bar{z}\right).$$

We cannot use Stoke's theorem here, as $\partial_{\bar{z}} \log h$ is not even globally well-defined. Instead, if we let $\{P_i\}_{i=1}^N$ be the zeroes and poles of a section ϕ, outside ε-discs D_i about P_i (only containing one zero or pole), we can write

$$\begin{aligned} F_{\bar{z}z} dz \wedge d\bar{z} &= -\partial_z \partial_{\bar{z}} \log ||\phi||_h^2 dz \wedge d\bar{z} \\ &= d\left((\partial_{\bar{z}} \log ||\phi||_h^2) d\bar{z}\right) \end{aligned}$$

as $||\phi||_h^2 = \phi\bar{\phi}h$, and so the log splits, but the extra terms involving $\log \phi$ and $\log \bar{\phi}$ vanish under the $\partial_{\bar{z}}$ and the ∂_z derivatives respectively. Then by Stokes' theorem we find that

$$\begin{aligned} \int_X F_{\bar{z}z} dz \wedge d\bar{z} &= \lim_{\varepsilon \to 0} \int_{X \setminus \cup_i D_i} d\left((\partial_{\bar{z}} \log ||\phi||_h^2) d\bar{z}\right) \\ &= \sum_{i=1}^N \lim_{\varepsilon \to 0} \oint_{\partial D_i} \partial_{\bar{z}} \log ||\phi||_h^2 d\bar{z} \\ &= \sum_{i=1}^N \lim_{\varepsilon \to 0} \oint_{\partial D_i} \partial_{\bar{z}} \log \bar{\phi} d\bar{z} \end{aligned}$$

where the $\log \phi$ vanishes under the $\partial_{\bar{z}}$ and the $\log h$ term vanishes when integrated against $d\bar{z}$, as it is smooth. As ϕ vanishes or blows up at each P_i, we can write $\phi(z) = (z - P_i)^{N_i} u(z)$ for some $u(z)$ holomorphic and non-zero at P_i. But then

$$\partial_z \log \phi = \frac{\partial_z \phi}{\phi} = \frac{N_i}{z - P_i} + \frac{\partial_z u(z)}{u(z)}$$

and so we have

$$\begin{aligned} \int_X F_{\bar{z}z} dz \wedge d\bar{z} &= \sum_{i=1}^N \lim_{\varepsilon \to 0} \oint_{\partial D_i} \frac{N_i}{\bar{z} - \bar{P}_i} + \overline{\left(\frac{\partial_z u(z)}{u(z)}\right)} d\bar{z} \\ &= \sum_{i=1}^N N_i \lim_{\varepsilon \to 0} \oint_{\partial D_i} \frac{d\bar{z}}{\bar{z} - \bar{P}_i} \\ &= -2\pi i \sum_{i=1}^N N_i, \end{aligned}$$

but then the first Chern class becomes:

$$c_1(L) = \frac{1}{2\pi} \int_X F_{\bar{z}z} dz \wedge d\bar{z} = \sum_{i=1}^N N_i,$$

but this is precisely the number of zeros minus the number of poles (counted with multiplicity). This proves the claim.

(c) The Riemann-Roch theorem states that

$$\dim H^0(X,L) - \dim H^0(X, L^{-1} \otimes K_X) = c_1(L) + \frac{1}{2} c_1(K_X^{-1})$$

where $K_X = \Lambda^{1,0}$ is the canonical bundle. Inserting $L = K_X^n$, we find that

$$\begin{aligned} \dim H^0(X, K_X^n) - \dim H^0(X, K_X^{1-n}) &= c_1(K_X^n) + \frac{1}{2} c_1(K_X^{-1}) \\ &= \left(\frac{1}{2} - n\right) c_1(K_X^{-1}) \end{aligned}$$

where we have used the homomorphism properties of the first Chern class.

(d) Let us use the above formula to deduce the value of $\dim H^0(X, K_X^n)$. There are three cases: $c_1(K_X^{-1}) > 0$, $c_1(K_X^{-1}) = 0$, and $c_1(K_X^{-1}) < 0$.

Consider first the case $c_1(K_X^{-1}) = 0$. It is clear that $c_1(K_X^n) = -nc_1(K_X^{-1}) = 0$, and hence the holomorphic sections of K_X^n must have no zeroes. This implies that the $\dim H^0(X, K_X^n) = 1$ for all n - to see this, note that given any two linearly independent sections f, g we can scale g to equal f at any one point. Taking the difference of f and this scaled g we obtain a section that is zero at that point, and hence this space must be one-dimensional.

Next consider $c_1(K_X^{-1}) < 0$. Then $c_1(K_X^n) = -nc_1(K_X^{-1})$ is greater than zero for positive n and less than zero for negative n. This implies that there are no holomorphic sections for negative n, i.e. $\dim H^0(X, K_X^n) = 0$ for $n < 0$. For positive n, the formula above yields:

$$\begin{aligned} \dim H^0(X, K_X^n) - \dim H^0(X, K^{1-n}X) &= (1-2n)(1-g) \\ \dim H^0(X, K_X^n) &= (1-2n)(1-g). \end{aligned}$$

Finally consider $c_1(K_X^{-1}) > 0$. Then $c_1(K_X^n) = -nc_1(K_X^{-1})$ is less than zero for positive n and greater than zero for negative n. This implies that there are no holomorphic sections for positive n, i.e. $\dim H^0(X, K_X^n) = 0$ for $n > 0$. For negative n, the formula above yields:

$$\begin{aligned} \dim H^0(X, K_X^n) - \dim H^0(X, K^{1-n}X) &= (1-2n) \\ \dim H^0(X, K_X^n) &= (1-2n). \end{aligned}$$

(e) Recall from class that we have a correspondence between complex structures and metrics, and hence it makes sense to attempt to construct the moduli space of Riemann surfaces by first considering the space of metrics. However, we have a natural action of both the Weyl group W and the group of diffeomorphisms of the surface Diff X on the space of metrics, in the form of $W \rtimes \text{Diff}\, X$. Hence we define the moduli space of a Riemann surface of genus h to be

$$\mathcal{M}_h = \{\text{space of metrics}\} / W \rtimes \text{Diff}\, X.$$

To study this moduli space, we instead study the structure of its tangent space, in order to determine $\dim \mathcal{M}_h$. In particular, we claim, by functoriality of the tangent space functor, that

$$T_{[g_{ij}]}\mathcal{M}_h = \{\delta g_{ij}\} / \{\delta\sigma g_{ij}\} + \{\nabla_i(\delta v)_j + \nabla_j(\delta v)_i\}$$

where g_{ij} is some equivalence class of metrics, the first is associated with the Weyl transform and the second term is associated with the tangent space of the group of diffeomorphisms (i.e.

smooth vector fields). We will not prove this claim here, as it is rather computational and because it was done in class. To compute the dimension of this tangent space we note that

$$T_{[g_{ij}]}\mathcal{M}_h = \left\{\delta g_{\bar{z}\bar{z}}(d\bar{z})^2\right\} / \left\{\partial_{\bar{z}}(\delta v^z)\right\} = \text{coker } \bar{\partial}|_{K_X^{-1}}$$

where the $\bar{\partial}$ operator takes

$$\Gamma(X, K_X^{-1}) \ni \delta v^z \longrightarrow \partial_{\bar{z}}(\delta v^z) \in \Gamma(X, \overline{K_X} \otimes K_X^{-1}).$$

Computing the dimension then simply reduces to computing $H^0(X, K_X^2)$ (as this gives us the "orthogonal complement"), which can now easily be done via Riemann-Roch and our earlier work. In particular, if $h = 0$, $c_1(K_X^2) = -2c_1(K_X^{-1}) = -4 < 0$ and hence $\dim H^0(X, K_X^2) = 0$, by the arguments of the previous part. Hence the tangent space (and thus the moduli space) is zero-dimensional. If $h = 1$, we are in the case where $c_1(K_X^{-1}) = 0$ and hence the tangent space (and the moduli space) is one-dimensional. Finally, if $h \geq 2$ then $c_1(K_X^{-1}) < 0$ and by the above work we find $\dim H^0(X, K_X^2) = -3(1-h) = 3h - 3$.

Problem 3

Let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X and let $h(z)$ be a metric on L and $g_{\bar{z}z}$ be a metric on K_X^{-1}.

(a) Let us define on $\Gamma(X, L)$ an L^2 inner product as

$$||\phi||^2 = \int_X \phi\bar{\phi}hg_{\bar{z}z}.$$

This is well-defined, as $\phi\bar{\phi}h$ yields a scalar, and $g_{\bar{z}z}$ is a $(1,1)$-form. Similarly, on $\Gamma(X, L \otimes \overline{K_X})$, we define the inner product as

$$||\psi||^2 = \int_X \psi\bar{\psi}h,$$

which is well-defined as the components in L cancel to yield, again, a $(1,1)$-form.

(b) Let the operator $\bar{\partial} : \Gamma(X, L) \to \Gamma(X, L \otimes \overline{K_X})$ be defined by $\bar{\partial}\phi = \partial_{\bar{z}}\phi$. In order to trace back to Riemann-Roch we must determine the formal adjoint of $\bar{\partial}$, i.e.

$$\langle \bar{\partial}\phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle,$$

where we note that these inner products are taken in different spaces. More explicitly,

$$\begin{aligned} \int \partial_{\bar{z}}\phi\bar{\psi}h &= \int \phi\overline{\bar{\partial}^\dagger\psi}hg_{\bar{z}z} \\ -\int \phi\overline{\partial_z(h\psi)} &= -\int \phi\overline{g_{\bar{z}z}^{-1}h^{-1}\partial_z(h\psi)}hg_{\bar{z}z}, \end{aligned}$$

where we have integrated by parts. This shows that

$$\bar{\partial}^\dagger\psi = -g^{z\bar{z}}h^{-1}\partial_z(h\psi) = -g^{\bar{z}z}\nabla_z\psi.$$

(c) Define the Laplacians $\Delta_+ = \bar{\partial}^\dagger \bar{\partial}$ and $\Delta_- = \bar{\partial}\bar{\partial}^\dagger$. We assume that the eigenvalues $\lambda_n^\pm$ of $\Delta_\pm$ are discrete and tend to infinity polynomially. Note that if $\lambda \neq 0$ is an eigenvalue of Δ_+, it is also an eigenvalue of Δ_-: if $\Delta_+\phi = \bar{\partial}^\dagger\bar{\partial}\phi = \lambda\phi$, then $\bar{\partial}\bar{\partial}^\dagger\bar{\partial}\phi = \bar{\partial}\lambda\phi$, i.e. $\Delta_-\bar{\partial}\phi = \lambda\bar{\partial}\phi$. The converse holds very similarly. The zero eigenvalues, on the other hand, are precisely the kernels - in other words, we have paired everything but the kernel and cokernel. We can write

$$\dim\ker\Delta_+ - \dim\ker\Delta_- = \dim\ker\bar{\partial} - \dim\ker\bar{\partial}^\dagger.$$

To see why, note that

$$\begin{aligned}\dim\ker\Delta_+ &= \left\{\phi \in \Gamma(X, L) \mid \bar{\partial}^\dagger\bar{\partial}\phi = 0\right\} \\ &= \left\{\phi \in \Gamma(X, L) \mid \langle\phi, \bar{\partial}^\dagger\bar{\partial}\phi\rangle = 0\right\} \\ &= \left\{\phi \in \Gamma(X, L) \mid \langle\bar{\partial}\phi, \bar{\partial}\phi\rangle = ||\bar{\partial}\phi||^2 = 0\right\}\end{aligned}$$

and similarly for Δ_-. Now we claim that this is in fact also equal to $\operatorname{tr} e^{-t\Delta_+} - \operatorname{tr} e^{-t\Delta_-}$, where the exponential of an operator just acts in the obvious way on basis eigenfunctions as multiplication by the exponential evaluated at the eigenvalue (here we are using completeness). Note:

$$\operatorname{tr} e^{-t\Delta_+} - \operatorname{tr} e^{-t\Delta_-} = \sum_{\lambda \text{ of } \Delta_+} e^{-t\lambda} - \sum_{\lambda \text{ of } \Delta_-} e^{-t\lambda} = \dim\ker\Delta_+ - \dim\ker\Delta_-,$$

as desired (we have implicitly used the pairing mentioned above). Hence we conclude that

$$\operatorname{tr} e^{-t\Delta_+} - \operatorname{tr} e^{-t\Delta_-} = \dim\ker\bar{\partial} - \dim\ker\bar{\partial}^\dagger.$$

(d) Note first that $H^0(X, L) = \ker\bar{\partial}|_{\Gamma(X,L)}$. Furthermore,

$$\begin{aligned}\ker\bar{\partial}^\dagger &= \left\{\psi \in \Gamma(X, L \otimes \overline{K_X}) \mid -g^{z\bar{z}}\nabla_z\psi = 0\right\} \\ &= \left\{\psi \in \Gamma(X, L \otimes \overline{K_X}) \mid \partial_z(h\psi) = 0\right\} \\ &= \left\{\psi \in \Gamma(X, L \otimes \overline{K_X}) \mid \partial_{\bar{z}}(h\bar{\psi}) = 0\right\},\end{aligned}$$

and thus we have (metric-dependent) isomorphism

$$\ker\bar{\partial}^\dagger \ni \psi \longleftrightarrow h\bar{\psi} \in \Gamma(X, K_X \otimes L^{-1}) = \ker\bar{\partial}|_{\Gamma(X, K_X \otimes L^{-1})},$$

which is precisely the second term of the left-hand side of the Riemann-Roch theorem:

$$\dim\ker\bar{\partial}^\dagger = \dim H^0(X, K_X \otimes L^{-1}).$$

Hence, from the part above, we find that

$$\operatorname{tr} e^{-t\Delta_+} - \operatorname{tr} e^{-t\Delta_-} = \dim H^0(X, L) - \dim H^0(X, K_X \otimes L^{-1}).$$

Problem 4

Let $E \to X$ be a holomorphic vector bundle of rank r over a complex manifold X of dimension n. Let $H_{\bar{\alpha}\beta}(z)$ be a Hermitian metric on E. Define the Chern unitary connection Δ on E by

$$\nabla_{\bar{j}}\phi^\alpha = \partial_{\bar{j}}\phi^\alpha \text{ and } \nabla_j\phi^\alpha = H^{\alpha\bar{\beta}}\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma).$$

(a) First note that these formulas make sense because the indices are proprerly contracted. However, we must show that they are globally well-defined as operators from $\Gamma(X, E) \to \Gamma(X, E \otimes \Lambda^{0,1})$ and $\Gamma(X, E) \to \Gamma(X, E \otimes \Lambda^{1,0})$. For the first definition, note that given a section ϕ^α (and transition matrices $\tau_{\mu\nu}$), the following quantity transforms as

$$\frac{\partial \phi^\alpha_\mu}{\partial \bar{z}_\mu} = \frac{\partial(\tau^\alpha_{\mu\nu\beta}\phi^\beta_\nu)}{\partial \bar{z}_\mu} = \tau^\alpha_{\mu\nu\beta}\frac{\partial \phi^\beta_\nu}{\partial \bar{z}_\mu} = \tau^\alpha_{\mu\nu\beta}\frac{\partial \phi^\beta_\nu}{\partial \bar{z}_\nu}\frac{\partial \bar{z}_\nu}{\partial \bar{z}_\mu}$$

where we have used the holomorphicity of E. In other words, $\partial_{\bar{z}}$ is a well-defined section of $\Gamma(X, E \otimes \Lambda^{0,1})$. Meanwhile, for the second definition, note first that $H_{\bar{\beta}\gamma}$ is a section of $\Gamma(X, \overline{E^{-1}} \otimes E^{-1})$, and hence $H_{\bar{\beta}\gamma}\phi^\gamma$ is a well-defined section of $\Gamma(X, \overline{E^{-1}})$. Then $\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma)$ is a well-defined section of $\Gamma(X, \overline{E^{-1}} \otimes \Lambda^{1,0})$ (this is the same argument as above, just with z instead of $\bar{z}$) and finally $H^{\alpha\bar{\beta}}\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma)$ is a well-defined section of $\Gamma(X, E \otimes \Lambda^{1,0})$, as desired.

(b) We can rewrite the second expression above as follows:

$$\begin{aligned} H^{\alpha\bar{\beta}}\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma) &= \delta^\alpha_\gamma \partial_j\phi^\gamma + (H^{\alpha\bar{\beta}}\partial_j H_{\bar{\beta}\gamma})\phi^\gamma \\ &= \partial_j\phi^\alpha + A^\alpha_{j\gamma}\phi^\gamma \end{aligned}$$

where we have defined $A^\alpha_{j\gamma} = H^{\alpha\bar{\beta}}\partial_j H_{\bar{\beta}\gamma}$. If we now define the curvature tensor $F^\alpha_{\bar{k}j\beta}$ by

$$[\nabla_j, \nabla_{\bar{k}}]\phi^\alpha = F^\alpha_{\bar{k}j\beta}\phi^\beta,$$

we find that

$$\begin{aligned} [\nabla_j, \nabla_{\bar{k}}]\phi^\alpha &= \nabla_j(\partial_{\bar{k}}\phi^\alpha) - \nabla_{\bar{k}}\left(H^{\alpha\bar{\beta}}\partial_j(H_{\bar{\beta}\gamma}\phi^\gamma)\right) \\ &= H^{\alpha\bar{\beta}}\partial_j\left(H_{\bar{\beta}\gamma}\partial_{\bar{k}}\phi^\gamma\right) - \partial_{\bar{k}}\left(\partial_j\phi^\alpha + A^\alpha_{j\gamma}\phi^\gamma\right) \\ &= A^\alpha_{j\gamma}\partial_{\bar{k}}\phi^\gamma + \partial_j\partial_{\bar{k}}\phi^\alpha - \partial_{\bar{k}}\partial_j\phi^\alpha - \partial_{\bar{k}}\left(A^\alpha_{j\gamma}\phi^\gamma\right) \\ &= -\partial_{\bar{k}}A^\alpha_{j\gamma}\phi^\gamma, \end{aligned}$$

and hence $F^\alpha_{\bar{k}j\beta} = -\partial_{\bar{k}}A^\alpha_{j\beta}$.

(c) Next consider the following $\operatorname{End} E = E \otimes E^*$-valued forms:

$$A^\alpha_\beta = A^\alpha_{j\beta}dz^j \text{ and } F^\alpha_\beta = F^\alpha_{\bar{k}j\beta}dz^j \wedge d\bar{z}^k.$$

We claim that $F = dA + A \wedge A$. To see this, we first compute (suppressing indices)

$$\begin{aligned} dA = d(A_j dz^j) &= (\partial_k A_j dz^k + \partial_{\bar{k}}A_j d\bar{z}^k) \wedge dz^j \\ &= \frac{1}{2}(\partial_k A_j - \partial_j A_k)dz^k \wedge dz^j - F_{\bar{k}j}d\bar{z}^k \wedge dz^j \\ &= \frac{1}{2}(\partial_k A_j - \partial_j A_k)dz^k \wedge dz^j + F \\ &= \frac{1}{2}\left(\partial_k(H^{-1}\partial_j H) - \partial_j(H^{-1}\partial_k H)\right)dz^k \wedge dz^j + F \\ &= \frac{1}{2}\left((-H^{-1}\partial_k H H^{-1})\partial_j H + H^{-1}\partial_k\partial_j H - (j \leftrightarrow k)\right)dz^k \wedge dz^j + F \\ &= \frac{1}{2}\left(-H^{-1}\partial_k H H^{-1}\partial_j H - (j \leftrightarrow k)\right)dz^k \wedge dz^j + F \\ &= \frac{1}{2}\left(-A_k A_j + A_j A_k\right)dz^k \wedge dz^j + F \\ &= -A \wedge A + F, \end{aligned}$$

where we have used the fact that $\partial_k H^{-1} = -H^{-1}\partial_k H H^{-1}$. Hence we obtain the desired identity. Finally, note that

$$
\begin{aligned}
dF + A \wedge F - F \wedge A &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A \\
&= dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A \\
&= 0,
\end{aligned}
$$

as desired.

(d) Take $\phi^\alpha \in \Gamma(X, E)$ and $\psi_\alpha \in \Gamma(X, E^*)$. We can compute the covariant derivative induced on the dual bundle (given a covariant derivative on E, i.e. a metric) by enforcing the Liebniz rule:

$$
\begin{aligned}
\psi_\alpha \partial_j \phi^\alpha + \phi^\alpha \partial_j \psi_\alpha &= (\partial_j \phi^\alpha + A^\alpha_{j\beta}\phi^\beta)\psi_\alpha + \phi^\alpha \nabla_j \psi_\alpha \\
\phi^\alpha \nabla_j \psi_\alpha &= \phi^\alpha \partial_j \psi_\alpha - A^\alpha_{j\beta}\phi^\beta \psi_\alpha \\
\phi^\alpha \nabla_j \psi_\alpha &= \phi^\alpha \partial_j \psi_\alpha - A^\gamma_{j\alpha}\phi^\alpha \psi_\gamma \\
\nabla_j \psi_\alpha &= \partial_j \psi_\alpha - A^\gamma_{j\alpha}\psi_\gamma,
\end{aligned}
$$

where we have switched dummy indices in order to isolate the covariant derivative. Next, take $T \in \Gamma(X, \mathrm{End}\, E)$, and enforce the Liebniz rule again:

$$
\begin{aligned}
\nabla_j(T)\phi &= \nabla_j(T\phi) - T(\nabla_j \phi) \\
&= \nabla_j(T^\beta_\alpha \phi^\alpha) - T^\beta_\alpha(\partial_j \phi^\alpha + A^\alpha_{j\gamma}\phi^\gamma) \\
&= \partial_j(T^\beta_\alpha \phi^\alpha) + A^\beta_{j\gamma}T^\gamma_\alpha \phi^\alpha - T^\beta_\alpha \partial_j \phi^\alpha - T^\beta_\alpha A^\alpha_{j\gamma}\phi^\gamma \\
&= \partial_j T^\beta_\alpha \phi^\alpha + A^\beta_{j\gamma}T^\gamma_\alpha \phi^\alpha - T^\beta_\alpha A^\alpha_{j\gamma}\phi^\gamma.
\end{aligned}
$$

Changing dummy indices, we find that

$$
\nabla_j T^\alpha_\beta = \partial_j T^\alpha_\beta + A^\alpha_{j\gamma}T^\gamma_\beta - T^\alpha_\gamma A^\gamma_{j\beta}.
$$

(e) Finally, let us use the covariant derivative now defined on $\mathrm{End}\, E$ to construct an operator d_A that differentiates on $\mathrm{End}\, E$-valued forms. This is fairly simple - in the definition of the exterior derivative, instead of using the usual derivative, we now substitute the covariant derivative associated to the connection A derived above. Let $\xi \in \Gamma(X, \mathrm{End}\, E \otimes \Lambda^p)$. Then

$$
\begin{aligned}
(d_A \xi)^\alpha_\beta &= \frac{1}{p!}\sum \nabla^A_j \xi^\alpha_{\beta, i_1, \dots, i_p} dz^j \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \\
&= \frac{1}{p!}\sum \left(\partial_j \xi^\alpha_\beta + A^\alpha_{j\gamma}\xi^\gamma_\beta - \xi^\alpha_\gamma A^\gamma_{j\beta}\right)_{i_1, \dots, i_p} dz^j \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \\
&= d\xi^\alpha_\beta + A^\alpha_{j\gamma} \wedge \xi^\gamma_\beta - \xi^\alpha_\gamma \wedge A^\gamma_{j\beta}.
\end{aligned}
$$

If F is the curvature form of a metric $H_{\bar{\alpha}\beta}$, then by the identity $dF + A \wedge F - F \wedge A = 0$, we find that

$$
d_A F = 0.
$$