Introduction to Algebraic Topology PSET 4

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Last updated: February 17, 2014

Proposition 1. *Hatcher exercise 1.1.10*

Proof. Consider two loops in $X \times Y$ based at (x_0, y_0). The first loop is purely in the X direction, denoted by $f : I \to X \times \{y_0\}$, while the second is purely in the Y direction, denoted by $g : I \to \{x_0\} \times Y$. We wish to find a homotopy between the loops $f \diamond g$ and $g \diamond f$. This can be done by continuously transporting the basepoint of f along g via $h_t : I \times I \to X \times Y$ given by

$$h_t(s) = \begin{cases} (x_0, g(2s)) & 0 \leq s \leq \frac{t}{2} \\ (f(2s-1), g(t)) & \frac{t}{2} \leq s \leq \frac{t+1}{2} \\ (x_0, g(2s-1)) & \frac{t+1}{2} \leq s \leq 1 \end{cases}.$$

This homotopy is continuous by continuity of f, g and it clearly takes $f \diamond g$ at $t = 0$ to $g \diamond f$ at $t = 1$. $\square$

Proposition 2. *Hatcher exercise 1.1.12*

Proof. Any morphism $\phi_* : \pi_1(S^1) \to \pi^1(S^1)$ is simply a morphism $\phi_* : \mathbb{Z} \to \mathbb{Z}$, which is determined by the image of its generator. Then, if $n = \phi_*(1)$, the map ϕ that induces ϕ_* is obviously the continuous map $\phi : S^1 \to S^1$ given by $\theta \mapsto n\theta$, as ϕ takes the $[\omega_1]$ to $[\omega_n]$, as required by ϕ_*. $\square$

Proposition 3. *Hatcher exercise 1.1.16*

Proof. Throughout this proof, we use freely Hatcher Proposition 1.17: if a space X retracts onto a subspace A, then the homomorphism $i_* : \pi_(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. Further, if A is a deformation retract of X, then i_* is an isomorphism. We also use the fact that the functor π_1 takes products to products.

(a) Let $X = \mathbb{R}^3$ with A any space homeomorphic to S^1. The existence of a retraction $r : X \to A$ would imply an injective morphism $\mathbb{Z} \hookrightarrow 0$ of the integers into the trivial group, which is absurd.

(b) Let $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$. The existence of a retraction $r : X \to A$ would imply an injective morphism $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{Z}$ as D^2 is contractible. This is of course impossible: if $(1, 0) \mapsto m$ for any $m, n \in \mathbb{Z}$ and $(0, 1) \mapsto n$, we find that $n(1, 0)$ maps to the same element that $m(0, 1)$ maps to; there are no injective group morphisms $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

(c) Let $X = S^1 \times D^2$ and A be the circle as shown in Hatcher. finish

(d) Let $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$. Of course, $\pi_1(D^2 \vee D^2)$ is the trivial group: any loop in D^2 is contractible and since any loop in $D^2 \vee D^2$ can be pinched off into the composition of two loops each entirely in a single copy of D^2, every loop in $D^2 \vee D^2$ is contractible as well.

For usch a retraction $r : X \to A$ to exist would imply an injective morphism from $\pi_1(S^1 \times S^1)$ to the trivial group, which is possible only if all loops in $S^1 \times S^1$ are contractible. This is clearly not the case, as one can consider a loop in just one of the copies of S^1, and hence no such retraction exists.

(e) Let X be a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$. It is clear that X is homotopy equivalent to S^1, as it deformation retracts to the diameter connecting the two identified points. Hence such a retraction would imply an injective morphism $\pi_1(S^1 \vee S^1) \hookrightarrow \mathbb{Z}$. But note that clearly $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} \hookrightarrow \pi_1(S^1 \vee S^1)$ and hence this is impossible, as it would imply an injective morphism of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$. Of course, this is easier to see via the fact that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

(f) Let X be the Möbius band and A be its boundary circle. finish

$\square$

Proposition 4. *Hatcher exercise 1.2.4*

Proof. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. It is clear that $\mathbb{R}^3 - X$ deformation retracts onto a $2n$-punctured S^2, where the punctures are antipodal. Note that in the case of $n = 1$, we can deformation retract $S^1 - \{N, S\}$ to the equatorial S^1. $\square$