Physics 6047
Problem Set 8, due 4/4/13

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1. Suppose we have a unitary operator $\hat{U}(\alpha)$ such that $\hat{U}(\alpha)^{-1}\hat{\phi}\hat{U}(\alpha) = \hat{\phi} + \alpha\delta\hat{\phi}$. Here, α is a parameter that describes the 'size' of the transformation, i.e. as $\alpha \to 0$, $\hat{U} \to 1$. Show that the generator for $\hat{U}$ (i.e. some $\hat{Q}$ such that $\hat{U} = e^{-i\alpha\hat{Q}}$) obeys: $i[\hat{Q}, \hat{\phi}] = \delta\hat{\phi}$. Note that you can also run this argument backward: given the Noether current and its associated charge due to some symmetry, we know how to form $\hat{U}$ out of it.

2. This problem is also more like notes, on showing that the generators of a group must obey a Lie algebra of some sort. Let us label the generators of the group as T^a, where a ranges from 1 to the total number of generators. Let us refer to the rotation angle associated with each generator as θ^a. We will use θ without any index to refer abstractly to the vector whose components are the θ^a's. The transformation effected by these generators and their associated angles is:

$$U(\theta) = 1 - i\theta^a T^a - \frac{1}{2}\theta_a\theta_b T^{ab} + \ldots . \tag{1}$$

This is an expansion of U for small angles. We have introduced the notation T^{ab} to denote whatever object that shows up in the second order term. The only requirement we will make is that it's symmetric $T^{ab} = T^{ba}$ since the combination $\theta_a\theta_b$ is symmetric under exchange of a and b. The fact that we have a *group* means

$$U(\bar{\theta})U(\theta) = U(g(\bar{\theta}, \theta)) , \tag{2}$$

where $\bar{\theta}$ and θ represents two in principle different set of angles, and g is yet another set of angles but should be a function of $\bar{\theta}$ and θ somehow. The function g can itself be Taylor expanded (recalling that g, just like θ, represents a set of angles and so it secretly carries an index):

$$g_a(\bar{\theta}, \theta) = \bar{\theta}_a + \theta_a + {g_a}^{bc}\bar{\theta}_b\theta_c + \ldots . \tag{3}$$

This expansion makes sense because we expect $g(0, 0) = 0$ $g_a(\bar{\theta}, 0) = \bar{\theta}_a$ and $g_a(0, \theta) = \theta_a$. We have introduced ${g_a}^{bc}$ to represent the coefficients of the second order terms in the expansion. By putting Eqs. (12) and (14) into Eq. (13), and comparing second order terms on both sides, *show that*:

$$T^b T^c = i{g_a}^{bc} T^a + T^{bc} . \tag{4}$$

Finally, use this to *show that*:

$$[T^b, T^c] = if^{bca}T^a \tag{5}$$

where $f^{bca} \equiv {g_a}^{bc} - {g_a}^{cb}$. One useful corollary of this derivation is that, because the angles are real, the structure constants f^{bca} are real.

3. Srednicki problem 24.3. For part b, you can simply use the canonical commutation relation between the φ's and their conjugate momenta.

4. Starting from Srednick's eq. (16.6) – an expression for V_3 – use dimensional regularization and the modified minimal subtraction scheme to derive eq. (27.18).

5. Using the modified minimal subtraction expressions for the one-loop correction to the propagator (Srednick's eq. 27.4), and the cubic vertex function V_3 (eq. 27.18), show that rerunning the arguments in chapter 20 gives eq. (27.19) – this is an expression in the high s, $|t|$, $|u|$ limit, with μ^2 dependence kept explicit, and ignoring other terms that are finite in the $m \to 0$ limit. You might find eq. 11.7, and footnote 1 of chapter 26 useful.

6. This problem functions more as notes. Let's work out the implication of having a propagator that has non-unity residue. To be concrete, let's consider the propagator according to modified minimal subtraction :

$$\tilde{\Delta}(k^2) = \frac{1}{k^2 + m^2 - \Pi(k^2)} \tag{6}$$

where we have suppressed $i\epsilon$ for simplicity. Suppose its pole is at $k^2 = -m_{\rm ph}^2$, i.e. $\tilde{\Delta}(k^2 = -m_{\rm ph}^2)^{-1} = 0$. Let's Taylor expand $\Pi(k^2) = \Pi(-m_{\rm ph}^2) + \Pi'(-m_{\rm ph}^2)(k^2 + m_{\rm ph}^2) + ...$ We know that $-m_{\rm ph}^2 + m^2 - \Pi(-m_{\rm ph}^2) = 0$. Thus, for k^2 close to $-m_{\rm ph}^2$, the propagator takes the form:

$$\tilde{\Delta}(k^2) \sim \frac{1}{(1 - \Pi'(-m_{\rm ph}^2))(k^2 + m_{\rm ph}^2)} \tag{7}$$

Thus, the residue is

$$R = \frac{1}{(1 - \Pi'(-m_{\rm ph}^2))} \tag{8}$$

which isn't equal to 1 in the modified minimal subtraction scheme. Recall that the propagator originates from the two-point correlation of ϕ

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} \tilde{\Delta}(k^2) e^{ik\cdot(x_1 - x_2)} \tag{9}$$

where $\tilde{\Delta}(k^2) \sim R/(k^2 + m_{\rm ph}^2)$ close to the pole at $k^2 = -m_{\rm ph}^2$. Recall that the above two-point function in operator language is really $\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle$. Rerun the Kallen-Lehmann argument, and *show that*

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \frac{1}{i} \int \frac{d^d k}{(2\pi)^d} |\langle k|\hat{\phi}(0)|0\rangle|^2 \frac{e^{ik\cdot(x_1 - x_2)}}{k^2 + m_{\rm ph}^2 - i\epsilon} + ... \tag{10}$$

where we have ignored terms that have to do with multi-particle or bound states. Thus, we conclude that $\langle k|\hat{\phi}(0)|0\rangle = R^{1/2}$, or $\langle k|\hat{\phi}(x)|0\rangle = R^{1/2} e^{-ik\cdot x}$ (up to an irrelevant phase). Note that when we write $\langle k|\hat{\phi}|0\rangle$, the k is on-shell, meaning $k^2 = -m_{\rm ph}^2$.

Next, look back at the derivation of the LSZ formula in problem set 5, you can see in equations (2), (3) and (4) that implicitly we assumed $\langle k|\hat{\phi}(x)|0\rangle = e^{-ik\cdot x}$. By the above reasoning, we thus need to correct our LSZ formula by multiplying the right hand side by a factor of $R^{-1/2}$ for each particle (ingoing or outgoing) i.e. for $2 \to 2$ scattering:

$$\begin{aligned}
&(2\pi)^d \delta^{(d)}(k_1 + k_2 - k_1' - k_2')\, i\mathcal{M} = \\
&R^{-n_{\rm tot.}/2} \int d^d x_2' i e^{-ik_2' x_2'} (-\Box_{2'} + m_{\rm ph}^2) \int d^d x_1' i e^{-ik_1' x_1'} (-\Box_{1'} + m_{\rm ph}^2) \int d^d x_2 i e^{ik_2 x_2} (-\Box_2 + m_{\rm ph}^2) \\
&\int d^d x_1 i e^{ik_1 x_1} (-\Box_1 + m_{\rm ph}^2)\ \langle 0|T\phi(x_2')\phi(x_1')\phi(x_2)\phi(x_1)|0\rangle
\end{aligned} \tag{11}$$

where $n_{\rm tot.}$ is the total number of ingoing and outgoing particles which in this case is 4.

Finally, *show that*, to compute the scattering amplitude $i\mathcal{M}$ in the case of $R \neq 1$ (as in for instance the modified minimal subtraction scheme), you can carry out the momentum-space Feynman rules as usual for scattering amplitude, but at the end need to multiply the resulting scattering amplitude by $R^{n_{\text{tot.}}/2}$ to get the correct $i\mathcal{M}$. Let me emphasize this is not a typo: the exponent that I want you to verify in the end is $+n_{\text{to.}}/2$, not $-n_{\text{to.}}/2$. This is why for example in Srednicki's eq. 27.19, he has a factor of R^2 on the right hand side.