Modern Algebra I: Problem Set 1

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Problem 1

We wish to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Pick an element $x \in A \cap (B \cup C)$. By definition, $x \in A$ and either $x \in B$ or $x \in C$. It is clear, that one of $(A \cap B)$ or $(A \cap C)$ must contain x. Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Conversely, pick an element $x \in (A \cap B) \cup (A \cap C)$. By definition, x is either in $(A \cap B)$ or $(A \cap C)$, i.e. x is either in A and B or in A and C. Hence $x \in A \cap (B \cup C)$, implying that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, and we are done.

Problem 2

(a) $P(\{0, 1\}) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$;

(b) $P(\varnothing) = \{\varnothing\}$;

(c) One can compute the cardinality of the power set of S by counting the number of ways one can choose subsets of S (of all possible sizes):

$$|P(S)| = \sum_{i=0}^{n} \binom{i}{n} = 2^n$$

by the binomial theorem (for $x = y = 1$).

Problem 3

Let $S = \{1, 2, 3\}$. Define $f : S \to S$ as the transposition (12), and $g : S \to S$ as the transposition (23). It's clear that $g(f(1)) = 3$ and $f(g(1)) = 2$, and hence $f \circ g \neq g \circ f$. For the case of $\mathbb{R}$, consider $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ and $g : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x + 1$. Note that $f(g(1)) = 4$ while $g(f(1)) = 2$.

Problem 4

(a) Suppose $g \circ f$ is injective; assume for the sake of contradiction that f is not injective. Then there exist a, b distinct such that $f(a) = f(b)$. Then $g(f(a)) = g(f(b))$, contradicting that $g \circ f$ is injective.

(b) Consider