Commutative algebra: notes

Nilay Kumar

Last updated: September 9, 2013

1 Class 2

Definition 1. Let A be a ring. Then we defined the **spectrum of** A, $\operatorname{Spec} A$ to be the set of primes of A. If $\phi : A \to B$ is a ring map, then we get $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ given by $q \mapsto \phi^{-1}(q)$. This works because if q is prime, then B/q is a domain, which implies that $A/\phi^{-1}(q)$ is a domain, and hence $\phi^{-1}(q)$ is a prime.

Remark. Abuse of notation: Often we write $A \cap q$ for $\phi^{-1}(q)$ even if ϕ is not injective.

Example 1. Consider $\operatorname{Spec}(\mathbb{C}[x])$. This is a UFD and thus a PID, and hence the primes are principal ideals generated by irreducibles, i.e. linear terms. Hence $\operatorname{Spec}(\mathbb{C}[x]) = \{(0), (x-\lambda); \lambda \in C\}$. Consider $\phi : \mathbb{C}[x] \to \mathbb{C}[y]$, given by $x \mapsto y^2$. Set $q_\lambda = (y-\lambda)$ and $p_\lambda = (x-\lambda)$. Then $\operatorname{Spec}(\phi)(q_\lambda) = p_{\lambda^2}$. Why is this? We have that $x - \lambda^2 \mapsto y^2 - \lambda^2 = (y+\lambda)(y-\lambda) \in q_\lambda$. Additionally, we have that $\operatorname{Spec}(\phi)\left((0)\right) = (0)$. Note that the fibers of $\operatorname{Spec}(\phi)$ are finite!

The goal of the next couple lectures is to show that the fibres of maps on spectra of a finite ring map are finite.

Let $\phi : A \to B$ be a ring map and $\mathfrak{p} \subset A$ a prime ideal. What is the fibre of $\operatorname{Spec}(\phi)$ over $\mathfrak{p}$? First of all, note that if $\mathfrak{q} \cap A = \mathfrak{p}$, then $\mathfrak{p}B \subset \mathfrak{q}$.

Lemma 1. *If $I \subset A$ is an ideal in a ring A then the ring map $A \to A/I$ induces via* $\operatorname{Spec}(-)$ *a bijection* $\operatorname{Spec}(A/I) \leftrightarrow \{\mathfrak{p} \in \operatorname{Spec}(A) | I \subset \mathfrak{p}\} =: V(I)$.

Proof. Fill this in. □

Remark. The **Zariski topology** has as closed subsets the sets $V(I)$.

Consider now (diagram 1). By staring at this diagram for a bit, we conclude that the fibre of $\operatorname{Spec}(\phi)$ are the set of primes $\mathfrak{q}' \subset B/\mathfrak{p}B$ which are $(0) \subset A/\mathfrak{p}$.

Lemma 2. *If K is a field, then* $\operatorname{Spec}(K)$ *has exactly one point. If K is the fraction field of a domain A, then* $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ *maps the unique point to* $(0) \in \operatorname{Spec}(A)$.

Proof. Fill this in. □

Next we wish to invert some elements in $B/\mathfrak{p}B$. This leads to a very general notion of localization.

Definition 2. Let A be a ring. A **multiplicative subset** of A is a subset $S \subset A$ such that $1 \in S$ and if $a, b \in S$, then $ab \in S$.

Definition 3. Given a multiplicative subset S, we can define the **localization of A with respect to S**, $S^{-1}A$, as the set of pairs (a, s) with $a \in A, s \in S$ modulo the equivalence relation $(a, s) \sim (a', s') \iff \exists s'' \in S$ such that $s''(as' - a's) = 0$ in A. Elements of $S^{-1}A$ are denoted $\frac{a}{s}$. Addition proceeds as usual. One checks that this is indeed a ring.

Lemma 3. *The ring map $A \to S^{-1}A$ given by $a \mapsto \frac{a}{1}$ induces a bijection* $\mathrm{Spec}(S^{-1}A) \leftrightarrow \{\mathfrak{p} \subset A | S \cap \mathfrak{p} = \varnothing\}$.

Proof. Fill this in. □

Note that any element of S becomes invertible in $S^{-1}A$ so not in any prime ideal of $S^{-1}A$. See diagram 2.

Example 2. Suppose $A = \mathbb{C}[x] \to B = \mathbb{C}[y]$ with $x \mapsto 5y^2 + 3y + 2$. Then $\mathrm{Spec}(\phi)^{-1}((x)) = \mathrm{Spec}\left((A/\mathfrak{p} - \{0\})^{-1}B/\mathfrak{p}B\right) = \mathrm{Spec}\left((\mathbb{C}^\times)^{-1}\mathbb{C}[y]/(5y^2 + 3y + 2)\right)$. There are two points.

Given $S \subset A$ multiplicative, and an A-module M, we can form an $S^{-1}A$-module

$$S^{-1}M = \left\{\frac{m}{s} | m \in M, s \in S\right\} / \sim$$

where the equivalence relation is the same as before. The construction $M \to S^{-1}M$ is a functor $\mathrm{Mod}_A \to \mathrm{Mod}_{S^{-1}A}$.

Lemma 4. *The localization function $M \to S^{-1}M$ is exact.*

Proof. □