Physics 6047
Problem Set 1, due 1/31/13
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Some fraction of this problem set functions as notes. It will be fairly typical of future ones.

1. In the first lecture, we discussed the following amplitude $\langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle$. If we were following strictly what you had learned in your non-relativistic QM class, we would have used $H = |\vec{k}|^2/(2m)$, where $\vec{k}$ is the momentum, and m the mass of the particle in question. You can convince yourself this would result in a non-zero amplitude even if $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$ are space-like separated. (You can find a discussion of this in Peskin & Schroeder 2.1.) But one might think this could be an artifact of using the non-relativistic expression for the energy H. This was why we computed instead:

$$\langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle = \int d^3k \langle \vec{x}_2 | \vec{k} \rangle \langle \vec{k} | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle \tag{1}$$

with the understanding $H|\vec{k}\rangle = \omega_k |\vec{k}\rangle$, where $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$ gives the correct relativistic form for the energy of the particle. By repeating the arguments in class (making sure you understand all the steps of the contour integral derivation), *show that*:

$$\langle \vec{x}_2 | e^{-iH(t_2-t_1)} | \vec{x}_1 \rangle = |\mathcal{N}|^2 \frac{4\pi i}{|\Delta \vec{x}|} \int_m^\infty z dz e^{-z|\Delta \vec{x}|} \sinh\left(\sqrt{z^2 - m^2} \Delta t\right) \tag{2}$$

where $\mathcal{N}$ is the normalization factor: $\langle \vec{k} | \vec{x} \rangle = \mathcal{N} e^{-i\vec{k}\cdot\vec{x}}$, and we assume $|\Delta \vec{x}|$ is non-zero. Since the integrand is positive definite (assuming $\Delta t > 0$), we see that we have a non-zero amplitude even for a space-like separation i.e. faster than speed of light propagation. We interpret this as a sign that we need a different sort of theory i.e. quantum field theory, in particular a theory that contains not merely single-particle states, but also multiple-particle states, due to the inevitable production of particles when we try to localize a particle to very high precision.

2. Show that

$$\int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 + m^2) \Theta(k^0 > 0) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} . \tag{3}$$

Note that on the left hand side, k^0 is to be integrated over. The step function $\Theta(k^0 > 0)$ forces it to be positive i.e. $\Theta = 1$ if $k^0 > 0$, and vanishes otherwise. The delta function $\delta(k^2 + m^2)$ forces it to be so-called "on-shell" or "on-mass-shell" i.e. it enforces $k^2 = -(k^0)^2 + |\vec{k}|^2 = -m^2$. On the right hand side, the integral over k^0 has disappeared, and ω_k is precisely the on-shell (and positive) value of k^0. Since the left hand side is manifestly Lorentz invariant, so is the right hand side. (By the way, this tells us another thing wrong with what we did in problem 1: the amplitude was not Lorentz invariant because the integration measure was not. Modifying the calculation accordingly wouldn't cure the problem of acausal propagation though.)

3. Suppose we have a Lagrangian (density) of the form $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ i.e. it depends on some scalar field $\phi(x)$ and its space-time (first) derivatives. (We loosely use x without an arrow on top to denote both space and time components i.e. x^μ.) Show that the classical equation of motion for ϕ – the one that follows from the principle of extremizing the action i.e. $\delta S = 0$ – is:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{4}$$

4. In the second lecture, we showed:

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{5}$$

By multiplying both sides by $\eta^{\gamma\alpha}$ (and summing over α), *show that*

$$\Lambda_\nu{}^\gamma \Lambda^\nu{}_\beta = \delta^\gamma{}_\beta \tag{6}$$

Note what we have done: starting from the Lorentz matrix $\Lambda^\nu{}_\beta$, we are saying by raising and lowering indices appropriately, we obtain its inverse i.e. $\Lambda_\nu{}^\gamma$ is equivalent to $(\Lambda^{-1})^\gamma{}_\nu$. *Check that* this actually does work in the example of a boost in the x-direction i.e. start from the Lorentz boost matrix, raise and lower indices as prescribed above, and show that you do get the inverse of the original boost matrix. *Check that* it also works in the example of a rotation, say around the z-axis.

5. Starting from $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = (2\pi)^3 2\omega_k \delta(\vec{k} - \vec{k}')$, show that

$$[\phi(t, \vec{x}), \dot{\phi}(t, \vec{x}')] = i\delta(\vec{x} - \vec{x}')\,, \tag{7}$$

follows from the expression for (the free) ϕ in terms of the annihilation and creation operators.

6. Show that the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right] \tag{8}$$

gives

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k (a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}) \tag{9}$$

once one expresses ϕ in terms of the creation and annihilation operators.

7. Srednicki problem 3.5. This is a problem of a complex scalar field, as opposed to the real scalar field we have been examining in class. In other words, for a real scalar field, the Fourier decomposition:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a_{\vec{k}} e^{ik\cdot x} + a^\dagger_{\vec{k}} e^{-ik\cdot x} \right] \tag{10}$$

makes sense, because $\phi^\dagger = \phi$. (Note k^0 in expressions like above is interpreted as on-shell i.e. $k^0 = \omega_k$.) But for a complex scalar field φ, we need to keep track of two different kinds of annihilation (or creation) operators $a_{\vec{k}}$ and $b_{\vec{k}}$:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} \left[a_{\vec{k}} e^{ik\cdot x} + b^\dagger_{\vec{k}} e^{-ik\cdot x} \right] \tag{11}$$

so that $\varphi^\dagger \neq \varphi$. *Add* to this problem a part (f): *show that* the equation of motion for φ (and its conjugate) implies

$$\partial_\mu j^\mu = 0 \tag{12}$$

where j^μ is defined as

$$j^\mu \equiv i\left[\varphi\partial^\mu\varphi^\dagger - \varphi^\dagger\partial^\mu\phi\right] . \tag{13}$$

Such an equation tells us j^μ is a conserved current, meaning that j^0 is the charge density and $\vec{j}$ is the current density. In other words, imagine integrating the above equation over some spatial volume V at some time t:

$$\partial_t\left(\int_V d^3x j^0\right) = -\int_V d^3x\vec{\nabla}\cdot\vec{j} = -\int_{\partial V} d\vec{S}\cdot\vec{j} \tag{14}$$

where we have used the Gauss law in the second equality and it gives the integrated current flux out of the boundary of the volume V. This is exactly what charge conservation should look like: the net charge in the volume $Q = \int_V d^3xj^0$ changes with time at a rate that is exactly determined by the flux of charges at the boundary. Thus, our complex scalar theory secretly has a conserved charge, which wasn't present for the real scalar theory. (The analog of j^μ is simply zero for a real scalar; the secret behind the complex scalar theory is the presence of a symmetry: rotating φ by a phase, a subject we will study at length later.) By taking the volume V to be infinite, *show that*

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a^\dagger_{\vec{k}}a_{\vec{k}} - b^\dagger_{\vec{k}}b_{\vec{k}}\right] . \tag{15}$$

This is very satisfying. We have discovered that the complex scalar theory contains 2 kinds of particles of the same mass m, one created by $a^\dagger_{\vec{k}}$, the other created by $b^\dagger_{\vec{k}}$. And they carry opposite charge! In other words, they are anti-particle of each other. (The real scalar field case has no such structure; sometimes, people would say in that case, the anti-particle is the same as the particle.) Thus, as promised, a relativistic quantum field theory which has a conserved charge would automatically contain anti-particles.

Finally, a question for you: why is it that for the real scalar field theory, the Lagrangian has this factor of 1/2, which is absent in the complex case? This is really pure convention, but this choice of normalization helps enforce certain conventions we would usually like to impose. Do you know what they are?