Commutative Algebra: Problem Set 13

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Problem 2

Consider the inclusion $\phi : \mathbb{R} \to \mathbb{C}$. We obtain an induced (surjective) map on spectra $f : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ that is clearly continuous as a map of topologies. As schemes, the morphism on sheaves is $f^{\#} : \mathcal{O}_{\operatorname{Spec} \mathbb{R}} \to f_* \mathcal{O}_{\operatorname{Spec} \mathbb{C}}$ but since the only open sets are all of $\mathbb{R}$ and all of $\mathbb{C}$, we get just the inclusion $\phi : \mathbb{R} \to \mathbb{C}$ on global sections. Note, however, that for there to be a right-inverse of f there must be a morphism of schemes $g : \operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{C}$. That there is a continuous map on topologies is clear, but there is no well-defined morphism on the structure sheaves, as this would require a ring map $\mathbb{C} \to \mathbb{R}$, which does not exist, as i cannot be sent to anything in $\mathbb{R}$ (lest it violate the homomorphism property).

Problem 5

Let $X_1, X_2, X_3, \ldots$ be a sequence of affine schemes. Let $F : \mathbf{Sch}^{op} \to \mathbf{Set}$ be the contravariant functor that takes a scheme T to the product $\Pi_i \operatorname{Mor}_{\mathbf{Sch}}(T, X_i)$. We wish to show that F is representable by an affine scheme. Note first that $\operatorname{Mor}_{\mathbf{Sch}}(T, X_i) = \operatorname{Mor}_{\mathbf{CRing}}(A_i, B)$ where A_i (resp. B) are the coordinate rings of X_i (resp. T). Hence, for F to be representable we must find an affine scheme Y with coordinate ring $\mathcal{O}_Y(Y)$ such that

$$\operatorname{Mor}_{\mathbf{Ring}}(\mathcal{O}_Y(Y), B) = \prod_i \operatorname{Mor}_{\mathbf{CRing}}(A_i, B).$$

But this is simply the definition of the coproduct in the category **CRing**: $\mathcal{O}_Y(Y) = \coprod_i A_i$ and hence F is representable by the affine scheme $\operatorname{Spec} \coprod_i A_i = \operatorname{Spec} \coprod_i \mathcal{O}_{X_i}(X_i)$.