Review of Linear Algebra

Throughout these notes, F denotes a field (often called the *scalars* in this context).

1 Definition of a vector space

Definition 1.1. A *F-vector space* or simply a *vector space* is a triple $(V, +, \cdot)$, where $(V, +)$ is an abelian group (the *vectors*), and there is a function $F \times V \to V$ (*scalar multiplication*), whose value at (t, v) is just denoted $t \cdot v$ or tv, such that

1. For all $s, t \in F$ and $v \in V$, $s(tv) = (st)v$. (An analogue of the associative law for multiplication.)

2. For all $s, t \in F$ and $v \in V$, $(s + t)v = sv + tv$. (Scalar multiplication distributes over scalar addition.)

3. For all $t \in F$ and $v, w \in V$, $t(v + w) = tv + tw$. (Scalar multiplication distributes over vector addition.)

4. For all $v \in V$, $1 \cdot v = v$. (A kind of identity law for scalar multiplication.)

It is a straightforward consequence of the axioms that $0v = 0$ for all $v \in V$, where the first 0 is the element $0 \in F$ and the second is $0 \in V$, that, for all $t \in F$, $t0 = 0$ (both 0's here are the zero vector), and that, for all $v \in V$, $(-1)v = -v$.

Example 1.2. (1) The n-fold Cartesian product F^n is an F-vector space, where F^n is a group under componentwise addition, i.e. $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$, and scalar multiplication is defined by $t(a_1, \dots, a_n) = (ta_1, \dots, ta_n)$.
(2) If X is a set and F^X denotes the group of functions from X to F, then F^X is an F-vector space. Here addition is defined pointwise: $(f + g)(x) = f(x) + g(x)$, and similarly for scalar multiplication: $(tf)(x) = tf(x)$.

(3) The polynomial ring $F[x]$ is an F-vector space under addition of polynomials, where we define $tf(x)$ to be the product of the constant polynomial $t \in F$ with the polynomial $f(x) \in F[x]$. Explicitly, if $f(x) = \sum_{i=0}^{n} a_n x^n$, then $tf(x) = \sum_{i=0}^{n} ta_n x^n$.
(4) More generally, if R is any ring containing F as a subring, then R becomes an F-vector space, using the fact that $(R, +)$ is an abelian group, and defining, for $t \in F$ and $r \in R$, scalar multiplication tr to be the usual multiplication in R applied to t and r. In this case, the properties (1)–(4) of a vector space become consequences of associativity, distributivity, and the fact that $1 \in F$ is the multiplicative identity in R. In particular, if E is a field containing F as a subfield, then E is an F-vector space in this way.

Definition 1.3. Let V be a vector space. An *F-vector subspace* of V is a subset $W \subseteq V$ is an additive subgroup of V which is closed under scalar multiplication (i.e. such that, for all $v \in W$ and for all $t \in F$, $tv \in W$). It then follows that $(W, +, \cdot)$ is also a vector space.

Example 1.4. $\{0\}$ and V are always vector subspaces of V. The set

$$P_n = \{\sum_{i=0}^{n} a_i x^i : a_i \in F\}$$

of polynomials of degree at most n, together with the zero polynomial, is a vector subspace of $F[x]$.

Finally, we have the analogue of homomorphisms and isomorphisms:

Definition 1.5. Let V_1 and V_2 be two F-vector spaces and let $f: V_1 \to V_2$ be a function (= map). Then f is *linear* or a *linear map* if it is a group homomorphism from $(V_1, +)$ to $(V_2, +)$, i.e. is additive, and satisfies: For all $t \in F$ and $v \in V_1$, $f(tv) = tf(v)$. The function f is a *linear isomorphism* if it is both linear and a bijection; in this case, it is easy to check that f^{-1} is also linear.

There is an analogue of vector spaces for more general rings:

Definition 1.6. Let R be a ring (always assumed commutative, with unity). Then an *R-module* M is a triple $(M, +, \cdot)$, where $(M, +)$ is an abelian group, and there is a function $R \times M \to M$, whose value at (r, m) is denoted rm or $r \cdot m$, such that:

1. For all $r, s \in R$ and $m \in M$, $r(sm) = (rs)m$.

2. For all $r, s \in R$ and $m \in M$, $(r+s)m = rm + sm$.
3. For all $r \in R$ and $m, n \in M$, $r(m+n) = rm + rn$.
4. For all $m \in M$, $1 \cdot m = m$.

Submodules and homomorphisms of R-modules are defined in the obvious way. For example, R^n is an R-module. Despite the similarity with the definition of vector spaces, general R-modules can look quite complicated. For example, a $\mathbb{Z}$-module is the same thing as an abelian group, and hence $\mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-module. For a general ring R, if I is an ideal of R, then both I and R/I are R-modules.

2 Linear independence and span

Let us introduce some terminology:

Definition 2.1. Let V be an F-vector space and let $v_1, \ldots, v_d \in V$ be a sequence of vectors. A *linear combination* of $v_1, \ldots, v_d$ is a vector of the form $t_1v_1 + \cdots + t_dv_d$, where the $t_i \in F$. The *span* of $\{v_1, \ldots, v_d\}$ is the set of all linear combinations of $v_1, \ldots, v_d$. Thus

$$\operatorname{span}\{v_1, \ldots, v_d\} = \{t_1v_1 + \cdots + t_dv_d : t_i \in F \text{ for all } i\}.$$

By definition (or logic), $\operatorname{span}\emptyset = \{0\}$.

We have the following properties of span:

Proposition 2.2. *Let $v_1, \ldots, v_d \in V$ be a sequence of vectors. Then:*

(i) $\operatorname{span}\{v_1, \ldots, v_d\}$ *is a vector subspace of V containing v_i for every i.*

(ii) *If W is a vector subspace of V containing $v_1, \ldots, v_d$, then*

$$\operatorname{span}\{v_1, \ldots, v_d\} \subseteq W.$$

In other words, $\operatorname{span}\{v_1, \ldots, v_d\}$ *is the smallest vector subspace of V containing $v_1, \ldots, v_d$.*

(iii) *For every $v \in V$,* $\operatorname{span}\{v_1, \ldots, v_d\} \subseteq \operatorname{span}\{v_1, \ldots, v_d, v\}$*, and equality holds if and only if* $v \in \operatorname{span}\{v_1, \ldots, v_d\}$.

Proof. (i) Taking $t_j = 0, j \neq i$, and $t_i = 1$, we see that $v_i \in \text{span}\{v_1, \dots, v_d\}$. Next, given $v = t_1v_1 + \cdots + t_dv_d$ and $w = s_1w_1 + \cdots + s_dw_d \in \text{span}\{v_1, \dots, v_d\}$,

$$v + w = (s_1 + t_1)v_1 + \cdots + (s_d + t_d)w_d \in \text{span}\{v_1, \dots, v_d\}.$$

Hence $\text{span}\{v_1, \dots, v_d\}$ is closed under addition. Also, $0 = 0 \cdot v_1 + \cdots + 0 \cdot v_d \in \text{span}\{v_1, \dots, v_d\}$, and, given $v = t_1v_1 + \cdots + t_dv_d \in \text{span}\{v_1, \dots, v_d\}$, $-v = (-t_1)v_1 + \cdots + (-t_d)v_d \in \text{span}\{v_1, \dots, v_d\}$ as well. Hence $\text{span}\{v_1, \dots, v_d\}$ is an additive subgroup of V. Finally, for all $v = t_1v_1 + \cdots + t_dv_d \in \text{span}\{v_1, \dots, v_d\}$ and $t \in F$,

$$tv = (tt_1)v_1 + \cdots + (tt_d)v_d \in \text{span}\{v_1, \dots, v_d\}.$$

Hence $\text{span}\{v_1, \dots, v_d\}$ is a vector subspace of V containing v_i for every i.

(ii) If W is a vector subspace of V containing $v_1, \dots, v_d$, then, for all $t_1, \dots, t_d \in F$, $t_iv_i \in W$. Hence $t_1v_1 + \cdots + t_dv_d \in W$. It follows that $\text{span}\{v_1, \dots, v_d\} \subseteq W$.

(iii) We always have $\text{span}\{v_1, \dots, v_d\} \subseteq \text{span}\{v_1, \dots, v_d, v\}$, since we can write $t_1v_1 + \cdots + t_dv_d = t_1v_1 + \cdots + t_dv_d + 0 \cdot v$.

Now suppose that $\text{span}\{v_1, \dots, v_d\} = \text{span}\{v_1, \dots, v_d, v\}$. Then in particular $v \in \text{span}\{v_1, \dots, v_d, v\} = \text{span}\{v_1, \dots, v_d\}$, by (i), and hence $v \in \text{span}\{v_1, \dots, v_d\}$, i.e. v is a linear combination of $v_1, \dots, v_d$. Conversely, suppose that $v \in \text{span}\{v_1, \dots, v_d\}$. Then by (i) $\text{span}\{v_1, \dots, v_d\}$ is a vector subspace of V containing $v_1, \dots, v_d$ and v and hence $\text{span}\{v_1, \dots, v_d, v\} \subseteq \text{span}\{v_1, \dots, v_d\}$. Thus $\text{span}\{v_1, \dots, v_d, v\} = \text{span}\{v_1, \dots, v_d\}$. □

Definition 2.3. A sequence of vectors $w_1, \dots, w_\ell$ such that

$$V = \text{span}\{w_1, \dots, w_\ell\}$$

will be said to *span* V.

Definition 2.4. An F-vector space V is a *finite dimensional vector space* if there exist $v_1, \dots, v_d \in V$ such that $V = \text{span}\{v_1, \dots, v_d\}$. The vector space V is *infinite dimensional* if it is not finite dimensional.

For example, F^n is finite-dimensional. But $F[x]$ is not a finite-dimensional vector space. On the other hand, the subspace P_n of $F[x]$ defined in Example 1.4 is a finite dimensional vector space, as it is spanned by $1, x, \dots, x^n$.

The next piece of terminology is there to deal with the fact that we might have chosen a highly redundant set of vectors to span V.

Definition 2.5. A sequence $w_1, \dots, w_r \in V$ is *linearly independent* if the following holds: if there exist real numbers t_i such that

$$t_1 w_1 + \cdots + t_r w_r = 0,$$

then $t_i = 0$ for all i. The sequence $w_1, \dots, w_r$ is *linearly dependent* if it is not linearly independent.

Note that the definition of linear independence does **not** depend **only** on the set $\{w_1, \dots, w_r\}$—if there are any repeated vectors $w_i = w_j$, then we can express 0 as the nontrivial linear combination $w_i - w_j$. Likewise if one of the w_i is zero then the set is linearly dependent.

By definition or by logic, the empty set is linearly independent. For a less obscure example, $e_1, \dots, e_n \in F^n$ are linearly independent since if $t_1 e_1 + \cdots + t_n e_n = 0$, then $(t_1, \dots, t_n)$ is the zero vector and thus $t_i = 0$ for all i. More generally, for all $j \leq n$, the vectors $e_1, \dots, e_j$ are linearly independent.

Lemma 2.6. *The vectors $w_1, \dots, w_r$ are linearly independent if and only if, given $t_i, s_i \in F$, $1 \leq i \leq r$, such that*

$$t_1 w_1 + \cdots + t_r w_r = s_1 w_1 + \cdots + s_r w_r,$$

then $t_i = s_i$ for all i.

Proof. If $w_1, \dots, w_r$ are linearly independent and if $t_1 w_1 + \cdots + t_r w_r = s_1 w_1 + \cdots + s_r w_r$, then after subtracting and rearranging we have $(t_1 - s_1) w_1 + \cdots + (t_r - s_r) w_r 0$. Thus by the definition of linear independence $t_i - s_i = 0$ for every i, i.e. $s_i = t_i$. Conversely, if the last statement of the lemma holds and if $t_1 w_1 + \cdots + t_r w_r = 0$, then it follows from

$$t_1 w_1 + \cdots + t_r w_r = 0 = 0 \cdot w_1 + \cdots + 0 \cdot w_r$$

that $t_i = 0$ for all i. Hence $w_1, \dots, w_r$ are linearly independent. $\square$

Clearly, if $w_1, \dots, w_r$ are linearly independent, then so is any reordering of the vectors $w_1, \dots, w_r$, and likewise any smaller sequence (not allowing repeats), for example $w_1, \dots, w_s$ with $s \leq r$. A related argument shows:

Lemma 2.7. *The vectors $w_1, \dots, w_r$ are* ***not*** *linearly independent if and only if we can write at least one of the w_i as a linear combination of the $w_j, j \neq i$.*

The proof is left as an exercise.

Definition 2.8. Let V be an F-vector space. The vectors $v_1, \dots, v_d$ are a *basis* of V if they are linearly independent and $V = \mathrm{span}\{v_1, \dots, v_d\}$. For example, the standard basis vectors $e_1, \dots, e_n$ are a basis for F^n. The elements $1, x, \dots, x^n$ are a basis for P_n.

Thus to say that the vectors $v_1, \dots, v_r$ are a basis of V is to say that every $x \in V$ can be uniquely written as $x = t_1 v_1 + \cdots + t_r v_r$ for $t_i \in F$.

Lemma 2.9 (Main counting argument)**.** *Suppose that $w_1, \dots, w_b$ are linearly independent vectors contained in $\mathrm{span}\{v_1, \dots, v_a\}$. Then $b \le a$.*

Proof. We shall show that, possibly after relabeling the v_i,

$$\begin{aligned}\mathrm{span}\{v_1, \dots, v_a\} &= \mathrm{span}\{w_1, v_2, \dots, v_a\} = \mathrm{span}\{w_1, w_2, v_3, \dots, v_a\} \\ &= \cdots = \mathrm{span}\{w_1, w_2, \dots, w_b, v_{b+1}, \dots, v_a\}.\end{aligned}$$

From this we will be able to conclude that $b \le a$.

To begin, we may suppose that $\{w_1, \dots, w_b\} \neq \emptyset$. (If $\{w_1, \dots, w_b\} = \emptyset$, then $b = 0$ and the conclusion $b \le a$ is automatic.) Moreover none of the w_i is zero. Given w_1, we can write it as a linear combination of the v_i:

$$w_1 = \sum_{i=1}^{a} t_i v_i.$$

Since $w_1 \neq 0$, $\{v_1, \dots, v_a\} \neq \emptyset$, i.e. $a \ge 1$, and at least one of the $t_i \neq 0$. After relabeling the v_i, we can assume that $t_1 \neq 0$. Thus we can solve for v_1 in terms of w_1 and the $v_i, i > 1$:

$$v_1 = \frac{1}{t_1} w_1 + \sum_{i=2}^{a} \left(-\frac{t_i}{t_1}\right) v_i.$$

It follows that $v_1 \in \mathrm{span}\{w_1, v_2, \dots, v_a\}$. Now using some of the properties of span listed above, we have

$$\mathrm{span}\{w_1, v_2, \dots, v_a\} = \mathrm{span}\{v_1, w_1, v_2, \dots, v_a\} = \mathrm{span}\{v_1, v_2, \dots, v_a\},$$

where the second equality holds since $w_1 \in \mathrm{span}\{v_1, v_2, \dots, v_a\}$.

Continuing in this way, write w_2 as a vector in $\mathrm{span}\{w_1, v_2, \dots, v_a\}$:

$$w_2 = t_1 w_1 + \sum_{i=2}^{a} t_i v_i.$$

For some $i \geq 2$, we must have $t_i \neq 0$, for otherwise we would have $w_2 = t_1 w_1$ and thus there would exist a nontrivial linear combination $t_1 w_1 + (-1) w_2 = 0$, contradicting the linear independence of the w_i. After relabeling, we can assume that $t_2 \neq 0$; notice that in particular we must have $a \geq 2$. Arguing as before, we may write

$$v_2 = -\frac{t_1}{t_2} w_1 + \frac{1}{t_2} w_2 + \sum_{i=3}^{a} \left(-\frac{t_i}{t_2} \right) v_i,$$

and thus we can solve for v_2 in terms of w_1, w_2, and $v_i, i \geq 3$ and so

$$\operatorname{span}\{w_1, v_2, \ldots, v_a\} = \operatorname{span}\{w_1, w_2, v_3, \ldots, v_a\}.$$

By induction, for a fixed $i < b$, suppose that we have showed that $i \leq a$ and that after some relabeling of the v_i we have

$$\begin{gathered} \operatorname{span}\{v_1, \ldots, v_a\} = \operatorname{span}\{w_1, v_2, \ldots, v_a\} = \\ = \operatorname{span}\{w_1, w_2, v_3, \ldots, v_a\} = \cdots = \operatorname{span}\{w_1, w_2, \ldots, w_i, v_{i+1}, \ldots, v_a\}. \end{gathered}$$

We claim that the same is true for $i + 1$. Write

$$w_{i+1} = t_1 w_1 + \cdots + t_i w_i + t_{i+1} v_{i+1} + \cdots + t_a v_a,$$

which is possible as

$$w_{i+1} \in \operatorname{span}\{v_1, \ldots, v_a\} = \operatorname{span}\{w_1, w_2, \ldots, w_i, v_{i+1}, \ldots, v_a\}.$$

At least one of the numbers $t_{i+1}, \ldots, t_a$ is nonzero, for otherwise $w_{i+1} = t_1 w_1 + \cdots + t_i w_i$, which would say that the vectors $w_1, \ldots, w_{i+1}$ are not linearly independent. In particular this says that $i+1 \leq a$. After relabeling, we may assume that $t_{i+1} \neq 0$. Then as before we can solve for v_{i+1} in terms of the vectors $w_1, \ldots, w_{i+1}, v_{i+2}, \ldots, v_a$. It follows that

$$\operatorname{span}\{w_1, w_2, \ldots, w_i, v_{i+1}, \ldots, v_a\} = \operatorname{span}\{w_1, w_2, \ldots, w_i, w_{i+1}, v_{i+2}, \ldots, v_a\}$$

and we have completed the inductive step. So for all $i \leq b$, $i \leq a$, and in particular $b \leq a$. $\square$

This rather complicated argument has the following consequences:

Corollary 2.10. (i) *Suppose that* $V = \operatorname{span}\{v_1, \ldots, v_n\}$ *and that* $w_1, \ldots, w_\ell$ *are linearly independent vectors in* V*. Then* $\ell \leq n$*.*

(ii) *If V is a finite-dimensional F-vector space, then every two bases for V have the same number of elements—call this number the* dimension *of V which we write as* $\dim V$ *or* $\dim_F V$ *if we want to emphasize the field F. Thus for example* $\dim F^n = n$ *and* $\dim P_n = n+1$.

(iii) *If $V = \mathrm{span}\{v_1, \dots, v_d\}$ then some subsequence of $v_1, \dots, v_d$ is a basis for V. Hence* $\dim V \leq d$, *and if* $\dim V = d$ *then $v_1, \dots, v_d$ is a basis for V.*

(iv) *If V is a finite-dimensional F-vector space and $w_1, \dots, w_\ell$ are linearly independent vectors in V, then there exist vectors*

$$w_{\ell+1}, \dots, w_r \in V$$

such that $w_1, \dots, w_\ell, w_{\ell+1}, \dots, w_r$ is a basis for V. Hence $\dim V \geq \ell$, *and if* $\dim V = \ell$ *then $w_1, \dots, w_\ell$ is a basis for V.*

(v) *If V is a finite-dimensional F-vector space and W is a vector subspace of V, then* $\dim W \leq \dim V$. *Moreover* $\dim W = \dim V$ *if and only if $W = V$.*

(vi) *If $v_1, \dots, v_d$ is a basis of V, then the function $f\colon F^d \to V$ defined by*

$$f(t_1, \dots, t_d) = \sum_{i=1}^{d} t_i v_i$$

is a linear isomorphism from F^d to V.

Proof. (i) Apply the lemma to the subspace V itself, which is the span of $v_1, \dots, v_n$, and to the linearly independent vectors $w_1, \dots, w_\ell \in V$, to conclude that $\ell \leq n$.

(ii) If $w_1, \dots, w_b$ and $v_1, \dots, v_a$ are two bases of V, then by definition $w_1, \dots, w_b$ are linearly independent vectors, and $V = \mathrm{span}\{v_1, \dots, v_a\}$. Thus $b \leq a$. But symmetrically $v_1, \dots, v_a$ is a sequence of linearly independent vectors contained in $V = \mathrm{span}\{w_1, \dots, w_b\}$, so $a \leq b$. Thus $a = b$.

(iii) If $v_1, \dots, v_d$ are not linearly independent, then, by Lemma 2.7, one of the vectors v_i is expressed as a linear combination of the others. After relabeling we may assume that v_d is a linear combination of $v_1, \dots, v_{d-1}$. By (iii) of Proposition 2.2, $\mathrm{span}\{v_1, \dots, v_{d-1}\} = \mathrm{span}\{v_1, \dots, v_d\}$. Continue in this way until we find $v_1, \dots, v_k$ with $k \leq d$ such that $\mathrm{span}\{v_1, \dots, v_k\} = \mathrm{span}\{v_1, \dots, v_d\}$ and such that $v_1, \dots, v_k$ are linearly independent. Then by definition $k = \dim V$ and $k \leq d$. Moreover $k = d$ exactly when $v_1, \dots, v_d$ are linearly independent, in which case $\mathrm{span}\{v_1, \dots, v_d\}$ is a basis.

(iv) If $\mathrm{span}\{w_1, \dots, w_\ell\} \neq V$, then there exists a vector, call it $w_{\ell+1} \in V$ with $w_{\ell+1} \notin \mathrm{span}\{w_1, \dots, w_\ell\}$. It follows that the sequence $w_1, \dots, w_\ell, w_{\ell+1}$

is still linearly independent: if there exist $t_i \in F$, not all 0, such that $0 = \sum_{i=1}^{\ell+1} t_i w_i$, then we must have $t_{\ell+1} \neq 0$ since $w_1, \dots, w_\ell$ are linearly independent. But that would say that $w_{\ell+1} \in \operatorname{span}\{w_1, \dots, w_\ell\}$, contradicting our choice. So $w_1, \dots, w_\ell, w_{\ell+1}$ are still linearly independent. We continue in this way. Since the number of elements in a linearly independent sequence of vectors in V is at most $\dim V$, this procedure has to stop after at most $\dim V - \ell$ stages. At this point we have found a linearly independent sequence which spans V and thus is a basis. To see the last statement, note that if $\operatorname{span}\{w_1, \dots, w_\ell\} \neq V$, then there exist $\ell + 1$ linearly independent vectors in V, and hence $\dim V \geq \ell + 1$.

(v) Choosing a basis of W and applying (iv) (since the elements of a basis are linearly independent) we see that it can be completed to a basis of V. Thus $\dim W \leq \dim V$. Moreover $\dim W = \dim V$ if and only if the basis we chose for W was already a basis for V, i.e. $W = V$.

(vi) A straightforward calculation shows that f is linear. It is a bijection by definition of a basis: it is surjective since the v_i span V, and it is injective since the v_i are linearly independent and by Lemma 2.6. □

Proposition 2.11. *If V is a finite dimensional F-vector space and W is a subset of V, then W is a vector subspace of V if and only if it is of the form* $\operatorname{span}\{v_1, \dots, v_d\}$ *for some $v_1, \dots, v_d \in V$.*

Proof. We have already noted that a set of the form $\operatorname{span}\{v_1, \dots, v_d\}$ is a vector subspace of V. Conversely let W be a vector subspace of V. The proof of (iv) of the above corollary shows how to find a basis of W. In particular W is of the form $\operatorname{span}\{v_1, \dots, v_d\}$. □

3 Linear functions and matrices

Let $f\colon F^n \to F^k$ be a linear function. Then

$$f(t_1, \dots, t_n) = f(\sum_i t_i e_i) = \sum_i t_i f(e_i).$$

We can write $f(e_i)$ in terms of the basis $e_1, \dots, e_k$ of F^k: suppose that $f(e_i) = \sum_{j=1}^k a_{ji} e_j = (a_{1i}, \dots, a_{ki})$. We can then associate to f an $k \times n$ matrix with coefficients in F as follows: Define

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}.$$

Then the columns of A are the vectors $f(e_i)$.

More generally, suppose that V_1 and V_2 are two finite dimensional vector spaces, and choose bases $v_1, \dots, v_n$ of V_1 and $w_1, \dots, w_k$ of V_2, so that $n = \dim V_1$ and $k = \dim V_2$. If $f\colon V_1 \to V_2$ is linear, then, given the bases $v_1, \dots, v_n$ and $w_1, \dots, w_k$ we again get a $k \times n$ matrix A by the formula: $A = (a_{ij})$, where a_{ij} is defined by

$$f(v_i) = \sum_{j=1}^{k} a_{ji} w_j.$$

We say that A is the matrix *associated to the linear map* f *and the bases* $v_1, \dots, v_n$ *and* $w_1, \dots, w_k$.

With the understanding that we have to choose bases to define the matrix associated to a linear map, composition of linear maps corresponds to multiplication of matrices. In particular, let $f\colon V \to V$ be a linear map from a finite dimensional vector space V to itself. In this case, it is simplest to fix one basis $v_1, \dots, v_n$ for V, viewing V as both the domain and range of f, and write $A = (a_{ij})$ where $f(v_i) = \sum_{j=1}^{k} a_{ji} v_j$. Again, having fixed the basis $v_1, \dots, v_n$ once and for all,linear maps from V to itself correspond to $n \times n$ matrices and composition to matrix multiplication. Finally, we can define the determinant $\det A$ of an $n \times n$ matrix with coefficients in F by the usual formulas (for example, expansion by minors). Moreover, the $n \times n$ matrix A is invertible $\iff$ $\det A \neq 0$, and in this case there is an explicit formula for A^{-1} (Cramer's rule).