Physics 6047
Problem Set 10, due 4/25/13
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1. Starting from the definitions of the Pauli matrices:

$$\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{1}$$

show that **a.** $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, **b.** $\sigma_i^T = \epsilon\sigma_i\epsilon$, **c.** $\bar\sigma^{\mu T} = -\epsilon\sigma^\mu\epsilon$, **d.** $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, **e.** $\gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0$, where the $\sigma^\mu \equiv (1, \vec\sigma)$, $\bar\sigma^\mu \equiv (1, -\vec\sigma)$, and

$$\gamma^\mu \equiv \begin{bmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{bmatrix} \quad , \quad \epsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \, . \tag{2}$$

2. A left-handed Weyl spinor ψ is supposed to transform like:

$$\psi_a \to \left(\delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}[S_L{}^{\mu\nu}]_a{}^b \right) \psi_b \, , \tag{3}$$

under a small Lorentz transformation $\Lambda = 1 + \delta\omega$, where

$$S_L{}^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma_k \quad \text{for rotation} \quad , \quad S_L{}^{k0} = \frac{i}{2}\sigma_k \quad \text{for boost} \, . \tag{4}$$

In class, we have shown that $\psi\epsilon\psi = \psi_a\epsilon^{ab}\psi_b$ is a scalar (which Srednicki writes as $-\psi\psi$). Show that $\psi^\dagger\bar\sigma^\mu\psi$ transforms as a vector i.e.

$$\psi^\dagger\bar\sigma^\mu\psi \to (1 + \delta\omega)^\mu{}_\nu \psi^\dagger\bar\sigma^\nu\psi \, , \tag{5}$$

for a rotation around the z-axis, and a boost along the z direction. Note that in all of the above expressions, I have suppressed the fact that the argument of ψ also gets transformed i.e. x on the left and $\Lambda^{-1}x$ on the right.

3. In class, starting from the Lagrangian for a left-handed Weyl spinor

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi + \frac{1}{2}m\psi\epsilon\psi - \frac{1}{2}m\psi^\dagger\epsilon\psi^\dagger \, , \tag{6}$$

we derived the equation of motion $\delta S/\delta\psi^\dagger = 0$:

$$i\bar\sigma^\mu\partial_\mu\psi - m\epsilon\psi^\dagger = 0 \, . \tag{7}$$

Show that its hermitian conjugate can be written as

$$i\sigma^\mu\partial_\mu\epsilon\psi^\dagger + m\psi = 0 \, , \tag{8}$$

and that it is equivalent to the equation of motion $\delta S/\delta\psi = 0$.

4. Defining the Dirac spinor using two Weyl spinors:

$$\Psi \equiv \begin{pmatrix} \chi \\ \epsilon\xi^\dagger \end{pmatrix} , \bar\Psi \equiv \Psi^\dagger\gamma^0 \, , \tag{9}$$

show that the Lagrangian

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi\,, \tag{10}$$

is equivalent to

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + m\chi\epsilon\xi - m\xi^\dagger\epsilon\chi^\dagger\,, \tag{11}$$

up to total derivatives (which vanish upon integration assuming vanishing boundary conditions).

5. Here, we want to think about Dirac and Majorana spinors without going through Weyl spinors. The starting point is the result discussed in problem 1 above, namely eq. (2) gives us a 4-dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. (The words: "representation of an algebra" merely mean an explicit set of matrices that obey the desired (anti-)commutation relations.) This automatically gives us a representation of the Lorentz algebra also, using $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ where $\Sigma^{\mu\nu}$ are the 6 Lorentz generators (3 boosts and 3 rotations). In other words, the Clifford algebra implies

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = i(\eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma}) - i(\eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\nu\sigma}\Sigma^{\mu\rho}) \tag{12}$$

which is the Lorentz algebra. As a check, *verify* that the above gives $[J_x, J_y] = iJ_z$, interpreting $\Sigma^{23} = J_x$, $\Sigma^{31} = J_y$ and $\Sigma^{12} = J_z$ – the angular momentum operators.

Having this representation means we have spinor objects such as Ψ, which has 4 components, and under a Lorentz boost/rotation Λ, transforms as:

$$\Psi(x) \to \Psi'(x) = e^{\frac{i}{2}\delta\omega_{\mu\nu}\Sigma^{\mu\nu}}\Psi(\Lambda^{-1}x) \tag{13}$$

where $\Sigma^{\mu\nu}$ is the 4-dimensional spinor representation of the transformation Λ, and $\delta\omega_{\mu\nu}$ represents the amount of the transformation e.g. $\delta\omega_{12}$ is the angle by which we rotate around the z-axis.

Now, Ψ is complex in general – we call it a Dirac spinor. We would like to introduce the concept of a 'real' spinor (i.e. a Majorana spinor) which has half the degrees of freedom. In the case of a scalar field ϕ, this is simple: a real scalar satisfies $\phi^* = \phi$. For a spinor, one might be tempted to declare something analogous $\Psi^* = \Psi$. But this won't work. This is because Ψ and Ψ^* transforms differently under Lorentz: whereas Ψ transforms by $e^{\frac{i}{2}\delta\omega_{\mu\nu}\Sigma^{\mu\nu}}$, Ψ^* transforms by $e^{-\frac{i}{2}\delta\omega_{\mu\nu}\Sigma^{\mu\nu *}}$ (i.e. simply taking the complex conjugate of eq. 13). Spinors that transform differently obviously can't be equated to each other!

Thus, for spinors, we are led to consider a more general definition of conjugation, let's try: $\Psi^C = B^{-1}\Psi^*$, where B^{-1} is some matrix. (I am paraphrasing Polchinski Vol. 2 Appendix B and using the same notation; but beware we are using different representations and so some of the expressions differ.) Once we find what this matrix is, we would declare a 'real' spinor – or more properly a Majorana spinor – obeys $\Psi^C = \Psi$. We call this the Majorana condition.

What are the properties we would like B^{-1} to have? First, we would like:

$$B\gamma^\mu B^{-1} = -\gamma^{\mu *} \tag{14}$$

(Strictly, speaking we can also have positive sign on the right, but our choice of B will turn out to correspond to negative sign, and in fact it seems to be the only choice in 4 space-time dimensions.) This is a good property to have because it implies

$$B\Sigma^{\mu\nu}B^{-1} = -\Sigma^{\mu\nu *} \tag{15}$$

Show that this means Ψ and Ψ^C transform in exactly the same way under a Lorentz transformation. The fact that they transform the same way makes it possible to contemplate situations in which they are in fact the same spinor. A second property we would like is $(\Psi^C)^C = \Psi$, which means

$$B^{-1}B^{-1*} = 1 \tag{16}$$

Show that both properties, eqs. (14) and (16), are realized by:

$$B^{-1} = i \begin{bmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{bmatrix} = B \tag{17}$$

Further *show that*

$$B^{-1} = \mathcal{C}\beta \tag{18}$$

where

$$\mathcal{C} = i \begin{bmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} = \begin{bmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \quad , \quad \beta = \gamma^0 \tag{19}$$

This is why Srednicki writes $\Psi^C = B^{-1}\Psi^*$ as

$$\Psi^C = \mathcal{C}\beta\Psi^* = \mathcal{C}\bar{\Psi}^T \tag{20}$$

A Dirac spinor is one where $\Psi^C \neq \Psi$, while a Majorana spinor is one where $\Psi^C = \Psi$. This is the analog of the scalar case: $\phi^* \neq \phi$ (complex); $\phi^* = \phi$ (real).

There is one further point to be made about (charge) conjugation: it has a connection with chirality. Recall the chirality operator $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The projection operators onto left- and right-handed chiralities (the two different Weyl representations that live inside a Dirac spinor) are: $P_L = (1-\gamma_5)/2$ and $P_R = (1+\gamma_5)/2$. *Show that*

$$B\gamma_5 B^{-1} = -{\gamma_5}^* = -\gamma_5 \, . \tag{21}$$

This implies

$$(P_L\Psi)^C = B^{-1}\,(P_L\Psi)^* = P_R B^{-1}\Psi^* = P_R\Psi^C \tag{22}$$

i.e. left-handed-projection followed by charge conjugation is the same as charge conjugation followed by right-handed-projection. If Ψ were a Majorana spinor, then the last equality yields $P_R\Psi$. Roughly speaking, charge conjugation flips chirality, i.e. for a Majorana spinor, its left- and right-handed parts are conjugates of each other, as should be clear from its explicit writing in the Weyl representation:

$$\Psi = \begin{bmatrix} \psi \\ \psi^\dagger \end{bmatrix} . \tag{23}$$

To conclude, for the complex and real scalars, the Lagrangian takes the respective form:

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \quad , \quad \mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \, ; \tag{24}$$

for the Dirac and Majorana spinors:

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \quad , \quad \mathcal{L} = \frac{1}{2}i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\bar{\Psi}\Psi = \frac{1}{2}i\Psi^T\mathcal{C}\gamma^\mu\partial_\mu\Psi - \frac{1}{2}m\Psi^T\mathcal{C}\Psi \, . \tag{25}$$

Finally, starting from the Majorana Lagrangian, *show that* the canonical position-momentum anti-commutator gives

$$\{\Psi_\alpha(t,\vec{x}), \bar{\Psi}_\beta(t,\vec{y})\} = {\gamma^0}_{\alpha\beta}\delta(\vec{x}-\vec{y}) \tag{26}$$

which is the same as that for the Dirac Lagrangian. However, *show* also that this implies

$$\{\Psi_\alpha(t,\vec{x}), \Psi_\beta(t,\vec{y})\} = (\mathcal{C}\gamma^0)_{\alpha\beta}\delta(\vec{x}-\vec{y})\,, \tag{27}$$

for the Majorana spinor, which is not true for the Dirac spinor!

6. In this problem, I ask you to work out a few things about solutions to the Dirac equation.

a. Let's recall the Dirac equation have the following plane wave solutions: $\Psi \propto u_\pm(\vec{p})e^{ip\cdot x}$ and $\Psi \propto v_\pm(\vec{p})e^{-ip\cdot x}$, where $[p_\mu\gamma^\mu + m]u_\pm = 0$, and $[-p_\mu\gamma^\mu + m]v_\pm = 0$. Following Srednicki, let's choose the following solutions in rest-frame of the particle:

$$u_+(\vec{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u_-(\vec{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
$$v_+(\vec{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad , \quad v_-(\vec{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{28}$$

Note that the (spin) angular momentum operator is

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{29}$$

One can see that $S^z u_\lambda = (\lambda/2)u_\lambda$ and $S^z v_\lambda = -(\lambda/2)v_\lambda$, where $\lambda = \pm 1$, i.e. these are precisely the spin up and down states (along the z-axis) of a spin 1/2 particle in its rest frame.

Let us boost these to some arbitrary frame i.e. let us boost in the $-\hat{p}$ direction, so that the particle moves in the $\hat{p}$ direction, say with velocity v. It is useful to define an "angle" ϕ such that $\cosh\phi = 1/\sqrt{1-v^2} = E/m$, and $\sinh\phi = v/\sqrt{1-v^2} = |\vec{p}|/m$. In an analogous manner to rotation, the boost transformation reads:

$$u_\lambda(\vec{p}) = e^{i(-\hat{p}_j)\Sigma^{0j}\phi}u_\lambda(\vec{0}) \tag{30}$$

where $\Sigma^{0j} = i[\gamma^0, \gamma^j]/4$, which gives

$$u_\lambda(\vec{p}) = \exp\left[\frac{\phi}{2}\begin{pmatrix} -\vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\right]u_\lambda(\vec{0})\,. \tag{31}$$

Show that this implies

$$u_\lambda(\vec{p}) = \left[\sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{\frac{E-m}{2m}}\begin{pmatrix} -\vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\right]u_s(\vec{0})\,. \tag{32}$$

You might find the following relations useful: $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$, $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$, $(\vec{\sigma}\cdot\hat{p})^2 = 1$, $\cosh\phi = 2\cosh^2(\phi/2) - 1 = 2\sinh^2(\phi/2) + 1$.

Let's pick $\hat{p} = \hat{z}$, *show that*:

$$u_+(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{E+m} - \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix} \quad , \quad u_-(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \\ \sqrt{E+m} - \sqrt{E-m} \end{pmatrix}$$
$$v_+(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \\ -\sqrt{E+m} + \sqrt{E-m} \end{pmatrix} \quad , \quad v_-(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{E+m} + \sqrt{E-m} \\ 0 \\ \sqrt{E+m} + \sqrt{E-m} \\ 0 \end{pmatrix} \tag{33}$$

In the relativistic limit $E \gg m$, these are particularly simple:

$$u_+ = v_- = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u_- = v_+ = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \tag{34}$$

Verify that these still have the same spin up and down interpretations as before i.e.

$$\vec{\sigma} \cdot \hat{p}\, u_\lambda = \lambda u_\lambda \quad , \quad \vec{\sigma} \cdot \hat{p}\, v_\lambda = -\lambda v_\lambda \, , \tag{35}$$

where $\hat{p} = \hat{z}$ in this particular example (though the expressions hold in general), and I have abused the notation a little bit i.e. what I call $\vec{\sigma} \cdot \hat{p}$ here should really be written as $\begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix}$.

This object $\sigma \cdot \hat{p}$ is known as the helicity operator, and λ (or -λ for v) is the helicity of the particle state in question. It tells us whether the spin is aligned or anti-aligned with the momentum. Helicity is a well-defined concept for a massless particle, corresponding to the extreme relativistic limit of what you have worked out. For a massive particle, helicity is a frame-dependent quantity: I can always change the helicity by boosting; in fact, in the rest-frame, it's not even defined.

Finally, *show that* in the relativistic limit, we also have

$$\gamma_5 u_\lambda = \lambda u_\lambda \quad , \quad \gamma_5 v_\lambda = -\lambda v_\lambda \tag{36}$$

Thus, we say that chirality (associated with γ_5) is the same as helicity for a massless particle.

b. Above, I asked you to work out the spinors in laborious details because it's sometimes useful to see their explicit forms, at various levels of boosting. However, there's a slicker way to see that helicity equals chirality for a massless particle. It goes as follows. I will do this for u, and the case for v is essentially the same except for a sign change. *Show that* the massless Dirac equation for u implies

$$\hat{p}_i \gamma^0 \gamma^i u_\lambda = u_\lambda \quad \text{or} \quad \hat{p}_i \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} u_\lambda = u_\lambda \tag{37}$$

Show that this implies

$$\gamma_5 \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} u_\lambda = u_\lambda \tag{38}$$

Thus, if u_λ is an eigenstate of the helicity operator with eigenvalue λ, it is also an eigenstate of the chirality operator with the same eigenvalue, and vice versa.