Physics 6047
Problem Set 2, due 2/7/13
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1. (a) Problem 2.2 of Srednicki; (b) problem 2.3 of Srednicki.

2. Problem 3.3 of Srednicki. At some point you might need to use the fact that $k \cdot (\Lambda^{-1} x) = (\Lambda k) \cdot x$. You can show this using a result from problem set 1.

3. Problem 3.4 of Srednicki.

4. Recall question 7 of problem set 1. It concerned a free complex scalar field theory: $S = \int d^4x \left[-\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi \right]$. I gave you an expression for the conserved current ($j^\mu = i[\varphi \partial^\mu \varphi^\dagger - \varphi^\dagger \partial^\mu \varphi]$) that arises from the symmetry $\varphi \rightarrow \varphi e^{-i\theta}$, where θ is some constant. Now that you know Noether's theorem, *derive* the conserved current. In other words, consider a small symmetry transformation and start from there. Feel free to try either, or both, of the Noether's theorem arguments I presented in class. You will find that the Noether current defined in class depends on the size of the symmetry transformation (e.g. θ in the case at hand) – this gives rise to an overall normalization which you can always remove i.e. given a conserved current, you can always rescale it (change its sign even) and define another conserved current.

5. Derive the Noether current corresponding to the symmetry $\phi \rightarrow \phi + b_\mu x^\mu$ (b_μ is constant) for the free massless scalar theory i.e. $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2$. Check that it is conserved using the ϕ equation of motion.

6. Consider the (real) scalar field theory:

$$S = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - V(\phi) \right] \tag{1}$$

where V is some function of ϕ (but not its derivatives).

(a). Show that the translation Noether currents (j^t for time-translation, j^x for spatial translation in the x direction, and so on) take the following form:

$$j^{t\nu} = \partial^t \phi \partial^\nu \phi - \eta^{t\nu} \left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right] \quad , \quad j^{x\nu} = \partial^x \phi \partial^\nu \phi - \eta^{x\nu} \left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right]$$
$$j^{y\nu} = \partial^y \phi \partial^\nu \phi - \eta^{y\nu} \left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right] \quad , \quad j^{z\nu} = \partial^z \phi \partial^\nu \phi - \eta^{z\nu} \left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right] , \tag{2}$$

where η is the Minkowski metric. It's very natural to group these four currents together into a single tensor – the energy momentum tensor –

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left[\frac{1}{2}(\partial\phi)^2 + V(\phi) \right] \tag{3}$$

Show that T^{00} gives you the correct Hamiltonian/energy density by comparing it against the Hamiltonian density you would have obtained by doing a Legendre transform starting from the above action. Recall that the conserved 'charge' is always an integral over space of the time-component of the conserved current. Thus, the conserved 'charge' corresponding to time-translation is $\int d^3x\, T^{00}$ which is indeed the energy; the conserved 'charge' corresponding to spatial translation in the i-th direction is $\int d^3x\, T^{i0}$ which is indeed the momentum in the i-th direction.

For those of you who have taken GR, note how the above $T^{\mu\nu}$ also beautifully agrees with the energy momentum tensor you would have obtained by taking derivative of the action with respect to the metric, if you simply couple the scalar theory minimally to gravity.

(b). Find the Noether currents corresponding to rotations and boosts. You should be able to express them in terms of the energy momentum tensor. If you have done the calculation correctly, you should find that the conserved 'charge' corresponding to rotation is indeed the angular momentum. The conserved 'charge' corresponding to boost though would look fairly unfamiliar.

7. Here are some notes to clarify the derivation of the LSZ formula. Srednicki's discussion is not very clear on some of the subtle points. My discussion follows Coleman's. You will find steps that I ask you to carry out in *italics*.

Let us start by defining the following operator:

$$\tilde{\phi}^\dagger_k(t) \equiv -i \int d^3x \left[e^{ikx} \partial_t \phi(t,\vec{x}) - \phi(t,\vec{x}) \partial_t e^{ikx} \right] \tag{4}$$

For free theory, we would have called $\tilde{\phi}^\dagger_k \to a^\dagger_k$ instead. But let us use $\tilde{\phi}^\dagger_k(t)$ to remind ourselves ϕ does not have to be a free field, and $\tilde{\phi}^\dagger(t)$ in general depends on time t.

I will take it as a given that one can create in-states by

$$|k_1\rangle_{\rm in} = \lim_{t\to-\infty} \tilde{\phi}^\dagger_{k_1}(t)|0\rangle \quad , \quad |k_2, k_1\rangle_{\rm in} = \lim_{t\to-\infty} \tilde{\phi}^\dagger_{k_2}(t)\tilde{\phi}^\dagger_{k_1}(t)|0\rangle \quad , \tag{5}$$

and create out-states by

$$|k'_1\rangle_{\rm out} = \lim_{t\to\infty} \tilde{\phi}^\dagger_{k'_1}(t)|0\rangle \quad , \quad |k'_2, k'_1\rangle_{\rm out} = \lim_{t\to\infty} \tilde{\phi}^\dagger_{k'_2}(t)\tilde{\phi}^\dagger_{k'_1}(t)|0\rangle \quad , \tag{6}$$

and also that the hermitian conjugate of $\tilde{\phi}^\dagger_k(t)$ annihilates the vacuum in far past and future:

$$\lim_{t\to\pm\infty} \tilde{\phi}_k(t)|0\rangle = 0 \tag{7}$$

See Srednicki p. 40, 41 for why these are reasonable statements to make, even for an interacting field. A formal derivation will use wave-packets. Note that I don't distinguish between in and out vacuum i.e. a state of vacuum in the far past will remain vacuum into the far future. Similarly, a state of a single particle of some momentum k in the far past should just remain the same state into the far future. Thus, technically, there is no distinction between in and out single particle states i.e. $|k\rangle_{\rm in} = |k\rangle_{\rm out}$. However, a state involving two particles in the far past will scatter into something else in the far future, and it's precisely this matrix element we are interested in: ${}_{\rm out}\langle k'_2, k'_1 | k_2, k_1\rangle_{\rm in}$.

Now, consider the following expression, which is the RHS of the LSZ formula

$$\begin{aligned} \text{RHS} \;=\; & \int d^4x'_2 i e^{-ik'_2 x'_2}(-\Box_{2'} + m^2) \int d^4x'_1 i e^{-ik'_1 x'_1}(-\Box_{1'} + m^2) \\ & \int d^4x_2 i e^{ik_2 x_2}(-\Box_2 + m^2) \int d^4x_1 i e^{ik_1 x_1}(-\Box_1 + m^2) \\ & \langle 0| T\phi(x'_2)\phi(x'_1)\phi(x_2)\phi(x_1)|0\rangle \end{aligned} \tag{8}$$

I have dropped the ˆ, but keep in mind ϕ is an operator. Consider the part that has to do with x_2':

$$\int d^4x_2' i e^{-ik_2'x_2'}(-\Box_{2'} + m^2)[\phi(x_2')] = \int dt_2' d^3x_2' i(e^{-ik_2'x_2'}\partial_{t_2'}^2[\phi(x_2')] - [\phi(x_2')]\partial_{t_2'}^2 e^{-ik_2'x_2'})$$
$$= \int dt_2' d^3x_2' i\partial_{t_2'}(e^{-ik_2'x_2'}\partial_{t_2'}[\phi(x_2')] - [\phi(x_2')]\partial_{t_2'} e^{-ik_2'x_2'}) = [\tilde{\phi}_{k_2'}(\infty)] - [\tilde{\phi}_{k_2'}(-\infty)] \quad (9)$$

where for the first equality, I have integrated by part in *space*, and use $\vec{k}_2'^2 + m^2 = \omega_{k_2'}{}^2$; for the last equality, I use the hermitian conjugate of Eq. (4). Note that I use the symbol $[\phi(x_2')]$ to denote what is in reality $[...\phi(x_2')...]$, meaning there are other operators surrounding $\phi(x_2')$. Convince yourself that in the above argument, time-ordering does not create problems.

Thus,

$$\begin{aligned}\text{RHS} \; = \; & \int d^4x_1' i e^{-ik_1'x_1'}(-\Box_{1'} + m^2)\int d^4x_2 i e^{ik_2x_2}(-\Box_2 + m^2)\int d^4x_1 i e^{ik_1x_1}(-\Box_1 + m^2) \\ & \Big[\langle 0|\tilde{\phi}_{k_2'}(\infty)\, T\phi(x_1')\phi(x_2)\phi(x_1)|0\rangle - \langle 0|T\phi(x_1')\phi(x_2)\phi(x_1)\,\tilde{\phi}_{k_2'}(-\infty)|0\rangle\Big] \qquad (10)\end{aligned}$$

The last term vanishes because of Eq. (7). Pushing through the same argument for x_1', we have

$$\begin{aligned}\text{RHS} &= \int d^4x_2 i e^{ik_2x_2}(-\Box_2 + m^2)\int d^4x_1 i e^{ik_1x_1}(-\Box_1 + m^2)\,\langle 0|\tilde{\phi}_{k_2'}(\infty)\tilde{\phi}_{k_1'}(\infty)\,T\phi(x_2)\phi(x_1)|0\rangle \\ &= \int d^4x_2 i e^{ik_2x_2}(-\Box_2 + m^2)\int d^4x_1 i e^{ik_1x_1}(-\Box_1 + m^2)\,{}_{\text{out}}\langle k_2', k_1'|\,T\phi(x_2)\phi(x_1)|0\rangle \qquad (11)\end{aligned}$$

Now, *show that* the analog of Eq. (9) for x_2, paying careful attention to signs, is

$$\int d^4x_2 i e^{ik_2x_2}(-\Box_2 + m^2)[\phi(x_2)] = [\tilde{\phi}_{k_2}^\dagger(-\infty)] - [\tilde{\phi}_{k_2}^\dagger(\infty)] \qquad (12)$$

FInally, integrating over x_1, *show that*

$$\begin{aligned}\text{RHS} &= \int d^4x_1 i e^{ik_1x_1}(-\Box_1 + m^2)\,\Big[{}_{\text{out}}\langle k_2', k_1'|\,\phi(x_1)\tilde{\phi}_{k_2}^\dagger(-\infty)|0\rangle - {}_{\text{out}}\langle k_2', k_1'|\,\tilde{\phi}_{k_2}^\dagger(\infty)\phi(x_1)|0\rangle\Big] \\ &= {}_{\text{out}}\langle k_2', k_1'|\,\tilde{\phi}_{k_1}^\dagger(-\infty)\tilde{\phi}_{k_2}^\dagger(-\infty)|0\rangle - {}_{\text{out}}\langle k_2', k_1'|\,\tilde{\phi}_{k_1}^\dagger(\infty)\tilde{\phi}_{k_2}^\dagger(-\infty)|0\rangle \\ &\quad - {}_{\text{out}}\langle k_2', k_1'|\,\tilde{\phi}_{k_2}^\dagger(\infty)\tilde{\phi}_{k_1}^\dagger(-\infty)|0\rangle + {}_{\text{out}}\langle k_2', k_1'|\,\tilde{\phi}_{k_2}^\dagger(\infty)\tilde{\phi}_{k_1}^\dagger(\infty)|0\rangle \qquad (13)\end{aligned}$$

The last two terms cancel because $\tilde{\phi}_{k_1}^\dagger(-\infty)|0\rangle = |k_1\rangle_{\text{in}} = |k_1\rangle_{\text{out}} = \tilde{\phi}_{k_1}^\dagger(\infty)|0\rangle$. The first term gives ${}_{\text{out}}\langle k_2', k_1'|k_2, k_1\rangle_{\text{in}}$. The second term is ${}_{\text{out}}\langle k_2', k_1'|k_2, k_1\rangle_{\text{out}}$, keeping in mind once again that $\tilde{\phi}_{k_2}^\dagger(-\infty)|0\rangle = |k_2\rangle_{\text{in}} = |k_2\rangle_{\text{out}} = \tilde{\phi}_{k_2}^\dagger(\infty)|0\rangle$.

We have thus successfully derived the LSZ formula:

$$\begin{aligned}&\int d^4x_2' i e^{-ik_2'x_2'}(-\Box_{2'} + m^2)\int d^4x_1' i e^{-ik_1'x_1'}(-\Box_{1'} + m^2)\int d^4x_2 i e^{ik_2x_2}(-\Box_2 + m^2) \\ &\quad \int d^4x_1 i e^{ik_1x_1}(-\Box_1 + m^2)\,\langle 0|T\phi(x_2')\phi(x_1')\phi(x_2)\phi(x_1)|0\rangle \\ &= {}_{\text{out}}\langle k_2', k_1'|k_2, k_1\rangle_{\text{in}} - {}_{\text{out}}\langle k_2', k_1'|k_2, k_1\rangle_{\text{out}} \\ &= {}_{\text{out}}\langle k_2', k_1'|k_2, k_1\rangle_{\text{in}} - (2\pi)^6 4\omega_{k_1}\omega_{k_2}\Big[\delta^{(3)}(\vec{k}_1 - \vec{k}_1')\delta^{(3)}(\vec{k}_2 - \vec{k}_2') + \delta^{(3)}(\vec{k}_1 - \vec{k}_2')\delta^{(3)}(\vec{k}_2 - \vec{k}_1')\Big] \\ &\equiv (2\pi)^4\delta^{(4)}(k_1 + k_2 - k_1' - k_2')\, i\mathcal{M} \qquad (14)\end{aligned}$$

The LSZ formula therefore does not merely give ${}_{\mathrm{out}}\langle k_2', k_1' | k_2, k_1 \rangle_{\mathrm{in}}$, but actually has the no/trivial-scattering-contribution subtracted out. The quantity $\mathcal{M}$ (or Srednicki's $\mathcal{T}$) is called the scattering amplitude. It multiplies a delta function for the total momentum (incoming - outgoing), because we expect the scattering process to conserve total momentum in general.

It is sometimes convenient to define an operator S such that

$$ {}_{\mathrm{out}}\langle k_2', k_1' | k_2, k_1 \rangle_{\mathrm{in}} = {}_{\mathrm{free}}\langle k_2', k_1' | S | k_2, k_1 \rangle_{\mathrm{free}} \tag{15} $$

where $|k_1, k_2\rangle_{\mathrm{free}}$ is simply a state of two free particles of momenta k_1 and k_2. This is called the S-matrix. The part of the S-matrix with no/trivial-scattering subtracted out can be represented very simply as

$$ {}_{\mathrm{out}}\langle k_2', k_1' | k_2, k_1 \rangle_{\mathrm{in}} - {}_{\mathrm{out}}\langle k_2', k_1' | k_2, k_1 \rangle_{\mathrm{out}} = {}_{\mathrm{free}}\langle k_2', k_1' | S - 1 | k_2, k_1 \rangle_{\mathrm{free}} \tag{16} $$

If you want to find out more about the distinctions between in, out and free states, read Weinberg volume I, 3.1.