Commutative Algebra: Problem Set 1

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Problem 5

Let k be a field and $A = k[x,y]/(x^2y^4 - x^4y^2 + 1)$. We wish to construct a finite injective map as in the Noether normalization lemma. As an algebra, A is generated by $\{x, y\}$. Using the trick from Noether normalization, we define $\tilde{x} = x - y^2$ in order to produce a monic polynomial. Inserting this into the polynomial F generating the ideal, we find that $1 + y^8 - y^{10} + 2y^6\tilde{x} - 4y^8\tilde{x} + y^4\tilde{x}^2 - 6y^6\tilde{x}^2 - 4y^4\tilde{x}^3 - y^2\tilde{x}^4$. In A, $F = 0$, and thus y is integral over $k[\tilde{x}]$. This implies that the map $\phi : k[\tilde{x}] \to A$ is finite (by lemma 3 in class). Injectivity of this map follows easily from what we did in class.

Problem 6

The prime ideals of $\mathbb{C}[x,y]/(xy)$ are in one-to-one correspondence with the prime ideals of $\mathbb{C}[x,y]$ that contain (xy). Since $\mathbb{C}[x]$ is a principal ideal domain, we may use the following lemma from undergraduate algebra:

Lemma 1. *Let R be a principal ideal domain. The prime ideals of $R[y]$ are precisely those of the following form:*

- (0)
- $(f(y))$ *where f is an irreducible polynomial,*
- $(p, f(y))$ *where $p \in R$ is prime and $f(y)$ is irreducible in $(R/p)[y]$*

It follows straightforwardly, then, that such prime ideals that contain (xy) are precisely $(x), (y), (x, y - \lambda), (x - \mu, y)$. A more geometric approach, of course would be to think of the x and y-axes, which would give us precisely this set.

Problem 7

Let k be a field. We wish to prove that $k[x,y]$ is not isomorphic to $k[x,y,z]$. Suppose for the sake of contradiction that there exists a (module) isomorphism $\phi : k[x,y] \to k[x,y,z]$. Then ϕ is clearly (module-)finite, i.e. there exists a set of generators $M \subset k[x,y,z]$ that $k[x,y]$-spans $k[x,y,z]$. It's clear that $1 = z^0 \in M$. The $k[x,y]$-span of 1 does not contain z, however, and hence we must have $z \in M$. By the same argument, we must have z^n for arbitrarily high powers $n \in \mathbb{Z}$. This is a contradiction - $k[x,y]$ cannot be isomorphic to $k[x,y,z]$.

Problem 8

Let k be a field. Suppose A is a k-algebra and f is a nonzerodivisor of A such that $k[x, y]$ is isomorphic to A_f as a k-algebra. Then, since the invertible elements of $k[x, y]$ are the field elements k, we must have that $f \in k$. Note that since $k \to A$ is injective, we must have that $f \in k \subset A$ and hence $A = A_f = k[x, y]$ since f is already invertible.

Problem 9

Let A be a ring and let f be an element of A. We wish to show that $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is isomorphic as an A-algebra to $A[x]/(fx - 1)$. Consider the following diagram:

$$
\begin{tikzcd}
 & A \arrow[dl, "\phi"'] \arrow[dr, "g"] & \\
A_f \arrow[rr, dashed, "\psi"] & & A[x]/(xf-1)
\end{tikzcd}
$$

Using the universal property of localization (see Atiyah-MacDonald pp. 37-38), since $g(f^k)$ is a unit in $A[x]/(xf - 1)$, $\ker g = 0$, and because every element of $A[x]/(xf - 1)$ is of the form $g(a)g(s)^{-1}$, there exists a unique isomorphism $\psi : A_f \to A[x]/(xf - 1)$.