Hartshorne Solutions

Matei Ionita and Nilay Kumar

Last updated: February 21, 2014

Problem I.3.1

(a) By the results of problem 1.1, we know that any conic in $\mathbb{A}^2$ can be written as either a variety Y defined by $y - x^2 = 0$ or a variety Z defined by $xy - 1 = 0$. We know that $A(Y) = k[x]$ and $A(Z) = k[x, x^{-1}]$. Note that $A(Y) \cong A(\mathbb{A}^1)$, and hence by Corollary 3.7, $Y \cong \mathbb{A}^1$ as affine varieties. It remains to show that Z is isomorphic to $\mathbb{A}^1 - \{0\}$. Note first that $xy - 1 = 0$ can be parametrized as (t, t^{-1}), which suggests the map $\phi : Z \to \mathbb{A}^1 - \{0\}$ given by $\phi(t, t^{-1}) = t$ as well as the reverse $\psi : \mathbb{A}^1 - \{0\} \to Z$ given by $\psi(x) = (x, x^{-1})$. It is easy to check ϕ and ψ are morphisms with $\psi \circ \phi = \mathrm{Id}_Z$ and $\phi \circ \psi = \mathrm{Id}_{\mathbb{A}^1 - \{0\}}$.

(b) Let B be a proper open subset of $\mathbb{A}^1$. By definition of the Zariski topology, we can write $B = \mathbb{A}^1 \setminus \{p_1, \ldots, p_n\}$ where p_i are a finite set of points in $\mathbb{A}^1$. The ring of regular functions of $\mathbb{A}^1$ is $\mathcal{O}(\mathbb{A}^1) = k[x]$. In B, however, polynomials that vanish only at any of the p_i are globally invertible, and hence $\mathcal{O}(B) = k[x, (x - p_1)^{-1}, \ldots, (x - p_n)^{-1}]$. These two rings are clearly not isomorphic, which completes the proof.

(c) In the projective plane, we can write a conic as $F(x, y, z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$, which can be rewritten under an appropriate change of variables as $x^2 + y^2 + z^2$(assuming char $k \neq 2$). Hence every conic in the projective plane is isomorphic, and it will suffice to show that there exists a conic that is isomorphic to $\mathbb{P}^1$. This follows From the result of exercise I.3.4: the 2-uple embedding $\rho_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism onto its image

do this

$$\rho_2(a_0, a_1) = (a_0^2, a_0 a_1, a_1^2),$$

which clearly traces out a conic $xz - y^2$.

(d) finish

(e) If an affine variety X is isomorphic to a projective variety Y, then we must have that $\mathcal{O}(X) = \mathcal{O}(Y) = k$. But for $k[x_1, \ldots, x_n]/I(X) = k$, $I(X)$ must be maximal. Hence $I(X) = (x_1 - a_1, \ldots, x_n - a_n)$, i.e. X is just a point.

Problem I.3.14

(a) Note first that ϕ is continuous, as the preimage of any closed subset $V \subset \mathbb{P}^n$ is the projective cone $\overline{C(V)}$, which is closed in $\mathbb{P}^{n+1}$. Furthermore, the point at which the line connecting any Q and P to the hypersurface (choose $x_0 = 0$ without loss of generality) is given by

$$\phi(Q) = [Q_1 - \frac{Q_0 P_1}{P_0} : \cdots : Q_{n+1} - \frac{Q_0 P_{n+1}}{P_0}],$$

where P_i and Q_i, are the ith components of P and Q, respectively (the coordinates are written as for a point in $\mathbb{P}^n$). It is easy to see that ϕ pulls back regular functions to regular functions: given $g/h : \mathbb{P}^n \to k$, $g(\phi(Q))/h(\phi(Q))$ is regular as well, since inserting $\phi(Q)$ (as above) will retain homogeneity as well as keep the denominator non-zero (as h has no zeroes).

(b) The twisted cubic is given parametrically by $[x : y : z : w] = [t^3 : t^2u : tu^2 : u^3]$. We wish to project from $P = [0 : 0 : 1 : 0]$ onto the hyperplane $z = 0$. This yields the points $[t^3 : t^2u : u^3] \in \mathbb{P}^2$. Note that these points satisfy the equation $x_0^2x_2 - x_1^3 = 0$. But this is precisely the projective closure of the cuspidal cubic $y^3 = x^2$.

Problem I.3.15

(a) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Consider the product $X \times Y \subset \mathbb{A}^{n+m}$ with the induced Zariski topology. Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subset Z_i\}$ for $i = 1, 2$. The irreducibility of Y guarantees that $X_1 \cup X_2 = X$: if there were an x for which $x \times Y$ were not contained in a Z_i, this would yield a covering of Y by closed sets $Z_1 \cap Y$ and $Z_2 \cap Y$. Now consider the inclusion map $\iota : X \to X \times Y$; if ι is continuous then X_i must be closed, as $\iota^{-1}(Z_i) = X_i$. But the inclusion is obviously continuous, as any closed set in $X \times Y$ is defined by the vanishing of polynomials $f_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_m)$, whose pullback to X is $f_\alpha(x_1, \ldots, x_n, 0, \ldots, 0)$, which is by definition a closed set of X. But if X_i are closed and cover X, either $X_1 = X$ or $X_2 = X$ and thus $Z_1 = X \times Y$ or $Z_2 = X \times Y$, i.e. $X \times Y$ is irreducible.

(b) Consider the map $\psi : A(X) \otimes_k A(Y) \to A(X \times Y)$ given by taking $(f \otimes g)(x, y)$ to $f(x)g(y)$. This map is clearly onto, as it produces the coordinate functions $x_1, \ldots, x_n, y_1, \cdots, y_m$.[1] Injectivity is less obvious.

(c)

(d) It suffices to show that $\dim A(X) \otimes_k A(Y) = \dim A(X) + \dim A(Y)$. By Noether normalization, $A(X)$ is module-finite over the polynomial ring $k[t_1, \ldots, t_{d_1}]$ and $A(Y)$ is module-finite over the polynomial ring $k[s_1, \ldots, s_{d_2}]$ with $d_1 = \dim A(X)$ and $d_2 = \dim A(Y)$. In other words, every element of $A(X)$ or $A(Y)$ is the solution to some polynomial over the above polynomial rings, respectively. Next note that $R = k[t_1, \ldots, t_{d_1}, s_1, \ldots, s_{d_2}]$ must inject into $A(X) \otimes_k A(Y)$ via a map ϕ. Recall that every element in the tensor product can be written as a sum of elementary tensors $x \otimes y$ with $x \in A(X), y \in A(Y)$. Hence every element in the tensor product must also solve some polynomial over the ring R, i.e. $A(X) \otimes_k A(Y)$ is module-finite over R and $\dim A(X) \otimes_k A(Y) = d_1 + d_2$, as desired.

Problem I.3.21

(a) It suffices to show that the addition and inversion maps are morphisms of varieties. But this follows from Lemma 3.6, as $\mu(a, b) = a + b$ and $\iota(a) = -a$ clearly define regular functions.

(b) Note that $\mathbb{G}_m$ is, as a variety, simply $\mathbb{A}^1 - \{0\}$, which in turn is isomorphic to an affine variety (c.f. problem I.3.1). Hence $\mathbb{G}_m$ is an affine variety, and the multiplication and inversion maps are morphisms again by Lemma 3.6.

[1] One might worry that generators may be missing from $A(X)$ or $A(Y)$ and hence that ψ may not produce all the generators of $A(X \times Y)$. This is actually not a problem: if $x_i \in I(X)$ then $x_i \in I(X \times Y)$ as well.

(c) We define the group operation $\cdot$ on $\mathrm{Hom}(X, G)$ as

$$(f \cdot g)(x) = \mu(f(x), g(x)),$$

where $f, g \in \mathrm{Hom}(X, G)$ and μ is the operation on G and inversion as

$$f^{-1}(x) = \iota(f(x)),$$

where ι is the inversion on G. Thus defined, $\mathrm{Hom}(X, G)$ becomes a group by virtue of the group structure on G.

(d) By part (c), $\mathrm{Hom}(X, \mathbb{G}_a)$ inherits a group structure from $\mathbb{G}_a$, while the group structure on $\mathcal{O}(X)$ is the usual one. Any $f \in \mathrm{Hom}(X, \mathbb{G}_a)$ defines a regular function on X, and hence $f \in \mathcal{O}(X)$. Conversely, any regular function $\tilde{f} \in \mathcal{O}(X)$ is a morphism from X to $\mathbb{G}_a = \mathbb{A}^1$ (by Lemma 3.1) and hence contained in $\mathrm{Hom}(X, \mathbb{G}_a)$. The set equality $\mathrm{Hom}(X, \mathbb{G}_a) = \mathcal{O}(X)$ clearly extends to a group isomorphism, as the additive structure is clearly preserved.

(e) By part (c), $\mathrm{Hom}(X, \mathbb{G}_m)$ inherits a group structure from $\mathbb{G}_m$, while the group of units H in $\mathcal{O}(X)$ is the group of invertible, globally regular functions on X. Just as in part (d), we have the setwise equality $\mathrm{Hom}(X, \mathbb{G}_m) = H$, which extends to a group isomorphism, as the multiplicative structure is preserved.