

# Notes on Topological and Differentiable Manifolds

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## 1 Elementary Topology

Let us begin with the definition of a topology:

**Definition 1.** A **topology** on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$ , called **open sets**, satisfying the following properties:

1.  $X$  and  $\emptyset$  are elements of  $\mathcal{T}$ .
2.  $\mathcal{T}$  is closed under finite intersections: If  $U_1 \dots U_n \in \mathcal{T}$ , then their intersection  $U_1 \cap \dots \cap U_n$  is in  $\mathcal{T}$ .
3.  $\mathcal{T}$  is closed under arbitrary unions: If  $U_1 \dots U_n \dots$  is any (finite or infinite) collection of elements of  $\mathcal{T}$ , then their union  $\cup_{\alpha} U_{\alpha}$  is in  $\mathcal{T}$ .

A pair  $(X, \mathcal{T})$  consisting of a set  $X$  and a topology  $\mathcal{T}$  on  $X$  is called a **topological space**. The elements of a topological space are usually called its **points**.

**Definition 2.** If  $X$  is a topological space and  $q \in X$ , a **neighborhood** of  $q$  is just an open set containing  $q$ . More generally, a neighborhood of a subset  $K \subset X$  is an open set containing  $K$ .

**Definition 3.** If  $X$  is a topological space and  $\{q_i\}$  is any sequence of points in  $X$ , we say that the sequence **converges** to  $q \in X$ , and  $q$  is the **limit** of the sequence, if for every neighborhood  $U$  of  $q$  there exists  $N$  such that  $q_i \in U$  for all  $i \geq N$ . We denote this as  $q_i \rightarrow q$  or  $\lim_{i \rightarrow \infty} q_i = q$ .

**Example 1.** Let  $Y$  be a trivial topological space (i.e. the only open sets are  $X$  and  $\emptyset$ ). Each point has only 1 neighborhood:  $X$  itself. Thus, any sequence can be entirely contained in the neighborhood  $X$ , and consequently, any sequence converges to any point in  $X$ .

**Example 2.** Let  $X$  be a discrete topological space (i.e. all every subset of  $X$  is open). Take any sequence of points  $\{q_i\}$ . If the sequence converges to  $q$ , every open set containing  $q$  must contain all but a finite elements of the sequence. By virtue of the discrete topology, there exists an open set that contains only  $q$ . Obviously, then, there must exist an  $N$  such that  $q_i = q$  for all  $i \geq N$ . Consequently, the only convergent sequences in  $X$  are the ones that are “eventually constant.”

**Definition 4.** If  $X$  and  $Y$  are topological spaces, a map  $f : X \rightarrow Y$  is said to be **continuous** if for every open set  $U \subset Y$ ,  $f^{-1}(U)$  is open in  $X$ .

**Lemma 1.** *Let  $X, Y, Z$  be topological spaces.*

1. *Any constant map  $f : X \rightarrow Y$  is continuous.*
2. *The identity map  $\text{Id} : X \rightarrow X$  is continuous.*
3. *If  $f : X \rightarrow Y$  is continuous, so is the restriction of  $f$  to any open subset of  $X$ .*
4. *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, so is their composition  $g \circ f : X \rightarrow Z$ .*

*Proof.* Let us begin with the constant map. Suppose  $f$  maps  $X$  to the constant  $\lambda \in Y$ . We wish to show that the preimage of  $f$  of every open set  $U$  in  $Y$  is open. There are two cases:  $U$  either does or does not contain  $\lambda$ . If it does,  $f^{-1}(U) = X$ ; otherwise,  $f^{-1}(U) = \emptyset$ . As both  $X$  and  $\emptyset$  are open sets,  $f$  is continuous.

The continuity of the identity map follows trivially from the fact that  $\text{Id}$  maps any open set back to the same open set.

To prove the third statement, take any open set  $U$  in  $Y$ .  $U$  can be written as a union of points in and outside  $f(V) \subset Y$ :  $U = U_i \cup U_o$ . We want to show that  $g^{-1}(U)$  is open in  $V$ . Since  $g^{-1}(U_o) = \emptyset$ , which is open, and  $g^{-1}(U_i) \subset V$  and is open in  $X$  by the continuity of  $f$ ,  $g^{-1}(U_o \cup U_i) = g^{-1}(U_o) \cup g^{-1}(U_i)$  is open in  $V$ .

To prove the fourth statement, it suffices to show that  $(g \circ f)^{-1}(U)$ , with  $U \subset Z$  open, is open in  $X$ . First note that  $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ . Since  $g$  is continuous,  $g^{-1}(U)$  is an open set in  $Y$ . Similarly,  $f^{-1}$  of an open set in  $Y$  is open in  $X$  as  $f$  is continuous, and we are done.  $\square$

**Lemma 2** (Local Criterion for Continuity). *A map  $f : X \rightarrow Y$  between topological spaces is continuous if and only if each point of  $X$  has a neighborhood on which (the restriction of)  $f$  is continuous.*

*Proof.* If  $f$  is continuous, each point of  $X$  has a neighborhood on which  $f$  is continuous; namely,  $X$  itself.

To prove the converse, suppose that each point of  $X$  has a neighborhood on which  $f$  is continuous - we wish to show that for any open set  $U \subset Y$ ,  $f^{-1}(U)$  is open in  $X$ . By continuity at each point, we know that any point  $x \in f^{-1}(U)$  has a neighborhood  $V_x$  on which  $f$  is continuous. In other words,  $(f|_{V_x})^{-1}(U) = f^{-1}(U) \cap V_x$  is open in  $X$  and is contained in  $f^{-1}(U)$ . As  $f^{-1}(U)$  is the union of such sets for all  $V_x$ , and these sets are open, it follows that  $f^{-1}(U)$  is open, and we are done.  $\square$

**Definition 5.** If  $X$  and  $Y$  are topological spaces, a **homeomorphism** from  $X$  to  $Y$  is defined to be a continuous bijective map  $\phi : X \rightarrow Y$  with continuous inverse. If there exists a homeomorphism between  $X$  and  $Y$ , we say that  $X$  and  $Y$  are **homeomorphic** or **topologically equivalent**. Sometimes this is abbreviated  $X \approx Y$ .

**Exercise 1.** Show that homeomorphisms are an equivalence relation.

*Proof.* To show that homeomorphisms are an equivalence relation, we show

- $X \approx X$ : The identity map  $\text{Id}$  is a homeomorphism from  $X$  to  $X$ .
- $X \approx Y \implies Y \approx X$ : There exists a homeomorphism from  $X$  to  $Y$ . Its inverse is clearly a homeomorphism from  $Y$  to  $X$ .
- $X \approx Y$  and  $Y \approx Z \implies X \approx Z$ : As the composition of two homeomorphisms is also a homeomorphism (from elementary set theory), the homeomorphism from  $X$  to  $Z$  is simply the composition of the homeomorphisms from  $Y$  to  $Z$  and from  $X$  to  $Y$ , respectively.

$\square$

**Example 3.** Any open ball in  $\mathbb{R}^n$  is homeomorphic to any other open ball. The homeomorphism can be constructed simply by composition translations  $x \mapsto x + x_0$  and dilations  $x \mapsto cx$ . This shows that size is not a topological property.

**Example 4.** If  $\mathbb{B}^n$  is the open unit ball, we can define  $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$  by

$$y = F(x) = \frac{x}{1 - |x|^2}.$$

The inverse is given by

$$x = F^{-1}(y) = \frac{2y}{1 + \sqrt{1 + 4|y|^2}}.$$

As both are continuous and bijective,  $F$  is a homeomorphism, so  $\mathbb{R}^n$  is homeomorphic to  $\mathbb{B}^n$ . This shows that boundedness is not a topological property.

**Example 5.** Take the surface of the unit sphere in  $\mathbb{R}^3$  and the surface of the cube of side 2, centered at the origin. There exists a homeomorphism between these two surfaces,

$$\phi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

whose inverse is given by

$$\phi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Thus, corners are not a topological property either.

**Example 6.** Now for an example of a continuous bijection that is not a homeomorphism by failure of its inverse to be continuous. Let  $X$  be the interval  $[0, 1) \subset \mathbb{R}$ , and let  $\mathbb{S}^1$  denote the unit circle in  $\mathbb{R}^2$  (both with the Euclidean metric topologies). Define a map  $a : X \rightarrow \mathbb{S}^1$  by  $a(t) = (\cos 2\pi t, \sin 2\pi t)$ . It is clear that the map is continuous and bijective. The inverse, however, is not continuous. To see why, take any neighborhood of the point  $(1, 0)$  - the inverse of  $a$  on this neighborhood will inevitably contain the point  $0 \in [0, 1)$ . Thus the preimage of open sets is not necessarily open, and thus  $a^{-1}$  is not continuous.

**Definition 6.** A map  $f : X \rightarrow Y$  is said to be an **open map** if for any open set  $U \subset X$ , the image set  $f(U)$  is open in  $Y$ . A map can be open but not continuous, continuous but not open, both, or neither.

**Definition 7.** We say that a continuous map  $f : X \rightarrow Y$  between topological spaces is a **local homeomorphism** if every point  $x \in X$  has a neighborhood  $U \subset X$  such that  $f(U)$  is an open subset of  $Y$  and  $f|_U : U \rightarrow f(U)$  is a homeomorphism.

**Exercise 2.** *Show that:*

1. *every local homeomorphism is an open map.*
2. *every homeomorphism is a local homeomorphism.*
3. *a bijective continuous open map is a homeomorphism.*

4. *a bijective local homeomorphism is a homeomorphism.*

*Proof.*

1. Take any open set  $U \subset X$ . For every  $x \in U$ , there exists some neighborhood  $V = U_x \cap U$  of  $x$  for which the local homeomorphism  $f(V)$  is open. Note that  $U$  is the union of all such  $V$  (for various  $x$ ) and thus  $f(U)$  is the union of all  $f(V)$ , and as the union of open sets must be an open set,  $f(U)$  must be open. Consequently,  $f$  is an open map.
2. Take  $f$  to be a homeomorphism from  $X$  to  $Y$ .  $f$  is trivially a local homeomorphism; the neighborhood needed by the definition is simply  $X$  itself.
3. Let  $f$  be a bijective, continuous, open map. To show that  $f$  is a homeomorphism, we must show that  $f^{-1}$  is a continuous map. In other words, we must show that the preimage of  $f$  on open sets of  $Y$  are open sets in  $X$ . This is true as  $f$  is open and bijective: open sets in  $X$  are taken to open sets in  $Y$  (open), and all open sets in  $Y$  are images of open sets in  $X$  (bijective).
4. Let us first show that any local homeomorphism  $f$  from  $X$  to  $Y$  is continuous. Take any open set  $V \subset Y$  whose preimage under  $f$  we call  $U$ . We wish to show that  $U$  is open. Take any point  $y \in V$ . For any  $x \in f^{-1}(y)$ , there is a neighborhood  $M_x \subset X$  of  $x$  that is homeomorphic to a neighborhood  $N_y$  of  $y$  in  $Y$ . This implies that  $U_x = M_x \cap U$  is homeomorphic to  $V_y = N_y \cap V$ . Take the union of all the sets  $U_x$  for every  $x$  in the preimage of  $y$  and call this  $W_y$ . Note that the preimage of  $V$  is the union of all such  $W_y$  for  $y \in V$ . This union is open, and we are done.  
Now it remains to show that the inverse of  $f$  (that exists via bijectivity) is continuous. Let  $U \subset X$  be open and  $V = (f^{-1})^{-1}(U) = f(U)$ . We wish to show that  $V$  is open. This follows trivially from the fact that  $f$  is an open map, and we are done.

□

Up until now, we have worked with topological spaces defined through open sets. There is a complementary notion that is just as important.

**Definition 8.** A subset  $F$  of a topological space  $X$  is said to be **closed** if its complement  $X \setminus F$  is open. It follows immediately from the definition of topological spaces that

1.  $X$  and  $\emptyset$  are closed.
2. Finite unions of closed sets are closed.
3. Arbitrary intersections of closed sets are closed.

A topology on a set  $X$  can be defined by describing the collection of closed sets, as long as they satisfy these three properties; the open sets are then just those sets whose complements are closed.

**Example 7** (Closed Sets).

- Any closed interval  $[a, b] \subset \mathbb{R}$  is a closed set, as are the half-infinite closed intervals  $[a, \infty)$  and  $(-\infty, b]$ .
- Any closed ball in a metric space is a closed set.
- Every subset of a discrete space is closed.

It is important to note that closed is *not* the same as “not open,” as sets can be both open and closed (such as  $X, \emptyset$ ), or neither open nor closed, such as  $[0, 1) \subset \mathbb{R}$ .

**Lemma 3.** *A map between topological spaces is continuous if and only if the inverse image of every closed set is closed.*

*Proof.* Assume  $f : X \rightarrow Y$  is continuous. Given  $V \subset Y$  is closed, and its preimage under  $f$  is  $U \subset X$ , we want to show that  $U$  is closed. By definition,  $Y \setminus V$  is open, and as by continuity of  $f$ , we have that its preimage  $X \setminus U$  is open. Thus, again by definition of closed sets,  $U$  must be closed.

Now assume that  $f : X \rightarrow Y$  is such that the preimage of any closed set  $V$  is a closed set in  $X$ . Take an open set  $V \subset Y$  whose preimage under  $f$  is  $U \subset X$ . The preimage of  $Y \setminus V$  is, of course  $X \setminus U$ , which is closed. Since  $X \setminus U$  is closed,  $U$  must be open, and we are done.  $\square$

**Definition 9.** Given any set  $A \subset X$ , we define several related sets as follows. The **closure** of  $A$  in  $X$ , denoted by  $\bar{A}$ , is the set

$$\bar{A} = \bigcap \{B \subset X : B \supset A \text{ and } B \text{ is closed in } X\}.$$

The **interior** of  $A$ , written  $\text{Int } A$  is

$$\text{Int } A = \bigcup \{C \subset X : C \subset A \text{ and } C \text{ is open in } X\}.$$

It is obvious that  $\bar{A}$  is closed and  $\text{Int } A$  is open. In words,  $\bar{A}$  is the “smallest closed set containing  $A$ ,” and  $\text{Int } A$  is “the largest open set contained in  $A$ .” We also define the **exterior** of  $A$ , written  $\text{Ext } A$ , as

$$\text{Ext } A = X \setminus \bar{A},$$

and the **boundary** of  $A$ , written  $\partial A$ , as

$$\partial A = X \setminus (\text{Int } A \cup \text{Ext } A)$$

It follows from the definitions that for any subset  $A \subset X$ , the whole space  $X$  is equal to the disjoint union of  $\text{Int } A$ ,  $\text{Ext } A$ , and  $\partial A$ . The set  $A$  always contains all of its interior points and none of its exterior points, and may contain all, some, or none of its boundary points.

**Lemma 4.** *Let  $X$  be a topological space and  $A \subset X$  any subset.*

1. *A point  $q$  is in the interior of  $A$  if and only if  $q$  has a neighborhood contained in  $A$ .*
2. *A point  $q$  is in the exterior of  $A$  if and only if  $q$  has a neighborhood contained in  $X \setminus A$ .*
3. *A point  $q$  is in the boundary of  $A$  if and only if every neighborhood of  $q$  contains both a point of  $A$  and a point of  $X \setminus A$ .*
4.  *$\text{Int } A$  and  $\text{Ext } A$  are open in  $X$ , while  $\partial A$  is closed in  $X$ .*
5.  *$A$  is open if and only if  $A = \text{Int } A$ .*
6.  *$A$  is closed if and only if it contains all its boundary points, which is true if and only if  $A = \text{Int } A \cup \partial A$ .*
7.  *$A = \bar{A} \cup \partial A = \text{Int } A \cup \partial A$ .*

*Proof.* Left to the reader. □

**Definition 10.** Given a topological space  $X$  and a set  $A \subset X$ , we say that a point  $q \in X$  is a **limit point** of  $A$  if every neighborhood of  $q$  contains a point of  $A$  other than  $q$  (which may or may not itself be in  $A$ ). If we let  $X = \mathbb{R}$  and  $A = \{1/n\}_{n=1}^{\infty}$ , for example, then 0 is the only limit point of  $A$ .

**Exercise 3.** *Show that a set  $A$  in a topological space is closed if and only if it contains all of its limit points.*

*Proof.*

□