

Notes on Topological and Differentiable Manifolds

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1 Elementary Topology

Let us begin with the definition of a topology:

Definition 1. A **topology** on a set X is a collection \mathcal{T} of subsets of X , called **open sets**, satisfying the following properties:

1. X and \emptyset are elements of \mathcal{T} .
2. \mathcal{T} is closed under finite intersections: If $U_1 \dots U_n \in \mathcal{T}$, then their intersection $U_1 \cap \dots \cap U_n$ is in \mathcal{T} .
3. \mathcal{T} is closed under arbitrary unions: If $U_1 \dots U_n \dots$ is any (finite or infinite) collection of elements of \mathcal{T} , then their union $\cup_{\alpha} U_{\alpha}$ is in \mathcal{T} .

A pair (X, \mathcal{T}) consisting of a set X and a topology \mathcal{T} on X is called a **topological space**. The elements of a topological space are usually called its **points**.

Definition 2. If X is a topological space and $q \in X$, a **neighborhood** of q is just an open set containing q . More generally, a neighborhood of a subset $K \subset X$ is an open set containing K .

Definition 3. If X is a topological space and $\{q_i\}$ is any sequence of points in X , we say that the sequence **converges** to $q \in X$, and q is the **limit** of the sequence, if for every neighborhood U of q there exists N such that $q_i \in U$ for all $i \geq N$. We denote this as $q_i \rightarrow q$ or $\lim_{i \rightarrow \infty} q_i = q$.

Example 1. Let Y be a trivial topological space (i.e. the only open sets are X and \emptyset). Each point has only 1 neighborhood: X itself. Thus, any sequence can be entirely contained in the neighborhood X , and consequently, any sequence converges to any point in X .

Example 2. Let X be a discrete topological space (i.e. all every subset of X is open). Take any sequence of points $\{q_i\}$. If the sequence converges to q , every open set containing q must contain all but a finite elements of the sequence. By virtue of the discrete topology, there exists an open set that contains only q . Obviously, then, there must exist an N such that $q_i = q$ for all $i \geq N$. Consequently, the only convergent sequences in X are the ones that are “eventually constant.”

Definition 4. If X and Y are topological spaces, a map $f : X \rightarrow Y$ is said to be **continuous** if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X .

Lemma 1. *Let X, Y, Z be topological spaces.*

1. *Any constant map $f : X \rightarrow Y$ is continuous.*
2. *The identity map $\text{Id} : X \rightarrow X$ is continuous.*
3. *If $f : X \rightarrow Y$ is continuous, so is the restriction of f to any open subset of X .*
4. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, so is their composition $g \circ f : X \rightarrow Z$.*

Proof. Let us begin with the constant map. Suppose f maps X to the constant $\lambda \in Y$. We wish to show that the preimage of f of every open set U in Y is open. There are two cases: U either does or does not contain λ . If it does, $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \emptyset$. As both X and \emptyset are open sets, f is continuous.

The continuity of the identity map follows trivially from the fact that Id maps any open set back to the same open set.

To prove the third statement, take any open set U in Y . U can be written as a union of points in and outside $f(V) \subset Y$: $U = U_i \cup U_o$. We want to show that $g^{-1}(U)$ is open in V . Since $g^{-1}(U_o) = \emptyset$, which is open, and $g^{-1}(U_i) \subset V$ and is open in X by the continuity of f , $g^{-1}(U_o \cup U_i) = g^{-1}(U_o) \cup g^{-1}(U_i)$ is open in V .

To prove the fourth statement, it suffices to show that $(g \circ f)^{-1}(U)$, with $U \subset Z$ open, is open in X . First note that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U)$ is an open set in Y . Similarly, f^{-1} of an open set in Y is open in X as f is continuous, and we are done. \square

Lemma 2 (Local Criterion for Continuity). *A map $f : X \rightarrow Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous.*

Proof. If f is continuous, each point of X has a neighborhood on which f is continuous; namely, X itself.

To prove the converse, suppose that each point of X has a neighborhood on which f is continuous - we wish to show that for any open set $U \subset Y$, $f^{-1}(U)$ is open in X . By continuity at each point, we know that any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous. In other words, $(f|_{V_x})^{-1}(U) = f^{-1}(U) \cap V_x$ is open in X and is contained in $f^{-1}(U)$. As $f^{-1}(U)$ is the union of such sets for all V_x , and these sets are open, it follows that $f^{-1}(U)$ is open, and we are done. \square

Definition 5. If X and Y are topological spaces, a **homeomorphism** from X to Y is defined to be a continuous bijective map $\phi : X \rightarrow Y$ with continuous inverse. If there exists a homeomorphism between X and Y , we say that X and Y are **homeomorphic** or **topologically equivalent**. Sometimes this is abbreviated $X \approx Y$.

Exercise 1. Show that homeomorphisms are an equivalence relation.

Proof. To show that homeomorphisms are an equivalence relation, we show

- $X \approx X$: The identity map Id is a homeomorphism from X to X .
- $X \approx Y \implies Y \approx X$: There exists a homeomorphism from X to Y . Its inverse is clearly a homeomorphism from Y to X .
- $X \approx Y$ and $Y \approx Z \implies X \approx Z$: As the composition of two homeomorphisms is also a homeomorphism (from elementary set theory), the homeomorphism from X to Z is simply the composition of the homeomorphisms from Y to Z and from X to Y , respectively.

\square

Example 3. Any open ball in \mathbb{R}^n is homeomorphic to any other open ball. The homeomorphism can be constructed simply by composition translations $x \mapsto x + x_0$ and dilations $x \mapsto cx$. This shows that size is not a topological property.

Example 4. If \mathbb{B}^n is the open unit ball, we can define $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$ by

$$y = F(x) = \frac{x}{1 - |x|^2}.$$

The inverse is given by

$$x = F^{-1}(y) = \frac{2y}{1 + \sqrt{1 + 4|y|^2}}.$$

As both are continuous and bijective, F is a homeomorphism, so \mathbb{R}^n is homeomorphic to \mathbb{B}^n . This shows that boundedness is not a topological property.

Example 5. Take the surface of the unit sphere in \mathbb{R}^3 and the surface of the cube of side 2, centered at the origin. There exists a homeomorphism between these two surfaces,

$$\phi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

whose inverse is given by

$$\phi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Thus, corners are not a topological property either.

Example 6. Now for an example of a continuous bijection that is not a homeomorphism by failure of its inverse to be continuous. Let X be the interval $[0, 1) \subset \mathbb{R}$, and let \mathbb{S}^1 denote the unit circle in \mathbb{R}^2 (both with the Euclidean metric topologies). Define a map $a : X \rightarrow \mathbb{S}^1$ by $a(t) = (\cos 2\pi t, \sin 2\pi t)$. It is clear that the map is continuous and bijective. The inverse, however, is not continuous. To see why, take any neighborhood of the point $(1, 0)$ - the inverse of a on this neighborhood will inevitably contain the point $0 \in [0, 1)$. Thus the preimage of open sets is not necessarily open, and thus a^{-1} is not continuous.

Definition 6. A map $f : X \rightarrow Y$ is said to be an **open map** if for any open set $U \subset X$, the image set $f(U)$ is open in Y . A map can be open but not continuous, continuous but not open, both, or neither.

Definition 7. We say that a continuous map $f : X \rightarrow Y$ between topological spaces is a **local homeomorphism** if every point $x \in X$ has a neighborhood $U \subset X$ such that $f(U)$ is an open subset of Y and $f|_U : U \rightarrow f(U)$ is a homeomorphism.

Exercise 2. *Show that:*

1. *every local homeomorphism is an open map.*
2. *every homeomorphism is a local homeomorphism.*
3. *a bijective continuous open map is a homeomorphism.*

4. *a bijective local homeomorphism is a homeomorphism.*

Proof.

1. Take any open set $U \subset X$. For every $x \in U$, there exists some neighborhood $V = U_x \cap U$ of x for which the local homeomorphism $f(V)$ is open. Note that U is the union of all such V (for various x) and thus $f(U)$ is the union of all $f(V)$, and as the union of open sets must be an open set, $f(U)$ must be open. Consequently, f is an open map.
2. Take f to be a homeomorphism from X to Y . f is trivially a local homeomorphism; the neighborhood needed by the definition is simply X itself.
3. Let f be a bijective, continuous, open map. To show that f is a homeomorphism, we must show that f^{-1} is a continuous map. In other words, we must show that the preimage of f on open sets of Y are open sets in X . This is true as f is open and bijective: open sets in X are taken to open sets in Y (open), and all open sets in Y are images of open sets in X (bijective).
4. Let us first show that any local homeomorphism f from X to Y is continuous. Take any open set $V \subset Y$ whose preimage under f we call U . We wish to show that U is open. Take any point $y \in V$. For any $x \in f^{-1}(y)$, there is a neighborhood $M_x \subset X$ of x that is homeomorphic to a neighborhood N_y of y in Y . This implies that $U_x = M_x \cap U$ is homeomorphic to $V_y = N_y \cap V$. Take the union of all the sets U_x for every x in the preimage of y and call this W_y . Note that the preimage of V is the union of all such W_y for $y \in V$. This union is open, and we are done.
Now it remains to show that the inverse of f (that exists via bijectivity) is continuous. Let $U \subset X$ be open and $V = (f^{-1})^{-1}(U) = f(U)$. We wish to show that V is open. This follows trivially from the fact that f is an open map, and we are done.

□

Up until now, we have worked with topological spaces defined through open sets. There is a complementary notion that is just as important.

Definition 8. A subset F of a topological space X is said to be **closed** if its complement $X \setminus F$ is open. It follows immediately from the definition of topological spaces that

1. X and \emptyset are closed.
2. Finite unions of closed sets are closed.
3. Arbitrary intersections of closed sets are closed.

A topology on a set X can be defined by describing the collection of closed sets, as long as they satisfy these three properties; the open sets are then just those sets whose complements are closed.

Example 7 (Closed Sets).

- Any closed interval $[a, b] \subset \mathbb{R}$ is a closed set, as are the half-infinite closed intervals $[a, \infty)$ and $(-\infty, b]$.
- Any closed ball in a metric space is a closed set.
- Every subset of a discrete space is closed.

It is important to note that closed is *not* the same as “not open,” as sets can be both open and closed (such as X, \emptyset), or neither open nor closed, such as $[0, 1) \subset \mathbb{R}$.

Lemma 3. *A map between topological spaces is continuous if and only if the inverse image of every closed set is closed.*

Proof. Assume $f : X \rightarrow Y$ is continuous. Given $V \subset Y$ is closed, and its preimage under f is $U \subset X$, we want to show that U is closed. By definition, $Y \setminus V$ is open, and as by continuity of f , we have that its preimage $X \setminus U$ is open. Thus, again by definition of closed sets, U must be closed.

Now assume that $f : X \rightarrow Y$ is such that the preimage of any closed set V is a closed set in X . Take an open set $V \subset Y$ whose preimage under f is $U \subset X$. The preimage of $Y \setminus V$ is, of course $X \setminus U$, which is closed. Since $X \setminus U$ is closed, U must be open, and we are done. \square

Definition 9. Given any set $A \subset X$, we define several related sets as follows. The **closure** of A in X , denoted by \bar{A} , is the set

$$\bar{A} = \bigcap \{B \subset X : B \supset A \text{ and } B \text{ is closed in } X\}.$$

The **interior** of A , written $\text{Int } A$ is

$$\text{Int } A = \bigcup \{C \subset X : C \subset A \text{ and } C \text{ is open in } X\}.$$

It is obvious that \bar{A} is closed and $\text{Int } A$ is open. In words, \bar{A} is the “smallest closed set containing A ,” and $\text{Int } A$ is “the largest open set contained in A .” We also define the **exterior** of A , written $\text{Ext } A$, as

$$\text{Ext } A = X \setminus \bar{A},$$

and the **boundary** of A , written ∂A , as

$$\partial A = X \setminus (\text{Int } A \cup \text{Ext } A)$$

It follows from the definitions that for any subset $A \subset X$, the whole space X is equal to the disjoint union of $\text{Int } A$, $\text{Ext } A$, and ∂A . The set A always contains all of its interior points and none of its exterior points, and may contain all, some, or none of its boundary points.

Lemma 4. *Let X be a topological space and $A \subset X$ any subset.*

1. *A point q is in the interior of A if and only if q has a neighborhood contained in A .*
2. *A point q is in the exterior of A if and only if q has a neighborhood contained in $X \setminus A$.*
3. *A point q is in the boundary of A if and only if every neighborhood of q contains both a point of A and a point of $X \setminus A$.*
4. *$\text{Int } A$ and $\text{Ext } A$ are open in X , while ∂A is closed in X .*
5. *A is open if and only if $A = \text{Int } A$.*
6. *A is closed if and only if it contains all its boundary points, which is true if and only if $A = \text{Int } A \cup \partial A$.*
7. *$A = \bar{A} \cup \partial A = \text{Int } A \cup \partial A$.*

Proof. Left to the reader. □

Definition 10. Given a topological space X and a set $A \subset X$, we say that a point $q \in X$ is a **limit point** of A if every neighborhood of q contains a point of A other than q (which may or may not itself be in A). If we let $X = \mathbb{R}$ and $A = \{1/n\}_{n=1}^{\infty}$, for example, then 0 is the only limit point of A .

Exercise 3. Show that a set A in a topological space is closed if and only if it contains all of its limit points.

Proof. It is clear from the previous lemma that every boundary point is also a limit point. In addition, it is clear that no limit point can also be outside of A . If A is closed, it must contain its boundary, and thus, since it contains its interior and its boundary, it contains all of its limit points. Conversely, if A contains all of its limit points, it contains its boundary, and is closed. \square

Definition 11. A subset A of a topological space X is said to be **dense** in X if $\bar{A} = X$.

Exercise 4. Show that a subset $A \subset X$ is dense if and only if every nonempty open set in X contains a point of A .

Proof. Suppose every nonempty open set in X contains a point of A . Every neighborhood of every point of X must then contain a point of A , and thus every point in X must be a limit point of A . $\bar{A} = \text{Int } A \cup \partial A$ contains all of its limit points, and thus $X = \bar{A}$ and A is dense in X .

Conversely, suppose that A is dense in X . Then, $\bar{A} = X$, i.e. X is equal to the union of the interior and the boundary of A . Any point in the interior or the boundary of a set has the property that all of its neighborhoods contain a point in the interior. Thus, since any open set in X is a neighborhood of any point in X , any open set in X must contain a point of A . \square

Definition 12. A map $f : X \rightarrow Y$ is said to be **closed** if it takes closed sets in X to closed sets in Y .

Exercise 5. Show that a bijective continuous map is a homeomorphism if and only if it is open if and only if it is closed.

Proof. Suppose f is a bijective continuous open map. We want to show that the inverse f^{-1} is continuous: the preimage of open sets in X under f^{-1} (i.e. the image of f) must be open. Since f is open, we are done. Conversely, if f is a homeomorphism, f must be bijective and continuous. It remains to show that f is an open map. This follows from the fact that f^{-1} is continuous, similar to above.

Suppose f is a bijective continuous closed map. We want to show that the inverse f^{-1} is continuous: the preimage of closed sets in X under f^{-1} (i.e. the image of f) must be closed (by the definition of continuity in terms of closed sets). Since f is closed, we are done. Conversely, if f is a homeomorphism, f must be bijective and continuous. It remains to show that f is an closed map. This follows from the fact that f^{-1} is continuous, similar to above, using the definition in terms of closed sets. \square