PROBLEM SET I

1. State Green's formula in the plane. Verify that if D is a piecewise C^1 boundary, and $f(z)$ is a C^1 function, then

$$\oint_{\partial D} f(z)dz = \int\int_D \partial_{\bar z} f \, d\bar z \wedge dz$$

2. Let $f(z)$ be a holomorphic function on $\Omega \setminus 0$, where Ω is a domain in $\mathbf{C}$ containing the point 0. Assume that f is bounded near 0, i.e., there exists $r > 0$ and C so that $|f(z)| \leq C$ for all z in the pointed disk centered at 0 of of radius r. Show that $f(z)$ extends to a holomorphic function on Ω. (Hint: set $g(z) = 0$ for $z = 0$, $g(z) = z^2 f(z)$ for $z \in \Omega \setminus z$. Show that $g(z)$ is holomorphic in Ω, and deduce the extension of $f(z)$ to $z = 0$ which would make f a holomorphic function on Ω.)

3. Let $f \in C^\infty([0,1])$, and consider the function $I(z)$ defined by

$$I(z) = \int_0^1 f(x)\, x^{z-1}\, dx.$$

(a) Show that $I(z)$ is a well-defined and holomorphic function of z for $\mathrm{Re}\, z > 0$.
(b) Show that $I(z)$ can be extended to a meromorphic function of z in the entire z-plane, with possibly poles at negative integers. Find the residues of $I(z)$ at these poles.

4. Let $f(z)$ be a holomorphic function in the disk $D_R(0)$ centered at 0 and of radius $R > 0$.
(a) Show that for any $0 < r < R$, we have

$$\int_{D_r(0)} |f(z)|^2 dxdy = \pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n+2}$$

where a_n are the coefficients of the Taylor expansion of $f(z)$ at 0, $f(z) = \sum_{n=0}^{\infty} a_n z^n$.
(b) Deduce that

$$|f(0)|^2 \leq \frac{1}{\pi R^2} \int_{D_R(0)} |f(z)|^2 dxdy.$$

(c) Let Ω be a bounded domain in $\mathbf{C}$. For each $\delta > 0$, let Ω_δ be the subset of points z_0 with $D_\delta(z_0) \subset \Omega$. Show that

$$\sup_{z \in \Omega_\delta} |f(z)|^2 \leq \frac{1}{\pi \delta^2} \int_\Omega |f(z)|^2 dxdy$$

(d) Deduce that if f_j is a sequence of holomorphic functions in Ω which converges with respect to the L^2 norm $||f_j||^2 = \int_\Omega |f_j(z)|^2 dxdy$, then the sequence f_j converges uniformly on any compact subset Ω' of Ω. Show that the limit $f(z)$ is holomorphic in Ω.