Physics 6047
Problem Set 3, due 2/14/13
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1. In class, we showed using the path integral, with its accompanying $i\epsilon$ prescription (to project onto ground state), that the two point function for the free (real) scalar is (after integrating over dk^0):

$$\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\Theta(t_1 > t_2)e^{ik\cdot(x_1-x_2)} + \Theta(t_2 > t_1)e^{-ik\cdot(x_1-x_2)}\right], \quad (1)$$

where k^0 should be understood as $\omega_k = \sqrt{\vec{k}^2 + m^2}$. To be completely explicit, $k \cdot (x_1 - x_2)$ means $-\omega_k(t_1 - t_2) + \vec{k}\cdot(\vec{x}_1 - \vec{x}_2)$. *Show that* you can obtain the same result using canonical quantization i.e. using the expansion of the free $\hat{\phi}$ in terms of the annihilation and creation operators. I hope you appreciate what a non-trivial check this is: that two completely different ways of thinking about a quantum theory – path integral on the one hand, operators on the other – give identical results on the two-point correlation function.

2. Problem 8.5 of Srednicki.

3. (a). Show that, for two space-like separated points, eq. (1) can be further integrated to give

$$\begin{aligned}\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = {} & \Theta(t_1 > t_2)\frac{1}{(2\pi)^2 x_{12}}\int_m^\infty \frac{z\,dz}{\sqrt{z^2 - m^2}}e^{-zx_{12}}\cosh\left(\sqrt{z^2 - m^2}\,t_{12}\right) \\ & + \Theta(t_2 > t_1)\frac{1}{(2\pi)^2 x_{21}}\int_m^\infty \frac{z\,dz}{\sqrt{z^2 - m^2}}e^{-zx_{21}}\cosh\left(\sqrt{z^2 - m^2}\,t_{21}\right), \quad (2)\end{aligned}$$

where $x_{12} = |\vec{x}_1 - \vec{x}_2| = |\vec{x}_2 - \vec{x}_1| = x_{21}$, and $t_{12} = t_1 - t_2 = -(t_2 - t_1) = -t_{21}$. You can proceed by essentially repeating the computation you did for question 1 of problem set 1, but keeping in mind that here we have an extra factor of $1/(2\omega_k)$ in the integrand. Note that this expression holds only for space-like separated points i.e. $x_{12} > |t_{12}|$. Note also the cosh does not care about the sign of t_{12}, a fact that will be relevant below.

(b). Let us consider a concrete example where $x_2 = (0, 0, 0, 0)$, and $x_1 = (a, b, 0, 0)$ i.e. x_2 is at the space-time origin, and x_1 is at time a and spatial position b (in the x-direction). We assume $|b| > |a|$ such that the two points are space-like separated. Now, let's consider a Lorentz boost to a different frame $x_1 \to x_1'$, $x_2 \to x_2'$. Under this boost, $x_2' = (0, 0, 0, 0)$, and $x_1' = (-a, b', 0, 0)$. In other words, this is a boost that exactly reverses the time order, flipping the time-coordinate of event 1 from a to $-a$. *Show that* such a boost gives $b' = b$, i.e. the spatial separation between the two events remain the same!

(c). A lot of you were worried about how an expression that involves time-ordering can be consistent with Lorentz invariance, when a boost can reverse time-ordering. We know implicitly our two-point correlation function must be Lorentz invariant since we obtained it from an integral over d^4k which was manifestly Lorentz invariant. But here we have a chance to check this explicitly. What does Lorentz invariance mean in the context of the two-point function? It means $\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle = \langle 0|T\hat{\phi}(x_1')\hat{\phi}(x_2')|0\rangle$. *Show that* the result in part (a) is consistent with exactly this statement, for x_1, x_2, x_1', x_2' taking the form in part (b).

(d). It is worth emphasizing that the result in part (a) tells us the two-point function does *not* in general vanish for space-like separated points. Lorentz invariance does not demand it to be zero. What has to be zero is that operators (observables) should commute at space-like separations. For instance, we expect $[\hat{\phi}(x_1), \hat{\phi}(x_2)] = 0$ for a space-like separation. Instead of showing this in general, let us set ourselves the more modest goal: *show that* $\langle 0|[\hat{\phi}(x_1), \hat{\phi}(x_2)]|0\rangle = 0$ for the free (real) scalar, for a space-like separation. This is easiest to do using canonical quantization i.e. method of question 1.

4. We learned in class that for the free (i.e. Gaussian) theory, all n-point correlation function can be expressed in terms of the two-point function e.g.

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle + \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle + \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle \, , \tag{3}$$

where we have used the shorthand $\langle \phi_1 \phi_2 \rangle$ for $\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0\rangle$, etc. *Show that* eq. (3) is indeed correct by taking derivatives of the generating function (and setting $J = 0$ afterwards). I won't ask you to derive analogous results for an arbitrary *even* n. But, *work out* the number of such Wick pairings for an arbitrary n-point function, assuming n is even. For instance, for $n = 2$, we say there are 3 such terms.

5. In class, we worked out the energy of a scalar field configuration sourced by a static J. The conclusion was that there's a certain potential energy associated with ϕ-exchange between two point sources, we called it $V(|\vec{x} - \vec{x}'|)$. Complete our discussion in class by *showing that*

$$V(|\vec{x} - \vec{x}'|) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2 + m^2} = \frac{1}{4\pi} \frac{e^{-m|\vec{x}-\vec{x}'|}}{|\vec{x} - \vec{x}'|} \tag{4}$$

6. Problem 8.7 of Srednicki.