Introduction to Differentiable Manifolds

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Last updated: February 3, 2013

Problem 1

The line that goes through $(0,0,1)$ and $(x,y,z) \in \mathbb{S}^2$ is given by $(0,0,1) + t(x,y,z-1)$ for $t \in \mathbb{R}$. This line intersects the $z=-1$ plane when $1+t(z-1)=-1$, i.e. $t = -\frac{2}{z-1}$. Therefore,

$$(0,0,1) - \frac{2}{z-1}(x,y,z-1) = (0,0,1) + (\frac{2x}{1-z}, \frac{2y}{1-z}, -1).$$

The line that goes through $(0,0,-1)$ and $(x,y,z) \in \mathbb{S}^2$ is given by $(0,0,-1) + t(x,y,z+1)$ for $t \in \mathbb{R}$. This line intersects the $z=1$ plane when $-1+t(z+1)=1$, i.e. $t = \frac{2}{z+1}$. Therefore,

$$(0,0,-1) + \frac{2}{z+1}(x,y,z+1) = (0,0,-1) + (\frac{2x}{1+z}, \frac{2y}{1+z}, 1).$$

Given the map $\psi \circ \phi^{-1}\left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) = \left(\frac{2x}{1+z}, \frac{2y}{1+z}\right)$, defined for $\left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) \in \mathbb{R}^2 - \{(0,0)\} = \phi(U \cap V)$, we can show that this transition map is smooth by taking

$$u = \frac{2x}{1-z} \text{ and } v = \frac{2y}{1-z}.$$

We then solve for x, y in terms of u, v

$$x = \frac{u(1-z)}{2} \text{ and } y = \frac{v(1-z)}{2},$$

and insert these into the equation of constraint,

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ u^2(1-z)^2 + v^2(1-z^2) + 4z^2 &= 4 \\ (u^2+v^2+4)z^2 - 2(u^2+v^2)z + (u^2+v^2-4) &= 0, \end{aligned}$$

which by the quadratic equation,

$$
\begin{aligned}
z &= \frac{2(u^2+v^2) \pm \sqrt{4(u^2+v^2)^2 - 4(u^2+v^2+4)(u^2+v^2-4)}}{2(u^2+v^2+4)} \\
&= \frac{u^2+v^2 \pm 4}{u^2+v^2+4} = \frac{u^2+v^2-4}{u^2+v^2+4}
\end{aligned}
$$

where we have dropped the impossible case $z = 1$. Inserting this value for z back into the expressions for x, y, we find

$$x = \frac{4u}{u^2+v^2+4} \text{ and } y = \frac{4v}{u^2+v^2+4}.$$

To determine what u, v are mapped to, we insert (x, y, z) into the values output by the transition map, and get as desired,

$$(u, v) \to \left(\frac{4u}{u^2+v^2}, \frac{4v}{u^2+v^2} \right). \tag{1}$$

To find the inverse transition map $\phi \circ \psi^{-1}$, we may simply insert (α, β) on the right hand side of Eq. (1), and solve for u, v:

$$\alpha = \frac{4u}{u^2+v^2} \text{ and } \beta = \frac{4v}{u^2+v^2}.$$

First note that $\frac{\alpha}{\beta} = \frac{u}{v}$. Additionally,

$$
\begin{aligned}
u^2\alpha - 4u - v^2\alpha &= 0 \\
\left(\alpha + \frac{\beta^2}{\alpha}\right) u^2 - 4u &= 0 \\
u\left(u(\alpha + \frac{\beta^2}{\alpha}) - 4\right) &= 0
\end{aligned}
$$

which yields (ignoring $u, v, = 0$ as above)

$$u = \frac{4\alpha}{\alpha^2+\beta^2} \text{ and } v = \frac{4\beta}{\alpha^2+\beta^2}.$$

Thus, it is clear that these two charts are smoothly compatible, as the transition functions are diffeomorphisms (compositions of smooth functions that are well-behaved).

Problem 2

Let $\mathfrak{U}_1$ be the atlas consisting of the above stereographic projections and $\mathfrak{U}_2$ be the atlas consisting of the 6 graphical coordinate charts. To show that $\mathfrak{U}_1 \cup \mathfrak{U}_2$ is an atlas, we must check that all of the charts are smoothly compatible with each other. Note, however, that due to the symmetry of the problem, we have to only two cases: one of the stereographic projections against the $z > 0$ chart and any one other graphical coordinate chart. This is justified because the cases for the two different stereographic projections are identical, because the $z < 0$ chart case is identical to that of $z > 0$ (except without the restriction on the south pole that is placed on the north) and because the projection treats all the "other" 4 charts functionally equivalent (since the sign in front of z does not change).

Let us first consider the case of the transition map between the north-pole stereographic projection and the $z > 0$ chart, $\psi \circ \phi^{-1}$. The expressions for x, y in terms of the stereographic u, v are identical to those above - the only thing that changes is the image:

$$\psi \circ \phi^{-1}\left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) = (x, y).$$

which, in terms of u, v is simply (from earlier),

$$\psi \circ \phi^{-1}(u, v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}\right).$$

This map's inverse is obviously

$$\phi \circ \psi^{-1}(x, y) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right).$$

As both the transition map and its inverse are smooth, these maps are smoothly compatible.

Now let us choose one other chart; namely, the $y > 0$ chart. Thus we have:

$$\psi \circ \phi^{-1}\left(\frac{2x}{1-z}, \frac{2y}{1-z}\right) = (x, z).$$

Using the same definitions of u, v as above, we can write

$$\psi \circ \phi^{-1}(u, v) = \left(\frac{4u}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4}\right).$$

Again, this map's inverse is obvious:

$$\phi \circ \psi^{-1}(x, z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z} \right).$$

Since the map and it's inverse are clearly smooth (we again don't worry about $z = 1$ as we are dealing with the stereographic projection from the north pole), these transition maps are smoothly compatible. Consequently, we have shown that the stereographic atlas and the graphical atlas of $\mathbb{S}^2$ are equivalent/compatible.

Problem 3

Let $\sim$ be the equivalence relation on $X = \mathbb{C}^{n+1} \setminus \{0\}$ with $z \sim \lambda z$, for any $z \in X$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Let $\mathbb{CP}^n = X \setminus \sim$ denote the set of equivalence classes and define the projection $\pi : X \to \mathbb{CP}^n$ as the map that takes each element of X to its equivalence class - i.e. the map that takes points in X to the linear subspace that they span. We declare, in the usual way, that a subset $U \subset \mathbb{CP}^n$ be open if and only if $\pi^{-1}(U)$ is open in X. Under π, then, $\mathbb{CP}^n$ forms a quotient space.

We can construct an atlas for $\mathbb{CP}^n$ almost exactly as we did for the real case. Consider the open set $U_i = \left\{ z \in X \middle| |z_i| > 0 \right\}$. Then, $V_i = \pi(U_i)$ is open. We define the map $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i(z) = \phi_i(z_1, \ldots, z_{n+1}) = \left(\frac{z^1}{z_i}, \ldots, \frac{z^{i-1}}{z_i}, \frac{z^{i+1}}{z_i}, \ldots, \frac{z^{n+1}}{z_i} \right).$$

Note that the map $\phi_i \circ \pi : X \to \mathbb{C}^n$ is essentially just a projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$, and is thus continuous. By the characteristic property of quotient maps, then, ϕ_i is continuous. As the inverse is given by

$$\phi_i^{-1}(z^1, \ldots, z^n) = [z^1, \ldots, z^{i-1}, 1, z^i, \ldots, z^n],$$

which is clearly continuous as well, ϕ_i is a homeomorphism. To show that $\{(U_i, \phi_i)\}$ forms an atlas, we must show that the transition maps are smoothly compatible. Take, without loss of generality, $i > j$, so

$$\phi_j \circ \phi_i^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j} \ldots, \frac{z^{i-1}}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j}, \ldots, \frac{z^n}{z^j} \right).$$

Since these charts overlap where $z^i \neq 0, z^j \neq 0$, this transition map is smooth and its inverse is as well. Thus, in general, the transition maps are diffeomorphisms, and $\mathbb{CP}^n$ forms a smooth manifold.

Problem 4

Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. The open sets of the quotient topology are those whose preimages $\pi^{-1}(U)$ are open in X. Note that this means that there exist open sets in M that contain two, one, or zero of the two origins (the appropriate unions of open sets in X are easily found). Let us first show that M is locally Euclidean. Let U_1 be the open set that contains all of M but one of the origins, and let U_2 be the open set that contains all of M but the other origin. Then, $\mathcal{U} = \{U_1, U_2\}$ is an open cover of M. Furthermore, it should be obvious that each of the open sets in this cover is homeomorphic to $\mathbb{R}$; in other words, any open subset of the line of two origins that contains only one origin is homeomorphic to an open set in the $\mathbb{R}$. Consequently, M is locally Euclidean.

We can show second countability fairly easily - each of the two open sets in the open cover $\mathcal{U}$ of M has a countable basis. Obviously, the union of these two bases is also countable, and generates the space, and we are done.

M, however, fails to be Hausdorff. Take, for example, the two origins. Every pair of open sets that each contain one of these origins will have points in common; i.e. any open set around each origin will always contain points on the "line," which is shared.

Problem 5

Let X be the disjoint union of uncountably many copies of $\mathbb{R}$. Note that the collection of the copies of $\mathbb{R}$ forms an open cover $\mathcal{U}$ of X. Each set in $\mathcal{U}$ is clearly homeomorphic to $\mathbb{R}$, which makes X locally Euclidean. Furthermore, X must be Hausdorff, simply because it is the disjoint union of Hausdorff spaces. It should also be clear that X is not second countable, as each copy of $\mathbb{R}$ has a countably number of basis sets, but the disjoint union of uncountably many copies has an uncountably infinite number of basis sets.

Problem 6

The map $\tilde{P}([x]) = [P(x)]$ takes a line in $\mathbb{RP}^n$, takes a point $x \in \mathbb{R}^{n+1} \backslash \{0\}$ on said line, transforms it smoothly according to P to a point $P(x) \in \mathbb{R}^{k+1} \backslash \{0\}$, and then returns the line generated by $P(x)$. We must be careful to check that this process is independent of the x chosen from $[x]$. If we take the point λx instead of x, we will be left with $[\lambda^d P(x)]$, which is simply $[P(x)]$.

$$
\begin{tikzcd}
\mathbb{R}^{n+1} \arrow[r, "P"] \arrow[d, "\pi_1"'] & \mathbb{R}^{k+1} \arrow[d, "\pi_2"] \\
\mathbb{RP}^n \arrow[r, "\tilde{P}"] & \mathbb{RP}^k
\end{tikzcd}
$$

This can also be seen from the diagram above. For the map to be well-defined, we must have $\tilde{P}(\pi_1(x)) = \pi_2(P(x))$ for any $x \in \mathbb{R}^{n+1}$. Starting with the left-hand side, we find

$$\tilde{P}([x]) = [P(x)],$$

which is exactly equal to $\pi_2(P(x))$ and we are done.

Now it remains to show that the map $\tilde{P}$ is smooth. To do this, we must move down and examine $\tilde{P}$ in subsets of $\mathbb{R}^n$ and $\mathbb{R}^k$. By definition, $\tilde{P}$ is smooth if $\psi \circ \tilde{P} \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^k$ is smooth. To do this, we take $U_j \subset \mathbb{RP}^n$, and $V_i \subset \mathbb{RP}^n$, which are mapped to subsets of Euclidean spaces by the charts ϕ_j and ψ_i respectively. Note that these subsets and charts are chosen in the usual way they are for projective spaces ($x_j \neq 0$, etc.). This is visualized in the diagram below.

$$
\begin{tikzcd}
\mathbb{RP}^n \arrow[r, "\tilde{P}"] & \mathbb{RP}^k \arrow[d, "\psi_i"] \\
\phi_j(U_j) \arrow[u, "\phi_j^{-1}"] & \psi_i(V_i)
\end{tikzcd}
$$

Take a point

$$\left(\frac{x_1}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_{n+1}}{x_j}\right) \in U_j.$$

This point is mapped by ϕ^{-1} to the equivalence class of points (i.e. the line)

$$[x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_{n+1}] \in \mathbb{RP}^n.$$

After $\tilde{P}$ acts on this equivalence class, we have, by definition of $\tilde{P}$'s action, we get

$$[P(x) = P\,(x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_{n+1})] \in \mathbb{RP}^k.$$

Suppose this falls into the ith chart in $\mathbb{RP}^k$; we take ψ_i of this line, which yields

$$\left(\frac{P_1(x)}{P_i(x)}, \cdots, \frac{P_j(x)}{P_i(x)}, \cdots, \frac{P_{i-1}(x)}{P_i(x)}, \frac{P_{i-1}(x)}{P_i(x)}, \cdots, \frac{P_{k+1}(x)}{P_i(x)}\right) \in \psi_i(V_i).$$

To determine whether this series of compositions is smooth, we must first rewrite the image in terms of $u_i = \frac{x_i}{x_j}$ (the input), with $i = 1, \cdots, n+1$ using the property $P(\lambda x) = \lambda^d P(x)$ (for $\lambda \neq 0$). We then have

$$\begin{aligned}
&\psi_i \circ \tilde{P} \circ \phi_j^{-1} (u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_{n+1}) \\
&\qquad = \left(\frac{P_1(x_j u)}{P_i(x_j u)}, \cdots, \frac{P_j(x_j u)}{P_i(x_j u)}, \cdots, \frac{P_{i-1}(x_j u)}{P_i(x_j u)}, \frac{P_{i-1}(x_j u)}{P_i(x_j u)}, \cdots, \frac{P_{k+1}(x_j u)}{P_i(x_j u)}\right) \\
&\qquad = \left(\frac{P_1(u)}{P_i(u)}, \cdots, \frac{P_j(u)}{P_i(u)}, \cdots, \frac{P_{i-1}(u)}{P_i(u)}, \frac{P_{i-1}(u)}{P_i(u)}, \cdots, \frac{P_{k+1}(u)}{P_i(u)}\right),
\end{aligned}$$

which is clearly a smooth map, by the smoothness of P.