MODERN ALGEBRA II SPRING 2013: FIFTH PROBLEM SET

1. Let $f(x) = x^2 + 3x + 2 = (x+1)(x+2) \in (\mathbb{Z}/6\mathbb{Z})[x]$. Find a root of $f(x)$ which is not -1 or -2, and use this new root to find a different factorization of $f(x)$ into linear factors.

2. Let R be the subring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of $\mathbb{R}$. Let $I = (6+\sqrt{2})$ be the principal ideal generated by $6+\sqrt{2}$. Follow the outline of Problem 7 in HW 4 to show that $R/I \cong \mathbb{Z}/34\mathbb{Z}$. Is I a maximal ideal? A prime ideal?

3. Let F be a field, and consider the ring $F[x]$.

   (i) Let I be the principal ideal $(x - r)$ in $F[x]$. Using the fact that every coset $p(x) + I$ contains a unique representative of the form a, where $a \in F$, conclude that $F[x]/I \cong F$, and in fact the homomorphism $\phi\colon F \to F[x]/I$ defined by $\phi(a) = a + I$ is an isomorphism. Note that this agrees with the fact that, if $\mathrm{ev}_r\colon F[x] \to F$ is the evaluation homomorphism, then $I = \operatorname{Ker} \mathrm{ev}_r$, so that $F[x]/I \cong \operatorname{Im} \mathrm{ev}_r = F$. Is I a prime ideal? A maximal ideal?

   (ii) Let I be the principal ideal (x^2) in $F[x]$. Show that every coset $p(x) + I$ contains a unique representative of the form $a_0 + a_1x$, where $a_0, a_1 \in F$. Write the coset $a_0 + a_1x + I$ in abbreviated form as $a_0 + a_1\alpha$, where $\alpha = x + I$, and describe addition and multiplication in $F[x]/I$ using this form. In other words, given two elements $a_0 + a_1\alpha, b_0 + b_1\alpha \in F[x]/I$, describe

   $$(a_0 + a_1\alpha) + (b_0 + b_1\alpha) \text{ and } (a_0 + a_1\alpha) \cdot (b_0 + b_1\alpha).$$

   In particular, what is α^2 in the ring $F[x]/I$ when written in the form $a_0 + a_1\alpha$? Is I a prime ideal? A maximal ideal?

   (iii) Let I be the principal ideal $(x^2 - 1)$ in $F[x]$. Show again that every coset $p(x) + I$ contains a unique representative of the form $a_0 + a_1x$, where $a_0, a_1 \in F$. Again write the coset $a_0 + a_1x + I$ in abbreviated form as $a_0 + a_1\alpha$, , where $\alpha = x + I$, and describe addition and multiplication in $F[x]/I$ using this form, in other words, given two elements $a_0 + a_1\alpha, b_0 + b_1\alpha \in F[x]/I$, describe

   $$(a_0 + a_1\alpha) + (b_0 + b_1\alpha) \text{ and } (a_0 + a_1\alpha) \cdot (b_0 + b_1\alpha).$$

In particular, what is α^2 when written in the form $a_0 + a_1\alpha$? Is I a prime ideal? A maximal ideal? Exhibit two nonzero elements of $F[x]/I$ whose product is 0.

(iv) Continuing with (iii), still with $I = (x^2 - 1)$ and now assuming that F is not of characteristic 2, consider the ring homomorphism $F[x] \to F \times F$ defined by $p(x) \mapsto (p(1), p(-1))$, in other words the homomorphism $(\mathrm{ev}_1, \mathrm{ev}_{-1})$. Show that I is in the kernel of both ev_1 and ev_{-1}, and that there is an induced ring homomorphism $\varphi\colon F[x]/I \to F \times F$, where $F \times F$ is viewed as a product ring. (In other words, $\varphi(p(x) + I) = (p(1), p(-1))$.) What is $\varphi(\alpha)$ (where as before $\alpha = x + I$)? Find elements $a_0 + a_1\alpha$ and $b_0 + b_1\alpha$ such that $\varphi(a_0 + a_1\alpha) = (1, 0)$ and $\varphi(b_0 + b_1\alpha) = (0, 1)$. (Where do we need to assume that characteristic $F \neq 2$?) Show that φ is surjective. Finally show that $\operatorname{Ker}\varphi = I$ and hence that φ is an isomorphism from $F[x]/I$ to $F \times F$.

4. Let F be a field, and let $E\colon F[x] \to F^F$ be the homomorphism from polynomials with coefficients in F to the ring of all functions from F to itself. Show that, if F is **infinite**, then E is injective but never surjective. (For both the injectivity and the failure to be surjective, you can use the fact that a nonzero polynomial $p(x) \in F[x]$ cannot be zero on infinitely many elements of F.)

   On the other hand, if F is **finite**, show that E is surjective but never injective. (If the elements of F are listed as $a_1, \dots, a_n$, first show that, for every i, there exists a polynomial $p_i \in F[x]$ such that $p_i(a_i) = 1$ and $p_i(a_j) = 0$ for $j \neq i$; in fact, you can take $p_i(x)$ to be a nonzero multiple of $(x - a_1)\cdots(x - a_{i-1})(x - a_{i+1})\cdots(x - a_n) = \prod_{j\neq i}(x - a_j)$. Then argue that every function from F to itself can be written as a linear combination $\sum_{i=1}^{n} c_i p_i(x)$ for appropriate $c_i \in F$.)