Introduction to Algebraic Topology PSET 8

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Proposition 1. *Hatcher exercise 2.1.11*

Proof. Let $\iota : A \to X$ be the inclusion of A into X, and $r : X \to X$ be the retract of X onto A. The composition $r \circ \iota : A \to A$ yields the identity $\mathrm{Id}_A : A \to A$. The induced maps on the homology are $(r \circ \iota)_* = r_* \circ \iota_* = \mathrm{Id} : H_n(A) \to H_n(A)$. This map is of course injective, which implies that $\iota_* : H_n(A) \to H_n(X)$ must be injective as well. □

Proposition 2. *Hatcher exercise 2.1.12*

Proof. Let us show that the relation of chain homotopy between chain maps is an equivalence relation. Consider $f_\#, g_\#, h_\# : C_n(A) \to C_{n+1}(B)$. The relation is clearly reflexive, as $f_\# \sim f_\#$ by the zero morphism $0 : C_n(A) \to C_{n+1}(B)$. Symmetry holds as follows: if $f_\# \sim g_\#$ via a chain homotopy h, then $g_\# \sim f_\#$ via the chain homotopy $-h$, because then

$$\begin{aligned} f_\# - g_\# &= \partial h + h\partial \\ g_\# - f_\# &= -(\partial h + h\partial) \\ &= \partial(-h) + (-h)\partial. \end{aligned}$$

Finally, the relation is transitive, because given $f_\# \sim g_\#$ via H_1 and $g_\# \sim h_\#$ via H_2, we can add the two commutation relations to obtain that

$$\begin{aligned} f_\# - h_\# &= \partial H_1 + H_1\partial + \partial H_2 + H_2\partial \\ &= \partial(H_1 + H_2) + (H_1 + H_2)\partial, \end{aligned}$$

as desired. □

Proposition 3. *Hatcher exercise 2.1.14*

Proof. □

Proposition 4. *Hatcher exercise 2.1.15*

Proof. Consider the exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

Exactness at B requires $\ker\beta = \mathrm{im}\ \alpha$, and hence α is surjective if and only if $\ker\beta = B$. Exactness at D requires $\ker\delta = \mathrm{im}\ \gamma$, and hence δ is injective if and only if $\mathrm{im}\ \gamma = 0$. Hence if (and only if) α is surjective and δ is injective then $\gamma = 0$ and $\beta = 0$ and the exactness at C (requiring that $\ker\gamma = \mathrm{im}\ \beta$) forces $C = 0$.

Hence for a good pair (X, A), we find that $H_n(X, A) = 0$ if and only if the inclusion $A \to X$ induces isomorphisms on all homology groups, as the long exact sequence of theorem 2.13 splits into sequences

$$0 \longrightarrow \tilde{H}_n(A) \xrightarrow{\iota_*} \tilde{H}_n(X) \longrightarrow 0$$

for all n. $\square$

Proposition 5. *Let A and B be chain complexes. A chain map $f : A \to B$ is a* chain homotopy equivalence *if there exists a chain map $g : B \to A$ such that $f \circ g \sim \mathrm{Id}_B$ and $g \circ f \sim \mathrm{Id}_A$ in the sense of chain homotopies.*

(a) Prove that if $f : A \to B$ is a chain homotopy equivalence, then f induces an isomorphism on homology.

(b) Give an example of chain complexes A and B with isomorphic homology but no chain homotopy equivalence between them. (Hint: let A be $\mathbb{Z}$ in two consecutive gradings and zero everywhere else.)

Proof.

(a) Recall that chain-homotopic maps induce the same homorphism on homology. Hence $(f \circ g)_* = f_* \circ g_* = (\mathrm{Id}_B)_* = \mathrm{Id}_{H_n(B)}$ and $(g \circ f)_* = g_* \circ f_* = (\mathrm{Id}_A)_* = \mathrm{Id}_{H_n(A)}$. As $\mathrm{Id}_{H_n(B)}$ is injective, $f_* : H_n(A) \to H_n(B)$ must be as well, and as $\mathrm{Id}_{H_n(A)}$ is surjective, f_* must be as well. Hence f_* is an isomorphism.

(b) Consider the map of chain complexes $f : A_\bullet \to B_\bullet$ given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & \mathbb{Z} \arrow[r, "\mathrm{Id}_{\mathbb{Z}}"] \arrow[d, "0"] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where each square clearly commutes. The homology groups of the two sequences are $H_\bullet(A) = 0$ and $H_\bullet(B) = 0$. However, there does not exist a chain homotopy equivalence between $A_\bullet$ and $B_\bullet$, as we now show. If there did exist one, there would exist a chain map $g : B_\bullet \to A_\bullet$ such that the appropriate compositions of f and g would be chain homotopic to Id_B and Id_A via some chain homotopy h. Of course, the only possible g is the zero morphism, and hence $g \circ f : A_\bullet \to A_\bullet$ is the zero map. Drawing the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\mathrm{Id}_{\mathbb{Z}}"] \arrow[d, "0"] & \mathbb{Z} \arrow[d, "0"] \arrow[dl, "h"'] \\
\mathbb{Z} \arrow[r, "\mathrm{Id}_{\mathbb{Z}}"] & \mathbb{Z}
\end{tikzcd}$$

we find that h must be an automorphism of $\mathbb{Z}$ such that $\mathrm{Id}_{\mathbb{Z}} \circ h + h \circ \mathrm{Id}_{\mathbb{Z}} = -\mathrm{Id}_{\mathbb{Z}}$. As the automorphisms of $\mathbb{Z}$ are $\mathrm{Id}_{\mathbb{Z}}, -\mathrm{Id}_{\mathbb{Z}}$, this equality cannot be satisfied and hence there does not exist a chain homotopy.

$\square$