Differentiable Manifolds Problem Set 8

Nilay Kumar

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Problem 1

Let m, n, k be positive integers with $k \leq \min(m, n)$. We wish to show that an $m \times n$ matrix M has rank greater than or equal to k if and only if there exists a $k \times k$ submatrix with nonzero determinant. First suppose that there exists a $k \times k$ submatrix with nonzero determinant. Then it is clear that there are either (at least, as there could be others) k linearly independent rows or columns of the matrix, as adding $m - k$ or $n - k$ components to a linearly independent vector keeps them linearly independent. Thus M has either at least k linearly independent rows or columns and thus has rank greater than or equal to k. Conversely, suppose M has rank greater than or equal to k, say $l \leq \min(m, n)$. Then, we can always rewrite M in its canonical form by changing bases, which as we showed in class, results in something that looks like (in block form):

$$\begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}$$

where I_l is the $l \times l$ identity. Of course, if $l = m$ or if $l = n$, some of the zero blocks will not be present. Clearly this matrix has a $k \times k$ submatrix that has nonzero determinant, as the top $k \times k$ submatrix of the I_l block has determinant 1. We proceed with a proof similar to one carried out in class for $k = 1$.

Given a matrix of rank k, we know we can find a $k \times k$ submatrix with non-zero determinant. Thus we can cover $M_k(m \times n)$ by **** WHAT? ****. It suffices to show that **** WHAT **?** is an embedded submanifold. Take some matrix E in **** WHAT? **** of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where A is the $k \times k$ submatrix. We can apply row-reduction to obtain in the lower-right block a term that must vanish for the rank of E to be k:

$$D - CA^{-1}B = 0_{(m-k)\times(n-k)}.$$

Hence, in order to realize **** WHAT? **** as a level set of a smooth map with full rank, we define a function $F : ?? \to M((m-k)\times(n-k))$ that takes some matrix of the form E to its $D - CA^{-1}B$.

Problem 2

Let $M(m \times n)$ be the space of real $m \times n$ matrices as a smooth manifold and $M_k(m \times n)$ be the set consisting of matrices with rank k. We wish to show that $M_k(m \times n)$ is an embedded submanifold of $M(m \times n)$ with codimension $(m-k) \times (n-k)$.

Problem 3

We wish to show that any closed subset of a compact space is compact. Take a compact space X and some open cover $\mathcal{U}$ of a compact subset A. It is clear that $\mathcal{U} \cup (X - A)$ is an open cover of X (as A is closed). But by the compactness of X, we find that this cover must have a finite subcover $\{U_1, \ldots, U_n\} \cup (X - A)$. It follows that $\{U_1, \ldots, U_n\}$ is an open over for A; thus, A must be compact, as this cover is finite.

Next, we wish to show that any compact subset A of a Hausdorff space X is closed. Take some sequence $\{a_i\}$ in A that converges to some $a \in X$. If we can show that $a \in A$, we are done, as this means A contains all of its limit points. By compactness, we know that $\{a_i\}$ has a subsequence $\{a_{i_k}\}$ that converges to $b \in A$. Suppose $a \neq b$. Then, by the Hausdorff property, we can find disjoint open sets U, V such that $a \in U$ and $b \in V$. By the definition of convergence, we know that far enough out in a_i we will be guaranteed to be in U, and far enough out in a_{i_k} we will be guaranteed to be in V. But since a_{i_k} is a subsequence of a_i, this means after some point, we will be both in U and V. This is a contradiction as U and V are disjoint. Hence $a = b$ and A must be closed.

Problem 4

Problem 5

Problem 6