Commutative Algebra: Problem Set 7

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Problem 2

Let us find a basis for $L(D)$ when $D = 2v_0 + 3v_1$. Recall the definition

$$L(D) = \left\{ f \in K^\times \mid (f) + D \geq 0 \right\}.$$

Since the principal divisor of f is just defined as $(f) = \sum_v v(f)v$, we see that the condition on $f \in L(2v_0 + 3v_1)$ allows poles of up to order 2 at 0 and poles of up to order 3 at 1. It is easy to see, then, that the following set of rational functions satisfy the condition:

$$\left\{ 1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x-1}, \frac{1}{(x-1)^2}, \frac{1}{(x-1)^3} \right\}.$$

Any product of these can be decomposed by partial fractions into these basis elements. It is clear that these are in fact a basis for $L(D)$ as a k-vector space (the dimension is 6, which falls into the bound we proved in class as $\deg D = 5$).

Next consider $D = 2v_0 + 2v_\infty$. Here we are allowed poles at 0 of up to order 2 and poles at infinity up to order 2. The basis is then

$$\left\{ 1, x, x^2, \frac{1}{x}, \frac{1}{x^2} \right\}.$$

For exactly the same reasons as above, we see that this gives us a basis for $L(D)$ as a k-vector space (again, the bound checks out).

Problem 3

For simplicity, let us consider the case of $k = \mathbb{C}$. Then any squarefree cubic may be written as $f(t) = \prod_{i=1}^3 (x - \lambda_i)$ for λ_i distinct. We wish to classify the discrete valuations on $K = \mathrm{Frac}\left(\mathbb{C}[x,y]/(y^2 - \prod_{i=1}^3 (x - \lambda_i)) \right)$ over $\mathbb{C}(x)$. Note first that we have a degree two extension and hence for every discrete valuation v on $\mathbb{C}(x)$ we must have finitely many extensions w_i on K of v such that $w_i = e_i v$. However, we also know that $\sum_i e_i = 2$, and hence there are only two possibilities: given an valuation v on $\mathbb{C}(x)$, there is either one valuation on K that restricts to v with ramification 2 or there are two valuations on K that restrict to v each with ramification 1. Let us proceed in cases.

Let w be a valuation on K. Suppose $w(x - \lambda_j) > 0$ for some j. Then, restricting to $\mathbb{C}(x)$, we see that $w|_{\mathbb{C}(x)} = \mathrm{ord}_{t=\lambda_j}$. But then we see that

$$2w(y) = w(y^2) = n\left(w(x - \lambda_1) + w(x - \lambda_2) + w(x - \lambda_3) \right) = n$$

for some integral $n \leq 2$. But this implies that $n = 2$ (given the bound earlier). Hence w is the only valuation lying over $\mathrm{ord}_{t=\lambda_j}$. Indeed, we can use this now to obtain exactly what w does to all elements in K, i.e. $w(y) = 1$, etc.

The other case is that $w(x - \lambda_j) < 0$ (as $w(x - \lambda_j) \neq 0$). Restricting to $\mathbb{C}(x)$ we see that $w|_{\mathbb{C}(x)} = \mathrm{ord}_{t=\infty}$. But then we see that

$$2w(y) = w(y^2) = m\left(w(x - \lambda_1) + w(x - \lambda_2) + w(x - \lambda_3)\right) = -3m$$

for some integral $m \leq 2$. But this implies that $m = 2$, as otherwise $w(y)$ will not be integral. Thus w is the only valuation lying over $\mathrm{ord}_{t=\lambda_j}$. Indeed, $w(y) = -3$ and we can deduce the value of w on all elements in K.

So far we have examined the valuations that lie over λ_i and ∞. But what about valuations lying over other points in $\mathbb{P}^1$? It turns out that there are two unique valuations lying over every other such point, i.e. these points in K are not ramified. To see this, let us appeal to some geometric reasoning. In terms of complex geometry, we see that $y^2 = \prod_{i=1}^{3}(x - \lambda_i)$ is simply a Riemann surface upon which y is holomorphic. It is easy graphically to see that this surface is a torus (and can be more rigorously shown via Abel's map and Jacobi inversion) and thus has one handle. But the number of ramified valuations in such a Riemann surface is simply four, as one can see in the diagrams below - these are precisely the λ_i and ∞. Hence the other points must each have two distinct valuations lying over them.