

A Note on Automorphisms of $\mathbb{P}_{\mathbb{A}}^n$

Nilay Kumar

February 7, 2014

Throughout this note, $\mathbb{A} = k$ denotes a field and \mathbb{A}^n denotes the affine n -space over k .

Projective spaces

Let us start by recalling the definition of projective space $\mathbb{P}_{\mathbb{A}}^n$ from last week.

Definition 1. We denote by $\mathbb{P}_{\mathbb{A}}^n$ the space of lines passing through the origin of \mathbb{A}^{n+1} . More precisely, let \mathbb{A}^{\times} act on $\mathbb{A}^{n+1} - \{0\}$ by scaling as $\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$ and define **projective n -space** to be the quotient $\mathbb{P}_{\mathbb{A}}^n = \mathbb{A}^{n+1} - \{0\} / \mathbb{A}^{\times}$ by this action.

Thinking of projective space as parametrizing a set of lines can be confusing at times, as it somewhat obscures the fact that $\mathbb{P}_{\mathbb{A}}^n$ is simply \mathbb{A}^n with extra “stuff” added at infinity. Let us thus think of projective space in terms of its coordinates.

Definition 2. Consider a point $p \in \mathbb{P}_{\mathbb{A}}^n$. Treating $\mathbb{P}_{\mathbb{A}}^n$ as a quotient space, we can think of p as the equivalence class of points on a line through \mathbb{A}^{n+1} . Suppose that this line passes through the point $\vec{x} = (x_0, \dots, x_n) \in \mathbb{A}^{n+1}$. We write **homogeneous coordinates** for p as

$$p = [x_0 : \dots : x_n] = [\lambda x_0 : \dots : \lambda x_n],$$

for any $\lambda \in \mathbb{A}^{\times}$. Note that these coordinates respect the action of \mathbb{A}^{\times} defining $\mathbb{P}_{\mathbb{A}}^n$ and hence are well-defined. Moreover, not all x_i can be zero.

With this coordinate system in hand, let us try to build an intuitive picture of projective space in the next few examples.

Example 1 (Real projective line). Let $\mathbb{A} = \mathbb{R}$ and consider $\mathbb{P}_{\mathbb{R}}^1$, the space of lines through the origin of \mathbb{R}^2 . It is clear that we can parametrize this space by the slopes of the lines everywhere except for the vertical line. This yields an \mathbb{R} 's worth of points, giving us the real line. If we now take the slope of the vertical line to be “infinity,” we obtain the whole projective space $\mathbb{P}_{\mathbb{R}}^1$. This is formalized by the homogeneous coordinates defined on $\mathbb{P}_{\mathbb{R}}^1$.

Consider a point $[x : y] \in \mathbb{P}_{\mathbb{R}}^1$. Suppose $y \neq 0$. Then, letting $\lambda = y^{-1}$,

$$[x : y] = y^{-1} \cdot [x, y] = [x/y : 1].$$

If we now note that x is free to range over all values in \mathbb{R} , we recover a copy of \mathbb{R} embedding in $\mathbb{P}_{\mathbb{R}}^1$. We can think of this as the set of lines with finite slopes. If $y = 0$, on the other hand, we can write

$$[x : 0] = x^{-1} \cdot [x : 0] = [1 : 0].$$

This implies that there is only one point in $\mathbb{P}_{\mathbb{R}}^1$ with $y = 0$, which can be thought of as the “point at infinity.” It is important to note that the point at infinity is fundamentally no “different” than the other points in $\mathbb{P}_{\mathbb{R}}^1$ – a concept that will become clear when we show that automorphisms of projective space can swap points at infinity with, say, the origin (in this case $[0 : 1]$). In this sense, we can think of the projective line as a disjoint union or a “compactification” $\mathbb{P}_{\mathbb{R}}^1 = \mathbb{R}^1 \sqcup \{\infty\}$ (with the topology extended to the quotient topology of $\mathbb{R}^2/\mathbb{R}^*$).

Example 2 (Real projective plane). Let $\mathbb{A} = \mathbb{R}$ and consider $\mathbb{P}_{\mathbb{R}}^2$, the space of lines through the origin of \mathbb{R}^3 . Using homogeneous coordinates, consider a point $[x : y : z] \in \mathbb{P}_{\mathbb{R}}^2$. If $z \neq 0$, we can scale points to be of the form $[x : y : 1]$, with x, y free to vary in \mathbb{R}^2 . This gives us a copy of $\mathbb{R}^2 \subset \mathbb{P}_{\mathbb{R}}^2$. Now suppose $z = 0$ and $y \neq 0$. In this subset, we can write points as $[x : 1 : 0]$, which gives us a copy of $\mathbb{R} \subset \mathbb{P}_{\mathbb{R}}^2$. This can be interpreted as a “line at infinity.” Finally, if $z = 0$ and $y = 0$, we obtain a single point $[1 : 0 : 0]$, interpreted as a “point at infinity.” As this exhausts all possibilities, we may write the projective plane as a disjoint union $\mathbb{P}_{\mathbb{R}}^2 = \mathbb{R}^2 \sqcup \mathbb{R}^1 \sqcup \{\infty\}$.

Example 3 (Riemann sphere). Let $\mathbb{A} = \mathbb{C}$ and consider $\mathbb{P}_{\mathbb{C}}^1$, the space of “complex lines” through the origin of \mathbb{C}^2 . Also known as the complex projective line, we can analyze $\mathbb{P}_{\mathbb{C}}^1$ in much the same way we dealt with the real projective line. Given a point $[z_1 : z_2] \in \mathbb{P}_{\mathbb{C}}^1$, we can consider two cases: either $z_2 = 0$, in which case we have a single point $[1 : 0]$, or $z_2 \neq 0$, in which case we get a complex line (or plane, depending on your terminology) of points $[z_1 : 1]$. In analogy to the real case, we can think of $\mathbb{P}_{\mathbb{C}}^1$ as the usual complex plane along with a point at infinity.

One way to visualize the complex projective line is through what is known as the Riemann sphere. Consider a sphere S^2 sitting tangent to the origin above the complex plane. It is easy to check that lines passing through both the north pole of the sphere and the complex plane give a bijection between $S^2 \setminus \{N\}$ and \mathbb{C} – this is known as the stereographic projection. This bijection can be extended to a bijection between the whole sphere S^2 and $\mathbb{P}_{\mathbb{C}}^1$ by treating the north pole as the point at infinity. This yields a nice visualization of the complex projective line.

Remark. The above examples hint at a pattern for real projective spaces: $\mathbb{P}_{\mathbb{R}}^n = \mathbb{R}^n \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \{\infty\}$. Indeed, the reader familiar with topology will note that this is precisely the usual cell decomposition for $\mathbb{P}_{\mathbb{R}}^n$. Similar statements apply for the complex case.

Automorphisms of projective space

Projective spaces will turn out to be an excellent backdrop for the classical algebraic geometry we will be discussing, and so, among other things, it will be useful to understand the automorphisms of projective spaces.

When discussing spaces such as \mathbb{R}^n or \mathbb{C}^n we typically treat them as finite-dimensional vector spaces, which places us firmly in the realm of linear algebra: objects are vector spaces and morphisms between them are linear maps. In this category, automorphisms are simply invertible linear maps from a vector space to itself. This set of automorphisms in fact forms a group, known as the general linear group GL_n , i.e. the group of $n \times n$ matrices with non-zero determinant.

Unfortunately, projective spaces $\mathbb{P}_{\mathbb{A}}^n$ do not live in the category of vector spaces, as there is no canonical way to add points. There is, however, a well-defined action of $GL_{n+1}(\mathbb{A})$ on $\mathbb{P}_{\mathbb{A}}^n$ given – just as in the case of vector spaces – by matrix multiplication.

Definition 3. We define the action of $GL_{n+1}(\mathbb{A})$ on $\mathbb{P}_{\mathbb{A}}^n$ by matrix multiplication,

$$\begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{00}x_0 + \cdots + a_{0n}x_n \\ \vdots \\ a_{n0}x_0 + \cdots + a_{nn}x_n \end{bmatrix},$$

for some $[x_0 : \cdots : x_n] \in \mathbb{P}_{\mathbb{A}}^n$.

Example 4. Consider the action of $GL_2(\mathbb{R})$ on $\mathbb{P}_{\mathbb{R}}^1$ given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

Geometrically, this automorphism “inverts” the projective line. It swaps the origin $[0 : 1]$ and infinity $[1 : 0]$ and sends points on the real line $[x : 1]$ to $[1 : x] = [x^{-1} : 1]$.

Example 5. Consider again the action of $GL_2(\mathbb{R})$ on $\mathbb{P}_{\mathbb{R}}^1$ but this time given by

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

More generally, any scalar matrix $\lambda \cdot \text{Id}$ (for $\lambda \in \mathbb{A}^\times$) will act on $\mathbb{P}_{\mathbb{A}}^n$ as the identity, due to the scaling laws of projective space.

The previous example highlights a redundancy in the action of $GL_{n+1}(\mathbb{A})$ on $\mathbb{P}_{\mathbb{A}}^n$: there are non-trivial elements of the group that act trivially on the space. In the interest of finding the group of self-maps that most accurately “represents” projective space, then, we should consider elements of $GL_{n+1}(\mathbb{A})$ only up to scalar multiples. This motivates the following definition.

Definition 4. Let $\lambda \cdot \text{Id}$ be the (normal) subgroup of scalar matrices in $GL_n(\mathbb{A})$. Define the **projective general linear** group of degree n , $PGL_n(\mathbb{A})$, by the quotient

$$PGL_n(\mathbb{A}) = GL_n(\mathbb{A}) / \lambda \cdot \text{Id}.$$

It is worth mentioning at this point that the notion of an automorphism of projective space has not yet been defined. Indeed, we have not even decided what kind of objects we should treat projective spaces as! In the context of curves in projective space however, it is useful to treat $\mathbb{P}_{\mathbb{A}}^n$ as a projective algebraic set. Recall that a projective algebraic set is a set of points in projective space cut out by the vanishing of a set of homogeneous polynomials¹; in particular, a projective algebraic curve is the set of points in $\mathbb{P}_{\mathbb{A}}^2$ satisfying $f(x_0, x_1, x_2) = 0$, for some homogeneous f . Then it is clear that choosing $f = 0$ yields the entire projective plane, from which it follows that projective spaces are projective algebraic sets.

When one introduces objects of a category (in this case, projective algebraic sets), one also introduces maps, or morphisms, between them. Unfortunately, we have not yet built up enough machinery to define morphisms between projective algebraic sets. It should be a welcome surprise, then, that in the case of $\mathbb{P}_{\mathbb{A}}^n$ we do not *need* to understand such morphisms, as the following theorem shows.

Theorem 1. *Let $\text{Aut}(\mathbb{P}_{\mathbb{A}}^n)$ be the group of automorphisms of $\mathbb{P}_{\mathbb{A}}^n$. Then*

$$\text{Aut}(\mathbb{P}_{\mathbb{A}}^n) = PGL_{n+1}(\mathbb{A}),$$

where the action of $PGL_{n+1}(\mathbb{A})$ on $\mathbb{P}_{\mathbb{A}}^n$ is by matrix multiplication as described above.

Hence we will stay away from the machinery of morphisms and instead view automorphisms of projective space simply as actions of $PGL_{n+1}(\mathbb{A})$, which are similar to the more familiar linear transformations of vector spaces. We will not prove the above theorem, as it requires a surprising amount of algebraic geometry².

¹Convince yourself that it makes sense to think about the *vanishing* of homogeneous polynomials in projective coordinates, but that the *value* of an arbitrary polynomial is not well-defined.

²For a proof in the scheme-theoretic case, see Hartshorne ch. II expm. 7.1.1. For the proof for $n = 1$ in the case of varieties, see Hartshorne ch. I ex. 6.6

Basic examples

Let us now turn to some examples, properties, and uses of automorphisms of projective space. In the context of projective algebraic curves, at least, automorphisms are natural in the sense that they preserve certain basic properties of curves.

Remark. Automorphisms preserve the degree of curves. Furthermore, they take singular points to singular points, i.e. preserve tangency, etc.³

Indeed, automorphisms act so naturally on curves that we often make use of them without stating it. Making a change of variables in the polynomial equations defining a curve, for example, is a basic example of an automorphism that we almost never explicitly acknowledge.

Example 6 (Parallel lines meet at infinity). The prototypical example of two curves intersecting is, of course, that of two lines that would otherwise be parallel in affine space. Consider the lines $y = 0$ and $y - 1 = 0$ in \mathbb{A}^2 . Extending the affine plane by adding points at infinity, we obtain the projective plane $\mathbb{P}_{\mathbb{A}}^2$. In the homogeneous coordinates $[x : y : z]$, we find that the equations for the lines become $y = 0$ and $y - z = 0$. Solving for the intersections, we find that z must be zero, while $x \in \mathbb{A}$ is free to vary. This yields the solution $[x : 0 : 0] = [1 : 0 : 0]$ (we are free to scale by x as it cannot vanish simultaneously with y and z). In other words, two “parallel” lines meet at the point at infinity given by the homogeneous coordinates $[1 : 0 : 0]$. Now let us apply the automorphism taking:

$$x \mapsto z, y \mapsto x, z \mapsto y.$$

I urge you to check that this is indeed an automorphism of projective space by constructing the appropriate matrix in the sense described in the previous section. The equations of our line now become $x = 0$ and $x - y = 0$, which intersect at the origin $[0 : 0 : 1]$. This automorphism has moved around the points of projective space, and in particular, taken the point at infinity to the origin. This allows us to better visualize the behavior of the two lines at infinity.

Example 7 (Asymptotes as tangents at infinity). For a slightly less trivial example, consider the hyperbola $x^2 - y^2 - 1 = 0$ in the affine plane \mathbb{R}^2 . It is not *a priori* clear what happens to the hyperbola at infinity. Homogenizing yields $x^2 - y^2 - z^2 = 0$, and we immediately see that there are two points on the hyperbola at infinity: $[1 : 1 : 0]$ and $[1 : -1 : 0]$. Is it now natural to ask how the asymptotes $y = x, y = -x$ behave at these points at infinity. One can check that each point is on one of the lines, but it is not clear whether this occurs as a simple intersection or as a tangency. To see what is happening visually, we can apply the automorphism from the last example, which gives us $z^2 - x^2 - y^2 = 0$. Dehomogenizing this (to study the affine behavior) gives us $x^2 + y^2 - 1 = 0$, a circle. Meanwhile, the lines $y - x = 0$ and $y + x = 0$ are sent to $x - z = 0$ and $x + z = 0$, which dehomogenize to $x = 1$ and $x = -1$. These two lines are clearly tangent to the circle, and hence we see, via application of an automorphism, that the asymptotes are tangent to the hyperbola at infinity.

Example 8 (Circle inversions). Consider the complex projective line $\mathbb{P}_{\mathbb{C}}^1$ with homogeneous coordinates $[z_0 : z_1]$ and the automorphism that takes $[z_0 : z_1] \mapsto [z_1 : z_0]$. For $z_1, z_2 \neq 0$, we find that

$$[z_0 : z_1] = [z_0/z_1 : 1] \mapsto [z_1 : z_0] = [z_1/z_0 : 1].$$

For $z_2 = 0$, $[1 : 0] \mapsto [0 : 1]$, i.e. the origin and the point at infinity are swapped.

³As we have not yet formally defined these terms, I will not prove these statements, as this will be done later in the seminar.

In terms of the geometry of the complex plane, this is a “geometric inversion” about the unit circle followed by complex conjugation. Those familiar with Euclidean plane geometry will recall that geometric inversion is a trick that allows one to invert the plane about a given circle. This inversion preserves the measure of angles (but not the direction), and hence is often used to make difficult problems in geometry more tractable. The complex inversion we have defined above, however, is geometric inversion followed by complex conjugation, but still preserves the non-trivial properties of geometric inversion.

In the category of vector spaces, automorphisms are uniquely determined by the image of a basis. Things are not as straightforward for projective space, as asking the same question is analogous to asking how many one-dimensional subspaces of \mathbb{A}^{n+1} determine an automorphism of $\mathbb{P}_{\mathbb{A}}^n$. Instead, one is forced to go back to the vector space \mathbb{A}^{n+1} underlying $\mathbb{P}_{\mathbb{A}}^n$, find the unique linear transformation in $GL_{n+1}(\mathbb{A})$, and descend to an element of $PGL_{n+1}(\mathbb{A})$. This motivates the following definition.

Definition 5. Let x_0, \dots, x_{n+1} be $n + 2$ points in $\mathbb{P}_{\mathbb{A}}^n$. We say that these points are in **general position** if no $n + 1$ of them lie in a hyperplane $\mathbb{P}_{\mathbb{A}}^{n-1}$. Equivalently these points are in general position if each subset of $n + 1$ points has representative vectors that are linearly independent in \mathbb{A}^{n+1} .

Example 9. Consider the projective line $\mathbb{P}_{\mathbb{A}}^1$. Hyperplanes $\mathbb{P}_{\mathbb{A}}^0$ are just points (can you see why?⁴) and thus it is impossible for two distinct points of $\mathbb{P}_{\mathbb{A}}^1$ to lie in the same hyperplane. Consequently, any three distinct points in $\mathbb{P}_{\mathbb{A}}^1$ are in general position.

On the other hand, four points in $\mathbb{P}_{\mathbb{A}}^2$, are in general position if no three of them are collinear in $\mathbb{P}_{\mathbb{A}}^2$. This is because three points being collinear in the projective plane is equivalent to the corresponding lines in \mathbb{A}^3 lying on the same plane through the origin (any choice of vector representatives will be dependent).

This brings us to the main result of the note.

Theorem 2 (First fundamental theorem of projective geometry). *Let x_0, \dots, x_{n+1} and y_0, \dots, y_{n+1} be in general position in $\mathbb{P}_{\mathbb{A}}^n$. Then there exists a unique automorphism $\Gamma : \mathbb{P}_{\mathbb{A}}^n \rightarrow \mathbb{P}_{\mathbb{A}}^n$ such that $\Gamma(x_i) = y_i$.*

Remark. A simple corollary of this theorem is that three points in $\mathbb{P}_{\mathbb{A}}^1$ determine an automorphism. The usual choice of three points is $[0 : 1]$, $[1 : 1]$, and $[1 : 0]$. In other words, there is a unique automorphism of $\mathbb{P}_{\mathbb{A}}^1$ sending three distinct points to 0, 1, and ∞ . Note, for example, that in the case of $\mathbb{A} = \mathbb{F}_2$, the finite field with two elements, these are the only three points in $\mathbb{P}_{\mathbb{F}_2}^1$, and hence $PGL_2(\mathbb{F}_2) \cong S_3$, the symmetric group on three letters.

Proof of theorem. We start with existence. Fix representatives v_0, \dots, v_{n+1} for x_0, \dots, x_{n+1} . By construction, the first $n + 1$ are independent (in \mathbb{A}^{n+1}), and hence we can write

$$v_{n+1} = \sum_{i=0}^n \lambda_i v_i,$$

for some $\lambda_i \in \mathbb{A}$. Furthermore, note that $\lambda_i \neq 0$ for all i , as otherwise, we would find a linear dependence between $n + 1$ of the vectors, contradicting general position. Hence if we rescale each

⁴We would define $\mathbb{P}_{\mathbb{A}}^0$ to be the space of lines in \mathbb{A}^1 – but of course there is only one such line, and hence $\mathbb{P}_{\mathbb{A}}^0$ is, as a space, a single point.

representative v_i (except for v_{n+1}) by λ_i^{-1} , we find that

$$v_{n+1} = \sum_{i=0}^n v_i.$$

Furthermore, these v_i are unique by linear independence – if $\sum_{i=0}^n v_i = \sum_{i=0}^n \mu_i v_i$, then $\mu_i = 1$. We can do the same for y_0, \dots, y_{n+1} by choosing representatives w_i such that

$$w_{n+1} = \sum_{i=0}^n w_i,$$

uniquely. Now, since v_0, \dots, v_n are linearly independent, they form a basis for \mathbb{A}^{n+1} , and hence determine a (unique) linear transformation $T : \mathbb{A}^{n+1} \rightarrow \mathbb{A}^{n+1}$ satisfying $T(v_i) = w_i$. By independence of w_0, \dots, w_n , we see that T is in fact invertible, i.e. $T \in GL_{n+1}(\mathbb{A})$. We denote by Γ the image of T under the quotient map $GL_{n+1}(\mathbb{A}) \twoheadrightarrow PGL_{n+1}(\mathbb{A})$. It is clear that Γ is the desired automorphism.

Uniqueness is straightforward: suppose $T' : \mathbb{A}^{n+1} \rightarrow \mathbb{A}^{n+1}$ provides another automorphism $\Gamma' : \mathbb{P}_{\mathbb{A}}^n \rightarrow \mathbb{P}_{\mathbb{A}}^n$ such that $\Gamma'(x_i) = y_i$. Then $T'(v_i) = \mu_i w_i$ for some scalar multiples $w_i \in \mathbb{A}$, and we find that

$$\mu_{n+1} w_{n+1} = T'(v_{n+1}) = \sum_{i=0}^n T'(v_i) = \sum_{i=0}^n \mu_i w_i.$$

By uniqueness of the expression $w_{n+1} = \sum_{i=0}^n w_i$, forcing $\mu_i / \mu_{n+1} = 1$, i.e. $T' = \mu_i / \mu_{n+1} T$, and hence T and T' must have the same image in PGL_{n+1} : $\Gamma' = \Gamma$. \square

Exercise 1. Show that $n+2$ points in $\mathbb{P}_{\mathbb{A}}^n$ are in general position if and only if they are the image under the quotient map $\mathbb{A}^{n+1} \twoheadrightarrow \mathbb{P}_{\mathbb{A}}^n$ of $\{e_0, \dots, e_n, e_0 + \dots + e_n\}$, where $\{e_i\}$ is a basis for \mathbb{A}^{n+1} (hint: see proof of theorem).

Acknowledgments

I would like to thank Ian Whitehead for running the undergraduate mathematics seminar that made this study possible, as well as for reviewing this note and providing me with some interesting examples.