Introduction to Algebraic Topology PSET 10

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Proposition 1. *Hatcher exercise 2.2.9*

Proof.

(a) Let X be the quotient of S^2 obtained by identifying the north and south poles to a point. X is homotopic to $S^2 \vee S^1$, and so fixing a cell complex on X yields the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where $d_1 = 0$ as the S^1 attached to the S^2 is a cycle, and $d_2 = 0$ as the degree of the map sending the two-cell comprising S^2 to the S^1 is zero (the map is not surjective). Hence $H_0(X) = H_1(X) = H_2(X) = \mathbb{Z}$.

(b) Let $X = S^1 \times (S^1 \vee S^1)$. We give X a cell complex in the obvious way, consisting of 2 2-cells, 3 1-cells, and 1 0-cell, which yields the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

where d_1 is 0, as all the 1-cells are loops. Similarly, $d_2 = 0$, as the boundary of each of the 2-cells traverses 1-cells in the usual way for the torus (by degree-counting). Hence we find that $H_0(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}^3$, and $H_2(X) = \mathbb{Z}^2$.

□

Proposition 2. *Hatcher exercise 2.2.22*

Proof. Given a cell structure on X, it suffices to show that $\tilde{X}$ can be given a cell structure with n times the number of i-cells as X, for all i. We proceed by induction. For the base case, it's clear that we can find $nc_0(X)$ 0-cells on $\tilde{X}$ by applying p^{-1} to the 0-cells on X. Moreover, $\tilde{X}$ can have no more than those 0-cells by the locally homeomorphic property. Now, as the induction step, we assume that $\tilde{X}$ has $nc_{i-1}(X)$ $(i-1)$-cells. Then, any i-cell in X is attached via a map $\phi_\alpha : D^i_\alpha \to X$. Contractibility of the disk allows us to use the lifting criterion (Hatcher Proposition 1.33) and thus there exists a lift $\tilde{\phi}_\alpha : D^i_\alpha \to \tilde{X}$ for each starting point of the lift. Since this is an n-sheeted cover there are n such starting points by the induction step, and hence there are n such lifts. This yields the desired cell structure. □

Proposition 3. *Hatcher exercise 2.2.28*

Proof.

(a) Let X be the space obtained from a torus T by attaching a Möbius band M via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Recall that $H_0(T) = H_2(T) = \mathbb{Z}, H_1(T) = \mathbb{Z}^2$, $H_0(M) = H_1(M) = \mathbb{Z}$, and $H_2(M) = 0$. Taking $T, M \subset X$ to be the appropriate subsets for the Mayer-Vietoris sequence and noting that $T \cap M = S^1$, we find the exact sequence

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus 0 \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{(1,0,2)} \mathbb{Z}^2 \oplus \mathbb{Z} \longrightarrow H_1(X) \longrightarrow \mathbb{Z} \longrightarrow \cdots$$

This yields the homology groups $H_2(X) = \mathbb{Z}, H_1(X) = \mathbb{Z}^2, H_0(X) = \mathbb{Z}$.

(b) Similar to part (a), using the fact that $H_0(\mathbb{RP}^2) = H_1(\mathbb{RP}^2) = \mathbb{Z}$ and $H_2(\mathbb{RP}^2) = 0$, we obtain the exact sequence

$$\cdots \longrightarrow 0 \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{(1,2)} \mathbb{Z}_2 \oplus \mathbb{Z} \longrightarrow H_1(X) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow \cdots$$

This yields the homology groups $H_2(X) = 0, H_1(X) = \mathbb{Z}_4, H_0(X) = \mathbb{Z}$.

$\square$

Proposition 4. *Hatcher exercise 2.2.31*

Proof. Consider the space $X \vee Y$ where the basepoints that are identified are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$. Then, taking $A = X \cup V$ and $B = Y \cup U$, we find that the interiors of A and B cover $X \vee Y$ and $A \cap B \subset X \vee Y$ deformation retracts to the basepoint, i.e. has trivial homology. The Mayer-Vietoris sequence is then simply

$$\cdots \longrightarrow 0 \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(X \vee Y) \longrightarrow 0 \longrightarrow \cdots$$

If we now note that $\tilde{H}_n(A) = \tilde{H}_n(X \cup V) = \tilde{H}_n(X)$ by contractibility of V and similarly for $\tilde{H}_n(B)$, we obtain isomorphisms $\tilde{H}_n(X) \oplus \tilde{H}_n(Y) \cong \tilde{H}_n(X \vee Y)$, as desired. $\square$

Proposition 5. *Hatcher exercise 2.2.32*

Proof. Let SX be the suspension of X, with $A, B \subset SX$ be neighborhoods of the two cones contained in SX. Clearly $A \cap B$ deformation retracts to X and A, B are contractible. The Mayer-Vietoris sequence is then simply

$$\cdots \longrightarrow 0 \longrightarrow H_n(SX) \longrightarrow H_{n-1}(X) \longrightarrow 0 \longrightarrow \cdots$$

which yields the isomorphisms $H_n(SX) \cong H_{n-1}(X)$, as desired. $\square$