Representation Theory: Final PSET

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Exercise 1. *Describe the invariants of the Weyl group action of type G_2 on $\mathbb{C}^2$.*

Solution. Recall that the Weyl group W_{G_2} of the group G_2 is the dihedral group of order 12, whose action on $\mathbb{C}^2$ is given by the usual plane rotations and reflections. By Chevalley-Shephard-Todd, we find that the invariant ring $A = \mathbb{C}[x,y]^{W_{G_2}}$ must be a polynomial ring generated by two fundamental invariants (as W_{G_2} is a reflection group of rank 2) $f(x,y)$ and $g(x,y)$ of degrees d_1 and d_2 respectively. Recall that the product $d_1 d_2$ must equal the order of the group, and the sum $d_1 + d_2 - 2$ must equal the number of reflections in W_{G_2}. Hence we find that the generators of the invariant ring A must be of degree 2 and 6. But then it is clear that the degree two generator must be $f = x^2 + y^2$, as the Weyl group action preserves the norm of a vector. Furthermore, the degree six generator must be $g = x^2(\sqrt{3}x/2 - y/2)^2(\sqrt{3}x/2 + y/2)^2$, as this polynomial is invariant under any permutation of the vertices of a hexagon (inscribed in the unit circle) and hence invariant under the W_{G_2} action. Hence $A = \mathbb{C}[x,y]^{W_{G_2}} = \mathbb{C}[f(x,y), g(x,y)]$. $\square$

Exercise 2. *Compute the Poincaré series of the algebra of invariants for S_3 acting on $\mathbb{C}^6 = \mathbb{C}^3 \otimes \mathbb{C}^2$ by permuting the coordinates in the first tensor factor.*

Solution. Recall Molien's formula for the Poincaré series of the algebra of invariants of $\mathbb{C}^6 = \operatorname{Specm} A$:

$$p_{A^{S_3}}(t) = \frac{1}{|S_3|} \sum_{\sigma \in S_3} \frac{1}{\det(\mathrm{Id} - t\sigma)}.$$

Note first that the summand is a class function, as $\det(\mathrm{Id} - t\tau\sigma\tau^{-1}) = \det(\tau(\mathrm{Id} - t\sigma)\tau^{-1}) = \det(\mathrm{Id} - t\sigma)$, and hence we can rewrite Molien's formula as a sum over conjugacy classes:

$$p_{A^{S_3}}(t) = \frac{1}{|S_3|} \sum_{[\sigma] \in S_3} \frac{|[\sigma]|}{\det(\mathrm{Id} - t[\sigma])}.$$

Fixing a basis for $\mathbb{C}^3 \otimes \mathbb{C}^2$ as $\{e_1 \otimes f_1, e_2 \otimes f_1, e_3 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2, e_3 \otimes f_3\}$, where e_i, f_i are the standard bases for $\mathbb{C}^3, \mathbb{C}^2$ respectively, we can write explicitly representatives of each conjugacy class of S_3:

$$\det(\mathrm{Id} - te) = \begin{vmatrix} 1-t & & & & & \\ & 1-t & & & & \\ & & 1-t & & & \\ & & & 1-t & & \\ & & & & 1-t & \\ & & & & & 1-t \end{vmatrix} = (1-t)^6,$$

$$\det(\mathrm{Id} - t(12)) = \begin{vmatrix} 1 & -t & & & & \\ -t & 1 & & & & \\ & & 1-t & & & \\ & & & 1 & -t & \\ & & & -t & 1 & \\ & & & & & 1-t \end{vmatrix} = (1-t)^2(1-t^2)^2,$$

$$\det(\mathrm{Id} - t(123)) = \begin{vmatrix} 1 & & -t & & & \\ -t & 1 & & & & \\ & -t & 1 & & & \\ & & & 1 & & -t \\ & & & -t & 1 & \\ & & & & -t & 1 \end{vmatrix} = (1-t^3)^2.$$

This yields

$$p_{A^{S_3}}(t) = \frac{1}{6}\left(\frac{1}{(1-t)^6} + \frac{3}{(1-t)^2(1-t^2)^2} + \frac{2}{(1-t^3)^2}\right).$$

$\square$

Exercise 3. *Recall that the character of an irreducible $GL(n)$-module with highest weight λ is called a Schur function:*

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\prod_{i<j}(x_i - x_j)}.$$

Expand $p_k s_\lambda$ in Schur functions, where $p_k = \sum x_i^k$.

Solution. Denoting the denominator of s_λ by Δ, we write

$$\begin{aligned}
\Delta p_k s_\lambda &= \sum_{l=1}^n x_l^k \det\left(x_i^{\lambda_j+n-j}\right) \\
&= \sum_{l=1}^n x_l^k \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_i^{\lambda_{\sigma(i)}+n-\sigma(i)} \\
&= \sum_{l=1}^n x_{\sigma^{-1}(l)}^k \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_i^{\lambda_{\sigma(i)}+n-\sigma(i)} \\
&= \sum_{l=1}^n \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_i^{\lambda_{\sigma(i)}+k\delta_{i,\sigma^{-1}(l)}+n-\sigma(i)} \\
&= \sum_{l=1}^n \sum_{\sigma\in S^n} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_i^{\lambda_{\sigma(i)}+k\delta_{\sigma(i),l}+n-\sigma(i)} \\
&= \sum_{l=1}^n \det\left(x_i^{\lambda_j+k\delta_{j,l}+n-j}\right).
\end{aligned}$$

In words, we find that multiplying s_λ by p_k yields a sum over l of determinants similar to the original determinant but now with an extra k added in the powers of the entries in the lth column. Unfortunately, these resulting determinants are not necessarily Schur functions, as the associated numbers $\lambda_j + k\delta_{j,l}$ may not form a partition (due to ordering issues). We fix this by rearranging the $\lambda_j + k\delta_{j,l} + n - j$ in decreasing order. Of course, if two terms in the sequence are equal, the determinant vanishes, so we may assume that for some $p \le l$ we have

$$\mu_{p-1} + n - p + 1 > \mu_l + n - l + r > \mu_p + n - p$$

and hence $\det\left(x_i^{\lambda_j+k\delta_{j,l}+n-j}\right) = (-1)^{l-p}\det\left(x_i^{\rho_j+n-j}\right)$, where ρ is the partition

$$\rho = (\lambda_1, \cdots, \lambda_{p-1}, \lambda_l + p - l + k, \lambda_p + 1, \cdots, \lambda_{l-1} + 1, \lambda_l, \cdots, \lambda_n).$$

It is now easy to check that $\theta = \rho - \lambda$ is a skew hook of length k. Now recall that the height $\operatorname{ht}(\theta)$ is one less than the number of rows θ spans, and hence we can write finally that

$$p_k s_\lambda = \sum_\rho (-1)^{\operatorname{ht}(\rho-\lambda)} s_\rho,$$

where the sum is over all partitions $\rho \supset \lambda$ such that $\rho - \lambda$ is a skew hook of length l. □

Exercise 4. *Show that*

$$s_{(\lambda_1,\dots,\lambda_n,0)}(x_1, \dots, x_n, 0) = s_\lambda(x_1, \dots, x_n).$$

This rule defines the Schur function in infinitely many variables, all but finitely many of which are zero.

Solution. Note that

$$s_{(\lambda_1,\dots,\lambda_n,0)}(x_1,\dots,x_n,x_{n+1}) = \frac{\begin{vmatrix} x_1^{\lambda_1+n} & \cdots & x_1^{\lambda_n+1} & 1 \\ \vdots & & \vdots & \vdots \\ x_n^{\lambda_1+n} & \cdots & x_n^{\lambda_n+1} & 1 \\ x_{n+1}^{\lambda_1+n} & \cdots & x_{n+1}^{\lambda_n+1} & 1 \end{vmatrix}}{\prod_{i<j}^{n+1}(x_i - x_j)}$$

and hence taking $x_{n+1} = 0$, we obtain

$$\begin{aligned} s_{(\lambda_1,\dots,\lambda_n,0)}(x_1,\dots,x_n,0) &= \frac{\begin{vmatrix} x_1^{\lambda_1+n} & \cdots & x_1^{\lambda_n+1} \\ \vdots & & \vdots \\ x_n^{\lambda_1+n} & \cdots & x_n^{\lambda_n+1} \end{vmatrix}}{\prod_i^n x_i \prod_{i<j}^n (x_i - x_j)} \\ &= s_\lambda(x_1,\dots,x_n), \end{aligned}$$

where in the last step we have factored out an x_i from the ith row of the determinant. □

Exercise 5. *Let $\mathbb{C}^\infty$ have a basis b_k for $k \in \mathbb{Z}$. Define the half-infinite wedge product $\Lambda^{\infty/2}\mathbb{C}^\infty$ of $\mathbb{C}^\infty$ to be the span of the vectors*

$$v_S = b_{s_1} \wedge b_{s_2} \wedge \cdots$$

where the set $S = \{s_1 > s_2 > \cdots\}$ contains finitely many positive integers and all but finitely many negative integers.

Let the matrix units $E_{ij} \in \mathfrak{gl}_\infty$ act on such monomials by the rules of linear algebra if $i \neq j$ and by

$$\left(\sum a_i \pi(E_{ii})\right) v_S = \left(\sum_{s\in S} a_s - \sum_{s\in\mathbb{Z}_{\le 0}} a_s\right) v_S$$

if $i = j$. Note that the right hand side yields a finite sum. Show that

$$[\pi(E_{ij}), \pi(E_{kl})] = \pi([E_{ij}, E_{kl}]) - (E_{ij}, [J, E_{kl}])$$

where $J = \sum_{k\le 0} E_{kk}$ and $(E_{ij}, E_{kl}) = \delta_{jk}\delta_{il}$ is the invariant bilinear form.

Solution. Let us first compute first the left-hand side. By the definition above, we see that

$$\pi(E_{ij}) = E_{ij} - \delta_{ij}\theta(i \le 0)\,\mathrm{Id},$$

and hence

$$[\pi(E_{ij}), \pi(E_{kl})] = [E_{ij}, E_{kl}]$$

as the identity commutes with everything. For the first term on right-hand side, we notice that

$$[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}$$

and so

$$\begin{aligned} \pi([E_{ij}, E_{kl}]) &= \delta_{kj}E_{il} - \delta_{il}\delta_{kj}\theta(i \le 0)\,\mathrm{Id} - \delta_{il}E_{kj} + \delta_{il}\delta_{kj}\theta(k \le 0)\,\mathrm{Id} \\ &= [E_{ij}, E_{kl}] - \delta_{il}\delta_{kj}(\theta(i \le 0) - \theta(k \le 0))\,\mathrm{Id}\,. \end{aligned}$$

Finally, if we compute the second term on the right-hand side, we have

$$\begin{aligned}(E_{ij}, [J, E_{kl}]) &= \left(E_{ij}, \sum_{m\leq 0}[E_{mm}, E_{kl}]\right)\\ &= \left(E_{ij}, \sum_{m\leq 0}\delta_{km}E_{ml} - \delta_{ml}E_{km}\right)\\ &= (E_{ij}, (\theta(k\leq 0) - \theta(l\leq 0))E_{kl})\\ &= \delta_{jk}\delta_{il}\left(\theta(k\leq 0) - \theta(i\leq 0)\right).\end{aligned}$$

Comparing this to the second term in the expression for $\pi([E_{ij}, E_{kl}])$ above, we obtain the required formula. □

Exercise 6. *Let v_λ be v_S for $S = \{\lambda_i - i + 1\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ as in Exercise 4. Show that the map that takes v_λ to the Schur function s_λ in infinitely many variables takes the operator*

$$\alpha_{-k} = \sum_{i\in\mathbb{Z}} E_{i,i-k}$$

to multiplication by the power-sum function p_k as in Exercise 3.

Proof. This is easy to see. Any given term $E_{j,j-k}$ in α_{-k} takes b_{j-k} and turns it into a b_j, i.e. raises it by k. Under the action of α_{-k}, we are thus left with a sum of finitely many terms (due to the "dense" negative basis vectors the action is zero on b_{-n-k} and lower), each of which now may be ordered incorrectly due to the transformation of b_{λ_j-j+1} to $b_{\lambda_j+n-j+1}$. Just as in exercise 3, however, the number of wedge flips needed to obtain the correct ordering is given by the height $\mathrm{ht}(\rho-\lambda)$ as defined previously, and hence we obtain precisely the multiplication by p_k. □