# Analysis I: Solutions to PSET 2

## Rudin 1.17

Take two vectors $x, y \in \mathbb{R}^k$. We claim that

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2,$$

which follows by the commutativity and distributivity of the dot product,

$$\begin{aligned}
|x+y|^2 + |x-y|^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y) \\
&= x\cdot x + 2x\cdot y + y\cdot y + x\cdot x - 2x\cdot y + y\cdot y \\
&= 2x\cdot x + 2y\cdot y \\
&= |x|^2 + |y|^2.
\end{aligned}$$

Interpreted geometrically, we find that the sum of the length-squareds of the diagonals of a parallelogram is equal to the sum of the squares of the side lengths.

## Rudin 2.2

We claim that the set of all algebraic numbers $A$ is countable. Consider the first the set of all polynomials $P$ with integer coefficients. The set $P$ admits a natural decomposition $P = \cup_n P_n$ where $P_n$ is the subset of polynomials with degree $n$. Any polynomial of degree $n$ is uniquely determined by its $n+1$ coefficients and hence is equivalent to the set of $(n+1)$-tuples of integers. This set is countable (c.f. Rudin Theorem 2.13) and hence $P_n$ is countable.

Given any polynomial $p \in P_n$, the set of its roots $R_p$ is finite (by the fundamental theorem of algebra) and hence we can write

$$A = \bigcup_{p\in P} R_p = \bigcup_{n\in\mathbb{N}} \left( \bigcup_{p\in P_n} R_p \right),$$

i.e. a countable union of a countable union of finite sets, which is countable (c.f. Rudin Theorem 2.12).

## Problem 3

We claim that the sets $(0,1) \subset \mathbb{R}, [0,1] \subset \mathbb{R}$, and $\mathbb{R}$ are equivalent. We show only two equivalences, using the fact that equivalence is an equivalence relation to obtain the third.

We know from precalculus that the function $\tan x : \mathbb{R} \to \mathbb{R}$, when restricted to $(-\pi/2, \pi/2)$ is bijective, with inverse $\tan^{-1} x$. This gives an equivalence between $(-\pi/2, \pi/2) \subset \mathbb{R}$ and $\mathbb{R}$. Moreover, $(-\pi/2, \pi/2)$ is equivalent to $(0,1)$ by the map $x \mapsto (x + \pi/2)/\pi$, which is obviously bijective. This establishes the equivalence of $(0,1)$ and $\mathbb{R}$.

Consider now the function $g : [0,1] \to (0,1)$ given piecewise as

$$g(x) = \begin{cases} 1/2, & x = 0 \\ 1/4, & x = 1 \\ 1/2^{n+2}, & x = 1/2^n \text{ for } n \in \mathbb{N} \\ x, & \text{otherwise.} \end{cases}$$

Intuitively, this function takes advantage of the "space" available in the infinite sequence $1/2^n$ by sending the "extra" points 0 and 1 to $1/2$ and $1/4$, respectively, and then pushing down what normally would have been sent to powers of $1/2$ by a factor of $1/4$. Note that this trick can be done with any sequence tending to zero, not just $1/2^n$. Let us show that $g$ is bijective; it will take some case work, due to the piecewise definition.

First surjectivity; suppose $y \in (0,1)$. If $y$ is $1/2$ or $1/4$ we note that $y$ must be $g(0)$ or $g(1)$. If $y$ is more generally $1/2^n$ (for $n \in \mathbb{N}$ greater than 2), we find that $y = g(1/2^{n-2})$. Otherwise, $g(y) = y$.

Injectivity is clear, but a little tedious; suppose $x, y \in [0,1]$ with $g(x) = g(y)$. If either of $x$ or $y$ is zero, it is clear that $g(x) = g(y)$ forces $x = y = 0$, as only 0 is sent to $1/2$. Now suppose neither $x$ nor $y$ is zero. Note that $x$ and $y$ must both either be a power of $1/2$ or not, as otherwise $g(x)$ could not possibly be equal to $g(y)$. If neither $x$ nor $y$ are powers of $1/2$, it follows from the definition of $g$ that $x = y$. If both $x$ and $y$ are powers of $1/2$, say $x = 1/2^n$ and $y = 1/2^m$, then $g(x) = g(y)$ implies that $n = m$, i.e. $x = y$.

This proves the equivalence of $[0,1]$ and $(0,1)$. As $(0,1)$ is equivalent to $\mathbb{R}$ as shown above, all three sets are equivalent.

## Problem 4

Let $\{E_n\}$ be a sequence of countable sets, and $S = \prod_n E_n$ be their Cartesian product. Suppose $S$ is countable; then there exists a bijection $f : \mathbb{N} \to S$.

Note that each $i \in \mathbb{N}$ is taken to an infinite sequence of elements $\{e_{ij}\}$, i.e.

$$\begin{aligned}
f(1) &= (e_{11}, e_{12}, \ldots) \\
f(2) &= (e_{21}, e_{22}, \ldots) \\
&\vdots \\
f(i) &= (e_{i1}, e_{i2}, \ldots) \\
&\vdots
\end{aligned}$$

Now choose $a = (a_1, a_2, \ldots) \in S$ such that for each $k \in \mathbb{N}$, $a_k \neq e_{kk}$. Surjectivity of $f$ implies that there exists some $\ell \in \mathbb{N}$ such that $f(\ell) = a$. This is, of course, a contradiction, as $a_k \neq e_{kk}$ for each $k$. Hence $S$ must be uncountable (as $S$ is clearly not finite).

Similarly, if each $E_n = \{0, 1\}$, we assume that there exists a bijection $f : \mathbb{N} \to S = \prod_n E_n$. Surjectivity of $f$ implies the existence of some $\ell \in \mathbb{N}$ mapped to $a = (a_1, a_2, \ldots) \in S$ where $a_k = 0$ if $e_{kk} = 1$ and $a_k = 1$ if $e_{kk} = 0$. This is again a contradiction, and hence $S$ must be uncountable (as $S$ is clearly not finite).