Notes on Topological and Differentiable Manifolds

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1 Elementary Topology

Let us begin with the definition of a topology:

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying the following properties:

1. X and $\varnothing$ are elements of $\mathcal{T}$.
2. $\mathcal{T}$ is closed under finite intersections: If $U_1 \ldots U_n \in \mathcal{T}$, then their intersection $U_1 \cap \cdots \cap U_n$ is in $\mathcal{T}$.
3. $\mathcal{T}$ is closed under arbitrary unions: If $U_1 \ldots U_n \ldots$ is any (finite or infinite) collection of elements of $\mathcal{T}$, then their union $\cup_\alpha U_\alpha$ is in $\mathcal{T}$.

A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a **topological space**. The elements of a topological space are usually called its **points**.

Definition 2. If X is a topological space and $q \in X$, a **neighborhood** of q is just an open set containing q. More generally, a neigborhood of a subset $K \subset X$ is an open set containing K.

Definition 3. If X is a topological space and $\{q_i\}$ is any sequence of points in X, we say that the sequence **converges** to $q \in X$, and q is the **limit** of the sequence, if for every neighborhood U of q there exists N such that $q_i \in U$ for all $i \geq N$. We denote this as $q_i \to q$ or $\lim_{i\to\infty} q_i = q$.

Example 1. Let Y be a trivial topological space (i.e. the only open sets are X and $\varnothing$). Each point has only 1 neighborhood: X itself. Thus, any sequence can be entirely contained in the neighborhood X, and consequently, any sequence converges to any point in X.

Example 2. Let X be a discrete topological space (i.e. all every subset of X is open). Take any sequence of points $\{q_i\}$. If the sequence converges to q, every open set containing q must contain all but a finite elements of the sequence. By virtue of the discrete topology, there exists an open set that contains only q. Obviously, then, there must exist an N such that $q_i = q$ for all $i \geq N$. Consequently, the only convergent sequences in X are the ones that are "eventually constant."

Definition 4. If X and Y are topological spaces, a map $f : X \to Y$ is said to be **continuous** if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

Lemma 1. *Let X, Y, Z be topological spaces.*

1. *Any constant map $f : X \to Y$ is continuous.*
2. *The identity map $\mathrm{Id} : X \to X$ is continuous.*
3. *If $f : X \to Y$ is continuous, so is the restriction of f to any open subset of X.*
4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f : X \to Z$.*

Proof. Let us begin with the constant map. Suppose f maps X to the constant $\lambda \in Y$. We wish to show that the preimage of f of every open set U in Y is open. There are two cases: U either does or does not contain λ. If it does, $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \varnothing$. As both X and $\varnothing$ are open sets, f is continuous.

The continuity of the identity map follows trivially from the fact that Id maps any open set back to the same open set.

To prove the third statement, take any open set U in Y. U can be written as a union of points in and outside $f(V) \subset Y$: $U = U_i \cup U_o$. We want to show that $g^{-1}(U)$ is open in V. Since $g^{-1}(U_o) = \varnothing$, which is open, and $g^{-1}(U_i) \subset V$ and is open in X by the continuity of f, $g^{-1}(U_o \cup U_i) = g^{-1}(U_o) \cup g^{-1}(U_i)$ is open in V.

To prove the fourth statement, it suffices to show that $(g \circ f)^{-1}(U)$, with $U \subset Z$ open, is open in X. First note that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U)$ is an open set in Y. Similarly, f^{-1} of an open set in Y is open in X as f is continuous, and we are done. $\square$

Lemma 2 (Local Criterion for Continuity)**.** *A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous.*

Proof. If f is continuous, each point of X a neighborhood on which f is continuous; namely, X itself.

To prove the converse, suppose that each point of X has a neighborhood on which f is continuous - we wish to show that for any open set $U \subset Y$, $f^{-1}(U)$ is open in X. By continuity at each point, we know that any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous. In other words, $(f|_{V_x})^{-1}(U) = f^{-1}(U) \cap V_x$ is open in X and is contained in $f^{-1}(U)$. As $f^{-1}(U)$ is the union of such sets for all V_x, and these sets are open, it follows that $f^{-1}(U)$ is open, and we are done. □

Definition 5. If X and Y are topological spaces, a **homeomorphism** from X to Y is defined to be a continuous bijective map $\phi : X \to Y$ with continuous inverse. If there exists a homeomorphism betwee X and Y, we say that X and Y are **homeomorphic** or **topologically equivalent**. Sometimes this is abbreviated $X \approx Y$.

Exercise 1. *Show that homeomorphisms are an equivalence relation.*

Proof. To show that homeomorphisms are an equivalence relation, we show

- $X \approx X$: The identity map Id is a homeomorphism from X to X.
- $X \approx Y \implies Y \approx X$: There exists a homeomorphism from X to Y. Its inverse is clearly a homeomorphism from Y to X.
- $X \approx Y$ and $Y \approx Z \implies X \approx Z$: As the composition of two homeomorphisms is also a homeomorphism (from elementary set theory), the homeomorphism from X to Z is simply the composition of the homeomorphisms from Y to Z and from X to Y, respectively.

□

Example 3. Any open ball in $\mathbb{R}^n$ is homeomorphic to any other open ball. The homeomorphism can be constructed simply by composition translations $x \mapsto x + x_0$ and dilations $x \mapsto cx$. This shows that size is not a topological property.

Example 4. If $\mathbb{B}^n$ is the open unit ball, we can define $F : \mathbb{B}^n \to \mathbb{R}^n$ by

$$y = F(x) = \frac{x}{1 - |x|^2}.$$

The inverse is given by

$$x = F^{-1}(y) = \frac{2y}{1 + \sqrt{1 + 4|y|^2}}.$$

As both are continuous and bijective, F is a homeomorphism, so $\mathbb{R}^n$ is homeomorphic to $\mathbb{B}^n$. This shows that boundedness is not a topological property.

Example 5. Take the surface of the unit sphere in $\mathbb{R}^3$ and the surface of the cube of side 2, centered at the origin. There exists a homeomorphism between these two surfaces,

$$\phi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

whose inverse is given by

$$\phi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Thus, corners are not a topological property either.