Notes on Topological and Differentiable Manifolds

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1 Elementary Topology

Let us begin with the definition of a topology:

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying the following properties:

1. X and $\varnothing$ are elements of $\mathcal{T}$.

2. $\mathcal{T}$ is closed under finite intersections: If $U_1 \dots U_n \in \mathcal{T}$, then their intersection $U_1 \cap \dots \cap U_n$ is in $\mathcal{T}$.

3. $\mathcal{T}$ is closed under arbitrary unions: If $U_1 \dots U_n \dots$ is any (finite or infinite) collection of elements of $\mathcal{T}$, then their union $\cup_\alpha U_\alpha$ is in $\mathcal{T}$.

A pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X is called a **topological space**. The elements of a topological space are usually called its **points**.

Definition 2. If X is a topological space and $q \in X$, a **neighborhood** of q is just an open set containing q. More generally, a neigborhood of a subset $K \subset X$ is an open set containing K.

Definition 3. If X is a topological space and $\{q_i\}$ is any sequence of points in X, we say that the sequence **converges** to $q \in X$, and q is the **limit** of the sequence, if for every neighborhood U of q there exists N such that $q_i \in U$ for all $i \geq N$. We denote this as $q_i \to q$ or $\lim_{i\to\infty} q_i = q$.

Example 1. Let Y be a trivial topological space (i.e. the only open sets are X and $\varnothing$). Each point has only 1 neighborhood: X itself. Thus, any sequence can be entirely contained in the neighborhood X, and consequently, any sequence converges to any point in X.

Example 2. Let X be a discrete topological space (i.e. all every subset of X is open). Take any sequence of points $\{q_i\}$. If the sequence converges to q, every open set containing q must contain all but a finite elements of the sequence. By virtue of the discrete topology, there exists an open set that contains only q. Obviously, then, there must exist an N such that $q_i = q$ for all $i \geq N$. Consequently, the only convergent sequences in X are the ones that are "eventually constant."

Definition 4. If X and Y are topological spaces, a map $f : X \to Y$ is said to be **continuous** if for every open set $U \subset Y$, $f^{-1}(U)$ is open in X.

Lemma 1. *Let X, Y, Z be topological spaces.*

1. *Any constant map $f : X \to Y$ is continuous.*

2. *The identity map $\mathrm{Id} : X \to X$ is continuous.*

3. *If $f : X \to Y$ is continuous, so is the restriction of f to any open subset of X.*

4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, so is their composition $g \circ f : X \to Z$.*

Proof. Let us begin with the constant map. Suppose f maps X to the constant $\lambda \in Y$. We wish to show that the preimage of f of every open set U in Y is open. There are two cases: U either does or does not contain λ. If it does, $f^{-1}(U) = X$; otherwise, $f^{-1}(U) = \varnothing$. As both X and $\varnothing$ are open sets, f is continuous.

The continuity of the identity map follows trivially from the fact that Id maps any open set back to the same open set.

To prove the third statement, take any open set U in Y. U can be written as a union of points in and outside $f(V) \subset Y$: $U = U_i \cup U_o$. We want to show that $g^{-1}(U)$ is open in V. Since $g^{-1}(U_o) = \varnothing$, which is open, and $g^{-1}(U_i) \subset V$ and is open in X by the continuity of f, $g^{-1}(U_o \cup U_i) = g^{-1}(U_o) \cup g^{-1}(U_i)$ is open in V.

To prove the fourth statement, it suffices to show that $(g \circ f)^{-1}(U)$, with $U \subset Z$ open, is open in X. First note that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since g is continuous, $g^{-1}(U)$ is an open set in Y. Similarly, f^{-1} of an open set in Y is open in X as f is continuous, and we are done. $\square$

Lemma 2 (Local Criterion for Continuity)**.** *A map $f : X \to Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which (the restriction of) f is continuous.*

Proof. If f is continuous, each point of X a neighborhood on which f is continuous; namely, X itself.

To prove the converse, suppose that each point of X has a neighborhood on which f is continuous - we wish to show that for any open set $U \subset Y$, $f^{-1}(U)$ is open in X. By continuity at each point, we know that any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous. In other words, $(f|_{V_x})^{-1}(U) = f^{-1}(U) \cap V_x$ is open in X and is contained in $f^{-1}(U)$. As $f^{-1}(U)$ is the union of such sets for all V_x, and these sets are open, it follows that $f^{-1}(U)$ is open, and we are done. □

Definition 5. If X and Y are topological spaces, a **homeomorphism** from X to Y is defined to be a continuous bijective map $\phi : X \to Y$ with continuous inverse. If there exists a homeomorphism betwee X and Y, we say that X and Y are **homeomorphic** or **topologically equivalent**. Sometimes this is abbreviated $X \approx Y$.

Exercise 1. *Show that homeomorphisms are an equivalence relation.*

Proof. To show that homeomorphisms are an equivalence relation, we show

- $X \approx X$: The identity map Id is a homeomorphism from X to X.
- $X \approx Y \implies Y \approx X$: There exists a homeomorphism from X to Y. Its inverse is clearly a homeomorphism from Y to X.
- $X \approx Y$ and $Y \approx Z \implies X \approx Z$: As the composition of two homeomorphisms is also a homeomorphism (from elementary set theory), the homeomorphism from X to Z is simply the composition of the homeomorphisms from Y to Z and from X to Y, respectively.

□

Example 3. Any open ball in $\mathbb{R}^n$ is homeomorphic to any other open ball. The homeomorphism can be constructed simply by composition translations $x \mapsto x + x_0$ and dilations $x \mapsto cx$. This shows that size is not a topological property.

Example 4. If $\mathbb{B}^n$ is the open unit ball, we can define $F : \mathbb{B}^n \to \mathbb{R}^n$ by

$$y = F(x) = \frac{x}{1 - |x|^2}.$$

The inverse is given by

$$x = F^{-1}(y) = \frac{2y}{1 + \sqrt{1 + 4|y|^2}}.$$

As both are continuous and bijective, F is a homeomorphism, so $\mathbb{R}^n$ is homeomorphic to $\mathbb{B}^n$. This shows that boundedness is not a topological property.

Example 5. Take the surface of the unit sphere in $\mathbb{R}^3$ and the surface of the cube of side 2, centered at the origin. There exists a homeomorphism between these two surfaces,

$$\phi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

whose inverse is given by

$$\phi^{-1}(x, y, z) = \frac{(x, y, z)}{\max(|x|, |y|, |z|)}.$$

Thus, corners are not a topological property either.

Example 6. Now for an example of a continuous bijection that is not a homeomorphism by failure of its inverse to be continuous. Let X be the interval $[0, 1) \subset \mathbb{R}$, and let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2$ (both with the Euclidean metric topologies). Define a map $a : X \to \mathbb{S}^1$ by $a(t) = (\cos 2\pi t, \sin 2\pi t)$. It is clear that the map is continuous and bijective. The inverse, however, is not continuous. To see why, take any neighborhood of the point $(1, 0)$ - the inverse of a on this neighborhood will inevitably contain the point $0 \in [0, 1)$. Thus the preimage of open sets is not necessarily open, and thus a^{-1} is not continuous.

Definition 6. A map $f : X \to Y$ is said to be an **open map** if for any open set $U \subset X$, the image set $f(U)$ is open in Y. A map can be open but not continuous, continuous but not open, both, or neither.

Definition 7. We say that a continuous map $f : X \to Y$ between topological spaces is a **local homeomorphism** if every point $x \in X$ has a neighborhood $U \subset X$ such that $f(U)$ is an open subset of Y and $f|_U : U \to f(U)$ is a homeomorphism.

Exercise 2. *Show that:*

1. *every local homeomorphism is an open map.*
2. *every homeomorphism is a local homeomorphism.*
3. *a bijective continuous open map is a homeomorphism.*

4. a bijective local homeomorphism is a homeomorphism.

Proof.

1. Take any open set $U \in X$. For every $x \in U$, there exists some neighborhood $V = U_x \cap U$ of x for which the local homeomorphism $f(V)$ is open. Note that U is the union of all such V (for various x) and thus $f(U)$ is the union of all $f(V)$, and as the union of open sets must be an open set, $f(U)$ must be open. Consequently, f is an open map.

2. Take f to be a homeomorphism from X to Y. f is trivially a local homeomorphism; the neighborhood needed by the definition is simply X itself.

3. Let f be a bijective, continuous, open map. To show that f is a homeomorphism, we must show that f^{-1} is a continuous map. In other words, we must show that the preimage of f on open sets of Y are open sets in X. This is true as f is open and bijective: open sets in X are taken to open sets in Y (open), and all open sets in Y are images of open sets in X (bijective).

4. Let us first show that any local homeomorphism f from X to Y is continuous. Take any open set $V \subset Y$ whose preimage under f we call U. We wish to show that U is open. Take any point $y \in V$. For any $x \in f^{-1}(y)$, there is a neighborhood $M_x \subset X$ of x that is homeomorphic to a neighborhood N_y of y in Y. This implies that $U_x = M_x \cap U$ is homeomorphic to $V_y = N_y \cap V$. Take the union of all the sets U_x for every x in the preimage of y and call this W_y. Note that the preimage of V is the union of all such W_y for $y \in V$. This union is open, and we are done.
   Now it remains to show that the inverse of f (that exists via bijectivity) is continuous. Let $U \subset X$ be open and $V = (f^{-1})^{-1}(U) = f(U)$. We wish to show that V is open. This follows trivially from the fact that f is an open map, and we are done.

□

Up until now, we have worked with topological spaces defined through open sets. There is a complementary notion that is just as important.

Definition 8. A subset F of a topological space X is said to be **closed** if its complement $X \setminus F$ is open. It follows immediately from the definition of topological spaces that

1. X and $\varnothing$ are closed.

2. Finite unions of closed sets are closed.

3. Arbitrary intersections of closed sets are closed.

A topology on a set X can be defined by describing the collection of closed sets, as long as they satisfy these three properties; the open sets are then just those sets whose complements are closed.

Example 7 (Closed Sets)**.**

- Any closed interval $[a, b] \subset \mathbb{R}$ is a closed set, as are the half-infinite closed intervals $[a, \infty)$ and $(-\infty, b]$.

- Any closed ball in a metric space is a closed set.

- Every subset of a discrete space is closed.

It is important to not that closed is *not* the same as "not open," as sets can be both open and closed (such as $X, \varnothing$), or neither open nor closed, such as $[0, 1) \subset \mathbb{R}$.

Lemma 3. *A map between topological spaces is continuous if and only if the inverse image of every closed set is closed.*

Proof. Assume $f : X \to Y$ is continuous. Given $V \subset Y$ is closed, and its preimage under f is $U \subset X$, we want to show that U is closed. By definition, $Y \setminus V$ is open, and as by continuity of f, we have that its preimage $X \setminus U$ is open. Thus, again by definition of closed sets, U must be closed.

Now assume that $f : X \to Y$ is such that the preimage of any closed set Y is a closed set in X. Take an open set $V \subset Y$ whose preimage under f is $U \subset X$. The preimage of $Y \setminus V$ is, of course $X \setminus U$, which are both closed. Since $X \setminus U$ is closed, U must be open, and we are done. $\square$

Definition 9. Given any set $A \subset X$, we define several related sets as follows. The **closure** of A in X, denoted by $\bar{A}$, is the set

$$\bar{A} = \bigcap \{B \subset X : B \supset A \text{ and } B \text{ is closed in } X\}.$$

The **interior** of A, written $\operatorname{Int} A$ is

$$\operatorname{Int} A = \bigcup \{C \subset X : C \subset A \text{ and } C \text{ is open in } X\}.$$

It is obvious that $\bar{A}$ is closed and $\operatorname{Int} A$ is open. In words, $\bar{A}$ is the "smallest closed set containing A," and $\operatorname{Int} A$ is "the largest open set contained in A." We also define the **exterior** of A, written $\operatorname{Ext} A$, as

$$\operatorname{Ext} A = X \setminus \bar{A},$$

and the **boundary** of A, written ∂A, as

$$\partial A = X \setminus (\operatorname{Int} A \cup \operatorname{Ext} A)$$

It follows from the definitions that for any subset $A \subset X$, the whole space X is equal to the disjoint union of $\operatorname{Int} A, \operatorname{Ext} A$, and ∂A. The set A always contains all of its interior points and none of its exterior points, and may contain all, some, or none of its boundary points.

Lemma 4. *Let X be a topological space and $A \subset X$ any subset.*

1. *A point q is in the interior of A if and only if q has a neighborhood contained in A.*
2. *A point q is in the exterior of A if and only if q has a neighborhood contained in $X \setminus A$.*
3. *A point q is in the boundary of A if and only if q every neighborhood of q contains both a point of A and a point of $X \setminus A$.*
4. $\operatorname{Int} A$ *and* $\operatorname{Ext} A$ *are open in X, while ∂A is closed in X.*
5. *A is open if and only if $A = \operatorname{Int} A$.*
6. *A is closed if and only if it contains all its boundary points, which is true if and only if $A = \operatorname{Int} U \cup \partial A$.*
7. *$A = \bar{A} \cup \partial A = \operatorname{Int} A \cup \partial A$.*

Proof. Left to the reader. $\square$

Definition 10. Given a topological space X and a set $A \subset X$, we say that a point $q \in X$ is a **limit point** of A if every neighborhood of q contains a point of A other than q (which may or may not itself be in A). If we let $X = \mathbb{R}$ and $A = \{1/n\}_{n=1}^{\infty}$, for example, then 0 is the only limit point of A.

Exercise 3. *Show that a set A in a topological space is closed if and only if contains all of its limit points.*

Proof. It is clear from the previous lemma that every boundary point is also a limit point. In addition, it is clear that no limit point can also be outside of A. If A is closed, it must contain its boundary, and thus, since it contains its interior and its boundary, it contains all of its limit points. Conversely, if A contains all of its limit points, it contains its boundary, and is closed. $\square$

Definition 11. A subset A of a topological space X is said to be **dense** in X if $\bar{A} = X$.

Exercise 4. *Show that a subset $A \subset X$ is dense if and only if every nonempty open set in X contains a point of A.*

Proof. Suppose every nonempty open set in X contains a point of A. Every neighborhood of every point of X must then contain a point of A, and thus every point in X must be a limit point of A. $\bar{A} = \operatorname{Int} A \cup \partial A$ contains all of its limit points, and thus $X = \bar{A}$ and A is dense in X.

Conversely, suppose that A is dense in X. Then, $\bar{A} = X$, i.e. X is equal to the union of the interior and the boundary of A. Any point in the interior or the boundary of a set has the property that all of it neighborhoods contain a point in the interior. Thus, since any open set in X is a neighborhood of any point in X, any open set in X must contain a point of A. $\square$

Definition 12. A map $f : X \to Y$ is said to be **closed** if it takes closed sets in X to closed sets in Y.

Exercise 5. *Show that a bijective continuous map is a homeomorphism if and only if it is open if and only if it is closed.*

Proof. Suppose f is a bijective continuous open map. We want to show that the inverse f^{-1} is continuous: the preimage of open sets in X under f^{-1} (i.e. the image of f) must be open. Since f is open, we are done. Conversely, if f is a homeomorphism, f must be bijective and continuous. It remains to show that f is an open map. This follows from the fact that f^{-1} is continuous, similar to above.

Suppose f is a bijective continuous closed map. We want to show that the inverse f^{-1} is continuous: the preimage of closed sets in X under f^{-1} (i.e. the image of f) must be closed (by the definition of continuity in terms of closed sets). Since f is closed, we are done. Conversely, if f is a homeomorphism, f must be bijective and continuous. It remains to show that f is an closed map. This follows from the fact that f^{-1} is continuous, similar to above, using the definition in terms of closed sets. $\square$

Definition 13. Suppose X is any set. A **basis** in X is a collection $\mathcal{B}$ of subsets of X satisfying the following conditions:

1. Every element of X is in some element of $\mathcal{B}$; in other words, $X = \bigcup_{B \in \mathcal{B}} B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Theorem 5. *Let $\mathcal{B}$ be a basis in a set X and let $\mathcal{T}$ be the collection of all unions of elements of $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X. This topology $\mathcal{T}$ is called the* ***topology generated by*** *$\mathcal{B}$.*

Before we prove this theorem, let us develop a parallel definition to that of an open set: given X and a collection $\mathcal{B}$ of subsets of X, we say that a subset $U \subset X$ satisfies the **basis criterion** with respect to $\mathcal{B}$ if for every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Lemma 6. *Suppose $\mathcal{B}$ is a basis in X. Then $\mathcal{T}$, defined as above, is precisely the set of all subsets of X that satisfy the basis criterion with respect to $\mathcal{B}$.*

Proof. Let $U \subset X$, and suppose first that U satisfies the basis criterion. Let

$$V = \bigcup \{B \in \mathcal{B} : B \subset U\}.$$

$V \in \mathcal{T}$ as it is a union of basis sets. If we can show that $U = V$, we will have $U \in \mathcal{T}$ and we will be done. Clearly, $V \subset U$, as V is a union of subsets of U. We want to show that $U \subset V$. For any point $x \in U$, since U satifsfies the basis criterion, there must exist a basis set $B \in \mathcal{B}$ such that $x \in B \subset U$. It follows that $x \in V$, and we are done.

Conversely, suppose that $U \in \mathcal{T}$. Consequently, U is a union of elements of $\mathcal{B}$. U satisfies the basis criterion, as each $x \in U$ satisfies $x \in B \subset U$ for some $B \in \mathcal{B}$. $\square$

Proof. We now prove the earlier theorem. We want to show that the collection $\mathcal{T}$ satisfies the conditions for a topology. Since $X = \bigcup_{B \in \mathcal{B}} B$, $X \in \mathcal{T}$. The empty set is as well, as it is the "union of no elements" of $\mathcal{B}$. A union of elements of $\mathcal{T}$ is a union of unions of elements of $\mathcal{B}$, and therefore is a union of elements of $\mathcal{B}$, and thus $\mathcal{T}$ is closed uner arbitrary unions. To show that $\mathcal{T}$ is closed under finite intersections, suppose first that $U_1, U_2 \in \mathcal{T}$. Then, for any $x \in U_1 \cap U_2$, the basis criterion says that there exist basis elements $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By the definition of the basis, however, we know that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Thus $U_1 \cap U_2$ satisfies the basis criterion, so it is again in $\mathcal{T}$. This shows that $\mathcal{T}$ is closed under pairwise intersections, and closure under finite intersections follows via induction. $\square$

Lemma 7. *Suppose X is a topological space, and $\mathcal{B}$ is a collections of open subsets of X. If every open subset of X satisfies the basis criterion with respect to $\mathcal{B}$, then $\mathcal{B}$ is a basis for the topology of X.*

Proof. If every open subset satisfies the basis criterion, the previous lemma tells us that the collection $\mathcal{T}$ is the collection of all open subsets of X, which does indeed form a topology. All that remains is to show that $\mathcal{B}$ is, in fact, a basis. The first requirement is that every point in X must be in a basis set. As X itself is an open set, and thus a basis set, every point is indeed in a basis set. The second requirement asserts that given basis (open) sets B_1 and B_2 and $x \in B_1 \cap B_2$, x must be in $B_3 \subset B_1 \cap B_2$. This follows from the definition of topology, which requires the intersection of two open sets to be an open set. $\square$

Exercise 6. *In each of the following cases, prove that the given set $\mathcal{B}$ is a basis for the given topology.*

- *M is a metric space with the metric topology, and $\mathcal{B}$ is the collection of all open balls in M.*
- *X is a set with the discrete topology, and $\mathcal{B}$ is the collection of all one-point subsets of X.*
- *X is a set with the trivial topology, and $\mathcal{B} = \{X\}$.*

Proof. By the previous lemma, for each case, we must show that the every open subset of the topology satisfies the basis criterion with respect to $\mathcal{B}$.

- We wish to show that for each point in any open set U in the metric space M, there exists an open ball in U that contains the point. Any ball with radius small enough such that $B \subset U$ will do the trick, and we are done.

- The collection of open sets forming the discrete topology X is the power set $\mathcal{P}(X)$. It is clear that for every such open set U, for all $x \in U$, there is a basis set in U containing x: namely, x's one-point basis set.

- The collection of open sets forming the trivial topology X is simply $\{X, \varnothing\}$. The given $\mathcal{B}$ has one basis set X. Obviously for each $x \in X$, said basis set contains x and is a subset of X. The same holds vacuously for the null set.

$\square$

Lemma 8. *Let X and Y be topological spaces and let $\mathcal{B}$ be a basis for Y. A map $f : X \to Y$ is continuous if and only if for every basis open set $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X.*

Proof. If f is continuous, by definition, the preimage of every basis open set is open in X. Conversely, suppose $f^{-1}(B)$ is open for every $B \in \mathcal{B}$. If V is an open set in Y, and $x \in U = f^{-1}(V)$, by the basis criterion, we know that there exists a basis set B such that $f(x) \in B \subset V$. Thus, $x \in f^{-1}(B) \subset U$, so every x has a neighborhood contained in U. The union of all such neighborhoods, then, is U, and consequently, U is open. $\square$

Definition 14. Let X be a topological space. A topological space M is said to be **locally Euclidean of dimension** n if every point $q \in M$ has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. Such a neighborhood is called a **Euclidean neighborhood** of q.

The following lemma shows that the "open subset" in the definition can be replaced by open ball, or $\mathbb{R}^n$.

Lemma 9. *A topological space M is locally Euclidean of dimension n if and only if either of the following properties holds:*

- *Every point of M has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$.*

- *Every point of M has a neighborhood homeomorphic to $\mathbb{R}^n$.*

Proof. As $\mathbb{R}^n$ is a Euclidean space, it immediately follows that if M is locally Euclidean of dimension n, one of the above two conditions must hold. For the converse proof, first note that the above two conditions are equivalent, as we showed earlier that any open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$. Thus, we need only prove the first condition.

Take any point $q \in M$ and let U be a neighborhood of q that admits a homeomorphism $\phi : U \to V$, where V is an open subset of $\mathbb{R}^n$. Since V is open, there must be some open ball B around $\phi(q)$ that is contained in V. Therefore, $\phi^{-1}(B)$ is a neighborhood of q homeomorphic to an open ball in $\mathbb{R}^n$, and we are done. $\square$

Definition 15. If M is locally Euclidean of dimension n, a homeomorphism from an open subset $U \subset M$ to an open subset of $\mathbb{R}^n$ is called a **coordinate chart** on U. We will call any open subset of M that is homeomorphic to a ball in $\mathbb{R}^n$ a **Euclidean ball** in M. The previous lemma shows that every point in a locally Euclidean space has a Euclidean ball neighborhood.

Note that the definition of locally Euclidean spaces makes sense even if $n = 0$. Since $\mathbb{R}^0$ is by convention a single point, the second condition of the previous lemma (that M is locally homeomorphic to the whole $\mathbb{R}^n$) implies that a space can be locally Euclidean of dimension 0 if and only if each point has a neighborhood that is homeomorphic to a one-point space. In other words: if and only if the space is discrete.

Definition 16. A topological space X is said to be a **Hausdorff space** if given any pair of distinct points $q_1, q_2 \in X$, there exist neighborhoods U_1 of q_1 and U_2 of q_2 with $U_1 \cap U_2 = \varnothing$.

Note that any open subset of a Hausdorff space is Hausdorff.

Lemma 10. *Let X be a Hausdorff space.*

1. *Every one-point set in X is closed.*

2. *If a sequence $\{x_i\}$ in X converges to a limit $x \in X$, the limit is unique.*

Proof. For the first part, take any one-point set $\{q\} \in X$. For any $p \neq q$, we are assured that there exist disjoint neighborhoods U_p of q and V_p of p. The complement of the one-point set is $X \setminus \{q\}$, which can be expressed as the union of the open sets V_p for every $p \in X \setminus \{q\}$.

To prove that the limits are unique, first assume for the sake of contradiction that the sequence converges to both x and x'. By the Hausdorff property, there exist disjoint neighborhoods U of x and U' of x'. By definition of convergence, there exist N, N' such that $i \geq N$ implies $x_i \in U$ and $i \geq N'$ implies $x_i \in U'$. But since U and U' must be disjoint, we reach a contradiction for when i is greater than both N and N'. $\square$

Exercise 7. *Show that the only Hausdorff topology on a finite set is the discrete topology.*

Proof. Suppose we have a finite, discrete topology X. It is clear that X is Hausdorff, as each point's one-point subset satisfies the Hausdorff condition of disjoint subsets.

Suppose we have a finite, Hausdorff topology X. We wish to show that X is the discrete topology. By the Hausdorff property and the above lemma, we know that every one-point set in X is closed. Consequently, for each point $q \in X$, the set $X \setminus \{q\}$ must be open. Label the points in X as $\{x_1 \cdots x_n\}$ and take the open subset $U = X \setminus \{x_1\} = \{x_2 \cdots x_n\}$. For each point in U, q, define ${}_q = X \setminus \{q\}$, which are all open. Note that the point x_1 is a member of V_q for all $q \in U$. Since V_q are open, it must be that $\bigcap_q V_q$ is open as well, and contains *only* x_1 (by virtue of how V_q was defined in terms of complements; drawing a picture of a set with 3 elements is a good way to visualize this). Thus, since x_1 was arbitrarily chosen in X, every one-point set in X is open; clearly, then, X is a discrete topology. $\square$

Definition 17. We say that a topological space is **second countable** if it admits a countable basis.

Definition 18. If X is a topological space and $q \in X$, a collection $\mathcal{B}_q$ of neighborhoods of q is called a **neighborhood basis** at q if every neighborhood of q contains some $B \in \mathcal{B}_q$. X is said to be **first countable** if there exists a countable neighborhood basis at each point.

Corollary 11. *Second countability implies first countability.*

Proof. Second countability states that there exists a countable basis for X. The collection of basis open sets containing any point $q \in X$ is, of course, a countable neighborhood basis for q. $\square$

Definition 19. If X is any topological space, a collection $\mathcal{U}$ of subsets of X is said to **cover** X, or be a cover of X, if every point in X is in one of the sets of $\mathcal{U}$. An **open cover** is a collection of open sets that covers X. Given any cover $\mathcal{U}$, a **subcover** of $\mathcal{U}$ is a subset of $\mathcal{U}$ that is still a cover.

Lemma 12. *If X is a second countable space, every open cover of X has a countable subcover.*

Proof. Let $\mathcal{B}$ be a countable basis for X, and let $\mathcal{U}$ be an arbitrary open cover of X. Let $\mathcal{B}'$ denote the subset of $\mathcal{B}$ consisting of those basis sets that are entirly contained in some element of $\mathcal{U}$. As a subset of a countable set is countable, $\mathcal{B}'$ must be a countable set. For each element $B \in \mathcal{B}'$, choose an element $U_b \in \mathcal{U}$ such that $B \subset U_B$. The collection $\{U_B : B \in \mathcal{B}'\}$ is a countable subset of $\mathcal{U}$. Now we want to show that it covers X, and we will be done.

Choose $x \in X$ arbitrary. Then $x \in U_0$ for some open $U_0 \in \mathcal{U}$, as $\mathcal{U}$ is an open cover. By the basis criterion, we know that there is some $B \in \mathcal{B}$ such that $x \in B \subset U_0$. Thus $B \in \mathcal{B}'$, and there exists a set $U_B \in \mathcal{U}'$ such that $x \in B \subset U_B$. This shows that $\mathcal{U}'$ is a cover. □

Any open subset U of a second countable space X is second countable: starting with a countable basis for X, simply throw away all the elements of the basis that do not lie in U; then it is easy to check that the remaining basis sets form a countable basis for the topology of U.

Definition 20. An **n-dimensional topological manifold** is a second countable Hausdorff space that is locally Euclidean of dimension n.

The most obvious example of an n-manifold is $\mathbb{R}^n$ itself. More generally, and open subset of $\mathbb{R}^n$ - or in fact of any n-manifold - is again an n-manifold, as the next lemma shows.

Lemma 13. *Any open subset of an n-manifold is an n-manifold.*

Proof. Let M be an n-manifold, and let V be an open subset of M. Any $q \in V$ has a neighborhood (in M) that is homeomorphic to an open subset of $\mathbb{R}^n$; the intersection of that neighborhood with V is still open, still homeomorphic to an open subset of $\mathbb{R}^n$, and lies in V, so V is locally Euclidean. Since any open subset of a Hausdorff space is Hausdorff and any open subset of a second countable space is second countable, V is an n-manifold. □

Definition 21. An n-dimensional **manifold with boundary** is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of the n-dimensional upper half space $\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Just as in the case of manifolds, we will call any homeomorphism from an open subset U of M to an open subset of $\mathbb{H}^n$ a **chart** on U.

The upper half space $\mathbb{H}^n$ is of course a manifold with boundary, as is any closed interval in $\mathbb{R}$, any closed disk in $\mathbb{R}^2$, or in fact a closed ball in any Euclidean space.

Definition 22. The boundary of $\mathbb{H}^n$ in $\mathbb{R}^n$ is the set of points where $x_n = 0$. If M is a manifold with boundary, a point that is in the inverse image $\partial\mathbb{H}^n$ under some chart is called a **boundary point** of M, and a point that is in the inverse image of $\operatorname{Int}\mathbb{H}^n$ is called an **interior point**. The **boundary** of M (the set of all of its boundary points) is denoted by ∂M; similarly, its **interior** is denoted by $\operatorname{Int} M$.

Since any open ball in $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{H}^n$, an n-manifold is automatically an n-manifold with boundary (with empty boundary), but the converse is not true: a manifold with boundary is a manifold if and only if its boundary is empty.