Introduction to Modern Analysis II

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1 Irrationality of e and π, transcendence of e

We begin by repeating the proof, from last term, that e is irrational. We define e by the series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Assuming e is rational, i.e. $e = m/n$ for some $m, n \in \mathbb{N}$, we can write

$$\frac{m}{n} = \sum_{k=0}^{n} \frac{1}{k!} + \sum_{k=n+1}^{\infty} \frac{1}{k!}.$$

Multiplying this by $n!$ yields

$$m(n-1)! = \sum_{k=0}^{n} \frac{n!}{k!} + \sum_{k=n+1}^{\infty} \frac{n!}{k!}.$$

The terms in the first sum on the right are integers, so moving it to the left hand side, we find that

$$\begin{aligned} m(n-1)! - \sum_{k=0}^{n} \frac{n!}{k!} &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \\ &= \frac{1}{n+1}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\ &= \frac{1}{n+1} \cdot \frac{1}{1-\frac{1}{n+1}} = \frac{1}{n} \\ &\leq 1. \end{aligned}$$

The sum on the right is positive, so the left hand side must be a positive integer less than 1, which is a contradiction. Hence e is irrational.

*Notes typeset by Nilay Kumar

Theorem 1 (Lambert, 1761)**.** *π is irrational.*

Proof (Niven, 1947). We show that π^2 is irrational, which is a slightly stronger statement, since if $\pi = u/v$ with $u, v \in \mathbb{N}$ then $\pi^2 = u^2/v^2$. For $n \in \mathbb{N}$, set

$$I_n = \int_0^1 p_n(x) \sin \pi x dx$$

where

$$p_n(x) = \frac{(x(1-x))^n}{n!}.$$

Since $0 \le p_n(x) \le 1/n!$ and $0 \le \sin \pi x \le 1$ for $0 \le x \le 1$, we see that

$$0 \le I_n \le \frac{1}{n!}. \tag{1}$$

Next we claim that for $k = 0, 1, 2, \ldots$, both $p_n^{(k)}(0)$ and $p_n^{(k)}(1)$ are integral: clearly $p_n^{(k)}(0) = 0$ for $k = 0, 1, \cdots, n-1$ due to the presence of the overall x^n. By the Binomial theorem,

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} v^k,$$

we get

$$p_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n+k}$$

so

$$\begin{aligned} p_n^{(n)}(x) &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n+k) \cdots (1+k) x^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{n+k}{n} x^k, \end{aligned}$$

a polynomial with integer coefficients. It follows that each $p_n^{(k)}$ with $k \ge n$ is a polynomial with integer coefficients and therefore each $p_n^{(k)}(0)$ is an integer. Since $p_n(x) = p_n(1-x)$ we have $p_n^{(k)}(x) = (-1)^k p_n^{(k)}(1-x)$, so $p_n^{(k)}(1) = (-1)^k p_n^{(k)}(0)$. Thus each $p_n^{(k)}(1)$ is an integer as well.

Furthermore, we claim that for each $n \in \mathbb{N}$ there are integers $c_0, c_1, \ldots, c_n$ such that

$$I_n = \frac{c_0}{\pi} + \frac{c_1}{\pi^3} + \cdots + \frac{c_n}{\pi^{2n+1}}.$$

This follows by integration by parts:

$$\begin{aligned} \int p_n(x) \sin \pi x dx &= -p_n(x) \frac{\cos \pi x}{\pi} + \int p_n'(x) \frac{\cos \pi x}{\pi} dx \\ &= -p_n(x) \frac{\cos \pi x}{\pi} + p_n'(x) \frac{\sin \pi x}{\pi^2} - \int p_n''(x) \frac{\sin \pi x}{\pi^2} dx \\ &= -p_n(x) \frac{\cos \pi x}{\pi} + p_n'(x) \frac{\sin \pi x}{\pi^2} + p_n''(x) \frac{\cos \pi x}{\pi^3} - p_n'''(x) \frac{\sin \pi x}{\pi^4} + \cdots \end{aligned}$$

following the same pattern, the last term involving $p_n^{(2n+1)}(x)$. Inserting $x = 1$ and $x = 0$ evaluates I_n, by the fundamental theorem of calculus. The terms with $\sin \pi x$ drop out as $\sin 0 = \sin \pi = 0$ and the terms with $\cos \pi x$ yield integers following the previous paragraph.

Finally, we suppose that $\pi^2 = a/b$ for $a, b \in \mathbb{N}$. It follows from above that

$$I_n = \frac{c_0 + c_1 a^{2n-1} b + \cdots + c_n b^{2n}}{\pi a^{2n}}.$$

Since I_n is positive, the numerator is a positive integer, and we can write $I_n \geq 1/(\pi a^{2n})$. Combining this inequality with Eq. (1), we find that

$$\frac{a^{2n}}{n!} \geq \frac{1}{\pi}$$

for all $n \in \mathbb{N}$, which is false as $\sum_{n=0}^{\infty} a^{2n}/n!$ converges. □

Definition 1. A real number α is **algebraic** if α is a root of some non-zero polynomial with integer coefficients. If $\alpha \in \mathbb{R}$ but α is not algebraic, then α is **transcendental**.

Theorem 2. *The set of all algebraic numbers is countable.*

Proof. The set of polynomials of degree less than or equal to n with integer coefficients is countable as there is a bijection between this set and $\mathbb{Z}^{n+1}$. Therefore the set of all polynomials with integer coefficients is countable since a countable union of countable sets is countable. Each non-zero polynomial of degree n has at most n roots. Therefore the set of all algebraic numbers, as a countable union of finite sets, is countable. □

Corollary 3. *Transcendental numbers exist.*

Proof. If not, every real number would be algebraic. But $\mathbb{R}$ is uncountable by Cantor's theorem. □

It is much harder to exhibit a transcendental number than it is to prove that they exist. Here is a famous example.

Theorem 4 (Hermite, 1873)**.** *e is transcendental.*

Lemma 5. *For $m = 0, 1, 2, \ldots,$*

$$\int_0^\infty x^m e^{-x} dx = m!.$$

Proof. For $m = 0$ we find that

$$\int_0^\infty e^{-x} = 1$$

and for $m > 0$, we integrate by parts to obtain

$$\int_0^\infty x^m e^{-x} dx = m \int_0^\infty x^{m-1} e^{-x} dx.$$

The result follows by induction. □

Corollary 6. *If $p \in \mathbb{N}$ and $c_{p-1}, c_p, \ldots, c_N \in \mathbb{Z}$, then*

$$\frac{1}{(p-1)!} \int_0^\infty \left(c_{p-1} x^{p-1} + c_p x^p + \cdots + c_N x^N\right) e^{-x} dx$$

is an integer equivalent to $c_{p-1} \mod p$.

Lemma 7 (Hermite)**.** *For $n, p \in \mathbb{N}$ and $k = 1, \ldots, n$ set*

$$H_{n,p} = \frac{1}{(p-1)!} \int_0^\infty x^{p-1} \left((x-1)\cdots(x-n)\right)^p e^{-x} dx$$

and

$$H_{n,k,p} = \frac{1}{(p-1)!} \int_0^\infty (x+k)^{p-1} \left((x+k-1)\cdots(x+k-n)\right)^p e^{-x} dx.$$

Then $H_{n,p}$ and $H_{n,k,p}$ are integers, with

$$H_{n,p} \equiv ((-1)^n n!)^p \mod p$$

and

$$H_{n,k,p} \equiv 0 \mod p$$

for $k = 1, \ldots, n$.

Proof. We use the corollary above. The polynomial $x^{p-1}\left((x-1)\cdots(x-n)\right)^p$ has integer coefficients and its lowest degree term is $((-1)^n n!)^p x^{p-1}$. However, for $k = 1, \ldots, n$, the polynomial $x^{p-1}\left((x+k-1)\cdots(x+k-n)\right)^p$ has integer coefficients and starts later than the x^{p-1} term so the coefficient of x^{p-1} is 0. $\square$

Hilbert's proof of Hermite's theorem (1893). If e is algebraic then

$$a_0 + a_1 e + \cdots + a_n e^n = 0$$

for some integers $a_0, a_1, \ldots, a_n$ not all 0. We may suppose $a_0 \neq 0$. It follows that for each positive integer p,

$$0 = \left(\sum_{k=0}^n a_k e^k\right) H_{n,p} = \sum_{k=0}^n a_k \frac{1}{(p-1)!} \int_0^\infty x^{p-1}\left((x-1)\cdots(x-n)\right)^p e^{-x} dx.$$

For $k = 1, \ldots, n$ we write the integral as $\int_0^k + \int_k^\infty$ and in $\int_k^\infty$ replace x by $x + k$ giving

$$a_0 H_{n,p} + \sum_{k=1}^n a_k H_{n,k,p} = -\sum_{k=1}^n a_k R_{n,k,p} \tag{2}$$

with

$$R_{n,k,p} = \frac{1}{(p-1)!} \int_0^k x^{p-1}\left((x-1)\cdots(x-n)\right)^p e^{k-x} dx.$$

By Hermite's lemma, the left side of Eq. (2) is an integer equivalent to $a_0\left((-1)^n n!\right)^p \mod p$. Therefore, if p is a prime greater than both n and $|a_0|$, then the left side of Eq. (2) is not equivalent to $0 \mod p$. In particular, the left side is a non-zero integer. The right side is small for large p: take

$$A = \sum_{u=1}^n |a_u|$$

and

$$B = \max_{[0,n]} |(x-1)\cdots(x-n)|.$$

Then

$$
\begin{aligned}
-\sum_{k=1}^{n} a_k R_{n,k,p} &\le \frac{A}{(p-1)!}\int_0^n x^{p-1}|(x-1)\cdots(x-n)|^p e^{n-x}dx \\
&\le \frac{Ae^n(nB)^p}{(p-1)!},
\end{aligned}
$$

since $\int_0^n e^{-x}dx < 1$, which goes to zero as p grows large. Since there are infinitely many primes, this yields a contradiction. □

2 Approximating roots

Omitted.

3 Laplace's asymptotic method

Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be continuous and non-negative with maximum M. We show first that

$$
\left(\int_a^b f^p\right)^{1/p} \to M \text{ for } p \to \infty
$$

and then, under stronger assumptions about f, we obtain Laplace's asymptotic formula for $\int_a^b f^p$ for $p \to \infty$.

Lemma 8. *For each $d > 0$, $d^{1/p} \to 1$ for $p \to \infty$.*

Proof. For $d > 0$ and $p > 0$, as $p \to \infty$,

$$
\log d^{1/p} = \frac{1}{p}\log d \to 0.
$$

Therefore

$$
d^{1/p} = e^{\log(d^{1/p})} \to e^0 = 1,
$$

where we have used the continuity of the exponential function at zero. □

Theorem 9. *For $f : [a,b] \to \mathbb{R}$ continuous and non-negative with $a < b$,*

$$
\left(\int_a^b f^p\right)^{1/p} \to M \text{ for } p \to \infty \tag{3}
$$

with $M = \max f$.

Proof. First we obtain an upper bound:

$$
\int_a^b f^p \le \int_a^b M^p = (b-a)M^p,
$$

so

$$
\left(\int_a^b f^p\right)^{1/p} \le (b-a)^{1/p}M.
$$

By the lemma above with $d = b - a$ we see that the first factor on the right approaches 1 for $p \to \infty$. Therefore, for each $\mu > 1$, we have

$$\left(\int_a^b f^p\right)^{1/p} \leq \mu M$$

for all sufficiently large p.

Next we obtain a lower bound: we have $f(c) = M$ for some $c \in [a, b]$. If $a < c < b$ then by continuity of f there must exist a $\delta > 0$ such that $f(x) > \sqrt{\lambda} M$ for $x \in [c - \delta, c + \delta] \subset [a, b]$ and $\lambda < 1$. It follows that

$$\int_a^b f^p \geq \int_{c-\delta}^{c+\delta} f^p \geq 2\delta(\sqrt{\lambda} M)^p,$$

so

$$\left(\int_a^b f^p\right)^{1/p} \geq (2\delta)^{1/p} \sqrt{\lambda} M.$$

Using the lemma above with $d = 2\delta$, the first factor on the right goes to 1 for $p \to \infty$. Therefore it is greater than or equal to $\sqrt{\lambda}$ for all sufficiently large p giving

$$\left(\int_a^b f^p\right)^{1/p} \geq \lambda M$$

for all sufficiently large p. If $c = a$ or $c = b$, we consider just $[a, a + \delta]$ or $[b - \delta, b]$ instead of $[c - \delta, c + \delta]$ and obtain the bound above as well.

Combining the upper and lower bounds, yields the desired formula, as $\lambda < 1$ and $\mu > 1$ can be chosen arbitrarily close to 1. $\square$

Theorem 10 (Laplace, 1774)**.** *For $a < b$ and $\phi : [a, b] \to \mathbb{R}$ continuous with continuous and negative second derivative on (a, b) and assuming ϕ has a maximum at some point $c \in (a, b)$,*

$$\int_a^b e^{p\phi(x)} dx \sim \sqrt{\frac{2\pi}{-p\phi''(c)}} e^{p\phi(c)} \tag{4}$$

for $p \to \infty$.[1]

Lemma 11.

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 \tag{5}$$

Proof. The square of the integral is readily evaluated:

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)^2 &= \int_{-\infty}^{\infty} e^{-\pi x^2} \int_{-\infty}^{\infty} e^{-\pi y^2} dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx dy \\
&= \int_0^{\infty} \int_0^{2\pi} e^{-\pi r^2} r d\theta dr \\
&= \int_0^{\infty} e^{-\pi r^2} 2\pi r dr \\
&= \int_0^{\infty} e^{-u} du = 1.
\end{aligned}$$

Since the integral is clearly positive and has square 1, it must be 1. $\square$

[1]Here for positive functions g, h defined on $(0, \infty)$, $g(p) \sim h(p)$ for $p \to \infty$ means that $g(p)/h(p) \to 1$ for $p \to \infty$.

We prove Laplace's theorem for the special case where $c = 0$, along with the additional assumptions that $\phi(0) = 0, \phi'(0) = 0, \phi''(0) = -1$. The general case is left for the reader in Exercise 1.

Proof of (special case of) Laplace's theorem. We are now supposing that ϕ is continuous on $[a, b]$ with $a < 0 < b$, has a maximum at zero, and has $\phi'' < 0$ on $[a, b]$, along with the special assumptions above. Of course, $\phi'(0) = 0$ follows from the maximum assumption. We wish to show that

$$\int_a^b e^{p\phi(x)} dx \sim \sqrt{\frac{2\pi}{p}}$$

for $p \to \infty$. Let's look first at an even more special case: $\phi(x) = -x^2/2$. In this case, the assumptions above are satisfied, and for $a < 0 < b$,

$$\begin{aligned}\int_a^b e^{p\phi(x)} dx &= \int_a^b e^{-px^2/2} dx \\ &= \sqrt{\frac{2\pi}{p}} \int_{a_1}^{b_1} e^{-\pi u^2} du\end{aligned}$$

with $a_1 = \sqrt{p/2\pi}a$ and $b_1 = \sqrt{p/2\pi}b$. Thus for $p \to \infty$ we have $a_1 \to -\infty$ and $b_1 \to \infty$ so the last integral approaches 1 by the previous lemma. This proves the theorem in this case.

We now return to the slightly more general case. Since $\phi(0) = 0$ and $\phi'(0) = 0$, Taylor's formula gives

$$\begin{aligned}\phi(x) &= \int_0^x \phi''(t)(x - t)dt \\ &= \int_0^x \phi''(0)(x - t)dt + \int_0^x \left(\phi''(t) - \phi''(0)\right)(x - t)dt \\ &= \frac{1}{2}\phi''(0)x^2 + \int_0^x \left(\phi''(t) - \phi''(0)\right)(x - t)dt\end{aligned}$$

The norm of the second term on the right is clearly bounded by $\max_{0\le t\le x} |\phi''(t)-\phi''(0)|\cdot x^2/2$. Since ϕ'' is continuous at zero, the maximum goes to zero for $x \to 0$. Thus $\phi(x) \sim \phi''(0)x^2/2 = -x^2/2$ for $x \to 0$. It follows that for each pair of real numbers λ, μ with $\lambda < 1 < \mu$, there is a $\delta > 0$ with $[-\delta, \delta] \subset [a, b]$ so that $-\mu x^2/2 \le \phi(x) \le -\lambda x^2/2$ for $|x| \le \delta$. Therefore, for all $p > 0$,

$$\int_{-\delta}^{\delta} e^{-\mu p x^2/2} dx \le \int_{-\delta}^{\delta} e^{p\phi(x)} dx \le \int_{-\delta}^{\delta} e^{-\lambda p x^2/2} dx,$$

which simplifies to

$$\sqrt{\frac{2\pi}{\mu p}} \int_{-\delta_\mu}^{\delta_\mu} e^{-\pi u^2} du \le \int_{-\delta}^{\delta} e^{p\phi(x)} dx \le \sqrt{\frac{2\pi}{\lambda p}} \int_{-\delta_\lambda}^{\delta_\lambda} e^{-\pi u^2} du,$$

with $\delta_\mu = \delta\sqrt{\mu p/2\pi}$ and $\delta_\lambda = \delta\sqrt{\lambda p/2\pi}$, both going to infinity as $p \to \infty$. It follows that the left and right integrals approach $\sqrt{2\pi/\mu p}$ and $\sqrt{2\pi/\lambda p}$ as $p \to \infty$ and hence $\int_{-\delta}^{\delta} e^{p\phi(x)} dx/\sqrt{2\pi/p}$ can be made arbitrarily close to 1 by first choosing δ sufficiently small, then choosing p sufficiently large. What about the rest of the integral?

Since ϕ decreases on $[0, b]$, we have $\int_\delta^b e^{p\phi(x)}dx \leq be^{p\phi(\delta)}$. For each $\delta > 0$ we have $\phi(\delta) < 0$, so

$$\frac{e^{p\phi(\delta)}}{\sqrt{2\pi/p}} = \frac{\sqrt{p}e^{p\phi(\delta)}}{\sqrt{\pi}} \to 0 \text{ for } p \to \infty.$$

Thus for each $\delta > 0$,

$$\frac{1}{\sqrt{2\pi/p}} \int_\delta^b e^{p\phi(x)}dx \to 0 \text{ for } p \to \infty.$$

Similarly for $[a, \delta]$. Putting it all together yields the desired result. □

Exercise 1.

1. *Show that the general case of Thm. 10 can be reduced to the case in which $c = 0$ by making the change of variables $x = c + u$, $\phi(x) = \psi(u)$.*

2. *Show that the case in (1) can be further reduced to the case in which $c = 0$ and $\phi(0) = 0$ by writing $\phi(x) = \phi(0) + \psi(x)$.*

3. *Show that the case in (2) can be further reduced to the case in which $c = 0$, $\phi(0) = 0, \phi'(0) = 0$, and $\phi''(0) = -1$ by making a change of variables $x = \xi u$ for some positive constant ξ.*

Theorem 12 (Generalizing Stirling's formula)**.** *As $p \to \infty$ for p real,*

$$\int_0^\infty x^p e^{-x}dx \sim \sqrt{2\pi p}\left(\frac{p}{e}\right)^p.$$

Proof. In the integral let $x = p(u + 1)$, giving

$$\int_0^\infty x^p e^{-x}dx = p^{p+1}e^{-p}\int_{-1}^\infty \left((u + 1)e^{-u}\right)^p du.$$

To obtain the result, it suffices to show that

$$\int_{-1}^\infty e^{p\phi(u)}du \sim \sqrt{\frac{2\pi}{p}}$$

for $p \to \infty$, with

$$\phi(u) = \log(u + 1) - u$$

for $u > -1$. This is left as the following exercise. □

Exercise 2. *Show how Laplace's theorem can be used to finish the proof above. Hint:*

$$\phi'(u) = \frac{1}{u + 1} - 1$$
$$\phi''(u) = -\frac{1}{(u + 1)^2}.$$

There are two slight difficulties: $\phi(u) \to -\infty$ as $u \to 0, u > 0$, and $b = \infty$. Show how to overcome these difficulties.