# Introduction to Algebraic Topology PSET 8

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Last updated: April 2, 2014

**Proposition 1.** *Hatcher exercise 2.1.11*

*Proof.* Let $\iota : A \to X$ be the inclusion of $A$ into $X$, and $r : X \to X$ be the retract of $X$ onto $A$. The composition $r \circ \iota : A \to A$ yields the identity $\mathrm{Id}_A : A \to A$. The induced maps on the homology are $(r \circ \iota)_* = r_* \circ \iota_* = \mathrm{Id} : H_n(A) \to H_n(A)$. This map is of course injective, which implies that $\iota_* : H_n(A) \to H_n(X)$ must be injective as well. $\square$

**Proposition 2.** *Hatcher exercise 2.1.12*

*Proof.* Let us show that the relation of chain homotopy between chain maps is an equivalence relation. Consider $f_\#, g_\#, h_\# : C_n(A) \to C_{n+1}(B)$. The relation is clearly reflexive, as $f_\# \sim f_\#$ by the zero morphism $0 : C_n(A) \to C_{n+1}(B)$. Symmetry holds as follows: if $f_\# \sim g_\#$ via a chain homotopy $h$, then $g_\# \sim f_\#$ via the chain homotopy $-h$, because then

$$\begin{aligned} f_\# - g_\# &= \partial h + h\partial \\ g_\# - f_\# &= -(\partial h + h\partial) \\ &= \partial(-h) + (-h)\partial. \end{aligned}$$

Finally, the relation is transitive, because given $f_\# \sim g_\#$ via $H_1$ and $g_\# \sim h_\#$ via $H_2$, we can add the two commutation relations to obtain that

$$\begin{aligned} f_\# - h_\# &= \partial H_1 + H_1\partial + \partial H_2 + H_2\partial \\ &= \partial(H_1 + H_2) + (H_1 + H_2)\partial, \end{aligned}$$

as desired. $\square$

**Proposition 3.** *Hatcher exercise 2.1.14*

*Proof.* Consider the sequence

$$0 \longrightarrow \mathbb{Z}_4 \xrightarrow{\phi} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \xrightarrow{\psi} \mathbb{Z}_4 \longrightarrow 0$$

with $\phi$ taking the generator of $\mathbb{Z}_4$ to $(2,1) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$, which is clearly a well-defined injective map. If we now quotient $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ by $H = \mathrm{im}\ \phi$, we obtain four cosets: $(0,0)H, (0,1)H, (1,0)H$, and $(1,1)H$, which is clearly isomorphic to $\mathbb{Z}_4$ (with $(0,1)H$ as the generator). This yields a short exact sequence.

Consider now more generally the sequence of groups

$$0 \longrightarrow \mathbb{Z}_{p^m} \xrightarrow{\phi} A \xrightarrow{\psi} \mathbb{Z}_{p^n} \longrightarrow 0$$

Note first that $A = \mathbb{Z}_{p^{m+n-k}} \oplus \mathbb{Z}_{p^k}$, for $k \le \min(m,n)$, fits into the sequence. Indeed, we let $\phi(1) = (p^{n-k}, 1)$, which is injective because the image along the first factor is injective. We now need to choose $\psi$ such that $\ker\psi = (jp^{n-k}, j)$ for $j \in \mathbb{Z}_{p^m}$. We claim that choosing $\psi(1,0) = 1$ fixes the map because $\psi(i,j) = i\psi(1,0) + j\psi(0,1)$ but since $\psi(jp^{n-k}, p) = 0$, we find that $\psi(0,1) = -p^{n-k}$. Hence, with this choice of $\psi(1,0)$, we find that $\psi(i,j) = i - p^{n-k}j$. It is easy to see that $\ker\psi = \text{im }\phi$. The image of $\psi$ is cyclic, generated by $\psi(1,0)$ (as $\psi(i,j) = (i - jp^{n-k})\psi(1,0)$), and has order $p^{m+n}/p^m$, and hence isomorphic to $\mathbb{Z}_{p^n}$. Note, however, that it is unclear that these are the *only* groups that fit into this sequence (though it might be possible to invoke the fundamental theorem of finitely generated abelian groups).

Finally, consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \phi\ } A \xrightarrow{\ \psi\ } \mathbb{Z}_n \longrightarrow 0$$

We claim that $A = \mathbb{Z} \oplus \mathbb{Z}_d$ fits into this sequence for $d|n$. Indeed, we define $\phi(1) = (1, n/d)$ and then $\psi(0,1) = 1$, which forces $\psi(i,j) = (j - in/d)\psi(0,1) = j - in/d$. The sequence is clearly exact, and im $\psi$ is cyclic. It suffices to compute the order of im $\psi$. This is done by counting the number of lattice points of $\mathbb{Z}^2$ contained in the parallelogram spanned by $(0,d)$ and $(1, n/d)$, which is simply $d \cdot n/d = n$. Hence we obtain $\mathbb{Z}_n$, as desired. Again, it is not clear that these are the *only* groups that fit into this sequence. ∎

**Proposition 4.** *Hatcher exercise 2.1.15*

*Proof.* Consider the exact sequence

$$A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \xrightarrow{\ \gamma\ } D \xrightarrow{\ \delta\ } E.$$

Exactness at $B$ requires $\ker\beta = \text{im }\alpha$, and hence $\alpha$ is surjective if and only if $\ker\beta = B$. Exactness at $D$ requires $\ker\delta = \text{im }\gamma$, and hence $\delta$ is injective if and only if im $\gamma = 0$. Hence if (and only if) $\alpha$ is surjective and $\delta$ is injective then $\gamma = 0$ and $\beta = 0$ and the exactness at $C$ (requiring that $\ker\gamma = \text{im }\beta$) forces $C = 0$.

Hence for a good pair $(X, A)$, we find that $H_n(X, A) = 0$ if and only if the inclusion $A \to X$ induces isomorphisms on all homology groups, as the long exact sequence of theorem 2.13 splits into sequences

$$0 \longrightarrow \tilde{H}_n(A) \xrightarrow{\ \iota_*\ } \tilde{H}_n(X) \longrightarrow 0$$

for all $n$. ∎

**Proposition 5.** *Let $A$ and $B$ be chain complexes. A chain map $f : A \to B$ is a* chain homotopy equivalence *if there exists a chain map $g : B \to A$ such that $f \circ g \sim \text{Id}_B$ and $g \circ f \sim \text{Id}_A$ in the sense of chain homotopies.*

*(a) Prove that if $f : A \to B$ is a chain homotopy equivalence, then $f$ induces an isomorphism on homology.*

*(b) Give an example of chain complexes $A$ and $B$ with isomorphic homology but no chain homotopy equivalence between them. (Hint: let $A$ be $\mathbb{Z}$ in two consecutive gradings and zero everywhere else.)*

*Proof.*

(a) Recall that chain-homotopic maps induce the same homorphism on homology. Hence $(f \circ g)_* = f_* \circ g_* = (\mathrm{Id}_B)_* = \mathrm{Id}_{H_n(B)}$ and $(g \circ f)_* = g_* \circ f_* = (\mathrm{Id}_A)_* = \mathrm{Id}_{H_n(A)}$. As $\mathrm{Id}_{H_n(B)}$ is injective, $f_* : H_n(A) \to H_n(B)$ must be as well, and as $\mathrm{Id}_{H_n(A)}$ is surjective, $f_*$ must be as well. Hence $f_*$ is an isomorphism.

(b) Consider the map of chain complexes $f : A_\bullet \to B_\bullet$ given by the first two rows of

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \mathbb{Z}_4 \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where each square clearly commutes. The homology groups of the two sequences are $H_\bullet(A) = 0$ and $H_\bullet(B) = 0$. However, there does not exist a chain homotopy equivalence between $A_\bullet$ and $B_\bullet$, as we now show. If there did exist one, there would exist a chain map $g : B_\bullet \to A_\bullet$ (between the second two rows) such that the appropriate compositions of $f$ and $g$ would be chain homotopic to $\mathrm{Id}_B$ and $\mathrm{Id}_A$ via some chain homotopy $h$. Of course, the only possible compositions are zero, and hence

$$\begin{tikzcd}
 & \mathbb{Z}_4 \arrow[r, "\pi"] \arrow[dl, "h_1"'] \arrow[d, "0"] & \mathbb{Z}_2 \arrow[dl, "h_2"] \\
\mathbb{Z}_2 \arrow[r, "\iota"] & \mathbb{Z}_4 &
\end{tikzcd}$$

there must exist $h_1, h_2$ such that $\mathrm{Id}_{\mathbb{Z}_4} = \iota \circ h_1 + h_2 \circ \pi$, but $h_1$ can only be the zero map or the quotient map and $h_2$ can only be the zero map or the inclusion map. It is easy to see that none of these combinations recover $\mathrm{Id}_{\mathbb{Z}_4}$.

$\square$