RTG Notes: Representation theory

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1 The irreducible finite representations of $\mathfrak{sl}_2\mathbb{C}$

Let V be an irreducible finite-dimensional representation of $\mathfrak{sl}_2$. Let us choose as a basis for $\mathfrak{sl}_2$ the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

and for now restrict our attention to the action of H. Since Lie algebra actions preserve Jordan decomposition (F-H theorem 9.20), the action of H on V is diagonalizable. Hence the eigenvectors of H in V span V and we have a decomposition $V = \bigoplus V_\alpha$ where the α run over a set of complex numbers, such that for any vector $v \in V_\alpha$ we have $Hv = \alpha v$, i.e. the V_α are invariant under the action of H. Next we must determine how X and Y act on these V_α. In particular, given a $v \in V_\alpha$, for which β is Xv contained in V_β?

$$\begin{aligned} HXv &= XHv + [H, X]v \\ &= X\alpha v + 2Xv \\ &= (\alpha + 2)Xv \end{aligned}$$

and thus if v is an eigenvector for H with eigenvalue α, then Xv is also an eigenvector H, with eigenvalue $\alpha + 2$. In other words, we can view X as a map from V_α to $V_{\alpha+2}$. The action of Y is similarly computed: $Y : V_\alpha \to V_{\alpha-2}$.

Consider the subspace of V given by the direct sum $V' = \bigoplus_{n\in\mathbb{Z}} V_{\alpha_0+2n}$. It's clear that V' is invariant under the action of H, X, Y, but since V is irreducible, we must have that $V = V'$. Furthermore, since V is finite-dimensional, this direct sum occurs over a finite subset of the complex numbers, $n, n-2, n-4, \ldots, p$. It is at the moment unclear whether n is an integer, and what the value of the lower bound, p is.

Now choose any vector $v \in V_n$. Since this is the highest subspace, $V_{n+2} = (0)$ and $Xv = 0$. Let us investigate what happens when we apply Y to this vector.

Lemma 1. *The vectors* $\{v, Yv, Y^2v, \ldots\}$ *span* V.

Proof. By the irreducibility of V, it suffices to show that the subspace W spanned by these vectors is invariant under the action of $\mathfrak{sl}_2$. It is immediate that W is invariant under both H and Y. Thus it suffices to check that $XW \subset W$; indeed we shall show by induction that

$$XY^m v = m(n - m + 1)Y^{m-1}v,$$

which would imply that X preserves W. For $m = 1$, using the fact that $Xv = 0$ and $[X, Y] = H$ we see that

$$\begin{aligned} XYv &= YXv + [X, Y]v \\ &= Hv = nv, \end{aligned}$$

as desired. Assuming the formula holds for m, we compute:

$$\begin{aligned}
XY^{m+1}v &= XY(Y^m)v = (YX + [X,Y])Y^m v \\
&= Y\left(mn - m^2 + m\right)Y^{m-1}v + HY^m v \\
&= (mn - m^2 + m)Y^m v + HY^m v \\
&= (mn - m^2 + m)Y^m v + (n - 2m)Y^m v \\
&= (mn - m^2 - m + n)Y^m v \\
&= (m+1)(n-m)Y^m v,
\end{aligned}$$

which completes the proof. □

Corollary 2. *Each of the eigenspaces V_α is one-dimensional.*

Proof. Suppose one of the V_α were more than one-dimensional. Then the set of vectors $\{v, Yv, Y^2v, \ldots\}$ will not span the whole space V, in contradiction to the above lemma. □

Incidentally, the representation V is completely characterized by the complex number n. Furthermore, since V is finite-dimensional, it must have a lower bound as well (as well as an upper bound). Suppose m is the smallest power of Y that annihilates v; then from the lemma we find that

$$0 = XY^m v = m(n - m + 1)Y^{m-1}v.$$

Hence, since $Y^{m-1}v \neq 0$, we must have that $n - m + 1$. This tells us several things. First of all, we see that n is, in fact, a non-negative integer. Additionally since, the eigenvalues jump by two, we see that the eigenvalues α of H on V form a string of integers differing by 2 and symmetric about the origin in $\mathbb{Z}$.

To summarize, we have found that there is a unique representation $V^{(n)}$ of $\mathfrak{sl}_2$ for each non-negative integer n. The representation $V^{(n)}$ is $(n+1)$-dimensional with H taking eigenvalues $n, n-2, \ldots, -n+2, -n$. This is very useful information: now, if we are given any representation V of $\mathfrak{sl}_2$ such that the eigenvalues of the action of H all have the same parity and occur with multiplicity one, it must be irreducible. Furthermore, given an arbitrary representation V of $\mathfrak{sl}_2$, the number of irreducible factors contained within it is exactly the sum of the multiplicities of 0 and 1 as eigenvalues of H.

Let's examine some of these irreducible representations. Take the trivial representation of $\mathfrak{sl}_2$ on $\mathbb{C}$ that sends every Lie algebra element to the zero endomorphism – this is clearly $V^{(0)}$. Consider next the standard representation on $\mathbb{C}^2$. If we take x and y be the standard basis for $\mathbb{C}^2$, we find that $H(x) = x, H(y) = -y$. This gives us the decomposition $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y = V_1 \oplus V_{-1}$. We can obtain the higher-dimensional irreducible representations by taking symmetric powers of the standard representation. Take for example, $W = \operatorname{Sym}^2 V = \operatorname{Sym}^2 \mathbb{C}^2$, which has basis $\{x^2, xy, y^2\}$:

$$\begin{aligned}
H(x^2) &= x\cdot Hx + Hx\cdot x = 2x\cdot x \\
H(xy) &= x\cdot Hy + Hx\cdot y = 0 \\
H(y^2) &= y\cdot Hy + Hy\cdot y = -2y\cdot y,
\end{aligned}$$

which yields $V^{(2)} = \mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2 = W_2 \oplus W_0 \oplus W_{-2}$.

We leave it as an exercise for the reader to determine the action of H on the basis of $\operatorname{Sym}^n V$ and show that the eigenvalues are precisely $n, n-2, \ldots, -n+2, -n$ and hence that $V^{(n)} = \operatorname{Sym}^n V$. In conclusion, then, any irreducible representation of $\mathfrak{sl}_2$ is a symmetric power of the standard representation $V = \mathbb{C}^2$.

2 The regular representation of $SL_2(\mathbb{C})$

Let us now turn to an example of how the results from the previous section can be useful. Consider the regular representation of the Lie group $SL_2(\mathbb{C})$, i.e. the space $R = C[SL_2(\mathbb{C})] = \mathbb{C}[a,b,c,d]/\{ad-bc-1\}$ of complex-valued functions on $SL_2(\mathbb{C})$. The action of $g \in SL_2$ on $f \in R$ (evaluated at $h \in SL_2$) is given by $\pi(g)(f)(h) = f(g^{-1}h)$. Let us determine the associated Lie algebra homomorphism via the following commutative diagram.

$$\begin{array}{ccc} SL_2 & \xrightarrow{\pi} & GL(R) \\ {\scriptstyle \exp}\uparrow & & {\scriptstyle \exp}\uparrow \\ \mathfrak{sl}_2 & \xrightarrow{\rho} & \mathfrak{gl}(R) \end{array}$$

Given some $f \in R, g = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in SL_2$, then, the diagram above shows that the action of H is:

$$\begin{aligned} \rho(H)(f)(g) &= \frac{d}{dt}\bigg|_{t=0} f\left(e^{-tH}g\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f\begin{pmatrix} e^{-t}x & e^{-t}y \\ e^{t}z & e^{t}w \end{pmatrix} \\ &= -x\frac{\partial f}{\partial a}(g) - y\frac{\partial f}{\partial b}(g) + z\frac{\partial f}{\partial c}(g) + w\frac{\partial f}{\partial d}(g). \end{aligned}$$

Note carefully that the action preserves the degree of the polynomial f, as it multiplies by a variable after taking a derivative. Similar computations for X and Y yield:

$$\begin{aligned} \rho(X)(f)(g) &= \frac{d}{dt}\bigg|_{t=0} f\left(e^{-tX}g\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f\begin{pmatrix} x-tz & y-tw \\ z & w \end{pmatrix} \\ &= -z\frac{\partial f}{\partial a}(g) - w\frac{\partial f}{\partial b}(g) \\ \rho(Y)(f)(g) &= \frac{d}{dt}\bigg|_{t=0} f\left(e^{-tY}g\right) \\ &= \frac{d}{dt}\bigg|_{t=0} f\begin{pmatrix} x & y \\ z-tx & w-ty \end{pmatrix} \\ &= -x\frac{\partial f}{\partial c}(g) - y\frac{\partial f}{\partial d}(g). \end{aligned}$$

Note that the action of X and Y do not preserve the degree, as $\rho(X)(cd) = 0$ and $\rho(Y)(ab) = 0$. The question now arises: how can we write this representation in terms of the irreducible representations that we computed in the previous section?

Take some monomial $a^i b^j c^k d^l$ in R. By successive applications of X one can reach the highest-weight vector $c^{i+k}d^{j+l}$. Let us rewrite, for convenience, this vector as $c^\alpha d^\beta$. Note, however, that the chain of eigenspaces specified by this highest weight vector is uniquely specified by the sum $n = \alpha + \beta$, because the eigenvalues associated with the eigenspaces are the same for c^2 as they are for cd or d^2. Hence, for a given n, we have some number of irreducible representations of $\mathfrak{sl}_2$: to keep track of the multiplicities we note that all the possible heighest weight vectors form the space

$\mathrm{Sym}^n(\mathbb{C}\cdot c \oplus \mathbb{C}\cdot d)$. Hence the regular representation decomposes as

$$R = \bigoplus_n \left(\mathrm{Sym}^n(\mathbb{C}\cdot c \oplus \mathbb{C}\cdot d) \otimes \mathrm{Sym}^n \mathbb{C}^2\right)$$
$$= \bigoplus_n \left(\mathrm{Sym}^n \mathbb{C}^2 \otimes \mathrm{Sym}^n \mathbb{C}^2\right)$$

We can now move back to the group level by noting that SL_2 is simply connected, and thus its representations are in one-to-one correspondence with the representations of $\mathfrak{sl}_2$ [FH91]. Hence, the direct sum above is the decomposition of the regular representation of SL_2.

Suppose now that instead of acting from the left, we have SL_2 acting from the right, i.e. given $g \in SL_2$ and $f \in R$ (evaluated at $h \in SL_2$),

$$f \cdot g = \pi(g)(f)(h)$$

3 Representations of $\mathfrak{sl}_3\mathbb{C}$

In the case of $\mathfrak{sl}_2$ we examined the action of the maximal torus, H, on the representation V. Things are not so simple in $\mathfrak{sl}_3$ (which is eight-dimensional), as the maximal torus is in fact a two-dimensional subspace $\mathfrak{h}$. Indeed, we can write the basis neatly in one place as:

$$\begin{pmatrix} H_1 & X_1 & X_3 \\ Y_1 & H_2 - H_1 & X_2 \\ Y_3 & Y_2 & -H_2 \end{pmatrix}.$$

Then the maximal torus is spanned by the basis elements H_1, H_2. Again, we must require that the action of the maximal torus on some representation V be diagonalizable, so we can get the eigenvectors to span V. Thankfully, since $[H_1, H_2] = 0$, we can use the fact that commuting diagonalizable matrices are simultaneously diagonalizable, and the eigenvectors under the action of $\mathfrak{h}$ span V.

Because we are no longer dealing with the action of one element of the Lie algebra, we have to be a little more careful about what we mean by eigenvalue and eigenvector. For one, by eigenvector, we mean any $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. Furthermore, we must label the eigenvalue with the action to which it belongs, which we will do as:

$$Hv = \alpha(H) \cdot v.$$

The reader can verify that α is linear in H and thus, when we refer to an eigenvalue, it will be to an element $\alpha \in \mathfrak{h}^*$ satisfying the above equation. Finally, by the eigenspace associated to the eigenvalue α, we mean the subspace of vectors v in V satisfying the above equation.

In light of this notation, it should be clear that any finite-dimensional representation V of $\mathfrak{sl}_3$ has a decomposition

$$V = \bigoplus V_\alpha$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$. Now the question arises: what elements of $\mathfrak{sl}_3$ will play the role of X and Y as before? The key lies in the commutation relations

$$[H, X] = 2X \quad \text{and} \quad [H, Y] = -2Y \ ,$$

i.e. X and Y are eigenvectors for the adjoint action of H on $\mathfrak{sl}_2$. In our present case, these are precisely the X_i, Y_j.

Finish writing this up?

4 A determinant identity

Consider the determinant $\det(\lambda I - A)$ where $A = \left(\begin{smallmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{smallmatrix}\right)$ is some 2×2 matrix. Explicit computation yields that

$$\begin{aligned} \det(\lambda I - A) &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - \lambda \operatorname{tr} A + \det A. \end{aligned}$$

We can further rewrite this identity using wedge powers. Recall that since A is a linear map $A : \mathbb{C}^2 \to \mathbb{C}^2$, we get a map $\Lambda^2 A : \Lambda^2\mathbb{C}^2 \to \Lambda^2\mathbb{C}^2$. But since $\Lambda^2\mathbb{C}^2$ is one-dimensional (write out the basis), $\Lambda^2 A$ must be the determinant map, as the determinant is the unique columnwise n-linear that is alternating and preserves the identity. Consequently,

$$\det(\lambda I - A) = \lambda^2 - \lambda \operatorname{tr}(\Lambda^1 A) + \operatorname{tr}(\Lambda^2 A).$$

One might ask how this identity generalizes to higher dimensions. It should be clear that in n dimensions, the λ^n term will have coefficient 1 and the λ^0 term will have coefficient $\operatorname{tr}(\Lambda^n \mathbb{C}^n)$. We will proceed by induction in order to show that

$$\det(\lambda I - A) = \lambda^n - \lambda^{n-1} \operatorname{tr}\left(\Lambda^1 A\right) + \lambda^{n-2} \operatorname{tr}\left(\Lambda^2 A\right) + \cdots + \lambda^0 \operatorname{tr}\left(\Lambda^n A\right)$$

Do the induction

5 B semi-invariants

Throughout this section, B is the subgroup of upper-triangular matrices in SL_n, and U is the subgroup of B with 1's along the diagonal (the Heisenberg group). We wish to compute the subalgebra of $\mathbb{C}[SL_n]^H = \mathbb{C}[SL_n/H]$ that is semi-invariant under the right-action of B. why?

Definition 1. Let a group G act on a vector space V. We say that $v \in V$ is semi-invariant under G if, for all $g \in G$

$$f \cdot g = \chi(g) f$$

where χ is an algebraic character $\chi : G \to \mathbb{C}^\times$ (and similarly for left actions).

Lemma 3. *Elements of $\mathbb{C}[SL_n/H]$ semi-invariant under B are also invariant under U.*

Proof. Suppose $f \in \mathbb{C}[SL_n/H]$ is semi-invariant under B, i.e. for $b \in B$

$$f \cdot b = \chi(b) f$$

where χ is a algebraic character, $\chi : B \to \mathbb{C}^\times$. It suffices to show that $\chi(u) = 1$ for $u \in U$. First note that $U \cong \mathbb{C}^{n(n-1)/2}$ and that χ restricted to U is a regular polynomial map. In particular, if $u = (u_1, \ldots, u_n)$, then $\chi(u)$ is a polynomial in $u_1, \ldots, u_n$. Since the polynomial maps into $\mathbb{C}^\times$, it must not vanish. By the maximum modulus principle, it follows that χ is constant. In particular, $\chi = 1$, as it is a homomorphism. $\square$

Theorem 4. *Elements of $f \in \mathbb{C}[SL_n/H]$ semi-invariant under B are the highest-weight vectors of $\mathbb{C}[SL_n/H]$ as a representation of SL_n.*

Proof. Suppose f is semi-invariant under B; by the above lemma, it must be invariant under the action of U: $f \cdot U = f$ This implies that for each $X \in \mathrm{Lie}(U)$, $f \cdot X = 0$. Incidentally, one can check that $\mathrm{Lie}(U)$ is simply the space of strictly upper-triangular matrices [Hal03], whose basis is precisely the space of raising operators X_i that we encounter in the representation theory of SL_n. That each X_i annihilates f implies that f must be a highest-weight vector in $\mathbb{C}[SL_n/H]$ as a SL_n-module (and conversely). □

right action! does that change much?

This result allows us to compute the B semi-invariants in $\mathbb{C}[SL_n/H]$ indirectly; namely by finding the highest-weight vectors in $\mathbb{C}[SL_n/H]$.

6 Wonderful compactification via symmetric varieties

This section serves primarily as notes from [DCP83], in which the authors take a semisimple adjoint group G along with an involution (automorphism of order 2) σ and construct a wonderful compactification of the symmetric variety G/G^σ, where G^σ is the subgroup of G invariant under the action of σ. Here we apply a similar method for our case of SL_n/H.

For the sake of concreteness, let us start with the simple example of $G = \mathrm{SL}_2$ and H the subgroup of diagonal matrices. Consider the involution $\sigma : \mathrm{SL}_2 \to \mathrm{SL}_2$, given by the conjugation

$$\sigma : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} x & -y \\ -z & w \end{pmatrix}$$

Note that $\mathrm{SL}^\sigma = H$ (in line with the master plan to wonderfully compactify SL_2/H). This involution descends to an involution on the Lie algebra $\mathfrak{sl}_2$. To see this, we compute the induced Lie algebra homomorphism via the commutative diagram

$$\begin{tikzcd} \mathrm{SL}_2 \arrow[r, "\sigma"] & \mathrm{SL}_2 \\ \mathfrak{sl}_2 \arrow[u, "\exp"] \arrow[r, "\rho"] & \mathfrak{sl}_2 \arrow[u, "\exp"'] \end{tikzcd}$$

which we call $\rho : \mathfrak{sl}_2 \to \mathfrak{sl}_2$. Given $A \in \mathfrak{sl}_2$, ρ acts as

$$\rho : A \mapsto \left.\frac{d}{dt}\right|_{t=0} \sigma(\exp(tA)).$$

In particular, given the conjugation σ earlier, we can compute the action of ρ on the basis $\{H, X, Y\}$ of $\mathfrak{sl}_2$:

$$\begin{aligned} \rho(H) &= \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} t & \\ & -t \end{pmatrix} = H \\ \rho(X) &= \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix} = -X \\ \rho(Y) &= \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} 1 & \\ -t & 1 \end{pmatrix} = -Y. \end{aligned}$$

In words, ρ negates the X and Y-axes in $\mathfrak{sl}_2$.

References

[DCP83] Corrado De Concini and Claudio Procesi. Complete symmetric varieties. In *Invariant theory*, pages 1–44. Springer, 1983.

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, 1991.

[Hal03] Brian Hall. *Lie groups, Lie algebras, and Representations: An Elementary Introduction*, volume 222 of *Graduate Texts in Mathematics*. Springer, 2003.