Introduction to Algebraic Topology PSET 6

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Proposition 1. *Hatcher exercise 1.3.10*

Proof. See drawing below. □

Proposition 2. *Hatcher exercise 1.3.18*

Proof. Normality of an abelian covering $p : \tilde{X} \to X$ implies that $G(\tilde{X}) = \pi_1(\tilde{X}, \tilde{x}_0)/H$, where $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then $H \supset [\pi_1(\tilde{X}, \tilde{x}_0), \pi_1(\tilde{X}, \tilde{x}_0)]$. Now, pick $\tilde{X}$ such that $H = [\pi_1(\tilde{X}, \tilde{x}_0), \pi_1(\tilde{X}, \tilde{x}_0)]$. By the Galois classification of covering spaces, then, $\tilde{X}$ covers all abelian covers of X, because any abelian cover will have associated subgroup containing the commutator subgroup, and $\tilde{X}$ is unique up to isomorphism. For $S^1 \vee S^1$, the universal abelian covering space must have deck group $\mathbb{Z} \oplus \mathbb{Z}$, and hence we can take the corresponding lattice $\mathbb{Z}^2$ as the covering space where we project the vertices to the wedged point, and the horizontal and vertical directions to a and b respectively. Similarly for $S^1 \vee S^1 \vee S^1$ – the universal abelian covering space is the lattice $\mathbb{Z}^3$. □

Proposition 3. *Let $n \geq 3$ be a natural number. Prove that the free group on two generators has a subgroup that is isomorphic to the free group on n generators.*

Proof. Let $\tilde{X}$ the wedge sum of a circle to n circles at n distinct points. The symmetry group of $\tilde{X}$ is clearly $\mathbb{Z}_n$, which acts as a covering space action on $\tilde{X}$. Hence the quotient, $X = \tilde{X}/\mathbb{Z}_n$, which is simply $S^1 \vee S^1$, is covered normally by $\tilde{X}$. Since $\tilde{X}$ is homotopy equivalent to $\vee^n S^1$ (with the free product taken n times), we find that $*^n\mathbb{Z} \hookrightarrow \mathbb{Z} * \mathbb{Z}$, as the fundamental group of the covering space must inject into the fundamental group of the base space. □

Proposition 4. *Hatcher exercise 1.3.20*

Proof. Consider the three-sheeted cover of the Klein bottle by the Klein bottle as depicted below. Recall that the Klein bottle has fundamental group $\langle a, b \mid abab^{-1} \rangle$. We see that the lift of b at x is a loop, whereas at y it is simply a path from y to z. Hence $b \in p_*(\pi_1(X, x))$ but $b \notin p_*(\pi_1(X, y))$ by Hatcher proposition 1.31, and thus the covering is not normal.

Next consider the universal cover $\mathbb{R}^2$ of the Klein bottle. If we tile the plane by squares as in Hatcher example 1.42. Consider the subgroup of translations of the plane isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ generated by $\langle a^3, a^2b^2 \rangle$ as depicted in the diagram below. It is clear that $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z} \cong T^2$, as a six-fold covering of the Klein bottle. Algebraically, we can confirm that this covering is not normal; consider conjugation by b. The relation $abab^{-1}$ implies that $bab^{-1} = a^{-1}$, and hence $ba^2b^2b^{-1} = a^{-2}b^2$, which is not an element of $\langle a^3, a^2b^2 \mid abab^{-1} \rangle$ (this is the image of the fundamental group of the torus in the fundamental group of the Klein bottle). We can check this topologically by noting that the lift of a^2b^2 at the point x is a loop, but the lift at the point y is just a path to z (which is not a loop on the torus). □

Proposition 5. *Hatcher exercise 1.3.24*

Proof. Note carefully that here we are not provided with the semilocally simply-connected condition (on X/G), and hence we cannot directly use the Galois correspondence provided by Hatcher theorem 1.38. Instead, we can only use the path-connectedness and local path-connectedness of X/G (which follows from the same for X, due to the nature of the quotient topology).

(a) We have a sequence of injections

$$\pi_1(X) \hookrightarrow \pi_1(Y) \hookrightarrow \pi_1(X/G)$$

and since $\pi_1(X) \lhd \pi_1(X/G)$, we find that $\pi_1(X) \lhd \pi_1(Y)$, i.e. $X \to Y$ is a normal covering. Denote by $H \leq G$ the deck group $G(X \to Y)$. Then, if we write out

$$X \xrightarrow{p_1} Y \xrightarrow{p_2} X/G$$

$$X \xrightarrow{q_1} X/H \xrightarrow{q_2} X/G$$

with $p \equiv p_2 \circ p_1 = q_2 \circ q_1 \equiv q$ (by construction) we find that

$$H \cong \frac{\pi_1(Y)}{p_{1*}(\pi_1(X))} \cong \frac{p_{2*}(\pi_1(Y))}{p_*(\pi_1(X))}$$

but also that

$$H \cong \frac{\pi_1(X/H)}{q_{1*}(\pi_1(X))} \cong \frac{q_{2*}(\pi_1(X/H))}{q_*(\pi_1(X))}.$$

This implies that $p_{2*}(\pi_1(Y)) \cong q_{2*}(\pi_1(X/H))$, which completes the proof by Hatcher proposition 1.37.

(b)

(c)

$\square$