Complex Analysis and Riemann Surfaces: Final

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Problem 1

Let $L \to (X, g_{\bar{k}j})$ be a holomorphic line bundle over a compact Kähler manifold. Let h be a smooth metric on L.

(a) Given the smooth metric h on L, we can define an L^2 inner product on sections $\phi, \psi \in \Gamma(X, L)$ as

$$\langle \phi, \psi \rangle = \int \phi \overline{\psi} h \omega^n / n!.$$

Similarly, for $\phi, \psi \in \Gamma(X, L \otimes \Lambda^{0,1})$, we can define the inner product to be

$$\langle \phi, \psi \rangle = \int \phi_{\bar{j}} \overline{\psi_{\bar{k}}} h g^{\bar{j}k} \frac{\omega^n}{n!},$$

and for $\phi, \psi \in \Gamma(X, L \otimes \Lambda^{0,2})$,

$$\langle \phi, \psi \rangle = \int \phi_{\bar{j}\bar{k}} \overline{\psi_{\bar{l}\bar{m}}} h g^{\bar{j}l} g^{\bar{k}m} \frac{\omega^n}{n!}.$$

Consider now, part of the Dolbeault complex,

$$\begin{tikzcd}
\Gamma(X, L) \arrow[r, bend left, "\bar{\partial}"] & \Gamma(X, L \otimes \Lambda^{0,1}) \arrow[l, bend left, "\bar{\partial}^\dagger"] \arrow[r, bend left, "\bar{\partial}"] & \Gamma(X, L \otimes \Lambda^{0,2}) \arrow[l, bend left, "\bar{\partial}^\dagger"]
\end{tikzcd}$$

Let us compute the first formal adjoint operator. By definition,

$$\langle \bar{\partial} \phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle,$$

for $\phi \in \Gamma(X, L)$ and $\psi \in \Gamma(X, L \otimes \Lambda^{0,1})$. Writing $\bar{\partial}\phi = \partial_{\bar{j}} \phi d\bar{z}^j$ and $\psi = \psi_{\bar{k}} d\bar{z}^k$, and using the inner product defined as above, we find that

$$\int \partial_{\bar{j}} \phi \overline{\psi_{\bar{k}}} h g^{k\bar{j}} \frac{\omega^n}{n!} = \int \phi \overline{\bar{\partial}^\dagger \psi} h \frac{\omega^n}{n!}.$$

Let us define $W^{\bar{j}} \equiv \overline{\psi_{\bar{k}} h g^{j\bar{k}}}$. Observe now that

$$\begin{aligned}
\left(\partial_{\bar{j}} \phi\right) W^{\bar{j}} &\equiv \left(\nabla_{\bar{j}} \phi\right) W^{\bar{j}} \\
&= \nabla_{\bar{j}} \phi^{\bar{j}} - \phi \left(\nabla_{\bar{j}} W^{\bar{j}}\right).
\end{aligned}$$

This will be useful when integrating by parts. Note that the nth wedge power of ω simplifies to yield

$$\int \left(\partial_{\bar{j}}\phi\right) W^{\bar{j}} \left(\det g_{\bar{q}p}\right) = \int \nabla_{\bar{j}} \left(\phi W^{\bar{j}}\right) \det g_{\bar{q}p} - \int \phi \left(\nabla_{\bar{j}} W^{\bar{j}}\right).$$

We claim that if the metric $g_{\bar{k}j}$ is Kähler, then $\int \nabla_{\bar{j}}(\phi W^{\bar{j}}) \det g_{\bar{q}p} = 0$. To see this, first define $V^{\bar{j}} = \phi W^{\bar{j}}$. Consider

$$\begin{aligned}
\left(\nabla_{\bar{j}} V^{\bar{j}}\right) \det g_{\bar{q}p} &= \left(\partial_{\bar{j}} V^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{j}\bar{k}} V^{\bar{k}}\right) \det g_{\bar{q}p} \\
&= \partial_{\bar{j}} \left(V^{\bar{j}} \det g_{\bar{q}p}\right) - V^{\bar{j}} \left(\partial_{\bar{j}} \det g_{\bar{q}p}\right) + \Gamma^{\bar{j}}_{\bar{j}\bar{k}} V^{\bar{k}} \det g_{\bar{q}p}.
\end{aligned}$$

Now note that

$$\partial_{\bar{j}} \det g_{\bar{q}p} = \left(\det g_{\bar{q}p}\right) g^{l\bar{n}} \partial_{\bar{j}} g_{\bar{n}l},$$

which comes from the fact that

$$\begin{aligned}
\delta \log\left(\det A\right) &= \sum \frac{\delta\lambda_j}{\lambda_j} \\
&= \operatorname{tr}\left(A^{-1}\delta A\right).
\end{aligned}$$

But now recall that for $g_{\bar{k}j}$ is Kähler if and only if $\Gamma^{\bar{j}}_{\bar{k}\bar{m}} = \Gamma^{\bar{j}}_{\bar{m}\bar{k}}$, and we see the last two terms in the expression above cancel. Hence the term picked up by integration by parts vanishes, and we are left with the equality

$$-\int \phi \nabla_{\bar{j}} \overline{\left(\psi_{\bar{k}} h g^{j\bar{k}}\right)} \frac{\omega^n}{n!} = \int \phi \overline{\left(-g^{j\bar{k}} \nabla_j \psi_{\bar{k}}\right)} h \frac{\omega^n}{n!}.$$

This yields the desired formula:

$$\boxed{\bar{\partial}^\dagger \psi = -g^{j\bar{k}} \nabla_j \psi_{\bar{k}}.}$$

A similar approach works for the other adjoint operator. Take $\phi \in \Gamma(X, L \otimes \Lambda^{0,1})$ and $\psi \in \Gamma(X, L \otimes \Lambda^{0,2})$. By definition, the formal adjoint is such that

$$\langle \bar{\partial}\phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle.$$

We can write $\psi = \frac{1}{2} \sum \psi_{\bar{l}\bar{m}} d\bar{z}^l \wedge d\bar{z}^m$ and

$$\begin{aligned}
\bar{\partial}\phi &= \sum \partial_{\bar{k}} \phi_{\bar{j}} d\bar{z}^k \wedge d\bar{z}^j \\
&= \frac{1}{2} \sum \left(\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}}\right) d\bar{z}^k \wedge d\bar{z}^j.
\end{aligned}$$

Then the above requirement thus becomes

$$\int_X \frac{1}{2} \left(\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}}\right) \overline{\psi_{\bar{l}\bar{m}}} h g^{l\bar{j}} g^{m\bar{k}} \omega^n / n! = \int_X \phi_{\bar{j}} \overline{(\bar{\partial}^\dagger \psi)_{\bar{k}}} h g^{k\bar{j}} \omega^n / n!.$$

Note now that

$$\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}} = \nabla_{\bar{k}}\phi_{\bar{j}} - \nabla_{\bar{j}}\phi_{\bar{k}}.$$

Now we can simplify the left-hand side by de-antisymmetrizing and integrating by parts:

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\int_X (\nabla_{\bar{k}}\phi_{\bar{j}} - \nabla_j\phi_{\bar{k}})\overline{\psi_{\bar{l}\bar{m}}}hg^{l\bar{j}}g^{m\bar{k}}\omega^n/n! \\
&= \int_X (\nabla_{\bar{k}}\phi_{\bar{j}})\overline{\psi_{\bar{l}\bar{m}}}hg^{l\bar{j}}g^{m\bar{k}}\omega^n/n! \\
&= \int_X \phi_{\bar{j}}\overline{(-g^{k\bar{m}}\nabla_k\psi_{\bar{l}\bar{m}})}g^{l\bar{j}}\omega^n/n!
\end{aligned}$$

Hence we can write the formal adjoint as

$$\boxed{(\bar{\partial}^\dagger\psi)_{\bar{l}} = -g^{k\bar{m}}\nabla_k\psi_{\bar{l}\bar{m}}.}$$

(b) Let $\Delta = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$ on $\Gamma(X, L\otimes\Lambda^{0,1})$. Set $\phi = \sum\phi_{\bar{j}}d\bar{z}^j \in \Gamma(X, L\otimes\Lambda^{0,1})$. First note that

$$\begin{aligned}
\left(\bar{\partial}\bar{\partial}^\dagger\phi\right) &= \bar{\partial}\left(-g^{j\bar{k}}\nabla_j\phi_{\bar{k}}\right) \\
&= \partial_{\bar{l}}\left(-g^{j\bar{k}}\nabla_j\phi_{\bar{k}}\right)d\bar{z}^l.
\end{aligned}$$

Hence, noting that the expression in parentheses is a section of a holomorphic bundle, the $\partial_{\bar{j}}$ is simply a covariant derviative, which commutes with the metric, and we can write

$$\left(\bar{\partial}\bar{\partial}^\dagger\phi\right)_{\bar{l}} = -g^{j\bar{k}}\nabla_{\bar{l}}\nabla_j\phi_{\bar{k}}.$$

Next note that

$$\bar{\partial}\phi = \frac{1}{2}\sum\left(\nabla_{\bar{k}}\phi_{\bar{j}} - \nabla_{\bar{j}}\phi_{\bar{k}}\right)d\bar{z}^k\wedge d\bar{z}^j$$

for $g_{\bar{k}j}$ Kähler. Hence we can write

$$\begin{aligned}
\left(\bar{\partial}^\dagger\bar{\partial}\phi\right)_{\bar{l}} &= -g^{k\bar{m}}\nabla_k(\bar{\partial}\phi)_{\bar{l}\bar{m}} \\
&= -g^{k\bar{m}}\nabla_k\left(\nabla_{\bar{m}}\phi_{\bar{l}} - \nabla_{\bar{l}}\phi_{\bar{m}}\right) \\
&= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + g^{k\bar{m}}\nabla_k\nabla_{\bar{l}}\phi_{\bar{m}}.
\end{aligned}$$

Summing the two terms of the Laplacian and switching appropriate dummy indices, we find that

$$\begin{aligned}
(\Box\phi)_{\bar{l}} &= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + g^{k\bar{m}}\nabla_k\nabla_{\bar{l}}\phi_{\bar{m}} - g^{k\bar{m}}\nabla_{\bar{l}}\nabla_{\bar{k}}\phi_{\bar{m}} \\
&= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + g^{k\bar{m}}[\nabla_k, \nabla_{\bar{l}}]\phi_{\bar{m}} \\
&= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + g^{k\bar{m}}\left(F_{\bar{l}k}\phi_{\bar{m}} + R^{\bar{p}}_{\bar{l}k\bar{m}}\phi_{\bar{p}}\right) \\
&= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + F^{\bar{m}}_{\bar{l}}\phi_{\bar{m}} + R^{\bar{m}}_{\bar{l}}\phi_{\bar{m}} \\
&= -g^{k\bar{m}}\nabla_k\nabla_{\bar{m}}\phi_{\bar{l}} + (F^{\bar{m}}_{\bar{l}} + R^{\bar{m}}_{\bar{l}})\phi_{\bar{m}}.
\end{aligned}$$

(c) Now suppose that $R^{\bar{m}}_{\bar{l}} + F^{\bar{m}}_{\bar{l}} \geq \epsilon\delta^{\bar{m}}_{\bar{l}}$. If we compute the inner product

$$\begin{aligned}
\langle\phi, \Delta\phi\rangle &= \int_X (\Delta\phi)_{\bar{j}}\overline{\phi_{\bar{k}}}hg^{k\bar{j}}\frac{\omega^n}{n!} \\
&= -\int_X g^{l\bar{m}}\nabla_l\nabla_{\bar{m}}\phi_{\bar{j}}\overline{\phi_{\bar{k}}}g^{k\bar{j}}h\frac{\omega^n}{n!} + \int_X (F^{\bar{m}}_{\bar{j}}\phi_{\bar{m}} + R^{\bar{m}}_{\bar{j}}\phi_{\bar{m}})\overline{\phi_{\bar{k}}}hg^{k\bar{j}}\frac{\omega^n}{n!}
\end{aligned}$$

The first term can be simplified as

$$
\begin{aligned}
-\int_X g^{l\bar{m}}\nabla_l\nabla_{\bar{m}}\phi_{\bar{j}}\overline{\phi_{\bar{k}}}g^{k\bar{j}}h\frac{\omega^n}{n!} &= -\int_X \nabla_l(g^{l\bar{m}}\nabla_{\bar{m}}\phi_{\bar{j}})\overline{\phi_{\bar{k}}}g^{k\bar{j}}h\frac{\omega^n}{n!} \\
&= \int_X \nabla_{\bar{m}}\phi_{\bar{j}}\overline{\nabla_{\bar{l}}\phi_{\bar{k}}}g^{l\bar{m}}g^{k\bar{j}}h\frac{\omega^n}{n!} \\
&= ||\nabla_{\bar{m}}\phi_{\bar{j}}||^2,
\end{aligned}
$$

and the second term can be simplified as

$$
\int_X (F_{\bar{j}}^{\bar{m}}\phi_{\bar{m}} + R_{\bar{j}}^{\bar{m}}\phi_{\bar{m}})\overline{\phi_{\bar{k}}}hg^{k\bar{j}}\frac{\omega^n}{n!} \geq \epsilon||\phi||^2
$$

Hence we find that $||\bar{\partial}\phi||^2 + ||\bar{\partial}^\dagger\phi||^2 \geq \epsilon||\phi||^2$, because $\langle\phi, \Delta\phi\rangle = \langle\phi, \bar{\partial}\bar{\partial}^\dagger\phi\rangle + \langle\phi, \bar{\partial}^\dagger\bar{\partial}\phi\rangle = ||\bar{\partial}^\dagger\phi||^2 + ||\bar{\partial}\phi||^2$ by construction of the adjoint.

(d) Define the domains of the operators $\bar{\partial}$ and $\bar{\partial}^\dagger$ by

$$
\text{Dom } \bar{\partial} = \{\phi \in L^2; \bar{\partial}\phi \in L^2\}
$$

and

$$
\text{Dom } \bar{\partial}^\dagger = \{\psi \in L^2; v \equiv \bar{\partial}^\dagger\psi \in L^2, \text{ and } \langle\bar{\partial}\phi, \psi\rangle = \langle\phi, v\rangle \text{ for } \phi \in \text{Dom } \bar{\partial}\},
$$

where on the right hand side, $\bar{\partial}$ and $\bar{\partial}^\dagger$ are taken in the sense of distributions. If we assume that the space of smooth sections is dense in Dom $\bar{\partial} \cap$ Dom $\bar{\partial}^\dagger$ with respect to the norm $||\phi|| + ||\bar{\partial}\phi|| + ||\bar{\partial}^\dagger\phi||$, the inequality above is preserved for the following reasons. Let ϕ_n be a sequence of smooth sections converging to ϕ in the intersection of the domains. Then, denoting the given norm by $||\cdot||_G$ and using the triangle inequality, we find that

$$
\begin{aligned}
||\phi||_G^2 &\leq ||\phi - \phi_n||_G^2 + ||\phi_n||_G^2 \\
&= ||\phi - \phi_n||_G^2 + ||\phi_n||^2 + ||\bar{\partial}\phi_n||^2 + ||\bar{\partial}^\dagger\phi_n||^2 \\
&\leq ||\phi - \phi_n||_G^2 + \left(1 + \frac{1}{\sqrt{\epsilon}}\right)\left(||\bar{\partial}\phi_n||^2 + ||\bar{\partial}^\dagger\phi_n||^2\right),
\end{aligned}
$$

and taking $n \to \infty$, we find $||\phi||^2 \leq 1/\sqrt{\epsilon}(||\bar{\partial}\phi_n||^2 + ||\bar{\partial}^\dagger\phi_n||^2)$, as desired.

(e) Let $u \in$ Dom $\bar{\partial}_0^\dagger$. We claim that if we have a decomposition $u = u_1 + u_2$ for $u_1 \in \ker\bar{\partial}_1$, $u_2 \perp \ker\bar{\partial}_1$, then $u_1 \in$ Dom $\bar{\partial}_1 \cap$Dom $\bar{\partial}_0^\dagger$. Note first that in the Dolbeault complex, range $\bar{\partial}_0 \subset \ker\bar{\partial}_1$ and so if u_2 is orthogonal to $\ker\bar{\partial}_1$ then u_2 is also orthogonal to range $\bar{\partial}_0$. In other words, $\langle\bar{\partial}_0\psi, u_2\rangle = 0$ for all $\psi \in$ Dom $\bar{\partial}_0$, and so we find that u_2 falls into Dom $\bar{\partial}_0^\dagger$ as defined above. This in turn implies that $u_1 \in$ Dom $\bar{\partial}_0^\dagger$.

(f) Now let $f \in L^2(X, L \otimes \Lambda^{0,1})$ satisfying $\bar{\partial}f = 0$. Consider the linear functional

$$
L(\bar{\partial}_0^\dagger u) = \langle u, f\rangle
$$

for all $u \in$ Dom $\bar{\partial}_0^\dagger$. This functional is not *a priori* well-defined, but turns out to be, as we will show below. Decompose $u = u_1 + u_2$ with $u_1 \in \ker\bar{\partial}_1, u_2 \perp \ker\bar{\partial}_1$. We can write

$$
\langle u, f\rangle = \langle u_1, f\rangle + \langle u_2, f\rangle = \langle u_1, f\rangle.
$$

Applying the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}|L(\bar{\partial}_0^\dagger u)|^2 &= |\langle u, f\rangle|^2 \\ &\leq ||u_1||^2 \cdot ||f||^2 \\ &\leq \frac{1}{\epsilon}(||\bar{\partial} u_1||^2 + ||\bar{\partial}^\dagger u_1||^2)||f||^2 \\ &= \frac{1}{\epsilon}||\bar{\partial}^\dagger u_1||^2 \cdot ||f||^2.\end{aligned}$$

and hence

$$|L(\bar{\partial}_0^\dagger u)| \leq \frac{1}{\sqrt{\epsilon}}||\bar{\partial}^\dagger u|| \cdot ||f||.$$

Note that this implies that the functional is well-defined because if we have u, u' with $\bar{\partial}_0^\dagger u = \bar{\partial}_0^\dagger u'$, we find that $|\langle u - u', f\rangle|^2 \leq 0$.

Now recall the Hahn-Banach theorem: let $V \subset B$ be a subspace of a Banach space and L a linear functional $V \ni v \mapsto L(v)$ with $|L(v)| \leq A||v||$ - then there exists an extension $\tilde{L}$ of L to all of B satisfying $|\tilde{L}v| \leq A||v||$ for all $v \in B$. Applying the Hahn-Banach theorem in our case, and assuming that the resulting extension is represented as an inner product with a section $u \in L^2(X, \Lambda)$, we find

$$\begin{aligned}\tilde{L}(\bar{\partial}_0^\dagger v) &= L(\bar{\partial}_0^\dagger v) \\ \langle \bar{\partial}_0^\dagger v, u\rangle &= \langle v, f\rangle,\end{aligned}$$

i.e. $\bar{\partial} u = f$ in the sense of distributions, and that

$$||u|| \leq \frac{1}{\sqrt{\epsilon}}||f||,$$

using the fact that $||u|| = ||\tilde{L}|| \leq \frac{1}{\sqrt{\epsilon}}||u|| \cdot ||f||/||u||$ from the Hahn-Banach theorem.

Problem 2

Let $E \to X$ be a smooth vector bundle over a smooth compact manifold X. Given a connection A on E, let $F = dA + A \wedge A$ be its curvature form. For each integer m, define the $2m$-forms $c_m(F)$ by

$$c_m(F) = \operatorname{tr}(F \wedge \cdots \wedge F)$$

with m factors of F on the right-hand-side.

(a) Let us show that $c_m(F)$ is always closed, for any connection A. Recall first the Bianchi identity

$$dF + A \wedge F - F \wedge A = 0.$$

Taking the differential, we find

$$\begin{aligned}dc_m(F) &= d\left(\operatorname{tr}(F \wedge \cdots \wedge F)\right) \\ &= \operatorname{tr}\left(dF \wedge \cdots F + \cdots F \wedge \cdots \wedge dF\right) \\ &= m \operatorname{tr}\left(dF \wedge F^{m-1}\right) \\ &= m \operatorname{tr}\left((F \wedge A - A \wedge F) \wedge F^{m-1}\right) \\ &= m \operatorname{tr}\left(F \wedge A \wedge F^{m-1}\right) - \operatorname{tr}\left(A \wedge F^m\right) \\ &= 0,\end{aligned}$$

where we have used the Bianchi identity and the cyclic property of the trace.

(b) Suppose $m = 3$. Let A and A_0 be any two connections. We wish to find a 5-form T_5 such that $dT_5 = c_3(F) - c_3(F_0)$. We can do this by defining $B = A - A_0$ and $A_t = A_0 + tB$ with $F_t = F(A_t)$ and noting that

$$\begin{aligned} c_3(A) - c_3(A_0) &= \int_0^1 \frac{d}{dt} \operatorname{tr}(F_t \wedge F_t \wedge F_t)\, dt \\ &= 3 \int_0^1 \operatorname{tr}\left(\dot{F}_t \wedge F_t \wedge F_t\right) dt, \end{aligned}$$

where we have used the cyclic property of the trace and permuted the wedge product appropriately. We can write

$$\dot{F}_t = d\dot{A}_t + \dot{A}_t \wedge A_t + A_t \wedge \dot{A}_t = dB + B \wedge A_t + A_t \wedge B$$

and

$$\begin{aligned} \operatorname{tr}(\dot{F}_t \wedge F_t \wedge F_t) &= \operatorname{tr}\left((dB + B \wedge A_t + A_t \wedge B) \wedge F_t \wedge F_t\right) \\ &= \operatorname{tr}\left(dB \wedge F_t \wedge F_t + B \wedge A_t \wedge F_t \wedge F_t + A_t \wedge B \wedge F_t \wedge F_t\right). \end{aligned}$$

Consider only the first term in the trace

$$\begin{aligned} dB \wedge F_t \wedge F_t &= d(B \wedge F_t \wedge F_t) + B \wedge dF_t \wedge F_t + B \wedge F_t \wedge dF_t \\ &= d(B \wedge F_t \wedge F_t) + B \wedge F_t \wedge A_t \wedge F_t - B \wedge A_t \wedge F_t \wedge F_t \\ &+ B \wedge F_t \wedge F_t \wedge A_t - B \wedge F_t \wedge A_t \wedge F_t \\ &= d(B \wedge F_t \wedge F_t) + B \wedge (-A_t \wedge F_t \wedge F_t + F_t \wedge F_t \wedge A_t). \end{aligned}$$

Inserting this back into the trace, we find that

$$\begin{aligned} c_3(A) - c_3(A_0) &= 3 \int_0^1 \operatorname{tr}\left(d(B \wedge F_t \wedge F_t\right) dt \\ &= d\left(\int_0^1 3 \operatorname{tr}(B \wedge F_t \wedge F_t) dt\right), \end{aligned}$$

i.e. $T_5 = 3 \int_0^1 \operatorname{tr}(B \wedge F_t \wedge F_t) dt$.

(c) Assume now that $E \to X$ is a holomorphic vector bundle over a compact complex manifold X. For each metric H on E, let A be the corresponding Chern unitary connection and let $F(H)$ be its curvature form. Let H and H_0 be two metrics and define $h \equiv H_0^{-1} H$ and D_{H_0} to be the covariant derivative with respect to H_0. Then we see that

$$\begin{aligned} F(H) &= -\bar{\partial}\left(H^{-1} \partial H\right) \\ &= -\bar{\partial}\left(h^{-1} H_0^{-1} \partial (H_0 h)\right) \\ &= -\bar{\partial}\left(h^{-1} \partial h + h^{-1} H_0^{-1} (\partial H_0) h\right) \\ &= -\bar{\partial}\left(h^{-1} (\partial h + H_0^{-1} (\partial H_0) h - h H_0^{-1} \partial H_0) + H_0^{-1} \partial H_0\right) \\ &= -\bar{\partial}(h^{-1} D_{H_0} h) + F(H_0) \end{aligned}$$

and hence

$$F(H) - F(H_0) = -\bar{\partial}(h^{-1} D_{H_0} h).$$

(d) Let $t \mapsto H(t)$ be a one-parameter family of metrics, $h(t) = H_0^{-1}H(t)$ and $t \mapsto F(H(t))$ be the corresponding family of curvature forms. The time-evolution of the curvature is given by differentiating the identity above

$$\begin{aligned}
\dot{F} &= \bar{\partial}\partial_t(h^{-1}D_{H_0}h) \\
&= -\bar{\partial}\partial_t\left(h^{-1}(\partial h + H_0^{-1}\partial H_0 h - hH_0^{-1}\partial H_0)\right) \\
&= -\bar{\partial}\partial_t\left(h^{-1}\partial h + h^{-1}H_0^{-1}\partial H_0 h\right) \\
&= -\bar{\partial}\partial_t\left(h^{-1}H_0^{-1}\partial(H_0 h)\right) \\
&= -\bar{\partial}\partial_t(H^{-1}\partial H) \\
&= -\bar{\partial}\left(-h^{-1}\dot{h}h^{-1}H_0^{-1}\partial(H_0 h) + h^{-1}H_0^{-1}\partial(H_0\dot{h})\right) \\
&= -\bar{\partial}\left(-h^{-1}\dot{h}(H^{-1}\partial H) + H^{-1}\partial(Hh^{-1}\dot{h})\right) \\
&= -\bar{\partial}\left(-h^{-1}\dot{h}(H^{-1}\partial H) + \partial(h^{-1}\dot{h}) + (H^{-1}\partial H)(h^{-1}\dot{h})\right) \\
&= -\bar{\partial}\partial_H(-h^{-1}\dot{h}).
\end{aligned}$$

This identity allows us to refine the expression above,

$$\begin{aligned}
c_3(H) - c_3(H_0) &= \int_0^1 \frac{d}{dt}\operatorname{tr}(F(H(t)) \wedge F(H(t)) \wedge F(H(t)))dt \\
&= 3\int_0^1 \operatorname{tr}\left(\dot{F} \wedge F \wedge F\right) dt \\
&= 3\int_0^1 \operatorname{tr}\left((-\bar{\partial}D_H(h^{-1}\dot{h})) \wedge F \wedge F\right) dt \\
&= -3\bar{\partial}D_H \int_0^1 \operatorname{tr}\left((h^{-1}\dot{h})F \wedge F\right) dt,
\end{aligned}$$

where we have used the Bianchi identity and that $\bar{\partial}F = 0$ in order to commute the partials out. Defining $\mathcal{B}_2 \equiv 3\int_0^1 \operatorname{tr}\left((h^{-1}\dot{h})F \wedge F\right)$, we can write simply

$$c_3(H) - c_3(H_0) = -\bar{\partial}\partial\mathcal{B}_2.$$

Problem 3

Let $E \to (X, \omega)$ be a holomorphic vector bundle over a compact Kähler manifold. Define the *slope* $\mu(E)$ by

$$\mu(E) = \frac{1}{\operatorname{rk}(E)\operatorname{Vol}_\omega(X)}\int_X \operatorname{tr} F \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

(a) It is easy to see that $\mu(E)$ does not depend on the metric H on E defining the curvature F. The integrand is the Chern form $c_1(F)$, which, up to cohomology is independent of the metric. As exact terms do not contribute to the integral (the ω are closed, so one can use integration by parts) the integrand is completely independent of the metric.

Furthermore, if ω is replaced by another Kähler metric ω' in the same cohomology class, we can write $\omega' = \omega + d\theta$. Integrating by parts, the extra terms go to zero, as ω is closed as is the Chern form that is being integrated.

(b) Suppose E admits a metric H satisfying the Hermitian-Einstein equation

$$\Lambda F - \mu(E) I = 0$$

and let E' be a holomorphic subbundle of E. We wish to show that $\mu(E') \leq \mu(E)$. We first choose a holomorphic frame $\{e_a\}_{a=1,\ldots,r}$ for E and $\{e_a\}_{a=1,\ldots,s}$ a frame for E, where $r = \text{rk } E$ and $s = \text{rk } E'$. Now recall that

$$\begin{aligned} F^{\alpha}_{\bar{k}j\beta} &= -\partial_{\bar{k}} \left(H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta} \right) \\ &= -H^{\alpha\bar{\gamma}} \partial_{\bar{k}} \partial_j H_{\bar{\gamma}\beta} + H^{\alpha\bar{\lambda}} \partial_{\bar{k}} H_{\lambda\mu} H^{\mu\bar{\nu}} \partial_j H_{\bar{\gamma}\beta}. \end{aligned}$$

If we work at each point and assume that $H_{\bar{\alpha}\beta} = \delta_{\alpha\beta}$ (which can be done at a point), then we can contract to find

$$\begin{aligned} F^{\alpha}_{\bar{k}j\beta} &= -\partial_{\bar{k}} \partial_j H_{\bar{\alpha}\beta} + \sum_{\mu=1}^{r} \partial_{\bar{k}} H_{\bar{\alpha}\mu} \partial_j H_{\bar{\mu}\beta} \\ (F')^{\alpha}_{\bar{k}j\beta} &= -\partial_{\bar{k}} \partial_j H_{\bar{\alpha}\beta} + \sum_{\mu=1}^{s} \partial_{\bar{k}} H_{\bar{\alpha}\mu} \partial_j H_{\bar{\mu}\beta} \end{aligned}$$

where the α, β range from 1 to r and 1 to s for the first and second lines respectively. Taking the difference, we find

$$(F')^{\alpha}_{\bar{k}j\beta} = F^{\alpha}_{\bar{k}j\beta} - \sum_{\mu=s+1}^{r} \partial_{\bar{k}} H_{\bar{\alpha}\mu} \partial_j H_{\bar{\mu}\beta},$$

which is positive (in the sense that contracting it with vectors appropriately will always yield a positive quantity. We can now compute the slope of the subbundle as

$$\begin{aligned} \mu(E') &= \frac{1}{s\text{Vol}_\omega(X)} \int_X \text{tr}_{E'} \left(g^{j\bar{k}} F^{\alpha}_{\bar{k}j\beta} - g^{j\bar{k}} \sum \partial_j H_{\bar{\alpha}\gamma \partial_{\bar{k}} H_{\gamma\bar{\beta}}} \right) \frac{\omega^n}{n!} \\ &= \frac{1}{s\text{Vol}_\omega(X)} \int_X \text{tr}_{E'} \left(\mu(E) \delta^{\alpha}_{\beta} - g^{j\bar{k}} \sum \partial_j H_{\bar{\alpha}\gamma \partial_{\bar{k}} H_{\gamma\bar{\beta}}} \right) \frac{\omega^n}{n!} \\ &\leq \frac{1}{s\text{Vol}_\omega(X)} \int_X \mu(E) s \frac{\omega^n}{n!} \\ &= \mu(E), \end{aligned}$$

and hence we find the necessary condition that $\mu(E') \leq \mu(E)$ if E admits a Hermitian-Einstein metric.

Problem 4

Let $E \to (X, \omega)$ be a holomorphic vector bundle over a compact Kähler manifold. Let H_0, H be metrics on E and let $\{e_a\}$ be a frame of E which is orthonormal with respect to H_0 and which diagonalizes the endomorphism $h = H_0^{-1} H$. Let A^a_b be the connection forms of the Chern unitary connection with respect to H_0, in the frame $\{e_a\}$, i.e. $De_a = e_b A^b_a$. Let $\{e^a\}$ be the dual frame, and let μ_a be the eigenvalues of h.

(a) Since e_a and e^a are dual, $D_j e^a = -A^a_{jb} e^b$. We compute

$$\begin{aligned} D_j h &= D_j(\mu_a e_a \otimes e^a) \\ &= (\partial_j \mu_a) e_a \otimes e^a + \mu_a (D_j e_a) \otimes e^a + \mu_a e_a \otimes (D_j e^a) \\ &= (\partial_j \mu_a) e_a \otimes e^a + \mu_a (e_b A^b_{ja} \otimes e^a) - \mu_a (e_a \otimes A^a_{jb} e^b) \\ &= (\partial_j \mu_a) e_a \otimes e^a + (\mu_a - \mu_b) A^b_{ja} e_b \otimes e^a \end{aligned}$$

where we have switched dummy indices in the last term.

(b) It follows immediately from above that

$$|D_j h|^2 = \sum_a |D\mu_a|^2 + \sum_{a,b} |\mu_a - \mu_b|^2 |A^b_a|^2$$

because the metric has been chosen such that the usual basis of endomorphisms ($e_a \otimes e^b$, for all pairs a, b running from 1 to rk E) is orthonormal (which can be checked manually via the Hilbert-Schmidt norm, since e_a, e^a are dual orthonormal frames).

(c) Since $h^{-1} = \mu_a^{-1} e_a \otimes e^a$, we can work out

$$\begin{aligned} h^{-1} Dh &= (\mu_a^{-1} e_a \otimes e^a) \left((\partial_j \mu_a) e_a \otimes e^a + (\mu_a - \mu_b) A^b_{ja} e_b \otimes e^a \right) \\ &= \mu_a^{-1} (\partial_j \mu_a) e_a \otimes e^a + \mu_b^{-1} (\mu_a - \mu_b) A^b_{ja} e_b \otimes e^a. \end{aligned}$$

where we have used that $(e_a \otimes e^b)(e_c \otimes e^d) = \delta^b_c e_a \otimes e^d$ (which follows from orthonormality of the frames). Then, noting that $\partial_{\bar{k}} h = (\partial_{\bar{k}} \mu_a) e_a \otimes e^a + (\mu_a - \mu_b) A^b_{ja} e_b \otimes e^a$, we can write, using the product rule, and the identities above,

$$\begin{aligned} (\partial_{\bar{k}} h) h^{-1} D_j h = (\partial_j \mu_a)(\partial_{\bar{k}} \mu_a) \mu_a^{-1} e_a \otimes e^a + (\partial_{\bar{k}} \mu_b) \mu_b^{-1} (\mu_a - \mu_b) A^b_{ja} e_b \otimes e^a \\ + (\partial_j \mu_a) \mu_b^{-1} (\mu_a - \mu_b) A^b_{\bar{k}a} e_b \otimes e^a + (\mu_b - \mu_d)(\mu_a - \mu_b) \mu_b^{-1} A^d_{\bar{k}b} A^b_{ja} e_d \otimes e^a. \end{aligned}$$

If we now take the trace and contract with $g^{j\bar{k}}$, the expression simplifies considerably (since the middle two terms go to zero via $a = b$), and we obtain:

$$\begin{aligned} g^{j\bar{k}} \operatorname{tr} \left((\bar{\partial}_{\bar{k}} h) h^{-1} D_j h \right) &= \mu_a^{-1} |D\mu_a|^2 - (\mu_b - \mu_a)^2 \mu_b^{-1} g^{j\bar{k}} A^a_{\bar{k}b} A^b_{ja} \\ &= \mu_a^{-1} |D\mu_a|^2 + (\mu_b - \mu_a)^2 \mu_b^{-1} |A^b_a|^2, \end{aligned}$$

where we have used the fact that $A^a_{\bar{k}b} = -\overline{A^b_{ka}}$.