Physics 6047
Problem Set 4, due 2/21/13
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1. In this problem, we wish to investigate the non-relativistic limit of the Klein-Gordon equation: $(-\Box + m^2)\hat{\phi} = 0$. Here, we will implicitly think of $\hat{\phi}$ as the complex scalar operator (though *complexity* is not strictly necessary). Recall that $\hat{\phi}$ can be written as a sum, or integral rather, of Fourier modes of momentum k. These Fourier modes oscillate in time with frequency k^0. Let's imagine a field configuration $\hat{\phi}$ that is composed of mostly low-momentum modes i.e. $|\vec{k}| \ll m$, i.e. $k^0 = \sqrt{\vec{k}^2 + m^2} \sim m$. Thus, it's useful to factor out the main time-dependence: $\hat{\phi}(t, \mathbf{x}) = e^{-imt}\hat{\psi}(t, \mathbf{x})$, where $\hat{\psi}$ should be understood to have small time derivatives i.e. $\partial_t \hat{\psi} \ll m\hat{\psi}$, and $\partial_t^2 \hat{\psi} \ll m\partial_t \hat{\psi}$. With this approximation, *show that* the Klein-Gordon equation reduces to

$$i\partial_t \hat{\psi} = -\frac{\vec{\nabla}^2}{2m}\hat{\psi} \,. \tag{1}$$

Should we be surprised we get Schrodinger equation out of Klein-Gordon equation? After all, the Schrodinger equation is supposed to describe the evolution of the state or wave function of some particle, while the Klein-Gordon equation is the equation of motion for some operator $\hat{\phi}$. You can think of the connection as follows. Suppose I am interested in some one-particle-state s (in the Heisenberg picture). One can think of $\langle 0|\hat{\phi}(x)|s\rangle \equiv f(x)$ as essentially the wavefunction. For instance, if s were a state of definite momentum p, $f(x) = e^{ip\cdot x}$ as you can check by expressing the free scalar operator in terms of creation and annihilation operators. This looks exactly like what you would expect for the wavefunction of a freely propagating particle. Since the Heisenberg operator $\hat{\phi}(x)$ obeys the Klein-Gordon equation, we expect $f(x)$ to obey the same (recall that the state $|s\rangle$ in Heisenberg picture does not evolve with time). Taking the non-relativistic limit of $f(t, \vec{x}) = e^{-imt}\tilde{\psi}(t, \vec{x})$ then gives us Schrodinger equation for $\tilde{\psi}$, with the interpretation that $|f|^2 = |\tilde{\psi}|^2$ gives us the probability of locating the particle at position $\vec{x}$ at time t. Note that in this way of thinking, f and $\tilde{\psi}$ are functions (wavefunctions in fact), not operators.

The above argument is the reason behind the (decidedly confusing) statement, progagated in old textbooks on relativistic quantum mechanics, that the Klein-Gordon equation (or Dirac equation for fermions) is the relativistic generalization of the Schrodinger equation – they would say the Schrodinger equation for the wavefunction should be generalized to its relativistic version, and then one should "second-quantize" by promoting the wavefunction to an operator. Our viewpoint, espoused above, goes in exactly the opposite direction.

The statements of the old textbooks are misleading, because one might falsely conclude that the Schrodinger equation no longer holds in relativistic quantum theory. The truth is: the Schrodinger equation remains valid in the relativistic theory, as long as one uses the correct Hamiltonian. To be more precise: you have already shown in problem set 2 question 3 that the Klein-Gordon equation is equivalent to $\partial_t \hat{\phi} = i[\hat{H}, \hat{\phi}]$, which implies $\hat{\phi}(t) = e^{iHt}\hat{\phi}(0)e^{-iHt}$ (suppressing the spatial dependence). The fact that operators in the Heisenberg picture evolve this way tells us that states in the Schrodinger picture evolve forward as $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$, which tells us $-i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$. In other words, the Schrodinger equation is entirely consistent with the Klein-Gordon equation, as long as you use the correct Hamiltonian.

2. In class, we discussed how the generating function can be written as the exponentiation of the sum of all connected diagrams. The proof is contained in equation 9.14 of Srednicki.

Justify the intermediate step:

$$\sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} = \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} . \tag{2}$$

3. Justify the symmetry factors associated with the bubble diagrams in Srednicki Fig. 9.1 and the top two diagrams of Fig. 9.2. You can do this either by brute force, or by counting what are called isomorphisms (the second method discussed in class).

4. Consider the theory described by the following generating functional:

$$Z[J] = \int D[\phi] \exp\left[i \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + J\phi \right) \right] . \tag{3}$$

Work out the bubble diagram that has only one interaction vertex (i.e. the ϕ^4 vertex), including the correct symmetry factor.