Differentiable Manifolds Problem Set 2

Nilay Kumar

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Problem 1

Let X be a Hausdorff and second countable topological space. We wish to show that any subspace Y of X is itself Hausdorff and second countable. Take any $p, q \in Y \subset X$. As X is Hausdorff, there exist disjoint open sets $U_p, U_q \in X$ containing p and q respectively. By definition of the subspace topology, the open sets in Y are of the form $U \cap Y$, where U are the open sets in X. Consequently, $U_p \cap Y$ and $U_q \cap Y$ are disjoint open sets in the topology on Y; as p, q were arbitrary, Y is Hausdorff.

Let us now show that Y is second countable. By second countability, we know that there exists a countable basis $\mathcal{B}_X$ of X. Take $\mathcal{B}_Y$ to be the collection of open sets $B_Y = Y \cap B_X$, where $B_X \in \mathcal{B}_X$. Note that for any two basis elements $C_X, D_X \in \mathcal{B}_X$, there exists a basis element E_X contained in $C_X \cap D_X$ by the basis criterion. It follows, then, that for the two basis elements $C_Y = Y \cap C_X$ and $D_Y = Y \cap D_X$ in $\mathcal{B}_Y$, the basis element $Y \cap E_X \in \mathcal{B}_Y$ is contained in $C_Y \cap D_Y$. It should be clear, then, that the basis $\mathcal{B}_Y$ generates the topology on Y, and since $\mathcal{B}_Y$ is necessarily smaller than $\mathcal{B}_X$, Y is second countable.

Problem 2

We wish to construct partition of unity for S^2 subordinate to our usual stereographic atlas $\{(U, \phi), (V, \phi)\}$. In other words, we wish to find smooth functions $u, v : S^2 : \mathbb{R}$ such that

- $\operatorname{supp} u \subset U$ and $\operatorname{supp} v \subset V$.
- $0 \leq u, v \leq 1$ and $u + v = 1$ everywhere on S^2.

Let us take (U, ϕ) to be the chart that excludes the north pole, and (V, ψ) to be the chart that excludes the south pole. The way to proceed is to use the bump function, $H : \mathbb{R}^2 \to \mathbb{R}$ that we defined in class. H has the properties that it is smooth, $H(x) \leq 1$ everywhere, $H \equiv 1$ on $|x| \leq 2$, and $\operatorname{supp} H$ is the set $|x| \leq 3$. Let us now define

$$\alpha = \begin{cases} H \circ \phi & \text{on } U \\ 0 & \text{north pole} \end{cases}$$
$$\beta = \begin{cases} H \circ \psi & \text{on } U \\ 0 & \text{south pole} \end{cases}$$

We must check that α and β are smooth. To do so, we check that the real functions $\alpha \circ \phi^{-1}, \alpha \circ \psi^{-1}$ and $\beta \circ \phi^{-1}, \beta \circ \psi^{-1}$ are smooth where they overlap (everywhere but the poles):

$$\alpha \circ \phi^{-1} = H \circ \phi \circ \phi^{-1} = H$$
$$\alpha \circ \psi^{-1} = H \circ \phi \circ \psi^{-1}.$$

The first is smooth by smoothness of H and the second is smooth by the smoothness of the transition maps of S^2. Exactly the same reasoning holds for β.

Note additionally that far enough out in the stereographic plane (i.e. close to the north/south pole), the bump function has no support. Consequently, there is a neighborhood about the north/south pole in which α (or β) are zero. Thus, $\operatorname{supp} \alpha \subset U$ and $\operatorname{supp} \beta \subset V$. However, it is not necessarily true that $\alpha + \beta = 1$, so we define functions from $S^2 \to \mathbb{R}$,

$$u = \frac{\alpha}{\alpha + \beta}$$
$$v = \frac{\beta}{\alpha + \beta}.$$

u and v are smooth by smoothness of α and β and the fact that $\alpha + \beta$ is never zero (this is not obvious, but is true by the specific "radius" of the bump function that we chose – the inner part of the bump function must always "wrap" at least past the equator). Additionally, $\operatorname{supp} u \subset U$ and $\operatorname{supp} v \subset V$ as u, v are zero wherever α, β are, and $u + v = 1$ everywhere on S^2. Finally, $0 \leq u, v \leq 1$ because α, β are (by the properties of the bump function), and because the quotients above are between α and something larger than α (and β and something larger than β). Consequently, u, v as defined above form a partition of unity for the sphere.

Problem 3

Let SL_n be the set of all $n \times n$ matrices with determinant 1. We wish to show that SL_n is a smooth manifold. Let us denote the determinant function by $F = \det : GL_n \to \mathbb{R}$; then $SL_n = F^{-1}(1)$. Recall that $F^{-1}(1)$ is an $(n^2 - 1)$-dimensional smooth manifold if 1 is a regular value of F. To check this, we must show that $\nabla F(x) \neq 0$ for all $x \in F^{-1}(1)$:

$$\begin{pmatrix} \frac{\partial F}{\partial \delta_{11}} & \frac{\partial F}{\partial \delta_{12}} & \cdots & \frac{\partial F}{\partial \delta_{n(n-1)}} & \frac{\partial F}{\partial \delta_{nn}} \end{pmatrix} (x) \neq 0$$

Here we have chosen to differentiate with respect to the n^2 basis matrices δ_{ij} of the tangent space, which are simply matrices with a 1 in the ith row and jth column and 0's everywhere else. In other words, for 1 to be a regular value, one of the $\partial F / \partial \delta_{ij}$ must be non-zero at x.

To compute one of these derivatives, we can use the formula given in Lee:

$$\begin{aligned} \left.\frac{\partial F}{\partial \delta_{ij}}\right|_x &= d(\det)_x(\delta_{ij}) = (\det x)\operatorname{tr}(x^{-1}\delta_{ij}) \\ &= \operatorname{tr}(x^{-1}\delta_{ij}) = \operatorname{tr}(x_{ij}^{-1}) \\ &= x_{ij}^{-1} \end{aligned}$$

All we require is for x_{ij}^{-1} to be non-zero for some i, j. This is clearly the case, because if it were not so, x^{-1} would be the zero matrix, which is not in SL_n. Thus 1 is a regular point of F, and SL_n is an $(n^2 - 1)$-dimensional smooth manifold.

Problem 4

Consider a map $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x, y, s, t) = (x^3 + y, x^3 + y^2 + s^2 + t^2 + y).$$

We wish to show that $(0, 1)$ is a regular value of F and that the level set $F^{-1}(0, 1)$ is diffeomorphic to S^2.

Let us first compute

$$dF = \begin{pmatrix} 3x^2 & 1 & 0 & 0 \\ 3x^2 & 2y + 1 & 2s & 2t \end{pmatrix}$$

Note that dF is not full rank if $y = s = t = 0$ (because the two rows will not be linearly independent). Note however, we are only interested in $F^{-1}((0,1))$, i.e. when

$$x^3 + y = 0$$
$$x^3 + y^2 + s^2 + t^2 + y = 1$$

but rank deficiency occurs only when $x^3 = 0 = 1$, which is clearly impossible. Consequently, $(0,1)$ is a regular value for F, and $M = F^{-1}((0,1))$ forms a 2-dimensional smooth manifold.

To show that $M = F^{-1}((0,1))$ is diffeomorphic to S^2, let us construct smooth coordinate charts for M and then show that there exists a diffeomorphism between M and S^2 with respect to this smooth structure. Using the relations between the 4 variables above, we can construct 6 coordinate charts as follows.

For $y < 0$ we define

$$\phi_1 : M \to \mathbb{R}^2$$
$$\phi_1(x, y, s, t) = (s, t)$$
$$\phi_1^{-1}(s, t) = \Big((1 - s^2 - t^2)^{1/6}, -(1 - s^2 - t^2)^{1/2}, s, t\Big)$$

and for $y > 0$ we define the analogous coordinate chart ϕ_2 with the appropriate sign change.

For $s < 0$ we define

$$\phi_3 : M \to \mathbb{R}^2$$
$$\phi_3(x, y, s, t) = (y, t)$$
$$\phi_3^{-1}(y, t) = \Big(-y^{1/3}, y, \sqrt{1 - y^2 - t^2}, t\Big)$$

and for $s > 0$ we define the analogous coordinate chart ϕ_4 with the appropriate sign change.

For $t < 0$ we define

$$\phi_5 : M \to \mathbb{R}^2$$
$$\phi_5(x, y, s, t) = (y, s)$$
$$\phi_5^{-1}(y, s) = \Big(-y^{1/3}, y, s, -\sqrt{1 - s^2 - t^2}\Big)$$

and for $t > 0$ we define the analogous coordinate chart ϕ_6 with appropriate sign change. It is straightforward but tedious to show that these charts are smoothly compatible.

Now, recall the coordinate charts for S^2 (with coordinates a, b, c). For $a < 0$ we define

$$\begin{aligned} \psi_1 : S^2 &\to \mathbb{R}^2 \\ \psi_1(a,b,c) &= (b,c) \\ \psi_1^{-1}(b,c) &= \left(-\sqrt{1-b^2-c^2}, b, c\right) \end{aligned}$$

and for $a > 0$ we define the analogous coordinate chart ψ_2 with appropriate sign change.

For $b < 0$ we define

$$\begin{aligned} \psi_3 : S^2 &\to \mathbb{R}^2 \\ \psi_3(a,b,c) &= (a,c) \\ \psi_3^{-1}(a,c) &= \left(a, -\sqrt{1-a^2-c^2}, c\right) \end{aligned}$$

and for $b > 0$ we define the analogous coordinate chart ψ_4 with appropriate sign change.

For $c < 0$ we define

$$\begin{aligned} \psi_5 : S^2 &\to \mathbb{R}^2 \\ \psi_5(a,b,c) &= (a,b) \\ \psi_5^{-1}(a,b) &= \left(a, b, -\sqrt{1-a^2-b^2}\right) \end{aligned}$$

and for $c > 0$ we define the analogous coordinate chart ψ_6 with appropriate sign change. These charts were shown to be smoothly compatible on the previous problem set.

Now define a map

$$\begin{aligned} f : M &\to S^2 \\ f(x,y,s,t) &= (y,s,t) \\ f^{-1}(a,b,c) &= (-a^{1/3}, a, b, c). \end{aligned}$$

We wish to show that this function is a diffeomorphism from M to S^2. To do this, we must check that $\phi_i \circ f \circ \psi_j^{-1}$ is smooth for all i, j. Fortunately, we need only check 4 combinations, as many of these charts have almost identical structure, and the analysis for these would proceed analogously:

$$\begin{aligned} \psi_1 \circ f \circ \phi_1^{-1}(s,t) = \psi_1 \left(-(1-s^2-t^2)^{1/2}, s, t\right) &= (s,t) \\ \phi_1^{-1} \circ f^{-1} \circ \psi_1(s,t) &= (s,t), \end{aligned}$$

which are clearly smooth as they are the identity, and

$$\psi_1 \circ f \circ \phi_3^{-1}(y,t) = \psi_1\left(y, -\sqrt{1-y^2-t^2}, t\right) = \left(-\sqrt{1-y^2-t^2}, t\right)$$
$$\phi_3 \circ f^{-1} \circ \psi_1^{-1}(b,c) = \phi_3\left((1-b^2-c^2)^{1/6}, -\sqrt{1-b^2-c^2}\right) = \left(-\sqrt{1-b^2-c^2}, c\right),$$

which are clearly smooth, as well (because the square roots are never zero in the appropriate charts). Thus, we have found a diffeomorphism between M and S^2, and $F^{-1}((0,1))$ is diffeomorphic to S^2.

Note that the subsets on which x and y can be solved as smooth functions of s and t were simply the subsets on which ϕ_1, ϕ_2 were defined, i.e. $y \neq 0$.

Problem 5

For each $n \in \mathbb{Z}$, we define the nth power map $p_n : S^1 \to S^1$ given in complex notation by $p_n(z) = z^n$. On each copy of S^1, we can take 4 graphical coordinate charts. Let us only work with two of these total 8 charts, one on each copy, (U, ϕ) and (V, ψ) respectively, where U, V are the upper hemispheres of the circles. The proofs for the other charts follow almost exactly as what follows. We have the charts

$$\phi(\cos\theta, \sin\theta) = \cos\theta$$
$$\phi^{-1}(\cos\theta) = \left(\cos\theta, \sqrt{1-\cos^2\theta}\right)$$

and

$$\psi(\cos\theta, \sin\theta) = \cos\theta$$
$$\psi^{-1}(\cos\theta) = \left(\cos\theta, \sqrt{1-\cos^2\theta}\right).$$

We now wish to show that the following composition is smooth:

$$\begin{aligned}\psi \circ p_n \circ \phi^{-1}(\cos\theta) &= \psi \circ p_n\left(\cos\theta, \sqrt{1-\sin^2\theta}\right) \\ &= \psi\left(\cos(n\theta), \sqrt{1-\sin^2(n\theta)}\right) = \cos(n\theta)\end{aligned}$$

It is well known that the function $\cos(n\theta)$ can be written smoothly in terms of $\cos\theta$ via the Chebyshev polynomials. Consequently, p_n is a smooth map from S^1 to S^1.

Now define the antipodal map $\alpha : S^n \to S^n$ such that $\alpha(x) = -x$. It should be clear that each copy of S^n will have $2(n+1)$ graphical coordinates. The charts are for each copy:

$$\phi_{2i} : S^n \to \mathbb{R}^n \text{ for } x^i > 0$$
$$\phi_{2i}(\vec{x}) = (x^1 \cdots \hat{x}^i \cdots x^{n+1})$$
$$\phi_{2i}^{-1}(x^1 \cdots \hat{x}^i \cdots x^n) = \left(x^1 \cdots + \sqrt{1 - \sum_{k \neq i} (x^k)^2} \cdots x^{n+1} \right)$$

and

$$\phi_{2i+1} : S^n \to \mathbb{R}^n \text{ for } x^i < 0$$
$$\phi_{2i+1}(\vec{x}) = (x^1 \cdots \hat{x}^i \cdots x^{n+1})$$
$$\phi_{2i+1}^{-1}(x^1 \cdots \hat{x}^i \cdots x^n) = \left(x^1 \cdots - \sqrt{1 - \sum_{k \neq i} (x^k)^2} \cdots x^{n+1} \right),$$

where we have paired the charts for convenience. Let us denote the coordinate charts for the second copy of S^n by ψ. Then, to check that α is smooth, we need only check that $\psi_{2j} \circ \alpha \circ \phi_{2i}^{-1}$ is smooth for all i, j. Of course, we are done if we can show smoothness for $i \neq j$ (we assume $i < j$ without loss of generality) and for $i = j$. We have for the first case

$$\begin{aligned} \psi_{2j} \circ \alpha \circ \phi_{2i}^{-1}(x^1 \cdots \hat{x}^i \cdots x^{n+1}) &= \psi_{2j} \left(-x^1 \cdots - \sqrt{1 - \sum_{k \neq i} (x^k)^2} \cdots - x^{n+1} \right) \\ &= \left(-x^1 \cdots \hat{x}^j \cdots \sqrt{1 - \sum_{k \neq i} (x^k)^2} \cdots - x^{n+1} \right), \end{aligned}$$

which is always smooth (the square root does not create problems as we are working in charts where the square root is never zero). If $i = j$, however, note that the coordinate removed is that which was added in, and so we have:

$$\begin{aligned} \psi_{2i} \circ \alpha \circ \phi_{2i}^{-1}(x^1 \cdots \hat{x}^i \cdots x^{n+1}) &= \psi_{2i} \left(-x^1 \cdots - \sqrt{1 - \sum_{k \neq i} (x^k)^2} \cdots - x^{n+1} \right) \\ &= \left(-x^1 \cdots \hat{x}^i \cdots - x^{n+1} \right), \end{aligned}$$

which is clearly smooth.

Now take the map $F : S^3 \to S^2$ given by $F(z, w) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of S^3 as the subset $\{(w, z) : |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$. We choose, of course, the typical graphical coordinate charts for S^2, and treat $S^3 \cong S^1_{\mathbb{C}}$. We then take the coordinate chart

$$\begin{aligned} \psi_1 : S^1_{\mathbb{C}} \to \mathbb{R}^3 &\text{ for } |w| \neq 1 \\ \psi_1(w, z) &= (|w|, \theta_w, \theta_z) \\ \psi_1^{-1}(|w|, \theta_w, \theta_z) &= (|w|e^{i\theta_w}, \sqrt{1 - |w|^2}e^{i\theta_z}), \end{aligned}$$

and examine the composition

$$\begin{aligned} \phi \circ F \circ \psi_1^{-1}(|w|, \theta_w, \theta_z) &= \phi \circ F \left(|w|e^{i\theta_w}, \sqrt{1 - |w|^2}e^{i\theta_z}\right) \\ = \phi \left(|w|\sqrt{1 - |w|^2}(e^{i(\theta_w - \theta_z)} + e^{-i(\theta_w - \theta_z)}), i|w|\sqrt{1 - |w|^2}(e^{i(\theta_w - \theta_z)} - e^{-i(\theta_w - \theta_z)}), 1 - 2|w|^2\right) \\ = \phi \left(|w|\sqrt{1 - |w|^2}2\cos(\theta_w - \theta_z), |w|\sqrt{1 - |w|^2}2\sin(\theta_w - \theta_z), 1 - 2|w|^2\right). \end{aligned}$$

Since ϕ is a graphical coordinate chart for S^2, all it will do is drop one of the coordinates. This composition is clearly smooth in $|w|, \theta_w, \theta_z$, in the chart. However, we need a chart to cover the cases for when $|w| = 1$ – all we must do is construct an identical chart, but now:

$$\begin{aligned} \psi_2 : S^1_{\mathbb{C}} \to \mathbb{R}^3 &\text{ for } |z| \neq 1 \\ \psi_2(w, z) &= (|z|, \theta_w, \theta_z) \\ \psi_2^{-1}(|z|, \theta_w, \theta_z) &= (|z|e^{i\theta_z}, \sqrt{1 - |z|^2}e^{i\theta_w}), \end{aligned}$$

from which the smoothness of F follows almost identically as above. As these two charts cover $S^1_{\mathbb{C}}$, and F is smooth with respect to the two manifolds' smooth structures, we are done.

Problem 6

Take our smooth manifold to be $\mathbb{R}$ and A to be the set $[1, 2)$, and $f(x)$ to be a constant c. Take the open subset U of $\mathbb{R}$ to be $(1-\varepsilon, 2)$ for some positive ε. It should be clear that $\operatorname{supp} f \supset [1, 2]$ and so it is not true that the support of f, even after weighted by a partition of unity, is contained in U.