Modern Geometry PSET 1

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Proposition 1. *Every topological manifold is path-connected if and only if it is connected.*

The following lemma is useful.

Lemma 1. *Let M^n be a topological manifold. Then M^n is locally path-connected.*

Proof. Given a point $x \in M$, it suffices to show that there exists an open path-connected neighborhood $U \ni x$. Let x lie in the chart (V, ψ); then $\psi(V)$ is an open set in $\mathbb{R}^n$. By the topology of $\mathbb{R}^n$, we can find an open ball $B \ni \psi(x)$ such that $B \subset \psi(V)$. Since B is clearly path-connected and open, so is $\psi^{-1}(B)$. Then $\psi^{-1}(B)$ is an open path-connected neighborhood of x, and we are done. □

We now prove the proposition.

Proof. Let C be a path-component of M. Given some point $p \in C$, path-connectedness implies that there exists a path-connected neighborhood U of p. It's clear that U must be contained in C, and hence C must be open. It is true in general that path-connectedness implies connectedness; it suffices to show that connectedness implies path-connectedness. Suppose M is connected, i.e. M consists of one (open) component. For some $p \in M$ consider the path component P containing p. Suppose P is not the only path-component in M. Then the set $\{P, M \setminus P\}$ would be separation of M, contradicting that M is connected. Hence P must be the only path-component of M, and since M is connected it follows that $P = M$, i.e. that M is path-connected. □

Proposition 2. *Consider the diagram*

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, dashed, "f"] \arrow[d, "\pi"'] & S^n \times \mathbb{R} \arrow[d, "p"] \\
\mathbb{RP}^n & S^n \arrow[l, "\pi|_{S^n}"]
\end{tikzcd}$$

where π is the quotient map and p is projection onto the first factor. Then the following are true:

(a) there exists a homeomorphism $f : \mathbb{R}^{n+1} \setminus \{0\} \to S^n \times \mathbb{R}$ such that the diagram commutes;

(b) the inverse image of an open set in $\mathbb{RP}^n$ in $S^n \times \mathbb{R}$ is

(c) $\mathbb{RP}^n$ is Hausdorff, second-countable, and compact.

Proof. (a) Define the map f by $\vec{v} \mapsto (\vec{v}/|\vec{v}|, \ln |\vec{v}|)$. It's clear that the map is bijective with inverse $f^{-1} : (\vec{w}, x) \mapsto e^x \vec{w}$ and that both f and f^{-1} are continuous. Hence f is a homeomorphism. To check that the diagram commutes, note that $p(f(\vec{v})) = \vec{v}/|\vec{v}|$, which when projected to $\mathbb{RP}^n$ simply yields $[\vec{v}]$, as needed.

(b) Geometrically, the map $\pi|_{S^n}$ simply identifies antipodal points on the sphere, and hence the inverse image of an open set in $\mathbb{RP}^n$ is two open sets on S^n (antipodally symmetric). The inverse image of this in $S^n \times \mathbb{R}$ gives us back these open sets but with fibers isomorphic to $\mathbb{R}$ attached to each point. Geometrically we can imagine a double-napped hypercone (with center at the center of the sphere) that intersects S^n in the shape of the open set of $\mathbb{RP}^n$.

(c) We showed in class that $\mathbb{RP}^n$ is locally Euclidean and constructed charts (U_i, ϕ_i). Let us first show that $\mathbb{RP}^n$ is second-countable. Take the collection of open sets $\{\pi^{-1}(U_i)\}$, which is an open cover of $\mathbb{R}^{n+1} \setminus \{0\}$. Pick a countable sub-cover of this collection and call $\mathcal{U}'$ the collection of open sets of $\mathbb{RP}^n$ whose preimage by π is this countable sub-cover. Then $\mathcal{U}'$ is a countable cover of $\mathbb{RP}^n$ by Euclidean balls of dimension n. Since each ball has a countable basis, the union of such bases gives us a countable cover for $\mathbb{RP}^n$.

Next we show that $\mathbb{RP}^n$ is Hausdorff. Pick two distinct points $x, y \in \mathbb{RP}^n$. The preimage of x in S^n is $\{x, -x\}$ and the preimage of y is $\{y, -y\}$. Consider S^n as embedded in $\mathbb{R}^{n+1}$ and define the following open sets of S^n:

$$U = S^n \cap B_\epsilon(x)$$
$$V = S^n \cap B_\epsilon(y).$$

for $\epsilon = \min(|x-y|, |x+y|)/2$. It's clear then that on the n-sphere, $U, V, -U$, and $-V$ are disjoint neighborhoods of $x, y, -x$ and $-y$. Note now that the preimage of the image of U and V under $\pi|_{S^n}$ are $-U \cup U$ and $-V \cup V$ respectively. By definition of the quotient topology, $\pi|_{S^n}(U)$ and $\pi|_{S^n}(V)$ are open in $\mathbb{RP}^n$. Additionally, they must be disjoint, because if there were a point common to both, the preimage of the point would fall into both $-U \cup U$ and $-V \cup V$, which is impossible as they are disjoint. Hence $\mathbb{RP}^n$ is Hausdorff.

That $\mathbb{RP}^n$ is compact follows from the fact that compactness is preserved by continuous maps (i.e. if X is compact then $f(X)$ is compact) applied to the quotient map $\pi|_{S^n}$ since the sphere is obviously compact.

□

Proposition 3. *If M, N, L are smooth manifolds with $f : M \to N$ and $g : N \to L$ smooth maps, then the composition $g \circ f : M \to L$ is smooth.*

Proof. Since g is smooth, we can find smooth charts (V, ψ) containing $f(p)$ and (W, ξ) containing $g(f(p))$ such that $g(V)$ is contained in W and the composition $\xi \circ g \circ \psi^{-1}$ is smooth. But now, by smoothness of f, we can find a chart (U, ϕ) containing p such that $f(U)$ is contained in V (as we discussed in class, the coordinate representation of a smooth map is smooth with respect to every pair of smooth charts). Hence the composition $\psi \circ f \circ \phi^{-1}$ is smooth as well. Composing these compositions, we find that $\xi \circ g \circ f \circ \phi^{-1}$ is smooth, and we conclude that the composition $g \circ f$ is smooth. □

Proposition 4. *Bump functions, etc.*

Proof. (a) It's clear that f is continuous and smooth on $\mathbb{R} \setminus \{0\}$. Additionally, f is continuous at $x = 0$ because $f(0) = 0$ and $\lim_{x \to 0} e^{-1/x^2} = 0$. Hence it suffices to compute the derivatives at all orders of f at $x = 0$.

Let us first prove by induction that for $x > 0$,

$$\frac{d^n f}{dx^n} = p_n(x) \frac{e^{-1/x^2}}{x^{3n}},$$

for some polynomial $p_n(x)$. The formula clearly holds for $n=0$. We assume it holds for $n-1$. Then

$$\begin{aligned}
\frac{d^n f}{dx^n} &= \frac{d}{dx}\left(p_{n-1}(x)\frac{e^{-1/x^2}}{x^{3(n-1)}}\right) \\
&= p'_{n-1}(x)\frac{e^{-1/x^2}}{x^{3(n-1)}} + p_{n-1}(x)\left(2e^{-\frac{1}{x^2}}x^{-3-3(-1+n)} - 3e^{-\frac{1}{x^2}}(-1+n)x^{-1-3(-1+n)}\right) \\
&= p'_{n-1}(x)\frac{e^{-1/x^2}}{x^{3(n-1)}} + p_{n-1}(x)\frac{e^{-1/x^2}}{x^{3n}}\left(2-3(n-1)x^2\right).
\end{aligned}$$

Combining the two terms yields an overall denominator of x^{3n} with a now-different polynomial out front. Hence we are done.

Now let us compute the nth derivative of f at zero:

$$\left.\frac{d^n f}{dx^n}\right|_{x=0} = \lim_{h\to 0}\frac{p_n(h)\frac{e^{-1/h^2}}{h^{3n}}}{h} = p_n(0)\lim_{h\to 0}\frac{e^{-1/h^2}}{h^{3n+1}} = 0.$$

Since the derivatives exist, they must be continuous as well. Hence f is smooth.

(b) Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x) = \frac{f(1-|x|)}{f(1-|x|)+f(|x|-1/2)}.$$

For $|x|>1$ the numerator vanishes, as desired and for $|x|<1/2$, note that the second term in the denominator vanishes and hence g is identically 1. It is also easy to see that when $|x|<1$, g is positive (by the properties of f). This function is smooth by composition on $\mathbb{R}\setminus\{0\}$. It is, in fact, also smooth at 0 as it is identically 1 in an open neighborhood of 0.

(c) Define $h:\mathbb{R}\to[0,1]$ by

$$h(x) = 1-\frac{f(1-x)}{f(1-x)+f(x)}.$$

Consider for now just the fraction: for $x<0$, the second term in the denominator vanishes and hence we get 1. For $x>1$, the numerator vanishes and we get 0. This is precisely the opposite of what we want. Hence we subtract it from 1. This function is smooth by composition.

(d) Take $\psi:\mathbb{R}^n\to\mathbb{R}$ given by

$$\psi(\vec{x}) = \frac{f(2\epsilon-|x|)}{f(2\epsilon-|x|)+f(|x|-\epsilon)},$$

for $\epsilon>0$. The numerator is zero for $|x|\geq 2\epsilon$ by the definition of f, and hence ψ is zero for $|x|\geq 2\epsilon$. For $\epsilon\leq|x|\leq 2\epsilon$ the second term in the denominator vanishes and hence ψ is identically 1, as desired. By composition, ψ is clearly smooth on $\mathbb{R}^n\setminus\{0\}$. At zero, however, it is also smooth, as ψ is identically 1 in an open neigborhood of zero.

(e) Consider $h(U)\subset\mathbb{R}^n$ where (U,h) is a coordinate chart. Pick balls $B(p),B'(p)$ such that $B(p)\subset B'(p)\subset h(U)$. We can construct a function (just as in the previous part of this problem) that is 1 on $B(p)$ and 0 outside $B'(p)$; call this function ψ. Then the function $\phi: M\to\mathbb{R}$ given by $\psi\circ h$ on U and 0 on $M\setminus U$ satisfies our requirements and is smooth because the coordinate representation is smooth (in U, this follows from the previous part, outside U the function is simply 0).

$\square$

Proposition 5. *The stereographic charts for S^2 are compatible.*

Proof. Let $U_- = S^2 \setminus \{0,0,1\}$ and $U_+ = S^2 \setminus \{0,0,-1\}$.

The line that goes through $(0,0,1)$ and $(x,y,z) \in S^2$ is given by $(0,0,1) + t(x,y,z-1)$ for $t \in \mathbb{R}$. This line intersects the plane $z = 0$ when $1 + t(z-1) = 0$, i.e. when $t = -1/(z-1)$. Hence we define

$$\phi_-(x,y,z) = \left(-\frac{x}{z-1}, -\frac{y}{z-1}\right)$$

as our chart (U_-, ϕ_-). Since $z \neq 1$ in U_-, ϕ_- is continuous. If we write $\phi_-(x,y,z) = (u,v)$, we can solve for z (via the quadratic formula, discarding the case $z = 1$) and then x, y in terms of u, v which yields the inverse

$$\phi_-^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$

which is clearly continuous as well. Hence ϕ_- is a homeomorphism.

The line that goes through $(0,0,-1)$ and $(x,y,z) \in S^2$ is given by $(0,0,-1) + t(x,y,z+1)$ for $t \in \mathbb{R}$. This line intersects the plane $z = 0$ when $-1 + t(z+1) = 0$, i.e. when $t = 1/(z+1)$. Hence we define

$$\phi_+(x,y,z) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right)$$

as our chart (U_+, ϕ_+). Since $z \neq -1$ in U_+, ϕ_+ is continuous. If we write $\phi_+(x,y,z) = (u,v)$, we can solve for z (via the quadratic formula, discarding the case $z = -1$) and then x, y in terms of u, v which yields the inverse

$$\phi_+^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{-u^2-v^2+1}{u^2+v^2+1}\right),$$

which is clearly continuous as well. Hence ϕ_+ is a homeomorphism.

Let us now check that the charts (U_-, ϕ_-) and (U_+, ϕ_+) are compatible, i.e. $\phi_+ \circ \phi_-^{-1} : \phi_-(U \cap V) \to \phi_+(U \cap V)$ and $\phi_- \circ \phi_+^{-1} : \phi_+(U \cap V) \to \phi_-(U \cap V)$ are smooth. We compute

$$\begin{aligned}
\phi_+ \circ \phi_-^{-1}(u,v) &= \phi_+\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) \\
&= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) \\
\phi_- \circ \phi_+^{-1}(u,v) &= \phi_-\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+b^2+1}, \frac{-u^2-v^2+1}{u^2+v^2+1}\right) \\
&= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right),
\end{aligned}$$

which are clearly smooth since $(u,v) \neq (0,0)$ in the overlap. $\square$

Proposition 6. *For $M \xrightarrow{\phi} N \xrightarrow{\phi} P$ smooth maps of manifolds, the push-forward is covariant, i.e. $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.*

Proof. Consider two functions $n : N \to \mathbb{R}, p : P \to \mathbb{R}$ and vectors $X \in T_qM, Y \in T_{\phi(q)}N$. The push-forward of the composition $(\psi \circ \phi)_* : T_qM \to T_{\psi(\phi(q))}P$ gives us, by definition,

$$(\psi \circ \phi)_*(X)(p) = X(p \circ \psi \circ \phi).$$

Next consider the composition of pushforwards. Again, by definition, we have

$$\phi_*(X)(n) = X(n \circ \phi)$$
$$\psi_*(Y)(p) = Y(p \circ \psi)$$

Composing these, $\psi_* \circ \phi_* : T_qM \to T_{\psi(\phi(q))}P$ we find:

$$\psi_*(\phi_*(X))(p) = \phi_*(X)(p \circ \psi) = X(p \circ \psi \circ \phi),$$

precisely as needed. □

Proposition 7. *If a non-empty m-manifold M is diffeomorphic to an n-manifold N, then $m = n$.*

Proof. Pick a point $p \in M$. We have a diffeomorphism $\phi : M \to N$, and hence a pushforward $\phi_* : T_pM \to T_{\phi(p)}N$ that is a vector space isomorphism. Note however that T_pM has dimension m and $T_{\phi(p)}N$ has dimension n, and hence m must equal n. □