Lie Groups PSET 3

Nilay Kumar

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Problem 1

Let V, W be irreducible representations of a Lie group G. Note that $V \otimes W^*$ is simply $\mathrm{Hom}(W; V)$ and $V \otimes V^*$ is simply $\mathrm{Hom}(V; V)$. With the restriction of G-equivariance, then, we see that the statement reduces to Schur's lemma. In other words, since both $\ker \phi$ and $\mathrm{im}\ \phi$ are invariant under the action of G, for some G-equivariant morphism ϕ, and V, W are irreducible, they must be either 0 or the whole space. Moreover, if $V = W$, ϕ must have an eigenvalue $\lambda \in \mathbb{C}$ as we are working over $\mathbb{C}$ and hence $\phi - \lambda I$ has a non-trivial kernel, which implies that it must be zero, i.e. $\phi = \lambda I$. Of course, this is simply isomorphic to $\mathbb{C}$.

We are given that V is an irreducible representation of a Lie algebra $\mathfrak{g}$, and we wish to show that V^*, the dual representation, is irreducible as well. Denote by (ρ, V) the original representation. Suppose the contrary: if V^* is reducible, there must exist a subspace $W^* \subset V^*$ invariant under the action of $\mathfrak{g}$. Now take the space $W = \{v \in V \mid w(v) = 1 \text{ for some } w \in W^*\}$. We obtain a contradiction if W is invariant under $\mathfrak{g}$. To show this, we must find a $w^* \in W^*$ such that $w^*(\rho(g)) = 1$. But we may simply choose the dual of v:

$$(\pi(g)v^*)\,(\rho(g)v) = v^*(v) = 1.$$

Hence V^* must be irreducible.

We can view the space of bilinear forms on V as $V^* \otimes V^* = \mathrm{Hom}(V; V^*)$. Then, by the above, we see that since both V, V^* are irreducible representations, either the maps (and hence the forms) must be zero, or they must be isomorphic to $\mathbb{C}$, i.e. one-dimensional. Note that the above statements held for group actions, but it is quite clear that nowhere did we use ideas specific to groups - Schur's lemma holds just as well for Lie algebra actions.

Problem 2

Consider the map $\pi : \mathbb{R} \to GL_2\mathbb{C}$ given by

$$t \mapsto \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$

It is easy to see that π is in fact a representation of $\mathbb{R}$ on $\mathbb{C}^2$, as it is a group homomorphism:

$$\pi(a+b) = \begin{pmatrix} 1 & a+b \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \pi(a)\pi(b).$$

Furthermore, all proper non-trivial subrepresentations are clearly one-dimensional, and hence can be found by computing the eigenvectors of the above matrix. The eigenvalues are clearly 1 and 1, and thus we solve

$$\begin{pmatrix} 0 & t \\ & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to find that the only subrepresentation is the one-dimensional space spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Note, however, that the orthogonal complement of this subspace - the space spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ - is not a subrepresentation:

$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix},$$

and hence, while π is a reducible representation, it is not completely reducible as it cannot be written as a direct sum of irreducibles.

Furthermore, this representation is not unitary with repsect to the standard Hermitian inner product on $\mathbb{C}^2$ defined by $\langle \vec{v}, \vec{w} \rangle = \sum_i v_i \bar{w}_i$. To see this, let e_1, e_2 be the two basis vectors (as decomposed above) and compute

$$\langle e_1, e_2 \rangle = 0$$

$$\langle \pi(a)e_1, \pi(a)e_2 \rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 1 \end{pmatrix} \rangle = a,$$

which are not equal in general.

Problem 3

Take $\omega \in (\mathfrak{g}^*)^{\otimes 3}$ given by

$$\omega(x, y, z) = ([x, y], z),$$

with the form symmetric and ad-invariant. We wish to show that ω is skew-symmetric and ad-invariant. To show skew-symmetry, we must show:

$$\begin{aligned} \omega(x, y, z) &= -\omega(y, x, z) \\ \omega(x, y, z) &= -\omega(z, y, x) \\ \omega(x, y, z) &= -\omega(x, z, y). \end{aligned}$$

The first identity follows trivially from the skew-symmetry of the Lie bracket. For the next identity we use ad-invariance of the bracket $([a, b], c) + (b, [a, c]) = 0$ to obtain

$$\omega(x, y, z) = ([x, y], z) = (z, [x, y]) = -(z, [y, x]) = -\omega(z, y, x).$$

The third identity is derived similarly:

$$\omega(x, y, z) = ([x, y], z) = -([y, x], z) = (x, [y, z]) = -(x, [z, y]) = -\omega(x, z, y).$$

Next we wish to show ad-invariance of ω:

$$\omega([w, x], y, z) + \omega(x, [w, y], z) + \omega(x, y, [w, z]) = 0.$$

We simply apply the symmetry and ad-invariance of the form as well as the Jacobi identity:

$$\begin{aligned} \omega([w, x], y, z) + \omega(x, [w, y], z) + \omega(x, y, [w, z]) &= ([[w, x], y], z) + ([x, [w, y]], z) + ([x, y], [w, z]) \\ &= ([[w, x], y] + [[y, w], x], z) + ([x, y], [w, z]) \\ &= -([[x, y], w], z) + ([x, y], [w, z]) \\ &= ([w, [x, y]], z) + ([x, y], [w, x]) \\ &= 0, \end{aligned}$$

as desired.

Problem 4

Let $G = SU(2)$. We can consider G as the group of unit quaternions sitting inside $\mathbb{R}^4$. From this point of view we can treat elements of $\mathbb{R}^4$ as quaternions and extend the action of G on itself (left-quaternionic-multiplication) to an action on all of $\mathbb{R}^4$ given by quaternionic left-multiplication. These yield orthogonal transformations, as multiplication by a unit quaternionic preserves the inner product Re $\bar{q}_1 q_2$ (since the inner product on $\mathbb{R}^4$ is precisely that defined on quaternions).

Next, let $\omega \in \Omega^3(G)$ be a left-invariant 3-form whose value at $1 \in G$ is defined by

$$\omega(x_1, x_2, x_3) = \operatorname{tr}([x_1, x_2]x_3)$$

where $x_i \in \mathfrak{g}$. Note carefully that the correspondence between $\mathfrak{su}_2$ and $\mathbb{H}$ is given by

$$w_0 + w_1 i + w_2 j + w_3 k \longleftrightarrow \begin{pmatrix} -iw_3 & -w_2 - iw_1 \\ w_2 - iw_1 & iw_3 \end{pmatrix}.$$

It follows that we can construct an orthonormal basis for $\mathfrak{su}_2$ in the context of quaternions to be $\{i, j, k\}$ (one can check this by computing against the inner product $\operatorname{tr}(a\bar{b})/2$). Let us now compute ω at the identity:

$$\begin{aligned} \omega(i,j,k) &= \operatorname{tr}([i,j]k) = \operatorname{tr}(ijk - jik) = 2\operatorname{tr}(k^2) \\ &= 2\operatorname{tr}\begin{pmatrix} -i & \\ & i \end{pmatrix}^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = -4. \end{aligned}$$

But this is precisely -4 times the value of the volume form (by definition) at the identity. By left-invariance, we see that this equality extends to the whole sphere (the volume form is left-invariant because pulling back by a diffeomorphism is by definition evaluation at pushforwards of vectors - but the pushforward is an isomorphism and takes basis to basis, and hence the volume form still evaluates to 1).

Finally, let us show that $\omega/8\pi^2$ is a bi-invariant form on G such that $\int_G \omega/8\pi^2 = 1$. Bi-invariance is obvious, as ω is proportional to the volume form on G, which is bi-invariant by its construction (the same argument holds as for left-invariance). By above, we have

$$\frac{1}{8\pi^2}\int_G \omega = -\frac{1}{2\pi^2}\int_G dV$$

and thus it suffices to show that $\int_{S^3} dV = 2\pi^2$ (the sign is a matter of orientation). But this is a simple calculation in analogy to computing the surface area of S^2:

$$\begin{aligned} \int_{S^3} dV &= \int_{-\pi/2}^{\pi/2} 4\pi r^2 d\theta = 4\pi \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta \\ &= 4\pi \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta = 2\pi^2, \end{aligned}$$

as desired.

Problem 5

Consider the Frobenius-Schur indicator,

$$I(\chi_V) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g^2).$$

We will use the following theorem to show that $I(\chi_V)$ takes values only in $\{-1,0,1\}$ for V irreducible.

Theorem. *An irreducible representation V is one and only one of the following:*

(i) Complex: χ_V is not real-valued; V does not have a G-invariant nondegenerate bilinear form;

(ii) Real: $V = V_0 \otimes \mathbb{C}$, a real representation; V has a G-invariant symmetric nondegenerate bilinear form;

(iii) Quaternionic: χ_V is real, but V is not real; V has a G-invariant skew-symmetric nondegenerate bilinear form.

Proof. Omitted. See [FH91], Theorem 3.37. □

To relate $I(\chi_V)$ to the spaces of bilinear forms on V we note first that $\chi_{V^*} = \bar{\chi}_V$ and hence by character theory (see [FH91], §2.1),

$$\chi_{\Lambda^2 V^*}(g) = \frac{\chi_{V^*}(g)^2 - \chi_{V^*}(g^2)}{2} = \frac{1}{2}\left(\overline{\chi_V(g)^2 - \chi_V(g^2)}\right)$$
$$\chi_{\mathrm{Sym}^2 V^*}(g) = \frac{\chi_{V^*}(g)^2 + \chi_{V^*}(g^2)}{2} = \frac{1}{2}\left(\overline{\chi_V(g)^2 + \chi_V(g^2)}\right).$$

But now note that

$$\chi_{\mathrm{Sym}^2 V^*}(g) - \chi_{\Lambda^2 V^*}(g) = \overline{\chi_V(g^2)},$$

which we can average over G to obtain

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Sym}^2 V^*}(g) - \frac{1}{|G|}\sum_{g\in G}\chi_{\Lambda^2 V^*}(g) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g^2)}.$$

Conjugating, we find that

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\mathrm{Sym}^2 V^*}(g)} - \frac{1}{|G|}\sum_{g\in G}\overline{\chi_{\Lambda^2 V^*}(g)}.$$

But note that the terms on the right are simply projecting onto the spaces of G-invariant symmetric and alternating bilinear forms, respectively (see [FH91], §2.4). Hence we obtain

$$I(\chi_V) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^2) = \dim\left(\mathrm{Sym}^2 V^*\right)^G - \dim\left(\Lambda^2 V^*\right)^G.$$

Using the theorem above, we now see that if V is complex then $I(\chi_V) = 0$, but if V is real then $I(\chi_V) = 1$, and if V is quaternionic then $I(\chi_V) = -1$. Now let us consider the G-equivariant endomorphisms of V, $\mathrm{Hom}_G(V;V)$, for each case above. By Schur's lemma, they must be invertible, which implies that we must have one of the three division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

References

[FH91] W. Fulton and J. Harris. *Representation Theory: A First Course*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, 1991.