Differentiable Manifolds Problem Set 6

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Last updated: April 8, 2013

Problem 1

Let $V_1, \ldots, V_k$ and W be finite-dimensional real vector spaces. We wish to show that there is a canonical isomorphism $V_1^* \otimes \cdots \otimes V_k^* \otimes W \cong L(V_1, \ldots, V_k; W)$. First consider the map $\Phi : V_1^* \times \cdots \times V_k^* \times W \to L(V_1, \ldots, V_k; W)$ given by

$$\Phi(\xi^1, \ldots, \xi^k, w)(v_1, \ldots, v_k, \omega) = \xi^1(v_1) \cdots \xi^k(v_k) w.$$

This right-hand side is clearly linear in each of the arguments and takes values in W, so it is indeed a member of $L(V_1, \ldots, V_k; W)$. Note that Φ is multilinear in $\xi^1, \ldots \xi^k, w$, as the right-hand side is simply multiplication. By the characteristic property of tensor product spaces, then, Φ descends uniquely to a linear map $\tilde{\Phi}$ from $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ to $L(V_1, \ldots, V_k; W)$:

$$\tilde{\Phi}(\xi^1 \otimes \cdots \otimes \xi^k, w)(v_1, \ldots, v_k, \omega) = \xi^1(v_1) \cdots \xi^k(v_k) w.$$

Furthermore, $\tilde{\Phi}$ takes the basis of $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ to the basis of $L(V_1, \ldots, V_k; W)$:

$$\tilde{\Phi}(e^1 \otimes \cdots \otimes e^k, E_1)(v_1, \ldots, v_k, w) = e^1(v_1) \cdots e^k(v_k) E_1.$$

Thus $\tilde{\Phi}$ must be an isomorphism (independent of the specific bases chosen), and we are done.

Problem 2

Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Suppose that $\omega = e^1 \otimes e^2 \otimes e^3$ can be written as the sum of an alternating tensor and a symmetric tensor,

$$\omega = e^1 \otimes e^2 \otimes e^3 = (\eta + \theta)_{ijk} \, e^i \otimes e^j \otimes e^k$$

with η antisymmetric, θ symmetric. By linear independence of the basis vectors of $T^3((\mathbb{R}^3)^*)$, the only non-vanishing term on the right-hand side must be that which contains a $e^1 \otimes e^2 \otimes e^3$, i.e. $(\eta+\theta)_{123} = 1$. Note, however, that $(\eta+\theta)_{312} = (\eta+\theta)_{123}$ must be zero. This contradicts the previous statment, and thus this tensor cannot be written as a sum of an alternating and a symmetric tensor.

Problem 3

We wish to show that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent on a finite-dimensional space if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$. Let us first assume that we have the set of covectors is linearly dependent. Then, since

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v_k) = \det(\omega^j(v_i)),$$

and since $\omega^j(v_i)$ is not full rank by assumption, the determinant is zero. Consequently the wedge product must be zero.

Conversely, let us assume that the wedge product is zero. It follows that one of the arguments must be a multiple of another. Clearly, then the set of covectors is linearly dependent.

Problem 4

1. Given an ordered k-tuple $(v_1, \ldots, v_k)$ of elements of V, we wish to show it's linearly dependent if and only if $v_1 \wedge \cdots \wedge v_k = 0$.

Problem 5

Problem 6