Introduction to Algebraic Topology PSET 7

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Proposition 1. *Hatcher exercise 2.1.1*

Proof. The Δ-complex obtained from the 2-simplex $[v_0, v_1, v_2]$ with edges $[v_0, v_1]$ and $[v_1, v_2]$ identified with order preserved is just a cone $C(S^1)$. $\square$

Proposition 2. *Hatcher exercise 2.1.5*

Proof. Let us compute the simplicial homology groups of the Klein bottle using the Δ-complex structure K described on page 102 of Hatcher. It's clear that there is one 0-simplex (v), three 1-simplices (a, b, c), and two 2-simplices (U, L). We obtain the chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

with $\partial_0 v = 0$, $\partial_1 a = \partial_1 b = \partial_1 c = 0$, and $\partial_2 U = a + b - c, \partial_2 L = a - b + c$. The Klein bottle is path-connected and hence $H_0^\Delta(K) = \mathbb{Z}$. Next note that $\ker \partial_2 = 0$ as no linear combination of U and L will map to 0 under ∂_2, and thus $H_2^\Delta(K) = 0$. Computing $H_1^\Delta(K)$ is a bit more tedious: $H_1^\Delta(K) = \ker \partial_1 / \mathrm{im}\ \partial_2 = \langle a, b, c\rangle / \langle a + b - c, a - b + c\rangle$. We can simplify this as follows:

$$\begin{aligned} \frac{\langle a, b, c\rangle}{\langle a + b - c, a - b + c\rangle} &= \frac{\langle a - b + c, b, c\rangle}{\langle a + b - c, a - b + c\rangle} \\ &= \frac{\langle b, c\rangle}{\langle 2b - 2c\rangle} = \frac{\langle b - c, c\rangle}{\langle 2(b - c)\rangle} = \frac{\langle d, c\rangle}{\langle 2d\rangle} \\ &= \mathbb{Z} \oplus \mathbb{Z}_2. \end{aligned}$$

Hence the non-trivial simplicial homology groups are $H_0^\Delta(K) = \mathbb{Z}$ and $H_1^\Delta(K) = \mathbb{Z} \oplus \mathbb{Z}_2$. $\square$

Proposition 3. *Find a Δ-complex structure for the orientable surface of genus two and compute the simplicial homology groups of this Δ-complex.*

Proof. Consider the Δ-complex Σ_2 obtained by taking an octagon (with sides identified appropriately) and drawing segments from a fixed vertex to the other vertices (see figure). This yields one 0-simplex (v), nine 1-simplices $(a, \ldots, i)$, and six 2-simplices $(\alpha, \ldots, \zeta)$ and the chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z}^6 \xrightarrow{\partial_2} \mathbb{Z}^9 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0.$$

Clearly $H_0^\Delta(\Sigma_2) = \mathbb{Z}$. More tedious is $H_1^\Delta(\Sigma_2)$:

$$\begin{aligned}
H_1^\Delta(\Sigma_2) &= \frac{\ker \partial_1}{\text{im } \partial_2} \\
&= \frac{\langle a, b, c, d, e, f, g, h, i \rangle}{\langle d + c - e, f + d - e, g + c - f, g + b - h, h + a - i, a + b - i \rangle} \\
&= \frac{\langle b, c, d, e, f, g, h, i \rangle}{\langle c + d - e, d - e + f, c - f + g, b + g - h, h - b \rangle} \\
&= \frac{\langle b, c, d, e, f, g, i \rangle}{\langle c + d - e, d - e + f, c - f + g, g \rangle} = \frac{\langle b, c, d, e, f, i \rangle}{\langle c + d - e, d - e + f, c - f \rangle} \\
&= \frac{\langle b, c, d, e, i \rangle}{\langle c + d - e, c + d - e \rangle} = \frac{\langle b, c, d, e, i \rangle}{\langle c + d - e \rangle} \\
&= \mathbb{Z}^4.
\end{aligned}$$

Finally, $H_2^\Delta(\Sigma_2) = \ker \partial_2 / \text{im } \partial_3 = \ker \partial_2$. It is straightforward to check that $\ker \partial_2 = \langle \alpha - \beta - \gamma + \delta + \varepsilon - \zeta \rangle$ (one can read this off of the figure via the the orientations of the 2-simplicies). Hence the non-trivial simplicial homology groups are $H_0^\Delta(\Sigma_2) = \mathbb{Z}, H_1^\Delta(\Sigma_2) = \mathbb{Z}^4, H_2^\Delta(\Sigma_2) = \mathbb{Z}$. □

Proposition 4. *Find a Δ-complex structure for S^3 and compute the simplicial homology groups of this Δ-complex.*

Proof. □

Proposition 5. *Hatcher exercise 2.1.8*

Proof. □