Complex Analysis and Riemann Surfaces: Midterm

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Problem 1

Consider the function $w = \sqrt{\prod_{j=1}^{5}(z - a_j)}$, where a_j (for $j = 1, \dots, 5$) are pairwise distinct complex numbers. Let us order a_j increasing in magnitude. Let us construct the Riemann surface of w. We start by considering the behavior of w in the complex plane $\mathbb{C}$. Note first that w is clearly well-defined and holomorphic in the disc $D_{|a_1|}(0)$ (of course, if $a_1 = 0$, this is not true), because we can write

$$w = \exp\left(\frac{1}{2}\sum_j \ln(z - a_j)\right),$$

where, as usual, we view the first logarithm with a branch cut along the ray incident with the origin and a_1. Note, however, that in the annulus $D_{|a_2|}(0) - \overline{D_{|a_1|}(0)}$, w is not holomorphic, as we obtain a discontinuity along a curve from a_1 to a_2 (the curve depends on the branch cut chosen for $\ln(z - a_1)$, so for our purposes we will keep it simple and pick such that the discontinuities are straight lines): along this discontinuity it is easy to see that w jumps to $-w$. One might expect to encounter a similar discontinuity in the annulus $D_{|a_3|}(0) - \overline{D_{|a_2|}(0)}$, but in fact, because both $\sqrt{z - a_1}$ and $\sqrt{z - a_2}$ take $w \mapsto -w$, the signs cancel out and hence w is in fact holomorphic in that domain. Going out a little further, we again will see a sign discontinuity (as the signs do not cancel out this time) in the annulus $D_{|a_4|}(0) - \overline{D_{|a_3|}(0)}$, none in $D_{|a_5|}(0) - \overline{D_{|a_4|}(0)}$, and a sign discontinuity starting from a_5 and going out to infinity.

Thus to maximize the domain of definition one which w is holomorphic, we take two copies of the "cut" complex plane that we have been discussing - call them $\mathbb{C}_1$ and $\mathbb{C}_2$ on which we define $w = \sqrt{\prod_{j=1}^{5}(z - a_j)}$ and $w = -\sqrt{\prod_{j=1}^{5}(z - a_j)}$ respectively. We can now glue the two planes together in a way such that w will, in fact, be continuous, as the sign changes will now be compensated for. Before we do this, let us simplify the visual picture by thinking in terms of stereographic projections - each copy of the plane can be seen as topologically just the Riemann sphere with some arcs and the north pole cut out. Expanding the cuts to holes continuously we see that gluing together $\mathbb{C}_1$ and $\mathbb{C}_2$ in this sense will yield a surface with 2 handles. We claim that the resulting topology Σ is in fact a one-dimensional complex manifold in the sense that it is locally homeomorphic to $\mathbb{C}$. In fact, we can even compactify by adding the north pole (as the point at ∞) as we did in class, resulting in the compact surface $\hat{\Sigma}$, which is also topologically a manifold. To see this, let us examine the (structure of) open neighborhoods of varous points in $\hat{\Sigma}$. There are four cases: away from any cuts, at an a_i, on a cut, and at ∞.

It is clear that points away from discontinuities admit neighbhorhoods that can be holomorphically (and in a 1-to-1 fashion) mapped to a disc in the complex plane. Next, consider the case where we have a point z_0 physically on the cut. Then we can simply take two half-discs (one from each copy) and glue them together - this clearly yields a disc. Furthermore, if z_0 is one of the points a_i, we see that there is an almost-full disc about z_0 contained in each copy $\mathbb{C}_i$. To glue these together we have to be a little more careful, and we construct our chart to send a z in the neighborhood of z_0 to $t = \sqrt{z - a_i}$ on $\mathbb{C}_1$ and $t = -\sqrt{z - a_i}$ on $\mathbb{C}_2$. In other words, we are taking these almost-full discs to half discs at the origin in $\mathbb{C}$ and gluing them together to obtain a full disc. Finally we must deal with the point at infinity. This is done almost identically as was done with the a_i, except we take a reciprocal for the coordinates to get a well-defined disc: $1/\sqrt{z}$ on $\mathbb{C}_1$ and $-1/\sqrt{z}$ on $\mathbb{C}_2$.

Note that we have two meromorphic functions defined on $\hat{\Sigma}$. The first is simply w, which flips sign under the involution that switches $\mathbb{C}_1$ and $\mathbb{C}_2$. Note that w has a pole of order 5 at infinity:

$$w = \sqrt{\prod_{j=1}^{5}(z - a_j)} = \sqrt{\prod_{j=1}^{5}\left(\frac{1}{t^2} - a_j\right)} = \frac{1}{t^5}\sqrt{\prod_{j=1}^{5}(1 - t^2 a_j)}.$$

Furthermore, w has a simple zero at each a_i, because near a_i we see that:

$$w = \sqrt{\prod_{j=1}^{5}(z - a_j)} = \sqrt{\prod_{j=1}^{5}(t^2 + a_i - a_j)} = t\sqrt{\prod_{j\neq i}^{5}(t^2 + a_i - a_j)},$$

which is an order one zero because what is left is holomorphic (in the local coordinate patch) and because the a_i are distinct. Similarly, the function z at infinity is locally $1/t^2$ and hence has a pole of order 2, and near zero is t^2 and hence has a zero of order 2.

Let us now show that the forms given by $\omega_1 = dz/w$ and $\omega_2 = zdz/w$ are holomorphic. First, near an a_i, since $z = t^2 + a_i$, we find that

$$\omega_1 = \frac{dz}{w} = \frac{2tdt}{t\sqrt{\prod_{j\neq i}(t^2 + a_i - a_j)}} = \frac{2dt}{\sqrt{\prod_{j\neq i}(t^2 + a_i - a_j)}},$$

which is clearly holomorphic in this local chart. On the other hand, near infinity we have that $z = 1/t^2$ and hence we get

$$\omega_1 = \frac{dz}{w} = \frac{-2dt/t^3}{\frac{1}{t^5}\sqrt{\prod_{j=1}^{5}(1 - t^2 a_j)}} = \frac{-2t^2dt}{\sqrt{\prod_{j=1}^{5}(1 - t^2 a_j)}},$$

which is holomorphic in the chart near infinity. Similarly, for ω_2, near a_i and ∞:

$$\omega_2 = \frac{zdz}{w} = \frac{t^2 \cdot 2tdt}{t\sqrt{\prod_{j\neq i}(t^2 + a_i - a_j)}} = \frac{2t^2dt}{\sqrt{\prod_{j\neq i}(t^2 + a_i - a_j)}}$$
$$\omega_2 = \frac{zdz}{w} = \frac{1/t^2 \cdot -2dt/t^3}{\frac{1}{t^5}\sqrt{\prod_{j=1}^{5}(1 - t^2 a_j)}} = \frac{-2dt}{\sqrt{\prod_{j=1}^{5}(1 - t^2 a_j)}},$$

both of which are clearly holomorphic. Hence, since these forms are holomorphic in their local coordinate charts, they define holomorphic forms on $\hat{\Sigma}$.

We can, in fact, construct a meromorphic form $\omega_P(z)$ on $\hat{\Sigma}$ which has a double pole at the point $P = a_i$ but is holomorphic everywhere else. To do so, note that near a_i, the function z takes the form $t^2 + a_i$; hence if we take a holomorphic form and multiply by $1/(z - z(a_i))$ the a_i's will cancel and yield a double pole at the coordinate $t = 0$, i.e. $P = a_i$. In particular, consider

$$\omega_P(z) = \frac{1}{z - z(a_i)}\omega_1(z) = \frac{1}{z - z(a_i)}\frac{dz}{w}.$$

In coordinates near a_i, it's clear (by looking at the equation for ω_1 in coordinates above) that we have a double pole. Near another a_j, we have neither a zero or a pole as the a_i are all distinct (and so the denominator out front does not blow up). Finally, at infinity, the factor out front becomes $1/\left(1/t^2 - a_i\right) = t^2/\left(1 - a_i t^2\right)$, which has a zero of order 2 at infinity (while the factor of ω_1 does not vanish as one can check above). Hence $\omega_P(z)$ is indeed a meromorphic form with a double pole at a_i.

Finally, note that although we have constructed a meromorphic form with a double pole at one point, it is impossible for a meromorphic form on $\hat{\Sigma}$ to have a single pole. This follows from the statement we proved in class that any meromorphic form ω on a compact Riemann surface must have residues that add up to zero; of course, if a form has only one pole, the residue will be nonzero, which is a direct contradiction. We proved this by applying Stoke's theorem the integral of ω about the boundary of the complement of the union of little neighborhoods around the poles (since ω is holomorphic there).

Problem 2

Let $\Lambda = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$ be the lattice generated by ω_1, ω_2, where ω_1, ω_2 are complex numbers which are linearly independent over $\mathbb{R}$ and let $\Lambda^\times = \Lambda \setminus 0$. Define the function $\sigma(z)$ on $\mathbb{C}$ by

$$\sigma(z) = z \prod_{\omega \in \Lambda^\times} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{z^2}{2w^2}}.$$

Let us first show that σ is well-defined in that it converges. To check if this product converges, we take a logarithm (picking a branch cut on $\mathbb{C}$):

$$\begin{aligned}
\log \sigma(z) &= \log z + \sum_{w \in \Lambda^\times} \left(\log\left(1 - \frac{z}{w}\right) + \frac{z}{w} + \frac{z^2}{2w^2}\right) \\
&= \log z + \sum_{w \in \Lambda^\times} \left(\frac{z}{w} - \frac{1}{2}\frac{z^2}{w^2} + O(\frac{1}{|w|^3}) + \frac{z}{w} + \frac{z^2}{2w^2}\right) \\
&= \log z + \sum_{w \in \Lambda^\times} O(\frac{1}{|w|^3}),
\end{aligned}$$

which clearly converges. Note that here we have used the Taylor expansion for the logarithm. One may inquire as to the zeroes of $\sigma(z)$ - if $z = 0$ or any term in the product is zero, the whole function $\sigma(z)$ goes to zero. This, of course, only happens when $z = w$, i.e. $z = 0 \mod \Lambda$. Next let us determine the transformation law for $\sigma(z)$ when $z \mapsto z + \omega_a$ for $a = 1, 2$. Recall how we defined the function

$$\zeta(z) = \frac{1}{z} + \sum_{w \in \Lambda^\times} \left(\frac{1}{z + w} - \frac{1}{w} + \frac{z}{w^2}\right).$$

From this, it is easy to see that we can write $\zeta(z) = \sigma'(z)/\sigma(z)$. But note that the periodicity condition for ζ we have that $\zeta(z + \omega_a) - \zeta(z) = \eta_a$, which becomes

$$\eta = \frac{\sigma'(z + \omega_a)}{\sigma(z + \omega_a)} - \frac{\sigma'(z)}{\sigma(z)} = \partial_z \log \sigma(z + \omega_a) - \partial_z \log \sigma(z).$$

Integrating both sides with respect to z and exponentiating, we find that

$$\sigma(z + \omega_a) = \sigma(z) e^{\eta_a z + c_a}$$

with c_a a constant of integration, which we can find by taking $z = -\omega_a/2$ and using the fact that ω is odd,

$$\sigma(\omega_a/2) = \sigma(-\omega_a/2)e^{-\eta_a\omega_a/2+c_a},$$

which yields

$$\sigma(z+\omega_a) = -\sigma(z)e^{\eta_a(z+\omega_a/2)}.$$

Let us now give another proof of Abel's theorem (for $\hat{\Sigma} = \mathbb{C}/\Lambda$). First recall our previous statement of Abel's theorem.

Theorem 1 (Abel's theorem)**.** *Let $P_1, \dots, P_M, Q_1, \dots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M A(P_i) = \sum_{i=1}^N A(Q_i)$.*

Recall that the Abel map takes $\mathbb{C}/\Lambda \ni p \mapsto A(p) = \int_{p_0}^p \omega$ where the value of the integral is taken modulo the lattice generated by $\oint_A \omega, \oint_B \omega$. Take $p_0 = 0$ and $\omega = dz$, which is a well-defined form, and if we take A to align with ω_2 and B to align with ω_1, we see that $\oint_A \omega = \oint_A dz = \omega_1$ and similarly $\oint_B \omega = \omega_2$. Hence the map simply takes p to $\int_0^p dz \mod \Lambda = p$ where p is viewed as a complex number.

Let us now restate Abel's theorem in the context of this fact.

Theorem 2 (Abel's theorem, v.2)**.** *Let $P_1, \dots, P_M, Q_1, \dots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M P_i = \sum_{i=1}^N Q_i$ mod Λ.*

Proof. Consider the function

$$f(z) = \frac{\prod_{i=1}^M \sigma(z-P_i)}{\prod_{i=1}^N \sigma(z-Q_i)}.$$

We should be a little careful to note that σ is a function not on the torus $\mathbb{C}/\Lambda$, but a function on $\mathbb{C}$ (it transforms under a lattice translation!). Hence we must be cognizant of the fact that P_i, Q_i here are some chosen representatives in $\mathbb{C}$ of the equivalence classes of the points P_i, Q_i. It should be clear that $f(z)$ is meromorphic with zeroes at every representative of each P_is and poles at every representative of each Q_i. The natural question, now, is whether this function extends to a function on the torus. To check this, let us see whether it is doubly periodic using what we know about σ:

$$\begin{aligned} f(z+\omega_a) &= f(z)\frac{\prod_{i=1}^M e^{\eta_a(z-P_i)}}{\prod_{i=1}^N e^{\eta_a(z-Q_i)}} \\ &= f(z)e^{-\eta_a\left(\sum_{i=1}^M P_i - \sum_{i=1}^N Q_i\right)}. \end{aligned}$$

Hence we wish to choose P_i, Q_i representatives such that the exponential becomes unity. By hypothesis, this can be done (by shifting one, if necessary). $\square$

Problem 3

Let now $\omega_1 = 1, \omega_2 = \tau$, with Im $\tau > 0$. Define the function

$$\theta_1(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(n+\frac{1}{2}\right)^2 \tau + 2\pi i \left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right).$$

The presence of an $(n+1/2)^2$ in the first term of the exponential dominates, as for large $|n|$ it is the leading order term, and it decays rapidly. More explicitly, we have the n^2 terms

$$|e^{\pi i n^2(\tau_1+i\tau_2)}| = |e^{\pi i n^2\tau_1}e^{-\pi n^2\tau_2}| = e^{-\pi n^2\tau_2},$$

where we have written $\tau=\tau_1+i\tau_2$. Due to this decay, it's clear that the sum converges (one might use the integral test) and hence $\theta_1(z|\tau)$ is holomorphic. Next let us show that $\theta_1(z|\tau)=0$ only at $z=0 \mod \Lambda$. Hence let us verify that θ_1 is odd; switching $z\mapsto -z$ yields in the exponent

$$\log\theta_1(z|\tau) = \pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right).$$

If we switch the indices $n\mapsto m$ such that $n+\frac{1}{2} = -\left(m+\frac{1}{2}\right)$, we find that the exponent is now

$$\log\theta_1(z|\tau) = \pi i\left(m+\frac{1}{2}\right)^2\tau + 2\pi i\left(m+\frac{1}{2}\right)\left(\left(z+\frac{1}{2}\right)-2\pi i\left(m+\frac{1}{2}\right)\right),$$

and hence θ_1 is odd. To see that this is indeed the only zero, we recall that we can compute an integral to count the number of zeroes enclosed (as θ_1 is holomorphic):

$$\begin{aligned}\oint_C \frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)}dz &= \oint_B\left(-\frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)}+\frac{\theta_1'(z+1,\tau)}{\theta_1(z+1,\tau)}\right)dz + \oint_A\left(\frac{\theta_1'(z,\tau)}{\theta_1(z,\tau)}-\frac{\theta_1'(z+\tau,\tau)}{\theta_1(z+\tau,\tau)}\right)dz\\ &= 2\pi i\oint_A dz = 2\pi i,\end{aligned}$$

where we have used the transformation properties of θ_1 that we will derive shortly. But recall that this integral gives us $2\pi i$ times the number of zeroes (for a holomorphic function) and hence $z=0$ mod Λ is in fact the only zero.

Let us compute two transformation properties of the θ_1 function that we used above (and will use below). First note that we can write

$$\theta_1(z) = e^{\pi i/2}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z\right)(-1)^n.$$

Then we can compute what happens under translations:

$$\begin{aligned}\theta_1(z+1) &= e^{\pi i/2}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z+2\pi i(n+\frac{1}{2})\right)(-1)^n\\ &= -e^{\pi i/2}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})z\right)(-1)^n\\ &= -\theta_1(z)\\ \theta_1(z+\tau) &= e^{\pi i/2}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2})^2+2\pi i(n+\frac{1}{2})(z+\tau)\right)(-1)^n\\ &= -e^{\pi i/2}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2})^2+2\pi i\tau(n+\frac{1}{2})+2\pi i(n+\frac{1}{2}+1)z-2\pi iz\right)(-1)^{n+1}\\ &= -e^{\pi i/2}e^{-\pi i\tau-2\pi iz}\sum_{n=-\infty}^{\infty}\exp\left(i\pi\tau(n+\frac{1}{2}+1)^2+2\pi i(n+\frac{1}{2}+1)z\right)(-1)^{n+1}\\ &= -e^{-\pi i\tau-2\pi iz}\theta_1(z)\end{aligned}$$

Let us now prove Abel's theorem using the machinery of θ-functions. In other words, $\sum_{i=1}^{N} A(P_i) = \sum_{j=1}^{N} A(Q_j)$ if and only if there exists an f meromorphic with zeros at P_i and poles at Q_j. Note first that we can write $\sum_i P_i = \sum_j Q_j + n + m\tau$. To construct f, let us try

$$f(z) = \frac{\prod_{i=1}^{N} \theta_1(z - P_i)}{\prod_{j=1}^{N} \theta_1(z - Q_j)}.$$

From the result about the zeros of θ_1 above, it is clear that this function should have the appropriate zero and pole behavior. It remains to check that θ_1 is well-defined on $\mathbb{C}/\Lambda$, i.e. doubly-periodic. First note that, using the periodicity properties of the θ_1 function,

$$\begin{aligned} f(z+1) &= \frac{\prod_{i=1}^{N} \theta_1(z+1-P_i)}{\prod_{j=1}^{N} \theta_1(z+1-Q_j)} = \frac{(-1)^N \prod_{i=1}^{N} \theta_1(z-P_i)}{(-1)^N \prod_{j=1}^{N} \theta_1(z-Q_j)} \\ &= f(z). \end{aligned}$$

Next,

$$\begin{aligned} f(z+\tau) &= \frac{\prod_{i=1}^{N} \theta_1\left(z+\tau-P_i\right)}{\prod_{j=1}^{N} \theta_1(z+\tau-Q_j)} = \frac{(-1)^N \prod_{i=1}^{N} \theta_1\left(z-P_i\right) e^{-2\pi i(z-P_i)-\pi i\tau}}{(-1)^N \prod_{j=1}^{N} \theta_1\left(z-Q_j\right) e^{-2\pi i(z-Q_j)-\pi i\tau}} \\ &= e^{2\pi i(\sum_j Q_j - \sum_i P_i)} f(z). \end{aligned}$$

Hence we may replace one of the points, say Q_1, by $\tilde{Q}_1 + n + m\tau$, thus enforcing $\sum_i P_i = \sum_j Q_j$ and the result follows.

Problem 4

Let $L \to X$ be a holomorphic line bundle over a compact Riemann surface X, defined by the transition functions $t_{\alpha\beta}(z)$ on $X_\alpha \cap X_\beta$, where $X = \cup_\alpha X_\alpha$ is a covering of X by holomorphic charts. A smooth metric h on L is a smooth section of $L^{-1} \otimes \overline{L}^{-1}$ that is strictly positive. Given any two metrics, h and h', it is clear that we can relate them by $h' = e^{-\phi}h$ for some smooth ϕ a scalar function on X. This follows simply because we may choose $\phi = -\log(h'/h)$. As h, h' are strictly positive everywhere, this is well-defined and ϕ is smooth.

Next let $F_{\bar{z}z}, F'_{\bar{z}z}$ be the curvatures of L with respect to the metrics h and h' respectively. Recall that the curvature is given explicitly as $F_{\bar{z}z} = -\partial_{\bar{z}}\Gamma = -\partial_{\bar{z}}\partial_z \log h$ in terms of the metric h. Writing it instead in terms of the curvature $h' = e^{-\phi}h$, we find that

$$F'_{\bar{z}z} = -\partial_{\bar{z}}\partial_z \log\left(e^{-\phi}h\right) = \partial_{\bar{z}}\partial_z\phi + F_{\bar{z}z}$$

If we now compute the Chern class $c_1(L)$ using each of these curvatures we find that

$$\begin{aligned} c_1(L) &= \frac{i}{2\pi}\int_X F_{\bar{z}z} dz \wedge d\bar{z} \\ c_1'(L) = \frac{i}{2\pi}\int_X F'_{\bar{z}z} dz \wedge d\bar{z} &= \frac{i}{2\pi}\int_X F_{\bar{z}z} + \partial_{\bar{z}}\partial_z\phi dz \wedge d\bar{z} \\ &= c_1(L) + \frac{i}{2\pi}\int_X \partial_{\bar{z}}\partial_z\phi dz \wedge d\bar{z} \\ &= c_1(L) + \frac{i}{2\pi}\int_X d\left(\partial_{\bar{z}}\phi d\bar{z}\right) \\ &= c_1(L) \end{aligned}$$

and hence the curvature is independent of the choice of metric.

Problem 5

Theorem 3 (Riemann-Roch). *Let X be a Riemann surface and $H^0(X, L)$ denote the space of holomorphic sections of a line bundle L over X. Furthermore let K_X be the canonical line bundle over X and $c_1(L)$ be the first Chern class of a line bundle L. Then*

$$\dim H^0(X, L) - \dim H^0(X, K_X \otimes L^{-1}) = c_1(L) + \frac{1}{2} c_1(K_X^{-1}).$$

To deduce that a line bundle L over X always admits non-trivial meromorphic sections, recall the construction of point bundles. For some $p \in X$ pick a coordinate system in a neighborhood X_0 of p, and set $X_\infty = X \setminus \{p\}$. Let L be $\{t_{0\infty}(z) = z \text{ on } X_0 \cap X_\infty\}$. This defines a holomorphic line bundle which admits a holomorphic section $1_p = 1$ on X_∞; z on X_0, and $1_p|_{X_0} = z = z \cdot 1 = t_{0\infty} \cdot 1_p|_{x_\infty}$. Hence we see that 1_p is a holomorphic section of $L \equiv [p]$ and has exactly one zero at p. In particular, $c_1(L) = 1$ here. Similarly, we may define, for any integer n, the bundle $[np]$ by the transition function $\{z^n\}$. It is easy to see that $c_1([np]) = n$. Let us now proceed to prove the existence of a non-trivial section. Pick a point p and consider the bundle $L \otimes [np]$. It's clear that $c_1(L \otimes [np]) = c_1(L) + n$. Then, by the Riemann-Roch theorem, we find that

$$\dim H^0(X, L \otimes [np]) - \dim H^0(X, L^{-1} \otimes [-np] \otimes K) = c_1(L) + n - \frac{1}{2}\chi(X)$$

where χ is the Euler characteristic as usual. But now we can take $n \gg 0$ arbitrarily large, and since dimensions are positive and the Chern class and Euler characteristic are independent of the point bundle chosen, we see that

$$\dim H^0(X, L \otimes [np]) > 0,$$

which implies that there exists a non-trivial holomorphic section of $L \otimes [np]$. Finally we note that multiplying this section by 1_{-np} yields a non-zero meromorphic section of L, and we are done.