Differentiable Manifolds Problem Set 6

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Problem 1

Let $V_1, \dots, V_k$ and W be finite-dimensional real vector spaces. We wish to show that there is a canonical isomorphism $V_1^* \otimes \cdots \otimes V_k^* \otimes W \cong L(V_1, \dots, V_k; W)$. First consider the map $\Phi : V_1^* \times \cdots \times V_k^* \times W \to L(V_1, \dots, V_k; W)$ given by

$$\Phi(\xi^1, \dots, \xi^k, w)(v_1, \dots, v_k, \omega) = \xi^1(v_1) \cdots \xi^k(v_k) w.$$

This right-hand side is clearly linear in each of the arguments and takes values in W, so it is indeed a member of $L(V_1, \dots, V_k; W)$. Note that Φ is multilinear in $\xi^1, \dots \xi^k, w$, as the right-hand side is simply multiplication. By the characteristic property of tensor product spaces, then, Φ descends uniquely to a linear map $\tilde{\Phi}$ from $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ to $L(V_1, \dots, V_k; W)$:

$$\tilde{\Phi}(\xi^1 \otimes \cdots \otimes \xi^k, w)(v_1, \dots, v_k, \omega) = \xi^1(v_1) \cdots \xi^k(v_k) w.$$

Furthermore, $\tilde{\Phi}$ takes the basis of $V_1^* \otimes \cdots \otimes V_k^* \otimes W$ to the basis of $L(V_1, \dots, V_k; W)$:

$$\tilde{\Phi}(e^1 \otimes \cdots \otimes e^k, E_1)(v_1, \dots, v_k, w) = e^1(v_1) \cdots e^k(v_k) E_1.$$

Thus $\tilde{\Phi}$ must be an isomorphism (independent of the specific bases chosen), and we are done.

Problem 2

Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Suppose that $\omega = e^1 \otimes e^2 \otimes e^3$ can be written as the sum of an alternating tensor and a symmetric tensor,

$$\omega = e^1 \otimes e^2 \otimes e^3 = (\eta + \theta)_{ijk}\, e^i \otimes e^j \otimes e^k$$

with η antisymmetric, θ symmetric. By linear independence of the basis vectors of $T^3((\mathbb{R}^3)^*)$, the only non-vanishing term on the right-hand side must be that which contains a $e^1 \otimes e^2 \otimes e^3$, i.e. $(\eta + \theta)_{123} = 1$. Note, however, that $(\eta + \theta)_{312} = (\eta + \theta)_{123}$ must be zero. This contradicts the previous statment, and thus this tensor cannot be written as a sum of an alternating and a symmetric tensor.

Problem 3

We wish to show that the covectors $\omega^1, \ldots, \omega^k$ are linearly independent on a finite-dimensional space if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Problem 4

Problem 5

Problem 6