ON THE WONDERFUL COMPACTIFICATION OF SL_n VIA WEIGHTED PROJECTIVE SPACE

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Abstract. Hello

1. Introduction

The wonderful compactification of a variety equipped with an action of an algebraic group G is a G-equivariant compactification such that the closure of each orbit is smooth. DeConcini and Procesi introduced the wonderful compactification of any complex semisimple group of adjoint type in [3] with a view towards problems of enumerative geometry. Since then, the wonderful compactification has taken on a significant role in various other areas of mathematics.

In this work, we construct and analyze the properties of a compactification of $G = \mathrm{SL}_n$. Here the usual technique of constructing a $G \times G$-equivariant embedding $\psi : G \to \mathbb{P}(\operatorname{End} V)$ (with V an irreducible G representation) fails; in particular, the center $Z(\mathrm{SL}_n)$ collapses to a single point in projective space. Hence we work instead with a generalization of projective space known as *weighted projective space*, denoted by $\mathbb{P}_w$. The crux of our approach lies in constructing an embedding $\psi : G \to \mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$ on which G acts equivariantly from the left.

This article is organized as follows.

In section 2, we state some useful definitions and well-known facts, as well as a brief summary of the properties of the wonderful compactification vis a vis [1] and [3]. Next, in section 3, we investigate some properties of the wonderful compactification of PGL_3. Finally, in section 4, we treat the main problem of SL_n, for which we construct a compactification and examine its properties.

2. Preliminaries

2.1. **Some definitions.** Throughout this article, G will denote either a complex semisimple group of adjoint type or SL_n, depending on the context, and $\mathfrak{g}$ will denote its Lie algebra. T will denote a fixed maximal rank-l torus ($\mathfrak{t}$ its Lie algebra) and R will denote the set of roots associated with $\mathfrak{g}$, $\mathfrak{t}$. W will denote the Weyl group of $(\mathfrak{g}, \mathfrak{t})$ and B a fixed Borel subgroup of G. This determines a set of simple roots $\Delta \subset R$ and a choice of positive and negative roots $R = R^+ \coprod R^-$. By I we will denote a subset of Δ. Define W_I to be the subgroup of W generated by I, and define $P_I = BW_IB$ to be the parabolic subgroup associated to I. We write the opposite parabolic as P_I^-. Each P_I admits a Levi decomposition, namely $P_I = L_I \ltimes U_I$ where L_I is the Levi subgroup and U_I is the unipotent radical of P_I. It will also be useful to consider the adjoint group $\tilde{L}_I = L_I / Z(L_I)$.

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2.2. **The wonderful compactification.** The following basic facts about the wonderful compactification of groups of adjoint type will be useful to keep in mind.

Theorem 2.1 (Brion and Kumar [1], Theorem 6.1.8)**.** *Let G be a complex semisimple group of adjoint type and take $\rho : G \to \operatorname{Aut} V$ to be a representation of highest weight λ. Consider the $(G \times G)$-module $\operatorname{End} V = V^* \otimes V$. Let $h \in \operatorname{End} V$ be the identity, with image $[h]$ in the projectivization $\mathbb{P} = \mathbb{P}(\operatorname{End} V)$. Define an embedding $\psi : G \to \mathbb{P}(\operatorname{End} V)$ given by $\psi(g) = [\rho(g)]$. We denote the closure of the embedding by $X = \overline{\psi(G)}$. Then the orbit orbit $(G \times G) \cdot [h]$ is isomorphic to G. Moreover, the following are true:*

1. *Denote by $\mathbb{P}_0$ the complement in $\mathbb{P}$ of $\{x_{11} = 0 : x \in \mathbb{P}\}$ and $X_0 = X \cap \mathbb{P}_0$. Then, X_0 is nonsingular.*
2. *X is covered by the $(G \times G)$-translates of X_0. In particular, X is nonsingular;*
3. *The boundary $\partial X = X - G$ is the union of l nonsingular prime divisors $X_1, \ldots, X_l$ with normal crossings;*
4. *For each subset $I \subset \{1, \ldots, l\}$, the intersection $X_I = \cap_{i \in I} X_i$ is the closure of a unique $(G \times G)$-orbit $\mathcal{O}_I$. Conversely, any $(G \times G)$-orbit in X equals $\mathcal{O}_I$ for a unique I. Further, $\overline{\mathcal{O}}_I \supseteq \mathcal{O}_J$ if and only if $I \subset J$;*
5. *X contains a unique closed orbit $Y = \mathcal{O}_{1,\ldots,l} = X_1 \cap \ldots \cap X_l$, which is isomorphic to $G/B \times G/B$;*
6. *X is independent of the choices of λ and V.*

Theorem 2.2 (DeConcini and Procesi [3], Section 5)**.** *Let I be a subset of $\{1, \ldots, l\}$ as above and consider the $(G \times G)$-orbit $\mathcal{O}_I$. There exists a distinguished point $x_I \in \mathcal{O}_I$ such that*

$$H_I \equiv \operatorname{Stab}_{x_I} = \{(lu, l'u') : u \in U_I, u' \in U_I^-, l \in L_I, l' \in L_I, l(l')^{-1} \in Z(L_I)\}.$$

Hence we may write $\mathcal{O}_I = (G \times G)/H_I$. Moreover, there exists a G-equivariant fibration $\pi_I : \mathcal{O}_I \to G/P_I \times G/P_I^-$ with fiber $\tilde{L}_I$.

For notational convenience, we will denote elements of $\mathcal{O}_I$ as cosets of the form $(g_1, g_2)H_I$ and elements of $G/P_I \times G/P_I^-$ as cosets of the form $(g_1 P_I, g_2 P_I^-)$.

2.3. **Weighted projective space.** To construct a wonderful compactification of SL_n we will use the construction known as weighted projective space, defined as follows.

Definition 2.3. The $(n-1)$-dimensional weighted projective space $\mathbb{P}_w$ with weights $w = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is the quotient space

$$\mathbb{P}_w = \mathbb{P}(a_1, \ldots, a_n) = \mathbb{C}^n - 0/(z_1, \ldots, z_n) \sim (\lambda^{a_1} z_1, \ldots, \lambda^{a_n} z_n),$$

for $\lambda \in \mathbb{C}^\times$. Alternatively, one can view weighted projective space as the quotient of $\mathbb{C}^n - 0$ by the action of left multiplication by the diagonal matrix:

$$\begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_n} \end{pmatrix}.$$

For an arbitrary choice of weights this action is not free in general and hence weighted projective spaces may have singularities.

2.4. **Representations of SL_3.** We briefly define the weights of the standard and adjoint representations of SL_3 below. For further discussion, refer to [4].

Let the subgroup H of the torus $T \in \mathrm{SL}_3$ be defined by:

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_2^{-1}t_1^{-1} \end{pmatrix},$$

for $t_1, t_2 \in \mathbb{C}^\times$. For the *standard representation* of SL_3, where $\rho_{std} : \mathrm{SL}_3 \to \mathrm{Aut}(\mathbb{C}^3)$, and SL_3 acts by left multiplication on $\mathbb{C}^3$, we define the weights $\omega_1, \omega_2, \omega_3 \in H^*$, with highest weight ω_1, as follows:

$$\omega_1(T) = t_1,\ \omega_2(T) = t_2,\ \omega_3(T) = t_2^{-1}t_1^{-1}.$$

For the *adjoint representation* of SL_3, with $\rho_{adj} : \mathrm{SL}_n \to \mathrm{Aut}(\mathfrak{sl}_3),\ (g,h) \mapsto gxh^{-1}$, for $x \in \mathfrak{sl}_3$, we define the weights $\alpha_1, \alpha_2, \alpha_3$, with highest weight vector α_3, as:

$$\alpha_3 = 2\omega_1 + \omega_2,\ \alpha_2 = \omega_1 + 2\omega_2,\ \alpha_1 = \omega_1 + \omega_2.$$

3. Left G-orbits in the wonderful compactification of PGL_3

It will be useful in the case of SL_n to decompose the $(G \times G)$-orbits X_I further into orbits under the left action of G. To this end, we first study this decomposition in the simpler case of PGL_3 where weighted projective space is not needed. The computation of the left G-orbit representatives follows fairly easily from the generalized Bruhat decomposition and from Theorem 2.2.

3.1. **Decomposition into left G-orbits.** Before computing left G-orbits in $\mathcal{O}_I$, we first prove the following lemma, which provides the orbit representatives in the product that $\mathcal{O}_I$ fibers over.

Lemma 3.1. *The elements (P_I, wP_I^-) and (P_I, vP_I^-) with distinct $w, v \in W_{P_I}\backslash W/W_{P_I}$ sit in different orbits of $G/P_I \times G/P_I^-$ under the left action of G.*

Proof. This amounts to showing that (P_I, wP_I^-) and (P_I, vP_I^-) are in the same orbit if and only if $w = v$ for $w, v \in W_{P_I}\backslash W/W_{P_I}$. Suppose there exists a $g \in G$ such that $g \cdot (P_I, wP_I^-) = (P_I, vP_I^-)$. This requires that $g \in P_I$ and $gwp^- = v$ for some $p^- \in P_I^-$. By the uniqueness of the generalized Bruhat decomposition,

$$G = \coprod_{w \in W_{P_I}\backslash W/W_{P_I}} P_I w P_I^-, \tag{3.1}$$

we see that $v = w$. The converse holds trivially. □

Moreover, $\mathcal{O}_I$ can be written as $G \times G/H_I$ where H_I is the stabilizer of the distinguished point $x_I \in \mathcal{O}_I$. Procesi and DeConcini compute explicitly in [3] that Using the equivariance of the fibration π, together with the above lemma, we find that $(1, w)H_I$ and $(1, v)H_I$ lie in different G-orbits of $\mathcal{O}_I$. We can now prove that the $(1, w)H_I$ are in fact G-orbit representatives using some simple algebra.

Theorem 3.2. *The elements $(1, w)H_I$ with $w \in W_{P_I}\backslash W/W_{P_I}$ are a set of representatives for the orbits in $\mathcal{O}_I$ under the left action of G.*

Proof. Consider $(g_1, g_2)H_I \in \mathcal{O}_I$. It suffices to show that this element falls into an orbit containing $(1, w)H_I$ for some w. The left action of $(g_1^{-1}, 1)$ yields $(1, g_1^{-1}g_2)H_I$. By the generalized Bruhat decomposition, we can write this as $(1, l_1u_1wl_2u_2)$ for $l_1, l_2 \in L_I$, $u_1 \in U_I, u_2 \in U_I^-$, and $w \in W_{P_I}\backslash W/W_{P_I}$. Absorbing u_2 into H_I and acting by $((l_1u_1)^{-1}, 1)$ yields $(u_1^{-1}l_1^{-1}, wl_2)$, which we can rewrite as $(l_1^{-1}u_3, wl_2)$ for some $u_3 \in U_I$ using the fact that $P_I = L_I \ltimes U_I$. We can absorb u_3 into H_I, and then write the point as $(l_1^{-1}l_2^{-1}, w)(l_2, l_2)H_I$ and absorb (l_2, l_2) into H_I. Finally, acting on this point by $(l_2l_1, 1)$ yields, as desired, the coset $(1, w)H_I$. □

4. The case of SL_n

In this section, we construct a compactification X of $G = \mathrm{SL}_n$ using weighted projective space. To do so, we will construct an embedding $\psi : \mathrm{SL}_n \to \mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$, with V a faithful representation of G. Then $X = \overline{\psi(G)}$ is a compactification equivariant under the left action of G. For ψ to be an embedding, the weights w of the weighted projective space must satisfy certain constraints. We derive these constraints and then examine the properties of the resulting compactification.

4.1. **Constructing the embedding.** Let $\rho : G \to \operatorname{Aut} V$ be a faithful representation. Define a map $\psi : G \to \mathbb{P} \equiv \mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$, given by $g \mapsto ([\rho(g)], [\rho(g)])$, where the square brackets denote equivalence classes. Next we equip $\mathbb{P}$ with a left action of G as:

$$g \cdot ([y_1], [y_2]) = ([\rho(g)y_1], [\rho(g)y_2]). \tag{4.1}$$

In particular, we can write y_1, y_2, and $\rho(g)$ as matrices, affording us the interpretation of the action as matrix multiplication. It is not *a priori* obvious that this action is well-defined; the following lemma derives constraints on the weights w for it to be so. For ease of notation, we also write the set of weights in matrix form:

$$w = \begin{pmatrix} a_1 & \dots & a_k \\ \vdots & \ddots & \vdots \\ a_{k^2-k-1} & \dots & a_{k^2} \end{pmatrix}.$$

Lemma 4.1. *The left-action of G is well-defined on the equivalence classes in $\mathbb{P}$ when*

$$w = \begin{pmatrix} a_1 & \dots & a_k \\ \vdots & \ddots & \vdots \\ a_1 & \dots & a_k \end{pmatrix},$$

where $a_1, \dots, a_k \in \mathbb{N}^k$.

Proof. Take $g \in G$ and $p = ([x], [y]) \in \mathbb{P}$. For the action to be well-defined it must be independent of the choice of representatives x and y. In other words, we require that, for $\tilde{x}, \tilde{y} \in \operatorname{End} V$ such that $[x] = [\tilde{x}]$ and $[y] = [\tilde{y}]$,

$$\begin{aligned} g \cdot ([x], [y]) &= g \cdot ([\tilde{x}], [\tilde{y}]) \\ ([\rho(g)x], [\rho(g)y]) &= ([\rho(g)\tilde{x}], [\rho(g)\tilde{y}]) \end{aligned} \tag{4.2}$$

This amounts to the statement that $[\rho(g)x] = [\rho(g)\tilde{x}]$ as elements of $\mathbb{P}(\operatorname{End} V)$ and that $[\rho(g)y] = [\rho(g)\tilde{y}]$ as elements of $\mathbb{P}_w(\operatorname{End} V)$. For ease of notation, we will prove the case $k = 2$; the proof generalizes easily to higher k.

We write the matrix elements of g as g_i, of x as x_i, and of y as y_i, with $g_i, x_i, y_i \in \mathbb{C}$ for $i = 1, \dots, k^2$. Since $[x] = [\tilde{x}] \implies \tilde{x} = \lambda x$ for some $\lambda \in \mathbb{C}^\times$, for the first condition to hold there must exist a $\mu \in \mathbb{C}^\times$ such that

$$\mu \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \lambda x_1 & \lambda x_2 \\ \lambda x_3 & \lambda x_4 \end{pmatrix}$$

$$\mu \begin{pmatrix} g_1x_1 + g_2x_3 & g_1x_2 + g_2x_4 \\ g_3x_1 + g_4x_3 & g_3x_2 + g_4x_4 \end{pmatrix} = \lambda \begin{pmatrix} g_1x_1 + g_2x_3 & g_1x_2 + g_2x_4 \\ g_3x_1 + g_4x_3 & g_3x_2 + g_4x_4 \end{pmatrix}.$$

Clearly we can pick $\mu = \lambda$. The second condition is a little more complicated; $[y] = [\tilde{y}] \implies \tilde{y} = \gamma \cdot y$ for some $\gamma \in \mathbb{C}^\times$, and so for the condition to hold there must exist a $\kappa \in \mathbb{C}^\times$ such that

$$\kappa \cdot \left(\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \right) = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} \gamma^{a_1} y_1 & \gamma^{a_2} y_2 \\ \gamma^{a_3} y_3 & \gamma^{a_4} y_4 \end{pmatrix}$$

$$\kappa \cdot \begin{pmatrix} g_1y_1 + g_2y_3 & g_1y_2 + g_2y_4 \\ g_3y_1 + g_4y_3 & g_3y_2 + g_4y_4 \end{pmatrix} = \begin{pmatrix} g_1\gamma^{a_1}y_1 + g_2\gamma^{a_3}y_3 & g_1\gamma^{a_2}y_2 + g_2\gamma^{a_4}y_4 \\ g_3\gamma^{a_1}y_1 + g_4\gamma^{a_3}y_3 & g_3\gamma^{a_2}y_2 + g_4\gamma^{a_4}y_4 \end{pmatrix}.$$

$$\begin{pmatrix} \kappa^{a_1}(g_1y_1 + g_2y_3) & \kappa^{a_2}(g_1y_2 + g_2y_4) \\ \kappa^{a_3}(g_3y_1 + g_4y_3) & \kappa^{a_4}(g_3y_2 + g_4y_4) \end{pmatrix} = \begin{pmatrix} g_1\gamma^{a_1}y_1 + g_2\gamma^{a_3}y_3 & g_1\gamma^{a_2}y_2 + g_2\gamma^{a_4}y_4 \\ g_3\gamma^{a_1}y_1 + g_4\gamma^{a_3}y_3 & g_3\gamma^{a_2}y_2 + g_4\gamma^{a_4}y_4 \end{pmatrix}.$$

Similarly as above, we can pick $\kappa = \gamma$, as long as $a_1 = a_3$ and $a_2 = a_4$. This extends obviously to higher dimensions.

Hence, for the action of G to be well-defined on $\mathbb{P}$, we must require that the weights along the columns are identical.

□

Remark 4.2. Under the above weight restriction, it is easy to see that two elements are in the same equivalence class of $\mathbb{P}_w(\operatorname{End} V)$ if one can be written as the other right-multiplied by the k-by-k diagonal matrix

$$\begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_k} \end{pmatrix}$$

for some $\lambda \in \mathbb{C}^\times$.

The next lemma provides the conditions for the injectivity of ψ.

Lemma 4.3. *The embedding ψ is injective if for some $p, q, r, s \in \{1, \dots, k\}$, $\gcd(a_p - a_q, a_r - a_q) = 1$.*

Proof. Suppose $\psi(g_1) = \psi(g_2)$ for some $g_1, g_2 \in G$. Then $([\rho(g_1)], [\rho(g_1)]) = ([\rho(g_2)], [\rho(g_2)])$ in $\mathbb{P}$. That is, for some $\lambda, \mu \in \mathbb{C}^\times$,

$$\rho(g_2)^{-1}\rho(g_1) = \begin{pmatrix} \mu & & \\ & \ddots & \\ & & \mu \end{pmatrix} = \begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_k} \end{pmatrix} \tag{4.3}$$

Since $\lambda^{a_i} = \mu$ for each $i \in \{1, \dots, k\}$, we have $\lambda^{a_p - a_q} = \mu\mu^{-1} = \lambda^{a_r - a_s} = 1$. The hypothesis requires that $\lambda = 1$, and hence that $\mu = 1$. Thus, $\rho(g_1) = \rho(g_2)$ in $\operatorname{End} V$, and since ρ is a faithful representation, $g_1 = g_2$. Consequently, ψ must be injective. □

The condition of Lemma 4.3 holds for any faithful V representation with the weight vector $w = (1,\dots,1,a,1,\dots,1,b,1,\dots,1)$, with $gcd(a-1,b-1) = 1$. We make use of this to compactify SL_3.

Example 4.1. We compactify SL_3 using the irreducible component of the tensor product of the adjoint representation and standard representation with highest weight vector $3\omega_1 + \omega_2$.

So $V \subset \mathfrak{sl}_3 \otimes \mathbb{C}^3$, and $\psi : \mathrm{SL}_3 \to \mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$.

Though we do not reproduce the computation, we specify the image of the torus $T \in \mathrm{SL}_3$ in V below.

For any $g \in \mathrm{SL}_3$, $\psi(g)$ is the pair $(\rho(g), \rho(g))$, where the two elements in the pair represent equivalence classes in their ambient projective spaces.

Then $\psi(T) = (\rho(T), \rho(T))$, where $T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_2^{-1}t_1^{-1} \end{pmatrix}$.

Since $\rho(T)$ is a 15×15 diagonal matrix, we write the diagonal entries as a row vector for the ease of the reader.

And $\rho(T) =$

$$\left[t_1^3t_2 : t_1^2t_2^2 : t_1 : t_1^2t_2^{-1} : t_1t_2^3 : t_2 : t_1 : t_1^{-1}t_2^2 : t_1^{-1}t_2^{-1} : t_2 : t_1^{-2} : t_2^{-2} : t_1^{-1}t_2^{-1} : t_1^{-3}t_2^{-2} : t^{-2}\right]$$

To investigate the closure of the image of SL_3, we compute the limit points of the one-parameter subgroups of the torus in $\mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$. A one-parameter subgroup of the torus in $\mathbb{P}(\operatorname{End} V) \times \mathbb{P}_w(\operatorname{End} V)$ is the pair $\left(\lambda^{(i,j)}(t), \lambda^{(i,j)}(t)\right)$, where $\lambda^{(i,j)}(t) =$

$$\left[t^{3i+j} : t^{2i+2j} : t^i : t^{2i-j} : t^{i+3j} : t^j : t^i : t^{-i+2j} : t^{-i-j} : t^j : t^{-2i} : t^{-2j} : t^{-i-j} : t^{-3i-2j} : t^{-2j}\right]$$

for $i, j, \in \mathbb{Z}$. Now, we compute $\lim_{x\to 0} \lambda^{(i,j)}(t)$ in both $\mathbb{P}(\operatorname{End} V)$ and $\mathbb{P}_w(\operatorname{End} V)$, for all possible values of i and j.

Let $w = (1,1,1,1,1,a,b,1,1,1,1,1,1,1,1)$ with $gcd(a-1,b-1) = 1$.The equivalence class of $\lambda(t)$ in $\mathbb{P}_w(\operatorname{End} V)$ is defined by scaling every entry by any $\mu \in \mathbb{C}^\times$, except for the sixth and seventh entires, t^i and t^j, which are scaled by μ^a and μ^b, respectively.

By inspection, for any values of i and j, neither coordinate will have the least power, that is, a negative power of greatest absolute value among the coordinates, except when $i = j = 0$.

Thus, the limit points of $\lambda^{(i,j)}(t)$ in $\mathbb{P}_w(\operatorname{End} V)$ are the same as the limit points in $\mathbb{P}(\operatorname{End} V)$, though they represent different equivalence classes. Consequently, we can completely determine the fan for $T \in \mathrm{SL}_3$.

The embedding for SL_3 can be generalized for any SL_n, as long as w assigns the weights a, b to coordinates of the torus that are not limit points in the character lattice.

4.2. **Properties of the embedding.** In essence, we hope to show that the tactic used to determine the weight vector w can be generalized for the compactification of SL_n. That is, we should be able to choose w so that the limit points of $\lambda(t)$ in $\mathbb{P}_w(\operatorname{End} V)$ are the same as those in $\mathbb{P}(\operatorname{End} V)$, and thus the fan for $Z = \overline{\psi(T)}$ will be complete and can be systematically determined.

Theorem 4.4. *Let $\psi : \mathrm{SL}_n \to \mathbb{P}(\mathrm{End}\, V) \times \mathbb{P}(\mathrm{End}\, V)$, where V is the irreducible component of $\mathfrak{sl}_\mathfrak{n} \otimes \mathbb{C}^n$, so $\rho : \mathrm{SL}_n \to V$ is a faithful representation. Then the embedding $Z = \overline{\psi(T)}$ is compact.*

Proof. We show this by appealing to the geometry of the root lattice Λ_w for the representation of SL_n on V. Note that $\dim \Lambda_R = n-1$, and $\mathrm{rank}\, T = n-1$, so the one-parameter subgroups are of the form $\Lambda^{\mathbf{v}}(t)$, where $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1})$.

We aim to choose $\alpha_1, \alpha_2 \in \Lambda_R$ so that $\langle \alpha_i, \mathbf{v}\rangle$, $\langle \alpha_j, \mathbf{v}\rangle < \langle \alpha_l, \mathbf{v}\rangle$, for all other roots $\alpha_l \in \Lambda_R$, where $\langle\ ,\ \rangle$ denotes the standard inner product. When we take the irreducible component of the $\mathfrak{sl}_\mathfrak{n} \otimes \mathbb{C}^n$, we can always choose such α_i, α_j, namely, those that are also roots of the standard representation. Then, w can be constructed so that $w = (1, \ldots, 1, a, \ldots, b, 1, \ldots, 1)$, with $gcd(a-1, b-1) = 1$ by Lemma **??**, so that for $\rho(T) \subset \mathbb{P}_w(\mathrm{End}\, V)$, the coordinates $t^{\langle \alpha_i, \mathbf{v}\rangle}, t^{\langle \alpha_j, \mathbf{v}\rangle}$ correspond to the weights a and b. Then, each limit point of $\lambda^{\mathbf{v}}(t)$ exists, so the fan of T is complete. Thus, by Theorem A.3, Z is compact.

□

Appendix A. Toric varieties

Here we state some useful facts about toric varieties. For a more complete treatment, see [2].

We use the following facts about toric varieties to examine the closure of our compactification of SL_n. For a toric variety X, we can study the orbit closures of the action of the torus through computing a fan in its character lattice. We begin by introducing some of the fundamental constructions of toric varieties.

Let M, N be $\mathbb{Z}$-lattices with associated vector spaces $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, with finite $S \subseteq N$.

(1) A *convex polyedral cone* $\sigma = \mathrm{Cone(S)} = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \right\} \subseteq N_\mathbb{R}$. A cone σ called *rational* if S $\subseteq$ N is finite.
(2) The *dual cone* $\sigma^v = \{m \in M_\mathbb{R} \mid \langle m, u\rangle \geq 0\ \forall u \in \sigma\} \subseteq M_\mathbb{R}$.
(3) The *hyperplane* associated to a vector $m \in M_\mathbb{R}$ is defined as $H_m = \{u \in N_\mathbb{R} \mid \langle m, u \rangle = 0\} \subseteq N_\mathbb{R}$.
(4) The *face* of a cone σ is $\tau = H_m \cap \sigma$, for some m $\in \sigma^v$. A cone is called *strongly convex* if the origin is a face.

Definition A.1. A *fan* Σ in $N_\mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_\mathbb{R}$ such that:

(1) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
(2) For all $\sigma \in \Sigma$, each face of σ is also in Σ.
(3) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

The generators of the cones of the fan consist of the limit points of the one-parameter subgroups of the torus. After computing these limit points, we can make use of the following results.

Definition A.2. Consider a fan $\Sigma \subseteq N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, where N is a lattice.

(1) Σ is *smooth* if every cone σ in Σ is smooth, or its minimal generators form part of a $\mathbb{Z}$-basis for N.
(2) Σ is *simplicial* if every cone σ in Σ is simplicial, or its minimal generators are linearly independent over $\mathbb{R}$.

(3) Σ is *complete* if its support $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma$ is all of $N_{\mathbb{R}}$.

Theorem A.3. *Let X_Σ be the toric variety defined by a fan $\Sigma \subseteq N_{\mathbb{R}}$. Then:*

(1) X_Σ is a smooth variety if and only if the fan Σ is smooth.
(2) X_Σ is an orbifold if and only if the fan Σ is simplicial.
(3) X_Σ is compact in the classical topology if and only if Σ is complete.

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