Honors Complex Variables
Lecture Notes

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1 Applications of Laurent series: classification of singularities

Suppose f has a (potential) isolated singularity at z_0. In other words, f is analytic in a deleted neighborhood of z_0, $D(Z_0; r)/\{z_0\}$, for some $r > 0$. We can write a Laurent expansion in $0 < |z - z_0| < r$:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$
$$\text{with } a_k = \frac{1}{2\pi i} \int_{\partial D(z_0;R)} \frac{f(z)}{(z - z_0)^k} dz$$

Let us see how we can use this to classify singularities. We have 3 cases:

1. No negative powers of $(z - z_0)$ appear in the series expansion: $a_k = 0\ \forall k < 0$. Then we call z_0 a **removable singularity**.
2. There is an $n > 0$ such that $a_{-n} \neq 0$ and $a_k = 0\ \forall k < -n$. Then we call z_0 a **pole** of $f(z)$.
3. There are infinitely many negative powers of $(z - z_0)$ in the expansion. Then we call z_0 an **essential singularity**.

1.1 Removable singularities

Let us first study removable singularities:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ for } 0 < |z - z_0| < r$$

Clearly f is also analytic at z_0 if we simply define $f(z_0) = a_0$. In this case we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ for } 0 \leq |z - z_0| < r.$$

Let us examine the example of $f(z) = \frac{\sin z}{z}$ for $0 < |z| < \infty$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$
$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Note that the series expansion for $f(z)$ can be defined for all z and thus 0 is a removable singularity of f.

Remark. If f has a removable singularity at z_0, then f is bounded around z_0. In other words, $\exists M > 0$ with $|f(z)| < M\ \forall 0 < |z - z_0| < r$. The converse is also true.

Theorem 1 (Riemann's theorem of removable singularities)**.** *Let z_0 be a potential isolated singularity of f. If*

$$\lim_{z \to z_0} f(z)(z - z_0) = 0$$

then f has a removable singularity at z_0.

Remark. Note also that if $|f(z)| \leq 1/|z - z_0|^{1-\varepsilon}$ for some positive ε, then f has a removable singularity at z_0. Consequently, this is a much stronger bound than that provided by the previous remark.

Proof. We want to show that $a_k = 0 \ \forall k < 0$ in the Laurent expansion. Take $k < 0$. Then, $k + 1 \leq 0$. As $\lim_{z \to z_0} f(z)(z - z_0) = 0$, we can find $\delta > 0$ such that $|f(z)(z - z_0)| < \varepsilon \ \ \forall z$ in $0 < |z - z_0| < \delta < r$. From the formula for the coefficients and the M-L formula, we now have that

$$\begin{aligned}|a_k| &= \Big|\frac{1}{2\pi i} \int_{\partial D(z_0;\delta)} \frac{f(z)(z - z_0)}{(z - z_0)^{k+2}} dz\Big| \\ &\leq \frac{1}{2\pi} \frac{\varepsilon}{\delta^{k+2}} \cdot \text{length}\, \partial D(z_0; \delta) = \varepsilon \delta^{-(k+1)} \\ &\leq \varepsilon \delta^0 = \varepsilon\end{aligned}$$

Thus we have that $|a_k| < \varepsilon$, $\forall \varepsilon > 0$, which yields that the coefficients must be identically zero for $k < 0$. □

1.2 Poles

Let us now examine poles. From the series expansion, there is an $n > 0$ such that $a_{-n} \neq 0$ but $a_k = 0$ for all $k < -n$. In this case, z_0 is called a pole of order n of f.

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$$

$$= \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

Definition 1. *The sum of negative powers*

$$P(z) = \sum_{k=-n}^{-1} a_k (z - z_0)^k$$

is called the ***principal part*** *of f at the pole z_0.*

Remark. Note that for our series, if we take $f(z) - P(z)$ we obtain the analytic function $\sum_{k=0}^{n} a_k (z - z_0)^k$. Incidentally, if $n = 1$, we call the pole **simple** and if $n = 2$, we call the pole **double**.

Lemma 1. *Suppose f is analytic in a region Ω and has a zero at a point $z_0 \in \Omega$ with $f \not\equiv 0$ in Ω. Then there exists in a neighborhood $U \in \Omega$ of z_0, a non-vanishing analytic function g on U and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$. Note that the number n is called the* ***order of the zero*** *z_0 of the function f (also called the multiplicty).*

Proof. In a small neighborhood $D(z_0; R)$ of z_0, f cannot be identically 0 by the uniqueness theorem. In this neighborhood let us expand:

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} a_k (z - z_0)^k \\ f(z_0) = 0 &\implies a_0 = 0 \end{aligned}$$

Therefore there must be an $n \geq 1$ such that $a_n \neq 0$, otherwise f would vanish identically. Thus we write:

$$\begin{aligned} f(z) &= a_n (z - z_0)^n + a_{n+1} (z - z_0)^{n+1} + \dots \\ &= (z - z_0)^n \left[a_n + a_{n+1} (z - z_0) + \dots \right] \end{aligned}$$

Let $g(z) = a_n + a_{n+1}(z - z_0) + \dots$. Clearly it is analytic in D. Additionally, since $\lim_{z \to z_0} g(z) = a_n \neq 0$, we have $|g(z)| \geq |a_n|/2$ whenever $|z - z_0| < r$. Thus let us take $U = D$.

Let us now show the uniqueness of n. Suppose $\exists n < m$ such that $f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$ where g and h are analytic in a neighborhood V of z_0, with $g, h \neq 0$ in V. When $z \neq z_0$ with $z \in V$, $g(z) = (z - z_0)^{m-n} h(z)$. Thus, $g(z_0) = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)^{m-n} h(z) \equiv 0$. This is a contradiction, and thus n must be unique. □

Theorem 2 (Characterization of a pole)**.** *Let z_0 be an isolated singularity of f. Then z_0 is a pole of $f(z)$ of order n:*

1. *iff $f(z) = g(z)/(z - z_0)^n$ where g is analytic and non-zero at z_0.*
2. *iff*
$$h(z) = \begin{cases} 1/f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$
is analytic at z_0 and has a zero of order n at z_0.
3. *iff $|f(z)| \to \infty$ when $z \to z_0$*

Proof. Let us attack the first claim. Suppose f has a pole of order n at z_0. Then, using the expansion for poles above,

$$f(z) = \frac{a_{-n} + a_{-n+1}(z - z_0) + \dots + a_{-1}(z - z_0)^{n-1} + a_0 (z - z_0)^n + \dots}{(z - z_0)^n} \equiv \frac{g(z)}{(z - z_0)^n}$$

It is clear that g is analtyic at z_0 and does not vanish, and so we reach the result. Let us now prove the converse. If g is analytic at z_0, and does not vanish, let us expand it into a power series about z_0:

$$g(z) = b_0 + b_1(z - z_0) + \dots \text{ with } g(z_0) = b_0 \neq 0$$

Hence,

$$\begin{aligned} f(z) = \frac{g(z)}{(z-z_0)^n} &= \frac{b_0 + b_1(z-z_0) + \dots}{(z-z_0)^n} \\ &= \frac{b_0}{(z-z_0)^n} + \frac{b_1}{(z-z_0)^{n-1}} + \dots \end{aligned}$$

Thus f has a pole of order n at z_0, and the first point is proved.

Let us now prove the second claim. Suppose f has a pole of order n at z_0. Then by the first point, we can represent

$$f(z) = \frac{g(z)}{(z-z_0)^n}$$

where g is analytic and non-zero at z_0. Take the function $M(z) = 1/g(z)$, which is also analytic at z_0. Then,

$$\frac{1}{f(z)} = \frac{(z-z_0)^n}{g(z)} = (z - z_0)M(z)$$

is analytic at z_0 and has a zero of order n at z_0. Conversely, if $1/f(z)$ has a zero of order n at z_0, by the above theorem,

$$\frac{1}{f(z)} = (z-z_0)^n K(z)$$

for some K analytic at z_0 and $K(z_0) \neq 0$. Hence,

$$f(z) = \frac{1/K(z)}{(z-z_0)^n} \equiv \frac{g(z)}{(z-z_0)^n}$$

Here $g(z) = 1/K(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Then, by the first point f has a pole of order n at z_0.

Let us now prove the third point. If f has a pole of order n at z_0 then by the first point, $f(z) = g(z)/(z-z_0)^n$ for some g analytic and nonvanishing at z_0. Thus,

$$|f(z)| = \frac{|g(z)|}{|z-z_0|^n} \to \infty.$$

Conversely, suppose that $f(z) \to \infty$ as $z \to z_0$. Obviously, $f(z) \neq 0$ for z near z_0. Thus we can define $h(z) = 1/f(z)$, which is analytic around z_0 and $h(z) \to 0$ as $z \to z_0$. Hence, by the Riemann's theorem proved above, $h(z)$ must have a removable singularity at z_0 and $h(z_0) = 0$. Since $h \neq 0$ in the neighborhood of z_0, by the above theorem on zeroes, we can write $h(z) = (z - z_0)^n g(z)$ for some $n > 0$ and g analytic in a neighborhood V of z_0 and $g \neq 0$ in V. Now,

$$f(z) = \frac{1}{h(z)} = \frac{1/g(z)}{(z - z_0)^n}$$

By the first point, f has a pole of order n at z_0, as $1/g(z)$ is analytic and non-zero in V. $\square$

We shall cover the topic of essential singularities next lecture.

1.3 Essential singularities

Remark. Note that the function

$$f(z) = e^{1/z}$$

has an essential singularity at 0.

Theorem 3 (Casorati-Weierstrass)**.** *If f is analytic in $A = D(z_0; r)\setminus\{z_0\}$ and has an essential singularity at z_0 then $f(A)$ is dense in $\mathbb{C}$. I.e. $f(A)$ intersects every disc in $\mathbb{C}$.*

Proof. Let us proceed by contradiction. Suppose that we can find a disc $D(w; \delta)$ such that $|f(z) - w| \geq \delta$ for all $0 < |z - z_0| < r$. Consider the function $g(z) = 1/(f(z)-w)$. g is analytic in A and $|g| \leq 1/\delta$ in A. Hence, by Riemann's theorem, z_0 is a removable singularity of g. If $g(z_0) \neq 0$ then

$$f(z) = \frac{1}{g(z)} + w$$

which implies that f is analytic at z_0, which is a contradiction. If $g(z_0)$ *is* zero, things are a little more complicated. We can write

$$g(z) = (z - z_0)^n h(z)$$

where h is analytic and non-zero at z_0. Note, then

$$\begin{aligned}(z - z_0)^n h(z) &= \frac{1}{f(z) - w}\\ f(z) - w &= \frac{1/h(z)}{(z - z_0)^n},\end{aligned}$$

which is a pole. This is a contradiction, and we are done. ☐

Theorem 4 (Picard's big theorem)**.** *In any neighborhood of an essential singularity, the function takes all values in the complex plane infinitely often, with possibly one exception.*

2 Meromorphic functions

Definition 2. *We call the function f **meromorhphic** in a domain D if f is analytic in D except at isolated poles.*

Remark. Quotients of meromorhphic functions are meromorhphic functions provided that the denominators are not identically zero.

Example. $1/\sin z$ is analytic in $\mathbb{C}$ except at isolated poles $z = k\pi$ with integral k. Consequently, $1/\sin z$ is meromorhphic on $\mathbb{C}$.

Example. Let $R(z)$ be a rational function

$$R(z) = \frac{P(z)}{Q(z)} = \frac{a\prod_{j=1}^{n}(z-z_j)^{m_j}}{b\prod_{k=1}^{p}(z-\omega_k)^{n_k}}$$

where P, Q are polynomials with no common zeros. R is meromorhpic in $\mathbb{C}$. It has a pole order n_k at ω_k and zeros of order m_j at z_j.

2.1 Singularities at infinity

Let f be analytic in $\{z : |z| > M\}$. This is a little difficult to work with, so consider instead the function $F(z) = f(1/z)$. This function is clearly analytic in the annulus $A = \{z : 0 < |z| < 1/M\}$. If F has a removable singularity or a pole or an essential singularity at 0, then we say that $f(z)$ has said **singularity at infinity**. Why have we done this? We see no singularities originally and are trying to make trouble for ourselves! Well, many functions in mathematics have a finite number of singularities, and it sometimes easier to concentrate on those, but nothing else, as those points can contain a lot of information about the function. Indeed, sometimes you can use singularities to solve problems that don't involve singularities. Here is an example:

Theorem 5. *Suppose f is entire, and maps any unbounded sequence to an unbounded sequence. Then f is a polynomial.*

Proof. There are three cases. The first case is that f has a removable singularity at infinity. If this is the case, then f is bounded at infinity. In other words, there exists a number $K > 0$ such that $|f(z)| \leq K$ for all $|z| \geq R$. By the maximum principle, $|f(z)| \leq K$ for all z. But if f is entire and bounded, it must be a constant. A constant function cannot map an unbounded sequence to and unbounded sequence, and thus f cannot have a removable singularity.

It could be that f has an essential singularity at infinity. Take $\omega \in \mathbb{C}$. By Carosati-Weierstrass, there is a sequence $\{z_k\}$ with $|z_k| \to \infty$ with $f(z_k) \in D$ for all k. This is not possible because the hypothesis asserts that $\{f(z_k)\}$ is unbounded.

The last case is that f has a pole at infinity. Then, $F(z) = f(1/z)$ has a pole at 0. The Laurent expansion at 0 yields

$$F(z) = f(1/z) = \frac{a_{-n}}{z^n} + \frac{a_{-n+1}}{z^{n-1}} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$$
$$f(z) = a_{-n}z^n + a_{-n+1}z^{n-1} + \cdots + a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

Note however, that as f is entire, the terms with powers of z^{-1} must vanish, and we are left with f being a polynomial. $\square$

Definition 3. *The **extended complex plane** is the union of the complex plane and the point ∞.*

Definition 4. *A function f is* ***meromorphic on the extended complex plane*** *if f is meromorphic and g has a removable singularity or a pole at ∞.*

Theorem 6. *If f is meromorphic in the extended complex plane, then f must be a rational function.*

Proof. The proof is available in the book. However, the basic idea is just to kill off all of f's zeros and poles. $\square$

3 Residues

Definition 5. *When z_0 is a pole of f and*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

in $A = \{z : 0 < |z - z_0| < r\}$, the coefficient a_{-1} is called the ***residue*** *of f at the pole z_0. Many notations are common, but we will denote residues by*

$$a_{-1} = \operatorname{Res}_{z_0} f(z).$$

Remark. In the principal part of f at z_0, $P(z) = \sum_{-n}^{-1} a_k (z - z_0)^k$, all the terms will have antiderivatives except for the a_{-1} term.

3.1 Computation of residues

One method of computing a residue is straight from the Laurent expansion,

$$a_k = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)dz}{(z - z_0)^{k+1}}$$

which yields

$$\operatorname{Res}_{z_0} f(z) = \frac{1}{2\pi i} \int_{|z-z_0|=R} f(z)dz.$$

If however, we know that $f(z)$ has a simple pole at z_0 (i.e. a pole of order 1), we can write

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

and thus the residue is found simply by computing the limit

$$\operatorname{Res}_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

Functions with poles of higher order, n, are a little bit trickier,

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

$$(z - z_0)^n f(z) = a_{-n} + \dots a_{-1}(z - z_0)(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \dots$$

To isolate a_{-1} we now must take $n-1$ derivatives to get rid of the negative n terms, and *then* take a limit to get rid of the positive n terms:

$$\operatorname{Res}_{z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left(\frac{d}{dz}\right)^{n-1} \left((z-z_0)^n f(z)\right).$$

Let us do a few examples before we state the main theorem concerning residues.

Example.

$$\operatorname{Res}_{-i} \frac{1}{1+z^2} = \lim_{z \to -i} (z+i) \frac{1}{(z+i)(z-i)} = -\frac{1}{2i}$$

as the singularity is obviously a pole or order 1 (upon factoring the denominator).

Example.

$$\begin{aligned}\operatorname{Res}_{-i} \frac{1}{(1+z^2)^2} &= \frac{1}{(2-1)!} \lim_{z \to -i} \frac{d}{dz} \frac{(z+i)^2}{(1+z^2)^2} = \lim_{z \to -i} \frac{d}{dz} \frac{1}{(z-i)^2} \\ &= \frac{-2}{(-2i)^3} = -\frac{1}{4i}\end{aligned}$$

Theorem 7 (Residue theorem)**.** *Suppose f is analytic in an open set Ω containing a piecewise smooth closed curve γ and its interior D except for the poles at $z_1 \ldots z_n$ in D. Then,*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k} f(z)$$

Proof. The proof is quite simple; first we let D_ε be the domain D with the usual small keyhole shapes that connect the boundary of D to small discs, D_j, about the poles. We take the limit where the width of the keyholes goes to zero and we are left only left with circular paths about the poles in the negative sense, and the path D in the positive sense. By Cauchy's theorem, since the function is analytic when the poles are excluded,

$$\begin{aligned}0 &= \int_{\gamma - \cup_{j=1}^{N} \partial D_j} f(z)dz \\ &= \int_\gamma f(z)dz - \sum_{j=1}^{N} \int_{\partial D_j} f(z)dz.\end{aligned}$$

But the second term can be expressed via the residue, as the other terms in the Laurent expansion of f vanish under integration, as remarked earlier, so

$$0 = \int_\gamma f(z)\, dz - 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{z_j} f(z),$$

and we are done. $\square$