Lie Groups PSET 2

Nilay Kumar

Last updated: October 7, 2013

Proposition 1. *Kirillov 2.15*

Proof. Let us first show that $\operatorname{End}_{\mathbb{H}} \mathbb{H}^n$ is naturally identified with the algebra of $n \times n$ quaternionic matrices. Let $\{e_\alpha\}$ be an orthonormal basis for $\mathbb{H}^n$ over $\mathbb{H}$ (the orthonormality condition will be useful later). Then we may write any $\vec{h} = \sum_\alpha e_\alpha h_\alpha$, as we are considering $\mathbb{H}^n$ as a right $\mathbb{H}$-module. Applying an endomorphism A, we find that $A(\vec{h}) = A\left(\sum_\alpha e_\alpha h_\alpha\right) = \sum_\alpha A(e_\alpha h_\alpha) = \sum_\alpha A(e_\alpha) h_\alpha$. Denoting the β component of $A(e_\alpha)$ by the quaternion $A_{\beta\alpha}$, we can rewrite

$$A(\vec{h}) = \sum_{\alpha,\beta} e_\beta A_{\beta\alpha} h_\alpha.$$

From this, it is clear that the endomorphism is characterized precisely by this $n \times n$ matrix $A_{\beta\alpha}$ of quaternions.

Now let us define an $\mathbb{H}$-valued form on $\mathbb{H}^n$ given by

$$(\vec{h}, \vec{h}') = \sum_i \bar{h}_i h_i'$$

and let $U(n, \mathbb{H})$ be the group of unitary quaternionic transformations:

$$U(n, \mathbb{H}) = \left\{ A \in \operatorname{End}_{\mathbb{H}} \mathbb{H}^n \middle| (A\vec{h}, A\vec{h}') = (\vec{h}, \vec{h}') \right\}.$$

It's clear that $U(n, \mathbb{H})$ is indeed a group - composing two transformations yields another such transformation, and the identity is contained in the group. The inverses are given by just the matrix inverses (that clearly exist because no $A \in U(n, \mathbb{H})$ has determinant 0):

$$(\vec{h}, \vec{h}') = (AA^{-1}\vec{h}, AA^{-1}\vec{h}') = (A^{-1}\vec{h}, A^{-1}\vec{h}').$$

Using the identification of endomorphisms with matrices that we established in the previous paragraph, let us find a matrix characterization for elements of $U(n, \mathbb{H})$:

$$\begin{aligned}
(A\vec{h}, A\vec{h}') &= \left(\sum_{\alpha,\beta} e_\beta A_{\beta\alpha} h_\alpha, \sum_{\gamma,\delta} e_\gamma A_{\gamma\delta} h_\delta' \right) \\
&= \sum_{\beta,\gamma} (e_\beta, e_\gamma) \left(\overline{\sum_\alpha A_{\beta\alpha} h_\alpha} \right) \left(\sum_\delta A_{\gamma\delta} h_\delta' \right) \\
&= \sum_{\alpha,\beta,\delta} \overline{A_{\beta\alpha} h_\alpha} A_{\beta\delta} h_\delta' = \sum_{\alpha,\beta,\delta} \overline{h_\alpha} \cdot \overline{A_{\beta\alpha}} A_{\beta\delta} h_\delta'.
\end{aligned}$$

But this is precisely $(\vec{h}, \vec{h}')$ if and only if the middle terms sum to the identity, i.e. $A^*A = 1$.

Define now a map $\phi : \mathbb{C}^{2n} \to \mathbb{H}^n$ given by $(z_1, \ldots, z_{2n}) \mapsto (z_1 + jz_{n+1}, \ldots, z_n + jz_{2n})$. If we treat $\mathbb{H}^n$ as a complex vector space via the scalar multiplication $z(h_1, \ldots, h_n) = (h_1 z, \ldots, h_n z)$, ϕ is in fact an isomorphism of complex vector spaces. To see this, it suffices to show that ϕ is injective, but this is obvious because $z_i + jz_{n+1} = a_i + ib_i + ja_{n+i} - kb_{n+i} = 0$ implies that the $a_i = a_{n+i} = b_i = b_{n+i} = 0$, and hence that $z_i = z_{n+i} = 0$, i.e. $\ker\phi = \{\vec{0}\}$. Next let us show that this isomorphism identifies $\operatorname{End}_{\mathbb{H}} \mathbb{H}^n$ with $\{A \in \operatorname{End}_{\mathbb{C}} \mathbb{C}^{2n} | \bar{A} = J^{-1}AJ\}$ where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

First note that the map $\vec{h} \mapsto \vec{h}j$ is identified with $\vec{z} \mapsto J\vec{\bar{z}}$, because quaternionic multiplcation by j is equivalent to multiplying $(z_i + jz_{n+i})j = z_i j + jz_{n+i}j = -\bar{z}_{n+i} + j\bar{z}_i$. finish this

Under the identification $\mathbb{C}^{2n} \cong \mathbb{H}^n$ above, the quaternionic form simplifies as follows. Let $h_i = z_i + jz_{n+i}$ and $h'_i = z'_i + jz'_{n+1}$. Then, by definition,

$$\begin{aligned}
(\vec{h}, \vec{h}') &= \sum_l (\bar{z}_l + \overline{jz_{n+l}})(z'_l + jz'_{n+l}) \\
&= \sum_l \left(\bar{z}_l z'_l + \bar{z}_{n+l} z'_{n+l} - \bar{z}_{n+l} j z'_l + \bar{z}_l j z'_{n+l}\right).
\end{aligned}$$

Anti-commuting the j and factoring it out of the last two terms, we see that the first term is the usual Hermitian inner product and the last terms are the standard bilinear skew-symmetric form in $\mathbb{C}^{2n}$ (multiplied by j). For the whole form to be preserved, each form must be preserved, and hence we see that $Sp(n) = Sp(n, \mathbb{C}) \cap SU(2n)$.

□

Proposition 2. *Kirillov 3.16*

Proof. We wish to show that $\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$. Note that (from Kirillov) $\mathfrak{sp}(n, \mathbb{C})$ is the set of $2n \times 2n$ complex matrices x such that $x + J^{-1}x^t J = 0$. As $Sp(n)$ is the intersection of $Sp(n, \mathbb{C})$ and $SU(2n)$, it's clear that $\mathfrak{sp}(n)$ is the set of matrices that satsify this condition in addition to the condition that $x^\dagger = -x$. If we now complexify and consider $\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n)$, the first condition $x + J^{-1}x^t J = 0$ still holds (by linearity) but the second no longer holds, as $(ix)^\dagger = -ix^\dagger = ix$ instead of $-ix$. This is precisely what we wanted to show. □

Proposition 3. *Kirillov 3.1*

Proof. Let $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Suppose $x \in \mathfrak{sl}(2, \mathbb{R})$ such that $\exp(x) = X$. It's clear that the eigenvalues of x must exponentiate to the eigenvalues of X, as eigenvalues of powers are simply powers of eigenvalues. Hence we see that if $\lambda_1, \lambda_2 \in \mathbb{C}$ are the eigenvalues of x then $e^{\lambda_1} = e^{\lambda_2} = -1$. This yields

$$\begin{aligned}
\lambda_1 &= (2n + 1)\pi i \\
\lambda_2 &= (2m + 1)\pi i
\end{aligned}$$

for some $m, n \in \mathbb{Z}$. The condition that $x \in \mathfrak{sl}(2, \mathbb{R})$ requires that $\lambda_1 + \lambda_2 = 0$, i.e. that $m = -(n+1)$. Hence the λ_1, λ_2 must be distinct, i.e. diagonalizable over the complex numbers. In other words, we have that $P^{-1}xP = \lambda$ where λ is diagonal with λ_1, λ_2. Exponentiating, we find that

$$P^{-1}\exp(x)P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moving the P's to the other side, we find that $\exp(x) = -I$, which is clearly not equal to X. Hence we obtain a contradiction - there can exist no such x. $\square$