Algebraic Topology I: PSET 4

Nilay Kumar*

Last updated: October 16, 2014

Problem 1

Let $X = S^2$ and $Y = S^3 \times \mathbb{CP}^\infty$. Recall that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$. Moreover, $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, and hence $\pi_n(Y) = \pi_n(S^3)$ for $n \geq 3$. Hence $\pi_n(X) \cong \pi_n(Y)$ for $n \geq 3$. Clearly $\pi_2(X) \cong \pi_2(Y) \cong \mathbb{Z}$ and $\pi_1(X) \cong \pi_1(Y) \cong 1$. We claim that even though X and Y have isomorphic homotopy groups, they are not homotopy equivalent. Indeed, suppose $f : X \to Y$ is a homotopy equivalence; then there exists some $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_{S^2}$. Note, however, that f can be made cellular up to homotopy $f \simeq \tilde{f}$, and hence $g \circ f \simeq g \circ \tilde{f} \simeq \mathrm{Id}_{S^2}$, where now $\mathrm{im}\, \tilde{f} \cap S^3 = *$. But this means that $\tilde{f}$ is (equal to a) map into only $\mathbb{CP}^\infty$; injectivity of $\tilde{f}_*$ for π_3 yields an injective map $\mathbb{Z} \to 0$, which is a contradiction.

Problem 2

Suppose X and Y are two weakly homotopy equivalent spaces. Recall that for any space X, there exists a CW complex Z such that $Z \to X$ is weak homotopy equivalence. We claim that moreover the composition $Z \to X \to Y$ is a weak homotopy equivalence. This is clear because $X \to Y$ induces an isomorphism on homotopy groups, as does $Z \to X$, and hence $Z \to X \to Y$ induces isomorphisms on homotopy groups as well. Thus $Z \to X \to Y$ must be a weak homotopy equivalence as well.

Problem 3

Let X be a topological space and $\alpha, \beta \in \pi_1(X)$. We claim that the Whitehead product $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in \pi_1(X)$, i.e. that the product yields the

*Collaborated with Matei Ionita, Sander Mack-Crane, and Kendric Schefers.

commutator of the loops. To see this, it suffices to examine carefully the map $S^1 \to S^1 \wedge S^1$ defining the product. We view

$$\begin{aligned} S^1 &= \partial(D^2) = \partial(D^1 \times D^1) \\ &= (\partial D^1 \times D^1) \cup_{S^0 \times S^0} (D^1 \times \partial D^1) \\ &= (S^0 \times D^1) \cup_{S^0 \times S^0} (D^1 \times S^0), \end{aligned}$$

and then map

$$(D^1 \times S^0) \twoheadrightarrow D^1 \twoheadrightarrow S^1 \overset{i_1}{\hookrightarrow} S^1 \vee S^1$$
$$(S^0 \times D^1) \twoheadrightarrow D^1 \twoheadrightarrow S^1 \overset{i_2}{\hookrightarrow} S^1 \vee S^1.$$

Graphically, we can view S^1 as a square; this union decomposes the square into two pairs of parallel lines and glues them together at the four corners. We note that the map $S^1 \to S^1 \vee S^1$ maps the generating loop of S^1 to $aba^{-1}b^{-1}$ by way of this decomposition and these maps, where a and b are the generators of the loops in $S^1 \vee S^1$. Hence the Whitehead product of α and β yields a loop performing $\alpha\beta\alpha^{-1}\beta^{-1}$, as desired.

Problem 4

Let X be a topological space and $\alpha \in \pi_n(X), \beta \in \pi_k(X)$, with the Whitehead product $[\alpha, \beta] \in \pi_{n+k-1}(X)$. Let us see what happens when we instead take $[\beta, \alpha]$. Since the product is given by first treating

$$S^{n+k-1} = \partial(D^n \times D^k) = (S^{n-1} \times D^k) \cup_{S^{n-1} \times S^{k-1}} (D^n \times S^{k-1}),$$

and mapping this into $S^n \vee S^k$ before mapping into the space via α, β, we find that swapping α and β is equivalent to swapping the order of the D^n and D^k in the expression $\partial(D^n \times D^k)$ above. This swap, we note, moves the first n coordinates to the right, which i.e. a composition of nk transpositions. Viewing D^n as the product of intervals I^n, each of these transpositions swaps two adjacent copies of I, and so restricting to the case of $\partial D^2 = S^1$, we find that the swap interchanges the two axes hence inverting the orientation on S^1. Thus each transposition yields an factor of -1 after composing with α or β and hence $[\beta, \alpha] = (-1)^{nk}[\alpha, \beta]$.

Problem 5

If we think about the sphere S^{n+k-1} as a boundary of the unit disk $D^{n+k} \subset \mathbb{R}^{n+k}$, we write

$$U = \{(x_1, \dots, x_{n+k}) \in S^{n+k-1} \mid x_1^2 + \cdots + x_n^2 \leq 1/2\}$$
$$V = \{(x_1, \dots, x_{n+k}) \in S^{n+k-1} \mid x_{n+1}^2 + \cdots + x_{n+k}^2 \leq 1/2\}.$$

It is clear that $U \cong D^n \times S^{k-1}$: we force the square of the sums of the first n coordinates to be less than or equal to $1/2$, which is D^n with a radius of $1/2$, which in turn forces the sum of the squares of the rest of the coordinates to be $1 - x_1^2 - \cdots - x_n^2$, a sphere of radius square root of said quantity. A similar argument works for $V \cong S^{n-1} \times D^k$. Now, it is clear that $U \cup V$ covers S^{n+k-1}; it suffices to show that $U \cap V = S^{n-1} \times S^{k-1}$. But the overlap is precisely when $x_1^2 + \cdots + x_n^2 = 1/2$ and $x_{n+1}^2 + \cdots + x_{n+k}^2 = 1/2$ (as the coordinates must sum to 1), which gives exactly the product of two spheres.

Problem 6

Recall that the Whitehead product $w : S^{n+k-1} \to S^n \vee S^k$ can be viewed as the attaching map for the cell e^{n+k} in the product $S^n \times S^k$. If we now suspend the product to obtain $\Sigma(S^n \times S^k)$, the attaching map for the e^{n+k+1} in $\Sigma(S^n \times S^k)$ is Σw. By Botvinnik's Claim 10.2, however, we know that this map is nullhomotopic, and hence, up to homotopy, the e^{n+k+1} cell's boundary is attached to the basepoint in $\Sigma(S^n \times S^k)$. But then it is clear that $\Sigma(S^n \times S^k)$ is (homotopic to) $\Sigma(S^n \vee S^k)$ together with a $(n+k+1)$-cell attached to the basepoint. But this is simply $S^{n+1} \vee S^{k+1} \vee S^{n+k+1}$, as desired.