Differentiable Manifolds Problem Set 2

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Problem 1

Let X be a Hausdorff and second countable topological space. We wish to show that any subspace Y of X is itself Hausdorff and second countable. Take any $p, q \in Y \subset X$. As X is Hausdorff, there exist disjoint open sets $U_p, U_q \in X$ containing p and q respectively. By definition of the subspace topology, the open sets in Y are of the form $U \cap Y$, where U are the open sets in X. Consequently, $U_p \cap Y$ and $U_q \cap Y$ are disjoint open sets in the topology on Y; as p, q were arbitrary, Y is Hausdorff.

Let us now show that Y is second countable. By second countability, we know that there exists a countable basis $\mathcal{B}_X$ of X. Take $\mathcal{B}_Y$ to be the collection of open sets $B_Y = Y \cap B_X$, where $B_X \in \mathcal{B}_X$. Note that for any two basis elements $C_X, D_X \in \mathcal{B}_X$, there exists a basis element E_X contained in $C_X \cap D_X$ by the basis criterion. It follows, then, that for the two basis elements $C_Y = Y \cap C_X$ and $D_Y = Y \cap D_X$ in $\mathcal{B}_Y$, the basis element $Y \cap E_X \in \mathcal{B}_Y$ is contained in $C_Y \cap D_Y$. It should be clear, then, that the basis $\mathcal{B}_Y$ generates the topology on Y, and since $\mathcal{B}_Y$ is necessarily smaller than $\mathcal{B}_X$, Y is second countable.

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6