Lie Groups PSET 7

Nilay Kumar

Last updated: February 2, 2014

Problem 1 (Kirillov 8.7)

Problem 2 (Kirillov 8.9)

Let V_n be the irreducible $(n+1)$-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. We wish to show that

$$V_n \otimes V_m \simeq \bigoplus V_k,$$

where the direct sum is over all $k \in \mathbb{Z}_+$ satisfying the *Clebsh-Gordan condition*:

$$|n-m| \leq k \leq n+m \text{ for } n+m-k \in 2\mathbb{Z}.$$

Recall first that the formal character for V_n is given

$$\mathrm{ch}(V_n) = x^n + x^{n-2} + \cdots + x^{-n+2} + x^{-n}.$$

Using the properties

$$\mathrm{ch}(V_n \otimes V_m) = \mathrm{ch}(V_n)\mathrm{ch}(V_m)$$
$$\mathrm{ch}\left(\bigoplus V_k\right) = \sum \mathrm{ch}(V_k)$$

we find that

$$\mathrm{ch}(V_n)\mathrm{ch}(V_m) = \sum \mathrm{ch}(V_k),$$

with the summation to be solved for. More explicitly, we have

$$\left(x^n + x^{n-2} + \cdots + x^{-n+2} + x^n\right)\left(x^m + x^{m-2} + \cdots + x^{-m+2} + x^{-m}\right)$$