QM for Mathematicians: PSET 9

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Problem 1

We wish to compute commutators of the form $[K_l, P_m]$ where K_l generate boosts and P_m generate translations. We can do this by following the derivation in the notes:

$$[X, Y] = \frac{d}{dt}\left(e^{tX} Y e^{-tX}\right)|_{t=0},$$

where the term in parentheses, using the property of semi-direct products becomes, for a translation $(a, 1) \in \mathcal{P}$ and a Lorentz transformation $(0, \Lambda) \in \mathcal{P}$,

$$\begin{aligned}(0, \Lambda)(a, 1)(0, \Lambda)^{-1} &= (0, \Lambda)(a, 1)(0, \Lambda^{-1}) \\ &= (0, \Lambda)(a + \Lambda^{-1}0, \Lambda^{-1}) \\ &= (0, \Lambda)(a, \Lambda^{-1}) \\ &= (\Lambda a, 1)\end{aligned}$$

If we take a $\Lambda = e^{tK_1}$, we find:

$$\frac{d}{dt}\Lambda a|_{t=0} = \frac{d}{dt}\begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} P_1 \\ P_0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\Lambda = e^{tK_2}$, we find:

$$\frac{d}{dt}\Lambda a|_{t=0} = \frac{d}{dt}\begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} P_2 \\ 0 \\ P_0 \\ 0 \end{pmatrix}.$$

For $\Lambda = e^{tK_3}$, we find:

$$\frac{d}{dt}\Lambda a|_{t=0} = \frac{d}{dt}\begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} P_3 \\ 0 \\ 0 \\ P_0 \end{pmatrix}.$$

Thus we see that in general we have $[K_j, P_0] = P_j$, $[K_j, P_j] = P_0$, and $[K_j, P_k] = 0$ if $j \neq k, k \neq 0$ (otherwise).

Problem 2

For the real scalar quantum field theory, we can write the momentum operator as:

$$\hat{P} = -i\int d^3x\dot{\phi}(x)\left(-i\nabla\right)\phi(x) = -\int d^3x\dot{\phi}(x)\nabla\phi(x)$$

using $\pi = \dot{\phi}$. We can now use the expansion:

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right) \\
\dot{\phi}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} i\omega_k\left(a_k e^{-ikx} - a_k^\dagger e^{ikx}\right) \\
\nabla\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} ik\left(-a_k e^{-ikx} + a_k^\dagger e^{ikx}\right)
\end{aligned}$$

Inserting these above yields:

$$\begin{aligned}
\hat{P} &= -\int d^3x\frac{d^3kd^3k'}{(2\pi)^3\sqrt{4\omega_k\omega_{k'}}}k'\omega_k\left(a_k e^{-ikx} - a_k^\dagger e^{ikx}\right)\left(-a_{k'}e^{-ik'x} + a_{k'}^\dagger e^{ik'x}\right) \\
&= -\int d^3x\frac{d^3kd^3k'}{(2\pi)^3\sqrt{4\omega_k\omega_{k'}}}k'\omega_k\left(-a_k a_{k'}e^{-ix(k+k')} - a_k^\dagger a_{k'}^\dagger e^{ix(k+k')} + a_k^\dagger a_{k'}e^{ix(k-k')} + a_k a_{k'}^\dagger e^{ix(k-k')}\right) \\
&= -\int \frac{d^3kd^3k'}{(2\pi)^3\sqrt{4\omega_k\omega_{k'}}}k'\omega_k\left(-a_k a_{k'} - a_k^\dagger a_{k'}^\dagger\right)\delta(k+k') + k'\omega_k\left(a_k^\dagger a_{k'} + a_k a_{k'}^\dagger\right)\delta(k-k') \\
&= -\int \frac{d^3k}{(2\pi)^3 2\omega_k}k\omega_k\left(-a_k a_{-k} - a_k^\dagger a_{-k}^\dagger\right) + k\omega_k\left(a_k^\dagger a_k + a_k a_k^\dagger\right)
\end{aligned}$$

Note that the first term vanishes as it is odd, and we are left with

$$\begin{aligned}\hat{P} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} k\omega_k \left(a_k^\dagger a_k + a_k a_k^\dagger\right)\\ &= \int \frac{d^3k}{(2\pi)^3} k\ a_k^\dagger a_k\end{aligned}$$

On the other hand, for the complex quantum field theory, we have

$$\begin{aligned}\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k e^{ikx} + b_k^\dagger e^{-ikx}\right)\\ \dot{\phi}(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} i\omega_k \left(-a_k e^{ikx} + b_k^\dagger e^{-ikx}\right)\\ \nabla\phi(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} ik \left(a_k e^{ikx} - b_k^\dagger e^{-ikx}\right)\\ \phi^\dagger(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(a_k^\dagger e^{-ikx} + b_k e^{ikx}\right)\\ \dot{\phi}^\dagger(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} i\omega_k \left(a_k^\dagger e^{-ikx} - b_k e^{ikx}\right)\\ \nabla\phi^\dagger(x) &= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} ik \left(-a_k^\dagger e^{-ikx} + b_k e^{ikx}\right).\end{aligned}$$

The momentum density is given by:

$$\mathcal{T}^{0i} = -\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\nabla\phi - \nabla\phi^\dagger\frac{\partial\mathcal{L}}{\partial\dot{\phi}^\dagger}$$

and so the momentum is found by integrating over space:

$$\hat{P} = -\int d^3x \left(\dot{\phi}^\dagger\nabla\phi + \nabla\phi^\dagger\dot{\phi}\right)$$

Let us look at the first term:

$$\begin{aligned}\hat{P}_1 &= \int \frac{d^3x d^3k d^3l}{(2\pi)^3\sqrt{4\omega_k\omega_l}} \omega_k l \left(a_k^\dagger e^{-ikx} - b_k e^{ikx}\right)\left(a_l e^{ilx} - b_l^\dagger e^{-ilx}\right)\\ &= \int \frac{d^3x d^3k d^3l}{(2\pi)^3\sqrt{4\omega_k\omega_l}} \omega_k l \left(a_k^\dagger a_l e^{-i(k-l)x} + b_k b_l^\dagger e^{i(k-l)x} - a_k^\dagger b_l^\dagger e^{-i(k+l)x} - b_k a_l e^{i(k+l)x}\right)\\ &= \int \frac{d^3k d^3l}{(2\pi)^3\sqrt{4\omega_k\omega_l}} \omega_k l \left(a_k^\dagger a_l \delta(k-l) + b_k b_l^\dagger \delta(k-l) - a_k^\dagger b_l^\dagger \delta(k+l) - b_k a_l \delta(k+l)\right)\\ &= \int \frac{d^3k}{(2\pi)^3 2} k \left(a_k^\dagger a_k + b_k^\dagger b_k\right)\end{aligned}$$

where in the last step we have discarded the odd term and in the remaining term made use of commutation relations. The term $\hat{P}_2$ contributes precisely the same term and thus the total momentum operator is given by:

$$\hat{P} = \int \frac{d^3k}{(2\pi)^3} k \left(a_k^\dagger a_k + b_k^\dagger b_k \right)$$

Problem 3

We have seen that we can construct a free theory of two real scalar fields that has an $SO(2)$ internal symmetry. We can instead consider a free theory of two complex scalar fields with Lagrangian:

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi + \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \psi^\dagger \psi - m^2 \phi^\dagger \phi.$$

Note that we can take some unitary two-by-two matrix U and define

$$\begin{pmatrix} \psi' \\ \phi' \end{pmatrix} = U \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

then we have that

$$\psi'^\dagger \psi' + \phi'^\dagger \phi' = \begin{pmatrix} \psi' & \phi' \end{pmatrix} \begin{pmatrix} \psi' \\ \phi' \end{pmatrix} = \begin{pmatrix} \psi & \phi \end{pmatrix} U U^\dagger \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \psi^\dagger \psi + \phi^\dagger \phi.$$

Hence, since the Lagrangian is dependent on the fields in this way (the derivatives can be ignored as U is constant in spacetime), it clearly has a $U(2)$ symmetry. To find the operators that give the Lie algebra action for this symmetry on the state space, let us first determine what the Lie algebra of $U(2)$ looks like. We have already seen in class that the elements of the algebra are two-by-two skew-Hermitian matrices, i.e. X such that $X^\dagger = -X$. It is fairly clear that any such matrix can be written as

$$X = \begin{pmatrix} ai & be^{-i\gamma} \\ -be^{i\gamma} & di \end{pmatrix}$$

for any real a, b, d, γ. We can split this up to find a basis for $\mathfrak{u}(2)$:

$$\begin{aligned} X &= a \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + b \begin{pmatrix} 0 & e^{-i\gamma} \\ -e^{i\gamma} & 0 \end{pmatrix} \\ &= a \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} + b\cos\gamma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - b\sin\gamma \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Going back to the group by exponentiating, we see that the first and second terms represents multiplying ψ and ϕ respectively by a phase, leaving the Lagrangian invariant. Exponentiating the third term yields a "rotation" of the fields into each other (it's the familiar generator for rotations), while the final term yields something that looks like a rotation but with a $+i$ and a $-i$ in front of the sine terms; a compex rotation if you will.