Lie Groups PSET 6

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Problem 1

Let $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{so}_4\mathbb{C}$ be the subalgebra consisting of matrices of the form $\begin{pmatrix} aJ & \\ & bJ \end{pmatrix}$ for $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ and $a, b \in \mathbb{C}$. We wish to show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We first show that the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$. Recall first that $\mathfrak{so}_4\mathbb{C}$ consists of antisymmetric matrices. The centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is the maximal subalgebra $C(\mathfrak{h})$ that commutes with $\mathfrak{h}$. Let us solve for the centralizer. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $C(\mathfrak{h})$. Then

$$\begin{aligned} 0 &= \begin{pmatrix} aJ & \\ & bJ \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} aJ & \\ & bJ \end{pmatrix} \\ &= \begin{pmatrix} a[J, A] & aJB - bBJ \\ bJC - aCJ & b[J, D] \end{pmatrix}. \end{aligned}$$

But A and D must be multiples of J as the whole matrix must be in $\mathfrak{g}$, and hence the diagonal vanishes. By antisymmetry, it now suffices to solve $aJB - bBJ = 0$ for B. This requires that

$$\begin{pmatrix} ab_3 + bb_2 & ab_4 - bb_1 \\ -ab_1 + bb_4 & -ab_2 - bb_3 \end{pmatrix} = 0,$$

which forces $B = 0$ if this it hold for all $a, b \in \mathbb{C}$. Hence we see that $C(\mathfrak{h}) = \mathfrak{h}$. It now suffices to show that every element of $\mathfrak{h}$ is semisimple, i.e. that we can obtain a root space decomposition. Let $H_j = \begin{pmatrix} & -h_j \\ h_j & \end{pmatrix}$ and $H_{jk} = \begin{pmatrix} H_j & \\ & H_k \end{pmatrix}$. Define $M = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $N = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, $Q = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$. Finally, take $X = \begin{pmatrix} & M \\ -M^\intercal & \end{pmatrix}$, $Y = \begin{pmatrix} & N \\ -N^\intercal & \end{pmatrix}$, $Z = \begin{pmatrix} & P \\ -P^\intercal & \end{pmatrix}$, and $W = \begin{pmatrix} & Q \\ -Q^\intercal & \end{pmatrix}$. Then,

$$\begin{aligned} [H_{jk}, X] &= i(h_j - h_k)X \\ [H_{jk}, Y] &= i(h_j + h_k)Y \\ [H_{jk}, Z] &= i(h_k - h_j)X \\ [H_{jk}, W] &= -i(h_j + h_k)Y, \end{aligned}$$

and we obtain a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_X \oplus \mathfrak{g}_Y \oplus \mathfrak{g}_Z \oplus \mathfrak{g}_W.$$

Problem 2

(a) Recall that for $\alpha, \beta \in R$ we took

$$(\beta, \alpha) = \langle \beta, H_\alpha \rangle = (H_\beta, H_\alpha).$$

We define the coroot $\alpha^\vee$ such that

$$\alpha^\vee = \frac{2H_\alpha}{(\alpha, \alpha)}$$

and hence we find that

$$(\alpha^\vee, \beta^\vee) = \frac{4}{(\alpha, \alpha)(\beta, \beta)}(\alpha, \beta)$$

and that

$$\langle \alpha^\vee, \beta \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that the inner product induced on the coroots is simply a positive multiple of the original inner product and hence it's clear that the coroots span E^*. Furthermore, $n_{\alpha^\vee \beta^\vee}$ is integral:

$$\begin{aligned} n_{\alpha^\vee \beta^\vee} &= \frac{2(\alpha^\vee, \beta^\vee)}{(\beta^\vee, \beta^\vee)} \\ &= \frac{2}{(\beta, \beta)} \frac{4(\alpha, \beta)}{(\alpha, \alpha)} \frac{(\beta, \beta)}{4} \\ &= \frac{2(\alpha, \beta)}{(\alpha, \alpha)}, \end{aligned}$$

which is integral as α, β are roots. Next we check that the coroots are closed under reflection. We define the reflection to act as

$$s_{\alpha^\vee}(\beta^\vee) = \beta^\vee - \frac{2(\alpha^\vee \beta^\vee}{(\alpha^\vee, \alpha^\vee)} \alpha^\vee,$$

and wish to show that $s_{\alpha^\vee}(\beta^\vee) \in R^\vee$. To do this, we use the definition of the coroot, i.e. we check for arbitrary $\gamma \in R$ (skipping a few steps):

$$\langle \gamma, s_{\alpha^\vee}(\beta^\vee) \rangle = \frac{2(\gamma, \beta)}{(\beta, \beta)} - \frac{4(\alpha, \beta)(\alpha, \gamma)}{(\alpha, \alpha)(\beta, \beta)}.$$

We claim that this is simply $\langle \gamma, (s_\alpha(\beta))^\vee \rangle$, which will show that the coroots are indeed closed under reflection:

$$\begin{aligned} \langle \gamma, (s_\alpha(\beta))^\vee \rangle &= \frac{2(\gamma, s_\alpha(\beta))}{(s_\alpha(\beta), s_\alpha(\beta))} \\ &= \frac{2(\beta, \gamma)(\alpha, \alpha) - 4(\alpha, \beta)(\alpha, \gamma)}{(\alpha, \alpha)(\beta, \beta)} \end{aligned}$$

which is precisely what we had above. Hence we see that the coroots in fact form a root system.

(b) Now let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R$ be the set of simple roots. We wish to show that the set $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset R^\vee$ is the set of simple roots of $R^\vee$. First note that given a $t \in E$ with $(t, \alpha) \neq 0$ for all $\alpha \in R$ we obtain the set of positive roots λ: $(\lambda, t) > 0$. We claim that $t^\vee$

yields a choice of positive coroots. This is simply because $(t^\vee, \alpha^\vee) > 0$ if (t, α) is (by the inner product defined above). Now note that

$$\begin{aligned}
C_+ &= \{\lambda^\vee \in E^* \mid (\lambda^\vee, \alpha^\vee) > 0 \;\forall \alpha^\vee \in R_+^\vee\} \\
&= \left\{\lambda^\vee \in E^* \mid \frac{4}{(\alpha,\alpha)(\lambda,\lambda)}(\alpha, \lambda) > 0\right\} \\
&= \{\lambda^\vee \in E^* \mid (\lambda, \alpha) > 0 \;\forall \alpha \in R_+\} \\
&= \{\lambda^\vee \in E^* \mid (\lambda, \alpha) > 0 \;\forall \alpha \in \Pi\} \\
&= \{\lambda^\vee \in E^* \mid (\lambda^\vee, \alpha^\vee) > 0 \;\forall \alpha \in \Pi\}
\end{aligned}$$

and since there are r such coroots (by dimensionality) we obtain a set of simple coroots $\Pi^\vee$.

Problem 3

(a) Let R be a reduced root system of rank 2, with simple roots α_1, α_2. By Kirillov Lemma 7.39, we know that the longest element of the Weyl group is the $w_0 \in W$ such that $w_0(C_+) = -C_+$. This is also geometrically rather obvious for the rank 2 case. Of course, since we know that reflections via simple roots moves a Weyl chamber to one adjacent to it, it will take precisely m reflections to move C_+ to $-C_+$, where each angle is π/m. And of course, the reflection must be a product of $s_1 s_2 s_1 \ldots$ as $s_1^2 = s_2^2 = 1$. This proves the result for the rank 2 systems.

(b) The first Coxeter relation $s_i^2 = 1$ is obvious simply be the properties of transpositions. We now wish to show that $(s_i s_j)^{m_{ij}} = 1$ where m_{ij} is determined by the angle between α_i, α_j in the same way as the previous part.

Problem 4

(a) Consider a complex semisimple Lie algebra $\mathfrak{g}$ with a root space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_\pm = \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$. It is obvious that $\mathfrak{n}_\pm$ are nilpotent, as the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and hence, in the $\mathfrak{n}_+$ case, successive commutators will keep increasing the root, but since the root system is finite, there is an upper bound, and hence we must obtain zero. The same argument holds for $\mathfrak{n}_-$ but with a lower bound.

(b) If we instead consider $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$, we find that $\mathfrak{b}$ is in fact solvable. To see this, note that any two elements of $\mathfrak{b}$ can be written $b_1 = e_1 + h_1, b_2 = e_2 + h_2$, and hence

$$\begin{aligned}
[b_1, b_2] &= [e_1, e_1] + [h_1, e_2] + [e_1, h_2] \\
&= [e_1, e_2] + a_{12} e_2 - a_{21} e_1,
\end{aligned}$$

but now further applications of commutators with elements of $\mathfrak{b}$ will simply yield commutators of e_i with e_j and hence by part (a) above, we are done.

Problem 5

Let G be a connected complex Lie group such that $\mathfrak{g}$ is semisimple. Fix a root decomposition of $\mathfrak{g}$.

(a) Choose $\alpha \in R$ and let $i_\alpha : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{g}$ be the embedding constructed in Kirillov Lemma 6.42. By Kirillov Theorem 3.41, this embedding can be lifted to a morphism $i_\alpha : SL(2, \mathbb{C}) \to G$. Let

$$S_\alpha = i_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}(f_\alpha - e_\alpha)\right) \in G.$$

Then $\operatorname{Ad} S_\alpha(h_\alpha) = -h_\alpha$ because

$$\begin{aligned}
\operatorname{Ad} S_\alpha(h_\alpha) &= \frac{d}{dt}\bigg|_{t=0} \left(i_\alpha \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} e^{t i_\alpha \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}} i_\alpha \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(i_\alpha \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} i_\alpha \left(e^{t \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}} \right) i_\alpha \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) \\
&= \frac{d}{dt}\bigg|_{t=0} i_\alpha \begin{pmatrix} e^{-t} & \\ & e^{t} \end{pmatrix} \\
&= i_\alpha \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = h_\alpha
\end{aligned}$$

Note also that if $h \in \mathfrak{h}$ with $\langle h, \alpha \rangle = 0$, then, by the Serre relations, since

$$[h_i, e_j] = a_{ij} e_j \text{ and } [h_i, f_j] = -a_{ij} f_j$$

where $a_{ij} = n_{\alpha_j, \alpha_i} = \langle \alpha_i^\vee, \alpha_j \rangle$, which is zero in our case, and hence h commutes with $S_\alpha = \exp(\pi/2(f_\alpha - e_\alpha))$ and thus we find that the adjoint action simply yields

$$\begin{aligned}
\operatorname{Ad} S_\alpha(h) &= \frac{d}{dt}\bigg|_{t=0} \left(S_\alpha e^{th} S_\alpha^{-1} \right) \\
&= \frac{d}{dt}\bigg|_{t=0} e^{th} = h,
\end{aligned}$$

as desired. This naturally induces the action on the dual, and hence we see that the action of S_α on $\mathfrak{g}^*$ preserves $\mathfrak{h}^*$ and that the restriction of $\operatorname{Ad} S_\alpha$ to $\mathfrak{h}^*$ coincides with the reflection s_α.

(b) An element of the Weyl group can, in general, be written as a product of s_{α_i} where the α_i are simple roots. By part (a) above we know that the action of $\operatorname{Ad} S_\alpha$ (when restricted to $\mathfrak{h}^*$) coincides with the reflection s_α, and hence if w is written as a product of simple reflections, we may simply compose multiple adjoint actions:

$$\begin{aligned}
\operatorname{Ad} S_\alpha \circ \operatorname{Ad} S_\beta &= s_\alpha \circ s_\beta \\
\operatorname{Ad} (S_\alpha S_\beta) &= s_\alpha \circ s_\beta
\end{aligned}$$

where we have used the homomorphism property of the Adjoint representation. Hence, for cases higher than 2, i.e. when w is written as a product of n simple reflections, we can simply use the homorphism property to find the element in G that acts as w.