Introduction to Algebraic Topology PSET 4

Nilay Kumar

Last updated: February 18, 2014

Proposition 1. *Hatcher exercise 1.1.10*

Proof. Consider two loops in $X \times Y$ based at (x_0, y_0). The first loop is purely in the X direction, denoted by $f : I \to X \times \{y_0\}$, while the second is purely in the Y direction, denoted by $g : I \to \{x_0\} \times Y$. We wish to find a homotopy between the loops $f \diamond g$ and $g \diamond f$. This can be done by continuously transporting the basepoint of f along g via $h_t : I \times I \to X \times Y$ given by

$$h_t(s) = \begin{cases} (x_0, g(2s)) & 0 \leq s \leq \frac{t}{2} \\ (f(2s-1), g(t)) & \frac{t}{2} \leq s \leq \frac{t+1}{2} \\ (x_0, g(2s-1)) & \frac{t+1}{2} \leq s \leq 1 \end{cases}.$$

This homotopy is continuous by continuity of f, g and it clearly takes $f \diamond g$ at $t = 0$ to $g \diamond f$ at $t = 1$. □

Proposition 2. *Hatcher exercise 1.1.12*

Proof. Any morphism $\phi_* : \pi_1(S^1) \to \pi^1(S^1)$ is simply a morphism $\phi_* : \mathbb{Z} \to \mathbb{Z}$, which is determined by the image of its generator. Then, if $n = \phi_*(1)$, the map ϕ that induces ϕ_* is obviously the continuous map $\phi : S^1 \to S^1$ given by $\theta \mapsto n\theta$, as ϕ takes the $[\omega_1]$ to $[\omega_n]$, as required by ϕ_*. □

Proposition 3. *Hatcher exercise 1.1.16*

Proof. Throughout this proof, we use freely Hatcher Proposition 1.17: if a space X retracts onto a subspace A, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. Further, if A is a deformation retract of X, then i_* is an isomorphism. We also use the fact that the functor π_1 takes products to products.

(a) Let $X = \mathbb{R}^3$ with A any space homeomorphic to S^1. The existence of a retraction $r : X \to A$ would imply an injective morphism $\mathbb{Z} \hookrightarrow 0$ of the integers into the trivial group, which is absurd.

(b) Let $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$. The existence of a retraction $r : X \to A$ would imply an injective morphism $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{Z}$ as D^2 is contractible. This is of course impossible: if $(1, 0) \mapsto m$ for any $m, n \in \mathbb{Z}$ and $(0, 1) \mapsto n$, we find that $n(1, 0)$ maps to the same element that $m(0, 1)$ maps to; there are no injective group morphisms $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

(c) Let $X = S^1 \times D^2$ and A be the circle as shown in Hatcher. The inclusion of A into X induces a map from $\mathbb{Z}$ to $\mathbb{Z}$ (injective by the retraction) that takes the generator to just the homotopy class of the knotted circle inside the solid torus. Of course, the knot can be homotoped away by pulling the ends through each other, and hence the map takes the generator to zero. But then the map must be the zero map, and hence not injective.

(d) Let $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$. Of course, $\pi_1(D^2 \vee D^2)$ is the trivial group: any loop in D^2 is contractible and since any loop in $D^2 \vee D^2$ can be pinched off into the composition of two loops each entirely in a single copy of D^2, every loop in $D^2 \vee D^2$ is contractible as well. For usch a retraction $r : X \to A$ to exist would imply an injective morphism from $\pi_1(S^1 \times S^1)$ to the trivial group, which is possible only if all loops in $S^1 \times S^1$ are contractible. This is clearly not the case, as one can consider a loop in just one of the copies of S^1, and hence no such retraction exists.

(e) Let X be a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$. It is clear that X is homotopy equivalent to S^1, as it deformation retracts to the diameter connecting the two identified points. Hence such a retraction would imply an injective morphism $\pi_1(S^1 \vee S^1) \hookrightarrow \mathbb{Z}$. But note that clearly $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} \hookrightarrow \pi_1(S^1 \vee S^1)$ and hence this is impossible, as it would imply an injective morphism of $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$. Of course, this is easier to see via the fact that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

(f) Let X be the Möbius band and A be its boundary circle. It is clear that X deformation retracts onto its central circle, and thus $\pi_1(X, x_0) = \mathbb{Z}$ and $\pi_1(A, x_0) = \mathbb{Z}$ for $x_0 \in A \subset X$. Drawing out the Möbius band as a square with left and right sides anti-identified, it is clear that the inclusion of the homotopy class of the boundary circle into $\pi_1(X, x_0)$ yields twice the generator, as the boundary circle goes around the core circle twice. But this would imply that the retraction induces a map that takes twice the generator to the generator (as the existence of a retraction implies that the inclusion composed with retraction yields the identity), which is impossible from $\mathbb{Z}$ to $\mathbb{Z}$.

□

Proposition 4. *Hatcher exercise 1.2.4*

Proof. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. It is clear that $\mathbb{R}^3 - X$ deformation retracts onto a $2n$-punctured S^2, where the punctures are antipodal. Note that in the case of $n = 1$, we can deformation retract $S^1 - \{N, S\}$ to a single equatorial S^1, as each puncture is contained by a circle. Similarly, for higher n, we can fix a basepoint on the sphere (that is not one of the punctured points) and draw loops about each punctured point, hence forming the wedge sum of $2n-1$ circles (as the last point one is automatically contained in the 'outside' of the others). Hence the $2n$-punctured S^2 can be deformation retracted onto the $2n-1$ wedge sum $S^1 \vee \cdots \vee S^1$, and we find that $\pi_1(\mathbb{R}^3 - X) = \mathbb{Z} * \cdots * \mathbb{Z}$, where the free product is taken $2n-1$ times. □

Proposition 5. *Hatcher exercise 1.2.8*

Proof. If we attach a circle of one of the tori to a circle of the other, the cell complex can be drawn as below. The generators are clearly a, b, c but due to the two attached 2-cells, the loops $aba^{-1}b^{-1}$ and $aca^{-1}c^{-1}$ are contractible. Hence the fundamental group, via Hatcher, has a presentation

$$\langle a, b, c \mid aba^{-1}b^{-1}, aca^{-1}c^{-1} \rangle.$$

Note that this is simply the group $\mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$, as $[a, b] = 1$ and $[a, c] = 1$. □