RTG Notes

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0 Introduction

0.1 Adam

What is algebraic geometry? It is the study of solutions to systems of polynomial equations.

Definition 1. An **algebraic variety** $V(f_1, \ldots, f_s$ is the set of points p such that $f_1(p) = f_2(p) = \cdots = 0$

One nice thing is that we can work over different fields and get different results. In fact, we can often get hard number theoretic problems. Take for example $V(x^n + y^n - z^n)$ over $\mathbb{Q}$ – analyzing this necessitates Fermat's last theorem!

Example 1. Let us, for example, find rational solutions to $x^2 + y^2 = z^2$. We can rewrite this as $X^2 + Y^2$ by a substitution. It's clear that we have a circle which goes through $(0, 1)$. If we assume there exists a rational point (p, q) also on the circle, it's clear that the line going from $(0, 1)$ and (p, q) has rational slope. Conversely, any line with rational slope will also hit some rational point on the circle. Thus we can parametrize the solutions to our problem by rational slopes m.

In light of this example, how do we work backwards from parametrizations (toric varieties) to varieties?

Example 2. Let $C = \left\{(t, t^2, t^3) \in \mathbb{R}^3 | t \in \mathbb{R}\right\}$. C is, in fact, an algebraic variety:

$$C = V(y - x^2, z - x^3).$$

Are there other choices? Well we could try to start with the parametrizations and try to eliminate the parameter. Consider the homomorphism

$\phi : \mathbb{R}[x, y, z] \to \mathbb{R}[t]$ given by $\phi(x) = t, \phi(y) = t^2, \phi(z) = t^3$. Clearly the set we want is $\ker \phi$, i.e. the set of polynomials in x, y, z that vanish when $x = t, y = t^2, z = t^3$. We've reduced the problem to determining the kernel of some ring homomorphism.

The general question is, given a homomorphism $\phi : \mathbb{R}[x_1, \dots, x_s] \to \mathbb{R}[r, s, t]$, how do we describe the kernel? Let's look at another example.

Example 3. Consider the parametrizations

$$\phi(x_1) = s^4, \phi(x_2) = s^3 t, \phi(x_3) = st^3, \phi(x_4) = t^4.$$

One equation is $x_1 x_4 - x_3 x_2$. However, it's clear that these problems can quickly get rather difficult.

These are the types of questions we will generally be talking about.

0.2 Pablo

0.3 Dan