Riemann Surfaces: Lecture Notes

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Class 10

Recall that we are studying function theory on the torus $\mathbb{C}/\Lambda$, where $\Lambda = \{m\omega_1 + n\omega_2; m, n \in \mathbb{Z}\}$. We had produced a candidate

$$\sigma(z) = z \prod_{\omega \in \Lambda^\times} \left(1 + \frac{z}{w}\right) e^{-\frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}}.$$

Let us check that this product converges by examining its logarithm:

$$\begin{aligned} \log\{\cdots\} &= \log(1 + \frac{z}{\omega}) - \frac{z}{\omega} + \frac{z^2}{2\omega^2} \\ &= \left(\frac{z}{\omega} - \frac{1}{2}\frac{z^2}{\omega^2} + \dots\right) - \frac{z}{\omega} + \frac{1}{2}\frac{z^2}{\omega^2}, \end{aligned}$$

which clearly converges. Hence $\sigma(z)$ is holomorphic for $z \in \mathbb{C}$. Recall that $\sigma'(z)/\sigma(z) = \zeta(z)$ and so $\partial_z \log \sigma(z + \omega_a) = \zeta(z + \omega_a)$. Thus we have (from before) that

$$\eta_a = \zeta(z + \omega_a) - \zeta(z) = \partial_z \log \sigma(z + \omega_a) - \partial_z \log \sigma(z),$$

which gives us periodicity information. Integrating and exponentiating, we see that

$$\sigma(z + \omega_a) = \sigma(z) e^{\eta_a z + c_a}$$

where c_a is the constant of integration, and taking $z = -\omega_a/2$, we find that

$$\sigma(\omega_a/2) = \sigma(-\omega_a/2) e^{-\eta_a \frac{\omega_a}{2} + c_a}.$$

It is easy to check, however, that σ is odd, and hence we find that

$$\sigma(z + \omega_a) = -\sigma(z) e^{\eta_a (z + \frac{\omega_a}{2})}.$$

So we have found that $\sigma(z)$ is holomorphic on $\mathbb{C}$ and that $\sigma(z) = 0$ if and only if $z = 0 \mod \Lambda$. Now that we have constructed such a σ, let us give another proof of Abel's theorem. First recall our previous statement of Abel's theorem.

Theorem 1 (Abel's theorem)**.** *Let $P_1, \dots, P_M, Q_1, \dots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M A(P_i) = \sum_{i=1}^N A(Q_i)$.*

Recall that the Abel map takes $\mathbb{C}/\Lambda \ni p \mapsto A(p) = \int_{p_0}^{p} \omega$ where the value of the integral is taken modulo the lattice generated by $\oint_A \omega, \oint_B \omega$. Take $p_0 = 0$ and $\omega = dz$, which is a well-defined form, and if we take A to align with ω_2 and B to align with ω_1, we see that $\oint_A \omega = \oint_A dz = \omega_1$ and similarly $\oint_B \omega = \omega_2$. Hence the map simply takes p to $\int_0^p dz \mod \Lambda = p$ where p is viewed as a complex number.

Let us now restate Abel's theorem.

Theorem 2 (Abel's theorem, v.2)**.** *Let $P_1, \ldots, P_M, Q_1, \ldots, Q_N$ be points in $\mathbb{C}$. Then there exists a meromorphic f with zeroes at P_i and poles at Q_i if and only if $M = N$ and $\sum_{i=1}^M P_i = \sum_{i=1}^N Q_i$* $\mod \Lambda$.

Proof. Consider the function

$$f(z) = \frac{\prod_{i=1}^M \sigma(z - P_i)}{\prod_{i=1}^N \sigma(z - Q_i)}.$$

We should be a little careful to note that σ is a function not on the torus $\mathbb{C}/\Lambda$, but a function on $\mathbb{C}$ (it transforms under a lattice translation!). Hence we must be cognizant of the fact that P_i, Q_i here are some chosen representatives in $\mathbb{C}$ of the equivalence classes of the points P_i, Q_i. It should be clear that $f(z)$ is meromorphic with zeroes at every representative of each P_is and poles at every representative of each Q_i. The natural question, now, is whether this function extends to a function on the torus. To check this, let us see whether it is doubly periodic using what we know about σ:

$$\begin{aligned} f(z + \omega_a) &= f(z) \frac{\prod_{i=1}^M e^{\eta_a (z - P_i)}}{\prod_{i=1}^N e^{\eta_a (z - Q_i)}} \\ &= f(z) e^{-\eta_a \left(\sum_{i=1}^M P_i - \sum_{i=1}^N Q_i \right)}. \end{aligned}$$

Hence we wish to choose P_i, Q_i representatives such that the exponential becomes unity. By hypothesis, this can be done (by shifting one, if necessary). □

Let us now return to Weierstrass theory. Given $\omega = dz$, we defined $\omega_0 = \mathcal{P}(z) dz$ which has a double pole at 0 and $\partial_z \log \sigma(z) = \zeta(z)$ and $\zeta'(z) = -\mathcal{P}(z)$. Now we can construct a form ω_{PQ} with residues $1, -1$ at P, Q respectively, by assigning $\omega_{PQ}(z) = (\zeta(z - P) - \zeta(z - Q)) \omega = \partial_z \log \frac{\sigma(z-P)}{\sigma(z-Q)} dz$. What Weierstrass theory tells us that we can write everything in terms of σ, our analog of z.

Jacobi theory: θ-functions

Consider again the torus $\mathbb{C}/\Lambda$, where we now normalize the lattice as $\Lambda = \{m + n\tau; m, n \in \mathbb{Z}\}$ with $\operatorname{Im} \tau > 0$ (by linear independence, it cannot be real). This simply corresponds to picking $\omega_1 = 1, \omega_2/\omega_1 = \tau$. Next define the **theta-function**

$$\theta(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

in which the structure of the lattice is explicitly clear (unlike in the Weierstrass theory). Let us examine its main properties.

First, note that $\theta(z|\tau)$ is holomorphic in $z \in \mathbb{C}$ because the series converges for all z; this is due to the term

$$|e^{\pi i n^2 (\tau_1 + i\tau_2)}| = |e^{\pi i n^2 \tau_1} e^{-\pi^2 n^2 \tau_2}| = e^{-\pi n^2 \tau_2}$$

for $\tau = \tau_1 + i\tau_2$, whose decay dominates due to the n^2. Next, notice that

$$\begin{aligned}
\theta(z+1|\tau) &= \theta(z|\tau) \\
\theta(z+\tau|\tau) &= e^{-\pi i\tau - 2\pi i z}\theta(z|\tau),
\end{aligned}$$

where the second is obtained by completing the square. Though θ is not invariant, its zeroes are.

Furthermore, we claim that $\theta(z|\tau)$ vanishes at exactly one point modulo lattice translates. It suffices to compute the integral $\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz$, as it yields $2\pi i$ times the difference in the number of zeroes and poles in a given region. We shall integrate over the curve C where C traverses the circumference of one lattice segment (i.e. the whole torus):

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = \oint_B \left(-\frac{\theta'(z|\tau)}{\theta(z|\tau)} + \frac{\theta'(z+1|\tau)}{\theta(z+1|\tau}\right) + \oint_A \left(\frac{\theta'(z|\tau)}{\theta(z|\tau)} - \frac{\theta'(z+\tau|\tau)}{\theta(z+\tau|\tau)}\right).$$

But these are just the shifts in the logarithmic derivative, and since $\partial_z \log\theta(z+\tau|\tau) = -2\pi i + \partial_z \log\theta(z|\tau)$ using the transformation rules above, we see that our integral simplifies to

$$\oint_C \frac{\theta'(z|\tau)}{\theta(z|\tau)}dz = 2\pi i \oint_A dz = 2\pi i.$$

Of course, since θ is holomorphic, it has no poles, and hence we see that we have one zero. The zero, in fact, occurs in the center: $\theta\left((1+\tau)/2|\tau\right) = 0$. To see this, consider the following function:

$$\begin{aligned}
\theta\left(z + \frac{1+\tau}{2}|\tau\right) &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i n\left(z + \frac{1+\tau}{2}\right)\right) \\
&= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+\frac{1}{2})^2\tau + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2})\right) \\
&= i\exp\left(-\pi i\frac{\tau}{4} - \pi i z\right)\theta_1(z|\tau)
\end{aligned}$$

where we have completed the square and defined the function θ_1. We claim that θ_1 is an odd function, which would imply that θ_1 vanishes at zero, which would prove the claim about the location of the zero. Hence let us verify that θ_1 is odd; switching $z \mapsto -z$ yields in the exponent

$$\log\theta_1(z|\tau) = \pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right).$$

If we switch the indices $n \mapsto m$ such that $n + \frac{1}{2} = -\left(m+\frac{1}{2}\right)$, we find that the exponent is now

$$\log\theta_1(z|\tau) = \pi i\left(m+\frac{1}{2}\right)^2\tau + 2\pi i\left(m+\frac{1}{2}\right)\left(\left(z+\frac{1}{2}\right) - 2\pi i\left(m+\frac{1}{2}\right)\right),$$

and hence θ_1 is odd. Now we see that the function we want is in fact $\theta_1(z|\tau)$ as it is odd, holomorphic, and has one zero.

We leave it as an exercise to show that

$$\sigma(z) = \omega_1\exp\left(\eta_1\frac{z^2}{\omega_1}\right)\frac{\theta_1\left(\frac{z}{\omega_1}|\tau\right)}{\theta_1'(0|\tau)}$$

Class 11

We claim that the theta-function theory is more powerful than what we have been using so far - to see this, let us prove Abel's theorem. Recall that the theorem states that there exists a meromorphic f with zeroes at P_i and poles and Q_j if and only if $N = M$ and $\sum_i A(P_i) = \sum_j A(Q_j)$. The idea is to express

$$f(z) = \frac{\prod_{i=1}^N \theta_1(x - P_i)}{\prod_{i=1}^N \theta_1(z - Q_i)}$$

and check double-periodicity. It is an exercise to check that

$$\begin{aligned}\theta_1(z+1|\tau) &= -\theta_1(z|\tau)\\ \theta_1(z+\tau|\tau) &= \exp\left(-\pi i\tau - 2\pi i(z+1/2)\right)\theta_1(z|\tau),\end{aligned}$$

from which periodicity follows easily. Of course, we must be careful to note that the P_i, Q_i used here are in fact chosen representatives.

Next let us define a meromorphic form

$$\omega_{PQ} = \partial_z \log \frac{\theta_1(z-P)}{\theta_1(z-Q)} dz,$$

which, it is easy to check, has poles at P, Q with opposite residues. Additionally, one can check that this expression is well-defined on the lattice, i.e. invariant under a shift. We leave it as a simple exercise to show that

$$\omega_P(z) = \partial_z^2 \log \frac{\theta_1(z-P|\tau)}{\theta_1'(0|\tau)} dz$$

is a meromorphic form with a double pole at P and is well-defined on the lattice.

But in fact, we can go even farther with this theta-function. Indeed, one attractive feature is that there exists a product expansion for $\theta(z|\tau)$.

Theorem 3. *We can expand*

$$\theta(z|\tau) = \prod_{n=1}^\infty (1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz})$$

where $q \equiv e^{\pi i\tau}$.

Proof. Define

$$T(z|\tau) = \prod_{n=1}^\infty (1-q^{2n})(1+q^{2n-1}e^{2\pi iz})(1+q^{2n-1}e^{-2\pi iz}).$$

We claim that $T(z|\tau)$ is equal to zero exactly when z is $(1+\tau)/2 \mod \Lambda$ and that the zeros are simple. This can be checked by some simple algebra. It's also easy to show that $T(z|\tau)$ is holomorphic in $\mathbb{C}$ and that $\tau(z+1|\tau) = T(z|\tau)$. Moreover

$$\begin{aligned}T(z+\tau|\tau) &= \prod_{n=1}^\infty (1-q^{2n}) \prod \left(1+q^{2n+1}e^{2\pi iz}\right)\left(1+q^{2n-3}e^{-2\pi iz}q^{-2}\right)\\ &= \prod_{n=1}^\infty (1-q^{2n}) \frac{\prod_{n=1}^\infty \left(1+q^{2n-1}e^{2\pi iz}\right)}{1+qe^{2\pi iz}} \prod_{n=1}^\infty \left(1+q^{2n-1}e^{-2\pi iz}\right) \quad = \frac{1-q^{-1}e^{-2\pi iz}}{1-qe^{2\pi iz}} T(z|\tau) = q^{-1}e^{-2\pi iz}.\end{aligned}$$

Recall that θ followd a similar condition. This shows that $\theta(z|\tau)/T(z|\tau) = c$, where c is a constant independent of z that can depend on τ. Next we claim that $c(\tau) = 1$. For this we show that there exists a c such that $c(\tau) = c(4\tau) = c(4^k\tau)$ and $c(\tau) = \lim_{k\to\infty} c(4^k\tau) = 1$, which shows the proof. Hence let us prove that $c(\tau) = c(4\tau)$ using $\theta(z|\tau) = C(\tau)T(z|\tau)$.

Take $z = 1/2$. Then $e^{2\pi i z = e^{\pi i}} \geq 1-$ and $\theta(1/2|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}(-1)^n$ but $T(1/2|\tau) = \prod_{n=1}^{\infty}(1-q^{2n)(1-q^{2n-1}}(1-q^{2n-1})$. and hence $c(\tau) = \sum e^{\pi i n \tau}(-1)^n / \prod(1-q^n)(1-q^{2n-1})$. Next take $z = 1/4$ $\square$

1 Semester 2

This semester we will start by describing the L^2 estimates of Hormander. Later we will delve into its applications, including the Kodaira embedding theorem, the lower bounds for the Bergman kernel, and the ideas of canonical metrics and stability.

1.1 Review

Let us recall some techniques from last semester. Let $X = \cup_\mu X_\mu$ be a complex n-manifold with X_μ coordinate charts. Hence each X_μ is homeomorphic (and thus biholomorphic) to $\mathbb{C}^n$ and $\Phi_\mu \circ \Phi_\nu^{-1}$ is holomorphic with invertible differentials. Let $E \to X$ be a holomorphic vector bundle. Recall what?
that a rank-r vector bundle is completely characterized by its transition functions $t^\alpha_{\mu\nu\beta}(z)$ (matrix valued in general) defined on $X_\mu \cap X_\nu$ with $1 \leq \alpha, \beta \leq r$. Note that the transition functions satisfy the cocycle condition. We denote by $\Gamma(X, E)$ the space of sections of E (recall that this means that $\phi^\alpha_\mu(z_\mu) = t^\alpha_{\mu\nu\beta}(z)\phi^\beta_\nu(z_\nu)$. For E to be holomorphic, it must have holomorphic transition functions.

Given a section $\phi \in \Gamma(X, E)$, we obtain a section $\bar\partial\phi \in \Gamma(X, E \otimes \Lambda^{0,1})$ via **covariant differentiation**. More explicitly, on X_μ, we write naively

$$\bar\partial\phi^\alpha \equiv \left(\frac{\partial}{\partial \bar z^j_\mu}\phi^\alpha_\mu\right)(z_\mu).$$

Fortunately, this is indeed a section. To see this, we note that $\phi^\alpha_\mu(z_\mu) = t^\alpha_{\mu\nu\beta}(z)\phi^\beta_\nu(z_\nu)$ and hence

$$\begin{aligned}\frac{\partial}{\partial \bar z^j_\mu}\phi^\alpha_\mu(z_\mu) &= t^\alpha_{\mu\nu\beta}(z)\frac{\partial}{\partial \bar z^j_\mu}\left(\phi^\beta_\nu(z_\nu)\right)\\ &= t^\alpha_{\mu\nu\beta}(z)\overline{\frac{\partial z^k_\nu}{\partial z^j_\mu}}\frac{\partial \phi^\beta_\nu}{\partial \bar z^k_\nu}(z).\end{aligned}$$

Thus we define $\Lambda^{0,1}$ to be the (antiholomorphic) vector bundle with transition functions $\overline{\partial z^k_\nu/\partial z^j_\mu}$.

Next, recall the definition of a **Hermitian metric** $H = H_{\bar\alpha\beta}(z)$ on a vector bundle E: we have $(H_\mu)_{\bar\alpha\beta}(z_\mu)$ on X_μ such that it is a positive-definite matrix for each z_μ satisfying

$$|\phi|^2_H \equiv (H_\mu)_{\bar\alpha\beta}\overline{\phi^\alpha_\mu}\phi^\beta_\mu = (H_\nu)_{\bar\gamma\delta}\overline{\phi^\gamma_\nu}\phi^\delta_\nu$$

on $X_\mu \cap X_\nu$. This quantity can be thought of as the length of the vector ϕ with respect to the metric H, which is by construction invariant of coordinate chart. Using metrics, we can introduce covariant derivatives of sections on a holomorphic vector bundle E with respect to a metric $H_{\bar\alpha\beta}$. Take $\phi \in \Gamma(X, E)$ and define on X_μ

$$(\nabla_j\phi)^\alpha \equiv H^{\alpha\bar\gamma}\partial_j(H_{\bar\gamma\beta}\phi^\beta_\mu),$$

where $H^{\alpha\bar{\gamma}}H_{\bar{\gamma}\beta} = \delta^\alpha_\beta$. It is easy to see that $\nabla\phi \in \Gamma(X, E \otimes \Lambda^{1,0})$, whose transition functions are $\partial z^k_\nu / \partial z^j_\mu$.

In summary, we write

$$\begin{aligned}
\nabla_{\bar{j}}\phi^\alpha &= \partial_{\bar{j}}\phi^\alpha \\
\nabla_j \phi^\alpha &= H^{\alpha\bar{\gamma}}\partial_j (H_{\bar{\gamma}\beta}\phi^\beta_\mu) \\
&= \partial_j \phi^\alpha + (H^{\alpha\bar{\gamma}}\partial_j H_{\bar{\gamma}\beta})\phi^\beta \\
&= \partial_j \phi^\alpha + A^\alpha_{j\beta}\phi^\beta,
\end{aligned}$$

where, in matrix notation, we have the **connection** $A_j = H^{-1}\partial_j H$. It is a priori not obvious that these two derivatives must commute. Indeed, we define the **curvature** F of the metric $H_{\bar{\alpha}\beta}$ on $E \to X$ to be

$$[\nabla_{\bar{j}}, \nabla_k]\phi^\alpha = -F^\alpha_{\bar{j}k\beta}\phi^\beta.$$

We leave it as an exercise that $[\nabla_{\bar{j}}, \nabla_{\bar{k}}] = 0$ and $[\nabla_j, \nabla_k] = 0$. More explicitly, we can write

$$\begin{aligned}
[\nabla_{\bar{j}}, \nabla_k]\phi^\alpha &= \nabla_{\bar{j}}\left(\nabla_k \phi^\alpha\right) - \nabla_k \left(\nabla_{\bar{j}}\phi^\alpha\right) \\
&= (\partial_{\bar{j}} A^\alpha_{k\beta})\phi^\beta,
\end{aligned}$$

and hence $F^\alpha_{\bar{j}k\beta} = -\partial_{\bar{j}} A^\alpha_{k\beta}$. In matrix notation, we can simply write

$$F_{\bar{j}k} = -\partial_{\bar{j}} A_k = -\partial_{\bar{j}}\left(H^{-1}\partial_k H\right).$$

We define the corresponding **curvature form** to be

$$F = \frac{i}{2\pi} F^\alpha_{\bar{j}k\beta}(z) dz^k \wedge d\bar{z}^j \in \Gamma(X, E \otimes E^* \otimes \Lambda^{1,1}) = \Gamma(X, \mathrm{End}(E) \otimes \Lambda^{1,1}),$$

i.e. a End(E)-valued $(1,1)$-form.

1.2 Bochner-Kodaira Formulas

Let X be a complex manifold, compact without boundary. Take $E \to X$ to be a holomorphic vector bundle on X. We consider the $\bar{\partial}$ **complex**:

$$\cdots \xrightarrow{\bar{\partial}} \Gamma(X, E \otimes \Lambda^{p,q}) \xrightarrow{\bar{\partial}} \Gamma(X, E \otimes \Lambda^{p,q+1}) \xrightarrow{\bar{\partial}} \cdots.$$

Let us be more precise. Consider some $\phi \in \Gamma(X, E \otimes \Lambda^{p,q})$. We can write explicitly:

$$\phi = \frac{1}{p!q!}\sum \phi_{\bar{j}_1,\ldots\bar{j}_q, i_1,\ldots,i_p}(z) dz^{i_p} \wedge \cdots \wedge dz^{i_1} \wedge d\bar{z}^{j_q} \wedge \cdots \wedge d\bar{z}^{j_i}.$$

Now what exactly do we mean by $\bar{\partial}$? We define

$$\begin{aligned}
\bar{\partial}\phi &\equiv \frac{1}{p!q!}\sum \left(\bar{\partial}\phi_{\bar{j}_1,\ldots\bar{j}_q, i_1,\ldots,i_p}\right) \wedge dz^{i_p} \wedge \cdots \wedge dz^{i_1} \wedge d\bar{z}^{j_q} \wedge \cdots \wedge d\bar{z}^{j_i} \\
&= \frac{1}{p!q!}\sum \left(\partial_{\bar{k}}\phi_{\bar{j}_1,\ldots\bar{j}_q, i_1,\ldots,i_p} d\bar{z}^k\right) \wedge dz^{i_p} \wedge \cdots \wedge dz^{i_1} \wedge d\bar{z}^{j_q} \wedge \cdots \wedge d\bar{z}^{j_i}.
\end{aligned}$$

We leave it as an exercise for the reader to check that this is well-defined (follows as per the usual de Rham exterior derivative).

Example 1. What is $\bar{\partial}$ on $\Gamma(X, E \otimes \Lambda^{0,0}) = \Gamma(X, E)$? By definition, $\bar{\partial}\phi = \partial_{\bar{k}}\phi^\alpha d\bar{z}^k$.

Example 2. What is $\bar{\partial}$ on $\Gamma(X, E \otimes \Lambda^{0,1})$? Given a section, we can write $\phi = \sum \phi^\alpha_{\bar{j}} d\bar{z}^j$. In this case,

$$\begin{aligned}
\bar{\partial}\phi^\alpha &= \sum \left(\bar{\partial}\phi^\alpha_{\bar{j}}\right) \wedge d\bar{z}^j \\
&= \sum \left(\partial_{\bar{k}}\phi^\alpha_{\bar{j}} d\bar{z}^k\right) \wedge d\bar{z}^j \\
&= \frac{1}{2} \sum \left(\partial_{\bar{k}}\phi^\alpha_{\bar{j}} - \partial_{\bar{j}}\phi^\alpha_{\bar{k}}\right) d\bar{z}^k \wedge d\bar{z}^j.
\end{aligned}$$

Hence one finds the coefficient $(\bar{\phi})_{\bar{j}\bar{k}} = \partial_{\bar{k}}\partial_{\bar{j}} - \partial_{\bar{j}}\partial_{\bar{k}}$.