Lie Groups and Representations: Notes

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Last updated: September 25, 2013

Theorem 1 (Fundamental theorem of Lie theory)**.** *The category of connected, simply-connected, Lie groups is equivalent to the category of Lie algebras.*

Proof. Omitted. □

One can consider as an example the case of $SO(3)$ and $SU(2)$, which have identical Lie algebras. Another example is $\mathfrak{sl}(2,\mathbb{R})$ (2×2 matrices with trace zero), which corresponds to $\widetilde{SL(2,\mathbb{R})}$, the universal cover of $SL(2,\mathbb{R})$. Indeed, to understand connected Lie groups, it suffices to understand Lie algebras as well as their theory of coverings. One direction of this theorem is easy to see, namely sending groups to algebras, but the difficulty arises in the other direction: how does one lift a Lie algebra homomorphism to a Lie group homomorphism? One (not completely obvious) way to do this is to use the Baker-Campbell-Hausdorff formula, which allows us to locally recover the group law from the Lie algebra.

Indeed, the Baker-Campbell-Hausdorff formula can be used to define the Lie bracket operation. Another way of doing this is of course to treat the Lie algebra elements as left-invariant vector fields, and to then use the usual Lie bracket of vector fields. Of course, for matrix groups we can obviously define the bracket as simply the commutator $[X, Y] = XY - YX$ (we can think of this more abstractly as the derivative of the adjoint representation of the group).

Definition 1. A **representation** (π, V) **of a group** G on a vector space V is a homomorphism $\pi : G \to GL(V)$.

Remark. For us, G will be a Lie group, and V will be a finite-dimensional vector space over a field, usually $\mathbb{C}$.

Definition 2. A **representation** (ϕ, V) **of a Lie algebra** $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V) = \operatorname{End} V$.

Remark. Given a representation (π, V) of G, we obtain a representation of $\mathfrak{g}$ simply by taking the derivative (π_*, V), with $\pi_*(X) = d/dt\pi(e^{tX})|_{t=0}$. However, we cannot necessarily go the other way: given a Lie algebra representation, we cannot always lift to a Lie group representation. If a representation of $\mathfrak{g}$ is π_* for some π a representation of G, then we call π **integrable**.

Note that for any Lie group G, we can construct an action of G on itself. This is the adjoint action given by $g \mapsto c(g) \in \operatorname{Diff}(G)$ where $c(g)h = ghg^{-1}$. Note however that $\operatorname{Diff}(G)$ is nonlinear and hence we obtain a nonlinear representation. Instead we can consider $g \mapsto (c(g))_* : TG \to TG$, which restricts at the identity to a map $\mathfrak{g} \to \mathfrak{g}$. Indeed, one can check that we get a representation this way.

Definition 3. The **Adjoint representation of the group** G is $(\mathrm{Ad}, \mathfrak{g})$ where $\mathrm{Ad}(g) = (c(g))_*(e)$.

We can go further and differentiate this representation to obtain a Lie algebra representation.

Definition 4. The **adjoint representation** of a Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ is $(\mathrm{ad}, \mathfrak{g})$ where $\mathrm{ad}(X) = \mathrm{Ad}'(e^{tX}) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(e^{tX})$.

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For matrix groups, it's clear that the Adjoint representation is just conjugation (via matrix multiplication), i.e. $Ad(g)M = gMg^{-1}$. For such we can use a trivialization and treat Lie algebra elements as matrices, i.e. given the left-invariant vector field $X_M = (g, gM)$, M is the associated element of the Lie algebra. Then the adjoint representation for matrix Lie algebras is given by the commutator $ad(X)M = [X, M]$. One can check this by computing

$$ad(X)(Y) = \frac{d}{dt}\Big|_{t=0} \left(e^{tX} Y e^{-tX}\right) = XY - YX$$

In general, the Adjoint and adjoint representations are interesting in that they are a distinguished feature of a given Lie group/algebra. This will be a key fact used when trying to classify algebras.

Example 1. Consider $G = SO(3)$ whose Lie algebra is $\mathrm{Lie}\, SO(3) = \mathbb{R}^3$, isomorphic to the space of anti-symmetric 3×3 real matrices. Choosing the usual basis L_i for $\mathfrak{so}_3$, where we have a 1 and -1 in each basis element, we see that we obtain, in some sense, the cross product on $\mathbb{R}^3$ via the bracket operation.

We can do something similar for $G = SU(2)$, and we find (as we know from the homework) that $\mathfrak{su}_2 = \mathfrak{so}_3$. Of course, this is a rather special case; in higher dimensions there are no such isomorphism in general.

So far we have considered Lie algebras that are real vector spaces. It turns out, however, that representations are simplest for complex vector spaces (as diagonalizability is easiest done). Hence we wish to study the "complexification" of $\mathfrak{g}$.

Definition 5. The **complexification of** $\mathfrak{g}$ is the complex vector space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. The Lie bracket of $\mathfrak{g}$ extends straightforwardly to a Lie bracket of $\mathfrak{g}_{\mathbb{C}}$.

Remark. One can think of $\mathfrak{g}_{\mathbb{C}}$ as a real Lie algebra with $\dim \mathfrak{g}_{\mathbb{C}} = 2 \dim \mathfrak{g}$. In general, when we write something like $\mathfrak{g} = \mathrm{Lie}\, G$, we will mostly treat G as a real manifold and assume that $\mathfrak{g}$ is real.

Definition 6. The Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ are **real forms** of a complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_2 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$.

Example 2. Take $\mathfrak{g}_1 = \mathfrak{sl}_2\mathbb{R}$, i.e. 2×2 traceless matrices. This is isomorphic to $\mathbb{R}^3$ as a vector space, but is not isomorphic to $\mathfrak{su}_2 = \mathfrak{so}_3$. If we complexify this Lie algebra, we obtain $\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$, which is the space of 2×2 traceless complex matrices, isomorphic to $\mathbb{C}^3 \cong \mathbb{R}^6$. So $\mathfrak{g}_1$ is a real form of $\mathfrak{sl}_2\mathbb{C}$; note that $SL(2, \mathbb{R})$ is a non-compact group.

Consider $\mathfrak{g}_2 = \mathfrak{su}_2$, which is the space of anti-hermitian 2×2 traceless matrices. If we take $\mathfrak{g}_2 \oplus i\mathfrak{g}_2$, we get both anti-hermitian and hermitian traceless matrices, i.e. all complex traceless matrices. Indeed, this is just $\mathfrak{sl}_2\mathbb{C}$ and hence $\mathfrak{g}_2$ is a real form of $SL(2, \mathbb{C})$. This is not isomorphic to $\mathfrak{g}_1$! Since $SU(2)$ is compact, $\mathfrak{g}_2$ is the "compact real form."

Example 3. Note that $\mathfrak{so}_3 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$. Indeed, we can define complex orthogonal groups, such as $SO(3, \mathbb{C})$a, that are the transformations of $\mathbb{C}^3$ that preserve $\langle \vec{z}, \vec{w} \rangle = z_i w^i$ (this is definitely not positive-definite!). Special to the dimension 3 is the fact that $SO(3, \mathbb{C}) \cong SL(2, \mathbb{C})$ (up to double cover issues). In general, we can define $SO(n, \mathbb{C})$ that give rise to complex Lie algebras that each have several real forms, including $\mathfrak{so}(p, q)$.