Commutative Algebra: Problem Set 12

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Last updated: December 4, 2013

Problem 4

We prove the sheaf condition for the structure $\mathcal{B}$-pre-sheaf $\mathcal{O}$ of an affine scheme $X = \operatorname{Spec} A$ where $\mathcal{B}$ is the basis of standard opens. We roughly follow Eisenbud and Harris (The Geometry of Schemes). We know already that any $\mathcal{B}$-sheaf on X extends uniquely to a sheaf on X, so it suffices to show that our $\mathcal{B}$-pre-sheaf is in fact a $\mathcal{B}$-sheaf. This amounts to showing first that if any two sections s_1 and s_2 become locally equal, i.e. when restricted to each of the standard opens X_{f_i}, then $s_1 = s_2$, and second that if for each i there exist $s_i \in \mathcal{O}(X_{f_i})$ such that for each pair i, j $s_i|_{X_{f_if_j}} = s_j|_{X_{f_if_j}}$, then there exists an $s \in \mathcal{O}(X)$ whose restriction to each X_{f_i} is precisely s_i.

Let us prove the first condition. For s_1, s_2 to become equal in X_{f_i} means that the difference $s_1 - s_2$ must be annihilated by some power of each f_i. Since the cover X_{f_i} of X is finite (see the quasicompactness problem below) $s_1 - s_2$ is annihilated by the ideal generated by all the f_i^N for some N. As this ideal clearly contains a power of the ideal generated by all the f_i, which in turn generate the unit ideal (again, see below), s_1 must equal s_2 in A as the difference is annihilated by all elements of the ring.

Next consider the second condition. Each $s_i \in \mathcal{O}(X_{f_i}) = A_{f_i}$ can of course be multiplied by a high enough power of f_i to yield an element $h_i \in A$. Again, by finiteness, one N suffices for all i and hence redefining $h_i = f_i^N s_i$, we find that

$$f_j^N h_i = f_j^N f_i^N s_i = f_j^N f_i^N s_j = f_i^N h_j$$

where the second equality follows from the hypothesis that each of pair of sections agrees on the overlap. Just as above, the f_i^N generate the unit ideal and so we have

$$1 = \sum_i e_i f_i^N$$

for some $e_i \in R$ (geometrically, this is simply a partition of unity). Let us consider the section $s \in \mathcal{O}(X)$ defined by

$$s = \sum_i e_i h_i.$$

Let us show that this is the desired global section, i.e. that it restricts to each X_{f_i} as s_i. Note that on X_{f_i} we have that

$$f_i^N s = \sum_j f_i^N e_j h_j = \sum_j e_j f_j^N h_i = h_i = f_i^N s_i.$$

But in this localization f_i is a unit, and hence cancelling f_i^N from both sides we find that $s|_{X_{f_i}}$, we do indeed get that $s = s_i$.

Problem 5

Let $(X, \mathcal{O}_X)$ be the usual affine scheme $X = \operatorname{Spec} \mathbb{Z}$. Consider the quotient topology Y where the points (2) and (3) are identified, and let ϕ be the natural projection. Since ϕ is continuous, we may construct the direct image sheaf $\mathcal{O}_Y = \phi_* \mathcal{O}_X$. Hence we obtain a ringed space $(Y, \mathcal{O}_Y)$. We claim that this is not a locally ringed space - to see this, we must compute the stalks of $\mathcal{O}_Y$. Note first that the (basis of) opens in Y containing $(\bar{2}) \equiv \phi(2) = \phi(3)$ are of the form $D(x)$ where $x \notin (\bar{2})$. At the level of the sheaf we see that:

$$\mathcal{O}_Y(D(x)) = \mathcal{O}_X(\phi^{-1}(D(x))) = \mathcal{O}_X(D(\phi^{-1}(x))) = \mathcal{O}_X(D(x)),$$

where we have written by abuse of notation $\phi^{-1}(x) = x$ (as we are excluded the trouble points). Now the stalk is the disjoint union of these opens, together with their sections in $\mathcal{O}_Y$, modulo sections that become equal on some small enough opens. Consider two opens $U = D(x_1), V = D(x_2) \subset Y$ containing $(\bar{2})$ with intersection $W = U \cap V = D(x_1 x_2)$. Using the above properties of $\mathcal{O}_Y$ we see that $\mathcal{O}_Y(U) = \mathbb{Z}_{x_1}, \mathcal{O}_Y(V) = \mathbb{Z}_{x_2}$, and $\mathcal{O}_Y(W) = \mathbb{Z}_{x_1 x_2}$. For a section $f \in \mathcal{O}_Y(U)$ and a section $g \in \mathcal{O}_Y(V)$ to restrict to the same section $h \in \mathcal{O}_Y(W)$ would require that $\psi(f) - \psi'(g)$ - with ψ and ψ' the localization maps - be annihilated by an element of the multiplicative set generated by $x_1 x_2$. As $\mathbb{Z}$ is an integral domain, we see that this is possible only if $\psi(f) = \psi'(g)$. Now suppose $f = a/c$ and $g = b/d$ for $a, b \in \mathbb{Z}$ and c, d in the respective multiplicative subsets; then, since ψ, ψ' are injections, we must have that $ad = bc$. We can assume that a and c are relatively prime as are b and d and hence we see that a must divide b and b must divide a, and similarly for d and c, so $a = b$ and $c = d$. Hence $f = g$, i.e. the equivalence relation on the stalk becomes trivial and we are left with only a union $\cup_{\mathfrak{p}} \mathbb{Z}[\mathfrak{p}^{-1}]$ over all the primes of $\mathbb{Z}$ except for 2 and 3 (due to the nature of the opens containing $(\bar{2})$). But this yields a ring with two maximal ideals (2) and (3). Hence we see that the stalk $\mathcal{O}_{Y,(\bar{2})}$ is not local, and that $(Y, \mathcal{O}_Y)$ is a ringed space but not a locally ringed space.

Problem 6

Let $(X, \mathcal{O}_X)$ be the locally ringed space given by $X = \operatorname{Spec} \mathbb{Z}_{(2)}$ with $\mathcal{O}_X$ the constant sheaf determined by $\mathbb{Z}_{(2)}$ and $(Y, \mathcal{O}_Y)$ be the locally ringed space given by $\operatorname{Spec} \mathbb{C}[[t]]$ with $\mathcal{O}_Y$ the constant sheaf determined by $\mathbb{C}[[t]]$. Note that these are indeed locally ringed spaces as the stalks of these sheaves are local. We check this for $\mathbb{Z}_{(2)}$ and the result follows similarly for $\mathbb{C}[[t]]$. The spectrum of $\mathbb{Z}_{(2)}$ is given $\operatorname{Spec} \mathbb{Z}_{(2)} = \{(0), (2)\}$ where topologically the point (0) is open along with $\varnothing$ and $\operatorname{Spec} \mathbb{Z}_{(2)}$. The stalk $\mathcal{O}_{X,(2)}$ is hence just $\mathcal{O}_X(\operatorname{Spec} \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}$ (by connectedness). For $\mathcal{O}_{X,(0)}$, note that the two open sets containing (0) are (0) and $\operatorname{Spec} \mathbb{Z}_{(2)}$, but the ring above each of these is $\mathbb{Z}_{(2)}$ and hence so is the stalk at (0).

Now consider the map of rings $\psi : \mathbb{Z}_{(2)} \hookrightarrow \mathbb{C}[[t]]$ and the induced map on spectra $\phi : \operatorname{Spec} \mathbb{C}[[t]] \to \operatorname{Spec} \mathbb{Z}_{(2)}$. As ψ is simply the inclusion, ϕ takes $(0) \mapsto (0)$ and $(t) \mapsto (0)$, which is continuous. Now consider the morphism Φ of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ given by the continuous map ϕ defined as above and the morphism of sheaves $\phi^\# : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ taking $\mathcal{O}_Y(V) \mapsto \mathcal{O}_X(\phi^{-1} V)$ for every open V in Y. Taking direct limits, we obtain the induced map on stalks (at a point p) $\phi_p : \mathbb{Z}_{(2)} \to \mathbb{C}[[t]]$, which is clearly not a local homomorphism, as the inverse image of the maximal (t) yields (0) instead of (2).

Problem 7

Recall that a space is quasicompact if every open cover has a finite subcover. Let us show that $\operatorname{Spec} R$ is a quasi-compact topological space for any R. Let $\operatorname{Spec} R = \cup_i X_i$ be an open cover. As the basis for the topology of $\operatorname{Spec} R$ is given by opens of the form $D(f), f \in R$, it is clear that we may refine the cover X_i to some union of standard opens $\operatorname{Spec} R = \cup_\alpha D(f_\alpha)$ for some $f_\alpha \in R$. But $\cup_\alpha D(f_\alpha)$ covers $\operatorname{Spec} R$ if and only if no prime contains all the f_α (right-to-left is obvious, left-to-right follows from the fact that the points of $D(f_\alpha)$ are in correspondence with the primes of R_{f_α}) which occurs if and only if the f_α span the unit ideal (one of seeing this is via Zorn's lemma). Hence we see that the $D(f_\alpha)$ cover $\operatorname{Spec} R$ if and only the f_α span the unit ideal. But now we can simply choose the finite set of f_i that generate the element 1 and construct a finite cover of X given by the union of those specific X_{f_i}. Hence we see that $\operatorname{Spec} R$ is quasicompact. Of course, as every standard open is of the form $\operatorname{Spec} R_f$, we find that every standard open is in fact quasicompact as well.