Algebraic Topology I: PSET 3

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Problem 1

Recall we have a sequence of abelian groups,

$$\cdots \longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots$$

where i_* and j_* are the homomorphisms induced by the inclusions $(A, x_0) \to (X, x_0)$ and $j : (X, x_0, x_0) \to (X, A, x_0)$ and $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ is the boundary homomorphism. We show that this sequence is exact.

First we show that $\operatorname{im} \partial = \ker i_*$. Suppose we have a $(n-1)$-spheroid in $\operatorname{im} \partial \subset \pi_{n-1}(A, x_0)$, i.e. the bottom face of relative spheroid $I^{n-1} \times I \to (X, A, x_0)$. Note that this map can in fact be viewed as a homotopy of I^{n-1} sent to A with boundaries to x_0 to I^{n-1} sent to x_0, as the upper face of the relative spheroid is sent to x_0. Hence the $(n-1)$-spheroid is contractible to x_0 and its image in $\pi_{n-1}(X, x_0)$ is zero. Conversely, any $(n-1)$-spheroid in $\pi_{n-1}(A, x_0)$ sent to 0 in $\pi_{n-1}(X, x_0)$ yields a homotopy $I^{n-1} \times I$ which can be viewed as a relative spheroid in $\pi_n(X, A, x_0)$ whose lower face is sent to the $(n-1)$-spheroid.

Showing that $\operatorname{im} i_* = \ker j_*$ is a little easier. Suppose we have a n-spheroid in $\pi_n(X, x_0)$ in $\operatorname{im} i_*$. As it sits entirely inside A, j_* sends it to 0 in $\pi_n(X, A, x_0)$. Conversely, if a n-spheroid in $\pi_n(X, x_0)$ is sent to 0 under j_*, there is a homotopy contracting the spheroid to live entirely inside A, i.e. in $\operatorname{im} i_*$.

Finally we show that $\ker \partial = \operatorname{im} j_*$. Suppose we have a relative n-spheroid in $\operatorname{im} j_*$, i.e. a cube whose boundary is sent to x_0; the boundary map clearly sends this to 0 in $\pi_{n-1}(A, x_0)$. Conversely, given a relative n-spheroid in $\pi_n(X, A, x_0)$ sent to 0 by the boundary map, there must exist

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a homotopy of the bottom face $I^{n-1} \to A$ of the spheroid to $I^{n-1} \to \{x_0\}$. Viewing this homotopy as a cube, we can attach it to the bottom of the spheroid, thus obtaining an n spheroid whose boundary is now completely sent to x_0, i.e. an element in $\pi_n(X, x_0)$. Applying j_* to this element yields a relative spheroid homotopic to the one we started with.

Problem 2

Consider pullback q^*M of the Möbius bundle M over S^1 by the quotient map $q : I \to S^1$, i.e. the bundle making the diagram

$$\begin{tikzcd}
q^*M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\
I \arrow[r, "q"] & S^1.
\end{tikzcd}$$

The bundle, by virtue of local trivality, exists (c.f. FFG p.55); by Feldbau's theorem, however, E is trivial.

Problem 3

Consider the commutative diagram of abelian groups,

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\phi_1"] & A_2 \arrow[r] \arrow[d, "\phi_2"] & A_3 \arrow[r] \arrow[d, "\phi_3"] & A_4 \arrow[r] \arrow[d, "\phi_4"] & A_5 \arrow[d, "\phi_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

where ϕ_2 and ϕ_4 are isomorphisms, ϕ_1 is surjective, and ϕ_5 is injective.

We show first that ϕ_3 is surjective. Fix $b \in B_3$. It is mapped to $b_4 \in B_4$, and subsequently to $0 \in B_5$. Injectivity of ϕ_5 forces the preimage under ϕ_5 to be 0, while using the bijectivity of ϕ_4 we obtain a_4 mapping to $b_4 \in B_4$ as well as $0 \in A_5$ by the commutativity of the fourth square. Exactness at A_4 now yields an $a_3 \in A$ mapping to $a_4 \in A_4$. Commutativity of the third square shows that $b - \phi_3(a) \in \mathrm{im}(B_2 \to B_3)$. This yields $b_2 \in B_2$ mapping to $b - \phi_3(a)$ which corresponds isomorphically to $a_2 \in A_2$. Commutativity of the second square now requires that if a_2 maps to $a \in A_3$, then $\phi_3(a_3) = b - \phi_3(a)$. We thus find that $\phi_3(a + a_3) = b$, as desired.

Injectivity is a bit easier. Suppose we have an $a \in A_3$ such that $\phi_3(a) = 0 \in B_3$. We obtain 0 again via the map $B_3 \to B_4$ which corresponds isomorphically to $0 \in A_4$. Commutativity of the third square requires a to map to

0 in A_4 and hence the existence of $a_2 \in A_2$ mapping to $a \in A_3$. The element $\phi_2(a_2)$ is mapped to $0 \in B_3$ by the commutativity of the second square, and hence there exists $b_1 \in B_1$ mapping to $\phi_2(a_2)$. Surjectivity of ϕ_1 implies the existence of a_1 such that $\phi_1(a_1) = b_1$, and the commutativity of the first square requires that a_1 is mapped to $a_2 \in A_2$. Exactness then forces a_2 to be mapped to $0 \in A_3$. However, we showed earlier that a_2 is mapped to $a \in A_3$, and hence $a = 0$.

FFG p.65 Exercise 7

Suppose that A is a retract of X. Recall that a retract r satisfies $r \circ i = \mathrm{Id}_A$ and hence since $r_* \circ i_* = \mathrm{Id}$, we find that i_* must be injective. For $n \geq 2$, this fact together with the long exact sequence yields the short exact sequence

$$0 \longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \longrightarrow 0$$

Notice that given a spheroid $f \in \pi_n(A, x_0)$, we can compose $r_*(i_*(f)) = \mathrm{Id}(f)$. This provides, via the usual splitting lemma, a splitting $\pi_n(X, x_0) \cong \pi_n(A, x_0) \oplus \pi_n(X, A, x_0)$.

FFG p.65 Exercise 8

Suppose A is contractible in X to a point $x_0 \in A$. It is not immediately obvious that the inclusion $\iota : (A, x_0) \to (X, x_0)$ is trivial, as the basepoint may move during the contraction of A to x_0. However, this is remedied by composing with the change of basepoint isomorphism β_h, and thus $\iota_* = \beta_h \circ \mathrm{const}_*(x_0) = 0$. The long exact sequence now yields short exact sequences

$$0 \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0) \longrightarrow 0.$$

Note that given an element $f \in \pi_{n-1}(A, x_0)$, a $(n-1)$-spheroid mapping into A, we can construct a relative n-spheroid in $\pi_n(X, A, x_0)$ with "bottom face" given by f, sitting in A, which under the boundary homomorphism clearly maps back to f. Applying the splitting lemma to this section, we find that

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \oplus \pi_{n-1}(A, x_0).$$

FFG p.68 Exercise 1

Recall the Hopf fibration $S^3 \to S^2$ with fiber S^1. The long exact sequence associated to the fibration splits via the identity $\pi_n(S^1) = 0$ (for $n \geq 2$, found via a covering space argument) to yield

$$0 \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow 0,$$

giving isomorphisms $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$. For $n < 3$, we find the short exact sequence

$$0 \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) = \mathbb{Z} \longrightarrow 0$$

The group $\pi_2(S^3)$ is trivial by an application of cellular approximation: as S^3 contains no two cells, any map $S^2 \to S^3$ is homotopic to the constant map at the zero cell of S^3 - this homotopy fixes basepoint, as cellular approximation guarantees that the zero cell is sent to the zero cell. Hence $\pi_2(S^3) = 0$ and $\pi_2(S^2) = \mathbb{Z}$.

FFG p.68 Exercise 3

Suppose we are given a fibration $E \to B$ with fiber F with the base contractible. Then $\pi_n(B) = 0$ for all n and the long exact sequence of the fibration splits into

$$0 \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow 0,$$

which yields isomorphisms $\pi_n(F) \cong \pi_n(E)$ for all n.

Similarly, if the fiber is contractible then $\pi_n(F) = 0$ for all n and the long exact sequence splits into

$$0 \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow 0,$$

providing isomorphisms $\pi_n(E) \cong \pi_n(B)$ for all n (even for $n = 1$, as the sequence extends to the zeroth homotopy groups, which happen to be zero).

FFG p.68 Exercise 4

Suppose the homotopy groups of the base as well as those of the fiber are finite. Then the long exact sequence can be spliced to yield short exact sequences

$$0 \longrightarrow \ker p_* \longrightarrow \pi_n(E) \longrightarrow \operatorname{im} p_* \longrightarrow 0$$

where p is the projection $E \to B$. As $\ker p_* = \operatorname{im} i_*$, where i is the inclusion $F \to E$, we find that $\ker p_*$ is finite as it is the homomorphic image of a finite group, and $\operatorname{im} p_*$ is finite as it is a subgroup of $\pi_n(B)$, a finite group. In particular, as $\pi_n(E)/\ker p_* = \operatorname{im} p_*$, we find that $|E| = |\ker p_*||\operatorname{im} p_*|$, but $|\operatorname{im} p_*| \leq |\pi_n(B)|$ and $|\ker p_*| = |\operatorname{im} i_*| \leq |\pi_n(F)|$.

FFG p.68 Exercise 5

Suppose the homotopy groups of the base and fiber are finitely generated (for each n). Then, as in the previous problem, the long exact sequence yields short exact sequences

$$0 \longrightarrow \ker p_* \longrightarrow \pi_n(E) \longrightarrow \operatorname{im} p_* \longrightarrow 0.$$

For $n > 1$, we find that $\ker p_* = \operatorname{im} i_*$ and $\operatorname{im} p_*$ are finitely generated (as subgroups of abelian groups are finitely generated. Hence $\pi_n(E)$ is finitely generated, with generators the union of the generators of these two groups (inside $\pi_n(E)$). The bound on ranks follows similarly to that of the bound in the last problem. For $n = 1$, it suffices to show that $\operatorname{im} p_*$ is finitely generated: we note that it is a subgroup of finite index in $\pi_1(B)$ as long as F has a finite number of path-components, and hence in this case is finitely generated.