MODERN ALGEBRA II SPRING 2013: FOURTH PROBLEM SET

1. Let R be a ring with $R \neq \{0\}$. Show that R is a field $\iff$ every ideal of R is either $\{0\}$ or R. (Note: we have already showed $\implies$. To see $\impliedby$, let $r \in R$, $r \neq 0$. What can you say about the principal ideal (r)?)

2. Let F be a field and let $\rho\colon F \to R$ be a ring homomorphism. Show that either ρ is injective or $R = \{0\}$ and hence $\rho(a) = 0$ for all $a \in F$.

3. Define the *ideal sum* $I + J$ to be the subset $\{r + s : r \in I, s \in J\}$. In other words, $I + J$ contains all sums of two elements, one of which is in I and the other is in J. Show that $I + J$ is an ideal containing I and J, and in fact every ideal of R containing both I and J must contain $I + J$.

4. Define the *ideal product* $I \cdot J$ to be the subset
$$\left\{\sum_{i=1}^{n} r_i s_i : r_i \in I, s_i \in J\right\}.$$
In other words, $I + J$ contains all finite sums of products of two elements, the first of which is in I and the second of which is in J. (There is no condition on the number n of such products we add up; it could be arbitrarily large.) Show that $I \cdot J$ is an ideal contained in $I \cap J$. (As we shall see, it is not always the case that $I \cdot J = I \cap J$.)

5. In the ring $\mathbb{Z}$, show that $r \in (n)$ if and only if n divides r. Describe concretely the ideals $(n) + (m)$; $(n) \cap (m)$; $(n) \cdot (m)$. In particular, give an example of two ideals I and J in $\mathbb{Z}$ such that $I \cdot J \neq I \cap J$.

6. By Part (i) of Problem 6 from the last problem set, we know that, if R is a subring of S and J is an ideal in S, then $I = R \cap J$ is an ideal in R. Show that, in this case, there is an injective homomorphism f from R/I to S/J, defined by $f(r + I) = r + J$. (Define f first as a homomorphism from R to S/J and then determine its kernel and use the fundamental theorem for homomorphisms.) Show that f is surjective $\iff$ for every $s \in S$, there exists an $r \in R$ such that $s \equiv r \bmod J$, i.e. $s - r \in J$.

7. Let R be the subring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ of $\mathbb{C}$. Let $I = (2 + 3i)$ be the principal ideal in $\mathbb{Z}[i]$ generated by $2 + 3i$.

(i) Show that I contains $2+3i$ and $-3+2i$, and in fact that the additive subgroup $(I,+)$ of the group $(\mathbb{Z}[i],+)$ is generated by $2+3i$ and $-3+2i$.

(ii) Show that $i \equiv -5 \mod I$, i.e. that $i+5 \in I$.

(iii) Show that the homomorphism $f\colon \mathbb{Z} \to \mathbb{Z}[i]/I$ defined by $f(n) = n+I$ is surjective.

(iv) Show that $13 \in \mathbb{Z} \cap I$.

(v) Show that $\mathbb{Z} \cap I = 13\mathbb{Z}$. (This may take some calculation.)

(vi) Conclude by the previous problem that $\mathbb{Z}[i]/I \cong \mathbb{Z}/13\mathbb{Z}$ as rings. Is I a maximal ideal? A prime ideal?

(Note: for those who took Modern Algebra I last semester, $\mathbb{Z}[i] \cong \mathbb{Z}\times\mathbb{Z}$ as (additive) groups and the ideal I, viewed as an additive subgroup of $\mathbb{Z}\times\mathbb{Z}$, is the subgroup generated by $(2,3)$ and $(-3,2)$. A homework problem from last semester then shows that the additive group $\mathbb{Z}[i]/I$ is isomorphic to $\mathbb{Z}/(\det A)\mathbb{Z}$, where A is the matrix $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ and hence $\det A = 13$.)