Physics 6047
Problem Set 9, due 4/11/13
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1. Srednicki problem 27.1.

2. Srednicki problem 28.1.

3. Take a look at the action for the Goldstone boson χ in eq. 32.9. There is an interaction term which is proportional to $\rho^2(\partial\chi)^2$, where ρ is the radial field. Compute the one-loop correction for the χ propagator, coming from the loop in ρ, and show that this does not renormalize the mass of χ. In other words, we say that the masslessness of the Goldstone boson is protected from quantum corrections.

4. This problem functions more as notes, and is loosely based on Srednicki's problem 29.2, though much expanded. You are asked to think through the idea of the renormalization group via two different approaches: one is Wilson's with a momentum cut-off (chapter 29), the other is dim. reg. (chapter 28).

a. Let us consider our old friend the ϕ^3 theory, with mass set to zero for simplicity:

$$Z = \int_{|k|<\Lambda_0} D[\phi] e^{i\int d^dx\,\left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{3!}g\phi^3\right]} \tag{1}$$

where Λ_0 indicates the cut-off of the theory. As discussed in class, to define this cut-off in a rigorous manner, we Euclideanize i.e. set $it = t_E$, such that

$$Z = \int_{|k|<\Lambda_0} D[\phi] e^{\int d^dx\,\left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{3!}g\phi^3\right]} \tag{2}$$

where x now represents $(t_E, \vec{x})$. In Wilson's approach, we replace $\phi \to \phi + \hat{\phi}$ where $\hat{\phi}$ denotes the high momentum contribution $\Lambda < |k| < \Lambda_0$, and ϕ the low momentum contribution. Thus the path integral becomes

$$Z = \int_{|k|<\Lambda} D[\phi] e^{\int d^dx\,\left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{3!}g\phi^3\right]} \int_{\Lambda<|k|<\Lambda_0} D[\hat{\phi}] e^{\int d^dx\,\left[-\frac{1}{2}(\partial\hat{\phi})^2 + \frac{1}{3!}g(\hat{\phi}^3 + 3\hat{\phi}^2\phi + 3\hat{\phi}\phi^2)\right]} \tag{3}$$

Let us focus on the terms generated by $\hat{\phi}^2\phi$. Among other things, $\hat{\phi}^2\phi$ (or powers thereof) generate corrections to the kinetic term $(\partial\phi)^2$ and the interaction term ϕ^3 i.e.

$$Z = \int_{|k|<\Lambda} D[\phi] e^{\int d^dx\left[-\frac{1}{2}Z_\phi^W(\partial\phi)^2 + \frac{1}{3!}Z_g^W g\phi^3\right] + \ldots} \tag{4}$$

Show that, by explicitly expanding out the $\hat{\phi}$ path integral:

$$\begin{aligned} Z_\phi^W - 1 &= \frac{d}{dk^2}\left[-\frac{1}{2}g^2\int_\Lambda^{\Lambda_0}\frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2(\ell-k)^2}\right]_{k^2=0} \\ &= \frac{d}{dk^2}\left[-\frac{1}{2}g^2\int_0^1 dx\int_\Lambda^{\Lambda_0}\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2+D)^2}\right]_{k^2=0} = \frac{\alpha}{6}\ln\frac{\Lambda_0}{\Lambda} \end{aligned} \tag{5}$$

where $D \equiv x(1-x)k^2$ and $\alpha \equiv g^2/(4\pi)^3$, *and that*:

$$Z_g^W - 1 = g^2 \int_\Lambda^{\Lambda_0} \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^6} = \alpha \ln \frac{\Lambda_0}{\Lambda} \tag{6}$$

The utility of the effective action displayed in Eq. (4) is that loop contributions from high momentum modes have already be included in the coefficients Z_ϕ^W and Z_g^W. Thus, in computing scattering amplitude, the tree amplitude computed using this action already includes the physics of the high momentum loops. Let us write down the tree-amplitude of a scattering process involving 3 particles. There is one technical detail that needs to be addressed before writing this down i.e. when we say particle, what particle, or more precisely, what normalization for the particle? It is best to define $\phi' = Z_\phi^{W\,1/2}\phi$ such that the kinetic term has the canonical normalization $-\frac{1}{2}(\partial\phi')^2$. The particle of interest is then the ϕ' particle which has the standard normalization of $\langle k|\phi'(x)|0\rangle = e^{-ikx}$, and the usual LSZ formula can be used for scattering (which now involves correlation function of ϕ'). [1] *Show that* the 3 particle tree-scattering amplitude for the canonically normalized ϕ' is, according to Eq. (4):

$$\mathcal{M} = g Z_g^W Z_\phi^{W\,-3/2} = g(1 + \frac{3}{4}\alpha \ln \frac{\Lambda_0}{\Lambda}) \tag{7}$$

Now, recall that g is the coupling of our original Λ_0 theory (Eq. 1), thus we can think of g and α as functions of Λ_0. The scattering amplitude $\mathcal{M}$ is the tree amplitude of the Λ effective theory, and it is natural to call it $g(\Lambda)$. Thus, squaring and divide by $(4\pi)^3$, we find

$$\alpha(\Lambda) \equiv \frac{\mathcal{M}^2}{(4\pi)^3} = \alpha(\Lambda_0)\left(1 + \frac{3}{2}\alpha(\Lambda_0) \ln \frac{\Lambda_0}{\Lambda}\right) \tag{8}$$

Show that

$$\frac{d\alpha(\Lambda_0)}{d\ln\Lambda_0} = -\frac{3}{2}\alpha(\Lambda_0)^2 \quad \text{or} \quad \frac{d\alpha(\Lambda)}{d\ln\Lambda} = -\frac{3}{2}\alpha(\Lambda)^2 \tag{9}$$

either by demanding $\alpha(\Lambda)$ be independent of Λ_0, or by differentiating both sides with respect to Λ. The fact that the scattering amplitude $\mathcal{M}$ is independent of Λ_0 is the defining feature of renormalization, i.e. observable is independent of the far UV cut-off, whose physics we have already integrated out. One can of course go on to compute loop correction to $\mathcal{M}$ according to our Λ effective theory, and basically one would just repeat the above story with a further lower scale.

b. We have already shown in chapter 28 that $d\alpha/d\ln\mu$ obeys the same equation as (9). But the relation between the two might not be so transparent. Thus, here I ask you to rederive $d\alpha/d\ln\mu$ using the method of dimensional regularization, with the modified minimal subtraction scheme.

Consider the theory

$$S = \int d^dx \left[-\frac{1}{2}(\partial\phi)^2 + \frac{1}{3!}g\tilde{\mu}^{\epsilon/2}\phi^3 - \frac{1}{2}(Z_\phi^c - 1)(\partial\phi)^2 + \frac{1}{3!}(Z_g^c - 1)g\tilde{\mu}^{\epsilon/2}\phi^3\right] \tag{10}$$

where we think of the last two terms as counter-terms i.e. they contain the right poles to cancel the poles we would encounter in computing the 1-loop propagator and the 1-loop cubic vertex. *Notice Z_ϕ^W and Z_g^W thus play very different roles from Z_ϕ^c and Z_g^c – the*

[1] The alternative would be to multiply the LSZ formula (using correlation function of ϕ pretending ϕ were canonically normalized) by a factor of $Z_\phi^{W\,-1/2}$ for each particle (i.e. $Z_\phi^{W\,-1/2}$ plays the role of $R^{1/2}$ in problem set 9).

former encodes the physics of high momentum modes; the latter removes the poles from high momentum modes i.e. superscript W for Wilson, c for counter-terms.

Suppose we are interested in the scattering process involving 3 particles, computed to 1-loop, with all external momenta sent to zero. One needs to do this in two steps. First, we need to consider the 1-loop propagator, which will affect our 3-particle scattering amplitude by its effect on the particle normalization. You have already worked out the 1-loop correction to the propagator (with counter-term):

$$\Pi(k^2) = \frac{1}{2} g^2 \tilde{\mu}^\epsilon \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 + D)^2} - (Z_\phi^c - 1) k^2 \tag{11}$$

where ℓ is Euclideanized, and $D \equiv x(1-x)k^2$. *Show that* this gives

$$Z_\phi^f \equiv 1 - \Pi'(k^2 = 0) = 1 + \frac{\alpha}{12} \left(\frac{2}{\epsilon} + \ln \mu^2 \right) + (Z_\phi^c - 1) \tag{12}$$

which is what we call R in problem set 9. Here, $\mu^2 \equiv 4\pi \tilde{\mu}^2 / e^\gamma$. In deriving the above, you will encounter a term D^0 with D ultimately vanishing – I was being sloppy and simply set $D^0 = 1$, perhaps you can think of a better justification. Note that Z_ϕ^c the counter-term factor is what Srednicki calls Z_ϕ in chapter 28, and it is chosen to exactly remove the $1/\epsilon$ pole, i.e. $Z_\phi^c - 1 = -\alpha/(6\epsilon)$. In other words:

$$Z_\phi^f = 1 + \frac{\alpha}{6} \ln \mu \quad , \quad Z_\phi^c = 1 - \frac{\alpha}{6\epsilon} \tag{13}$$

We have learned that to compute the 3-particle-scattering amplitude, we are to multiply the standard LSZ formula (involving correlation function of ϕ) by $(Z_\phi^f)^{-3/2}$.

Next, we need to compute the 1-loop contribution to the cubic vertex (with counter term), at zero external momenta. *Show that* it is given by

$$Z_g^f - 1 \equiv \frac{V_3^{\rm loop}}{(g \tilde{\mu}^{\epsilon/2})} = g^2 \tilde{\mu}^\epsilon \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{\ell^6} + (Z_g^c - 1) = \frac{\alpha}{2} \left(\frac{2}{\epsilon} + \ln \mu^2 \right) + (Z_g^c - 1) \tag{14}$$

where ℓ is Euclideanized. Under the minimal subtraction scheme, Z_g^c is chosen to remove the $1/\epsilon$ pole i.e.

$$Z_g^f = 1 + \alpha \ln \mu \quad , \quad Z_g^c = 1 - \frac{\alpha}{\epsilon} \tag{15}$$

Putting everything together, *show that* the 3-particle-scattering amplitude, to 1-loop, is

$$\mathcal{M} = (Z_\phi^f)^{-3/2} Z_g^f g \tilde{\mu}^{\epsilon/2} = g(1 + \frac{3}{4} \alpha \ln \mu) \tag{16}$$

where we can safely put $\epsilon = 0$. We thus have

$$\frac{\mathcal{M}^2}{(4\pi)^3} = \alpha(1 + \frac{3}{2} \alpha \ln \mu) \tag{17}$$

It is easy to see that for this to be independent of μ. We have $d\alpha / d\ln \mu = -3\alpha^2/2$, consistent with the result of the Wilsonian approach, i.e. we are to think of g or α as a function of μ here, just like we think of g or α there as a function of Λ_0. You might be puzzled by the fact that we have logarithm of μ which is a dimensionful quantity. Presumably, if we were to compute the amplitude for something more physical i.e. non-zero external momenta, we will see that we have logarithm of μ divided by some combination of those momenta.

Note that in chapter 28, Srednicki derived this RG equation for α in a different way. Essentially, without thinking through what the one-loop scattering amplitude is, he suggested looking at

$$g_0 \equiv (Z_\phi^c)^{-3/2} Z_g^c g \tilde{\mu}^{\epsilon/2} = g\left(1 - \frac{3\alpha}{4\epsilon}\right)\left(1 + \frac{\epsilon}{2}\ln\tilde{\mu}\right) \tag{18}$$

and insisted g_0 be independent of μ. This looks kind of like $\mathcal{M}$ but is not the same: here, we are interested in only the counter-terms, and here it is important not to set $\epsilon = 0$. Amazingly, by rewriting this as $g_0^2/(4\pi)^3 = \alpha[1 - 3\alpha/(2\epsilon)][1 + \epsilon(\ln\mu + \text{const.})]$, and differentiating with respect to $\ln\mu$, one recovers $d\alpha/d\ln\mu = -\epsilon\alpha - 3\alpha^2/2$, where now one can safely set $\epsilon = 0$.