MODERN ALGEBRA II SPRING 2013: EIGHTH PROBLEM SET

1. Let F be a field and let n be a positive integer. Show that, if the characteristic of $F = p > 0$ and p divides n, then $x^n - 1$ has a multiple root in some extension field E of F. In fact, show that $x^n - 1$ is the p^{th} power of a polynomial. Show however that if the characteristic of $F = p$ and p does not divide n, or if the characteristic of F is 0, then the polynomial $x^n - 1$ does not have a multiple root in some extension field E of F.

2. (Derivatives via formal difference quotients). Let F be a field.

   (i) By direct computation, show that $\dfrac{y^n - x^n}{y - x}$ is a polynomial in x and y, i.e. is an element of $F[x, y]$.

   (ii) Let $f(x) \in F[x]$. Using (i), show that $\dfrac{f(y) - f(x)}{y - x}$ is a polynomial $Q_f(x, y)$ in x and y.

   (iii) Show that, for all $c \in F$ and $f, g \in F[x]$, $Q_{cf}(x, y) = cQ_f(x, y)$ and that $Q_{f+g}(x, y) = Q_f(x, y) + Q_g(x, y)$. By the usual trick of adding and subtracting an appropriate term, show that $Q_{fg}(x, y) = f(y)Q_g(x, y) + Q_f(x, y)g(x)$.

   (iv) Viewing $Q_f(x, y)$ as an element of $F[x, y]$, evaluate $Q_f(x, x)$ in case $f(x) = x^n$. Conclude that the formal derivative $Df(x)$, as we have defined it in class, is equal to $Q_f(x, x)$ for every $f(x) \in F[x]$. Use (iii) to prove the sum and product rules.

3. Let F be a field, and suppose that F is a subring of a ring R. Let $r \in R$. For $r \in R$, let $M_r : R \to R$ be multiplication by r, i.e. $M_r(s) = rs$. Show that M_r is an F-linear map from R to R. Finally, show (i) M_r is injective $\iff$ r is not a zero divisor, and (ii) M_r is surjective $\iff$ r is a unit $\iff$ M_r is an isomorphism (of F-vector spaces).

4. Consider the field $\mathbb{Q}(\sqrt{2})$, viewed as a vector space of dimension 2 over $\mathbb{Q}$. Let $r + s\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, and define the multiplication map $M_{r+s\sqrt{2}} : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ as in the previous problem by

   $$M_{r+s\sqrt{2}}(\alpha) = (r + s\sqrt{2}) \cdot \alpha.$$

   In other words, $M_{r+s\sqrt{2}}$ is multiplication by $r + s\sqrt{2}$.

(i) Referring to the handout on linear algebra if necessary, in terms of the basis $\{1, \sqrt{2}\}$ of $\mathbb{Q}(\sqrt{2})$, express the $\mathbb{Q}$-linear map $M_{r+s\sqrt{2}}$ as a 2×2 matrix with entries in $\mathbb{Q}$.

(ii) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q})$ (i.e. A is a 2×2 matrix with entries in $\mathbb{Q}$), what is the condition that $A = M_{r+s\sqrt{2}}$ (with respect to the basis $\{1, \sqrt{2}\}$) for some $r, s \in \mathbb{Q}$?

(iii) What is $\det M_{r+s\sqrt{2}}$? Argue without any computation that

$$\det M_{r+s\sqrt{2}} \neq 0$$

as long as $r + s\sqrt{2} \neq 0$.

(iv) Calculate the inverse matrix $M^{-1}_{r+s\sqrt{2}}$, and show that it is of the form $M_{t+u\sqrt{2}}$ for some $t, u \in \mathbb{Q}$. Use this to give an explicit formula for $(r + s\sqrt{2})^{-1}$.

5. Consider the field $\mathbb{Q}(\sqrt[3]{2})$, viewed as a vector space of dimension 3 over $\mathbb{Q}$. Let $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \in \mathbb{Q}(\sqrt{2})$, and define the multiplication map $M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2} \colon \mathbb{Q}(\sqrt[3]{2}) \to \mathbb{Q}(\sqrt[3]{2})$ as in the problems above by

$$M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}(\alpha) = (a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2) \cdot \alpha.$$

In other words, $M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}$ is multiplication by $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$.

(i) Using the basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$ of $\mathbb{Q}(\sqrt[3]{2})$, express the $\mathbb{Q}$-linear map $M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}$ as a 3×3 matrix with entries in $\mathbb{Q}$.

(ii) If $A \in M_3(\mathbb{Q})$ (i.e. A is a 3×3 matrix with entries in $\mathbb{Q}$), what is the condition that $A = M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}$ (with respect to the basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$) for some $a, b, c \in \mathbb{Q}$?

(iii) What is $\det M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}$? Argue without any computation that

$$\det M_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2} \neq 0$$

as long as $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \neq 0$.

(iv) (If you know Cramer's rule for the inverse of a 3×3 matrix.) Calculate the inverse matrix $M^{-1}_{a+b\sqrt[3]{2}+c(\sqrt[3]{2})^2}$, and show that it is of the form $M_{d+e\sqrt[3]{2}+f(\sqrt[3]{2})^2}$ for some $d, e, f \in \mathbb{Q}$. Use this to give an explicit formula for $(a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2)^{-1}$.