Analysis I: Solutions to PSET 4

Problem 1

Suppose for the sake of contradiction that $E \cup F$ is disconnected, i.e. there exist opens $A, B \subset M$ such that $A \cup B = E \cup F$ and $A \cap B = \varnothing$. Then $(A \cap E)$ and $(B \cap E)$ provide a separation for E. Since E is connected, we find, say, that $A \cap E = E$ and $B \cap E = \varnothing$. Since B is nonempty, $B \cap F \neq \varnothing$, but since $(A \cap F)$ and $(B \cap F)$ provide a separation for F, we find that $B \cap F = F$ and $A \cap F = \varnothing$. But note now that since $E \cap F$ is nonempty, any element in $E \cap F$ is neither in A nor in B and hence not in $A \cup B$. But this contradicts that $A \cup B = E \cup F$, whence such a separation cannot exist.

Rudin 2.16

We show that E^c is open in $\mathbb{Q}$, i.e. that it can be written as the intersection of $\mathbb{Q}$ and an open in $\mathbb{R}$. Since $\sqrt{2}$ and $\sqrt{3}$ are irrational, we find that

$$E^c = \{p \in \mathbb{Q} \mid p^2 < 2 \text{ or } p^2 > 3\}.$$

But clearly

$$\begin{aligned} E^c &= \mathbb{Q} \cap \{x \in \mathbb{R} \mid x^2 < 2 \text{ or } x^2 > 3\} \\ &= \mathbb{Q} \cap (-\infty, -\sqrt{3}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{3}, \infty), \end{aligned}$$

and hence E^c is open. This proves that E is closed in $\mathbb{Q}$. Moreover, E is obviously bounded by, say -2 below and 2 above. However, E cannot be compact in $\mathbb{Q}$, as otherwise Rudin theorem 2.33 would imply that E is compact in $\mathbb{R}$, but E is clearly open in $\mathbb{R}$ as $E = \mathbb{Q} \cap \{x \in \mathbb{R} \mid 2 < x^2 < 3\}$ is an intersection of two opens. Finally, note that E is open in $\mathbb{Q}$ as, again, it is the intersection of two opens.

Problem 3

Since K is compact, it must be closed and bounded. Rudin's theorem 2.28 implies that $\sup K \in K$. The analogous argument holds for $\inf K$: let

$y = \inf K$. If $y \notin K$ then for every $h > 0$ there exists a point $x \in K$ such that $y < x < y + h$ otherwise $y + h$ would be a lower bound of K. This implies that y is a limit point of K. Since K is closed, $y \in K$.

Problem 4

We present two proofs. The first proof is as follows. Let $\{(a_n, b_n)\} \subset A \times B$ be any infinite sequence. The sequence $\{a_n\} \subset A$ is an infinite subsequence of A and hence by compactness of A we can find a convergent subsequence $a_{n_k} \to a_0$. Now consider the sequence $\{b_{n_k}\} \subset B$; the compactness of B implies the existence of a convergent subsequence $b_{n_{k_j}} \to b_0$. Now it is straightforward to see that the subsequence $\{(a_{n_{k_j}}, b_{n_{k_j}})\}$ of our original sequence must converge to (a_0, b_0). Hence $A \times B$ is compact in $M \times N$.

The second proof is as follows. Let $\{U_a\}_{\alpha \in I}$ be any open cover of $A \times B$. Every point $(a, b) \in A \times B$ is contained in some open U_α and hence we can find a $\delta(a, b) > 0$ such that $B_\delta(a) \times B_\delta(b) \subset U_\alpha$. We will use the compactness of A and B to eliminate all but finitely many of these.

Fix $b_0 \in B$ and consider the union $\cup_{a \in A \times \{b_0\}} B_\delta(a)$ with δ as above, which forms an open cover of A. Compactness yields a finite subcover $\{B_{\delta_1}(a_1), \ldots, B_{\delta_k}(a_k)\}$ of $A \times \{b_0\}$; denote this set by V_{b_0}. Repeating this process to obtain V_b for all $b \in B$, we obtain open covers of $A \times \{b\}$. For each of these open covers V_b, denote by $\varepsilon_b = \min(\delta_1, \ldots, \delta_k)$.

Now, the union $\cup_{b \in B} B_{\varepsilon_b}(b)$ is an open cover of $\{a\} \times B$ for each a, and compactness of B yields a finite subcover $\{B_{\varepsilon_{b_0}}, \ldots, B_{\varepsilon_{b_n}}(b_n)\}$. Then we find that $\{B_{\varepsilon_i}(a_i) \times B_{\varepsilon_{b_j}}(b_j)\}_{i,j=1,\cdots,n}$ is a finite open cover of $A \times B$. By our choice of ε_i each of these opens is contained in an open U_α of our given open cover. Hence we find a finite subcover of $\{U_\alpha\}$ by choosing only such U_α.