Lie Groups PSET 4

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Problem 1

Let $V = \mathbb{C}^2$ be the standard two-dimensional representation of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ and let $\operatorname{Sym}^k V$ be the symmetric power of V.

1. Let us write explicitly the action of $e, f, h \in \mathfrak{sl}_2\mathbb{C}$ on $\operatorname{Sym}^k V$ in the natural basis $e_1^i \cdot e_2^{k-i}$. Note first that we can write

$$h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, f = \begin{pmatrix} & \\ 1 & \end{pmatrix},$$

and hence the action of $\mathfrak{sl}_2\mathbb{C}$ on V, given by matrix multiplication, is simply as follows:

$$\begin{aligned} he_1 = e_1, he_2 = -e_2 \\ ee_1 = 0, ee_2 = e_1 \\ fe_1 = e_2, fe_2 = 0. \end{aligned}$$

Recall that Lie algebra actions act as derivations on tensor product representations. Of course, the kth symmetric product of V is simply a symmetrized tensor product and it is easy to see that Lie algebra actions act on derivations across this symmetric product as well. In particular, for $\operatorname{Sym}^2 V$:

$$\begin{aligned} X(a \cdot b) &= \frac{1}{2} X(a \otimes b + b \otimes a) \\ &= \frac{1}{2}(Xa \otimes b + a \otimes Xb + Xb \otimes a + b \otimes Xa) \\ &= \frac{1}{2}(Xa \otimes b + b \otimes Xa) + \frac{1}{2}(a \otimes Xb + Xb \otimes a) \\ &= Xa \cdot b + a \cdot Xb. \end{aligned}$$

This clearly extends to $\operatorname{Sym}^k V$ by induction. Using this fact, we can now compute the action of $\mathfrak{sl}_2\mathbb{C}$ on $\operatorname{Sym}^k V$ (supressing the symmetric product $\cdot$ for clarity):

$$\begin{aligned} h(e_1^i e_2^{k-i}) &= (2i - k) e_1^i e_2^{k-i} \\ e(e_1^i e_2^{k-i}) &= (k - i) e_1^{i+1} e_2^{k-i-1} \\ f(e^i e_2^{k-i}) &= i e_1^{i-1} e_2^{k-i+1}. \end{aligned}$$

These follow simply by induction on the action of $\mathfrak{sl}_2\mathbb{C}$ on V together with the derivation property.

2. Consider in particular the case of $k = 2$, i.e. $W = \mathrm{Sym}^2 \mathbb{C}^2$ as a $\mathfrak{sl}_2\mathbb{C}$ representation. We claim that W is in fact isomorphic to the adjoint representation A of $\mathfrak{sl}_2\mathbb{C}$. Typically one would look for an $\mathfrak{sl}_2\mathbb{C}$-equivariant linear isomorphism, from W to A, but in this case, we can explicitly write out the actions of $\mathfrak{sl}_2\mathbb{C}$ on the respective bases and show that they are identical. In particular, we find (using the rules derived above for W and using the commutation relations for A)

$$\begin{aligned} h(e_1^2) &= 2e_1^2 \\ h(e_1e_2) &= 0 \\ h(e_2^2) &= -2e_2^2 \\ e(e_1^2) &= 0 \\ e(e_1e_2) &= e_1^2 \\ e(e_2^2) &= 2e_1e_2 \\ f(e_1^2) &= 2e_1e_2 \\ f(e_1e_2) &= e_2^2 \\ f(e_2^2) &= 0 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} [h,e] &= 2e \\ [h,h] &= 0 \\ [h,f] &= -2f \\ [e,e] &= 0 \\ [e,h] &= -2e \\ [e,f] &= h \\ [f,e] &= -h \\ [f,h] &= -2f \\ [f,f] &= 0 \end{aligned}$$

Up to scaling h and f by constant factors of 2 and -1, we see that these representations are identical.

3. By the results of Kirillov problem 4.1, we see that representations of $\mathrm{SO}_3\mathbb{R}$ are precisely those representations of $\mathfrak{sl}_2\mathbb{C}$ satisfying $e^{\pi i \rho(h)} = \mathrm{id}$. Using our result above for the way $\rho(h)$ looks, we see that we obtain the identity if and only if k is even; hence the representation lifts only for k even.

Problem 2

We wish to show that $\Lambda^n\mathbb{C}^n \cong \mathbb{C}$ as a representation of $\mathfrak{sl}_n\mathbb{C}$. Note first that $\mathbb{C}$ is the trivial representation of $\mathfrak{sl}_n\mathbb{C}$ and has one basis vector, i.e. the basis of the Lie algebra acts as zero. Let us show that the same happens for $\Lambda^n\mathbb{C}^n$. Note first that the nth exterior power of an n-dimensional vector space is one-dimensional as well, as there is only one linearly independent n-form, i.e. $\omega = e_1 \wedge \cdots \wedge e_n$. If we write out e_i as the usual basis column vectors, we see that the matrix multiplication action of every off-diagonal basis element on ω is zero. This is simply because the off-diagonal interaction converts an e_i into an e_j for $i \neq j$, and hence the wedge product yields zero (using the derivation property). Similarly, the action of any of the diagonal Lie algebra basis elements, h_i, contains both a 1 and a -1, and hence when acted on the wedge product, produces two terms that cancel each other out. Thus we see that these two representations are equivalent. More generally, we can note that the action of the Lie algebra brings yields the trace of the chosen operator, but the Lie algebra consists of traceless matrices.

The same reasoning does not hold for $\mathfrak{gl}_n\mathbb{C}$, as the Lie algebra is the space of all $n \times n$ matrices, and hence there are basis elements that will act non-trivially on ω, i.e. basis matrices with a single 1 somewhere on the diagonal. Indeed, the trace is not necessarily zero.

Problem 3

If we let G be the group of symmetries of the cube, it will act on functions on the cube $f \in V$ as

$$(gf)(\sigma) = f(g^{-1}\sigma).$$

We wish to show that the action of G commutes with the action of A. First note that if we denote by g_1, g_2, g_3 the rotations by $\pi/2$ we may write the action of A as an average over actions by different generators (and inverses); we must be careful to note, however that the precise expression for A depends on where the function is being evaluated. For example, we might show

$$((A \circ g_1) f)(\sigma) = ((g_1 \circ A) f)(\sigma).$$

More explicitly, we can start with the left-hand side:

$$((A \circ g_1) f)(\sigma) = Af(g_1^{-1}\sigma) = \frac{1}{4}\left(f(\sigma) + f(g_1^{-1}g_1^{-1}\sigma) + f(g_3 g_1^{-1}\sigma) + f(g_3^{-1}g_1^{-1}\sigma)\right)$$

whereas the right-hand side yields:

$$\begin{aligned} ((g_1 \circ A) f)(\sigma) &= \frac{g_1}{4}\left(f(g_1\sigma) + f(g_1^{-1}\sigma) + f(g_2\sigma) + f(g_2^{-1}\sigma)\right) \\ &= \frac{1}{4}\left(f(\sigma) + f(g_1^{-1}g_1^{-1}\sigma) + f(g_1^{-1}g_2\sigma) + f(g_1^{-1}g_2^{-1}\sigma)\right) \end{aligned}$$

These two are in fact equivalent, as the second two terms on the left yield the same cube faces as do the second two terms on the right. One can show this similarly for other combinations (not just for g_1, as well as for inversion z, which is not generated by g_i).

Let $z = -I \in G$; it's clear that z swaps all opposing faces. Writing any $f \in V$ in terms of its values as (a, b, c, d, e, f), we can symmetrize and antisymmetrize:

$$(a, b, c, d, e, f) = \left(\frac{a+b}{2}, \frac{a+b}{2}, \ldots\right) + \left(\frac{a-b}{2}, -\frac{a-b}{2}, \ldots\right).$$

This clearly yields a direct sum decomposition, as this is unique.

Let us now show that V_+ can be decomposed into a direct sum $V_+ = V_+^0 \oplus V_+^1$ where

$$V_+^0 = \left\{ f \in V_+ \mid \sum_\sigma f(\sigma) = 0 \right\}, \qquad V_+^1 = \mathbb{C} \cdot 1.$$

We may write, using symmetry, $f \in V_+$ as three components $f = (a, b, c)$. This three-dimensional space can be decomposed into those above by noting that

$$(a, b, c) = (x, y, -x-y) + (t, t, t)$$

if $t = (a+b+c)/3$. Furthermore, these subspaces are invariant under the action of G, as if $\sum f(\sigma) = 0$, clearly $\sum f(g^{-1}\sigma) = 0$ as well, and if the function is constant, then rotation has no effect.

The eigenvalue of A on V_+^1 is clearly 1, as if the function is constant, averaging the neighbors will simply yield back that constant. On the other hand, the eigenvalue of A on V_- is zero, due to the complete antisymmetry. Finally, the eigenvalue on V_+ is $-1/2$, because

$$A(x, y, -x-y)|_x = \frac{1}{4}\left(2y + 2(-x-y)\right) = -\frac{x}{2}.$$

Problem 4

Recall that we have a basis for $\mathfrak{so}_3\mathbb{R}$ given by $\{J_x, J_y, J_z\}$. Using the isomorphism $\mathfrak{su}_2 \cong \mathfrak{so}_3\mathbb{R}$, we deduce that the complexification

$$\mathfrak{so}_3\mathbb{C} \cong \mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}.$$

More explicitly, we can define the isomorphism $\psi : \mathfrak{so}_3\mathbb{C} \to \mathfrak{sl}_2\mathbb{C}$ as

$$\begin{aligned} J_x &\mapsto -\frac{1}{2}(e+f) \\ J_y &\mapsto \frac{1}{2}(f-e) \\ J_z &\mapsto -\frac{i}{2}h \end{aligned}$$

where $\{e, f, h\}$ is the usual basis for $\mathfrak{sl}_2\mathbb{C}$. It is easy to check that the Lie bracket is preserved. Now consider a finite-dimensional complex representation V of $\mathfrak{so}_3\mathbb{R}$. By Kirillov 4.4, V can be viewed uniquely as a representation of $\mathfrak{so}_3\mathbb{C}$ as well; let $\pi : \mathfrak{so}_3\mathbb{C} \to \mathfrak{gl}V$ denote this representation. Now the composition $\pi \circ \psi^{-1}$ gives V the structure of an $\mathfrak{sl}_2\mathbb{C}$ representation. Applying Kirillov's theorem 4.60, we find a weight decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V[n].$$

To compute the weights of each of these weight subspaces note that $(\pi \circ \psi^{-1}(h))v = \pi(2iJ_z)v$, which must be nv for all $v \in V[n]$ and hence we see that every finite-dimensional representation of $\mathfrak{so}_3\mathbb{R}$ admits such a weight decomposition where

$$V[n] = \left\{ v \in V \mid J_z v = \frac{in}{2} v \right\}.$$

Again by Kirillov problem 4.1, we know that representations of $\mathfrak{sl}_2\mathbb{C}$ lift to representations of $\mathrm{SO}_3\mathbb{R}$ if and only if $e^{\pi i \pi(\psi^{-1}((h))} = \mathrm{Id}$. In our case, this reduces to $e^{\pi i \pi(2iJ_z)}$. But since J_z acts as $in/2$ on each subspace, it's clear the the exponential yields Id if and only if each of the weights is even.

Problem 5

Let G be a finite group, and H a subgroup. Let (ρ, W) be a representation of H and $\mathrm{Ind}_H^G W$ be the induced representation. We wish to prove Frobenius reciprocity

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) = \mathrm{Hom}_H(\mathrm{Res}_H^G V, W).$$

where V is an arbitrary representation of G and $\mathrm{Res}_H^G V$ is the restriction of V to a representation of H. Note that for finite groups, induction and coinduction are identical, and hence we may prove

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G W, V) = \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$$

instead. Recall first that by the definition of induction, we may write

$$\mathrm{Ind}_H^G W = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

Given a $\phi \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G V)$, let us define the associated $\tilde{\phi} \in \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V)$ on each $\sigma \cdot W$ as

$$\sigma \cdot W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\phi} \mathrm{Res}_H^G V \xrightarrow{g_\sigma} V$$

for some representative $g_\sigma \in g_\sigma H$.

We must check that $\tilde{\phi}$ is indeed G-equivariant and independent of g_σ and that the map $\phi \mapsto \tilde{\phi}$ is bijective. Independence of representative follows simply because ϕ is H-linear. To show equivariance, it suffices to show that $\tilde{\phi}(gg_\sigma \cdot w) = g\tilde{\phi}(g_\sigma \cdot w)$. First write $gg_\sigma = g_\tau h$ for $g_\tau \in G, h \in H$. Note that right-hand side can be rewritten $gg_\sigma\phi(w)$ simply by definition of $\tilde{\phi}$. The left-hand side, meanwhile, obeys:

$$\tilde{\phi}\left((g_\tau h) \cdot w\right) = \tilde{\phi}(g_\tau \cdot w') = g_\tau \phi(w') = g_\tau h \phi(w)$$

using the definitions and the H-equivariance of ϕ. But this is precisely the right-hand side, and we are done. Injectivity follows from noting that if there exists another $\tilde{\psi}$ that is associated to ϕ then it must agree with ϕ when evaluated on W, call the restriction ψ. But then we may extend ψ using the map above, and we will simply get back $\tilde{\phi}$. This also shows surjectivity, as and $\tilde{\psi}$ can be restricted and then said restriction will be sent to $\tilde{\psi}$.