An Introduction to Automorphisms of $\mathbb{P}^n$

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Recall the definition of projective space $\mathbb{P}^n$ from last week.

Definition 1. We denote by $\mathbb{P}^n$ the space of lines passing through the origin of $\mathbb{A}^{n+1}$. More precisely, let $\mathbb{A}^*$ act on $\mathbb{A}^{n+1} - \{0\}$ by scaling as $\lambda \cdot (x_0, \cdots, x_n) = (\lambda x_0, \cdots, \lambda x_n)$ and define **projective n-space** to be the quotient $\mathbb{P}^n = \mathbb{A}^{n+1} - \{0\}/\mathbb{A}^*$ by this action.

Thinking of projective space as parametrizing a set of lines can be confusing at times, as it somewhat obscures the fact that $\mathbb{P}^n$ is simply $\mathbb{A}^n$ with extra "stuff" added at infinity. Let us thus think of projective space in terms of its coordinates.

Definition 2. Consider a point $p \in \mathbb{P}^n$. Treating $\mathbb{P}^n$ as a quotient space, we can think of p as the equivalence class of points on a line through $\mathbb{A}^{n+1}$. Suppose that this line passes through the point $\vec{x} = (x_0, \ldots, x_n) \in \mathbb{A}^{n+1}$. We write **homogeneous coordinates** for p as

$$p = [x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n],$$

for any $\lambda \in \mathbb{A}^*$. Note that these coordinates respect the action of $\mathbb{A}^*$ defining $\mathbb{P}^n$ and hence are well-defined. Moreover, not all x_i can be zero.

With this coordinate system in hand, let us try to build an intuitive picture of projective space in the next few examples.

Example 1 (Real projective line)**.** Let $\mathbb{A} = \mathbb{R}$ and consider $\mathbb{P}^1_{\mathbb{R}}$, the space of lines through the origin of $\mathbb{R}^2$. It is clear that we can parametrize this space by the slopes of the lines everywhere except for the vertical line. This yields an $\mathbb{R}$'s worth of points, giving us the real line. If we now take the slope of the vertical line to be "infinity," we obtain the whole projective space $\mathbb{P}^1_{\mathbb{R}}$. This is formalized by the homogeneous coordinates defined on $\mathbb{P}^1_{\mathbb{R}}$.

Consider a point $[x : y] \in \mathbb{P}^1_{\mathbb{R}}$. Suppose $y \neq 0$. Then, letting $\lambda = y^{-1}$,

$$[x : y] = y^{-1} \cdot [x, y] = [x/y : 1].$$

If we now note that x is free to range over all values in $\mathbb{R}$, we recover a copy of $\mathbb{R}$ embedding in $\mathbb{P}^1_{\mathbb{R}}$. We can think of this as the set of lines with finite slopes. If $y = 0$, on the other hand, we can write

$$[x : 0] = x^{-1} \cdot [x : 0] = [1 : 0].$$

This implies that there is only one point in $\mathbb{P}^1_{\mathbb{R}}$ with $y = 0$, which can be thought of as the "point at infinity." It is important to note that the point at infinity is fundamentally no "different" than the other points in $\mathbb{P}^1_{\mathbb{R}}$ – a concept that will become clear when we show that automorphisms of projective space can swap points at infinity with, say, the origin (in this case $[0 : 1]$). In this sense, we can think of the projective line as a disjoint union or a "compactification" $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (with the topology extended to the quotient topology of $\mathbb{R}^2/\mathbb{R}^*$).

Example 2 (Real projective plane)**.** Let $\mathbb{A} = \mathbb{R}$ and consider $\mathbb{P}^2_{\mathbb{R}}$, the space of lines through the origin of $\mathbb{R}^3$. Using homogeneous coordinates, take a point $[x : y : z] \in \mathbb{P}^2_{\mathbb{R}}$.