Representation Theory PSET 4

Nilay Kumar

Last updated: April 28, 2014

Exercise 1. *What is the Poincaré series of a free commutative algebra with generators of degrees* $d_1, \ldots, d_n$*? What if the generators anticommute?*

Solution. Note that if A and B are two graded k-algebras, then clearly $p_{A\otimes_k B}(t) = p_A(t)p_B(t)$. Hence we first compute the Poincaré series of $k[x]$ where x has degree d. We can write

$$p_{k[x]}(t) = t^0 + t^d + t^{2d} + \cdots = \frac{1}{1-t^d}.$$

Therefore we find that the Poincaré series of $k[x_1, \ldots, x_n]$, where x_i has degree d_i, is given by the product

$$p(t) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}.$$

Now suppose, on the other hand, that the generators anticommute. Returning to the case of $k[x]$, we find that the Poincaré series is truncated to $t^0 + t^d$. Taking a tensor product, we find that

$$p(t) = \prod_{i=1}^{n} (t^0 + t^{d_i}).$$

□

Exercise 2. *Let* $G \subset GL(\mathbb{C}^n)$ *be a finite group and* $A = \mathbb{C}[x_1, \ldots, x_n]^G$ *its algebra of invariants graded by the usual degree. Then*

$$p_A(t) = \frac{1}{|G|} \sum_{g\in G} \det(1-tg)^{-1}.$$

Proof. The algebra of invariants is simply the invariant subalgebra $(S^\bullet V)^G$ of the symmetric algebra. Consider the averaging operator $T = \sum_{g\in G} g/|G|$, which projects to the invariant subalgebra. Therefore, the trace $\operatorname{tr} T$ yields the dimension of the invariant subalgebra. Hence we find that

$$\dim(S^d V)^G = \operatorname{tr}_{S^d V} T = \frac{1}{|G|} \sum_{g\in G} \operatorname{tr}_{S^d V} g = \frac{1}{|G|} \sum_{g\in G} \chi_{S^d V}(g).$$

The Poincaré series is then

$$p_A(t) = \frac{1}{|G|} \sum_{g\in G} \sum_{d} t^d \chi_{S^d V}(g).$$

To simplify this expression, suppose $\dim V = n$ and for a given g, $\gamma_1, \ldots, \gamma_n$ are the eigenvalues of g. Then

$$\chi_{S^d V}(g) = \sum_{k_1+\cdots+k_n=d} \gamma_1^{k_1} \cdots \gamma_n^{k_n},$$

which allows us to write

$$\begin{aligned}\sum_d t^d \chi_{S^d V}(g) &= \sum_d \sum_{k_1+\cdots+k_n=d} t^d \gamma_1^{k_1} \cdots \gamma_n^{k_n} \\ &= \prod_{j=1}^n \sum_{k_j} (\gamma_j t)^{k_j} \\ &= \prod_{j=1}^n \frac{1}{1-\gamma_j t} \\ &= \frac{1}{\det(1-gt)}.\end{aligned}$$

$\square$