Introduction to Algebraic Topology: Class Notes

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1 Class 1

Remark. The goal of algebraic geometry is to develop algebraic tools to study topological spaces. To this end, all maps are assumed to be continuous unless otherwise stated.

1.1 Homotopy

Definition 1. A **deformation retract** of a space X onto a subspace A is a family of maps $f_t : X \to X$ for $t \in I = [0, 1]$ such that

1. $f_0 = \mathrm{Id}_X$
2. $f_1(X) = A$
3. $f_t|_A = \mathrm{Id}_A$ for all t
4. f_t is continuous as a map $X \times I \to X$ sending $(x, t) \mapsto f_t(x)$.

Example 1. Consider the annulus $X = \{x \in \mathbb{C} \mid 1/2 \leq |x| \leq 3/2\}$ with the subspace $A = \{x \in \mathbb{C} \mid |x| = 1\}$. We can construct a deformation retract of X onto A as

$$f_t(x) = \frac{x}{1 - t + t|x|}.$$

Definition 2. Given $f : X \to Y$, we define the **mapping cylinder** M_f to be the quotient space

$$(X \times I) \sqcup Y / \sim$$

where $(x, 1) \sim f(x)$. Note that M_f deformation retracts to Y by sliding each point (x, t) along $\{x\} \times I \subset M_f$ to $\{x\} \times \{1\} = f(x) \in Y$.

Example 2. If $f : X \to Y$ is an inclusion, say of the circle into the plane, then the mapping circle is simply a cylinder attached to the plane.

Definition 3. A **homotopy** is a family of maps $f_t : X \to Y$ for $t \in I$ such that $F : X \times I \to Y$ given by $F(x, y) = f_t(x)$ is continuous. We say that two maps f_0 and f_1 from $X \to Y$ are **homotopic** if there exists a homotopy f_t connecting them. We write this as $f_0 \simeq f_1$.

Definition 4. A **retraction** of X onto A is a map $r : X \to X$ such that

1. $r(X) = A$
2. $r|_A = \mathrm{Id}_A$

In this case, A is called a **retract** of X. Note that $r^2 = r$, as r is the identity on its image.

does this character-ize r?

Example 3. A deformation retract is a homotopy between the identity map to a retraction.

Example 4. Let X be any space with $x_0 \in X$. Consider the map $f : X \to X$ given by $x \mapsto x_0$. This is clearly a retract.

One might ask when there exists a deformation retract $X \to \{x_0\}$. For example, given an annulus and some x_0 inside, does there exist a deformation retract? The answer is in fact no, and later we develop some tools to show this. Similarly for the disjoint union of two disks, with x_0 contained in one of them.

Definition 5. Let $A \subset X$. A homotopy $f_t : X \to Y$ such that $f_t|_A$ is independent of t. Then we say that f_t is a **homotopy relative to** A.

Example 5. For example, a deformation retract of X onto A is a homotopy rel A.

Example 6. Consider the unit disk $X = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with $f_t(z) = ze^{2\pi i t}$. In this case, f_t is a homotopy rel O. Note that this is a homotopy from the identity to itself.

Remark. Let $f_t : X \to X$ be a deformation retract of X onto A. Let $r = f_1$ be the resulting retract and let $i : A \to X$ be the inclusion. Then $r \circ i = \mathrm{Id}_A$ and $i \circ r \simeq \mathrm{Id}_X$ (via the homotopy f_t). In this sense r and i are inverses up to homotopy.

Definition 6. A map $f : X \to Y$ is a **homotopy equivalence** if there exists a $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. Then we say that X and Y are **homotopy equivalent** and that they have the same **homotopy type** written $X \simeq Y$.

Definition 7. A space that is homotopy equivalent to a point is called **contractible**. This is equivalent to asking that the identity map be homotopic to a constant map, i.e. **nullhomotopic**.

Example 7. The spaces $\mathbb{R}^n$ and D^n are contractible, while S^n is not.

Remark. It will be useful to think of the sphere as $S^n = D^n/\partial D^n$.

1.2 Operations on spaces

We are of course already familiar with products and quotients.

Definition 8. Given a space X, we define the **cone of** X CX to be $X \times I/X \times \{0\}$. The prototypical example is given by $S^1 \times I/S^1 \times \{0\}$.

Definition 9. Given a space X, we define the **suspension of** X SX to be $X \times \{0\}$ collapsed to a point and $X \times \{1\}$ collapsed to another point. Note that suspension is a functor, and we can suspend maps as well.