Algebraic Topology I: PSET 5

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Problem 1

Recall that $\pi_0(X) = [*, X]$ is a set with cardinality the number of path-components of X. Thus if we wish to define a $X = K(G, 0)$ space, i.e. a space with $\pi_0(X) = G$ (as sets) and all higher homotopy groups zero, X must be a space with $|G|$ path components. The obvious choice is to take the group G itself, with the discrete topology.

Problem 2

(a) Let $f : A \to B$ be a morphism of complexes. We define $k : \ker f$ to be a morphism of complexes $k : K \to A$ satisfying the usual universal property for kernels. More explicitly, we take $K_i = \ker f_i$ and $d_i^K : K_i \to K_{i-1}$ to take $a \in K_i$ to the unique element in K_{i-1} that is sent to $d_i^A k(a)$. This is well-defined as each map $k_i : K_i \to A_i$ is injective. Let us show that K is universal with respect to the property that $K \to A \to B$ is the zero morphism. Suppose we have another map $h : H \to A$ such that $H \to A \to B$ is zero. Then the universal property of each K_i with respect to H_i, A_i, and B_i yields unique maps $l_i : H_i \to K_i$. Pictorially,

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we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1} \arrow[r] \arrow[d, dashed] \arrow[dd, bend left] & H_i \arrow[r] \arrow[d, dashed] \arrow[dd, bend left] & H_{i-1} \arrow[r] \arrow[d, dashed] \arrow[dd, bend left] & \cdots \\
\cdots \arrow[r] & K_{i+1} \arrow[r] \arrow[d] & K_i \arrow[r] \arrow[d] & K_{i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & A_i \arrow[r] \arrow[d] & A_{i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r] & B_i \arrow[r] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

and it now suffices to show that the l_i form a morphism $l : H \to K$ of complexes. This is done by checking that the squares in the top row commute. By injectivity of k_{i-1}, it suffices to show that $k_{i-1}d_i^K l_i = k_{i-1}l_{i-1}d_i^H$. By the commutativity of the squares in the second row, we find that $k_{i-1}d_i^K l_i = d_i^A k_i l_i$, but using the fact that $H \to A$ is a morphism of complexes, $k_{i-1}l_{i-1}d_i^H = d_i^A k_i l_i$ and hence we find that $k_{i-1}d_i^K l_i = k_{i-1}l_{i-1}d_i^H$, as desired.

Similarly, we define $q : \operatorname{coker} f$ to be a morphism of complexes $q : B \to Q$ satisfying the usual universal property for cokernels. Its construction is exactly dual to above, using the cokernels of each f_i as the groups in each degree, and with differentials chosen analogously.

(b) Let $h : A \to B$ be a homotopy of complexes. We claim that $f = dh + hd : A \to B$ is a morphism of complexes. Each f_i, as a sum of compositions of homomorphisms, is a group homomorphism $f_i : A_i \to B_i$. It now suffices to show that the relevant squares commute:

$$\begin{aligned}
d_i^B f_i &= d_i^B(d_{i+1}^B h_i + h_{i+1}d_i^A) \\
&= d_i^B h_{i-1} d_i^A \\
f_{i-1}d_i^A &= (d_i^B h_{i-1} + h_{i-2}d_{i-1}^A)d_i^A \\
&= d_i^B h_{i-1} d_i^A,
\end{aligned}$$

where we have used the fact that $d^2 = 0$, and hence $d_i^B f_i = f_{i-1}d_i^A$, as desired.

(c) The commutativity of squares guaranteed by any morphism of complexes ensures an induced morphism of homology groups. Suppose $f = dh +$

hd is nullhomotopic via a homotopy h. Then f induces the zero map $HA \to HB$ if and only if the f maps $\ker d_i^A$ into $\operatorname{im} d_{i+1}^B$ for all i. But $f = h_{i-1}d_i^A + d_{i+1}^B h_i$, and the first term annihilates $\ker d_i^A$. Clearly, then f maps $\ker d_i^A$ into $\operatorname{im} d_{i+1}^B h_i$, as desired. More generally, homotopic morphisms induce the same map on homology, as if $f \sim g$ then $f - g \sim 0$, and thus $(f-g)_* = f_* - g_* = 0$, implying that $f_* = g_*$.

(d) Suppose we have maps of complexes $A \xrightarrow{f} B \xrightarrow{g} C$ with f nullhomotopic. Then the composition gf must be nullhomotopic as well, since

$$\begin{aligned}(gf)_i &= g_i(d_{i+1}^B h_i + h_{i-1}d_i^A)\\ &= g_i d_{i+1}^B + g_i h_{i-1} d_i^A\\ &= d_{i+1}^C g_{i+1} h_i + g_i h_{i-1} d_i^A\\ &= d_{i+1}^C k_i + k_{i-1} d_i^A,\end{aligned}$$

where we take $k_i = g_{i+1}h_i$ as our homotopy. An analogous argument follows for if instead of f, we take g to be nullhomotopic.

(e) Let $f, g : A \to B$ be nullhomotopic. Then

$$\begin{aligned}f &= dh + hd\\ g &= dh' + h'd\\ f \pm g &= (dh + hd) \pm (dh' + h'd)\\ &= d(h \pm h') + (h \pm h')d,\end{aligned}$$

using the homomorphism property, and hence $f \pm g$ is nullhomotopic as well. Putting these properties together, we find that the set of nullhomotopic maps constitutes an ideal in $\mathrm{Kom}(\mathbb{Z})$, as left- and right-composition preserve nullhomotopy.

Problem 3

Suppose a complex $A \in \mathrm{Kom}(\mathbb{Z})$ is contractible, i.e. its identity map is nullhomotopic. Then, in $\mathrm{Com}(\mathbb{Z})$, its identity map is equal to the zero map; as the identity map must be bijective, this implies that A must be isomorphic to the zero complex in $\mathrm{Com}(\mathbb{Z})$. Conversely, if it is isomorphic in $\mathrm{Com}(\mathbb{Z})$ to the zero complex, its identity map is homotopic to the zero map in $\mathrm{Kom}(\mathbb{Z})$, thus making it nullhomotopic.

Now consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

as a complex of abelian groups. As it is exact, it is clearly acyclic. It is easy to see, however, that it is not contractible; indeed, suppose it is. Then there must exist a homotopy h

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] \arrow[dl, "h_2"] & \mathbb{Z}_4 \arrow[r] \arrow[d] \arrow[dl, "h_1"] & \mathbb{Z}_2 \arrow[r] \arrow[d] \arrow[dl, "h_0"] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{Id}_i = d_{i+1}h_i + h_{i-1}d_i$. Applying this here for $i = 2$, we find that $\mathrm{Id}_{\mathbb{Z}_2} = 0 + d_2h_1$. This is clearly impossible, as the only possible map $h_1 : \mathbb{Z}_4 \to \mathbb{Z}_2$ is the quotient, and hence the map on the right-hand side is not bijective. Hence this complex is not contractible.

Problem 4

We know that any contractible complex of vector spaces is acyclic, so it suffices to prove that an acyclic complex is contractible. Recall from class that a complex of free abelian groups with finitely generated homology groups decomposes into direct sum of complexes $0 \to W \xrightarrow{\mathrm{Id}} W \to 0, 0 \to \mathbb{Z} \to 0, 0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$. Since in our case the complex is acyclic, we find that the complex decomposes into complexes of the form $0 \to W \xrightarrow{\mathrm{Id}} W \to 0$, as otherwise it would have non-zero homology. But each of these is obviously contractible as the identity map can be written as $\mathrm{Id} = dh + hd$ where h is the homotopy shown in the diagram below:

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "\mathrm{Id}"] \arrow[dl, "0"] & W \arrow[r] \arrow[dl, "\mathrm{Id}"] & 0 \arrow[dl, "0"] \\
0 \arrow[r] & W \arrow[r, "\mathrm{Id}"] & W \arrow[r] & 0.
\end{tikzcd}$$

Hatcher 2.1.1

The quotient of the 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices yields a Möbius strip.

Hatcher 2.1.4

The given space X is composed of one 0-simplex v, three 1-simplicies a, b, c, and one 2-simplex T. This yields the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^1 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^1 \longrightarrow 0 \longrightarrow \cdots$$

whose homology we compute:

$$\begin{aligned} H^0(X) &= \mathbb{Z}/\langle v - v\rangle = \mathbb{Z} \\ H^1(X) &= \mathbb{Z}^3/\langle a + b - c\rangle = \mathbb{Z}^2 \\ H^2(X) &= 0, \end{aligned}$$

since $dT = a + b - c, da = db = dc = v - v = 0$.

Hatcher 2.1.5

The Klein bottle, with the given simplicial structure, has one 0-simplex v, three 1-simplices a, b, c, and two 2-simplices U, L. This yields the chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^1 \longrightarrow 0 \longrightarrow \cdots$$

whose homology we compute

$$\begin{aligned} H^0(X) &= \mathbb{Z} \\ H^1(X) &= \mathbb{Z}^3/\langle a + b - c, a - b + c\rangle = \mathbb{Z}^3/\langle 2a, a - b + c\rangle \\ &= \mathbb{Z} \oplus \mathbb{Z}_2 \\ H^2(X) &= 0, \end{aligned}$$

since $dU = a + b - c, dL = a - b + c, da = db = dc = v - v = 0$.

Hatcher 2.1.15

Let $A \to B \to C \to D \to E$ be an exact sequence. If $C = 0$ then exactness at B implies surjectivity of $A \to B$ and exactness at D implies injectivity of $D \to E$. Conversely, suppose $A \to B$ is surjective and $D \to E$ is injective. The first implies that $B \to C$ is the zero map and the latter implies that the map $C \to D$ is the zero map. But now exactness at C implies that $C = 0$.

Hatcher 2.1.16

(a) Suppose A meets every path-component of X. Any map $\sigma : \Delta^0 \to X$ lands in a path-component X^i. By hypothesis there exists a path connecting the point $\operatorname{im}\sigma$ to a point in A. This path is a map $\Delta^1 \to X^i$ (a relative 1-chain in X) whose boundary is the map σ above. Hence we have an exact sequence

$$C_1(X, A) \longrightarrow C_0(X, A) \longrightarrow 0,$$

and $H_0(X, A) = 0$. Conversely, if $H_0(X, A) = 0$, every map $\sigma : \Delta^0 \to X^i$ is the boundary of a map $\Delta^1 \to X^i$ which provides a path from $\operatorname{im}\sigma$ to A. Thus A meets every path-component X^i of X.

(b) We have the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_1(A) \arrow[r] & H_1(X) \arrow[r] & H_1(X, A) \arrow[dll] & \\ & H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X, A) \arrow[r] & 0. \end{tikzcd}$$

Recall that $H_0(A)$ and $H_0(X)$ are sets whose elements correspond to the path-components of A and X respectively and the map $H_0(A) \to H_0(X)$ takes path-components of A to the path-component of X in which A sits. Hence $H_0(A) \to H_0(X)$ is injective if and only if there is at most one path-component of A in each path-component of X. Now by Hatcher 2.1.15, $H_1(A) \to H_1(X)$ is surjective and there is at most one path-component of A in each path-component of X if and only if $H_1(X, A) = 0$.

Hatcher 2.1.17

(a) Let $X = S^2$ and A be a finite subset of points. Then we obtain a long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(X,A) \arrow[dll] & \\
& 0 \arrow[r] & 0 \arrow[r] & H_1(X,A) \arrow[dll] & \\
& \mathbb{Z}^{|A|} \arrow[r] & \mathbb{Z} \arrow[r] & H_0(X,A) \arrow[r] & 0.
\end{tikzcd}
$$

From this, it is evident that $H_2(X,A) = \mathbb{Z}$, $H_1(X,A)$ injects into $\mathbb{Z}^{|A|}$, and via the previous problem, since A meets the path-components of X, $H_0(X,A) = 0$. But then we must have that $H_1(X,A) = \mathbb{Z}^{|A|-1}$.

Now let $X = S^1 \times S^1$ and A be a finite subset of points. Then we obtain a long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(X,A) \arrow[dll] & \\
& 0 \arrow[r] & \mathbb{Z}^2 \arrow[r] & H_1(X,A) \arrow[dll] & \\
& \mathbb{Z}^{|A|} \arrow[r] & \mathbb{Z} \arrow[r] & H_0(X,A) \arrow[r] & 0
\end{tikzcd}
$$

Again we can conclude that $H_2(X,A) = \mathbb{Z}$ and $H_0(X,A) = 0$. Now $H_1(X,A)$ must be of the form $\mathbb{Z}^n \oplus \mathbb{Z}_m$, but it is clear that there is no torsion term, as it must map to zero under ∂, placing it in $\ker \partial$ and hence $\operatorname{im} j_*$. This is impossible since $\mathbb{Z}^2$ has no torsion, and hence $H_1(X,A)$ is free, i.e. $H_1(X,A) = \mathbb{Z}^2 \oplus \operatorname{im} \partial = \mathbb{Z}^2 \oplus \mathbb{Z}^{|A|-1} = \mathbb{Z}^{|A|+1}$.

(b) Recall that $H_n(X,A) \cong \tilde{H}_n(X/A)$. Clearly $X/A \cong T^2 \vee T^2$ and hence

$$H_n(X,A) = \tilde{H}_n(T^2 \vee T^2) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2).$$

Thus

$$
\begin{aligned}
H_0(X, A) &= 0 \\
H_1(X, A) &= \mathbb{Z}^4 \\
H_2(X, A) &= \mathbb{Z}^2.
\end{aligned}
$$

Similarly, since $X/B \sim T^2 \vee S^1$, we find that

$$H_n(X, B) = \tilde{H}_n(T^2 \vee S^1) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(S^1).$$

Thus

$$
\begin{aligned}
H_0(X, B) &= 0 \\
H_1(X, B) &= \mathbb{Z}^3 \\
H_2(X, B) &= \mathbb{Z}.
\end{aligned}
$$