Quantum Mechanics for Mathematicians: Problem Set 8

Due Monday, March 25

Problem 1: Compute the propagator

$$G(x', t', x, t) = \langle 0|\hat{\psi}(x', t')\hat{\psi}^\dagger(x, t)|0\rangle$$

for the free non-relativistic particle of mass m (for $t' > t$). First do this in momentum space, showing that

$$\widetilde{G}(k', t', k, t) = e^{-i\frac{k^2}{2m}(t'-t)}\delta(k' - k)$$

then Fourier transform to find the position space result

$$G(x', t', x, t) = (\frac{m}{i2\pi(t' - t)})^{\frac{3}{2}} e^{\frac{m}{i2\pi(t'-t)}(x'-x)^2}$$

Show also that

$$\lim_{t \to t'} G(x', t', x, t) = \delta(x' - x)$$

Problem 2: For non-relativistic quantum field theory of a free particle, including interaction with a potential, show that there is a particle number operator $\hat{N}$ that is conserved (commutes with the Hamiltonian) and generates a $U(1)$ symmetry.

Problem 3: For non-relativistic quantum field theory of a free particle in three dimensions, find the "classical" angular momentum functions L_j on the phase space of complex valued functions on $\mathbf{R}^3$. Show that these functions Poisson-commute with the Hamiltonian function. Find the corresponding quantized operators $\hat{L}_j$, show that these commute with the Hamiltonian operator, and satisfy the Lie algebra commutation relation

$$[\hat{L}_1, \hat{L}_2] = \hat{L}_3$$

Problem 4: Show that the Lie algebra $\mathfrak{so}(4, \mathbf{C})$ is $\mathfrak{sl}(2, \mathbf{C}) \times \mathfrak{sl}(2, \mathbf{C})$. Within this Lie algebra, identify the sub-Lie algebras of the groups $Spin(4)$, $Spin(1, 3)$ and $Spin(2, 2)$.