Physics 6047
Problem Set 7, due 3/28/13
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This is two weeks' worth of problems. Start early!

1. In this problem, you will compute the $O(g^2)$ correction to the propagator $\Pi(k^2)$ for the $g\phi^3$ theory, by several different methods. The goal is to check that they all yield the same finite result.

a. First, it is useful to rewrite the result obtained in class using dimensional regulation. This will help us compare against the result obtained by other methods. We started from:

$$\Pi(k^2) = \frac{1}{2}g^2 \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2} - (Ak^2 + Bm^2) \tag{1}$$

where $D = x(1-x)k^2 + m^2$. We have from class, by setting $d = 6 - \epsilon$:

$$\Pi(k^2) = -\frac{1}{2}\frac{g^2}{(4\pi)^3}\left(\frac{k^2}{6} + m^2\right)\left(\frac{2}{\epsilon} + 1 - \gamma + \ln 4\pi\right) - (Ak^2 + Bm^2) + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D \ln D \tag{2}$$

Recall that the dimension of g changes as we vary d i.e. $g \sim \text{mass}^{(6-d)/2}$. Thus, for a non-zero ϵ, to keep g dimensionless, it is useful to introduce the notation $g \to g\tilde{\mu}^{\epsilon/2}$, following Srednicki, where $\tilde{\mu}$ is some kind of mass.

Show that $\Pi(k^2)$ can be rewritten as:

$$\Pi(k^2) = -\frac{1}{2}\frac{g^2}{(4\pi)^3}\left(\frac{k^2}{6} + m^2\right)\left(\frac{2}{\epsilon} + \ln\left[\frac{4\pi\tilde{\mu}^2}{e^{\gamma-1}}\right]\right) - (Ak^2 + Bm^2) + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D \ln D \tag{3}$$

Note that both $\tilde{\mu}^2$ and D have mass^2 dimension. To make the expression look nicer, one could divide both by m^2 under the logarithms - you can convince yourself the result is the same as the above expression. This is what Srednicki did. We will just leave the expression as is.

We then went on to show that imposing $\Pi(-m^2) = \Pi'(-m^2) = 0$, by adjusting A and B, gives the finite result

$$\Pi(k^2) = \frac{1}{2}\frac{g^2}{(4\pi)^3}\left(-\frac{1}{6}(k^2 + m^2) + \int_0^1 dx D \ln(D/D_0)\right) . \tag{4}$$

where $D_0 = -x(1-x)m^2 + m^2$.

b. An alternative way is to differentiate Eq. (1) twice with respect to k^2. Compute the resulting $\Pi''(k^2)$ for $d = 6$. This is manifestly finite. Then, integrate this twice to obtain back $\Pi(k^2)$, but take care to choose the integration constants to match the $\Pi(-m^2) = \Pi'(-m^2) = 0$ conditions. Show that you recover Eq. (4). This way of proceeding completely bypasses the identification (and then removal) of divergences, and looks very clean. However, for reasons that will become clear later when we discuss renormalization group, it is actually useful to know what these divergences are. Therefore, let us next investigate two other ways to regularize.

c. Perform the integral

$$\int \frac{d^6\bar{q}}{(2\pi)^6}\frac{1}{(\bar{q}^2 + D)^2} = \frac{\pi^3}{(2\pi)^6}\int_0^{\Lambda_c} d|\bar{q}|\, |\bar{q}|^5 \frac{1}{(|\bar{q}|^2 + D)^2} \tag{5}$$

where I have switched to spherical coordinates (π^3 is the solid angle of the 5-sphere), and I have introduced a UV cut-off Λ_c to the magnitude of the vector $\bar{q}$. Putting the result into Eq. (1), show that

$$\Pi(k^2) = \frac{1}{2}\frac{g^2}{(4\pi)^3}\frac{\Lambda_c^2}{2} - \frac{1}{2}\frac{g^2}{(4\pi)^3}\left(\frac{k^2}{6} + m^2\right)\ln\left[\frac{\Lambda_c^2}{\sqrt{e}}\right] - (Ak^2 + Bm^2) + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D \ln D \quad (6)$$

You should check my algebra as there're likely to be mistakes! Comparing this against the dim. reg. expression Eq. (3), you can see that the $1/\epsilon$ divergence is given here by the quadratic divergence Λ_c^2, and the logarithmic divergence $\ln\Lambda_c^2$. Alternatively, the somewhat mysterious $\ln\tilde{\mu}^2$ term can also be thought of as $\ln\Lambda_c^2$ in disguise. Show that imposing $\Pi(-m^2) = \Pi'(-m^2) = 0$ results in Eq. (4). This way of regularizing is the crudest (i.e. sometimes it doesn't work, as you will see later), but also the most physical, in that we are saying our theory is good only up to some maximum momentum Λ_c. After we adjust the coefficients A and B to impose the Π and Π' conditions, we are left with a $\Pi(k^2)$ that is finite, and independent of the cut-off Λ_c i.e. independent of the details of the UV, or high energy, physics.

d. Perform the integral

$$\int \frac{d^6\bar{q}}{(2\pi)^6}\frac{1}{(\bar{q}^2 + D)^2} \to \int \frac{d^6\bar{q}}{(2\pi)^6}\left[\frac{1}{(\bar{q}^2 + D)}\frac{\Lambda^2}{\bar{q}^2 + \Lambda^2}\right]^2 \quad (7)$$

This is known as Pauli-Villars regularization, where Λ is some large ($\gg \sqrt{D}$), but finite mass scale. The integration will be allowed to extend to arbitrarily high momentum, in which case there is sufficient power of $\bar{q}$'s downstairs for the integral to converge. For small momentum $\bar{q} \ll \Lambda$, the factor of $\Lambda^2/(\bar{q}^2 + \Lambda^2) \sim 1$, and so the Pauli-Villars factor is not altering the integrand. Thus, Λ plays the role of a UV cut-off. To do this integral, I suggest you switch to spherical coordinates, and allow $|\bar{q}|$ to integrate from 0 to X, where X is some very large momentum (i.e. $X \gg \Lambda$). A useful rewrite of the integrand is

$$\frac{1}{(\bar{q}^2 + D)}\frac{\Lambda^2}{\bar{q}^2 + \Lambda^2} = \left[\frac{1}{(\bar{q}^2 + D)} - \frac{1}{\bar{q}^2 + \Lambda^2}\right]\frac{\Lambda^2}{\Lambda^2 - D} \quad (8)$$

You can convince yourself that all the X dependent terms cancel out upon doing the integral. Show that the integral Eq. (7) yields

$$\frac{1}{(4\pi)^3}\left[\frac{\Lambda^2}{2} - D\ln\Lambda^2 + D\ln D + \frac{3}{2}D + O\left(\frac{D}{\Lambda^2}\right)\right] \quad (9)$$

Put this into Eq. (1) and show that

$$\Pi(k^2) = \frac{1}{2}\frac{g^2}{(4\pi)^3}\frac{\Lambda^2}{2} - \frac{1}{2}\frac{g^2}{(4\pi)^3}\left(\frac{k^2}{6} + m^2\right)\ln\left[\frac{\Lambda^2}{e^{3/2}}\right] - (Ak^2 + Bm^2) + \frac{1}{2}\frac{g^2}{(4\pi)^3}\int_0^1 dx D \ln D \quad (10)$$

This looks very similar to Eq. (6). Finally, by the same argument there, imposing the Π and Π' conditions would yield Eq. (4).

2. Consider the theory

$$S = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \right] \tag{11}$$

Work out the order λ correction to the propagator. Think about what kind of counter-term you need, and work out its (infinite and finite) contributions using dimensional regularization.

3 Srednicki problem 16.1. Ignore the question about $O(\lambda)$ contribution to C.

4. Srednicki problem 17.1.

5. Srednicki problem 20.1.

6. We have shown in class the Ward identity:

$$-\frac{\partial}{\partial x^\mu}\langle J^\mu(x)\phi(y)\rangle = i\delta(x-y)\langle\delta\phi(y)\rangle \tag{12}$$

with the understanding that under the transformation $\phi \to \phi + \delta\phi(x)\rho(x)$, the action changes by $\delta S = \int d^dx J\cdot\partial\rho$.

Let's remind ourselves how we derived the charge commutator in class. Imagine integrating the above equation in the following manner: $\int d^{d-1}x \int dx^0$, where the spatial integral is over everywhere, and the time integral is from $y^0 - \Delta$ to $y^0 + \Delta$. (We will eventually take $\Delta \to 0$.) On the right hand side, we simply get $i\langle\delta\phi(y)\rangle$. On the left hand side, we will imagine the current J^μ to decay away at spatial infinity such that there's no spatial boundary term. (In fact even this assumption can be relaxed...) There are time boundary terms at $y^0 \pm \Delta$:

$$\begin{aligned} &-\int_{y^0-\Delta}^{y^0+\Delta} dx^0 \frac{\partial}{\partial x^0}\langle\int d^{d-1}x J^0(x)\phi(y)\rangle = \\ &-\langle\left[\int d^{d-1}x J^0(\vec{x}, x^0 = y^0+\Delta)\right]\phi(y)\rangle + \langle\phi(y)\left[\int d^{d-1}x J^0(\vec{x}, x^0 = y^0-\Delta)\right]\rangle \end{aligned} \tag{13}$$

Note how I flip the order of $\left[...\right]$ and ϕ in the two terms. The order doesn't matter in the context of path integrals. But recall now that path integrals of products of fields become expectation values of time-ordered products of operators. Thus, putting everything together, we see that

$$-\langle[\hat{Q}, \hat{\phi}]\rangle = i\langle\delta\hat{\phi}\rangle \tag{14}$$

where I have put a $\hat{}$ on top to remind myself I am now talking about expectation values of operators, and I have used the definition $Q = \int d^{d-1}x J^0$, and I have gone ahead to let $\Delta \to 0$ so that I can think of $\hat{Q}$, $\hat{\phi}$ and $\delta\hat{\phi}$ as all residing at the same point. Lastly, I can define contour of integration in the path integral in such a way that the expectation value is between any state I want (not just the vacuum), or if you'd like, I can insert extra operators (in far past or future) to represent states I want to create out of the vacuum. Therefore, we claim eq. (14) holds for any state, and thus we can write an operator statement:

$$\delta\hat{\phi} = i[\hat{Q}, \hat{\phi}] \tag{15}$$

A familiar example of this: think of the Hamiltonian as the charge operator $\hat{Q}$, which indeed does generate time-translation on an operator $\hat{\phi}$ in the above manner. Another example: momemtum as the generator of spatial translation.

What I want you to do is this: derive eq. (14) in a different way. Integrate eq. (12) over space as before, but don't integrate over time. Instead, evaluate the remaining time derivative, but be careful to take into account you are taking time derivative of a time-ordered product of operators. The time derivative hitting the step-functions involved in the definition of time-ordered products will give you the desired commutator.