HINT FOR PROBLEM (4) OF DIFFERENTIABLE MANIFOLDS HOMEWORK #2

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We first recall the statement of the problem:
(4) Consider a map $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x, y, s, t) = (x^3 + y, x^3 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1)$ is a regular value of F and that the level set $F^{-1}((0, 1))$ is diffeomorphic to S^2. Find the subset of $F^{-1}((0, 1))$ on which x and y can be solved as smooth functions of s and t.

Here is what I discussed in class:

Theorem 1. *Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a smooth map. Let $c \in \mathbb{R}^m$ and $F^{-1}(c) \neq \varnothing$. Suppose that for all $a \in F^{-1}(c)$, $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is of rank m (full rank). Then, $F^{-1}(c)$ is an n dimensional smooth manifold and c is called a **regular value** of F.*

Recall that DF_a is simply the $m \times (n+m)$ matrix of partial derivatives of F. The corresponding implicity function theorem says that if the determinant of the upper left $m \times m$ submatrix of DF_a is not zero, then $x_1 \cdots x_m$ can be locally solved in terms of $x_{m+1} \cdots x_{m+n}$.

Example 1. *Let's look at the simple case $m = 2$, $n = 1$ when F is a linear map:*

$$F_1(x_1, x_2, x_3) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$F_2(x_1, x_2, x_3) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3.$$

Thus $DF = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$. It's clear that if the 2×2 submatrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has non-zero determinant, and we set $F_1 = c_1, F_2 = c_2$, we will be able to solve for x_1 and x_2 in terms of x_3.

Hint for Problem (4): Denote $S = F^{-1}((0, 1))$. To check that $(0, 1)$ is a regular value, we compute

$$\begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial s} & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial s} & \frac{\partial F_2}{\partial t} \end{bmatrix}$$

and show that at any point in S, the two rows are linearly independent. We can then cover S by six coordinate charts as in the S^2 case (Problem 2 in the first homework assignment). In each coordinate chart, two of the coordinate variables can be solved in term of the other two. Note that the function $f(x) = x^{1/3}$ is smooth as long as one keeps away from $x = 0$ (thus on either $\{x > 0\}$ or $\{x < 0\}$).