Introduction to Algebraic Topology Midterm II

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Proposition 1. *Problem 1*

Proof. Let $G = \langle a, b \mid aba^{-1}b \rangle$ be the fundamental group of the Klein bottle as in Figure 1.

(a) Consider the subgroup H generated by a^3 and b^2. We wish to find a covering space $p : \tilde{K} \to K$ such that $\tilde{K}$ is associated to H by the Galois correspondence. Consider the Klein bottle $\tilde{K}$ depicted in Figure 2, which is a sixfold cover of K. If the fundamental group of $\tilde{K}$ is generated by x and y, with the relation $xyx^{-1}y$, we find that under p (defined in the obvious way along the dashed lines) $p_*(x) = a^3$ and $p_*(y) = b^2$, and hence $p_*(\pi_1(\tilde{K}, \tilde{x})) = H$ as desired (by uniqueness via the Galois correspondence).

(b) Next, consider the three-fold cover $p_1 : \tilde{K}_1 \to K$ lying between $\tilde{K}$ and K given by the diagram in Figure 3. Note that topologically K_1 is again a Klein bottle. Using the same logic as above, if we define p_1 following the dashed lines as indicated in the figure, we find that if x, y are the generators of $\pi_1(\tilde{K}_1, \tilde{x}_1)$ with relation $xyx^{-1}y$, then $(p_1)_*(x) = a^3$ and $(p_1)_*(y) = b$. Hence $(p_1)_*(\pi_1(\tilde{K}_1, \tilde{x}_1))$ is the subgroup of $\pi_1(K, x)$ generated by a^3, b.

(c) Similarly, consider the two-fold cover $p_2 : \tilde{K}_2 \to K$ lying between $\tilde{K}$ and K given by the diagram in Figure 4. This space is topologically again a Klein bottle. Defining p_2 accordingly, we find that if x, y are the generators of $\pi_1(\tilde{K}_2, \tilde{x}_2)$ with relation $xyx^{-1}y$ then $(p_2)_*(x) = a$ and $(p_2)_*(y) = b^2$. Hence $(p_2)_*(\pi_1(\tilde{K}_2, \tilde{x}_2))$ is the subgroup of $\pi_1(K, x)$ generated by a, b^2.

(d) Recall the algebraic fact that any subgroup of index two is normal; this immediately shows that q_1 and p_2 are normal covering spaces, as both these covering maps induce index two subgroups in the fundamental groups of their base spaces. It remains to check p, p_1, and q_2. Note that checking normality of a covering reduces to checking normality of the associated image subgroup under the covering space map, which in turn can be reduced to checking that conjugation of the generators (by the homomorphism property of conjugation) returns an element of the subgroup. Hence for p_1 we must check that conjugating a^3 and b by a and b yield elements in the subgroup generated by a^3 and b; this is easy via the relation $aba^{-1}b$: ba^3b^{-1} is already in the subgroup while $aba^{-1} = b^{-1}$ which is also contained in the subgroup. Similarly, q_2 is shown to be a normal covering by noting that $ab^2a^{-1} = b^{-2}$ and $b^2a^3b^{-2}$ are both contained in the appropriate subgroup. Finally, p is a normal covering, as $ab^2a^{-1} = b^{-2}$ and

$$ba^3b^{-1} = b(b^{-1}a^3b^{-1})b^{-1}$$
$$= a^3b^{-2}$$

are in the appropriate subgroup, where I have used the identity $a^3ba^{-3}b = 1$, verified as follows.

$$a^3b = a^2b^{-1}a,$$
$$a^{-3}b = ba^{-1}b^2a^{-1}b^2a^{-1}b^2,$$
$$a^3ba^{-3}b = a^2b^{-1}aba^{-1}b^2a^{-1}b^2a^{-1}b^2$$
$$= ab^2a^{-1}b^2$$
$$= b^{-2}b^2 = 1.$$

Of course, this is much more easily seen topologically, by the fact that the change of basepoint homomorphism on lifts takes loops to loops.

Thus we find that every covering in this diagram is normal, which makes computing deck groups relatively easy. Recall that in the case of a normal covering, the deck group is isomorphic to the quotient of the fundamental group of the base space by the subgroup H associated to the covering. First note $G(q_1) = \mathbb{Z}_2$ as we are forcing the generator $b \in \pi_1(\tilde{K}_1, \tilde{x}_1)$ to square to zero (this is also clear by eyeballing the picture – one can swap the upper and lower halves of the six-fold covering). Similarly, $G(q_2) = \mathbb{Z}_3$ as we are forcing the generator $a \in \pi_1(\tilde{K}_2, \tilde{x}_2)$ to cube to zero (again, topologically this is simply permuting the thirds of the six-fold covering). Next, $G(p_1) = \mathbb{Z}_3$ as we are forcing the generator $a \in \pi_1(K, x)$ to cube to zero and $G(p_2) = \mathbb{Z}_2$ as we are forcing the generator $b \in \pi_1(K, x)$ to square to zero. Finally, we find that $G(p) = \mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$ as we are forcing $a \in \pi_1(K, x)$ to cube to zero and $b \in \pi_1(K, x)$ to square to zero.

□

Proposition 2. *Problem 2*

Proof.

1. See Figure 5 for a drawing of the simplicial structure chosen for $X = S^2 \vee S^1 \vee S^1$. Recall from Hatcher that the simplicial homology for the torus $T^2 = S^1 \times S^1$ is as follows: $H_0^\Delta(T^2) = \mathbb{Z}, H_1^\Delta(T^2) = \mathbb{Z}^2, H_2^\Delta(T^2) = \mathbb{Z}$ and zero for higher groups. Associated to X we obtain the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^5 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{\partial_0} 0$$

with $\partial_0 x = \partial_0 y = \partial_0 z = 0, \partial_1 a = y - x, \partial_1 b = z - y, \partial_1 c = z - x$, and $\partial_2 U = \partial_2 L = a + b - c$. Of course, $H_0^\Delta(X) = \mathbb{Z}$ by path-connectedness,

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle a + b - c, d, e \rangle}{\langle a + b - c \rangle} = \mathbb{Z}^2,$$

and $H_2^\Delta(X) = \langle U - L \rangle = \mathbb{Z}$. Hence the homology groups of the torus and X are isomorphic.

2. The universal cover of the torus is of course $\mathbb{R}^2$, while the universal cover $\mathcal{U}X$ of X is sketched in Figure 6 (this was previously done on a homework). By contractibility, $H_n^\Delta(\mathbb{R}^2) = 0$ for all n. On the other hand, since in $\mathcal{U}X$ we have spheres touching the universal cover of $S^1 \vee S^1$ at every intersection point, $H_2^\Delta(\mathcal{U}X) = \ker \partial_2$ which is clearly non-zero, since the difference $U - L$ of the two 2-simplices comprising a sphere is send by ∂_2 to zero (as shown in the previous part). Hence the homologies of $\mathbb{R}^2$ and $\mathcal{U}X$ are not isomorphic.

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