Commutative Algebra: Problem Set 4

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Problem 5

Let $A = k[x, y, z]/(x^2y^2z^2, x^3y^2z)$. We wish to compute the dimension of A at the maximal ideal (x, y, z). Since $x^2y^2z^2$ is a nonzerodivisor in $k[x, y, z]_{(x,y,z)}$, quotienting out by it yields a ring with dimension two (by lemma 46). Next note that after quotienting out by x^3y^2z we still have a chain of primes $(x) \subset (x, y) \subset (x, y, z)$ and hence the dimension is two.

Problem 6

Let $A = k[x, y, z]/(x^3 - y^2, x^5 - z^2, y^4 - z^3)$. We wish to compute the dimension of A at (x, y, z).

Problem 7

Let k be a field. Let $f \in k[x, y]$ be a polynomial and $a, b \in k$ be elements such that $f(a, b) = 0$. Let $\mathfrak{m} = (x - a, y - b)$ be the corresponding maximal ideal in the ring $A = k[x, y]/(f)$. By construction, $A_\mathfrak{m}$ is a local ring. We wish to check that it is regular, i.e. that the maximal ideal has exactly $\dim A_\mathfrak{m}$ generators. In this case, it's clear that $\dim A_\mathfrak{m}$ is one (via theorems from class) and hence the ideal $(x - a, y - b)$ must collapse to a principal ideal in $A_\mathfrak{m}$. In other words, there must be some way of solving f to write $x - a$ in terms of $y - b$ or vice versa. Of course, this is formally possible via the implicit function theorem as long as one of $\partial f/\partial a, \partial f/\partial b$ is non-zero, as desired.

Problem 9

Consider $f = xy^2 + x^2y = xy(y + x)$, which has zeros at $x = y = 0$ and at $x = y$. We compute $\partial_x f = y^2 + 2xy$ and $\partial_y f = 2yx + x^2$; the only singular point, then, is $(0, 0)$. Next consider $f = x^2 - 2x + y^3 - 3y^2 + 3y$; we compute $\partial_x f = 2x - 2$ and $\partial y f = 3y^2 - 6y + 3 = 3(y - 1)^2$. The singular point is then $(1, 1)$, as $f(1, 1) = 0$. Finally, consider $f = x^n + y^n + 1$, which has $\partial_x f = nx^{n-1}, \partial_y f = ny^{n-1}$. These derivatives are never zero except at $(0, 0)$, which is not a root of f, and hence f has no singular points.

Problem 10

Let k an algebraically closed field. Let $f \in k[x, y]$ be a squarefree polynomial of degree 1. In other words, $f = ax + by + c$. Clearly f can only have singular points if it is a constant, which contradicts the degree being 1, and hence has no singular points. Next consider degree 2: $f = ax^2 + bxy + cy^2 + dx + ey + f$. In this case, solving for the singular points involves simultaneously solving a linear system, which yields a single point. For degree 3, we solve two quadratics, and hence we expect 4 singular points. In this way, we might guess that in general for degree d we have, at maximum, $(d - 1)^2$ singular points.