Representation Theory PSET 2

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Exercise 1. *Describe the group P/Q for classical simple Lie algebras and also for G_2 and F_4.*

Proof. □ finish

Exercise 2. *Let B be a nondegenerate bilinear form (symmetric or antisymmetric) on $\mathbb{C}^n$ and let G be its group of automorphisms. Describe the closed G-orbits on the variety of complete flags in $\mathbb{C}^n$.*

Proof. Consider first the case of a nondegenerate symmetric bilinear form. The group G is then the orthogonal group, and any flag in $\mathbb{C}^n$ is simply an ordered basis of vectors. The orthogonal group clearly acts transitively on such bases, and hence the flag variety decomposes into a single (closed) orbit under the action of the orthogonal group.

For case of a nondegenerate antisymmetric bilinear form, □ finish

Exercise 3. *Let $g \in GL_n$ be a unipotent operator. Describe the Zariski closure of the cyclic group generated by g. Conclude, in particular, that any algebraic group consisting of unipotent operators is connected.*

Proof. We work over an algebraically closed field k of characteristic zero. Then we note that any nilpotent matrix N can be exponentiated to a unipotent matrix $\exp N$. Conversely, any unipotent matrix can be written as $U = 1 + A$ where A is nilpotent; more precisely, one can consider the logarithm of U to obtain $N = \log U$ such that $\exp N = U$. In particular, this correspondence gives us an isomorphism between the variety of nilpotent matrices and the variety of unipotent matrices, as the exponential and logarithm power series terminate to become polynomials. Hence, given any unipotent element $g \in GL_n$, we can write it as $g = \exp N$, and the cyclic subgroup generated by g becomes the group $\exp(mN)$ for $m \in \mathbb{Z}$. Consider now the map $\phi : k \to GL_n$ given by $x \mapsto g^x$, which is regular as the components of g^x are simply polynomials for g unipotent – this is evident from the Taylor series for x^c for $c \in k$. The preimage of the closure $\phi^{-1}(\overline{\langle g \rangle})$ must be a Zariski-closed set in k containing $\mathbb{Z}$. Of course, the only closed set in k containing an infinite number of points is k itself. Hence the closure of the cyclic subgroup $\langle g \rangle$ must be of the form g^x for $x \in \mathbb{C}$. From this characterization, it is clear that any algebraic group consisting of unipotent operators is connected, as every operator is in the component of the identity (take $x = 0$). □

Exercise 4. *Let $G \subset GL_n$ be a solvable Lie subgroup, not necessarily connected. Is it true that G is triangular in a suitable basis?*

Proof. Consider the subgroup $S_2 \subset GL_n$ embedded as a 2×2 block in the upper-left corner with 1's down the diagonal. The group S^2 is of course solvable, but it is clear that there exists no basis in which it is triangular: if there were, it would no longer contain permutation matrices. □

Exercise 5. *For G as above, let $\bar{G} \subset GL_n$ be the Zariski closure of G. Is it true that $\bar{G}$ is solvable?*

Proof. In this exercise, I reference Borel's *Linear Algebraic Groups*. Note first that it is not *a priori* obvious that $\bar{G}$ is a group. However, it can be shown that the Zariski closure $\bar{G}$ is precisely the intersection of the closed subgroups of GL_n containing G (c.f. Borel I.2.1a). Now, as G is solvable, successive group commutators $\mathcal{D}^{n+1} = (\mathcal{D}^n G, \mathcal{D}^n G)$ (with $\mathcal{D}^0 G = G$) must vanish after a finite n. To show that $\bar{G}$ is solvable, we must study the commutator $(\bar{G}, \bar{G})$. We claim that the closure of $(\bar{G}, \bar{G})$ is precisely the closure of (G, G). To see this, consider the map $c : GL_n \times GL_n \to GL_n$ given by $c(x, y) = xyx^{-1}y^{-1}$. Since $G \times G$ is dense in $\bar{G} \times \bar{G}$ we find that $c(G \times G)$ must be dense in $c(\bar{G} \times \bar{G})$. It follows then that $\overline{c(G \times G)} = \overline{c(\bar{G} \times \bar{G})}$, but this is precisely the statement that $\overline{(G, G)} = \overline{(\bar{G}, \bar{G})}$. Finally, we note that $(\bar{G}, \bar{G})$ is in fact closed (c.f. Borel Proposition I.2.3), and hence $(\bar{G}, \bar{G}) = \overline{(G, G)}$. By a simple induction, then, we find that $\mathcal{D}^N \bar{G} = 0$ for some large N, which shows that $\bar{G}$ is solvable. $\square$