Notes on Algebra II

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1 Reducibility

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Let F be a field, and $F[x]$ be the ring of polynomials over F. Recall we have already shown that every ideal in $F[x]$ is principal, and that there exists a unique gcd of two non-zero polynomials. Additionally, we showed that if f and g are two relatively prime polynomials, then $f|gh \implies f|h$.

Definition 1. A polynomial $p(x) \in F[x]$ is **irreducible** if $\deg p(x) > 0$, i.e. p is not zero and not a unit, and if $p = fg$ implies that one of f, g is a unit and the other is a unit times p. In words, $p(x)$ is irreducible if it does not factor into a product of two polynomials with strictly smaller (non-zero) degree. A polynomial is said to be **reducible** if it is not irreducible.

Example 1. (Reducibility)

(i) Any linear polynomial $x + a$ is obviously irreducible.

(ii) Any quadratic polynomial is clearly reducible if and only if it has two linear factors. This is equivalent to the polynomial having a root, as long division will yield the second factor.

(iii) Similarly, a cubic polynomial is reducible if and only if it has a root.

(iv) For higher degrees, the existence of a root is not equivalent to reducibility, as we will see in the next example.

Example 2. (Simple examples)

- $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$, as it has no roots in $\mathbb{Q}$. It is, however, reducible in $\mathbb{R}[x]$: $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$.
- x^2+1 is irreducible in $\mathbb{R}[x]$ but reducible in $\mathbb{C}[x]$: $x^2+1 = (x-i)(x+i)$.

- $x^3 - 2$ is irreducible in $\mathbb{Q}[x]$, but reducible in $\mathbb{R}[x]$, where we can write it as a product of $x - \sqrt[3]{2}$ and an irreducible quadratic.
- $x^4 - 4 = (x^2 - 2)(x^2 + 2)$ is reducible in $\mathbb{Q}[x]$ but has no roots!

In fact, it is generally a hard problem to determine whether an arbitrary polynomial $f(x) \in \mathbb{Q}[x]$ is irreducible. Note, however, that we can think of irreducibility in analogy to that for natural numbers, as the following dichotomy illustrates.

Remark. If $p(x) \in F[x]$ is irreducible, then for any polynomial $f \in F[x]$, either $p|f$ or p and f are relatively prime.

Proof. Let $d = \gcd(p, f)$. By definition, d divides p. However, as p is irreducible, d must either be a unit or d must be cp for c a unit. In the first case, since the gcd of p and f is a unit, p and f must be relatively prime. In the second case, since $d = cp$ by construction divides f, p must divide f. □

Corollary 1. *If $p \in F[x]$ is irreducible and $p|fg$, then either $p|f$ or $p|g$.*

Proof. By the above remark, either $p|f$ or p and f are relatively prime. If $p|f$, we are done. Otherwise, p is relatively prime to f, and by what we showed last class, $p|g$. □

Theorem 2 (Unique factorization of polynomials)**.** *Let $f(x) \in F[x]$ with $\deg f(x) > 0$. Then there exist k irreducible polynomials in $F[x]$ such that*

$$f(x) = \prod_{i=1}^{k} p_i(x).$$

Additionally, if it is also true that $f(x) = \prod_{i=1}^{l} q_i$, then $k = l$, and after some reordering, there exist nonzero constants such that $q_i = c_i p_i$.

In other words, for any polynomial with degree greater than zero, there always exists a unique factorization into a product of irreducible polynomials.

Proof. Let us first show existence. We proceed by complete induction on the degree of f. If $\deg f = 1$, f is irreducible, and we are done. Otherwise, we assume that the theorem holds for all degrees less than n. Let $\deg f = n$. If f is irreducible, we are done. Otherwise, $f = g_1 g_2$ with $\deg g_1 < n$ and $\deg g_2 < n$. By the inductive hypothesis, g_1 and g_2 are products of irreducible polynomials, and thus f must be as well, and we are done.

The real muscle of this theorem comes in the form of uniqueness. Suppose $f = \prod_{i=1}^{k} p_i = \prod_{j=1}^{l} q_j$, with p_i, q_j reducible. We proceed by induction on k. If $k = 1$, $p_1 = q_1 \cdots q_l$. Clearly, then, $p_1 | q_1 \cdots q_l$, and thus (by induction over the statement at the beginning of lecture), p_1 must divide q_i for some i. But the q_i are irreducible and p_1 is not a constant, so $p_1 = cq_i$ for some unit c. If we now reorder terms, we can assume that $i = 1$ and we can cancel:

$$\begin{aligned} p_1 = cq_1 &= q_1 q_2 \cdots q_l \\ c &= q_2 \cdots q_l. \end{aligned}$$

But this is impossible, as the product of q's has degree greater than zero. Consequently, l must be 1, and thus $p_1 = q_1$ and we have shown that $k = l$. The general case is similar; we write $p_1 \cdots p_k = q_1 \cdots q_l$. Then $p_1 | q_1 \cdots q_l$, and so for some i, $p_1 = cq_i$. After reordering, we can write

$$\begin{aligned} cq_1 p_2 \cdots p_k &= q_1 \cdots q_l \\ cp_2 \cdots p_k &= q_2 \cdots q_l, \end{aligned}$$

and by induction, we know that $k - 1 = l - 1$. Reordering, we can write $p_= c_i q_i$ for $i = 2 \cdots k$, and we are done. □

Note that the irreducible factors need not be distinct.

Theorem 3. *Let F be a field. Let I be an ideal in $F[x]$. Then the following are equivalent:*

(i) I is a maximal ideal.

(ii) I is a prime ideal and $I \neq \{0\}$.

(iii) $I = (p)$, where p is a irreducible polynomial.

Proof. Let us first show that $(i) \implies (ii)$. Say I is maximal. Then, I must be prime. Additionally, I cannot be the zero ideal, as it is not maximal, and so we are done.

Showing $(ii) \implies (iii)$ is a little trickier. Suppose I is a prime ideal with $I \neq \{0\}$. We want to show that the ideal is generated by an irreducible element. Since every ideal in $F[x]$ is principal, $I = (p)$ for some $p \in F[x]$. Let us show that p is irreducible. First note that p cannot be a unit, because otherwise $1 \in (p)$ which implies that $(p) = F[x]$, which is not possible for prime ideals. Furthermore, $p \neq 0$, as I is assumed not to be the zero ideal.

To show that p is irreducible, we need to show that if $p = fg$ then one of f, g is a unit and the other is a unit times p. So take $p = fg$. Then, $fg \in (p) = I$. Since I is prime, either $f \in I$ or $g \in I$. Take the first case, $f \in (p)$. Then, $f = hp$ for some $h \in F[x]$, and so $p = hpg \implies 1 = hg$, i.e. h, g are units, and thus f is a unit times p. Thus, p is irreducible.

Finally, we show that $(iii) \implies (i)$. Let $I = (p)$, with p irreducible. We wish to show that I is maximal, i.e. $(p) \neq F[x]$ and if $(p) \subset J$ then either $J = (p)$ or $J = F[x]$. First note that $(p) \neq F[x]$ because $\deg p > 1$ and so it can't generate constants. Next, since J is necessarily a principal ideal, $J = (f)$, for some $f \in F[x]$. If $(p) \subset (f)$, then $p \in (f)$, so $p = fg$ for some $g \in F[x]$. But p is irreducible, so either f is a unit, in which case $J = (f) = F[x]$, or $f = cp$, for c a unit, in which case $J = (f) = (p)$. Hence, I is maximal. □

This theorem is quite handy in constructing interesting fields, as the following corollary shows.

Corollary 4. *$F[x]/(f)$ is a field if and only if f is irreducible.*

Proof. This follows from above theorem and the fact that $F[x]/(f)$ is a field if and only if (f) is a maximal ideal. □

This allows us to show that certain rings are, in fact, fields – something that may not have been obvious – or, in fact, to find wholly new fields.

Example 3.

- $\mathbb{Q}[x]/(x^2-2)$ is a field, as x^2-2 is irreducible in $\mathbb{Q}[x]$, and its elements, by what we know about long division, are of the form $c + d\alpha$, where $\alpha = x + (x^2 - 2)$. In addition, $\alpha^2 = 2$.
- $\mathbb{R}[x]/(x^2+1)$ is a field, as x^2+1 is irreducible in $\mathbb{R}[x]$, and its elements are of the form $c + d\alpha$ where $\alpha = x + (x^2 + 1)$ satisfies $\alpha^2 = -1$.
- $\mathbb{Q}[x]/(x^3-2)$ is a field, as x^3-2 is irreducible in $\mathbb{Q}[x]$, and its elements are of the form $c + d\alpha + e\alpha^2$, where $\alpha = x + (x^3 - 2)$ satisfies $\alpha^3 = 2$. We often rewrite the elements as $c + d\sqrt[3]{2} + e\sqrt[3]{2}^2$.
- Take the finite field $\mathbb{F}_2$ and the polynomial $x^2 + x + 1 \in \mathbb{F}_2$. Since the only members of $\mathbb{F}_2$ are 0 and 1, it should be clear that this polynomial has no roots, and thus is irreducible in $\mathbb{F}_2[x]$. Consequently, $E = \mathbb{F}_2[x]/(x^2 + x + 1)$ is a field. Its elements are of the form $c + d\alpha$, where

of course $c, d \in \mathbb{F}_2$ and $\alpha = x+(x^2+x+1)$, which satisfies the property that $\alpha^2 = -\alpha - 1 = \alpha + 1$. E has four elements (since c and d can each take 2 values).

2 Field extensions

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In general, a problem in algebra is to enlarge the domain of discourse, i.e. $\mathbb{R} \to \mathbb{C}$, so that one can solve equations that were hitherto unsolvable. So given a polynomial $f(x) \in F[x]$, we wish to find a root of $f(x)$. Maybe there is not root of $f(x)$ in F, so we wish to enlarge F. A simple idea that we saw earlier was that if R is any ring with $f(x) \in R[x]$, there was a way to enlarge R. We consider $R[x]/(f(x))$, which always has a root $x+(f(x)) = \alpha$. By construction, $f(\alpha) = 0$.

There is a problem with this – we don't know much about the algebraic structure of this new quotient ring, $R[x]/I$. The solution is: if R is a field, and $f(x)$ is irreducible, then $R[x]/I$ is good, i.e. it is a field, as we saw last lecture. But really, even if $f(x)$ is reducible, we should think in analogy to the world of $\mathbb{Z}/n\mathbb{Z}$, where $n > 0$. The full details of this analogy are fleshed out in the file `analogy.pdf`.

Theorem 5. *Let $f(x) \in F[x]$ and suppose $\deg f(x) > 0$, i.e. f is nonconstant. Then, there exists a field E that contains (a subfield isomorphic to) F, and an element $\alpha \in E$ such that $f(\alpha) = 0$.*

Proof. Take $f(x)$ and find an irreducible factor (we know these exist from last time) $p(x)$. We consider $E = F[x]/(p(x))$; E is a field. Additionally, there is a map $F \to E$ that sends $a \in F \mapsto a + (p(x))$. This map is injective (why?), and we identify F with the image subfield. We know that if $\alpha = x + (p(x))$, then $p(\alpha) = 0$. But $p(x)|f(x)$, so $f(\alpha) = 0$ as well, and we are done. □

Corollary 6. *Let $f(x) \in F[x]$, $\deg f(x) > 0$. Then there exists a field E such that, in $E[x]$, $f(x)$ is a product of linear factors.*

Proof. We apply the above theorem to find E_1 and $\alpha_1 \in E_1$ such that $f(\alpha_1) = 0$ ($F \leq E_1$). In $E_1[x]$, $f(x) = (x - \alpha_1)g_1(x)$, where $\deg g_1(x) = \deg f(x) - 1$. Informally speaking, now all we have to do is to keep going! We find E_2 with $E_1 \leq E_2$ such that we can write $g_1(x) = (x - \alpha_2)g_2(x)$ with $\alpha_2 \in E_2$ with $\deg g_2 = \deg f(x) - 2$. We continute until we run out of degrees, and clearly we have factored f into linear factors. □

Let us now switch perspectives. Let's consider the situation where E and F are fields, and $F \leq E$, i.e. F is a subfield of E. We also say that E is an **extension field** of F. Note that we used the machinery of prime and maximal ideals to construct extensions. Now, given an extension, let us use these tools to analyze these fields.

Consider E an extension field of F, and let $\alpha \in E$. Look at $\mathrm{ev}_\alpha : F[x] \to E$. Note that $\operatorname{Im} \mathrm{ev}_\alpha = F[\alpha] \leq E$. At this point, all we know is that $F[\alpha]$ is an integral domain. We claim that there are exactly 2 possibilities.

1. $\ker \mathrm{ev}_\alpha = \{0\}$, i.e. that if $f(x) \in F[x]$, $f(x) \neq 0$, then $f(\alpha) \neq 0$. In this case, we say that α is **transcendental** over F. Additionally, $\mathrm{ev}_\alpha : F[x] \to E$ is injective, which suggests that it extends to an injection $F(x) \to E$, whose image we call $F(\alpha)$. Elements of this image have the form $f(\alpha)/g(\alpha)$ where $f, g \in F[x]$ and $g \neq 0$. This is the smallest subfield of E containing F and α.

   A famous example is that $\pi \in \mathbb{R}$ is transcendental over $\mathbb{Q}$ (Lindemann, 1880). The same holds for e. Note carefully, that $\pi \in \mathbb{R}$ is not transcendental over $\mathbb{R}$, as π is a root of $x - \pi$.c w

2. $\ker \mathrm{ev}_\alpha \neq \{0\}$, i.e. there exists an $f(x) \in F[x]$, $f \neq 0$ such that $f(\alpha) = 0$. In this case, we say that α is **algebraic** over F. What can we say here? An incredible amount, it turns out.

   First note that $\ker \mathrm{ev}_\alpha$ is a principal ideal in $F[x]$: $\ker \mathrm{ev}_\alpha = (p(x))$. Additionally, we know that $F[\alpha] = \operatorname{Im} \mathrm{ev}_\alpha \cong F[x]/\ker \mathrm{ev}_\alpha$. But since $F[x]$ is an integral domain (subring of a field), $(p(x))$ is a prime ideal that is not $\{0\}$. This means that $(p(x))$ is a maximal ideal and $p(x)$ is irreducible (by theorem proved last time). Consequently, $F[x]/(p(x)) = F[x]/\ker \mathrm{ev}_\alpha$ is a field, and $F[\alpha]$ is a field. Recall the example of $\mathbb{Q}[\sqrt[3]{2}]$ being a field. Thus, we now write $F[\alpha] = F(\alpha)$, which is the smallest subfield of E containing F and α.

   In particular, there is a unique monic generator of $(p(x)) = \ker \mathrm{ev}_\alpha$. It is denoted $\mathrm{irr}(\alpha, F, x)$, which is read "the irreducible polynomial for α over F." It satisfies $\mathrm{irr}(\alpha, F, \alpha) = 0$. Let us do a few examples:

   Example 4.

$$\mathrm{irr}(\frac{1}{2}, \mathbb{Q}, x) = x - \frac{1}{2}$$
$$\mathrm{irr}(\sqrt[3]{2}, \mathbb{Q}, x) = x^3 - 2$$
$$\mathrm{irr}(\sqrt[3]{2}, \mathbb{Q}(\sqrt[3]{2}), x) = x - \sqrt[3]{2}$$
$$\mathrm{irr}(\sqrt[3]{2}, \mathbb{Q}(\sqrt{2}), x) = ?$$

Remark. Note that if $f(x) \in F[x]$ is any polynomial such that $f(\alpha) = 0$, then the $\text{irr}(a, F, x) | f(x)$.

Proof. $f(\alpha) = 0 \iff f(x) \in \ker \text{ev}_\alpha = (\text{irr}(\alpha, F, x))$. By definition, then, $\text{irr}(\alpha, F, x) | f(x)$. □

For example, take $f(x) \in \mathbb{R}[x]$. If $f(i) = 0$, then $x^2 + 1 | f(x)$.

Mostly we will be working with the algebraic case, as the transcendental case belongs in a separate course.

Definition 2. Given $F \leq E$ and $\alpha \in E$ algebraic over F, we define the **degree** of α over F as $\deg \text{irr}(\alpha, F, x)$.

For example, $\deg_{\mathbb{Q}} \sqrt[3]{2} = 3$ and $\deg_{\mathbb{R}} \sqrt[3]{2} = 1$ and $\deg_{\mathbb{R}} i = 2$.

Definition 3. Let $F \leq E$. Then we say that E is a **simple extension** of F if $E = F(\alpha)$ for some $\alpha \in E$.

Roughly speaking, this means that we can extend F to E by throwing in only one more element – i.e. we can do this if α is transcendental. Take, for example, $\mathbb{Q}(\sqrt{2})$. $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. Then, $\mathbb{Q}(\sqrt{2}) \leq \mathbb{Q}(\sqrt{2})(\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then, $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2} + \sqrt{3})$, one can find that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. This shows that simple extensions are not always obviously simple.

2.1 interlude : F-vector spaces

Let us take a detour through vector spaces.

Let F be a field. An **F-vector space** is an abelian group $(V, +)$ and a function $F \times V \to V$ called **scalar multiplication**, which we write as av such that:

1. $a(bv) = (ab)v$
2. $a(v + w) = av + aw$
3. $(a + b)v = av + bv$
4. $1 \cdot v = v$

A very useful example is $V = F^n = \{(a_1 \cdots a_n) : a_i \in F\}$, i.e. the Cartesian product of F with itself n times. We define addition componentwise, and multiply the scalar through each component, as usual. It is easy to

check that F^n is a vector space. The $n = 0$ case is allowed, as it is the zero vector space with only 0.

Another example is the space of functions $X \to F$ on any set X, which we denote by F^X. Functions are added pointwise as usual, and scalar multiplication is done pointwise as well.

The important example for us is actually a bit unexpected. Suppose E is an extension field of F. Then E is an F-vector space. E is already an abelian group and scalar multiplication is defined in the ordinary sense of multiplication. The rest of the axioms follow straightforwardly. Now, this is not as strange as it might look. The complex numbers, for example, are an extension field of the reals, and we are used to going back and forth between numbers/vectors: $a + bi \iff (a, b)$. Similarly, $\mathbb{Q}(\sqrt[3]{2})$ is a $\mathbb{Q}$-vector space as $a + b\sqrt[3]{2} + c\sqrt[3]{2}^2 \iff (a, b, c)$. Furthermore, $\mathbb{R}$ is a $\mathbb{Q}$-vector space (but a really big one).

One might ask, why does one require F to be a field? We can define a similar structure for a ring.

Definition 4. If R is a commutative ring with unity, then an **R-module** M is an abelian group $(M, +)$ with a scalar multiplication $R \times M \to M$ that satisfies the properties defined above for F-vector spaces.

These are, however, more interesting for algebra in general, and not so much for our case, where we will extensively use the field properties.

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What we wish to do with these ideas is to use linear algebraic ideas to understand more deeply field extensions.

Let's talk about a few basic notions.

Definition 5. A **vector subspace** of an F-vector space V is a subgroup W of $(V, +)$ such that for all $a \in F$,$w \in W$, $aw \in W$. W then becomes an F-vector space in its own right.

Definition 6. $F : V_1 \to V_2$ is a **linear map** if

1. f is a homomorphism of abelian groups
2. f preserves scalar multiplication: $a, b \in F, v \in V_1$, $f(av) = af(v)$

A **linear isomorphism** is just a bijective linear map. Its inverse is linear as well.

Definition 7. Let V be an F-vector space and let $v_1, \ldots v_n \in V$. Then a **linear combination** of these vectors is an element of v of the form $a_1v_1 +$

$\ldots + a_n v_n$. We define the **span** of this set of vectors as the set of all such linear combinations. It should be clear that the span of a set of vectors is a vector space.

Definition 8. V is **finite dimensional** if there exists a set of vectors in V whose span equals V.

Note, for example, that F^n is finite dimensional (via the standard basis), but $F[x]$ is not. However, $F[x]$, does have many interesting finite dimensional subspaces. If we define P_n to be the set of polynomials in $F[x]$ with degree n or less, it forms a vector subspace spanned by $1, x, x^2, \ldots, x^n$.

Definition 9. A set of vectors $v_1, \ldots, v_n \in V$ are **linearly independent** if the only linear combination of them that yields zero is where the coefficients in the linear combination are all zero.

Theorem 7. *If V is an F-vector space, and $v_1, \ldots, v_n \in V$ are linearly independent, and $w_1, \ldots, w_m$ span V, then $n \leq m$.*

Definition 10. $v_1, \ldots v_n$ is a **basis** of V if these vectors are linearly independent, and they span V.

Theorem 8. *If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases of V, then $n = m$. In this case, we define this number $n = \dim_F V$.*

Proof. By the counting theorem above, $n \leq m$ and $m \leq n$, so $n = m$. □

The main example that will be important for us to consider is as follows. Let $f(x) \in F[x]$ with $\deg f(x) = n$ and $f \notin F$. Consider $F \leq F[x]/(f(x))$. In fact, we know that every element in the coset ring is uniquely of the form $g(x) + (f(x))$ where g is zero or $\deg g(x) < n$. This says that the cosets $1 + (f(x)), x + (f(x)), \ldots, x^{n-1} + (f(x))$ are an F-basis for $F[x]/(f(x))$. This implies that $\dim_F F[x]/(f(x)) = n$.

Corollary 9. *Let $F \leq E$ and α algebraic over F. Let $\deg_F \alpha = \deg \operatorname{irr}(\alpha, F, x)$. Then $F(\alpha)$ is a finite dimensional F-vector space: $\dim_F F(\alpha) = \deg_F \alpha = \deg \operatorname{irr}(\alpha, F, x)$.*

We have already seen a few examples: $\dim_{\mathbb{R}} \mathbb{C} = 2$ and $\dim_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2}) = 3$. Let's recall some more important linear algebra facts.

Theorem 10. *Let V be a finite-dimensional F-vector space. Then,*

1. *Any set $v_1, \ldots, v_n \in V$ that spans V contains a subset which is a basis.*

2. *Any set $v_1, \ldots, v_n \in V$ which is linearly independent can be completed to a basis.*

3. *If W is a vector subspace of V, then $\dim W \leq \dim V$. If $\dim W = \dim V$, then $W = V$.*

4. *If V_1 and V_2 are two finite-dimensional vector spaces with bases given by v_n, w_m, and $f : V_1 \to V_2$ is a linear map, then*

$$f(v_i) = \sum_{j=1}^{m} a_{ji} w_j$$

where $A = (a_{ij})$ is an $m \times n$ matrix. f determines and is determined by A.

Remark. Suppose F is a finite field, with q elements. Suppose V is a finite-dimensional F-vector space of dimension n, i.e. there exists a basis $v_1, \ldots, v_n$ of V where every vector can be written uniquely in terms of this basis. It should be clear that the number of elements in V is q^n. In fact, any finite dimensional vector space of dimension n is isomorphic to F^n.

In particular, if F itself is finite, its characteristic must be a prime p, and $\mathbb{F}_p \leq F$. Thus, any finite field is an $\mathbb{F}_p$-vector space. Since F is also finite-dimensional, the number of elements in F is simply p^k.

Definition 11. Let E be an extension field of F. Then E is a **finite extension** of F if E is a finite-dimensional F-vector space. In this case, we define $\dim_F E = [E : F]$, the **degree of E over F**.

Example 5. $\mathbb{C}$ is a finite extension of $\mathbb{R}$, as $[\mathbb{C} : \mathbb{R}] = 2$. Similarly, $\mathbb{Q}(\sqrt{2})$ is a finite extension of $\mathbb{Q}$ with $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$. Again, $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. However, $\mathbb{R}$ is not a finite extension of $\mathbb{Q}$, as $\mathbb{R}$ is an infinite-dimensional $\mathbb{Q}$-vector space.

Theorem 11. *Let $E = F(\alpha)$ be a simple extension of F. Then, E is a finite extension of F if and only if α is algebraic over F. In this case, $[E : F] = \deg_F \alpha = \deg \operatorname{irr}(\alpha, F, x)$.*