# Honors Complex Variables
# Lecture Notes

Nilay Kumar

# 1 Applications of Laurent series: classification of singularities

Suppose $f$ has a (potential) isolated singularity at $z_0$. In other words, $f$ is analytic in a deleted neighborhood of $z_0$, $D(Z_0; r)/\{z_0\}$, for some $r > 0$. We can write a Laurent expansion in $0 < |z - z_0| < r$:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$
$$\text{with } a_k = \frac{1}{2\pi i} \int_{\partial D(z_0;R)} \frac{f(z)}{(z - z_0)^k} dz$$

Let us see how we can use this to classify singularities. We have 3 cases:

1. No negative powers of $(z - z_0)$ appear in the series expansion: $a_k = 0\ \forall k < 0$. Then we call $z_0$ a **removable singularity**.
2. There is an $n > 0$ such that $a_{-n} \neq 0$ and $a_k = 0\ \forall k < -n$. Then we call $z_0$ a **pole** of $f(z)$.
3. There are infinitely many negative powers of $(z - z_0)$ in the expansion. Then we call $z_0$ an **essential singularity**.

## 1.1 Removable singularities

Let us first study removable singularities:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ for } 0 < |z - z_0| < r$$

Clearly $f$ is also analytic at $z_0$ if we simply define $f(z_0) = a_0$. In this case we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \text{ for } 0 \leq |z - z_0| < r.$$

Let us examine the example of $f(z) = \frac{\sin z}{z}$ for $0 < |z| < \infty$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$
$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Note that the series expansion for $f(z)$ can be defined for all $z$ and thus 0 is a removable singularity of $f$.

*Remark.* If $f$ has a removable singularity at $z_0$, then $f$ is bounded around $z_0$. In other words, $\exists M > 0$ with $|f(z)| < M\ \forall 0 < |z - z_0| < r$. The converse is also true.

**Theorem 1** (Riemann's theorem of removable singularities). *Let $z_0$ be a potential isolated singularity of $f$. If*

$$\lim_{z \to z_0} f(z)(z - z_0) = 0$$

*then $f$ has a removable singularity at $z_0$.*

*Remark.* Note also that if $|f(z)| \leq 1/|z - z_0|^{1-\varepsilon}$ for some positive $\varepsilon$, then $f$ has a removable singularity at $z_0$. Consequently, this is a much stronger bound than that provided by the previous remark.

*Proof.* We want to show that $a_k = 0 \ \forall k < 0$ in the Laurent expansion. Take $k < 0$. Then, $k + 1 \leq 0$. As $\lim_{z \to z_0} f(z)(z - z_0) = 0$, we can find $\delta > 0$ such that $|f(z)(z - z_0)| < \varepsilon \ \ \forall z$ in $0 < |z - z_0| < \delta < r$. From the formula for the coefficients and the M-L formula, we now have that

$$\begin{aligned} |a_k| &= \Big| \frac{1}{2\pi i} \int_{\partial D(z_0;\delta)} \frac{f(z)(z - z_0)}{(z - z_0)^{k+2}} dz \Big| \\ &\leq \frac{1}{2\pi} \frac{\varepsilon}{\delta^{k+2}} \cdot \text{length}\, \partial D(z_0; \delta) = \varepsilon \delta^{-(k+1)} \\ &\leq \varepsilon \delta^0 = \varepsilon \end{aligned}$$

Thus we have that $|a_k| < \varepsilon$, $\forall \varepsilon > 0$, which yields that the coefficients must be identically zero for $k < 0$. □

## 1.2 Poles

Let us now examine poles. From the series expansion, there is an $n > 0$ such that $a_{-n} \neq 0$ but $a_k = 0$ for all $k < -n$. In this case, $z_0$ is called a pole of order $n$ of $f$.

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$$

$$= \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

**Definition 1.** *The sum of negative powers*

$$P(z) = \sum_{k=-n}^{-1} a_k (z - z_0)^k$$

*is called the* ***principal part*** *of $f$ at the pole $z_0$.*

*Remark.* Note that for our series, if we take $f(z) - P(z)$ we obtain the analytic function $\sum_{k=0}^{n} a_k (z - z_0)^k$. Incidentally, if $n = 1$, we call the pole **simple** and if $n = 2$, we call the pole **double**.

**Lemma 1.** *Suppose $f$ is analytic in a region $\Omega$ and has a zero at a point $z_0 \in \Omega$ with $f \not\equiv 0$ in $\Omega$. Then there exists in a neighborhood $U \in \Omega$ of $z_0$, a non-vanishing analytic function $g$ on $U$ and a unique positive integer $n$ such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$. Note that the number $n$ is called the* ***order of the zero*** *$z_0$ of the function $f$ (also called the multiplicty).*

*Proof.* In a small neighborhood $D(z_0; R)$ of $z_0$, $f$ cannot be identically 0 by the uniqueness theorem. In this neighborhood let us expand:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
$$f(z_0) = 0 \implies a_0 = 0$$

Therefore there must be an $n \geq 1$ such that $a_n \neq 0$, otherwise $f$ would vanish identically. Thus we write:

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \dots \\ &= (z - z_0)^n \left[a_n + a_{n+1}(z - z_0) + \dots\right] \end{aligned}$$

Let $g(z) = a_n + a_{n+1}(z - z_0) + \dots$. Clearly it is analytic in D. Additionally, since $\lim_{z \to z_0} g(z) = a_n \neq 0$, we have $|g(z)| \geq |a_n|/2$ whenever $|z - z_0| < r$. Thus let us take $U = D$.

Let us now show the uniqueness of $n$. Suppose $\exists n < m$ such that $f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$ where $g$ and $h$ are analytic in a neighborhood $V$ of $z_0$, with $g, h \neq 0$ in $V$. When $z \neq z_0$ with $z \in V$, $g(z) = (z - z_0)^{m-n} h(z)$. Thus, $g(z_0) = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)^{m-n} h(z) \equiv 0$. This is a contradiction, and thus $n$ must be unique. □

**Theorem 2** (Characterization of a pole)**.** *Let $z_0$ be an isolated singularity of $f$. Then $z_0$ is a pole of $f(z)$ of order $n$:*

1. *iff $f(z) = g(z)/(z - z_0)^n$ where $g$ is analytic and non-zero at $z_0$.*
2. *iff*
$$h(z) = \begin{cases} 1/f(z) & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$
*is analytic at $z_0$ and has a zero of order $n$ at $z_0$.*
3. *iff $|f(z)| \to \infty$ when $z \to z_0$*

*Proof.* Let us attack the first claim. Suppose $f$ has a pole of order $n$ at $z_0$. Then, using the expansion for poles above,

$$f(z) = \frac{a_{-n} + a_{-n+1}(z - z_0) + \dots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \dots}{(z - z_0)^n} \equiv \frac{g(z)}{(z - z_0)^n}$$

It is clear that $g$ is analtyic at $z_0$ and does not vanish, and so we reach the result. Let us now prove the converse. If $g$ is analytic at $z_0$, and does not vanish, let us expand it into a power series about $z_0$:

$$g(z) = b_0 + b_1(z - z_0) + \dots \text{ with } g(z_0) = b_0 \neq 0$$

Hence,

$$\begin{aligned} f(z) &= \frac{g(z)}{(z-z_0)^n} = \frac{b_0 + b_1(z - z_0) + \dots}{(z-z_0)^n} \\ &= \frac{b_0}{(z-z_0)^n} + \frac{b_1}{(z-z_0)^{n-1}} + \dots \end{aligned}$$

Thus $f$ has a pole of order $n$ at $z_0$, and the first point is proved.

Let us now prove the second claim. Suppose $f$ has a pole of order $n$ at $z_0$. Then by the first point, we can represent

$$f(z) = \frac{g(z)}{(z-z_0)^n}$$

where $g$ is analytic and non-zero at $z_0$. Take the function $M(z) = 1/g(z)$, which is also analytic at $z_0$. Then,

$$\frac{1}{f(z)} = \frac{(z-z_0)^n}{g(z)} = (z - z_0)M(z)$$

is analytic at $z_0$ and has a zero of order $n$ at $z_0$. Conversely, if $1/f(z)$ has a zero of order $n$ at $z_0$, by the above theorem,

$$\frac{1}{f(z)} = (z - z_0)^n K(z)$$

for some $K$ analytic at $z_0$ and $K(z_0) \neq 0$. Hence,

$$f(z) = \frac{1/K(z)}{(z-z_0)^n} \equiv \frac{g(z)}{(z-z_0)^n}$$

Here $g(z) = 1/K(z)$ is analytic at $z_0$ and $g(z_0) \neq 0$. Then, by the first point $f$ has a pole of order $n$ at $z_0$.

Let us now prove the third point. If $f$ has a pole of order $n$ at $z_0$ then by the first point, $f(z) = g(z)/(z - z_0)^n$ for some $g$ analytic and nonvanishing at $z_0$. Thus,

$$|f(z)| = \frac{|g(z)|}{|z-z_0|^n} \to \infty.$$

Conversely, suppose that $f(z) \to \infty$ as $z \to z_0$. Obviously, $f(z) \neq 0$ for $z$ near $z_0$. Thus we can define $h(z) = 1/f(z)$, which is analytic around $z_0$ and $h(z) \to 0$ as $z \to z_0$. Hence, by the Riemann's theorem proved above, $h(z)$ must have a removable singularity at $z_0$ and $h(z_0) = 0$. Since $h \neq 0$ in the neighborhood of $z_0$, by the above theorem on zeroes, we can write $h(z) = (z - z_0)^n g(z)$ for some $n > 0$ and $g$ analytic in a neighborhood $V$ of $z_0$ and $g \neq 0$ in $V$. Now,

$$f(z) = \frac{1}{h(z)} = \frac{1/g(z)}{(z - z_0)^n}$$

By the first point, $f$ has a pole of order $n$ at $z_0$, as $1/g(z)$ is analytic and non-zero in $V$. $\square$

We shall cover the topic of essential singularities next lecture.