DIFFERENTIABLE MANIFOLDS HOMEWORK 1

(1) Consider the stereographic coordinates on
$$S^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$
   (a) Sow the projection from the north pole $(0, 0, 1)$ sends a point $(x, y, z) \in S^2$ to $(\frac{2x}{1-z}, \frac{2y}{1-z}, -1)$ on the $z = -1$ plane.
   (b) Show the projection from the south pole $(0, 0, -1)$ sends a point $(x, y, z) \in S^2$ to $(\frac{2x}{1+z}, \frac{2y}{1+z}, 1)$ on the $z = 1$ plane.

   We then identify both the $z = -1$ plane and $z = 1$ plane with $\mathbb{R}^2$, and define the atlas $\mathfrak{A}_1 = \{(U, \phi), (V, \psi)\}$ in which
$$U = S^2 - \{(0, 0, 1)\} \text{ and } \phi(x, y, z) = (\frac{2x}{1-z}, \frac{2y}{1-z}),$$
   and
$$V = S^2 - \{(0, 0, -1)\} \text{ and } \psi(x, y, z) = (\frac{2x}{1+z}, \frac{2y}{1+z}).$$
   The map $\psi \circ \phi^{-1}(\frac{2x}{1-z}, \frac{2y}{1-z}) = (\frac{2x}{1+z}, \frac{2y}{1+z})$ is defined for $(\frac{2x}{1-z}, \frac{2y}{1-z}) \in \mathbb{R}^2 - \{(0, 0)\} = \phi(U \cap V)$.
   (c) Set $u = \frac{x}{z}, v = \frac{y}{z}$ and solve x, y, z in terms of u and v (note that $x^2 + y^2 + z^2 = 1$). Show that the map $\psi \circ \phi^{-1}$ is given by
$$(u, v) \mapsto (\frac{4u}{u^2 + v^2}, \frac{4v}{u^2 + v^2}) \text{ for } (u, v) \neq (0, 0).$$
   (d) Find $\phi \circ \psi^{-1}$ and its domain of definition, and show that it is a diffeomorphism.

(2) We can use graphical coordinates for S^2 and define another atlas $\mathfrak{A}_2$ which consists of 6 coordinate charts. Show that $\mathfrak{A}_2$ is compatible with (or equivalent to) $\mathfrak{A}_1$ given by the stereographic projection, i.e. $\mathfrak{A}_1 \cup \mathfrak{A}_2$ is an atlas.

(3) The complex projective space $\mathbb{CP}^n$ is the set of all (complex) 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$. Show that $\mathbb{CP}^n$ can be considered as a quotient space of $\mathbb{C}^{n+1}\backslash\{0\}$ by $\mathbb{C}\backslash\{0\}$. Construct an atlas analogous to the one we constructed for $\mathbb{RP}^n$ in class.

(4) Do Problem 1-1 on page 29 of [1].

Date: due Feb. 5, 2013 in class .

(5) Do Problem 1-2 on page 30 of [1].
(6) Let $P : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{k+1}\backslash\{0\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}^{n+1}\backslash\{0\}$. Show that the map $\tilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\tilde{P}([x]) = [P(x)]$ is a well-defined smooth map.

References

[1] Lee, John M. *Introduction to Smooth Manifolds,* Second edition, Graduate Texts in Mathematics, 218, Springer-Verlag, New York, 2012.