Complex Geometry: Midterm

Nilay Kumar

Last updated: March 21, 2014

Problem 1

Let $(X, g_{\bar{k}j})$ be a compact complex manifold, equipped with a Hermitian metric $g_{\bar{k}j}$, which is not necessarily Kähler. Define the torsion tensor

$$T^j_{lp} = g^{j\bar{k}}\partial_l g_{\bar{k}p} - g^{j\bar{k}}\partial_p g_{\bar{k}l}$$

and let $\omega = \frac{i}{2} g_{\bar{k}j} dz^j \wedge d\bar{z}^k$.

(a) Now let V^j be a smooth vector field. We may write, via the product rule,

$$\begin{aligned}
\int_X \nabla_j V^j \omega^n &= \int_X \left(\partial_j V^j + \Gamma^j_{jk} V^k\right) \omega^n \\
&= \int_X \left(\partial_j V^j + \Gamma^j_{jk} V^k\right) \det g \cdot \wedge_{l=1}^n dz^l \wedge_{m=1}^n d\bar{z}^m \\
&= \int_X \left(\partial_j (V^j \det g) - V^j(\partial_j \det g)\right) \cdot \wedge_l dz^l \wedge_m d\bar{z}^m + \int_X \Gamma^j_{jk} V^k \omega^n.
\end{aligned}$$

Next, note that

$$\partial_j \log \det g = \sum_i \frac{\partial_j \lambda_i}{\lambda_i} = \operatorname{tr}\left(g^{-1}\partial_j g\right),$$

where λ_i are the eigenvalues of g but the left side is simply $\partial_j \log \det g = \partial_j(\det g)/\det g$, which yields the formula

$$\partial_j \det g = (\det g)\operatorname{tr}\left(g^{-1}\partial_j g\right) = (\det g) g^{\bar{a}b}\partial_j g_{b\bar{a}} = (\det g)\Gamma^b_{jb}.$$

Inserting this into the integral above and noting that the first term of the first integral vanishes (as it is exact), we find that

$$\begin{aligned}
\int_X \nabla_j V^j \omega^n &= \int_X \Gamma^j_{jk} V^k \omega^n - \int_X \Gamma^b_{jb} V^j \omega^n \\
&= \int_X \left(\Gamma^j_{jk} - \Gamma^j_{kj}\right) V^k \omega^n \\
&= \int_X T^j_{jk} V^k \omega^n,
\end{aligned}$$

as desired.

(b) Let us derive the following integration by parts formula:

$$\int_X (\nabla_j \phi_{\bar{K}I})\psi^{j\bar{K}I}\omega^n = -\int_X \phi_{\bar{K}I}(\nabla_j \psi^{j\bar{K}I})\omega^n + \int_X \phi_{\bar{K}I}\psi^{j\bar{K}I}T^p_{jp}\omega^n,$$

where ω is any Hermitian metric. This follows immediately from the previous part: write

$$(\nabla_j \phi_{\bar{K}I})\psi^{j\bar{K}I} = \nabla_j(\phi_{\bar{K}I}\psi^{j\bar{K}I}) - \phi_{\bar{K}I}\nabla_j \psi^{j\bar{K}I},$$

and if we denote $V^j = \phi_{\bar{K}I}\psi^{j\bar{K}I}$, the integral on the left becomes

$$\begin{aligned}
\int_X (\nabla_j \phi_{\bar{K}I})\psi^{j\bar{K}I}\omega^n &= \int_X \nabla_j V^j \omega^n - \int_X \phi_{\bar{K}I}(\nabla_j \psi^{j\bar{K}I})\omega^n \\
&= \int_X T^p_{jp} V^k \omega^n - \int_X \phi_{\bar{K}I}(\nabla_j \psi^{j\bar{K}I})\omega^n \\
&= \int_X T^p_{jp}\phi_{\bar{K}I}\psi^{j\bar{K}I}\omega^n - \int_X \phi_{\bar{K}I}(\nabla_j \psi^{j\bar{K}I})\omega^n,
\end{aligned}$$

as desired.

Problem 2

Let X be a complex manifold, and define the operator $\bar{\partial}$ on forms of type (p,q) by

$$\bar{\partial}\left(\frac{1}{p!q!}\sum_{\bar{J}I}\phi_{\bar{J}I}dz^I \wedge d\bar{z}^J\right) = \frac{1}{p!q!}\sum_{\bar{J}I}\frac{\partial \phi_{\bar{J}I}}{\partial \bar{z}^k}d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J.$$

Note that, as written, $(\bar{\partial}\phi)_{\bar{k}\bar{J}I} = (q+1)\partial_{\bar{k}}\phi_{\bar{J}I}$. If we antisymmetrize, this becomes

$$\begin{aligned}
(\bar{\partial}\phi)_{\bar{k}\bar{J}I} &= (\bar{\partial}\phi)_{\bar{k}\bar{j}_q\cdots\bar{j}_1 I} \\
&= \sum_{\sigma\in S_{q+1}} \operatorname{sgn}(\sigma)\partial_{\sigma(\bar{k})}\phi_{\sigma(j_q)\cdots\sigma(j_1)I} \\
&= \sum_{\sigma\in S_{q+1}} \operatorname{sgn}(\sigma)\partial_{\sigma(\bar{k})}\phi_{\sigma(J)I} \\
&= \sum_{\sigma\in S_{q+1}} \operatorname{sgn}(\sigma)\left(\nabla_{\sigma(\bar{k})}\phi_{\sigma(J)I} - \Gamma^{\bar{L}}_{\sigma(\bar{k})\sigma(J)}\phi_{\bar{L}I}\right),
\end{aligned}$$

where the multi-index $\bar{L}$ is being summed over. The Kähler condition on the metric g gives us, in this case, that $\Gamma^{\bar{L}}_{\sigma(\bar{k})\sigma(J)} = \Gamma^{\bar{L}}_{\sigma(J)\sigma(\bar{k})}$. In the summation above, then, we can pair the terms to obtain $(q+1)!/2$ pairs in which the connection terms cancel by the Kähler condition. If we then de-antisymmetrize, we find that, as desired,

$$\bar{\partial}\left(\frac{1}{p!q!}\sum_{\bar{J}I}\phi_{\bar{J}I}dz^I \wedge d\bar{z}^J\right) = \frac{1}{p!q!}\sum_{\bar{J}I}\nabla_{\bar{k}}\phi_{\bar{J}I}d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J.$$

If $g_{\bar{k}j}$ were not Kähler, then in the $(0,1)$ case, we would find that for $\phi = \sum \phi_k d\bar{z}^k$,

$$\begin{aligned}(\bar{\partial}\phi)_{\bar{i}\bar{j}} &= \sum (\bar{\partial}\phi_{\bar{k}}) \wedge d\bar{z}^k = \sum (\partial_{\bar{l}}\phi_{\bar{k}} d\bar{z}^l) \wedge d\bar{z}^k \\ &= \frac{1}{2}\sum \left(\partial_{\bar{l}}\phi_{\bar{k}} - \partial_{\bar{k}}\phi_{\bar{l}}\right) d\bar{z}^l \wedge d\bar{z}^k \\ &= \frac{1}{2}\sum \left(\nabla_{\bar{l}}\phi_{\bar{k}} - \nabla_{\bar{k}}\phi_{\bar{l}} + \Gamma^{\bar{m}}_{\bar{l}\bar{k}}\phi_{\bar{m}} - \Gamma^{\bar{m}}_{\bar{k}\bar{l}}\phi_{\bar{m}}\right) d\bar{z}^l \wedge d\bar{z}^k\end{aligned}$$

and hence the formula would become

$$\bar{\partial}\left(\sum_{\bar{i}} \phi_{\bar{i}} d\bar{z}^i\right) = \sum_{\bar{i}} (\nabla_{\bar{l}}\phi_{\bar{i}} + \Gamma^{\bar{m}}_{\bar{l}\bar{i}}\phi_{\bar{m}}) d\bar{z}^l \wedge d\bar{z}^i.$$

Problem 3

Let $(X, g_{\bar{k}j})$ be a compact Kähler manifold, and consider the operator $\bar{\partial}$ as defined on (p,q)-forms as in the previous problem. Let $L \to X$ be a holomorphic line bundle over X, with metric h.

(a) Suppose we have $\phi, \psi \in \Gamma(X, L \otimes \Lambda^{0,2})$. We can then define an inner product (with summations supressed)

$$\langle \phi, \psi \rangle \equiv \frac{1}{4}\int_X \phi_{\bar{j}\bar{i}} \overline{\psi_{\bar{k}\bar{l}}} g^{k\bar{j}} g^{\bar{i}l} h \omega^n / n!.$$

Note that this is clearly linear in the first argument and conjugate-symmetric (moreover, the indices match up appropriately). Positive-definiteness follows from the positive-definiteness of h and g. Similarly, given $\phi, \psi \in \Gamma(X, L \otimes \Lambda^{0,3})$, we can define an inner product

$$\langle \phi, \psi \rangle \equiv \frac{1}{36}\int_X \phi_{\bar{i}\bar{j}\bar{k}} \overline{\psi_{\bar{l}\bar{m}\bar{p}}} g^{\bar{i}l} g^{\bar{j}m} g^{\bar{k}p} h \omega^n / n!.$$

(b) Now consider part of the the $\bar{\partial}$ complex:

$$\begin{tikzcd} \Gamma(X, L \otimes \Lambda^{0,1}) \arrow[r, bend right, "\bar{\partial}"'] & \Gamma(X, L \otimes \Lambda^{0,2}) \arrow[l, bend right, "\bar{\partial}^\dagger"'] \arrow[r, bend right, "\bar{\partial}"'] & \Gamma(X, L \otimes \Lambda^{0,3}) \arrow[l, bend right, "\bar{\partial}^\dagger"'] \end{tikzcd}$$

Let us compute the formal adjoints $\bar{\partial}^\dagger : \Gamma(X, L \otimes \Lambda^{0,2}) \to \Gamma(X, L \otimes \Lambda^{0,1})$ and $\bar{\partial}^\dagger : \Gamma(X, L \otimes \Lambda^{0,3}) \to \Gamma(X, L \otimes \Lambda^{0,2})$.

For the first, take $\phi \in \Gamma(X, L \otimes \Lambda^{0,1})$ and $\psi \in \Gamma(X, L \otimes \Lambda^{0,2})$. By definition, the formal adjoint is such that

$$\langle \bar{\partial}\phi, \psi \rangle = \langle \phi, \bar{\partial}^\dagger \psi \rangle.$$

We can write $\psi = \frac{1}{2}\sum \psi_{\bar{l}\bar{m}} d\bar{z}^l \wedge d\bar{z}^m$ and

$$\begin{aligned}\bar{\partial}\phi &= \sum \partial_{\bar{k}}\phi_{\bar{j}} d\bar{z}^k \wedge d\bar{z}^j \\ &= \frac{1}{2}\sum \left(\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}}\right) d\bar{z}^k \wedge d\bar{z}^j.\end{aligned}$$

Then the above requirement thus becomes

$$\int_X \frac{1}{2}\left(\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}}\right) \overline{\psi_{\bar{l}\bar{m}}} h g^{l\bar{j}} g^{m\bar{k}} \omega^n / n! = \int_X \phi_{\bar{j}} \overline{(\bar{\partial}^\dagger \psi)_{\bar{k}}} h g^{k\bar{j}} \omega^n / n!.$$

Note now that

$$\partial_{\bar{k}}\phi_{\bar{j}} - \partial_{\bar{j}}\phi_{\bar{k}} = \nabla_{\bar{k}}\phi_{\bar{j}} - \nabla_{\bar{j}}\phi_{\bar{k}}.$$

Now we can simplify the left-hand side by de-antisymmetrizing and integrating by parts:

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\int_X (\nabla_{\bar{k}}\phi_{\bar{j}} - \nabla_{j}\phi_{\bar{k}})\overline{\psi_{\bar{l}\bar{m}}}hg^{l\bar{j}}g^{m\bar{k}}\omega^n/n! \\
&= \int_X (\nabla_{\bar{k}}\phi_{\bar{j}})\overline{\psi_{\bar{l}\bar{m}}}hg^{l\bar{j}}g^{m\bar{k}}\omega^n/n! \\
&= \int_X \phi_{\bar{j}}\overline{(-g^{k\bar{m}}\nabla_k\psi_{\bar{l}\bar{m}})}g^{l\bar{j}}\omega^n/n!
\end{aligned}$$

Hence we can write the formal adjoint as

$$(\bar{\partial}^\dagger\psi)_{\bar{l}} = -g^{k\bar{m}}\nabla_k\psi_{\bar{l}\bar{m}}.$$

The computation for the next formal adjoint is similar. We take $\phi \in \Gamma(X, L\otimes\Lambda^{0,2})$ and $\psi \in \Gamma(X, L\otimes\Lambda^{0,3})$. We obtain the equality, as above

$$\int_X (\nabla_{\bar{l}}\phi_{\bar{m}\bar{p}})\overline{\psi_{\bar{i}\bar{j}\bar{k}}}hg^{\bar{l}i}g^{\bar{m}j}g^{\bar{p}k}\omega^n/n! = \int_X \phi_{\bar{m}\bar{p}}\overline{(\bar{\partial}^\dagger\psi)_{\bar{i}\bar{j}}}hg^{\bar{m}i}g^{\bar{p}j}\omega^n/n!.$$

We can now integrate by parts (note that we have switched to covariant derivatives on the left, just as above, by noting that the connection terms drop out in the antisymmetrized expression, vis a vis problem 2) to obtain

$$\begin{aligned}
\text{LHS} &= -\int_X \phi_{\bar{m}\bar{p}}(\nabla_{\bar{l}}\overline{\psi_{\bar{i}\bar{j}\bar{k}}})hg^{\bar{l}i}g^{\bar{m}j}g^{\bar{p}k}\omega^n/n! \\
&= \int_X \phi_{\bar{m}\bar{p}}\overline{(-\nabla_l\psi_{\bar{i}\bar{j}\bar{k}}g^{l\bar{i}})}hg^{\bar{m}j}g^{\bar{p}k}\omega^n/n!,
\end{aligned}$$

from which we conclude that the formal adjoint can be written

$$(\bar{\partial}^\dagger\psi)_{\bar{i}\bar{j}} = -g^{l\bar{k}}\nabla_l\psi_{\bar{k}\bar{i}\bar{j}}.$$

(c) Define $\Delta \equiv \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}$. With the expressions for the formal adjoints as above, we can compute explicitly the action of Δ. Take some $\phi \in \Gamma(X, L\otimes\Lambda^{0,2})$. The first term becomes

$$\begin{aligned}
\left(\bar{\partial}\bar{\partial}^\dagger\phi\right)_{\bar{i}\bar{j}} &= \nabla_{\bar{i}}(\bar{\partial}^\dagger\phi)_{\bar{j}} - \nabla_{\bar{j}}(\bar{\partial}^\dagger\phi)_{\bar{i}} \\
&= \nabla_{\bar{i}}(-g^{k\bar{m}}\nabla_k\phi_{\bar{j}\bar{m}}) - \nabla_{\bar{j}}(-g^{k\bar{m}}\nabla_k\phi_{\bar{i}\bar{m}}) \\
&= -g^{k\bar{m}}\nabla_{\bar{i}}\nabla_k\phi_{\bar{j}\bar{m}} + g^{k\bar{m}}\nabla_{\bar{j}}\nabla_k\phi_{\bar{i}\bar{m}} \\
&= -g^{l\bar{k}}\nabla_{\bar{i}}\nabla_l\phi_{\bar{j}\bar{k}} + g^{l\bar{k}}\nabla_{\bar{j}}\nabla_l\phi_{\bar{i}\bar{k}}.
\end{aligned}$$

On the other hand, the second term is

$$\begin{aligned}
\left(\bar{\partial}^\dagger\bar{\partial}\phi\right)_{\bar{i}\bar{j}} &= -g^{l\bar{k}}\nabla_l\left(\nabla_{\bar{k}}\phi_{\bar{i}\bar{j}} - \nabla_{\bar{i}}\phi_{\bar{j}\bar{k}} - \nabla_{\bar{j}}\phi_{\bar{k}\bar{i}}\right) \\
&= -g^{l\bar{k}}\nabla_l\nabla_{\bar{k}}\phi_{\bar{i}\bar{j}} + g^{l\bar{k}}\nabla_l\nabla_{\bar{i}}\phi_{\bar{j}\bar{k}} + g^{l\bar{k}}\nabla_l\nabla_{\bar{j}}\phi_{\bar{k}\bar{i}}.
\end{aligned}$$

Adding the two terms, we find a Bochner-Kodaira formula for Δ:

$$(\Delta\phi)_{\bar{i}\bar{j}} = -g^{l\bar{k}}\nabla_l\nabla_{\bar{k}}\phi_{\bar{i}\bar{j}} + g^{l\bar{k}}[\nabla_l, \nabla_{\bar{i}}]\phi_{\bar{j}\bar{k}} + g^{l\bar{k}}[\nabla_l, \nabla_{\bar{j}}]\phi_{\bar{k}\bar{i}}.$$

Recall from class that we can rewrite the commutator as a sum of curvatures

$$\begin{aligned}
[\nabla_l, \nabla_{\bar{i}}]\phi_{\bar{j}\bar{k}} &= F_{\bar{i}l}\phi_{\bar{j}\bar{k}} + R_{\bar{i}l\bar{j}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + R_{\bar{i}l\bar{k}}^{\bar{m}}\phi_{\bar{j}\bar{m}} \\
[\nabla_l, \nabla_{\bar{j}}]\phi_{\bar{k}\bar{i}} &= F_{\bar{j}l}\phi_{\bar{i}\bar{k}} + R_{\bar{j}l\bar{i}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + R_{\bar{j}l\bar{k}}^{\bar{m}}\phi_{\bar{i}\bar{m}}
\end{aligned}$$

Inserting these into the formula for the Laplacian, we find that

$$\begin{aligned}
(\Delta\phi)_{\bar{i}\bar{j}} = &-g^{l\bar{k}}\nabla_l\nabla_{\bar{k}}\phi_{\bar{i}\bar{j}} + g^{l\bar{k}}F_{\bar{i}l}\phi_{\bar{j}\bar{k}} + g^{l\bar{k}}R_{\bar{i}l\bar{j}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + g^{l\bar{k}}R_{\bar{i}l\bar{k}}^{\bar{m}}\phi_{\bar{j}\bar{m}} \\
&+ g^{l\bar{k}}F_{\bar{j}l}\phi_{\bar{i}\bar{k}} + g^{l\bar{k}}R_{\bar{j}l\bar{i}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + g^{l\bar{k}}R_{\bar{j}l\bar{k}}^{\bar{m}}\phi_{\bar{i}\bar{m}} \qquad (1)
\end{aligned}$$

(d) Given the above expression for the Laplacian in terms of the appropriate curvatures, we now reach a vanishing result. In particular, let L be a positive line bundle (i.e. h has positive curvature). Given a Kähler metric ω on X, we consider the Δ operator on $\Gamma(X, L^m \otimes \Lambda^{0,2})$. If we consider the inner product

$$\langle \Delta\phi, \phi\rangle = \int_X (\Delta\phi)_{\bar{i}\bar{j}}\overline{\phi_{\bar{k}\bar{l}}}g^{k\bar{j}}g^{l\bar{i}}h\frac{\omega^n}{n!}.$$

Looking at equation (1), we see that the first term in the integrand will (after integrating by parts and thus losing the minus sign) become $||\bar{\nabla}\phi||^2$, just as in the simpler case done in class. The rest of the integral,

$$\int_X g^{l\bar{k}}\left(mF_{\bar{i}l}\phi_{\bar{j}\bar{k}} + R_{\bar{i}l\bar{j}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + R_{\bar{i}l\bar{k}}^{\bar{m}}\phi_{\bar{j}\bar{m}} + mF_{\bar{j}l}\phi_{\bar{i}\bar{k}} + R_{\bar{j}l\bar{i}}^{\bar{m}}\phi_{\bar{m}\bar{k}} + R_{\bar{j}l\bar{k}}^{\bar{m}}\phi_{\bar{i}\bar{m}}\right)\overline{\phi_{\bar{p}\bar{q}}}g^{\bar{i}p}g^{\bar{j}q}h\frac{\omega^n}{n!},$$

can be forced positive by choosing $m \gg 1$ sufficiently large, by the positivity of the line bundle (and hence F). This forces the inner product $\langle\Delta\phi, \phi\rangle$ to be positive, and hence we find that $\Delta\phi$ has trivial kernel, as the inner product is *strictly* positive.

Problem 4

Let $f(z)$ be a holomorphic function in a neighborhood of 0 in $\mathbb{C}^n$. Consider the plurisubharmonic function $\phi(z) = (n+5)\log|z|^2$ (we showed in class that functions of this form are plurisubharmonic). Now suppose that

$$\int_{B_\rho(0)} |f(z)|^2 e^{-\phi(z)} = \int_{B_\rho(0)} |f(z)|^2|z|^{-2(n+5)} < \infty,$$

where we have chosen ρ such that $B_\rho(0) \subset U$. Holomorphicity of $f(z)$ allows us to express it as

$$f(z) = \sum_{|\alpha|\leq n+5} a_\alpha z^\alpha + E(z),$$

where $|E(z)| \leq C|z|^{n+6}$. Applying the triangle inequality, we find that

$$\int \left|\sum_{|\alpha|\leq n+5} a_\alpha z^\alpha\right|^2 |z|^{-2(n+5)} \leq \int \left(|f|^2 + |E(z)|^2\right)|z|^{-2(n+5)} < \infty.$$

The left-hand side can be computed in polar coordinates to be

$$\int_0^\rho \int_{S^{2n-1}} \left| \sum_{|a| \le \gamma} a_\alpha r^{|\alpha|} \omega^\alpha \right|^2 r^{-2(n+5)} r^{2n-1} dr d\sigma(\omega) < \infty$$

$$\int_0^\rho \sum_{|\alpha| \le n+5} |a_\alpha|^2 r^{2|\alpha|} r^{2n-1-2(n+5)} \left(\int_{S^{2n-1}} |\omega^\alpha|^2 d\sigma(\omega) \right) dr < \infty$$

Note that the integral diverges unless $a_\alpha = 0$ for $2|\alpha| + 2n - 1 - 2(n+5) \le -1$, or equivalently $|\alpha| \le 5$. Hence we find that $f(z)$ vanishes of order 5 at the origin, as $a_\alpha = 0$ for $|\alpha| \le 5$.