PROBLEM SET III

1. Let $\Lambda = \{m\omega_1 + n\omega_2;\ m, n \in \mathbf{Z}\}$ be a non-degenerate lattice in $\mathbf{C}$. Let $\zeta(z)$ be the Weierstrass ζ-function.

(a) Show that $\zeta(z+\omega_a) = \zeta(z) + \eta_a$ for a constant η_a, $a = 1, 2$.

(b) Show that the constants η_a satisfy the identity $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$.

2. Consider the same lattice, normalized with $\omega_1 = 1$, and $\omega_2 = \tau$, with $\mathrm{Im}\,\tau > 0$. For each $\delta = (\delta', \delta'')$, with δ' and δ'' real, define the θ-function with "characteristics" δ by

$$\theta[\delta](z|\tau) = \sum_{n\in\mathbf{Z}} e^{i\pi(n+\delta')^2\tau + 2\pi i(n+\delta')(z+\delta'')}$$

(The function $\theta[\delta](z|\tau)$ reduces to the function $\theta_1(z|\tau)$ discussed in class when $\delta = [\frac{1}{2}\,\frac{1}{2}]$.)

(a) Show that, if $\theta(z|\tau)$ is the usual θ-function, then

$$\theta[\delta](z|\tau) = e^{i\pi(\delta')^2\tau + 2\pi i\delta'(z+\delta'')}\theta(z+\delta'\tau+\delta''|\tau)$$

(b) Derive the following transformation laws for $\theta[\delta](z|\tau)$ under shifts by lattice elements

$$\theta[\delta](z+m+n\tau|\tau) = \theta[\delta](z|\tau)e^{-i\pi n^2\tau - 2\pi i n(z+\delta'') + 2\pi i\delta' m}$$

(c) Show that

$$\theta[\delta'+n, \delta''+m](z|\tau) = \theta[\delta', \delta''](z|\tau)e^{2\pi i\delta' m}$$

(d) Show that $\theta[\delta](z|\tau)$ is an even function of z for $\delta = [0\,0]$, $[\frac{1}{2}\,0]$, $[0\,\frac{1}{2}]$, and is an odd function of z for $\delta = [\frac{1}{2}\,\frac{1}{2}]$. Where are the zeroes of $\theta[\delta](z|\tau)$?

3. Let $\wp(z)$ and $\sigma(z)$ be the Weierstrass functions defined by the lattice Λ. Show that they are related to the Jacobi θ-functions by

$$\sigma(z) = e^{\frac{1}{2}\eta_1 z^2}\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}$$

and by

$$\wp(z) = -\frac{d^2}{dz^2}\log\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} + c(\tau)$$

where $c(\tau)$ is a constant depending only on τ. Determine $c(\tau)$.