Differentiable Manifolds Problem Set 7

Nilay Kumar

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Problem 1

We wish to show that for any $(e^{2\pi ix}, e^{2\pi iy})$ on the torus, and for any $\varepsilon > 0$, there exists an $n \in \mathbb{Z}$ such that

$$|(e^{2\pi ix}, e^{2\pi iy}) - (e^{2\pi i(x+n)}, e^{2\pi i\alpha(x+n)})| < 2\pi\varepsilon,$$

where the second term is on the curve γ. This will show that we can get arbitrarily close to any point on the torus. This specific choice allows simplifies the distance to

$$|1 - e^{2\pi i(\alpha x+\alpha n-y)}| < 2\pi\varepsilon$$
$$|e^{2\pi i(\alpha x+\alpha n-y)} - e^{2\pi im}| < 2\pi\varepsilon$$

which, using the trick from the book yields

$$|e^{2\pi i(\alpha x+\alpha n-y)} - e^{2\pi im}| \leq |\beta + \alpha n - m| < \varepsilon$$

where we've defined $\beta = \alpha x - y$. In other words, we want to show that $-\beta$ can be well approximated by $\alpha n - m$ for some integers n, m. This holds by an analog to Dirichlet's Approximation Theorem, which I can't quite figure out how to prove.

Problem 2

Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space.

(a) Let us for now consider the case where $n = 1$ and extrapolate from there. Let $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ be the quotient map generating the projective space. In the 2nd coordinate chart, we have that:

$$\hat{\pi}\left(x_1 + iy_1, x_2 + iy_2\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)$$

where we are implicitly working with reals. We can now compute the differential:

$$d\hat{\pi} = \begin{pmatrix} \frac{x_2}{x_2^2+y_2^2} & \frac{y_2}{x_2^2+y_2^2} & \frac{x_1(x_2^2+y_2^2)-(x_1x_2+y_1y_2)2x_2}{(x_2^2+y_2^2)^2} & \frac{y_1(x_2^2+y_2^2)-(x_1x_2+y_1y_2)2y_2}{(x_2^2+y_2^2)^2} \\ -\frac{y_2}{x_2^2+y_2^2} & \frac{x_2}{x_2^2+y_2^2} & \frac{y_1(x_2^2+y_2^2)-(y_1x_2+x_1y_2)2x_2}{(x_2^2+y_2^2)^2} & -\frac{x_1(x_2^2+y_2^2)-(y_1x_2+x_1y_2)2y_2}{(x_2^2+y_2^2)^2} \end{pmatrix}$$

Note that the first 2-by-2 block is clearly non-singular (the determinant is 1) and thus the matrix is full-rank and surjective (and smooth, since in this chart $x_2, y_2 \neq 0$). Thus π is a submersion in the case where $n = 1$. If we go to higher n, we will have, in the ith chart,

$$\hat{\pi}(z_1, \ldots, z_{n+1}) = \left(\frac{z_1}{z_i}, \ldots, \hat{z}_i, \ldots, \frac{z_{n+1}}{z_i}\right).$$

If we now compute the differential, we will find, similar to the $n = 2$ case, that there is a $2n$-by-$2n$ minor on the left that will be block diagonal (and look similar to that above depending on the coordinate chart chosen) whose determinant will always be 1. Thus, for any n, the differential will be surjective (and smooth), and we have a submersion.

(b) We wish to show that $\mathbb{CP}^1$ and S^2 are diffeomorphic; first note that both can be described with the usual two coordinate charts, call them (U_1, f_1) and (U_2, f_2) for $\mathbb{CP}^1$ and (V_1, g_1) and (V_2, g_2) for S^2. Let us define $\Phi_1 : U_1 \to V_1$ such that $\Phi_1 = g_1^{-1} \circ \mathrm{Id} \circ f_1$ and $\Phi_2 : U_2 \to V_2$ such that $\Phi_2 = g_2^{-1} \circ \mathrm{Id} \circ f_2$. Clearly, since g_i, f_i, Id are bijective and smooth with smooth inverse, Φ_i are diffeomorphisms from U_i to V_i. Consequently, we have diffeomorphisms from charts to charts - now we must put them together to get a diffeomorphism from $\mathbb{CP}^1$ to S^2. Define Φ to be Φ_1 on U_1 and Φ_2 on U_2. We must check that Φ_1 and Φ_2 agree where they overlap, i.e. $\Phi_1(p) = \Phi_2(p)$ whenever $p \in U_1 \cap U_2$. Pushing definitions, we find that this is equivalent to:

$$f_1 \circ f_2^{-1} = g_1 \circ g_2^{-1}.$$

For the sphere, we know we have $(u, v) \mapsto (4u/(u^2 + v^2), 4v/(u^2 + v^2))$, which in complex notation goes as $z \mapsto 1/z$. For the complex projective space, we do the computation:

$$f_1 \circ f_2^{-1}(z_2/z_1) = f_1(z_2, z_1) = z_1/z_2$$

and again we get $z \mapsto 1/z$. Consequently, the transition maps agree and we have a diffeomorphism Φ from $\mathbb{CP}^1$ to S^2.

Problem 3

Let M be a nonempty smooth compact manifold. Suppose there exists a smooth submersion $F : M \to \mathbb{R}^k$ for some $k > 0$. Since F is smooth, it must be continuous. Consequently, the image $N = F(M) \subset \mathbb{R}^k$ must be compact, as continuous maps take compact to compact. Note, however, that every smooth submersion is an open map, and since M is open, N must be open as well. But if N is both open and closed in $\mathbb{R}^k$, it must be either $\mathbb{R}^k$ itself or the null set $\varnothing$. As the image of a map, N cannot possibly be the null set, but it cannot be $\mathbb{R}^k$ either as N is compact and $\mathbb{R}^k$ is not (for $k > 0$). Thus we reach a contradiction - no such smooth submersion can exist.

Problem 4

Let $S : V \to W$ and $T : W \to X$ be linear maps. Suppose S and T are both injective. Then, $\ker S = \{0\}$ and $\ker T = \{0\}$. Clearly, then, $\ker T \circ S = \{0\}$ as well, because the only element that T maps to $0 \in X$ is $0 \in W$, and in turn, the only element of V sent by S to $0 \in W$ is $0 \in V$. Thus $T \circ S$ is injective.

Suppose now that S and T are both surjective. In other words, $\text{Im } S = W$ and $\text{Im } T = X$. Clearly $T \circ S$ must be surjective as well - given an $x \in X$, we can find a $w \in W$ such that $T(w) = x$ by surjectivity of T, and a $v \in V$ such that $S(v) = w$ by surjectivity of S.

Next suppose that $T \circ S$ is surjective. Take $x \in X$. Can we find a $w \in W$, such that $T(w) = x$? Using the surjectivity of $T \circ S$, we can find a $v \in V$ such that $T(S(v)) = x$. We simply take $w = S(v)$ and thus $T(w) = T(S(v)) = x$, and so T is surjective as well. However, S need not be surjective; consider, for example, $V = \mathbb{R}^2, W = \mathbb{R}^3, X = \mathbb{R}$ with S the inclusion map and T the projection map. $T \circ S$ is clearly surjective, as it takes $\mathbb{R}^2$, injects it into $\mathbb{R}^3$, and then projects out 2 dimensions onto $\mathbb{R}$ - the overall effect is simply a projection that drops one dimension. T is surjective, as it is a projection, but S is clearly not surjective, as $\mathbb{R}^3$ has higher dimension than $\mathbb{R}^2$.

Finally, suppose that $T \circ S$ is injective, i.e. if $T(S(v_1)) = T(S(v_2))$, then $v_1 = v_2$. Now let $S(v_1) = S(v_2)$. Then, applying T on both sides yields $T(S(v_1)) = T(S(v_2))$, which implies that $v_1 = v_2$, and thus S must be injective as well. However, T need not be injective; consider, for example $V = \mathbb{R}, W = \mathbb{R}^3, X = \mathbb{R}^2$ with S the inclusion map and T the projection map. $T \circ S$ is clearly injective, as it is simply the inclusion of $\mathbb{R}$ into $\mathbb{R}^2$, but T is not injective, as it is a projection onto $\mathbb{R}^2$.

Problem 5

Suppose V, W, X are finite-dimensional vector spaces, and $S : V \to W$ and $T : W \to X$ are linear maps.

(a) By the rank-nullity theorem we have that $\dim V = \text{rank } S + \text{null } S$. Since $\text{null} S \geq 0$, it is clear that $\text{rank } S \leq \dim V$. Note that equality is obtained only when $\text{null } S = 0$, i.e. when $\ker S = \{0\}$ and S is injective.

(b) Since $\text{rank } S = \dim \text{Im } S$ and $\text{Im } S \subset W$, it's clear that $\text{rank } S \leq \dim W$. Note that if S is surjective, then $\text{rank } S = \dim W$ because $\text{Im} S = W$. Conversely, if $\text{rank } S = \dim W$, then $\text{Im } S = W$, i.e. S is surjective.

(c) Let $\dim V = \dim W$. If S is injective, $\text{null } S = 0$ and thus $\dim W = \dim V = \text{rank } S$ by the rank-nullity theorem, and consequently, S is surjective as well, making it an isomorphism. If S is surjective, on the other hand, $\text{rank } S = \dim W$ and so by the rank-nullity theorem, $\text{null } S = \dim V - \text{rank } S = 0$, i.e. S is injective as well, making it an isomorphism.

(d) By the rank-nullity theorem we know that $\dim V = \text{rank } S + \text{null } S$ and that $\dim V = \text{rank } T \circ S + \text{null } T \circ S$. This implies that

$$\text{rank } T \circ S = \text{rank } S + \text{null } S - \text{null } T \circ S.$$

Note, however, that $\text{null } S - \text{null } T \circ S \leq 0$; this is because $T(\ker S) = 0$ and the fact that there may be vectors in S not in $\ker S$ that are sent to the $\ker T$, so thus the nullity of $T \circ S$ must be greater than or equal to the nullity of S. Hence, $\text{rank } T \circ S \leq \text{rank } S$. Note that equality only happens when there are no vectors in S that are sent to $\ker T$ other than those in $\ker S$, i.e. $\text{Im } S \cap \ker T = \{0\}$.

(e) Since $T \circ S$ is the composition of T with S, and since not necessarily all of $\text{Im } S$ is sent by T to non-zero elements in X (the kernel might be non-zero), we must have that $\text{rank } T \circ S \leq \text{rank } T$. We have equality only when what is nothing is lost due to S, i.e. $\text{Im } S + \ker T = W$.

(f) By the previous part, we know that $\text{rank } T \circ S \leq \text{rank } T$ with equality only if $\text{Im } S + \ker T = W$. But note that if S is an isomorphism, then $\text{Im } S = W$, and thus we must have equality $\text{rank } T \circ S = \text{rank } T$.

(g) By part (d) we know that rank $T \circ S \leq$ rank S with equality only if Im $S \cap \ker T = \{0\}$. But note that if T is an isomorphism, then $\ker T = \{0\}$, and thus the condition is satisfied (as Im S will always contain 0) and we have equality, rank $T \circ S =$ rank S.