Quantum Mechanics for Mathematicians: Problem Set 7

Due Wednesday, February 20

Problem 1: Show that the Lie algebras of $Spin(n)$ and $SO(n)$ are the same by showing that the quadratic elements

$$\frac{1}{2}\gamma_j\gamma_k$$

for $j < k$ of the Clifford algebra $Cliff(n, \mathbf{R})$ satisfy the same commutation relations as the L_{jk} (elementary antisymmetric matrices).

Problem 2: Show that conjugation by an exponential of the quadratic Clifford algebra element of the previous problem gives a rotation in the $j-k$ plane.

Problem 3: Using the construction of spinors given in class, consider the cases of the Clifford algebra in 4 or 6 dimensions, corresponding to the fermionic oscillator in 2 or 3 variables.

- The Hamiltonian operator generates a $U(1)$ action on the spinors. What is it explicitly? This $U(1)$ is a subgroup of the *Spin* group (in 4 or 6 dimensions respectively), and so acts not just on spinors, but on vectors as a rotation. What is the rotation on vectors?
- For a rotation by an angle θ in the $j-k$ plane in 4 or 6 dimensions, what are the elements of the *Spin* group that correspond to this, and how do they act on the fermionic oscillator states? Do this by expressing things in terms of annihilation and creation operators and their action on the spinors, thought of as a fermionic oscillator state space.

Problem 4: Use the anti-commuting variable analog of the Bargmann-Fock construction construct spinors in even dimensions as spaces of functions of anti-commuting variables. Find the inner product on such spinors that is the analog of the one constructed using an integral in the bosonic case. Show that the operators a_{Fj} and $a_{Fj}^\dagger$ are adjoints with respect to this inner product.