Notes on Differentiable Manifolds

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Last updated: February 26, 2013

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Recommended textbooks:

- John Lee: *Introduction to Smooth Manifolds* (2012)
- Spivak: *Differential Geometry: A Comprehensive Introduction*
- L. Tu: *An Introduction to Manifolds* (2008 E-book available)

Problem sets will be assigned every one/two weeks through email. Some homework problems will be taken from Lee (second edition). There will most likely be two midterms and a final, all in-class.

Office hours are on Fridays from 11am to 12pm.

1 Introduction

Definition 1. A function f defined on $\mathbb{R}^n$ is C^k for a positive integer k if $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_l}}$ exists and is continuous for any positive integer $l \leq k$, where $1 \leq i_1 \cdots i_l \leq n$. f is C^∞ if it is C^k for any positive integer k.

Example 1. $f(x) = x^{1/3}$ for $x \in \mathbb{R}$ is C^0 but not C^1.

Example 2. $f(x) = x^{1/3} + k$ for $x \in \mathbb{R}$ is C^k but not C^{k+1}, for $k \geq 1$.

Definition 2. A **coordinate chart** (U, ϕ) on a topological space X is an open set $U \subset X$ together with a map $\phi : U \to \mathbb{R}^n$ such that ϕ is a homeomorphism onto $\phi(U)$, an open set in $\mathbb{R}^n$. In other words, (U, ϕ) gives each $p \in U$ a coordinate.

Example 3. Let $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Let $U = \{z > 0\} \cap S^2$ be the upper hemisphere. x, y, z are not good coordinates in that they are not free - they are constrained to the surface. Note that if we define $\phi : U \to \mathbb{R}^2$ such that it takes $(x, y, z) \to (x, y)$, we have a projection map and we now have free coordinates (z can be computed). This is a **graphical coordinate chart**. We can work similarly with the lower hemisphere. But what about the equator? We can build similar charts for the equator by projecting onto different planes. It is clear, however, that to cover every point, we will need a total of 6 graphical charts to cover S^2.

This is nice because we can now do calculus on the open sets that ϕ maps us to in Euclidean space.

Example 4 (Stereographic projection of S^2)**.** Use a different model for S^2. Consider $\{(x, y, z) | x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}\} \subset \mathbb{R}^3$, the sphere of radius 1/2 centered at $(0, 0, \frac{1}{2})$. Note that the south pole's coordinates are $(0, 0, 0)$ and the north pole's are $(0, 0, 1)$. Imagine that there is a light source at the north pole, which projects through the sphere onto the xy-plane. If the line hits the point (x, y, z) on the sphere, we can solve for the point at which it hits the xy-plane. The line is given by $(0, 0, 1) + t(x, y, z - 1)$ for $t \in \mathbb{R}$. Solving this for where $z = 0$ yields $t = \frac{1}{1-z}$. The point is then $(0, 0, 1) + \frac{1}{1-z}(x, y, z-1) = (\frac{x}{1-z}, \frac{y}{1-z}, 0)$. This gives a coordinate chart (U, ϕ) with $U = S - \{(0, 0, 1)\}$ (as the chart is undefined there) and $\phi : U \to \mathbb{R}^2$ that maps $(x, y, z) \to (\frac{x}{1-z}, \frac{y}{1-z})$.

In order to cover the south pole as well, we can perform stereographic projection from the south pole onto $z = 1$ plane. The corresponding line is now given by $(0, 0, 0) + t(x, y, z)$ which has $z = 1$ for $t = \frac{1}{z}$, and the corresponding point is $(\frac{x}{z}, \frac{y}{z}, 1)$. This gives us a (V, ψ) where $V = S - \{(0, 0, 0)\}$ with $\psi : V \to \mathbb{R}^2$ mapping $(x, y, z) \to (\frac{x}{z}, \frac{y}{z})$.

Example 5. Let X be the set of all lines on $\mathbb{R}^2$. X is a topological space (check this!). Take the set $U = \{$ lines of the form $y = mx + c\}$, which is the collection of all non-vertical lines. To cover the vertical lines, we can have $V = \{$ lines of the form $x = \bar{m}y + \bar{c}\}$, the collection of all non-horizontal lines. We now define $\phi : U \to \mathbb{R}^2$ that maps $y = mx + c \to (m, c)$ and $\psi : V \to \mathbb{R}^2$ that maps $x = \bar{m}y + \bar{c} \to (\bar{m}, \bar{c})$.

Notice that for this example and the stereographic projection example, there are instances where the charts overlap. For the sake of consistency, we want the coordinate charts to be compatible with one another. For example, a function that is differentiable in one chart should be differentiable in the other as well.

Definition 3. Given a coordinate chart (U, ϕ) and a function f defined on U, we can consider $f \circ \phi^{-1}$ as a function on $\phi(U)$ and differentiate $f \circ \phi^{-1}$. Suppose (V, ψ) is another coordinate chart and that $U \cap V \neq \varnothing$. Now we can consider $f \circ \psi^{-1}$ as a function to do calculus with. Now the question is: is $f \circ \phi^{-1}$ differentiable the same as $f \circ \psi^{-1}$ differentiable? This should be the case! We are considering a function on an abstract space, and the coordinates should respect properties such as differentiablity. So let us write $f \circ \psi^{-1} \circ (\psi \circ \phi^{-1})$, which is differentiable if the term in the parentheses is differentiable. We can do the same, but with ϕ and ψ switched. Thus, we want to make sure that both $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are differentiable. These are called **transition maps** of these coordinate charts. Two coordinate charts are **smoothly compatible** if their transition maps are diffeomorphisms (or if, trivially, they don't intersect). These diffeomorphisms are from $\phi(U \cap V)$ to $\psi(U \cap V)$ or vice versa.

Returning to the previous manifold of lines, suppose we have a line labeled by (m, c) with $m \neq 0$. This can be expressed in the other chart as $(m^{-1}, -m^{-1}c)$. It turns out, that for $m \neq 0$, these transition maps are differentiable. For the stereographic projection of S^2, it is the same, just a little trickier.

Definition 4. An **atlas** $\mathcal{A}$ for a topological space X is a collection of coordinate charts that covers X such that any two charts in $\mathcal{A}$ are smoothly compatible.

Example 6. Take the set of all lines through the origin in $\mathbb{R}^3$. Since only the direction matters, each line might be represented by a non-zero vector. This is the same as the quotient space $\mathbb{R}^3 \setminus \{(0,0,0)\} / \sim$ with $(x_1, x_2, x_3) \sim \lambda(x_1, x_2, x_3)$. Thus it is equipped with the quotient topology; i.e.$\Pi : \mathbb{R}^3 \;\; \{(0,0,0)\} \to \mathbb{RP}^2$ is continuous. We use $[x_1 x_2 x_3]$ to denote the equivalence class of (x_1, x_2, x_3). On the open set (check this) $U_1 = \{x_1 \neq 0\}$, we can use the coordinate chart $\phi_1 : U_1 \to \mathbb{R}^2$ that takes $(x_1, x_2, x_3) \to (x_2/x_1, x_3/x_1)$. However, we have not covered the whole set, so we repeat this process for $x_2 \neq 0$ and $x_3 \neq 0$. We claim that this set of charts is an atlas. It is obvious that these charts covers the space. We must now check that the transition maps are smoothly compatible. For example, we must check that $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism. For $[x_1, x_2, x_3] \in U_1 \cap V_2$, it's clear that $\phi_2 \circ \phi_1^{-1}(x_2/x_1, x_3/x_1) = (x_1/x_2, x_3/x_2)$. To show that this is a diffeomorphism, we choose coordinates (u, v) on $\mathbb{R}^2$ and write the function in terms of these coordinates: $u = x_2/x_1$ and $v = x_3/x_1$. We know that $x_1 \neq 0, x_2 \neq 0$ because of our domain. Thus, $\phi_2 \circ \phi_1^{-1} = (1/u, v/u)$. Again, by our domains, $u \neq 0$, and thus this map is differentiable. The inverse is found by writing $(1/u, v/u)$ as (p, q) which yields $(u, v) = (1/p, q/p)$ which is differentiable as well. One can check that this map is also one-to-one and onto, and we are done.

In general, it turns ou that $\mathbb{RP}^n$ requires $n + 1$ coordinate charts.

Definition 5. Two atlases are compatible (or equivalent) if their union is another atlas.

Definition 6. A **differentiable (smooth) structure** on a topological space X is an equivalence class of atlases (a maximal atlas).

Example 7. Take the curve $\mathcal{C} : y = x^{\frac{1}{3}}$ in $\mathbb{R}^2$. Recall that this curve has a vertical tangent at $x = 0$. This curve has the subspace topology. We can consider the graphical coordinate charts $\phi_1 : (x, y) \to x$ and take $U_1 = \mathcal{C}$. We can define an atlas $\mathcal{A}_1 = \{(U_1, \phi_1)\}$. We can also take $\phi_2 : (x, y) \to y$ and $\mathcal{A}_1 = \{(U_2, \phi_2)\}$. Are these equivalent? Consider the height function h on $\mathcal{C}$ (the y coordinate). Consider $h \circ \phi_1^{-1}(x) = x^{\frac{1}{3}}$. Consider instead $h \circ \phi_2^{-1}(y) = y$. But h is only differentiable on the second chart! Thus these atlases are not equivalent.

Example 8. Take the real line. $f(x) = x^{1/3}$ is not differentiable. Suppose we make a new coordinate system that assigns each point its cubic root. The points, of course go home all happy, but now f is differentiable!

In order to define what a "smooth manifold" we must impose global topological conditions to rid ourselves of certain pathological examples.

Definition 7. A topological space X is **Hausdorff** if any two points can be separated by disjoint open sets; i.e. $\forall p, q \in X$, there exist U, v open such that $p \in U, q \in V$ and $U \cap V = \varnothing$.

Definition 8. A topological space X is **second countable** if it has a countable basis of open sets.

Recall that a basis $\mathcal{B}$ is a subset of the collection of all open sets such that any open set can be written as a union of members of $\mathcal{B}$.

Example 9. Note that $\mathbb{R}^n$ is second countable because we can take a basis $\mathcal{B}$ that is the collection of all open balls with rational centers and rational radii. Additionally, it is clear that $\mathbb{R}^n$ is Hausdorff.

Note that a subspace of a Hausdorff, second-countable topological space is itself Hausdorff and second countable. Thus any subset of $\mathbb{R}^n$ is Hausdorff and second countable. Pathological spaces:

- the disjoint union of uncountably many copies of $\mathbb{R}$ is not second countable.
- a real line with two origins is not Hausdorff.

Definition 9. A **differentiable/smooth manifold** is a Hausdorff, second countable topological space with a smooth (differentiable) structure.

Example 10. Suppose $U \subset \mathbb{R}^n$ is open with $F : U \to \mathbb{R}^m$ that is C^∞. The graph of F is $\Gamma = \{(x, y)|x \in U, y = F(x)\} \subset \mathbb{R}^n \times \mathbb{R}^m$ is a differentiable manifold. Γ is Hausdorff and second countable because $\Gamma \subset \mathbb{R}^{n+m}$. We can take $\mathcal{A} = \{(\Gamma, \Pi)\}$ where $\Pi : \Gamma \to U$ maps $(x, F(x)) \to x$ (check that this is a homeomorphism).

Example 11. Take $S^2 = \left\{(x, y, z)|x^2 + y^2 + z^2 = 1\right\} \subset \mathbb{R}^3$. This is Hausdorff and second countable. We can take the 6 graphical coordinate charts $\{x > 0\}, \{x < 0\}, \{y > 0\}, \{y < 0\}, \{z > 0\}, \{z < 0\}$, with the appropriate homeomorphisms (projections, in this case) to map from hemispheres to the planes "below" them. Take for example the maps ϕ_1, ϕ_2 that map from the upper and lower hemispheres to the x-y plane respectively. Also, take ϕ_3 to be the map for $\{y > 0\}$ to the x-z plane. Then, on $U_1 \cap U_1$ we have that $\phi_3 \circ \phi_1^{-1} : (x, y) \to (x, z) = \sqrt{1 - x^2 - y^2}$. Clearly this is differentiable as long as $z \neq 0$. Indeed, all the charts turn out to be smoothly compatible, and we have a manifold. This manifold is what we call a **level set** of $F(x, y, z) = x^2 + y^2 + z^2$.

In general, though, what about $\{x|F(x) = c\} \subset \mathbb{R}^n$? Is it a manifold? (Indeed, it turns out that a set of equations $F_i(x_1 \cdots x_m) = c_n$ has solutions that often form a manifold.)

Note: take $U \subset \mathbb{R}^n$ open. Take $F : U \to \mathbb{R}$ continuous for any $c \in \mathbb{R}$. Then, $F^{-1}(c)$ is a closed subset of U. In fact, F^{-1} is always Hausdorff and second countable as a subset of $\mathbb{R}^n$. When is $F^[- 1](c)$ a differentiable manifold? Let us first ask: when can we solve one variable in terms of the others? In other words, given $F(x, y, z) = c$, can we write, for example, $z = f(x, y)$?

Example 12. Suppose F is linear: $F(x, y, z) = ax + by + dz$. On $F(x, y, z)$, we can solve x in terms of y, z as long as $a = \frac{\partial F}{\partial x} \neq 0$.

Theorem 1 (Implicit function theorem)**.** *Suppose U is open in $\mathbb{R}^n$ and $F : U \to \mathbb{R}$. Suppose $F(a) = c$, with $a \in U$, and $\frac{\partial F}{\partial x^n}(a) \neq 0$. Then, there exists a neighborhood V of a and a unique function $f(x_1 \cdots x_{n-1})$ such that $\frac{\partial F}{\partial x_n} \neq 0$ on V and*

1. $V \cap F^{-1}(c) = V \cap \{(x_1 \cdots x_{n-1}) | x_n = f(x_1 \cdots x_n\}$
2. $\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} / \frac{\partial F}{\partial x_n}$ *for* $i = 1 \cdots n-1$

Proof. Go over this in your own time/past notes. □

Theorem 2. *Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is C^∞. Take $c \in \mathbb{R}$ and $F^{-1} \neq \varnothing$ and suppose $\nabla F(x) \neq 0$ for all $x \in F^{-1}(c)$. c is called a* ***regular value*** *of f. Then, $F^{-1}(c)$ is an $(n-1)$ dimensional smooth manifold.*

Proof. $F^{-1}(c)$ is Hausdorff and second countable. Now we wish to produce a differentiable structure. Consider $\tilde{U}_i = \left\{ \frac{\partial F}{\partial x_i} \neq 0 \right\} \cap F^{-1}(c)$. U_i is open in $F^{-1}(c)$ and $F^{-1}(c) \subset \cup_{i=1}^n U_i$ by the assumption that the gradient is nonzero on $F^{-1}(c)$. For each point $a \in U_i$, by the implicit function theorem, there exists a neighborhood U_a of a (may assume $U_a \subset U_i$ by taking an intersection) such that $U_a \cap F^{-1}(c)$ is the graph of a function $\{x_i = f_i(x_1 \cdots x_{i+1} \cdots x_n)\}$ and this f_i is unique.

We take the collection of all these U_a $a \in F^{-1}(c)$ as coordinate charts. When two coordinate charts overlap, if they belong to the same U_i, by uniqueness of f_i the transition map must be the identity map. But when two coordinate charts overlap but belong to different U_i, the transition maps consist going from one projection to another (or unprojecting), which is simply a matter of dropping (or adding back) certain x_i. Since, by the implicit function theorem, we may write x_i smoothly in terms of the other coordinates, the transition maps are smooth. □

For $F(x, y, z) = x^2 + y^2 + z^2$, the gradient is $\nabla F = (2x, 2y, 2z)$. Thus, the sphere is a differentiable manifold as the choice of $c = 1$ yields a gradient that is not the zero vector.

Remark (invariance of dimension): If a smooth manifold is connected, then any coordinate chart is homeomorphic to an open subset of $\mathbb{R}^n$ for a fixed n, this n is called the dimension of the manifold.

Theorem 3. *Take U open in $\mathbb{R}^{n+m}$ and $F : U \to \mathbb{R}^m$ smooth. Let $c \in \mathbb{R}^m$ and $F^{-1}(c) \neq \varnothing$. Suppose that for all $a \in F^{-1}(c)$, $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is of rank m (full rank). Then, $F^{-1}(c)$ is a n dimensional smooth manifold and c is called a* ***regular value*** *of F. Recall that DF_a is simply the $m \times (n+m)$*

matrix of partial derivatives of F_i. The corresponding implicity function theorem says that if the determinant of the $m \times m$ submatrix of DF_a is not zero, then $x_1 \cdots x_m$ can be locally solved in terms of $x_{m+1} \cdots x_{m+n}$.

Example 13. Let's look at the simple case $m = 3$, $n = 2$:

$$F_1(x_1, x_2, x_3) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$F_2(x_1, x_2, x_3) = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

It's clear that if the 2×2 left submatrix has non-zero determinant, and we set $F_1 = c_1, F_2 = c_2$, we will be able to solve for x_3 in terms of x_1, x_2.

(January 31, 2012)

Definition 10. Let M be a smooth manifold. $f : M \to \mathbb{R}$ is a **smooth function** if for all $p \in M$ there exists a coordinate chart (U, ϕ) containing p such that $f \circ \phi^{-1}$ is C^∞ on $\phi(U)$. Note that there is no ambiguity in terms of overlap of charts, as a smooth manifold's charts are smoothly compatible.

Definition 11. Let M, N be smooth manifolds. A map $F : M \to N$ is a **smooth map** if for any point $p \in M$, there exists a coordinate chart $(U, \phi$ containing p and a coordinate chart (V, ψ) containing $F(p)$, such that $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is smooth. Check that this definition is well-defined.

Definition 12. $F : M \to N$ is a **diffeomorphism** if both F and F^{-1} are differentiable.

Example 14. There are two different differentiable structures on $\mathbb{R}$. One of these simply consists of the atlas $\{(\mathbb{R}, \mathrm{Id})\}$. However, we can also use $\big\{(\mathbb{R}, \psi(x) = x^3)\big\}$. These are different because the cubic root function, for example, is differentiable only under the second structure. Now consider F from the first to the second structure, F being the cubic root function. We claim that both $\psi \circ F \circ \mathrm{Id}^{-1}$ and $\mathrm{Id} \circ F^{-1} \circ \psi^{-1}$ are identity functions. Thus F, by the definitions above, is a diffeomorphism between these two differentiable manifolds. These two differentiable structures are equivalent under diffeomorphism.

Consider the vector space (over $\mathbb{R}$) of smooth functions of M. What constitutes this vector space? Can we, for example, approximate a characteristic function by smooth functions? Furthermore, suppose we are given k points on M - is there a smooth function that has a given value at each of these points? Note that if we can find this, by taking k to infinity, and setting all but one of the given values to 0, we will have shown that the vector space is infinite dimensional.

Definition 13. A function f is **real analytic** at a point p if f is equal to its Taylor expansion at p in a neighborhood of p.

Theorem 4. *If an analytic function is zero on an open set, it is zero everywhere as long as M is connected.*

Note that this theorem shows that analytic functions could not be used to approximate a characteristic functions, as they have to be zero in some open set.

Thankfully, there are smooth functions that are not analytic. Take, for example,

$$f(x) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

One can show that $f^n(0) = 0$ for any positive integer n, which means that the Taylor series of f at $t = 0$ is identically zero but f is "not" the zero function in any neighborhood of $t = 0$. One can show (via, say, l'Hopital)

$$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t} = \lim_{t \to 0} \frac{e^{-1/t}}{t} = 0$$

and repeat this for any higher derivatives.

Lemma 5 (Existence of cut-off and bump functions)**.**

Definition 14. The support of a function is defined to be

$$\text{supp }(g) = \overline{\{x|g(x) \neq 0\}}$$

Theorem 6 (Existence of partition of unity)**.** *Let M be a smooth manifold and $\mathcal{O}$ is an open cover of M. Then there exist C^∞ functions $\phi_\alpha : M \to [0, 1]$ for $\alpha \in A$ (where A is an index set) such that*

1. *The set of supports $\{\text{ supp } \phi_\alpha\}_{\alpha \in A}$ is locally finite. In other words, for any point $p \in M$, there exists a neighborhood of p that intersects a finite number of* $\text{supp } \phi_\alpha$.
2. $\sum_\alpha \phi_\alpha(p) = 1$ *for all $p \in M$*
3. *For all $\alpha \in A$, there exists $U \in \mathcal{O}$ such that* $\text{supp } \phi_\alpha \in U$.

We will not prove this, as it is rather long and difficult. However, this theorem is important as it will allow us to move from talking about local properties to global properties.

** FINISH : cutoff/bump functions **

Example 15.

$$\mathbb{R} = (-\infty, 2.5) \cup (0.5, \infty)$$

Take $\phi_1(t) = \frac{f(2-t)}{f(2-t)+f(t-1)}$ and $\phi_2(t) = \frac{f(t-1)}{f(2-t)+f(t-1)}$. These satisfy the above theorem, as there are a finite number of them, they clearly add up to 1, and each one's support is contained in a different set of the open cover.

Theorem 7. *Suppose A is a closed subset and U is an open subset of M with $A \subset U$. Then there exists a smooth bump function $\psi : M \to \mathbb{R}$, $0 \leq \psi \leq 1$ such that $\psi \equiv 1$ on A and $\operatorname{supp} \psi \subset U$.*

Proof. Let $U_0 = I$, $U_1 = M \setminus A$. Then, $\{U_1, U_2\}$ is an open cover, and there exist ψ_α partition of unity. $\operatorname{supp} \psi_\alpha \subset U$ or $\operatorname{supp} \psi_\alpha \subset M \setminus A$ for each α. Consider $\psi_1 = \sum_{\operatorname{supp} \psi_\alpha \subset M \setminus A} \psi_\alpha$. Then, $\psi_1 \equiv 0$ on A. Claim: $\psi_0 = 1 - \psi_1$ is the desired function, as $\psi_0 \equiv 0$ on A. On the other hand, $\psi_0 = \sum_{\operatorname{supp} \psi_\alpha \subset U} \psi_\alpha$, meaning $\operatorname{supp} \psi_0 \subset U$. □

Theorem 8. *Suppose $A \subset U \subset M$ with A closed and U open and $f : A \to \mathbb{R}$ is a function that can be extended to a smooth function in a neighborhood of A. Then, there exists a smooth function $\tilde{f} : M \to \mathbb{R}$, such that $\tilde{f}|_A = f|_A$ and $\operatorname{supp} \tilde{f} \subset U$.*

The proof of this theorem is very similar to that of the above theorem, and will likely be in a future homework.

2 Tangent spaces

(February 5, 2013)

Example 16. Let $F : \mathbb{R}^n \to \mathbb{R}$ be smooth and $c \in \mathbb{R}$ be a regular value of f. The tangent space of $f^{-1}(c)$ at $a \in f^{-1}(c)$ is the $(n-1)$ dimensional affine space passing through a and orthogonal to $\nabla f(a)$. The formula for the tangent space, in this case, is simply $\nabla f \cdot (x - a)$. For the case of S^2, for example, we have $(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$.

Note that here, the manifold and its tangent space are embedded in an ambient Euclidean space, but we want to be able to discuss tangent spaces "in vacua," so to speak. First, recall that we can view directional derivatives as tangent vectors.

Definition 15. Given $v \in \mathbb{R}^n$ and a smooth function defined near a, we define the **directional derivative** of f at a in the direction of v to be:

$$D_v|_a f = \frac{d}{dt}|_{t=0} f(a + tv) = \sum_{i=1} v_i \frac{d}{dx^i}|_a f$$

In other words, $D_v|_a$ assigns a number to each smooth function. In this sense, the directional derivative lives in the dual space, because when we consider them as operators, they form a vector space that is spanned by $\frac{d}{dx^i}|_a$. Note that for $v = 0$, the operator will always return 0.

To define tangent spaces intrinsically, we will have to extract certain abstract properties of these directional derivatives.

Definition 16. X is a **derivation** at a if X is a linear map $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ that satisfies:

- $X(cf) = cX(f)$
- $X(f + g) = X(f) + X(g)$
- $X(fg) = f(a)X(g) + X(f)g(a)$

for $f, g \in C^\infty(\mathbb{R})^n$. This definition can easily be extended to manifolds.

Remark. A directional derivative at a is a derivation at a. Additionally, the set of derivations at a forms a real vector space:

- $(cX)(f) = c(Xf)$

- $(X_1 + X_2)(f) = X_1(f) + X_2(f)$
- the zero element is the zero map from $C^\infty(\mathbb{R}^n) \to \mathbb{R}$

It is not quite obvious that the vector space of derivations is indeed finite-dimensional, as the space of directional derivatives is.

Theorem 9. *The vector space of directional derivatives at a and the vector space of derivations at a are isomorphic as real vector spaces.*

We will need two lemmas to prove this theorem.

Remark. In fact, only the local behavior of a smooth function near a "matters." We should define a derivation as a linear map from the space of germs of functions at a.

Definition 17. A **germ** of a function at $a \in \mathbb{R}^n$ is a pair (f, U) with $a \in U$, such that f is differentiable in U. We define also an equivalence relation $(f, U) \sim (g, V)$ if $f \equiv g$ on $U \cap V$.

Lemma 10. *Let c denote a constant function with value c. Then we have $X(c) = 0$ for any derivation.*

Proof. Using linearity and the third property,

$$X(c) = cX(1) = cX(1 \cdot 1) = c\,(1X(1) + 1X(1) = 2cX(1))$$

This property is true only for 0, and we are done. $\square$

Lemma 11. *Suppose f is differentiable in a neighborhood U of a, then there exists an $\varepsilon > 0$ such that $B_a(\varepsilon) \subset U$ and differentiable functions g_i in $B_a(\varepsilon)$ such that*

$$f(x) = f(a) + \sum_{i=1}^{n} g_i(x)(x^i - a^i)$$

in $B_a(\varepsilon)$ and

$$g_i(a) = \frac{\partial}{\partial x^i}|_a f$$

Remark. Note that in the one-dimensional case, $f(x) = f(a) + (x - a)g(x)$ and $g(a) = f'(a)$,

$$g(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & : g \neq a \\ f'(a) & : x = a \end{cases}$$

Now we prove the theorem.

Proof. Any directional derivative is a derivation, so if suffices to prove the inclusion map from the space of derivatives to the space of derivations is an isomorphism. This map is linear (exercise), so now we prove that the map is injective and surjective.

If $\sum_{i=1}^n v_i \frac{\partial}{\partial x^i}|_a f = 0$ for any $f \in C^\infty(\mathbb{R}^n)$, we want $v_i = 0$ to show that the nullspace is zero. Note that here we can use the fact that $\frac{\partial x^j}{\partial x^i} = \delta^{ij}$: take $f = x^j$, which will give us precisely $v_j = 0$ for each j. Thus, this map is injective.

If a derivation X corresponds to a directional derivative $\sum_{i=1}^n v_i \frac{\partial}{\partial x^i}|_a$, what are the v_i's? In this case, again in analogy to typical vector spaces, we want $X(x^j) = \sum_{i=1}^n v_i \frac{\partial}{\partial x^i}|_a x^j = v_j$. Consequently, take $\sum_{i=1}^n X(x^i) \frac{\partial}{\partial x^i}|_a$ and we claim $X = \sum_{i=1}^n X(x^i) \frac{\partial}{\partial x^i}|_a$ as derivations, or $X(f) = \sum_{i=1}^n X(x^i) \frac{\partial}{\partial x^i}|_a f$ for any $f \in C^\infty(\mathbb{R}^n)$. We apply the second of the above lemmas and the product rule, and write f in the corresponding form

$$
\begin{aligned}
X(f) = X\left(f(a) + \sum_{i=1}^n g_i(x)(x^i - a^i)\right) &= \sum_{i=1}^n X\left(g_i(x)(x^i - a^i)\right) \\
= \sum_{i=1}^n g_i(a) X(x^i - a^i) &= \sum_{i=1}^n X(x^i) \frac{\partial}{\partial x^i}|_a f
\end{aligned}
$$

□

Definition 18. Let M be a smooth manifold and $p \in M$. The **tangent space** of M at p, T_pM, is the vector space of all derivations at p. Recall that $X : C^\infty(M) \to \mathbb{R}$ is linear and $X(fg) = f(p)X(g) + g(p)X(f)$.

Remark. Suppose $F : M \to N$ is differentiable and $f : N \to \mathbb{R}$ is a differentiable function. Then we can **pullback** f by F:

$$F^*(f) = f \circ F$$

Suppose $X \in T_pM$ is a derivation. Then we can **pushforward** X by F. $F_*(X)$ is a derivation at $F(p)$ so $F_*(X) \in T_{F(p)}N$. Then,

$$F_*(X)(f) = X(F^* f) = X(f \circ F)$$

In other words, F_* takes the derivation on T_pM to a derivation on $T_{F(p)}N$ that can act on a function on N.

(February 7, 2013)

Recall for $\mathbb{R}^n$, $T_p\mathbb{R}^n$ is the real space spanned by $\frac{\partial}{\partial x^i}|_p$. For a coordinate chart (U, ϕ) on M, we can identify U with $\phi(U)$ and take $\frac{\partial}{\partial x^i}|_{\phi(p)}$ as a basis for T_pM. Check the textbook for how this is done precisely via pushforwards (partial derivatives are pushed forward from the tangent space of the Euclidean space to derivations on the tangent space of the manifold).

What if two coordinate charts (U, ϕ) and (V, ψ) overlap?

$$\tilde{x}^j = (\psi \circ \phi^{-1})^j(x^1, \cdots, x^n)$$

Take, for example a point in $\mathbb{R}^2$ – one can choose polar coordinates and cartesian coordinates. For a given point, we can simply relate these coordinates by the usual $x = r\cos\theta$ and $y = r\sin\theta$. The partial derivatives of the coordinates can be written via the chain easily

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

and similarly for $\frac{\partial}{\partial y}$. In general, it should be clear that

$$\frac{\partial}{\partial x^i}|_p = \frac{\partial(\psi \circ \phi^{-1})^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}|_p,$$

where we have used the Einstein summation convention.

Let us now consider in more detail pushforwards of derivations. Take $F : \mathbb{R}^n \to \mathbb{R}^m$ differentiable. If $g : \mathbb{R}^m \to \mathbb{R}$ and X is a derivation at $p \in \mathbb{R}^n$, we consider the pushforward F_*X, which is a derivation at $F(p) \in \mathbb{R}^m$ that operates of g. In terms of local coordinates, what is $F_*(\frac{\partial}{\partial x^i}|_p)$? If x^α and y^α are our coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $y^\alpha = F^\alpha(x^1 \cdots x^n)$

$$\begin{aligned} F_*(\frac{\partial}{\partial x^i}|_p)(g) &= \frac{\partial}{\partial x^i}|_p g(F) = \frac{\partial}{\partial x^i}|_p g\left(F^1(x)\cdots F^m(x)\right) \\ &= \frac{\partial g}{\partial y^\alpha}\left(F(p)\right)\frac{\partial F^\alpha}{\partial x^i}|p \end{aligned}$$

and so we write

$$F_*(\frac{\partial}{\partial x^i}|_p) = \frac{\partial F^\alpha}{\partial x^i}|_p \frac{\partial}{\partial y^\alpha}|_{F(p)}$$

Now we see more concretely, with the coordinates, how the pushforward takes derivations in the tangent space of one manifold the the tangent space of another.

3 Tangent bundles

So now we have at each point, a vector space of the same dimension as the base manifold. Now consider putting all of these tangent spaces together.

Definition 19. The **tangent bundle** TM of a manifold M is, as a set, the disjoint union of all the tangent spaces T_pM of M:

$$TM = \coprod_{p \in M} T_pM$$

Naturally, as TM is a set defined based off of a manifold, the question arises: what is the topology of TM?

Consider $T\mathbb{R}$; with each point on the line, we associate another, tangent, line. One can do this in many ways. On $\mathbb{R}^2$ one can construct a topology from the following metric:

$$d\left((x_0, y_0), (x_1, y_1)\right) = |y_0| + |x_0 - x_1| + |y_1|$$

In this topology, distances are defined by projecting down to the x-axis, moving along it, and then moving back up. This is analogous to how one might define a topology on lines intersecting $\mathbb{R}$, since the tangent spaces can't naturally "talk" to each other: one projects down to the base space, moves along, and then goes to the second tangent space. Incidentally, it turns out that $T\mathbb{R} = \mathbb{R}^2$ quite naturally.

Example 17. Take $S^2 = \left\{(x, y, z) | x^2 + y^2 + z^2 = 1\right\}$. The tangent space at a point consists of all vectors orthogonal to the sphere at that point. Therefore,

$$TS^2 = \left\{(x, y, z, u, v, w) \subset \mathbb{R}^6 | x^2 + y^2 + z^2 = 1 \text{ and } xu + yv + zw = 0\right\},$$

where the second equation enforces the orthogonality. One can check that this is a differentiable manifold of dimension 4 by viewing it as a level set in $\mathbb{R}^6$ with two constraints (at a regular value).

More generally, if c is a regular value of f, $f^{-1}(c) \subset \mathbb{R}^n$,

$$T(f^{-1}(c)) = \left\{(a, x) | a \in f^{-1}(c) \text{ and } \nabla f(a) \cdot x = 0\right\} \subset \mathbb{R}^{2n},$$

which can be shown to be a differentiable manifold.

Note, however, that this approach has a disadvantage: we required the sphere to be embedded in Euclidean space, which is not necessarily always the case. We must find a more intrinsic definition of the tangent bundle, which is independent of the embedding.

First note that there is always a canonical projection map $\pi : TM \to M$. We wish to define a differentiable structure on TM such that the projection is differentiable (topologically, one could simply make this map continuous and TM would have a pullback topology).

Theorem 12. *Let M be an n-dimensional differentiable manifold. There exists a natural differentiable structure on TM such that TM is a $(2n)$-dimensional smooth manifold and that the projection map $\pi : TM \to M$ is differentiable.*

Proof. It is left as an exercise to show that this structure will be Hausdorff and second-countable. Suppose (U, ϕ) is a coordinate chart on M and x^i, $i = 1 \cdots n$ are coordinates on U (more accurately, on $\phi(U)$). Take $\left(\pi^{-1}(U), \tilde{\phi}\right)$ as a coordinate chart for TM. Any element in $\pi^{-1}(U)$ is a derivation at $p \in U$. But any derivation at p is of the form $v^i \frac{\partial}{\partial x^i}|_p$, and so

$$v^i \frac{\partial}{\partial x^i}|_p \xrightarrow{\tilde{\phi}} \left(x^1(p) \cdots x^n(p), v^1 \cdots v^n\right) \in \phi(U) \times \mathbb{R}^n.$$

In other words, we give 2 coordinates: the first is of the base manifold and the second set are coordinates on the tangent space, so $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. We must now make sure that the transition maps are smoothly compatible, i.e. that $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a smooth map. Suppose we have $\tilde{x}^j$ coordinates on $\psi(V)$. Then,

$$\tilde{\phi}^{-1}\left(x^1(p) \cdots x^n(p), v^1 \cdots v^n\right) = v^i \frac{\partial}{\partial x^i}|_p = v^i \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}|_p,$$

so we have two different expressions for this derivation, depending on which chart we are in. Now let us compute:

$$\begin{aligned}\psi \circ \tilde{\phi}^{-1}\left(x^1 \cdots x^n, v^1 \cdots v^n\right) &= \tilde{\psi}\left(v^i \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}|_p\right) \\ &= \left(\tilde{x}^1 \cdots \tilde{x}^n, v^i \frac{\partial \tilde{x}^1}{\partial x^i} \cdots v^i \frac{\partial \tilde{x}^n}{\partial x^i}\right)\end{aligned}$$

Now it suffices to show that this is a C^∞ map from an open set of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$. Note carefully that the $\tilde{x}$ are functions of x coordinates; but $\tilde{x}$ is smooth (because it originates from the transition map of the base manifold and because it is independent of v^i). The second set of coordinates is smooth because it's linear in the v^i and because, again, $\tilde{x}$ is smooth. $\square$

(February 12, 2013)

Definition 20. A C^∞ **vector field** Y on M is a C^∞ map $Y : M \to TM$ such that $\pi \circ Y = \mathrm{Id}_M$ or $Y(p) \in T_pM$

In other words, we assign a derivation to each point p on the base manifold with the stipulation that the assigned derivation be in the tangent space "over" p. That means, that we can write, in the local coordinate chart:

$$Y = Y^i(p)\frac{\partial}{\partial x^i}\bigg|_p$$

where $Y^i(p)$ are C^∞ functions on the coordinate chart. We have assigned a tuple of smooth functions at each point on the manifold - this should remind you of vector fields that we are used to working with on Euclidean spaces. Of course, if p is in another chart as well, it can be written as

$$Y = \tilde{Y}^i(p)\frac{\partial}{\partial \tilde{x}^i}\bigg|_p.$$

Clearly we must have

$$\begin{aligned} Y^i(p)\frac{\partial}{\partial x^i}\bigg|_p &= Y^i(p)\frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}\bigg|_p \\ \tilde{Y}^j(p) &= Y^i(p)\frac{\partial \tilde{x}^j}{\partial x^i}(p) \end{aligned}$$

Example 18. On $\mathbb{R}^2 \setminus (0,0)$ we can take

$$\begin{aligned} Y_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ Y_2 &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \\ \tilde{Y}_1 &= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \\ \tilde{Y}_2 &= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right). \end{aligned}$$

Note that the second two are unit vector fields and thus cannot be extended continuously to the point 0, as their length remains constant. The first two, on the other hand shrink as we approach 0, and thus *can* be extended to become vector fields on $\mathbb{R}^2$.

Let us now try to push this vector field on $\mathbb{R}^2$ to a vector field on S^2 with the north pole delted using the stereographic projection. Taking the usual stereographic chart ϕ, we can use ϕ^{-1} to push forward Y_1 and Y_2 so they become vector fields on S^2 with the north pole deleted. The question now arises: can we extend this vector field to the north pole? Well, note that $\left(\phi^{-1}\right)_* Y_1$ (push forwards of rays going out to infinity) is tangent to a longitude line of the sphere. What we must examine, then, is the length of the vector field along this line in the vicinity of the north pole – in fact, it could be infinite, finite, or zero! But what about $\left(\phi^{-1}\right)_* Y_2$? This is the push forward of circles about the origin. This will yield tangents to a latitude line of the sphere, and again the question is whether or not the field goes to zero as we approach the north pole.

This question of whether the vector field can be globally defined on S^2 is left as a future homework problem.

Example 19. Consider $T = \mathbb{R}^2 \setminus \sim$ with

$$(x, y) \sim (x + 1, y)$$
$$(x, y) \sim (x, y + 1).$$

The first relation quotients out points into vertical lines, and the second relation quotients out points into horizontal lines, and thus it can be seen that the only relevant space is a square of side length 1 with the bottom-left corner at the origin. Note, however, that one must be careful to associate the opposite edges with each other – in fact, what we get is called a flat torus.

We can show that this is a differentiable manifold. Indeed, another way to define this is to consider

$$T = \left\{(x, y, z, w)|x^2 + y^2 = \frac{1}{2}, z^2 + w^2 = \frac{1}{2}\right\} \subset \mathbb{R}^4.$$

But notice that x, y are not globally defined functions on T. $\frac{\partial}{\partial x}$, on the other hand, is a well-defined global vector field, as the position of the vector does not matter, so a horizontal vector on the left edge of the square can be unambiguously identified with itself on the right edge of the square. The same holds for $\frac{\partial}{\partial y}$. These derivative vector fields are non-zero everywhere on T, unlike what we saw at the poles of S^2.

Definition 21. A **global derivation** on M is a linear map over $\mathbb{R}$, $Y : C^\infty(M) \to C^\infty(M)$ such that $Y(fg)(p) = f(p)(Yg)(p) + g(p)(Yf)(p)$.

Theorem 13. *Every global derviation comes from a global smooth vector field.*

Definition 22. Let $\mathcal{T}(M)$ be the set of all smooth vector fields on M. Again, this is a real vector space. We can also multiply a smooth vector field by a smooth function, so in fact, $C^\infty(M)$ has a ring structure. Thus, $\mathcal{T}(M)$ can be considered as a module over $C^\infty(M)$. In fact, $\mathcal{T}(M)$ can be identified with the vector space of global derivations on M.

Suppose X and Y are smooth vector fields, and thus global derivations, $f \to Xf$. The question is: can we take a second derivative? Is YX a global derivation?

$$\begin{aligned} YX(fg) &= Y(gXf + fXg) \\ &= (Yg) \cdot Xf + g \cdot YXf + (Yf) \cdot Xg + f \cdot YXg \end{aligned}$$

The second and the fourth terms are the requirements to be a global derivation, and thus there are two extra terms and YX is not necessarily a global derivation. Note that the extra terms are symmetric with respect to X and Y, which we can use to obtain a global derivation. Consider

$$(XY - YX)(fg) = g \cdot (XY - YX)f + f \cdot (XY - YX)g,$$

which *is* a global derivation.

Definition 23. The **Lie bracket**, or bracket, is a binary operation on $\mathcal{T}(M)$ that satisfies:

- linearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- skew-symmetry: $[X, Y] = -[Y, X]$
- Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$

Consequently, if $X, Y \in \mathcal{T}(M)$, XY is not necessarily in $\mathcal{T}(M)$, but $[X, Y]$ is. This makes $\mathcal{T}(M)$ a infinte-dimensional **Lie algebra**.

Note that in local coordinates, if $X = v^i \frac{\partial}{\partial x^i}$ and $Y = w^j \frac{\partial}{\partial x^j}$,

$$[X, Y] = \left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(February 14, 2013)

Recall that last time we showed that X is a smooth vector field if and only if X is a global derivation.

Now suppose $F : M \to N$ is smooth. Recall we have the pushforward $F_* : T_pM \to T_{F(p)}(M)$, such that if $Y \in T_p(M)$ and $f : N \to \mathbb{R}$,

$$(F_*Y)f = Y(f \circ F).$$

This operation on derivations is purely local – it does not extend to a map on $\mathcal{T}(M)$.

Example 20. Consider a map from an interval to a manifold, $\gamma : (-\varepsilon, \varepsilon) \to M$, which sweeps out a curve on M. On the interval, one can consider $\frac{\partial}{\partial t}$ as a vector field. The pushforward is then given,

$$\gamma_* \frac{\partial}{\partial t} = \frac{\partial \gamma}{\partial t}$$

and is, of course, defined *only* on the image of γ.

Definition 24. Let Y be a vector field on M and Z a vector field on N. Let $F : M \to N$ be smooth. Then, Y and Z are **F-related**, if

$$F_*Y(p) = Z(F(p)).$$

Theorem 14. *Suppose V_1 and V_2 are smooth vector fields on M, W_1 and W_2 are smooth vector fields on N, and V_i and W_i are F-related for $i = 1, 2$. Then, $[V_1, V_2]$ is F-related to $[W_1, W_2]$.*

In particular, if F is a diffeomorphism, then F_* can be defined globally on $\mathcal{T}(M)$, and then $W_i = F_*V_i$, and the statement is equivalent to $F_*[V_1, V_2] = [F_*V_1, F_*V_2]$ and so F_* is a (Lie algebra) isomorphism of $\mathcal{T}(M)$ and $\mathcal{T}(N)$.

What is the geometric meaning of the Lie bracket of two vector fields? We are going to interpret it as the Lie derivative of Y along X. To make this more concrete, consider a curve γ, parameterized by t. One can write, assuming $\gamma(0) = p$ and $\gamma'(0) = X$,

$$Xf = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t}.$$

However, if we are taking a derivative of Y, it doesn't make much sense to use this definition, as derivations along points on a curve live in completely different spaces!

Example 21. Let $M = \mathbb{R}^n$ and $X = v$, a constant vector field. Let Y be another smooth vector field on $\mathbb{R}^n$.

$$D_v Y(p) = \lim_{t \to 0} \frac{Y(p + tv) - Y(p)}{t} = \lim_{t \to 0} \frac{Y(p) - Y(p - tv)}{t},$$

but this is a very special case. How can we generalize this?

One way to think about this is to think about the "flow" generated by X. Take a point p on the flow line (i.e. tangent at each point is simply X) of X. If we take ϕ to be the "flow function," one can take points $\phi(t, p)$ and $\phi(-t, p)$ on this curve and look at the tangents, $Y(\phi(-t, p))$ and $Y(p)$. We then pushforward $Y(\phi(-t, p))$ by the flow to a vector at p:

$$(\phi_t)_* (Y(\phi(-t, p))) \in T_p M$$

A flow generated by a vector field

Definition 25. A C^∞ map $\phi : (-\varepsilon, \varepsilon) \times M \to M$ is a **flow generated by** X if $\phi(t, \cdot) : M \to M$ is a diffeomorphism for each $t \in (-\varepsilon, \varepsilon)$ and $\phi(0, p) = \phi(p)$, i.e. $\phi(0, \cdot) : M \to M$ is the identity map. Additionally,

$$\phi_* \left(\frac{\partial}{\partial t} \right) (t, p) = X(\phi(t, p))$$

We write $\phi_t(\cdot) = \phi(t, \cdot)$.

Example 22. Take $M = \mathbb{R}^2$ with the standard coordinates. Take $X(x, y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. What is the flow generated by X? First note that the vectors are always perpendicular to the position vector, and thus are tangents to circles of any radius.

Take any point $p = (a, b)$ on $\mathbb{R}^2$. We look for a curve $\phi(t, p)$ with p fixed – we can write it as $\gamma(t) = (x(t), y(t))$. By the above definition, we must have $x(0) = a, y(0) = b$. Additionally, we have $\gamma'(t) = (x'(t), y'(t)) = (-y(t), x(t))$. This is a system of differential equations with an initial condition. It should be clear that

$$\begin{aligned} x''(t) &= -x(t) \\ y''(t) &= -y(t), \end{aligned}$$

so we will have some combination of sines and cosines that satisfies the initial conditions:

$$\begin{aligned} x(t) &= a \cos t - b \sin t \\ y(t) &= b \cos t + a \sin t \end{aligned}$$

This is for a fixed (a, b), but we want to do this for any $x, y = p$:

$$\phi(t, x, y) = (x \cos t - y \sin t, y \cos t + x \sin t,$$

which is simply a rotation (seen a bit clearer in matrix form). One can check that ϕ_t is a diffeomorphism.

Theorem 15. *Any smooth vector field X of compact support gives a smooth family of diffeomorphisms that is defined for $t \in (-\infty, \infty)$.*

Proof. We will not prove this theorem, but it is important to note that one will have to use the existence and uniqueness theorem for solutions to a first-order system of ordinary differential equation, as well as prove that the solutions are smoothly dependent on initial conditions and the vector field. Indeed, uniqueness of solutions implies that

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

□

In fact, ϕ_t actually forms an infinite-dimensional, one-parameter group of diffeomorphisms, called the diffeomorphism group.

Note that since $\phi_t(\phi_{-t}(p)) = \phi_0(p) = p$. In particular, $(\phi_t)_* : T_{\phi_{-t}(p)}M \to T_pM$. In fact, we define

$$L_X Y(p) = \lim_{t \to 0} \frac{Y(p) - (\phi_t)_*(Y(\phi_{-t}(p)))}{t}$$

and we claim that $[X, Y] = L_X Y$. The proof is not difficult. We apply it to a smooth function f. Rewrite the numerator as:

$$\begin{aligned} &\lim_{t \to 0} \frac{Y(p)f - Y(\phi_{-t}(p))f + Y(\phi_{-t}(p))f - Y(\phi_{-1}(p))(f \circ \phi_t)}{t} \\ =&\lim_{t \to 0} \frac{Yf(p) - Yf(\phi_{-t}(p)) + Yf(\phi_{-t}(p)) - Y(f \circ \phi_t)(\phi_{-t}(p))}{t} \end{aligned}$$

Note that the first two terms are just $XYf(p)$, and if $\phi_{-t}(p) = q$, then the last two terms can be written $Y(f - f \circ \phi_t)(q)$, and we have

$$= XYf(p) - YX(\phi_{-t}(p)) = XY - YX$$

Where do vector fields come from?

February 19, 2013

One can take any smooth flow, $\phi : (-\varepsilon, \varepsilon) \times M \to M$. Then, $\phi_*(\partial/\partial t)$ is a vector field on M. Recall that on $\mathbb{R}^n$ there is actually a gradient vector field associated with any smooth function f:

$$\nabla f = \big(\partial f/\partial x^1 \cdots \partial f/\partial x^n\big)$$

which has the nice property that it's always perpendicular to a level set. Now given a smooth function f on a smooth manifold M, is $(\partial f/\partial x^1 \cdots \partial f/\partial x^n)$ a smooth vector field?

Recall that given local coordinates of two overlapping charts x^i and $\tilde{x}^j$, one can write a derivation as

$$\begin{aligned} v = v^i \frac{\partial}{\partial x^i} &= \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j} \\ &= \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial x^i}, \end{aligned}$$

which means the coordinates of the derivation must transform as

$$v^i = \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j}$$

for all $i = 1 \cdots n$. The question, then, is – given the gradient vector field above, does it transform similarly? In other words, how do $\nabla f = \big(\partial f/\partial x^1 \cdots \partial f/\partial x^n\big)$ and $\nabla f = \big(\partial f/\partial \tilde{x}^1 \cdots \partial f/\partial \tilde{x}^n\big)$ relate? It should be clear that we have

$$\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^i}.$$

It appears that this object transforms as the inverse, since $\partial \tilde{x}^j/\partial x^i \cdot \partial x^i/\partial \tilde{x}^k = \delta^j_k$. In general, a matrix is not equal to its inverse, and so ∇f is actually not a vector field on a general manifold.

This is similar to the total differential from multivariable calculus:

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

We will see that the dx_i obejcts are sections of the cotangent bundle, and not just a convenient notational tool. But let us first discuss a more general concept of a bundle.

4 Vector bundles

Definition 26. A real **vector bundle** of rank k over M smooth is a pair (E, π), in which E is a smooth manifold of dimension $n+k$ and π is a smooth projection map $\pi : E \to M$ such that:

1. For any $p \in M$, $\pi^{-1}(p)$ is a k-dimensional real vector space
2. (Existence of local trivialization) For any $p \in M$ there exists a neighborhood U of p, and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the following diagram commutes. In other words, $\pi^{-1}(q)$ is sent to $\{q\} \times \mathbb{R}^k$ by Φ, for any $q \in U$, so fibers are sent to fibers.
3. For each $q \in U$, the restriction of Φ to $\pi^{-1}(q)$ is a linear isomorphism between $\pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k$. (i.e. $\Phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^k$ is linear, one-to-one, and onto)

Let's compare this abstract definition to the tangent bundle.

Example 23. Take the tangent bundle TM of M. Recall that we had the coordinate chart for TM, $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ that mapped derivations to a tuple containing the point at which that derviation's tangent space is attached and the derivation's coordinates in the tangent space. Furthermore, $v^i = \tilde{v}^j \partial x^i / \partial \tilde{x}^j$, when changing coordinates. This Jacobian of this transformation is a linear isomorphism.

Definition 27. A **global section** of (E, π) is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. We can define **local sections** $\sigma : U \subset M \to E$, as well.

A section simply assigns a point in the fiber above $p \in M$ to p. In this sense, as one moves around on M, the section, in some sense, forms a "graph".

Note that a local trivialization gives local sections of the bundle.

Example 24. Take $E = TM$. Then $\pi^{-1}(U) \to U \times \mathbb{R}^n$ maps $v^i \partial/\partial x^i|_p \to (p, v^i)$. We claim that for any $v_0 \in \mathbb{R}^n$, $\Phi^{-1}(p, v_0)$ gives a local section. In fact, if $v_0 = (1, 0, \cdots, 0)$, then $\Phi^{-1}(p(1, 0, \cdots, 0)) = \partial/\partial x^1|_p$, which is a local section.

Definition 28. A vector bundle (E, π) is **trivial** is a global trivialization exists, i.e. there exists a diffeomorphism $\Phi : E \to M \times \mathbb{R}^k$ such that $\Phi|_{\pi^{-1}(q)}$ is a linear isomorphism between $\pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k$. This essentially replaces U by M in the definition of local trivialization.

Example 25 (Product bundle)**.** Take $S^1 \times \mathbb{R}$ (cylinder). This is a trivial bundle over S^1, by definition.

Example 26. Claim: the total space of an infinite Möbius strip is a non-trivial, rank 1, vector bundle over S^1.

First, observe that if a bundle is trivial, then there exists a global section that is nowhere zero. Take any nonzero vector $v_0 \in \mathbb{R}^k$ and consider $\Phi^{-1}(p, v_0)$, since Φ is a linear isomorphism on each fiber, $\Phi^{-1}(p, v_0) \neq 0 \in \pi^{-1}(p)$.

Now suppose there exists a global section that is nowhere zero (i.e. the Mobius band is trivial). However, if one draws the rectangle from which the Mobius band is created, it is clear that we must cross zero somewhere, as we identify positive values on the left edge with negative values on the right edge.

Definition 29. Suppose two local trivializations overlap. On $\pi^{-1}(U) \cap \pi^{-1}(V)$, we obtain the **transition function**

$$\Psi \circ \Phi^{-1} : U \cap V \times \mathbb{R}^k \to U \cap V \times \mathbb{R}^k.$$

and $\Psi \circ \Phi^{-1}$ restricts to a linear isomorphism on each fiber. Because Φ and Ψ send fibers to fibers,

$$\Psi \circ \Phi^{-1}(p, v) = (p, f(p, v)),$$

where $f(p, v)$ is a linear isomorphism for each p. But $\Psi \circ \Phi^{-1}$ is smooth, so $f(p, v) = \tau(p)v$, where τ is a linear isomorphism that is smooth as p varies. $\tau(p)$ is a linear isomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$ that depends smoothly on $p \in U \cap V$, i.e. it is in $GL(k)$. Thus we get a smooth map from $U \cap V \to GL(k)$.

February 21, 2013

Local trivialization of a vector bundle yields a local section – take any $v_0 \in \mathbb{R}^k$ and consider $\Phi^{-1}(p, v_0)$. In the case of the tangent bundle, this just sends $v^i \partial/\partial x^i \to (p, v^i)$, so, for example, $\Phi^{-1}(p, (1, 0, 0, \cdots) = \partial/\partial x^i|_p$. We can take a basis $e_1 \cdots e_k$ for $\mathbb{R}^k$, and then $\Phi^{-1}(p, e_k)$ for all k form a basis at each $E_p = \pi^{-1}(p)$; i.e. bases are preserved due to the linear isomorphism property. $\Phi^{-1}(e_k)$ is known as a **local frame**.

Now, what if we wish to talk about morphisms over the category of vector bundles? The most natural map, just from the definition of vector bundles, is as follows.

Definition 30. Suppose (E, π) over M and (E', π') over M' are two smooth vector bundles. A **bundle map** from E to E' is a pair of C^∞ maps F :

$E \to E'$ and $f : M \to M'$ such that F covers f, i.e. the diagram commutes and

$$\pi' \circ F = f \circ \pi,$$

(in other words, F sends fibers to fibers), and $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map.

A **bundle isomorphism** is a bundle map where the C^∞ maps are diffeomorphisms, and the linear map is a linear isomorphism.

Thus, whenever we talk about classifying bundles, we are really talking bundles upto bundle isomorphisms.

Example 27. A smooth map $f : M \to N$ induces a bundle map F on the tangent bundle via f_*:

$$F(X_p) = f_*(X_p)$$

which is linear, as,

$$f_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial f^\lambda}{\partial x^i} \frac{\partial}{\partial y^\alpha}.$$

We denote the smooth sections of E by $\Gamma(E)$. We claim, then, that in the case of $M = N$ and the identity map between the two, there exists a smooth section $\mathcal{F} : \Gamma(E) \to \Gamma(E')$. Taking any $\sigma \in p(E)$, we define $\mathcal{F}(\sigma)(p) = F(\sigma(p)) \in E'_p$, where F is a map from $TM \to TN$. If f is a diffeomorphism from $M \to M$, f induces a bundle map on $\mathcal{T}(M)$. Notice that $\Gamma(E)$ is again a real vector space, and is in fact, a module over $C^\infty(M)$.

Question: Does every linear map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ come from a bundle map $F : E \to E'$?

Example 28. Consider $C^\infty(M)$ as sections of the trivial bundle $M \times \mathbb{R}$. Take $f \in C^\infty(M)$ and define the map $\mathcal{T}(M) = \Gamma(TM) \to C^\infty(M) = \Gamma(M \times \mathbb{R})$ by $X \to Xf$. We claim that this comes from a bundle map $F : TM \to M \times \mathbb{R}$ that sends $X_p \to (Xf)(p)$, since the directional derivative is linear.

However, if we fixed X instead, and consider the map from $C^\infty(M) \to C^\infty(M)$ that sends $f \to Xf$. We claim that this does not come from any bundle map. Suppose there is such a bundle map, $F(f(p)) = Xf(p)$. This is not necessarily linear, as a function that is zero need not have a derivative that is zero.

Theorem 16. *Suppose E and E are both smooth vector bundles over M. A linear map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ comes from a bundle map $F : E \to E'$ if and only if $\mathcal{F}$ is linear over $C^\infty(M)$. We consider both $\Gamma(E)$ and $\Gamma(E')$ as modules over $C^\infty(M)$, i.e. we need to require that $\mathcal{F}$ is a module morphism $F : \Gamma(E) \to \Gamma(E')$ such that $F(u\sigma) = uF(\sigma)$ for any smooth function u.*

Proof. We sketch the proof.

If $\mathcal{F}$ is from a bundle map F, then $\mathcal{F}(\sigma)(p) = F(\sigma(p))$. Let u be any smooth function. Then, $\mathcal{F}(u\sigma)(p) = F(u\sigma(p)) = F(u(p)\sigma(p))$. But F is linear on each fiber, so we just get $u(p)F(\sigma(p)) = u(p)\mathcal{F}(\sigma)(p) = (u\mathcal{F}(\sigma))(p)$. On the other hand, if $\mathcal{F}$ is linear over $C^\infty(M)$, we show that it must come from a bundle map. To define F_p on any $v \in E_p$, take a section $\sigma \in \Gamma(E)$ such that $\sigma(p) = v$. Then, $F_p(v) = F_p(\sigma(p)) = \mathcal{F}(\sigma)(p)$. What if we choose a different $\tilde{\sigma}$, such that $\sigma(p) = \tilde{\sigma}(p)$? We need to show $\mathcal{F}(\sigma)(p) = \mathcal{F}(\tilde{\sigma}(p)$ if this is true, i.e. that if $\sigma(p) = 0$, then $\mathcal{F}(\sigma)(p)$. Heuristically, if $\sigma(p) = 0$, we write it as $\sigma = u\sigma'$ such that $u(p) = 0$. Then, $\mathcal{F}(\sigma)(p) = \mathcal{F}(u\sigma')(p) = u\mathcal{F}(\sigma')(p) = u(p)\mathcal{F}(\sigma')(p) = 0$ by linearity. □

5 Cotangent Bundles

We have already talked about tangent bundles, whose sections are vector fields. Now we turn to the dual of the tangent bundle. Let V be a finite-dimensional real vector space and $V^* = \{\omega | \omega : V \to \mathbb{R} \text{ linear}\}$ be the real vector space of linear functionals on V. Take any basis $\{E_i\}_{i=1\cdots n}$ of V. Then we can consider the dual basis $\left\{\varepsilon^i\right\}_{i=1\cdots n}$ of $V*$ defined by $\varepsilon^i(E_j) = \delta^i_j$. We claim that this is a basis; we must show that this set spans $V*$ and is linearly independent. Given any $\omega \in V*$ and any $x = x^i E_i \in V$, we simply apply

$$\varepsilon^j(x) = \varepsilon^j(x^i E_i) = x^i \varepsilon^j(E_i) = x^j$$

and then take

$$\omega(x) = \omega(x^i E_i) = x^i \omega(E_i) = \omega(E_i)\varepsilon^i(x)$$

Since this is true for all x, we see that $\omega = \omega(E_i)\varepsilon^i$ and so this set spans all linear functionals. Next suppose that $\sum_{i=1}^n a_i \varepsilon^i = 0$; we wish to show that $a_i = 0$. We apply this sum to an element in the original space:

$$\sum_{i=1}^n a_i \varepsilon^i(E_j) = a_j = 0$$

for all j, and thus, what we called the dual basis spans the space of functionals and is linearly independent, and thus is a basis.

Indeed, there exists a non-canonical isomorphism from V to $V*$ when working with the dual basis, as we can just send basis vectors to basis vectors. But we can even talk about the double dual, $V^{**} = (V^*)^*$, and the claim is that this is isomorphic to V independent of any basis. Take the map f from V to V^{**} such that $f(x)(\omega) = \omega(x)$. It suffices to show that the kernel of this map is non-zero because $\dim V = \dim V^{**}$ and for any linear map between two vector spaces of the same dimension, the map is injective if and only if it is surjective (this follows fairly easily from the rank-nullity theorem). Well what does it mean for $f(x) = 0$? This is equivalent to saying that $\omega(x) = 0$ for all ω. Then, $\varepsilon^j(x^i E_i) = 0$, which implies that $x^j = 0$ for all j, and we are done.

Definition 31. If $A : V \to W$ is a linear map, then the **dual map** $A^* : W^* \to V^*$ is defined as follows. Given $\omega \in W^*$, $x \in V$, we define

$$A^*\omega(x) = \omega(A(x)).$$

One can check that this map is linear: $A^*(c_1\omega_1 + c_2\omega_2) = c_1 A^*\omega_1 + c_2 A^*\omega_2$. Although this definition is basis-independent, we are free to choose $\{E_i\}_{i=1\cdots n}$ for V, $\{\varepsilon^j\}$ for V^*, $\left\{\tilde{E}_\alpha\right\}_{\alpha=1\cdots m}$ for W, and $\{\tilde{\varepsilon}^\alpha\}$ for W^*. Then, $A(E_i) = A_i^\alpha \tilde{E}_\alpha$ and $A^*(\tilde{\varepsilon}^\alpha) = B_i^a \varepsilon^i$. How are the matrices A and B related? In general, A is an $m \times n$ matrix whereas B is an $n \times m$ matrix, so it seems natural for them to be transposes. Indeed, we are used to dual vectors' matrix representations being transposes, i.e. row vectors instead of column vectors. Let us prove this. Assume $A(E_i) = A_i^\alpha \tilde{E}_\alpha$ and consider $A^*(\tilde{\varepsilon}^\alpha(E_j)) = \tilde{\varepsilon}^\alpha(A(E_j)) = \tilde{\varepsilon}^\alpha(A_j^\beta \tilde{E}_\beta) = A_j^\beta \tilde{\varepsilon}^\alpha(\tilde{E}_\beta) = A_j^\alpha$. But since $A^*(\tilde{E}^\alpha)(E_i) = B_j^\alpha \varepsilon^j(E_i) = B_i^\alpha$ and so $A_i^\alpha = B_i^\alpha$, and looking at the way the maps are operated, one finds the transpose property.

Let us now apply these concepts to vector bundles. In essence, we will simply dualize each fiber in the vector bundle.

Definition 32. Let E be a vector bundle of rank k over M. Define $E^* = \sqcup_{p \in M} E_p^*$ and a projection $\pi : E^* \to M$ that sends the fiber to the point at which the fiber is attached to the base manifold. Given $U \subset M$, we consider $\pi^*(U)$ and define the local trivialization as follows. Consider a map from $(\pi^*)^{-1}(U) \to U \times \mathbb{R}^k$. This is sending elements of E_p^*, say ω_p, to $(p, \omega_p(E_1) \cdots \omega_p(E_k))$ where E_i is the local frame coming from the local trivialization of E, i.e. $E_i(p) = \Phi^{-1}(p, e_i)$.

Let us apply this procedure to the tangent bundle and denote the dual bundle by T^*M. A section of T^*M is called a one-form, and the space of smooth sections of T^*M is denoted by $\Gamma(T^*M)$.

Example 29. Recall, that given a smooth function f on M, one can consider the gradient ∇f. We previously showed that this n-tuple of functions does not yield a section of the tangent bundle, as it transforms incorrectly. But, as we shall see, the gradient is a section of the cotangent bundle, i.e. a one-form. In fact, it is better to consider the total differential, *df* such that $df(x_p) = (Xf)_p$. This is in analogy to how directional derivatives can be written as the gradient dotted into a vector – since this yields a number at any point, it suggests that the gradient is, in fact, some sort of covector.

How does a section of T^*M transform? Well, for the tangent bundle, the local trivialization sends derivations that can be represented by $v^i\partial/\partial x^i|_p$ to $(p, v^1 \cdots v^n)$. Thus we have local frames that look like $\Phi^{-1}(p, e_i) = \partial/\partial x^i|_p$. What are the dual local frames to $\partial/\partial x^i|_p$? We look for ω such that $\omega^j(\partial/\partial x^i|_p) = \delta^j_i$. We take x^j and defined dx^j as $dx^j(X_p) = X(x^j)_p$. We apply this: $dx^j(\partial/\partial x^i|_p) = \delta^j_i$. Thus, $dx^1 \cdots dx^n$ form a local frame of T^*M on U and the local trivialization is given by $v_i dx^j|_p \mapsto (p, v_1 \cdots v_n)$. Notice that they transform differently from $\partial/\partial x^i$:

$$\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}$$
$$dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$$