# MODERN ALGEBRA II SPRING 2013: SEVENTH PROBLEM SET

1. Let $E = \mathbb{Q}(\sqrt{5}, \sqrt{7})$ and let $\alpha = 2\sqrt{5} - \sqrt{7} \in E$. Show that $[E : \mathbb{Q}] = 4$ and that $E = \mathbb{Q}(\alpha)$. What is $\operatorname{irr}(\alpha, \mathbb{Q}, x)$? Find two different bases for $E$ as a $\mathbb{Q}$-vector space, reflecting the fact that $E = \mathbb{Q}(\sqrt{5}, \sqrt{7})$ and that $E = \mathbb{Q}(\alpha)$.

2. Assuming that $x^4 - 2$ is irreducible over $\mathbb{Q}$, show that $[\mathbb{Q}(i, \sqrt[4]{2}) : \mathbb{Q}] = 8$ and find an explicit basis for $\mathbb{Q}(i, \sqrt[4]{2})$ as a $\mathbb{Q}$-vector space. If $\alpha = i + \sqrt[4]{2}$, show by explicit computation that $\alpha$ is the root of a degree 8 polynomial in $\mathbb{Q}[x]$. (In fact, $\deg_{\mathbb{Q}} \alpha = 8$, hence $\mathbb{Q}(i, \sqrt[4]{2}) = \mathbb{Q}(\alpha)$, but this is somewhat messy for us to show at the moment.)

3. Let $F$ be a field of characteristic not equal to 2. Suppose that $E$ is a finite extension field of $F$ and that $[E : F] = 2$. Show that there exists an element $\alpha \in E$ such that $\alpha \notin F$, $\alpha^2 = a \in F$, and $E = F(\alpha)$. In other words, $E$ is of the form $F(\sqrt{a})$ for some $a \in F$ which is not a square in $F$.

4. Let $F$ be a field, and suppose that $F$ is a subring of an integral domain $R$. Thus $R$ is a vector space over $F$. Suppose that $R$ is a **finite dimensional** vector space over $F$. Show that $R$ is a field. (Hint: if $r \in R$ and $r \neq 0$, the vectors $1, r, r^2, \dots$ are not linearly independent. Write down a linear combination of these vectors which is equal to zero and factor out the largest power of $r$ which divides it.)

5. Let $E$ be a finite extension of a field $F$, and suppose that the degree $[E : F] = t$ is a prime number. Show that $E$ is a simple extension of $F$, and in fact that $E = F(\alpha)$ for every $\alpha \in E$ such that $\alpha \notin F$.

6. Let $F$ be a field and let $E = F(\alpha)$ be a finite extension field of $F$ such that $[E : F] = \deg_F \alpha$ is odd. Show that, if $\alpha \notin F$, then $\alpha^2 \notin F$, and in fact more generally that $F(\alpha^2) = F(\alpha)$. (Hint: $F(\alpha^2) \leq F(\alpha)$. What are the possibilities for $[F(\alpha) : F(\alpha^2)]$, i.e. for $\operatorname{irr}(\alpha, F(\alpha^2), x)$?) Thus for example $\mathbb{Q}((\sqrt[3]{2})^2) = \mathbb{Q}(\sqrt[3]{2})$; this also follows from the problem above (or via direct computation).

7. Let $F$ be a field and $E$ an extension field of $F$. Suppose that $\alpha \in E$ and $\beta \in E$ are both algebraic over $F$, and that $\deg_F \alpha = n$, $\deg_F \beta = m$. Show that $\beta$ is algebraic over $F(\alpha)$, and that $\deg_{F(\alpha)} \beta \leq m$. Conclude that $[F(\alpha, \beta) : F] \leq nm$, and hence that $\deg_F(\alpha + \beta) \leq nm$, and

similarly $\deg_F(\alpha\beta) \leq nm$. Thus for example $\sqrt{3} + \sqrt[3]{5}$ is the root of some irreducible polynomial in $\mathbb{Q}[x]$ of degree at most 6.

8. Let $F$ be a field and $E$ an extension field of $F$. Suppose that $\alpha \in E$ and $\beta \in E$ are both algebraic over $F$, and that $\deg_F \alpha = n$, $\deg_F \beta = m$, with $n$ and $m$ relatively prime. Using the previous problem, show that the degree of $F(\alpha, \beta)$ over $F$ is $nm$. Using this, compute the degree $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}]$.