Introduction to Algebraic Topology PSET 11

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Proposition 1. *Hatcher exercise 2.2.43*

Proof.

(a) Given a chain complex of free abelian groups C_n, we wish to show that it splits as a direct sum of subcomplexes $0 \to L_{n+1} \to K_n \to 0$ with at most two nonzero terms. First note that for any n the first isomorphism theorem yields the exact sequence

$$0 \longrightarrow \ker \partial_n \longrightarrow C_n \longrightarrow \text{im } \partial_n \longrightarrow 0.$$

This exact sequence splits, as we can find a section s of $\partial_n : C_n \to \text{im } \partial_n$. Indeed, choose a basis $\{e_\alpha\}$ for im ∂_n (as a subgroup of a free abelian group it must be free abelian as well), and define $s : \text{im } \partial_n \to C_n$ by taking $s(e_\alpha)$ to be any element $c_\alpha \in C_n$ such that $\partial_n c_\alpha = e_\alpha$. Hence we can write $C_n \cong \ker \partial_n \oplus \text{im } \partial_n$, which now allows us to decompose the chain complex C_n. Indeed, we take K_n above to be $\ker \partial_n$ and L_{n+1} to be im ∂_{n+1}:

$$\begin{array}{ccccccccccc}
 & & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & 0 \\
 & & & & 0 & \longrightarrow & \text{im } \partial_1 & \longrightarrow & \ker \partial_0 & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & \text{im } \partial_2 & \longrightarrow & \ker \partial_1 & \longrightarrow & 0 & & \\
0 & \longrightarrow & \text{im } \partial_3 & \longrightarrow & \ker \partial_2 & \longrightarrow & 0 & & & & \\
 & & & & \vdots & & & & & &
\end{array}$$

(b) Now suppose that the groups C_n are finitely generated. Then the map $L_{n+1} \to K_n$ is a linear transformation of finite-dimensional vector space. Note now that $L_{n+1} \subset K_n$ as C_n is a chain complex and $\partial^2 = 0$ requires that im $\partial_{n+1} \subset \ker \partial_n$. Now, without loss of generality, we can write im $\partial_n \cong \mathbb{Z}^j$ and $\ker \partial_n \cong \mathbb{Z}^k$ for $j \leq k$. By an appropriate change of basis, we can write the map ∂_n as taking each basis vector in $\mathbb{Z}^j$ to a multiple of a basis vector in $\mathbb{Z}^k$. This yields a further splitting of the complex into summands of the form $0 \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$, where the first type generate the subspace of $\mathbb{Z}^k$ not in the image of the map $L_{n+1} \to K_n$ and the latter type generate the image.

(c) Let X be a CW complex with finitely many cells in each dimension. The chain complex C_n obtained by cellular homology satisfies the conditions necessary for the results of part (b) to hold. From this chain complex we can compute the homology groups $H_n(X)$. If we now consider instead the chain complex C_n with coefficients in G (again obtained by cellular homology)... I'm not sure how to relate the $\mathbb{Z}$ case to the G case.

$\square$

Proposition 2. *Hatcher exercise 2.3.1*

Proof. Let $T_n(X, A)$ denote the torsion subgroup of $H_n(X, A; \mathbb{Z})$. Consider the functors $(X, A) \mapsto T_n(X, A)$, with the obvious induced homomorphisms $T_n(X, A) \to T_n(Y, B)$ and boundary maps $T_n(X, A) \to T_{n-1}(A)$. Consider the CW pair $(\mathbb{RP}^2, A)$, where A is the union of the zero- and one-cells. It is easy to see that $\tilde{H}_2(\mathbb{RP}^2) = 0, \tilde{H}_1(\mathbb{RP}^2) = \mathbb{Z}_2, \tilde{H}_0(\mathbb{RP}^2) = 0$, $\tilde{H}_2(A) = 0, \tilde{H}_1(A) = \mathbb{Z}, \tilde{H}_0(A) = 0$, and since $X/A = S^2$, we have that $\tilde{H}_2(X/A) = 0, \tilde{H}_1(X/A) = \mathbb{Z}, \tilde{H}_0(X/A) = 0$. Working now instead with the torsion functor, we obtain via the second axiom the sequence $0 \to \mathbb{Z}_2 \to 0$, which is not exact, which is a contradiction. If, instead, we were to work with the mod-torsion functor $MT_n(X, A) = H_n(X, A; \mathbb{Z})/T_n(X, A)$, we would find the sequence $0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$, which is again not exact.

$\square$

Proposition 3. *Hatcher exercise 2.3.4*

Proof. We proceed by induction on the number of wedge summands. Consider first the CW pair $(X_1 \vee X_2, X_1)$, i.e. $n = 2$. We obtain a long exact sequence

$$\cdots \xrightarrow{\partial} \tilde{h}_n(X_1) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X_2) \xrightarrow{\partial} \tilde{h}_{n-1}(X_1) \xrightarrow{i_*} \cdots .$$

From the retraction $q : X_1 \vee X_2 \to X_1$ we find that i_* in the diagram above is injective, which yields a number of short exact sequences

$$0 \xrightarrow{\partial} \tilde{h}_n(X_1) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X_2) \xrightarrow{\partial} 0,$$

which, due to the existence of the retraction splits as $\tilde{h}_n(X) \cong \tilde{h}_n(X_1) \oplus \tilde{h}_n(X_2)$.

Now suppose we have that $\tilde{h}_n(\vee_i^n X_i) \cong \oplus_i^n \tilde{h}_n(X_i)$. Then consider the CW pair $(\vee_i^n X_i \vee X_{n+1}, X_{n+1})$. By exactly the same reasoning as above (using the retraction from the wedge sum to X_{n+1}) we find that $\tilde{h}_n(\vee_i^n X_i \vee X_{n+1}) \cong \oplus_i^n \tilde{h}_n(X_i) \oplus \tilde{h}_n(X_{n+1})$, which concludes the proof. $\square$

Proposition 4. *Hatcher exercise 3.1.6*

Proof.

(a) Consider the usual simplicial structure for the torus T^2 (Hatcher p. 102) consisting of 1 0-simplex v, 3 1-simplices a, b, c, and 2 2-simplices U, L. Recall that we obtain the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{a+b-c} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

with homology $H_0(T^2) = \mathbb{Z}, H_1(T^2) = \mathbb{Z}^2, H_2(T^2) = \mathbb{Z}$. The corresponding cochain complex is

$$0 \xleftarrow{\delta_2} \mathbb{Z}^2 \xleftarrow{\delta_1} \mathbb{Z}^3 \xleftarrow{\delta_0} \mathbb{Z} \longleftarrow 0$$

where $\delta_0 = \partial_1^* = 0$ and $\delta_1 = \partial_2^*$. Clearly $H^0(T^2) = \mathbb{Z}$. Next, note that $H^1(T^2) = \ker \delta_1 / \text{im } \delta_0 = \ker \delta_1 = \ker \partial_2^*$. We can write a basis for C_1^* as $\{\alpha, \beta, \gamma\}$ dual to a, b, c respectively. For any $\phi \in C_1^*$, we find that $\delta_1 \phi = \phi \circ \partial_2$, but the image of ∂_2 is $a+b-c$, and hence $\ker \delta_1 = \langle \alpha - \gamma, \beta - \gamma \rangle$ and $H^1(T^2) = \mathbb{Z}^2$. Finally, $H^2(T^2) = \mathbb{Z}^2 / \text{im } \delta_1 = \mathbb{Z}$ by rank-nullity.

Now consider the case of $\mathbb{Z}_2$ coefficients. We obtain the chain complex

$$0 \longrightarrow \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

where $\partial_1 = 0$ and $\partial_2 = a+b+c$. The homology is given $H_0(T^2;\mathbb{Z}_2) = \mathbb{Z}$, $H_1(T^2;\mathbb{Z}_2) = \mathbb{Z}_2^3/\mathbb{Z}_2 = \mathbb{Z}_2^2$, $H_2(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$. The associated cochain complex is given

$$0 \xleftarrow{\delta_2} \mathbb{Z}_2^2 \xleftarrow{\delta_1} \mathbb{Z}_2^3 \xleftarrow{\delta_0} \mathbb{Z}_2 \longleftarrow 0$$

where $\delta_0 = 0$. Clearly $H^0(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$. Next, $H^1(T^2;\mathbb{Z}_2) = \ker\delta_1 = \ker\partial_2^*$. As above, we can write a basis for C_1^* to be $\{\alpha,\beta,\gamma\}$ dual to a,b,c respectively. Since the image of ∂_2 is $a+b+c$, we find that $H^1(T^2;\mathbb{Z}_2) = \langle\alpha+\beta,\beta+\gamma\rangle = \mathbb{Z}_2^2$. Finally, $H^2(T^2;\mathbb{Z}_2) = \mathbb{Z}_2^2/\text{im }\delta_2 = \mathbb{Z}_2$ by rank-nullity.

(b) Consider the usual simplicial structure for $\mathbb{RP}^2$ (Hatcher p. 102) consisting of 2 0-simplices v,w, 3 1-simplices a,b,c, and 2 2-simplices U,L. We obtain the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

where $\partial_1 a = \partial_1 b = w - v, \partial_1 c = 0$, $\partial_2 U = -a+b+c, \partial_2 L = a-b+c$. The homology is given $H_0(\mathbb{RP}^2) = \mathbb{Z}, H_1(\mathbb{RP}^2) = \mathbb{Z}_2, H_2(\mathbb{RP}^2) = 0$. The associated cochain complex is

$$0 \xleftarrow{\delta_2} \mathbb{Z}^2 \xleftarrow{\delta_1} \mathbb{Z}^3 \xleftarrow{\delta_0} \mathbb{Z}^2 \longleftarrow 0.$$

By definition, $H^0(\mathbb{RP}^2) = \ker\delta_0 = \ker\partial_1^*$. Note that $\partial_1^*\nu$ is a morphism that maps $a \mapsto -1, b \mapsto -1, c \mapsto 0$ and $\partial_1^*\omega$ is a morphism that maps $a \mapsto 1, b \mapsto 1, c \mapsto 0$ and hence $H^0(\mathbb{RP}^2) = \langle\nu+\omega\rangle = \mathbb{Z}$. Next, $H^1(\mathbb{RP}^2) = \ker\delta_1/\text{im }\delta_0 = \ker\delta_1/\mathbb{Z}$. Note that $\partial_2^*\alpha$ is a morphism that maps $U \mapsto -1, L \mapsto 1$, $\partial_2^*\beta$ is a morphism that maps $U \mapsto 1, L \mapsto -1$, and $\partial_2^*\gamma$ is a morphism that maps $U \mapsto 1, L \mapsto 1$. Thus $H^1(\mathbb{RP}^2) = \mathbb{Z}/\langle\alpha+\beta\rangle = 0$. Finally, $H^2(\mathbb{RP}^2) = \langle\mu,\lambda\rangle/\text{im }\delta_1$. By above, we find that $H^2(\mathbb{RP}^2) = \langle\mu,\lambda\rangle/\langle\mu-\lambda,\mu+\lambda\rangle = \langle\mu-\lambda,\lambda\rangle/\langle\mu-\lambda,\mu+\lambda\rangle = \mathbb{Z}_2$.

Now consider the case of $\mathbb{Z}_2$ coefficients. We obtain the chain complex

$$0 \longrightarrow \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^2 \longrightarrow 0$$

where $\partial_1 a = \partial_1 b = v+w, \partial_1 c = 0$ and $\partial_2 U = \partial_2 L = a+b+c$. The homology is easily found $H_0(\mathbb{RP}^2;\mathbb{Z}_2) = H_1(\mathbb{RP}^2;\mathbb{Z}_2) = H_2(\mathbb{RP}^2;\mathbb{Z}_2) = \mathbb{Z}_2$. The associated cochain complex is

$$0 \xleftarrow{\delta_2} \mathbb{Z}_2^2 \xleftarrow{\delta_1} \mathbb{Z}_2^3 \xleftarrow{\delta_0} \mathbb{Z}_2^2 \longleftarrow 0.$$

By definition, $H^0(\mathbb{RP}^2;\mathbb{Z}_2) = \ker\partial_1^*$. The morphisms $\partial_1^*\nu = \partial_1^*\omega$ map $a \mapsto 1, b \mapsto 1, c \mapsto 0$ and thus $H^0(\mathbb{RP}^2;\mathbb{Z}_2) = \langle\nu+\omega\rangle = \mathbb{Z}_2$. Next, $H^1(\mathbb{RP}^2;\mathbb{Z}_2) = \ker\delta_1/\text{im }\delta_0 = \ker\delta_1/\mathbb{Z}_2$. Since $\delta_1\alpha = \delta_1\beta = \delta_1\gamma$ take $U \mapsto 1, L \mapsto 1$, $\ker\delta_1 = \langle\alpha+\beta,\beta+\gamma\rangle = \mathbb{Z}_2^2$, and hence $H^1(\mathbb{RP}^2;\mathbb{Z}_2) = \mathbb{Z}_2$. Finally, $H^2(\mathbb{RP}^2;\mathbb{Z}_2) = \ker\delta_2/\text{im }\delta_1 = \mathbb{Z}_2^2/\mathbb{Z}_2 = \mathbb{Z}_2$.

(c) Consider the usual simplicial structure for the Klein bottle K (Hatcher p. 102) consisting of 1 0-simplex, 3 1-simplices, and 2 2-simplices. We obtain the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\partial_1 = 0, \partial_2 U = a + b - c, \partial_2 L = -a + b + c$. Clearly $H^0(K) = \mathbb{Z}$. Next, $H^1(K) = \ker \delta_1$ and since $\delta_1 \alpha$ maps $U \mapsto 1, L \mapsto -1$, $\delta_1 \beta$ maps $U \mapsto 1, L \mapsto 1$, and $\delta_1 \gamma$ maps $\mapsto -1, L \mapsto 1$, we find that $H^1(K) = \langle \alpha + \gamma \rangle = \mathbb{Z}$. Finally, $H^2(K) = \ker \delta_2 / \mathrm{im}\ \delta_1 = \mathbb{Z}^2/\mathbb{Z}^2 = 0$.

Now consider the case of $\mathbb{Z}_2$ coefficients. We obtain the chain complex

$$0 \longrightarrow \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^3 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The associated cochain complex is given

$$0 \xleftarrow{\delta_2} \mathbb{Z}_2^2 \xleftarrow{\delta_1} \mathbb{Z}_2^3 \xleftarrow{\delta_0} \mathbb{Z}_2 \longleftarrow 0.$$

Again, $H^0(K;\mathbb{Z}_2) = \mathbb{Z}_2$, but now $H^1(K;\mathbb{Z}_2) = \mathbb{Z}_2^2$ as $\ker \delta_1 = \langle \alpha + \beta, \beta + \gamma \rangle$. Finally, $H^2(K;\mathbb{Z}_2) = \mathbb{Z}_2$.

$\square$

Proposition 5. *Hatcher exercise 3.1.8*

Proof.

(a) Consider the pair $(X, A) = (D^n, S^{n-1})$, so $X/A = S^n$. The long exact sequence of cohomology groups for the pair (X, A) has every third term $\tilde{H}^i(D^n)$ zero since D^n is contractible and the chain and cochain groups are zero. Hence we find that $\tilde{H}^{i-1}(S^{n-1}) \cong \tilde{H}^i(S^n)$. The result follows by induction as usual. Similarly, using the Mayer-Vietoris sequence for reduced cohomology with $X = S^n$ and A, B the northern and southern hemispheres, we obtain $\tilde{H}^i(S^{n-1};G) \cong \tilde{H}^{i+1}(S^n;G)$, as the split term goes to zero by contractibility.

(b) Let A be a closed subspace of X that is a deformation retract of some neighborhood V. We obtain a commutative diagram

$$\begin{array}{ccccc}
H^n(X,A;G) & \longleftarrow & H^n(X,V;G) & \longrightarrow & H^n(X-A,V-A;G) \\
\uparrow q^* & & \uparrow q^* & & \uparrow q^* \\
H^n(X/A,A/A;G) & \longleftarrow & H^n(X/A,V/A;G) & \longrightarrow & H_n(X/A-A/A,V/A-A/A;G)
\end{array}$$

Note that in the long exact sequence of the triple (X, V, A), the groups $H^n(V, A; G)$ are zero for all n because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \cong (A, A)$ and $H^n(A, A; G) = 0$. Hence the upper left map is an isomorphism. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, and hence the same argument shows that the lower left map is an isomorphism. The remainder of the horizontal maps are isomorphisms via excision for cohomology. Finally, the right-hand vertical map q^* is an isomorphism since q restricts to a homeomorphism on the complement of A. It follows by the commutativity of the diagram, now, that $H^n(X, A; G) \cong H^n(X/A, A/A; G) \cong \tilde{H}^n(X/A; G)$.

(c) Consider the short exact sequence

$$0 \longleftarrow H^n(A;G) \xleftarrow{i^*} H^n(X;G) \longleftarrow H^n(X,A;G) \longleftarrow 0.$$

Suppose A is a retract of X, i.e. we have a map $r : X \to A$ such that $r \circ i = \mathrm{Id}_A$, and hence $(r \circ i)^* = i^* \circ r^* = \mathrm{Id}_{H^n(A;G)}$. The splitting lemma now implies that $H^n(X;G) \cong H^n(A;G) \oplus H^n(X,A;G)$, as desired.

$\square$