An introduction to special relativity:

a geometric approach

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Ask questions!

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The situation before 1900

- Maxwell unifies electricity and magnetism, light is a wave
- Waves need a medium in which to propogate: ether
  - negligible density and interaction with matter
- Ether existed solely for E&M
  - Fizeau's experiments (1850's): ether must be dragged along by moving fluids according to n!
  - Michelson-Morley (1886): no detected ether wind

Galilean transformations

Observation

Newtonian mechanics is invariant under the Galilean transformations:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t$$
$$t' = t$$

- This is not true of Maxwell's equations!
- Electromagnetic phenomena are instead Lorentz invariant

Einstein's options

1. Maxwell's equations were incorrect; they must be adjusted to be invariant under Galilean transformations
2. Galilean relativity applied to classical mechanics, but E&M had a preferred reference frame given by the ether
3. There existed a relativity principle for both mechanics and E&M but it was not Galilean, implying that mechanics was in need of revision

Criticism and verification

- SR was a radical shift in physical perspective
- Many physicists claimed that SR
  - was too abstract and mathematical
  - did not agree with experiment
  - was not interally consistent
- Extensive testing:
  - Michelson-Morley (isotropy), Kennedy-Thorndike (speed), Ives-Stillwell (time dilation)
  - Multitude of more and more precise experiments
  - Particle accelerators

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Postulate 1

Postulate of relativity (Galileo)

The laws of nature and the results of all experiments performed in a given inertial frame of reference are independent of the translational motion of the system as a whole.

- There is no absolute velocity!
- An inertial frame is an unaccelerated coordinate system for space and time that observes events as (t, x, y, z)

Postulate 2

Postulate of the constancy of the speed of light (Einstein)

The speed of light is finite and independent of the motion of its source or the motion of the inertial observer. In other words, two different inertial observers measuring the speed of the same photon will each find it moving at $c = 3 \times 10^8$ m/s relative to themselves, regardless of the relative state of motion.

- Completely invalidates Galilean addition of velocities and replaces it with Lorentz transformations
- Counter-intuitive but experimentally verified
- Implies that inertial observers' coordinates are somehow "different"

Measuring time in meters

Definition

We define 1 m of time to be the time it takes for light to travel one meter. Under this definition, the speed of light becomes

$$c = 1.$$

Note: c is dimensionless - the units cancel!

It is not *a priori* obvious that we should think of space and time as similar quantities

Spacetime diagrams

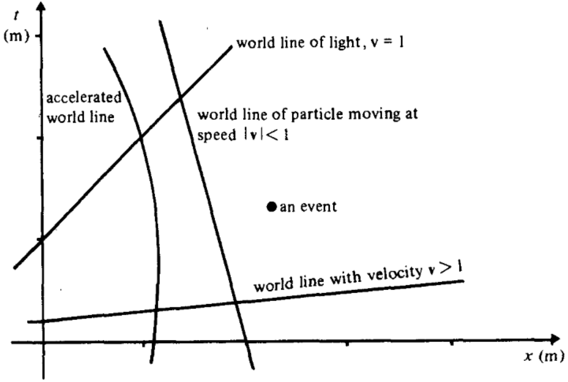


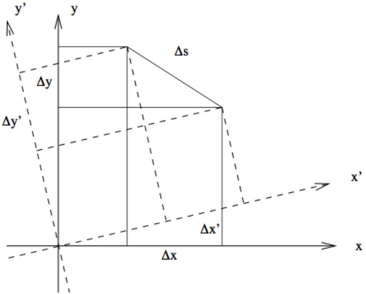
Fig. 1.1 A spacetime diagram in natural units.

$$\text{slope} = \frac{dt}{dx} = \frac{1}{v}$$

The geometry of Newtonian mechanics

In Newtonian mechanics we "consider all of space at a single moment in time." This allows us to think about the geometry of space as independent of motion:

$$\begin{aligned}\Delta s_N^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (\Delta x')^2 + (\Delta y')^2\end{aligned}$$



The geometry of spacetime

Theorem

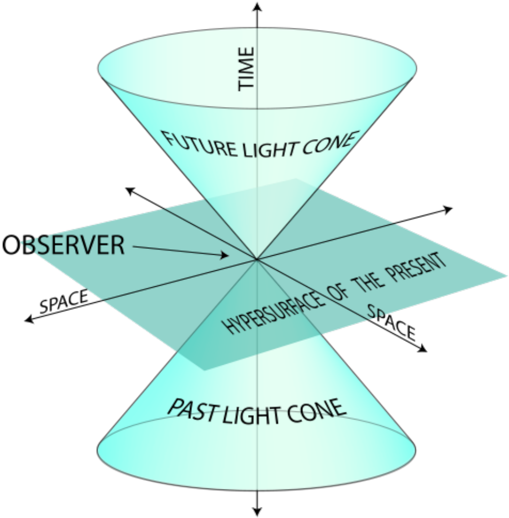
Under the postulates of special relativity, the **spacetime interval**

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

between two events $\mathbf{x}$ and $\mathbf{x}'$ is independent of the inertial frame.

- Note: the usual Newtonian Δs_N^2 is no longer invariant
- The global geometry of spacetime is 4D, non-Euclidean!

Light cone


TIME
FUTURE LIGHT CONE
OBSERVER
SPACE
HYPERSURFACE OF THE PRESENT
SPACE
PAST LIGHT CONE

Transformation symmetries

If inertial observers do not agree on distances and times, what's the conversion rate? Equivalently - what transformations leave the spacetime interval unchanged?

Question

Given two events $\mathbf{x_1}, \mathbf{x_2}$, since the spacetime interval is a function $\Delta s^2(\mathbf{x_1}, \mathbf{x_2})$, we wish to find all such (linear) transformations Λ such that

$$\Delta s^2(\mathbf{x_1}, \mathbf{x_2}) = \Delta s^2(\Lambda \cdot \mathbf{x_1}, \Lambda \cdot \mathbf{x_2})$$

The Lorentz group

- The set of such transformations is known as the **Lorentz group**, $\mathrm{O}(3,1)$, and the transformations are known as Lorentz transformations
- The Lorentz group was recognized mathematically by Poincaré just months before Einstein derived them from the postulates of SR
- An event $\mathbf{x}$ in on inertial frame will look like $\Lambda \cdot \mathbf{x}$ in another, where Λ is a Lorentz transformation that depends on the relative orientation and velocity of the frames

Lorentz boost in the x-direction

Explicitly, one can characterize a Lorentz "boost" in the x-direction to a frame with relative velocity v as:

$$
\begin{aligned}
t' &= \gamma(t - vx) \\
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
\gamma &= \frac{1}{\sqrt{1 - v^2}}
\end{aligned}
$$

More general transformations rapidly become much more tedious.

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Time dilation

Consider a box moving with uniform velocity $\mathbf{v}$ in a lab. Denote by $\mathcal{L}$ the lab frame, whose coordinate system is given (t, x, y, z), and $\mathcal{O}$ the frame in which the box is at rest, whose coordinate system is given (τ, x', y', z'). We can write:

$$\begin{aligned} -(\Delta\tau)^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -(\Delta t)^2 + v^2(\Delta t)^2 \end{aligned}$$

and thus

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2}} = \gamma\Delta\tau.$$

Time dilation: sanity checks

Moving clocks appear to run slower! The lab $\mathcal{L}$ sees its own clock tick γ times for every once that it sees $\mathcal{O}$'s clock tick.

$$\Delta t = \gamma \Delta \tau > \Delta \tau$$

The Newtonian limit $v \to 0$, $\Delta t \to \Delta \tau$ checks out.
In the extreme relativistic limit $v \to 1$, $\Delta t \to \infty$!

Length contraction

Consider now an observer moving at a velocity v with respect to the box, supressing all but one spatial dimensions. Suppose the box has rest length L_0 (i.e. $\Delta x' = L_0$). Then

$$\frac{L^2}{v^2} = \frac{L_0^2}{v^2} - L_0^2$$
$$L = L_0/\gamma$$

In other words, the moving observer find that the length of the box has diminished by a factor of γ!

Failure of simultaneity

`https://upload.wikimedia.org/wikipedia/commons/7/78/Relativity_of_Simultaneity_Animation.gif`

Modern physics

- Special relativity has become a touchstone of modern physics; only those theories consistent with relativity may be considered - Lagrangians must be manifestly covariant
- SR (and by extension, GR) motivated much interdisciplinary research: mathematical physics is now highly geometric (spacetime, bundles) and algebraic (symmetries, representations) in nature $\rightarrow$ gauge theories like the Standard Model
- This has allowed researchers to attack questions in fundamental physics from a variety of perspectives

References

J.D. Jackson. *Classical Electrodynamics*, (1999).

B. Schutz. *A course in general relativity*, (2009).