Analysis I: Solutions to PSET 6

Problem 1

(a) Recall from Rudin Theorem 3.3 that the limit of a product is the product of the limits if the individual limits exist and hence

$$\lim_{n\to\infty} \sqrt[n]{2n} = \left(\lim_{n\to\infty} \sqrt[n]{2}\right)\left(\lim_{n\to\infty} \sqrt[n]{n}\right).$$

From Rudin Theorem 3.20, we find that $\lim_{n\to\infty} \sqrt[n]{2} = 1$ and $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Thus $\lim_{n\to\infty} \sqrt[n]{2n} = 1$.

(b) Invoking Rudin Theorem 3.20(d) for $p = 1$ and $\alpha = 2$, we find that

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

Problem 2

(a) Fix $\varepsilon > 0$. We wish to find some $N \in \mathbb{N}$ such that for $n > N$ we have $|a_n - a| < \varepsilon$ given that $\lim_{n\to\infty} x_n = a$. The sequence x_n must be bounded, and hence $|x_n - a| < M$ for some $M \in \mathbb{R}$ for all $n \in \mathbb{N}$. Moreover, convergence implies that there exists an $N' \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/2$ for all $n > N'$. Note that (for $N' < n$)

$$\begin{aligned}
|a_n - a| &= \left|\frac{(x_0 - a) + \cdots + (x_n - a)}{n+1}\right| \\
&\leq \frac{1}{n+1}\sum_{k=0}^{N'} |x_k - a| + \frac{1}{n+1}\sum_{k=N'+1}^{n} |x_k - a| \\
&\leq \frac{M(N'+1)}{n+1} + \frac{\varepsilon(n - N')}{2(n+1)}.
\end{aligned}$$

To have $|a_n - a| < \varepsilon$,

$$\begin{aligned}
2\varepsilon(n+1) &> 2M(N'+1) + \varepsilon(n - N') \\
\varepsilon n &> 2M(N'+1) - \varepsilon N' - 2\varepsilon \\
n &> 2M(N'+1)/\varepsilon - (N'+2)
\end{aligned}$$

Indeed, for $n > 2M(N'+1)/\varepsilon - (N'+2)$ (assuming N' is smaller than this quantity),

$$\begin{aligned}
|a_n - a| &= \left|\frac{(x_0 - a) + \cdots + (x_n - a)}{n+1}\right| \\
&\le \frac{1}{n+1}\sum_{k=0}^{N'} |x_k - a| + \frac{1}{n+1}\sum_{k=N'+1}^{n} |x_k - a| \\
&\le \frac{M(N'+1)}{n+1} + \frac{\varepsilon(n-N')}{2(n+1)} \\
&< \frac{\varepsilon/2}{2M(N'+1) - \varepsilon(N'+1)}\big(2M(N'+1) + \varepsilon(n-N')\big) \\
&< \frac{\varepsilon/2}{2M(N'+1) - \varepsilon(N'+1)}\big(2M(N'+1) + 2M(N'+1) - 2\varepsilon(N'+1)\big) \\
&< \varepsilon,
\end{aligned}$$

as desired. Thus we take $N \equiv \max\{N', 2M(N'+1)/\varepsilon - (N'+2)\}$.

(b) Consider the sequence $x_n = (-1)^n$. The average a_n is 0 if n is odd and $1/(n+1)$ if n is even. The sequence a_n converges (use the Archimedean property) but the sequence x_n clearly does not.

Rudin 3.16(a)

By the arithmetic-geometric mean inequality,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \ge \sqrt{\alpha},$$

and hence $x_{n+1}^2 \ge \alpha$. Now

$$x_n - x_{n+1} = \frac{1}{2}\left(x_n - \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(\frac{x_n^2 - \alpha}{x_n}\right) \ge 0,$$

which implies that $x_n \ge x_{n+1}$, i.e. the sequence decreases monotonically. Now by Rudin Theorem 3.14, x_n converges, as it is bounded below (by $\sqrt{\alpha}$ as above). Taking $x = \lim_{n\to\infty} x_n$, the recurrence relation becomes

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right),$$

and $x = \sqrt{\alpha}$ (as the negative solution is absurd).

Rudin 3.7

It suffices to show, by Rudin's theorem 3.22, that for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for $m \geq n \geq N$,

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon.$$

Fix $\varepsilon > 0$. Since both $\sum a_n$ and $\sum n^{-2}$ are convergent series (c.f. Rudin theorem 3.28) there exists an $N \in \mathbb{N}$ such that for $m \geq n \geq N$,

$$\sum_{k=n}^{m} a_k \leq \varepsilon$$
$$\sum_{k=n}^{m} \frac{1}{k^2} \leq \varepsilon.$$

Using the Cauchy-Schwarz inequality (Rudin theorem 1.35),

$$\left| \sum_{k=n}^{m} \frac{\sqrt{\alpha_k}}{k} \right|^2 \leq \left(\sum_{k=n}^{m} a_k^2 \right) \cdot \left(\sum_{k=n}^{m} \frac{1}{k^2} \right) \leq \varepsilon^2$$

and hence

$$\left| \sum_{k=n}^{m} \frac{\sqrt{\alpha_k}}{k} \right| \leq \varepsilon$$

for $m \geq n \geq N$ as desired.