Modern Geometry I: PSET 1

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Problem 1

(a) We first show that $(S^n - \{N\}, \pi_1)$ and $(S^n - \{S\}, \pi_2)$ are charts for S^n. To do this, we write π_1 and π_2 in coordinates. Suppose a point p on S^n is given by coordinates in $\mathbb{R}^{n+1}$ as $(p_1, \ldots, p_{n+1})$. The line from the north pole to p can be written parametrically as $(p_1, \ldots, p_{n+1} - 1)t + (0, \ldots, 1)$, and hence intersects the hyperplane $x_{n+1} = 0$ when $t = 1/(1 - p_{n+1})$. Then

$$\pi_1(p_1, \ldots, p_{n+1}) = \frac{1}{1 - p_{n+1}}(p_1, \ldots, p_n),$$

which is of course continuous on the open $S^n - \{N\}$ (this set is open in the subspace topology as it is the complement of a point). Conversely, given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, the corresponding point on the sphere can be found by solving for the t for which $|(x_1 - x_1 t, \ldots, x_n - x_n t, t)| = 1$. The quadratic formula reveals that this occurs when $t = (\sum_i x_i^2 - 1)/(\sum_i x_i^2 + 1)$. Denote this value of t by $\tau(x)$. Then the inverse is given

$$\pi_1^{-1}(x_1, \ldots, x_n) = (x_1 - \tau(x)x_1, \ldots, x_n - \tau(x)x_n, \tau(x)),$$

which is continuous. Hence π_1 is a homeomorphism and $(S^n - \{N\}, \pi_1)$ provides a chart for S^n. Similarly, we find that

$$\pi_2(p_1, \ldots, p_{n+1}) = \frac{1}{1 + p_{n+1}}(p_1, \ldots, p_n),$$

which is continuous on the open $S^n - \{S\}$. We can compute the inverse just as above, and π_2 becomes a homeomorphism, and $(S^n - \{S\}, \pi_2)$ provides a chart.

It remains to show that these two stereographic charts are compatible. Consider the composition $\pi_2 \circ \pi_1^{-1} : \pi_1(S^n - \{N, S\}) \to \pi_2(S^n - \{N, S\})$.

We write

$$\begin{aligned}
\pi_2(\pi_1^{-1}(x_1,\ldots,x_n)) &= \pi_2(x_1-\tau(x)x_1,\ldots,x_n-\tau(x)x_n,\tau(x)) \\
&= \frac{1}{\tau(x)}(x_1-\tau(x)x_1,\ldots,x_n-\tau(x)x_n) \\
&= \left(\frac{x_1}{\tau(x)}-x_1,\ldots,\frac{x_n}{\tau(x)}-x_n\right).
\end{aligned}$$

This composition is clearly smooth, as $\tau(x)$ is non-zero on the domain of definition of $\pi_2\circ\pi_1^{-1}$, to wit, x is not zero. For the sake of brevity, we do not describe the computation of the transition function $\pi_1\circ\pi_2^{-1}$, but an identical argument shows that it is differentiable. Hence the two transition functions are diffeomorphisms, and the above two charts form a smooth atlas for S^n.

(b) Consider now the inclusion map $\iota : S^n \to \mathbb{R}^{n+1}$. The inclusion is a homeomorphism onto its image as it is a bijective continuous map (onto its image) from a compact topological space to a (subset of a) Hausdorff space. Moreover, ι is a smooth map, as $\pi_i^{-1}\circ\iota\circ\mathrm{Id}_{\mathbb{R}^{n+1}}$ is smooth, as given in the previous part. It remains to show that ι is an immersion. In coordinates, this amounts to showing that the Jacobian of $i\circ\pi^{-1}$ has rank n. We can compute, for example, that

$$\frac{\partial}{\partial x_j}(x_i-\tau(x)x_i) = \delta_{ij}-\delta_{ij}\tau(x)-\frac{4x_ix_j}{(\sum_k x_k^2+1)^2},$$

and hence

$$d\iota = \frac{1}{(\sum_k x_k^2+1)^2}\begin{pmatrix} 2(\sum_k x_k^2+1)-4x_1^2 & \cdots & -4x_1x_n \\ \vdots & \ddots & \vdots \\ -4x_1x_n & \cdots & 2(\sum_k x_k^2+1)-4x_n^2 \\ 4x_1 & \cdots & 4x_n \end{pmatrix}$$

Suppose, now, that the columns of this matrix are linearly dependent, i.e. there exist c_k (for $1<k<n+1$) such that $c_1(d\iota)_{k1}+\ldots+c_n(d\iota)_{kn}=0$. Imposing this condition forces $\sum_k c_kx_k = 0$ due to the last row, and hence the first row yields $2c_1(\sum_k x_k^2+1)=0$, forcing $c_1=0$, and similarly for the following $n-1$ rows. Hence the columns are independent, and $d\iota$ has rank n.

Problem 2

Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $F(x, y, z) = (x^2 - y^2, xy, zx, yz)$ and denote by f the restriction of F to $S^2 \subset \mathbb{R}^3$. Note that $F(x, y, z) = F(-x, -y, -z)$ and thus f descends to a map $\tilde{f} : \mathbb{P}^2_{\mathbb{R}} = S^2/\{\pm 1\} \to \mathbb{R}^4$. In other words, the diagram

$$\begin{tikzcd} S^2 \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathbb{R}^4 \\ & \mathbb{P}^2_{\mathbb{R}} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutes, where π is the natural quotient map. The map $\tilde{f}$ is smooth, as it takes, in coordinates (say where $x \neq 0$)

$$(\alpha, \beta) \mapsto \frac{1}{1 + \alpha^2 + \beta^2} \left(1 - \alpha^2, \alpha, \beta, \alpha\beta\right).$$

Moreover, to show that $\tilde{f}$ is injective, it suffices to show that f is two-to-one (compatible with the quotient map). This is done as follows. Suppose $f(x, y, z) = f(\alpha, \beta, \gamma)$, i.e.

$$\begin{aligned} x^2 - y^2 &= \alpha^2 - \beta^2 \\ xy &= \alpha\beta \\ xz &= \alpha\gamma \\ yz &= \beta\gamma. \end{aligned}$$

Dividing, we find that $r \equiv x/\alpha = \beta/y = \gamma/z$ (the cases where α, y, or z are zero fall out easily). The first equation yields

$$\begin{aligned} r^2\alpha^2 - \frac{\beta^2}{r^2} &= \alpha^2 - \beta^2 \\ (r^2 - 1)(r^2\alpha^2 + \beta^2) &= 0, \end{aligned}$$

which forces $r = \pm 1$. In other words $f(x, y, z) = f(\alpha, \beta, \gamma)$ if and only if $\vec{x} = \pm\vec{\alpha}$. As these points are identified in projective space, we find that $\tilde{f}$ is indeed injective. Thus, as a bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism, we find that $\tilde{f}$ is a homeomorphism onto its image. It remains now to show that $\tilde{f}$ is an immersion. This is done using the coordinate representation of $\tilde{f}$ as

$$\tilde{f}_{x \neq 0}(\alpha, \beta) = \frac{1}{1 + \alpha^2 + \beta^2}(1 - \alpha^2, \alpha, \beta, \alpha\beta).$$

$$d\tilde{f}_{x\neq 0}(\alpha,\beta) = \frac{1}{(1+\alpha^2+\beta^2)^2}\begin{pmatrix} -2\alpha(2+\beta^2) & 2\beta(\alpha^2-1) \\ 1-\alpha^2+\beta^2 & -2\alpha\beta \\ -2\alpha\beta & 1-\beta^2+\alpha^2 \\ \beta(1-\alpha^2+\beta^2) & \alpha(1-\beta^2+\alpha^2) \end{pmatrix}.$$

Computing the determinant of the minor consisting of the second and third rows yields

$$\begin{aligned} (1-\alpha^2+\beta^2)(1-\beta^2+\alpha^2)+4\alpha^2\beta^2 &= 0 \\ 1-2\alpha^2\beta^2-\alpha^4-\beta^4 &= 0 \\ \alpha^2+\beta^2 &= 1. \end{aligned}$$

This minor thus has rank 2 away from the unit circle. On the unit circle, we compute the determinant of the minor consisting of the second and fourth rows

$$\begin{aligned} \alpha(1-\beta^2+\alpha^2)(1-\alpha^2+\beta^2)+2\alpha\beta^2(1-\alpha^2+\beta^2) &= 0 \\ \alpha(1-\alpha^2+\beta^2)(1+\alpha^2+\beta^2) &= 0 \\ 4\alpha\beta^2 &= 0. \end{aligned}$$

Hence the differential is of rank 2 away from the four points $(\pm 1, 0)$ and $(0, \pm 1)$. Finally, we compute the determinant of the minor consisting of the first and the fourth rows

$$\begin{aligned} -2\alpha^2(2+\beta^2)(1-\beta^2+\alpha^2)-2\beta^2(\alpha^2-1)(1-\alpha^2+\beta^2) &= 0 \\ -2(1-\beta^2)(2+\beta^2)(2-2\beta^2)+4\beta^6 &= 0 \\ -4(1-\beta^2)^2(2+\beta^2)+4\beta^6 &= 0. \end{aligned}$$

It is clear that $\beta = 0, \pm 1$ are not solutions, and hence $d\tilde{f}$ is rank 2 everywhere. A similar computation in other charts reveals that $d\tilde{f}$ is injective everywhere. Consequently, $\tilde{f}$ is an injective immersion homeomorphic to its image, and thus a smooth embedding of $\mathbb{P}^2_{\mathbb{R}}$ into $\mathbb{R}^4$.

Problem 3

Let Y_r be the set of points in $\mathbb{R}^3$ at a distance $r > 0$ from the unit circle in the xy-plane. Let $A = \{r \in (0,\infty) \mid Y_r \text{ is a smooth submanifold of } \mathbb{R}^3\}$.

(a) We consider three cases: $r < 1, r = 1$, and $r > 1$. For $r = 1$, it is clear that Y_1 is not locally Euclidean: let U be an open neighborhood of Y_1 at the origin. Removing the point at the origin yields a disconnected open neighborhood; if a homeomorphism to an open subset of $\mathbb{R}^n$ were to exist, it would have to be disconnected after removing a point, which is clearly impossible. For $r < 1$, we obviously obtain a smooth submanifold by the preimage theorem stated in class; one can write a level set expression $((1-\sqrt{x^2+y^2})^2+z^2-r^2=0)$ and the differential is easily checked to be surjective. Finally, for $r > 1$, Y_r is clearly homeomorphic to the sphere, and thus forms a topological manifold. However, Y_r cannot be a smooth submanifold or $\mathbb{R}^3$, for the following reason. If it were, i.e. we were to have a smooth inclusion $\iota : Y_r \to \mathbb{R}^3$, the rank of $d\iota$ would be three (as we can find three linearly independently vectors $(\sqrt{r^2-1}, 0, 1), (-\sqrt{r^2-1}, 0, 1)$, and $(0, \sqrt{r^2-1}, 1)$ tangent to Y_r at $(0, 0, \sqrt{r^2-1})$), which is absurd. In other words, the dimension of the tangent space of Y_r at the points $(0, 0, \pm\sqrt{r^2-1})$ would not be 2. Thus, $A = (0, 1)$.

(b) Recall the stereographic charts π_1 and π_2 for the sphere. The smooth structure of $S^1 \times S^1$ is given by four charts, $\pi_i \times \pi_j$, for $1 \leq i, j \leq 2$. Now define $\iota : S^1 \times S^1 \to \mathbb{R}^3$ as

$$\iota(x, y, \alpha, \beta) = (x + rx\alpha, y + ry\alpha, r\beta),$$

where (x, y) are the coordinates on the first circle and (α, β) are the coordinates on the second circle (as embedding in $\mathbb{R}^2$), and $r \in A = (0, 1)$. It is clear that $\iota(S^1 \times S^1) = Y_r$; it suffices to check that ι is an immersion (a smooth embedding is a diffeomorphism onto its image). Working in the chart given by $\pi_1 \times \pi_1$, we find that

$$(\pi_1 \times \pi_1)^{-1}(x, y) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}, \frac{2y}{y^2+1}, \frac{y^2-1}{y^2+1} \right),$$

and hence

$$\begin{aligned}\iota \circ (\pi_1 \times \pi_1)^{-1}(x, y) = \Bigg(&\frac{2x}{x^2+1} + \frac{4rxy}{(x^2+1)(y^2+1)}, \\ &\frac{x^2-1}{x^2+1} + \frac{2ry(x^2-1)}{(x^2+1)(y^2+1)}, \frac{r(y^2-1)}{y^2+1} \Bigg).\end{aligned}$$

Denote the differential of this map by M. We compute:

$$M = \begin{pmatrix} \frac{1-x^2}{(1+x^2)^2}\left(2 + \frac{4ry}{y^2+1}\right) & \frac{4rx(1-y^2)}{(x^2+1)(y^2+1)^2} \\ \frac{4x}{(x^2+1)^2}\left(1 + \frac{2ry}{y^2+1}\right) & \frac{2r(x^2-1)(1-y^2)}{(x^2+1)(y^2+1)^2} \\ 0 & \frac{4ry}{(y^2+1)^2} \end{pmatrix}$$

The determinant of the bottom 2×2 minor vanishes when either x or y are zero. However, we see by inspection that $(x, 0), (0, y)$, and $(0, 0)$ for $x \neq 0, y \neq 0$ all yield matrices of rank 2. This proves that $S^1 \times S^1 \cong Y_r$.