Notes on the D_4 extension $E = \mathbb{Q}(\sqrt[4]{2}, i)$ of $\mathbb{Q}$

Elements of D_4: 1, (1234), $(1234)^2 = (13)(24)$, $(1234)^3 = (1432)$; (13), (24), (12)(34), (14)(23).

Subgroups of D_4: $\{1\}$ (order 1), D_4 (order 8). The three subgroups of order 4, all automatically normal:

$$\begin{aligned} H_1 &= \langle (1234) \rangle \\ H_2 &= \{1, (13)(24), (12)(34), (14)(23)\} \\ H_3 &= \{1, (13)(24), (13), (24)\}. \end{aligned}$$

The five subgroups of order 2: $\langle (13)(24) \rangle$, $\langle (13) \rangle$, $\langle (24) \rangle$, $\langle (12)(34) \rangle$, $\langle (14)(23) \rangle$. Of these, only $\langle (13)(24) \rangle$ is normal (it is the center of D_4).

The fixed fields: Label the roots of $x^4 - 2$ as

$$\alpha_1 = \sqrt[4]{2}; \qquad \alpha_2 = i\sqrt[4]{2}; \qquad \alpha_3 = -\sqrt[4]{2}; \qquad \alpha_4 = -i\sqrt[4]{2},$$

corresponding to the labeling of elements of D_4 above. Then the fixed field of $\{1\}$ is $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and the fixed field of D_4 is $\mathbb{Q}$. As for the subgroups of order 2, they correspond to subfields K of E such that $[K : \mathbb{Q}] = 4$. For example, it is clear that $\sqrt[4]{2} \in E^{\langle (24) \rangle}$ and hence by counting degrees that

$$E^{\langle (24) \rangle} = \mathbb{Q}(\sqrt[4]{2}).$$

Likewise $E^{\langle (13) \rangle} = \mathbb{Q}(i\sqrt[4]{2})$. As for $E^{\langle (13)(24) \rangle}$, note that $\sqrt{2} = (\sqrt[4]{2})^2 = (-\sqrt[4]{2})^2$ is fixed by (13)(24), and also i is fixed by (13)(24) since if $\sigma(\sqrt[4]{2}) = -\sqrt[4]{2}$ and $\sigma(i\sqrt[4]{2}) = -i\sqrt[4]{2}$, then

$$\sigma(i) = \sigma(i\sqrt[4]{2}/\sqrt[4]{2}) = \sigma(i\sqrt[4]{2})/\sigma(\sqrt[4]{2}) = (-i\sqrt[4]{2})/(-\sqrt[4]{2}) = i.$$

Thus $\mathbb{Q}(\sqrt{2}, i) \subseteq E^{\langle (13)(24) \rangle}$, so again by counting degrees they are equal. As for $E^{\langle (12)(34) \rangle}$, note that $\sqrt[4]{2} + i\sqrt[4]{2} = \alpha_1 + \alpha_2 \in E^{\langle (12)(34) \rangle}$. In particular, this forces $\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) \neq F$. While it may not be obvious how to compute the degree $[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}]$, note that

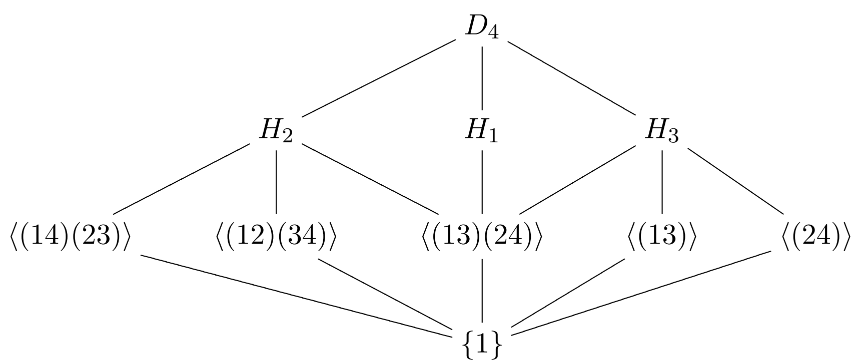
$$(\sqrt[4]{2} + i\sqrt[4]{2})^2 = (1+i)^2(\sqrt[4]{2})^2 = 2i\sqrt{2}.$$

Thus $[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}(i\sqrt{2})] = 2$ since $\sqrt[4]{2} + i\sqrt[4]{2} \notin \mathbb{Q}(i\sqrt{2})$, and since $[\mathbb{Q}(i\sqrt{2}) : \mathbb{Q}] = 2$ since $i\sqrt{2} = \sqrt{-2}$, it follows that

$$[\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) : \mathbb{Q}(i\sqrt{2})][\mathbb{Q}(i\sqrt{2}) : \mathbb{Q}] = 4.$$

Hence, again by counting degrees, $E^{\langle(12)(34)\rangle} = \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2})$. Similarly, $E^{\langle(14)(23)\rangle} = \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$.
Finally, there are the 3 fields E^{H_1}, E^{H_2}, E^{H_3}. A computation shows that $i \in E^{H_1}$, hence $E^{H_1} = \mathbb{Q}(i)$. As for the others, clearly $E^{H_2} = E^{\langle(13)(24)\rangle} \cap E^{\langle(12)(34)\rangle}$. Since $E^{\langle(13)(24)\rangle} = \mathbb{Q}(\sqrt{2}, i)$ and $i\sqrt{2} \in E^{\langle(12)(34)\rangle}$, $i\sqrt{2} \in E^{H_2}$ and hence $E^{H_2} = \mathbb{Q}(i\sqrt{2})$. The other equality $E^{H_3} = \mathbb{Q}(\sqrt{2})$ is similar.

Picture of the subgroups of D_4:



Picture of the intermediate subfields between E and $\mathbb{Q}$: