Commutative Algebra: Problem Set 3

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Problem 7

Define the ring $A = k[f_1, f_2, \ldots]/(f_1^2, f_2^2, \ldots)$ and consider the ideal $I = A$. Consider the ideal I^n for some $n > 0$. Let $g = f_1 + f_2 + \ldots + f_n$ in I. Then g^n is an element of I^n, but note that g^n clearly contains the term $n!(f_1 f_2 \cdots f_n) \neq 0$ and hence I^n cannot be nilpotent.

Problem 8

Since B is a finite-type A algebra (A,B domains), we can write $B = A[x_1, \ldots, x_n]/I$ where I is an ideal of $A[x_1, \ldots, x_n]$. By finiteness of the extension $K \subset L$ (and since obviously $A \subset K, B \subset L$), $x_i \in L$ is algebraic over K. Hence each x_i solves a polynomial of degree n_i with coefficients in K. By clearing denominators, we can obtain an associated set of polynomials with coefficients in A; let us denote by a_i the leading coefficient of the polynomial for x_i and let $\alpha = a_1 \cdots a_n$ be the product. We can now localize both A and B about the multiplicative set generated by α. We now have that $x_i \in B_\alpha$ are integral over A_α. Now (by lemma 10.33.5 of the Stacks project, Tag 02JJ) we see that the map $A_\alpha \to B_\alpha$ is finite, and by exactness of localization, injective. This yields the following diagram:

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d] & \operatorname{Spec} B_\alpha \arrow[l, hook'] \arrow[d, two heads] \\ \operatorname{Spec} A & \operatorname{Spec} A_\alpha \arrow[l, hook'] \end{tikzcd}$$

It follows from the properties of primes of localization that the image of the map $\operatorname{Spec} A_\alpha \hookrightarrow \operatorname{Spec} A$ is $D(\alpha)$. But since $\operatorname{Spec} B_\alpha \to \operatorname{Spec} A_\alpha$ is a surjection and $\operatorname{Spec} B_\alpha \hookrightarrow \operatorname{Spec} B$, $\operatorname{Spec} A_\alpha \hookrightarrow \operatorname{Spec} A$ are injections, it follows that the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ must also contain $D(\alpha)$. Since $D(\alpha)$ is open, we are done.

Problem 9

Consider the evaluation homomorphism $k[x, y] \xrightarrow{\phi} k[t]$ that takes $P(x, y) \mapsto P(f(t), g(t))$ for the given $f(t), g(t)$. Let us suppose for the sake of contradiction that there does not exist a $P \in k[x, y]$ such that $\phi(P) = 0$. In other words, ϕ is injective. Suppose $f(t) = a_n t^n + \ldots + a_0$. We can show that $t \in k[t]$ is in fact integral over $k[x, y]$, as it solves the equation

$$\begin{aligned} \phi(x) &= f(t) \\ 0 &= t^n + \frac{a_{n-1}}{a_n} t^{n-1} + \ldots + \frac{a_0 - \phi(x)}{a_n}. \end{aligned}$$

Hence ϕ must be finite. This implies that $\operatorname{Spec}\phi$ is surjective, which implies that $\dim k[t] = \dim k[x,y]$. This is a contradiction; thus ϕ cannot be injective, and there must exist a $P(x,y) \in k[x,y]$ such that $\phi(P) = 0$.

Problem 10

Let $A = \mathbb{Z}/4\mathbb{Z}$. It's clear that A is Artinian as the ring is finite. Additionally, A is clearly not an algebra of finite-type over a field.

Problem 11