DIFFERENTIABLE MANIFOLDS HOMEWORK 3

DUE MAR. 5, 2013 IN CLASS

(1) Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Suppose $(\tilde{x}, \tilde{y})$ are given by

$$\tilde{x} = x, \tilde{y} = f(x, y)$$

for a smooth function f defined on $\mathbb{R}^2$. Find the condition on f so that $\tilde{x}$ and $\tilde{y}$ become another global smooth coordinates on $\mathbb{R}^2$. Show that in general

$$\frac{\partial}{\partial x}|_p \neq \frac{\partial}{\partial \tilde{x}}\,|_p$$

even though the coordinate functions x and $\tilde{x}$ are identically equal.

(2) Recall that in problem (1) of homework 1, the transition map $\psi \circ \phi^{-1}$ is given by

$$(u, v) \mapsto (\tilde{u}, \tilde{v}) = (\frac{4u}{u^2 + v^2}, \frac{4v}{u^2 + v^2}) \text{ for } (u, v) \neq (0, 0).$$

Show that the vector filed $u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ can be extend smoothly to the whole S^2. What about the vector field $v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$?

(3) Consider $T = \mathbb{R}^2/\sim$ in which the equivalence relation is given by $(x, y) \sim (x + 1, y)$ and $(x, y) \sim (x, y + 1)$. Show that T is a differentiable manifolds and that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are global smooth vector fields on T.

(4) Let x, y be the standard coordinates on $\mathbb{R}^2$ and X is a vector field given by $X(x, y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. Find a C^∞ map $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, such that $\phi(0, x, y) = (x, y)$ and $\frac{\partial \phi}{\partial t}(t, x, y) = X(\phi(t, x, y))$.

(5) Let X be the smooth vector field on S^2 which is the extension of $u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$ from $U = S^2 - \{(0, 0, 1)\}$ to the whole S^2. Find the one-parameter group of diffeomorphisms ϕ_t generated by X. What is the behavior of ϕ_t near the two poles?

(6) 8-20 on page 202 of [1].

(7) 10-13 on page 270 of [1].

Date: Feb. 19, 2013.

References

[1] Lee, John M. *Introduction to Smooth Manifolds,* Second edition, Graduate Texts in Mathematics, 218, Springer-Verlag, New York, 2012.