Algebraic Topology I: PSET 1

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Problem 1

(a) We construct a CW complex for $\mathbb{R}^2$ as follows. Let X^0 contain a point for every pair of integers. Then, for every pair of horizontally or vertically neighboring 0-cells, attach a line segment with endpoints identified to said 0-cells. Finally, for every square of side length 1 line segment, attach a 2-cell such that the boundary of the 2-cell is identified with the square.

(b) The CW complex for the punctured plane, $\mathbb{R}^2 - \{(0,0)\}$ is similar to that of $\mathbb{R}^2$. Start with the CW complex from (a) except remove all cells in the square of side length one in the first quadrant with bottom-left corner $(0,0)$. In this square, line the borders with a square annulus of some size (adding 0,1, and 2-cells as necessary). Repeat this process with smaller and smaller square annuli such that we obtain a countable number of annuli (each with radius, say, $1/n$). Note that this complex does not "contain the origin."

Hatcher 0.5

The space X deformation retracts to $x \in X$. Let $U \subset X$ is an open containing x. We wish to find an open $V \subset U$ containing x such that the inclusion $V \hookrightarrow U$ is nullhomotopic. We are provided with a homotopy $F = f_t : X \times I \to X$ taking Id_X to $\{x\}$; hence it suffices to find a neighborhood V which, under f_t, is contained in U for all time t. To do so, consider the open set $F^{-1}(U) \subset X \times I$ containing the slice $\{x\} \times I$. We can find an open neighborhood $V \ni x$ such that the cylinder $V \times I$ is contained $F^{-1}(U)$: take an open cover of $\{x\} \times I$ (by basis opens of the product topology). Compactness of I yields a finite subcover, from which the intersection of the

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opens in the second components yields an open $V \subset U$ containing x such that $V \times I \subset F^{-1}(U)$. This implies that, for all t, $f_t(V)$ is contained in U, and hence V is the desired open.

Hatcher 0.6a

Suppose x_0 is a point in the segment $[0,1] \times \{0\}$; then there exists a deformation retract taking X to x_0 as follows. For $t \in [0, 1/2]$ we collapse the teeth of the comb via $(x, y) \mapsto (x, y(1-2t))$. Then, for $t \in [1/2, 1]$, we collapse the remaining line segment onto the chosen point x_0 via $(x, 0) \mapsto (x + 2(x - x_0)(t - 1/2), 0)$. Note that x_0 is fixed throughout, and hence this provides a deformation retraction.

There is no deformation retraction taking X to a point not on the segment $\{0, 1\} \times \{0\}$. We show this by contradiction. If there is, and U is an open about said point, there must exist an open $V \subset U$ such that the inclusion $V \hookrightarrow U$ is nullhomotopic, i.e. contractible in U. In this case, U is the disjoint union of countably many line segments; any open subset will be the disjoint union of countably many intervals, which is clearly not contractible (this can be shown by noting that no map $X \times I \to X$ could continuously move disconnected components to a single component).

Hatcher 0.8

Suppose $n = 2m$ for $m > 1$. Then the construction of the house with n rooms is simply m copies of the house with 2 rooms glued in the obvious way. The case of $n = 2m + 1$ is a little subtler. Take the house with $2m$ rooms and add another "roof" as shown in my diagram and add a tunnel that traverses the top 2 rooms (with vertical support walls as necessary), connecting the outside to the third room from the top. In both cases, these structures are clearly contractible, as one can imagine "unscooping" the space created in the rooms backwards along the tunnels to obtain a space homeomorphic to a ball.

Hatcher 0.9

Let X be a contractible space and $r : X \to A$ be a retract of X to the subspace A. There exists a homotopy $F = f_t : X \times I \to X$ from the identity map on X to the map that takes X to a point $x \in X$. The composition $r \circ f_t$, when restricted to A provides a homotopy from Id_A to the map that takes A to $r(x) \in A$, and hence A is contractible.

Hatcher 0.17

(a) Let $f : S^1 \to S^1$ be a continuous map and consider the associated mapping cylinder M_f. We can give M_f a CW structure as follows. The zero-skeleton consists of two points, x and $f(x)$. The one-skeleton consists of three lines, one forming a circle at x and another forming a circle at $f(x)$, and the last, snaking down to connect x to $f(x)$ as the "graph" of x. Finally, we attach a single two cell to this one-skeleton, completing the cylinder.

(b) Denote the Möbius strip by M and the annulus by A. Consider the retractions $f : M \to S^1$ and $g : A \to S^1$. We can construct the mapping cylinders $X_1 = S^1 \coprod_f M$ and $X_2 = S^1 \coprod_g A$. These spaces clearly deformation retract onto M and A respectively. Gluing these spaces together via the identity map on S^1 in each of these spaces, we obtain a space that deformation retracts onto M and A. Note that this is a CW complex as we can give the Möbius strip and the annulus CW structures.

Hatcher 0.20

Consider the disk at which the Klein bottle intersects itself. As it is a subcomplex, we can contract it to a point without changing the homotopy type of the Klein bottle. We can deform the Klein bottle into a sphere-like shape and replace both the inward and outward flutes with the north and south pole of this sphere and extend the pinched point into inward and outward line segments. Shifting the endpoints of these line segments yields $S^1 \vee S^1 \vee S^2$ (see diagram).

Hatcher 0.23

Let $X = A \cup B$ be a union of two contractible subcomplexes with contractible intersection $A \cap B$. We use Hatcher Proposition 0.17 repeatedly to show that X is contractible:

$$\begin{aligned} A \cup B &\approx (A \cup B)/(A \cap B) \\ &= A/(A \cap B) \vee B/(A \cap B) \\ &\approx A \vee B \\ &\approx A \\ &\approx \{\text{pt}\}, \end{aligned}$$

as every CW pair satisfies the HEP.