Commutative Algebra: Problem Set 1

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Problem 5

Let k be a field and $A=k[x,y]/(x^2y^4-x^4y^2+1)$. We wish to construct a finite injective map as in the Noether normalization lemma. As an algebra, A is generated by $\{x,y\}$. Using the trick from Noether normalization, we define $\tilde{x}=x-y^2$ in order to produce a monic polynomial. Inserting this into the polynomial F generating the ideal, we find that $1+y^8-y^{10}+2y^6\tilde{x}-4y^8\tilde{x}+y^4\tilde{x}^2-6y^6\tilde{x}^2-4y^4\tilde{x}^3-y^2\tilde{x}^4$. In A, F=0, and thus y is integral over $k[\tilde{x}]$. This implies that the map $\phi:k[\tilde{x}]\to A$ is finite (by lemma 3 in class). Injectivity of this map follows easily from what we did in class.

Problem 6

The prime ideals of $\mathbb{C}[x,y]/(xy)$ are in one-to-one correspondence with the prime ideals of $\mathbb{C}[x,y]$ that contain (xy). Since $\mathbb{C}[x]$ is a principal ideal domain, we may use the following lemma from undergraduate algebra:

Lemma 1. Let R be a principal ideal domain. The prime ideals of R[y] are precisely those of the following form:

- (0)
- (f(y)) where f is an irreducible polynomial,
- (p, f(y)) where $p \in R$ is prime and f(y) is irreducible in (R/p)[y]

It follows straightforwardly, then, that such prime ideals that contain (xy) are precisely (x), (y), $(x, y - \lambda)$, $(x - \mu, y)$. A more geometric approach, of course would be to think of the x and y-axes, which would give us precisely this set.

Problem 7

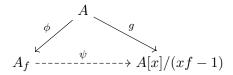
Let k be a field. We wish to prove that k[x,y] is not isomorphic to k[x,y,z]. Suppose for the sake of contradiction that there exists a (module) isomorphism $\phi: k[x,y] \to k[x,y,z]$. Then ϕ is clearly (module-)finite, i.e. there exists a set of generators $M \subset k[x,y,z]$ that k[x,y]-spans k[x,y,z]. It's clear that $1 = z^0 \in M$. The k[x,y]-span of 1 does not contain z, however, and hence we must have $z \in M$. By the same argument, we must have z^n for arbitrarily high powers $n \in \mathbb{Z}$. This is a contradiction - k[x,y] cannot be isomorphic to k[x,y,z].

Problem 8

Let k be a field. Suppose A is a k-algebra and f is a nonzerodivisor of A such that k[x,y] is isomorphic to A_f as a k-algebra. Then, since the invertible elements of k[x,y] are the field elements k, we must have that $f \in k$. Note that since $k \to A$ is injective, we must have that $f \in k \subset A$ and hence $A = A_f = k[x,y]$ since f is already invertible.

Problem 9

Let A be a ring and let f be an element of A. We wish to show that $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is isomorphic as an A-algebra to A[x]/(fx-1). Consider the following diagram:



Using the universal property of localization (see Atiyah-MacDonald pp. 37-38), since $g(f^k)$ is a unit in A[x]/(xf-1), ker g=0, and because every element of A[x]/(xf-1) is of the form $g(a)g(s)^{-1}$, there exists a unique isomorphism $\psi: A_f \to A[x]/(xf-1)$.