# Hartshorne Solutions

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## Problem 3.1

(a) By the results of problem 1.1, we know that any conic in $\mathbb{A}^2$ can be written as either a variety $Y$ defined by $y - x^2 = 0$ or a variety $Z$ defined by $xy - 1 = 0$. We know that $A(Y) = k[x]$ and $A(Z) = k[x, x^{-1}]$. Note that $A(Y) \cong A(\mathbb{A}^1)$, and hence by Corollary 3.7, $Y \cong \mathbb{A}^1$ as affine varieties. It remains to show that $Z$ is isomorphic to $\mathbb{A}^1 - \{0\}$. why

(b) Let $B$ be a proper open subset of $\mathbb{A}^1$. By definition of the Zariski topology, we can write $B = \mathbb{A}^1 \setminus \{p_1, \ldots, p_n\}$ where $p_i$ are a finite set of points in $\mathbb{A}^1$. The ring of regular functions of $\mathbb{A}^1$ is $\mathcal{O}(\mathbb{A}^1) = k[x]$. In $B$, however, polynomials that vanish only at any of the $p_i$ are globally invertible, and hence $\mathcal{O}(B) = k[x, (x - p_1)^{-1}, \ldots, (x - p_n)^{-1}]$. These two rings are not isomorphic.

(c) In the projective plane, we can write a conic as $F(x, y, z) = ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2$, which can be rewritten under an appropriate change of variables as $x^2 + y^2 + z^2$. Hence every conic in the projective plane is isomorphic, and it will suffice to show that there exists a conic that is isomorphic to $\mathbb{P}^1$. This is done by noting that the 2-uple of $\rho_2 : \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism onto its image (c.f. problem 3.4), and that

$$\rho_2(a_0, a_1) = (a_0^2, a_0 a_1, a_1^2),$$

which clearly traces out a conic $xz - y^2$.

(d) This is obvious from the cell decomposition $\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^0$ – one cannot construct a bijection between $\mathbb{A}^2$ and $\mathbb{P}^2$.

(e) If an affine variety $X$ is isomorphic to a projective variety $Y$, then we must have that $\mathcal{O}(X) = \mathcal{O}(Y) = k$. But for $k[x_1, \ldots, x_n]/I(X) = k$, $I(X)$ must be maximal. Hence $I(X) = (x_1 - a_1, \ldots, x_n - a_n)$, i.e. $X$ is just a point.

## Problem 3.14

(a) Note first that $\phi$ is continuous, as the preimage of any closed subset $V \subset \mathbb{P}^n$ is the projective cone $\overline{C(V)}$, which is closed in $\mathbb{P}^{n+1}$. Furthermore, the point at which the line connecting any $Q$ and $P$ to the hypersurface (choose $x_0 = 0$ without loss of generality) is given by

$$\phi(Q) = [Q_1 - \frac{Q_0 P_1}{P_0} : \cdots : Q_{n+1} - \frac{Q_0 P_{n+1}}{P_0}],$$

where $P_i$ and $Q_i$, are the $i$th components of $P$ and $Q$, respectively (the coordinates are written as for a point in $\mathbb{P}^n$). It is easy to see that $\phi$ pulls back regular functions to regular functions: given $g/h : \mathbb{P}^n \to k$, $g(\phi(Q))/h(\phi(Q))$ is regular as well, since inserting $\phi(Q)$ (as above) will retain homogeneity as well as keep the denominator non-zero (as $h$ has no zeroes).

(b) The twisted cubic is given parametrically by $[x : y : z : w] = [t^3 : t^2u : tu^2 : u^3]$. We wish to project from $P = [0 : 0 : 1 : 0]$ onto the hyperplane $z = 0$. This yields the points $[t^3 : t^2u : u^3] \in \mathbb{P}^2$. Note that these points satisfy the equation $x_0^2x_2 - x_1^3 = 0$. But this is precisely the projective closure of the cuspidal cubic $y^3 = x^2$.

## Problem 3.15

(a) Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Consider the product $X \times Y \subset \mathbb{A}^{n+m}$ with the induced Zariski topology. Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X \mid x \times Y \subset Z_i\}$ for $i = 1, 2$. The irreducibility of $Y$ guarantees that $X_1 \cup X_2 = X$: if there were an $x$ for which $x \times Y$ were not contained in a $Z_i$, this would yield a covering of $Y$ by closed sets $Z_1 \cap Y, Z_2 \cap Y$. Furthermore, the $X_i$ must be closed. Hence either $X_1 = X$ or $X_2 = X$ and thus $Z_1 = X \times Y$ or $Z_2 = X \times Y$, i.e. $X \times Y$ is irreducible.

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(b) Consider the map $A(X) \otimes_k A(Y) \to A(X \times Y)$ given by taking $(f \otimes g)(x, y)$ to $f(x)g(y)$. This map is clearly onto, as it produces the coordinate functions $x_1, \ldots, x_n, y_1, \cdots, y_m$.