A Rigorous Framework for Profit and Loss Optimization via Linear Signal Constructions from Exogenous Variables

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Executive Summary

- We consider an asset with a specific price series $(\mathsf{price}_t)_t$ and a set of exogenous *stationnary* and *homoscedastic* variables $(X_{1,t}, X_{2,t}, \ldots, X_{n,t})$.
- The objective is to derive a signal capable of generating an 'optimal' Profit and Loss (PnL), under two linear assumptions:
  - We will consider the signal as a linear combination of the exogenous variables:

$$\mathsf{signal}_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} + \ldots + \beta_n X_{n,t}$$

  - At each time step t, a predetermined number of shares of the asset will be purchased. The positions taken will be proportional to both the signal and the price (a negative value would correspond to a short position.) This approach establishes a linear relationship between Profit and Loss (PnL) and the trading signal, which can be mathematically represented as:

$$\mathsf{PnL}_t = \mathsf{signal}_{t-1} \times (\mathsf{price}_t - \mathsf{price}_{t-1})$$

- This allows us to express PnL as a linear combination of the transformed variables $\tilde{X}_t = X_{t-1} \times (\mathsf{price}_t - \mathsf{price}_{t-1})$:

$$\mathsf{PnL}_t = \beta^{\mathrm{T}} \tilde{X}_t$$

- A parametric expression of the empiric Sharpe Ratio is then derived:

$$\mathcal{L}(\beta) = \frac{\overline{PnL}}{\sigma(PnL)} = \frac{\beta^{\mathrm{T}}\mu}{\sqrt{\beta^{\mathrm{T}}\Sigma\beta}}$$

  With μ_i and $\Sigma_{i,j}$ the empiric mean and covariance of the transformed variables $(\tilde{X}_{i,t})_i$

- The optimality criterium is to maximimize this empiric sharpe ratio. Transforming this into an optimization problem, the beta that maximizes this empirical Sharpe Ratio is:

$$\hat{\beta} = \operatorname{argmax} \mathcal{L}(\beta) = \frac{\Sigma^{\text{-1}}\mu}{\sqrt{\mu^{\mathrm{T}}\Sigma^{\text{-1}}\mu}}$$

  These optimal coefficients, upon examination, are seen to create a signal with the highest correlation to the asset's price variations, constrained by a low variance in this correlation.

- This model can then be utilized as a Machine Learning (ML) model for finance strategies: trained on a dataset composed of the price and exogenous variable of the last τ days $t-\tau, ..., t-1$, and used to create the present-time signal at t.
- Additional engineering techniques, (mainly feature engineering of the exogenous variables,, specific regularization techniques and corrective factor) are employed to develop a enhance the quantitative finance strategies based on this model.
- This strategy was tested on a specific asset ('IEF', a widely traded Exchange Traded Fund (ETF) that mirrors the performance of U.S. Treasury bonds of maturities: 1-3 years.) yielding qualitative results with a strategy exhibiting good metrics (effective sharpe ratio of 1.2 backtested under the period 2000-2023)

Introduction

This study embarks on the development of a robust mathematical framework with the intent to optimize the Profit and Loss (PnL) of a trading strategy. Central to our approach is the construction of a linear signal from a carefully selected set of exogenous variables. The focal point of this strategy is an asset which price is represented by a time series $(\mathsf{price}_t)_t$. The ultimate aim is to devise a signal that not only responds to market dynamics but also maximizes a predefined objective function: the Sharpe Ratio. This ratio, a key measure of risk-adjusted return, is critical in evaluating the effectiveness of trading strategies.

In the methodology proposed, the signal is formulated as a linear combination of exogenous variables, expressed as $\mathsf{signal}_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} + \ldots + \beta_n X_{n,t}$. The strategy advocates for holding a position, $\mathsf{pos}_t = \mathsf{price}_t \times \mathsf{signal}_t$, congruent with the signal magnitude, effectively aligning the number of shares held at time t with signal_t. The PnL is computed based on the fluctuations in the asset's price and the positions maintained over time, leading to the formulation: $\mathsf{PnL}_t = \mathsf{pos}_{t-1} \frac{\mathsf{price}_t - \mathsf{price}_{t-1}}{\mathsf{price}_{t-1}}$. Based on the assumptions about the form of the signal and the position-taking strategy, we can derive the form of the daily PnL as $\mathsf{PnL}_t = \beta^{\mathrm{T}} X_{t-1}(\mathsf{price}_t - \mathsf{price}_{t-1})$.

The crux of this research lies in optimizing the coefficients β_i to maximize the Sharpe Ratio of the PnL through convex optimization methods. Through a blend of meticulous mathematical analysis and empirical validation using real-world market data, this paper elucidates a structured approach towards the systematic construction and optimization of trading signals. This contribution is not only a technical advancement in the field of quantitative finance but also a practical guide for the development of efficient trading strategies.

1 Problem Formulation

Consider an asset whose price evolves according to the time series $(\mathsf{price}_t)_t$. An investment strategy for this asset entails the creation of a time series $(\mathsf{pos}_t)_t$, contingent on time, representing a position on this asset, i.e., the number of shares purchased of this asset multiplied by its price. Once a position pos_t is determined, it is straightforward to compute the PnL: indeed, we have a asset value's variation term $(\mathsf{pos}_t - \mathsf{pos}_{t-1})$ which corresponds to the value variation of the positions and a cashflow term: $\mathsf{price}_t \times (\frac{\mathsf{pos}_{t-1}}{\mathsf{price}_{t-1}} - \frac{\mathsf{pos}_t}{\mathsf{price}_t})$ which corresponds to encashing (resp. disbursing) the value difference of the positions sold (resp. bought) during the period.

Hence, the sum of these two terms is: $\boxed{\mathsf{PnL}_t = \mathsf{pos}_{t-1} \dfrac{\mathsf{price}_t - \mathsf{price}_{t-1}}{\mathsf{price}_{t-1}}}$

Subsequently, this time series pos_t is primarily determined by constructing a principal signal in the form of a time series signal_t. Thus, in the general case, $\mathsf{pos}_t = f_t(\mathsf{price}_t, \mathsf{signal}_t)$. It then becomes essential to ascertain both f_t and signal_t to optimize the PnL. This signal can be constructed in numerous ways, either technically or by utilizing fine economic relations between the asset price and certain exogenous variables. In our case, we will focus on the creation of a signal with a position-taking form as follow :

$$\text{Ansatz: } \boxed{\mathsf{pos}_t = \mathsf{price}_t \times \mathsf{signal}_t}$$

which corresponds to the case where one decides to hold exactly signal_t shares of the asset at any time t. Therefore we can deduce the relation between PnL_t, price_t and signal_t :

$$\boxed{\mathsf{PnL}_t = \mathsf{signal}_{t-1} \times (\mathsf{price}_t - \mathsf{price}_{t-1})} \tag{1}$$

Now let's consider a set of *stationnary* exogenous variables denoted as $X_{1,t}, X_{2,t}, \ldots, X_{n,t}$ and create a signal as a linear combination of these variables. Mathematically, the sought-after signal is:

$$\text{Ansatz: } \boxed{\mathsf{signal}_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} + \ldots + \beta_n X_{n,t}} \tag{2}$$

(That we will note with matrix notation: $\mathsf{signal}_t = \beta^{\mathrm{T}} X_t$ with: $X_{0,t} = 1$ for the intercept)

Multiplying equation (2) by $(\mathsf{price}_t - \mathsf{price}_{t-1})$ and then substituting relation (1) into yields:

$$\boxed{\mathsf{PnL}_t = \beta^{\mathrm{T}} \tilde{X}_t} \text{ where } \tilde{X}_{i,t} = (\mathsf{price}_t - \mathsf{price}_{t-1}) X_{i,t-1}$$

In the ensuing section, a methodology for ascertaining the optimal coefficients β is delineated. The subsequent phase involves the identification of the optimal set of coefficients $\beta_0, \beta_1, \ldots, \beta_n$ that maximize a commonly used metrics of the PnL: the Sharpe Ratio.

2 Mathematical Analysis and Optimization

Now that a parametric expression for our PnL has been established, our goal is to define a metric for optimizing our PnL. The Sharpe Ratio, attributed to William F. Sharpe, acts as a gauge for evaluating the risk-adjusted performance of a trading strategy. It is computed as the mean return of the strategy minus the risk-free rate (in practice in Quant Finance, and in this paper, the risk-free rate is set to 0), divided by the standard deviation of the return, thus rendering a normalization for volatility. This ratio is invaluable as it not only elucidates the return potential but also encapsulates the inherent risk, thereby offering a comprehensive measure of a strategy's efficacy and robustness.

The objective function we aim to maximize is thus defined as the Sharpe Ratio of the PnL, computed over the expectancy and standard deviation:

$$\mathcal{L} = \frac{\mathbb{E}[\mathsf{PnL}]}{\sigma[\mathsf{PnL}]}$$

2.1 Optimization methodology

Using the notations $\mu^{\mathrm{T}} = \left[\overline{\tilde{X}_{0,t}}, \quad \overline{\tilde{X}_{1,t}}, \quad \ldots, \quad \overline{\tilde{X}_{n,t}} \right]$ for the empiric mean of $\tilde{X}_t$ and $\Sigma = \left[\mathrm{cov}(\tilde{X}_{i,t}, \tilde{X}_{j,t})_{i,j}\right]$ its empiric covariance matrix, we have for a PnL that would have been found for a certain β:

$$\overline{\mathsf{PnL}} = \beta^{\mathrm{T}}\mu \text{ and } \sigma_{emp.}(\mathsf{PnL}) = \sqrt{\beta^{\mathrm{T}}\Sigma\beta}$$

Thus, we can compute the empiric objective function:

$$\mathcal{L}(\beta) = \frac{\beta^{\mathrm{T}}\mu}{\sqrt{\beta^{\mathrm{T}}\Sigma\beta}}$$

For an invertible Σ, the closed-form solution for the β that optimize $\mathcal{L}$ is[1]:

$$\boxed{\hat{\beta} = \frac{\Sigma^{-1}\mu}{\sqrt{\mu^{\mathrm{T}}\Sigma^{-1}\mu}}} \tag{3}$$

Finally the Optimal Linear Signal extracted from the exogenous variables can be computed by: $\hat{\beta}^{\mathrm{T}} X_t$

How to Interpret These Results?

A 'good signal' is defined as a time series s_t whose 1-day lagged series s_{t-1} exhibits a high correlation with the asset's price variation ($\mathsf{price}_t - \mathsf{price}_{t-1}$). The methodology we present aims to construct a linear signal that maximally captures the most of this correlation using n exogenous variables.

In effect, because the time series X_t are stationary, μ_i are estimator of $\mathbb{E}[(\mathsf{price}_t - \mathsf{price}_{t-1}) \times X_{i,t}]$, and actually represents the empirical covariance between the series $(X_{i,t-1})_t$ and the price changes ($\mathsf{price}_t - \mathsf{price}_{t-1}$).

Furthermore, following the logic of the 'bias/variance trade-off', a corrective term involving Σ^{-1} is introduced to optimally reduce the variance of the PnL generated by this method. The 'bias/variance trade-off' is mecanically set to maximize the sharpe ratio.

[1] We can compute the gradient of $\mathcal{L}$ which is null when $\Sigma\beta \propto \mu$:

$$\nabla\mathcal{L}(\beta) = \frac{\mu(\beta^{\mathrm{T}}\Sigma\beta) - \Sigma\beta}{(\beta^{\mathrm{T}}\Sigma\beta)^{3/2}}$$

The coefficient of proportionnarity is free since $\mathcal{L}(|cst| \times \beta) = \mathcal{L}(\beta)$ we have choosen a positive one that gives: $\beta^{\mathrm{T}}\Sigma\beta = 1$ for a unitary standard deviation of the PnL and a positive expectancy.

Remarks:

- The presence of non-stationarity in the X_t variable introduces bias, predominantly captured by the intercept term β_0. Essentially, the mean contribution of each variable influences the intercept, potentially assigning it an irrelevant value that fails to capture information effectively.
- Heteroscedasticity, characterized by differing variances among the exogenous variables, can introduce bias in the model. This phenomenon occurs because variables with larger amplitude variations are perceived by the model as being more significant compared to others. Such disproportionate emphasis on certain variables due to their variance can skew the model's performance, leading to biased outcomes.

Then, a good practice is to standardize the exogenous variables X_t before using them in the model. It has several advantages, mainly *(i)* force them into stationnarity and *(ii)* force them into homoscedasticity.

2.2 Regularization techniques

L2 regularization: For non-invertible covariance matrix Σ, L2 regularization can be utilized to ensure invertibility by adding a scaled identity matrix, resulting in:

$$\tilde{\Sigma} = \Sigma + \lambda Id$$

However, to preserve parameter comparability and interpretability, an alternative transformation can be employed:

$$\tilde{\Sigma} = \frac{(\Sigma + \lambda \frac{||\Sigma||}{n} Id)}{(1+\lambda)}$$

L1 regularization: Moreover, when confronting potential redundancy among exogenous variables, L1 regularization emerges as a viable solution. It modifies the objective function to:

$$\tilde{\mathcal{L}}(\beta) = \mathcal{L}(\beta) - \gamma|\beta|$$

thereby promoting sparsity in the parameter vector β. No closed-form solution is available in this case, necessitating numerical optimization to find: $\hat{\beta} = \text{argmax}\, \tilde{\mathcal{L}}$. To maintain parameter comparability and interpretability an alternative formulation can be considered:

$$\tilde{\mathcal{L}}(\beta) = \mathcal{L}(\beta) - (\gamma \times \max \mathcal{L}) \times |\beta|$$

This transformation scales the regularization term relative to the maximum of the objective function, ensuring a balanced influence on the optimization process.

PCA regularization: Another approach involves the use of PCA (Principal Component Analysis) before calculating μ and Σ, in order to select only the top k principal components of $\tilde{X}_t$. Caution is required: the PCA should be performed on $\tilde{X}_t$ rather than X_t. This is because the valuable information resides in the covariance of the transformed variables $\tilde{X}_{i,t} = (\mathsf{price}_t - \mathsf{price}_{t-1})X_{i,t-1}$, not in the covariance of the original variables. Therefore, conducting PCA within the OLS model is preferable rather than prior to it.

Denoting μ_i and σ_i^2, the empiric mean and variance of the i^{th} principal component. Since the principals components are uncorrellated, Σ is diagonal and $\hat{\beta}$ is computed with :

$$\hat{\beta}_i = \frac{\mu_i}{\sigma_i^2} / \sqrt{\sum_{j=1}^{k} \frac{\mu_j^2}{\sigma_j^2}}$$

The signal regularized is: $\hat{\beta}^{\mathrm{T}} \Pi X_t$ with Π the $(k \times n+1)$ matrix of the projector onto the top k principal components of $\tilde{X}_t$[2]

[2]It is important to note that Π is the projector onto the top k principal components of $\tilde{X}_t$, and not of X_t

Statistical Significance Regularization: For each beta coefficient, we compute the p-value of β_i and retain only those coefficients $\hat{\beta}_i$ that meet the criterion:

$$\mathbb{P}[\beta_i = 0|\hat{\beta}_i] < p_{\text{threshold}}$$

Under null hypothesis and assuming that $\tilde{X}_t$ adheres to a Gaussian distribution, subsequent calculations[3] reveal that $\sqrt{\tau}\hat{\beta}_i \times \hat{\beta}^{\mathrm{T}}\mu$ follows a Student's t-distribution with $\tau - 1$ degrees of freedom, where τ denotes the length of the time series. This understanding enables the computation of the aforementioned p-value.

Practical use of the ML model

In practical terms, this model is applicable in the development of a trading strategy. This process entails the training of the model over a span of τ days, utilizing datasets $(X_{t-\tau}, ..., X_{t-1})$ and $(\mathsf{price}_{t-\tau}, ..., \mathsf{price}_{t-1})$. From this training dataset, $\hat{\beta}$ is deduced through the prescribed methodology. Then, a signal for the subsequent day, t, is formulated as $\hat{\beta}^{\mathrm{T}} X_t$.

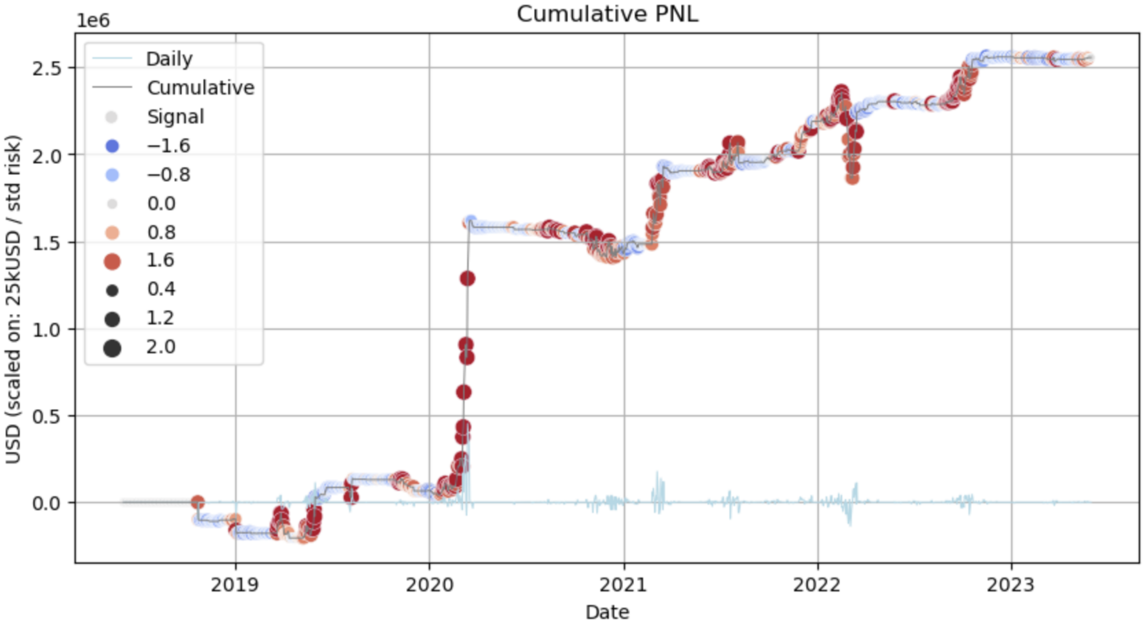
This results in a machine learning model able to at generating trading signals from exogenous variables. The model is parametric, encompassing parameters such as the training period τ, and a regularization parameter for each regularization technique.

3 Empirical Application to a Trading Strategy

The strategies empirically tested in this document is a day-to-day strategies. We consider an asset with an opening price denoted by price_t, reflecting the Open price of the asset in the market. A signal signal_t is generated using the proposed method. At the market opening we buy[4], signal_t shares of the asset, under the condition that the signal is sufficient, establishing a position of: $\mathsf{pos}_t = \mathsf{signal}_t \times \mathsf{price}_t$ if: $\mathsf{signal}_t > s_0$ else: 0. Trading is conditional upon the signal surpassing a specific threshold s_0 which is determined as a fraction of its standard deviation computed over a rolling window.

Traded Products: 'IEF', a widely traded Exchange Traded Fund (ETF) that mirrors the performance of U.S. Treasury bonds of maturities: 1-3 years. **Exogenous variables:** Prices of U.S. treasury bonds of differing maturities and various macros, *that have been feature-engineered.* **Open source data from Yahoo finance.**

The graph below corroborates the signal's efficacy: intervals of augmented signals (illustrated in red) are congruent with noticeable ascents in the PnL.

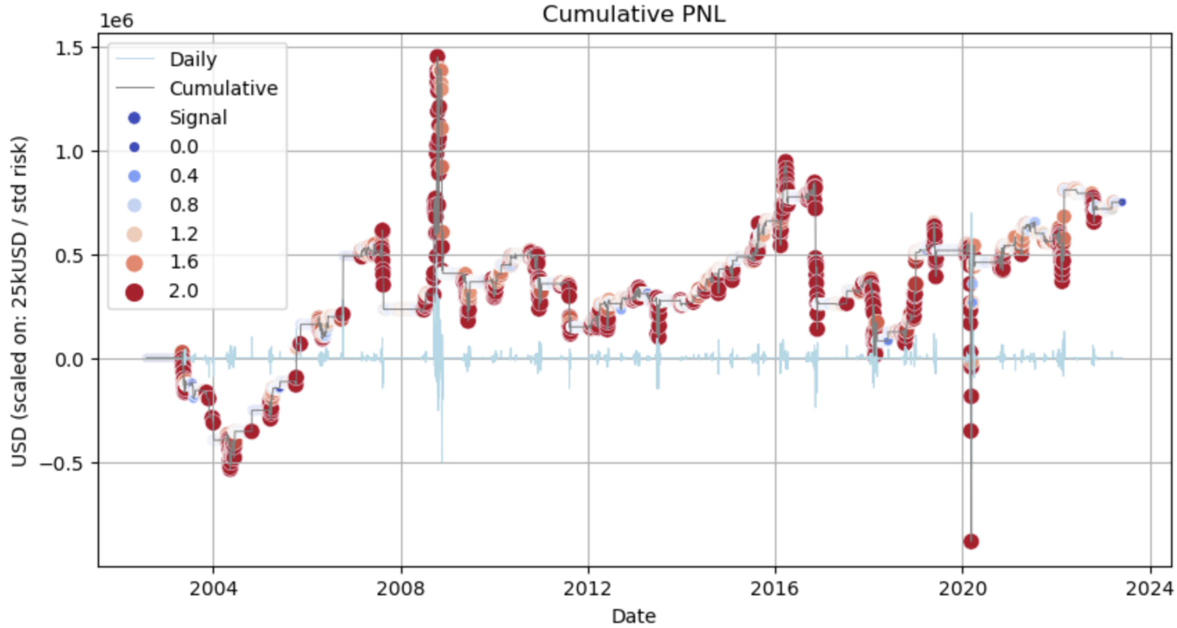


[3] The demonstration is straightforward in the case where Σ is diagonal because we have : $\sqrt{\tau}\hat{\beta}_i \times \hat{\beta}^{\mathrm{T}}\mu = \sqrt{\tau}\frac{\mu_i}{\sigma_i}$

[4] Note: The strategy operates under the assumption that the asset can be acquired at its opening price on day t, without any transaction fees.

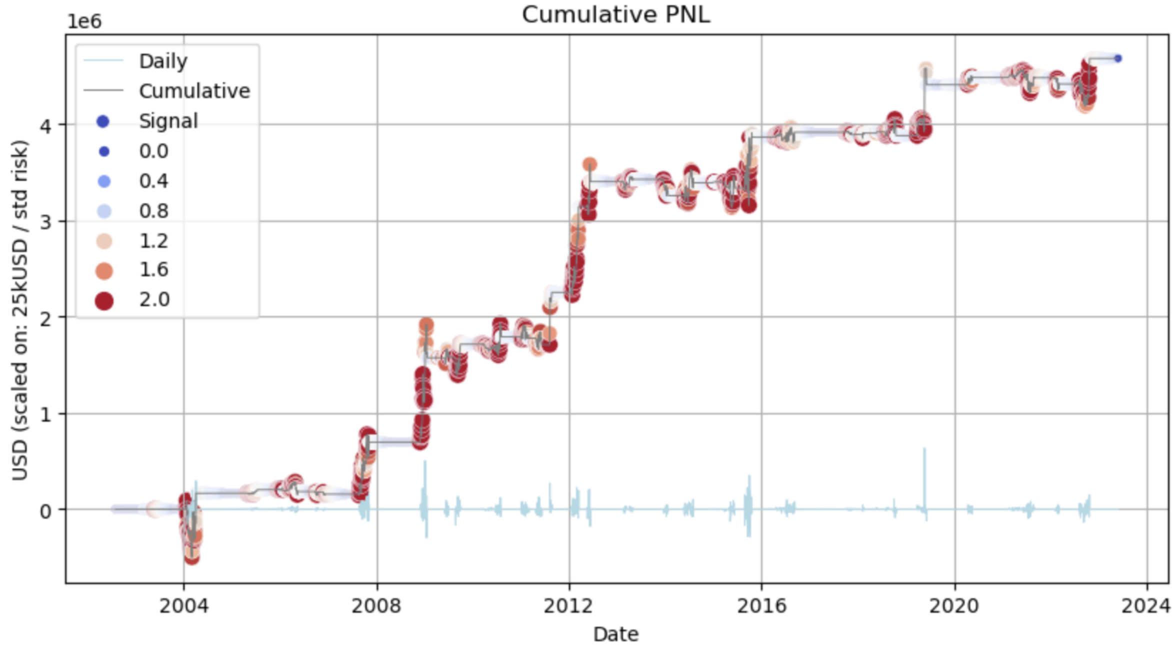
Quantitative Metrics: Sharpe Ratio: 1.25 / effective[5]: 2.1 ;
Turnover: 45.9%; Bips: 18.9

However, it is observed that over a longer period, we are subject to fluctuations: at times, the signal misinterprets the direction. The primary reason is that the signal can detect that the period is conducive to generating PnL, but it remains highly uncertain about which direction to take. Consequently, during each of these periods, the signal can choose the wrong direction, leading to frequent monetary losses in the strategy, which adversely affects the metrics. As a result, we obtain rather mediocre outcomes, as shown in the following figure.



Quantitative Metrics: Sharpe Ratio: -0.09 / effective: -0.46 ;

Nevertheless, as depicted in the graph below, two corrective techniques have been identified that drastically improve the results: [again, we have choosen not to disclose these techniques]

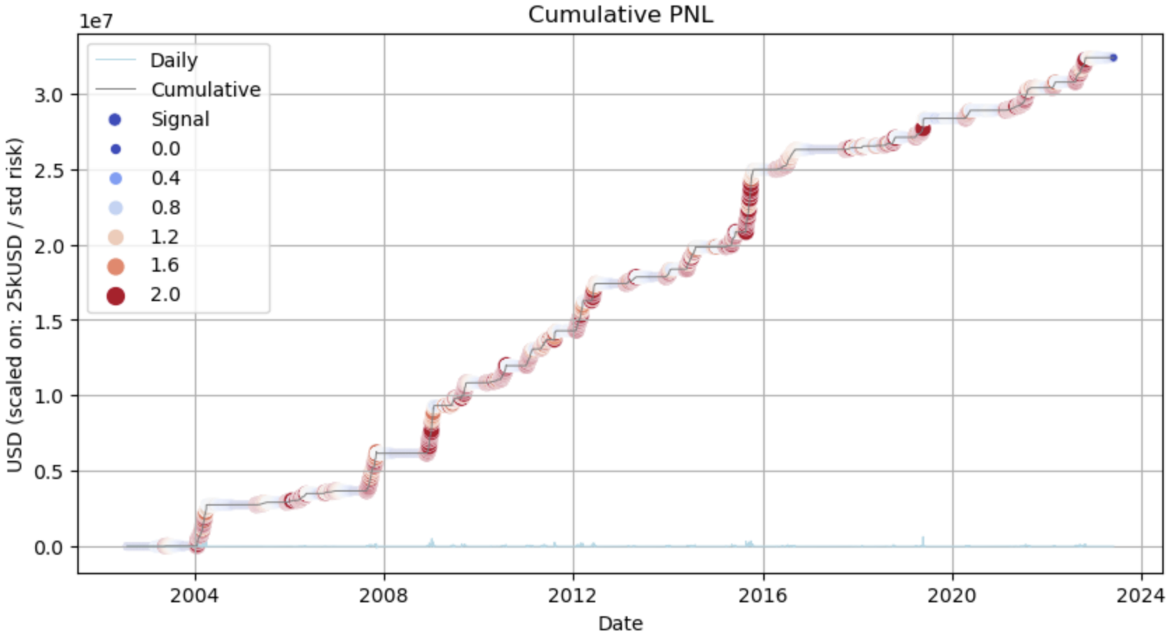


Quantitative Metrics: Sharpe Ratio: 0.55 / effective: 1.19 ;
Turnover: 37.3% ; Bips: 11.9 / effective: 51.1

[5]Effective metrics are computed over the days where and effective trading has been made i.e. at days t where $\mathsf{pos}_t \neq 0$

Overall, testing the model with a range of parameters, we obtain Sharpe ratios ranging from 0.5 to 1.1, and effective Sharpe ratios between 0.9 and 2.1 for this period.

Sanity check: If we could predict the PnL that will be realized today with the optimal linear signal, could we create an even more effective correction? We have tested this look-ahead scenario to ensure the quality of our signal. With the look-ahead correction of $\mathsf{signal}_t = \mathrm{sg}(\mathsf{PnL}_{t+1}) * \hat{\beta}^{\mathrm{T}} X_t$, we produce unrealistic performance, thereby confirming the quality of our signal.



Quantitative Metrics: Sharpe Ratio: 3.78 / effective: 9.47.
Warning: Look-ahead scenario, not a valid strategy.

This empirical analysis reinforces the algorithm's potential for generating a robust trading strategy, while also providing valuable insights into the selection and efficacy of exogenous variables.

Remark on Overfitting

A critical concern in this context is overfitting. With at most an order of magnitude of $1000 - 10\,000$ data points, the available dataset is quite limited, inherently posing a constant risk of overfitting. The key to the success of such a machine learning model lies in its ability to effectively implement regularization. This is the rationale behind proposing a variety of complementary regularization methods in the previous section. Beyond feature engineering, a significant portion of the work has been dedicated to constructing and optimizing relevant regularization techniques to mitigate this issue. Still, this problem should be kept in mind as a important risk using this model for a real trading strategy.

Conclusion

This study delineates a comprehensive framework for systematically constructing and optimizing trading strategies, anchored in a solid mathematical approach. The core innovation resides in the formulation of a signal linearly derived from selected exogenous variables. By integrating a linear combination of these variables into a strategic position-taking model, we have demonstrated a method that effectively constructs a signal responsive to market dynamics and predictive of future trends.

The methodology involved formulating the trading signal as a linear combination of a set of exogenous variables, which was subsequently employed to ascertain the position in the asset. The consequent PnL computation, based on the asset's price variations and the held positions over time, underscores the robustness of the proposed framework. This was further validated through empirical application, using publicly accessible market data, to demonstrate the efficacy of the model in constructing a durable trading strategy. Looking forward, the study opens avenues for further development, which include:

- **Generalized Time Steps:** In the 'How to Interpret These Results' section, we have shown that this methods constructs a time series s_t such that its 1-day lagged series s_{t-1} is strongly correlated with the asset price variations $(\mathsf{price}_t - \mathsf{price}_{t-1})$. This concept can be extended to t_s-days lagged time series s_{t-t_s}, examining their correlation with $(\mathsf{price}_t - \mathsf{price}_{t-t_s})$. The optimal t_s intervals could be determined using methods that study seasonality, such as Fourier analysis.

- **Generalized Linear Signal:** Expanding the signal model to a more generalized form: $\mathsf{signal}_t = f_{\text{act.}}(\beta^{\mathrm{T}} X_t)$, leading to a new expression for PnL and optimizing the objective function $\mathcal{L}(\beta)$ accordingly. Optimize the objective function $\mathcal{L}(\beta)$ based on this new formulation. Begin with a $f_{\text{act.}}$ as a logit activation function inspired by Logistic Regression to create a signal that can take values in [0,1], and can be used as a mitigateur of other signals. Explore generalization with various activation functions used as hyperparameters.

- **Continuous Hyperparameter Selection:** Implementing a dynamic system for adjusting hyperparameters every 'training-size' days, based on the parameters that maximized metrics in the previous period. Note that it will significantly increase the computational cost.

- **Enhance Corrective Terms with Command & Control Theory:** A primary limitation of this model is its relatively low reactivity, as it updates only every 'training-size' days. An approach grounded in command & control theory could significantly enhance the model's responsiveness, enabling real-time adjustments. The 'corrective factor' introduced in the 'Empirical Application to a Trading Strategy' section represents a rudimentary form of this adaptive correction. Investigating the application of command and control theory, especially the use of Kalman filters, to refine the model's corrective terms emerges as a promising avenue for development. Such an advancement would not only add dynamism to the model but also improve its accuracy and adaptability in response to evolving market conditions.

- **Stacking Linear Signal:** Drawing on stacking regressors from machine learning to create a new, stacked signal by utilizing the Optimal Linear Signal with already efficient signals as Exogenous Variables, potentially enhancing the signals' strength and depth.

- **Develop a Beta Neutral Strategy:** One of the limitations of this strategy is that it focuses on a single asset, relying on its price fluctuations to generate PnL. Structurally, it cannot be beta neutral as it entails taking an uncompensated position in the market. However, it is conceivable to modify the strategy to include multiple assets, ensuring that the sum of positions is zero, thereby aiming for beta neutrality. Despite this, the outcomes from such an adaptation have not been satisfactory and are not presented in this document. Further investigation and refinement are required to determine the feasibility of this approach.

In essence, this study not only lays a foundation for traders and financial analysts seeking to augment their strategies through quantitative methods but also sets the stage for continuous innovation and advancement in systematic trading strategies, effectively balancing profit pursuit with prudent risk management.