Furuta Pendulum

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1-Introduction:

The Furuta Pendulum, a rotational inverted pendulum, was invented in 1992 at Tokyo Institute of Technology by Katsuhisa Furuta and his colleagues. Often used to demonstrate nonlinear control laws, the inverted pendulum consists of a driving arm which rotates in the horizontal plane and the pendulum attached to that arm, which is free to rotate in the vertical plane. The objective of this project is to design and implement a state-feedback control system that will balance the pendulum in the upright, vertical position. This requires deriving a mathematical model of the Furuta Pendulum and a controller for the system which can be inputted into MATLAB. Coppliasim will then be used to create a simulated version of the system that can be linked back to MATLAB. Together, MATLAB will act as the controller of the Coppeliasim simulation.

Figure 1: Furuta Pendulum

2-Modeling:

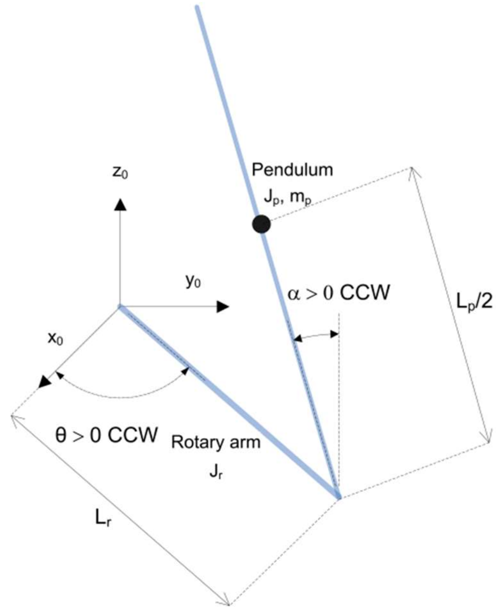

Figure 2: Furuta Pendulum Conventions

The diagram shown in Figure 2 above lists the conventions of the system. Based upon the variables listed, the angle θ lies between the rotary arm and the pendulum and the angle α describes the inverted pendulum angle. The angle α will be equal to zero when perfectly upright and balanced. Both angles are derived taking into account that a counterclockwise rotation will result in a positively increasing angle. A counterclockwise rotation occurs when the control voltage is positive.

Modeling of the system begins by transcribing the physical model into a series of equations that can accurately portray the movement of the inverted pendulum. These equations of motion describe the motion of the rotary arm and the pendulum with respect to the servo motor voltage. The equations of motion can be derived using the Lagrange method. This method is a procedure used for finding the local maxima and minima of a function containing several variables and is often used for complicated systems. The general form for the equation that describes the system's dynamics can be described by the equation below.

$$Q_i = \frac{\partial^2 L}{\partial t \partial q_s} - \frac{\partial L}{\partial q_i} \qquad Eq. (1)$$

Take into account that Qi is used to represent the non-conservative forces applied to the system with respect to the generalized coordinates. The generalized coordinates for the system pertain to the angle θ and the angle α with respect to time. As such, the velocities pertaining to these angles are described as follows.

$$\dot{q}(t)^T = \left[\frac{\partial\theta(t)}{\partial t}\frac{\partial\alpha(t)}{\partial t}\right] \qquad Eq.(2)$$

Using the defined generalized coordinates and velocities it is possible to derive the Euler-Lagrange equations for the rotary pendulum arm.

$$Q_1 = \frac{\partial^2 L}{\partial t\partial\dot{\theta}} - \frac{\partial L}{\partial\alpha} \qquad Eq.(3)$$

$$Q_2 = \frac{\partial^2 L}{\partial t\partial\dot{\alpha}} - \frac{\partial L}{\partial\alpha} \qquad Eq.(4)$$

The Langrangian of the system is modeled as the difference of the total kinetic energy of the system and the total potential energy of the system. Q1 will describe the generalized force acting on the rotary arm while Q2 will be used to describe the generalized forces acting on the pendulum.

$$Q_1 = \tau - B_r\dot{\theta} \qquad Eq.(5)$$

$$Q_2 = -B_p\dot{\alpha} \qquad Eq.(6)$$

Q1 above takes into account the torque and the viscous friction torque of the rotary arm while Q2 takes into account the viscous damping coefficient of the pendulum. Using the equations derived thus far, the nonlinear equations of motion for the rotary inverted pendulum are as follows.

$$\left(m_pL_r^2 + \frac{1}{4}m_pL_p^2 - \frac{1}{4}m_pL_p^2\cos^2(\alpha) + J_r\right)\ddot{\theta} - \left(\frac{1}{2}m_pL_pL_r\cos(\alpha)\right)\ddot{\alpha}$$
$$+\left(\frac{1}{2}m_pL_p^2\sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2}m_pL_pL_r\sin(\alpha)\right)\dot{\alpha}^2 = \tau - B_r\dot{\theta} \qquad Eq.(7)$$

$$-\frac{1}{2}m_pL_pL_r\cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4}m_pL_p^2\right)\ddot{\alpha} - \frac{1}{4}m_pL_p^2\cos(\alpha)\sin(\alpha)\dot{\theta}^2$$

$$-\frac{1}{2}m_pL_pgsin(\alpha) = -B_p\dot{\alpha} \qquad Eq.(8)$$

The control variable for this system is the input servo motor voltage. Opposing the applied torque is the viscous friction torque, otherwise known as the viscous damping force. The viscous damping force is the only force acting upon the link between the pendulum and the rotary arm. The torque acting upon the rotary arm is applied at the base of the rotary arm that is generated by the servo motor. The equation for torque is derived below.

$$\tau = \frac{\eta_g K_g \eta_m (V_m - k_m\dot{\theta})}{R_m} \qquad Eq.(9)$$

The system is now transcribed into a mathematical model that accurately portrays the motion of the system. However, the equations of motion must be transitioned into linearized equations. The linearized versions of the equations of motion are needed to be able to form a matrix with the addition of Eq. (9). Only then can these equations in matrix form be transitioned into a state-space model and inputted into MATLAB. The general linearized function is as follows.

$$f_{lin} = f(z_0) + \left(\frac{\partial f(z)}{\partial z_1}\right)\Bigg|_{z=z_0}(z_1 - a) + \left(\frac{\partial f(z)}{\partial z_2}\right)\Bigg|_{z=z_0}(z_2 - b) \qquad Eq.(10)$$

The linearized functions for the pendulum and the rotary arm are derived based upon Eq. (10) and can be seen in the Appendix. The state-space model can now be derived from the linearized equations of motion described above. Taking into consideration the generalized form for a linear state space equation, the input of the system will be based upon the angles θ and α and their respective derivatives while the output will be based upon the position of the servo and the link angles. The computation of the state space model can be found in the Appendix.

$$\dot{x} = Ax + Bu \qquad Eq.(11)$$

$$y = Cx + Du \qquad Eq.(12)$$

3-Sensor Calibration:

This project does not include the physical implementation of the inverted pendulum and therefore does not need to be calibrated. Any physical system should be tested to ensure that the rotary arm angle and pendulum angle increases positivity when rotated counterclockwise as dictated by an applied positive voltage. These assumptions are based upon the model conventions declared above.

4-Controller Design and Simulations:

4.1-Balance Control:

The conventions of the system listed above are not satisfactory to design the control system for the Furuta Pendulum. The needed information is listed below in Table 1. Table 1 describes the necessary specification pertaining to the control design and time-response requirements for the system.

Table 1: Control Specification

Specification	Value
Damping Ratio	$\delta = 0.7$
Natural Frequency	$w_n = 4\, rad/s$
Max Pendulum Angle Deflection	$\lvert\alpha\rvert < 15\, deg$
Max Control Voltage	$\lvert V_m\rvert < 10\, V$

Complimentary to the specifications of Table 1 above, the pendulum deflection and control effort requirements are to be satisfied when the rotary arm is tracking a 20 degree angle square wave.

Designing the controls begins with determining the stability of the system. If a system is unstable, the transient response and steady state errors become moot points. A system's stability relies upon its poles. The poles of a system are the roots of a system's characteristic equation. The open loop poles for the system can be found with further analysis of the matrix created above in Modeling. The open loop poles can be referenced in the Appendix. Given the values pertaining to the open loop poles, the system can be classified as unstable due to the fact that one of the poles is located within the right-hand plane. An unstable system can be described as a natural response that approaches infinity as the time approaches infinity. A system can be declared as stable if the system's closed loop transfer function contains poles only in the left half-plane. Unlike an unstable system, a stable system can be described as a natural response that approaches zero instead of infinity as the time approaches infinity.

The method of pole placement allows the closed loop poles to be tailored, based upon the chosen specifications, to any individual system so that it can be made stable. This method cannot be implemented to any system that is not deemed controllable. Using a rank test, it is possible to determine if a system is controllable. The system will be deemed controllable if the rank of its controllability matrix equals the number of the states in the system.

$$rank(T) = n \qquad Eq. (13)$$

In determining the controllability matrix, the use of companion matrices can simplify the process.

$$A = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -a_1 & \cdots & -a_n \end{bmatrix} \qquad Eq. (14)$$

$$B = \begin{bmatrix} 0 \\ \ldots \\ 1 \end{bmatrix} \qquad Eq. (15)$$

$$T = [B \quad \ldots \quad BA^n] \qquad Eq. (16)$$

Now that the system has been deemed controllable it is possible to implement the pole placement method. Given the control law stated below, the state-space equations derived above in Modeling can be altered to determine the closed loop poles that can satisfy the specifications mentioned above in Table 1.

$$u = -Kx \qquad Eq. (17)$$

The desired poles of the system can now be calculated through the use of the companion matrices. The resulting closed loop poles are composed of four poles: two poles are complex conjugate dominant poles and the remaining poles are placed along the real-axis to the left of the dominant poles.

$$p = -\sigma \pm j\omega_d \qquad Eq. (18)$$

The final step to the balance control is the feedback control loop. The feedback loop can balance the rotary pendulum as illustrated below.

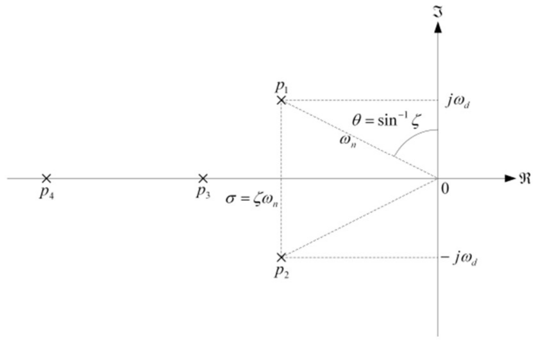

Figure 3: Desired closed-loop pole locations

The feedback loop must be set up while taking into account that the balance control should be enabled only once the pendulum is swung up to an upright vertical position. The controller should therefore be designed to fit Eq. (19) below.

$$u = K(x_d - x) \qquad Eq. (19)$$

The variable xd above relates to the reference state that is dependent upon the desired rotary arm angle. above represents the angle at which the controller should be engaged, and K represents the gain.

4.2-Swing-up control:

The concept of the swing up control is reliant on a nonlinear, energy based control scheme. The swing up control is implemented to get the pendulum from a downward vertical position to the upward position that is needed to initiate the balance control. Beginning with modeling the dynamics, the pivot acceleration of the pendulum can be derived below.

$$J_p\ddot{\alpha} + \frac{1}{2}m_p g L_p \sin(\alpha) = \frac{1}{2}m_p L_p u cos(\alpha) \qquad Eq. (20)$$

The acceleration modeled above is proportional to the torque of the rotary arm, as expressed below.

$$\tau = m_r L_r u \qquad Eq. (21)$$

Besides taking into account the dynamics of the system, the swing up control will need to take into account energy control. The purpose of energy control is to control the pendulum in such a way that it forces the friction to become constant. In order for the swing-up to be successful it is important to be able to create constant amplitude oscillations. If the arm angle is kept constant and the pendulum is given an initial position it will allow the pendulum to swing with a constant amplitude. Friction can cause a damping effect on the oscillations and therefore must be controlled. Derived from potential and kinetic energy concepts, the swing-up controller can be modeled as follows.

$$u = sat_{u_{max}}\left(\mu(E - E_r) sign(\dot{\alpha}\cos(\alpha))\right) \qquad Eq. (22)$$

The variables above describe the following quantities: represents the turnable control gain and satumax represents the function that saturates the control signal at the maximum acceleration of the pendulum pivot. The pivot acceleration can be translated into a servo voltage and further transformed by substituting the torque-acceleration relationship into it.

$$V_m = \frac{R_m m_r L_r u}{\eta_g K_g \eta_m k_t} + K_g k_m \dot{\theta} \qquad Eq. (23)$$

Combining the balance control and the swing-up control it is possible to create a system that can successfully describe both.

$$u = \begin{cases} K(x_d - x) & |x_2| < \epsilon \\ sat_{u_{max}}(\mu(E - E_r) sign(\dot{\alpha}\cos(\alpha)) & \end{cases} \qquad Eq. (24)$$

5-Controller Implementation: (Optional)

6-References:

Benjamin Seth Cazzolato, Zebb Prime, "On the Dynamics of the Furuta Pendulum", *Journal of Control Science and Engineering*, vol. 2011, Article ID 528341, 8 pages, 2011. https://doi.org/10.1155/2011/528341

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