

Financial Economics - Asset Pricing and Portfolio Selection

Lecture 3

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3 Equilibrium, Efficiency, and Representative Investors

The asset pricing formula

$$E[u'(\tilde{w})(\tilde{R}_i - \tilde{R}_j)] = 0 \quad (3.1)$$

described in the previous chapter involves the marginal utility of an individual investor and the individual investor's wealth (consumption). To describe asset prices in a useful way, we need a stochastic discount factor involving aggregate variables. In this chapter, we discuss some conditions under which this is possible.

There are two basic routes to replacing individual variables with aggregate ones: (i) we can define an aggregate ("representative") investor for whom a formula such as (3.1) holds, or (ii) if the formula is linear, we can simply add across investors. This chapter is concerned with the first route. The second will appear in various contexts in later chapters (see, for example, the proof of the CAPM in Section 6.3).

The basic questions we ask in this chapter are (ia) when does there exist a function u such that (3.1) holds for aggregate wealth \tilde{w} , and (ib) when the function u exists, how does it relate to individual utility functions and endowments? This is the problem of "aggregation." When aggregation is possible, we say that there is a "representative investor."

The question of aggregation is intimately linked to the issue of economic efficiency, which is obviously of interest in its own right. The key result in this chapter (Section 3.5) is that there is a representative investor at any allocation that is both a competitive equilibrium and Pareto optimal. It is reasonable to assume that market forces lead to a competitive equilibrium allocation. The remaining question is what conditions on the market are sufficient for a competitive equilibrium to be Pareto optimal. An answer is that it is sufficient for markets to be complete (Section 3.6) or for investors to have linear risk tolerance $\tau(w) = A + Bw$, with the B coefficient being the same for all investors (Section 3.7).

*These notes are based on *Asset Pricing and Portfolio Choice Theory* by Kerry Back. We thank Professor Back for his kindly and enormous support. Copy is not allowed without any permission from the author.

A stronger concept of aggregation is “Gorman aggregation,” which means that the utility function of the representative investor and equilibrium prices are independent of the initial distribution of wealth across investors. Gorman aggregation is possible for all asset payoff distributions if (and only if) investors have linear risk tolerance with the same B coefficient. This is discussed in Section 3.7.

Section 3.8 is a bit of a digression from the main topic of Pareto optimality and representative investors. It shows that two-fund separation holds if the asset payoffs have a factor structure with a single factor, the residuals are mean-independent of the factor, and the market portfolio is well diversified. This is independent of the preferences of investors. This complements the result of Section 2.4, discussed again in Section 3.7, that two-fund separation holds for all asset payoff distributions if investors have linear risk tolerance with the same B coefficient. It is also shown in Section 3.8 that investors hold mean-variance efficient portfolios and the Capital Asset Pricing model holds under the factor model assumption. These topics are treated in more detail in Chapters 5 and 6, respectively.

We will suppose except in Section 3.9 that beginning-of-period consumption is already determined and focus on the investment problem. Section 3.9 shows that the results also hold when investors choose both beginning-of-period consumption and investments optimally. We will also assume that all investors agree on the probabilities of the different possible states of the world.

3.1 Pareto Optima

Suppose there are H investors, indexed as $h = 1, \dots, H$, with utility functions u_h . We suppose there is a given (random) aggregate wealth \tilde{w}_m (“ m ” for “market”), and we ask how to allocate the aggregate wealth to investors so that it is impossible to make any investor better off without making another investor worse off. An allocation with the property that it is impossible to improve the welfare of any investor without harming the welfare of another is called a “Pareto optimal” allocation. By “better off” or “worse off,” we mean in an ex ante sense, that is, before the state of the world that determines the level of aggregate wealth is realized. After the amount of aggregate wealth is known, it is obviously impossible to Pareto improve on any allocation, because any reallocation would reduce the wealth of some investor, making him worse off (given strictly monotone utilities). Thus, Pareto optimality - in our model with a single consumption good - is about efficient risk sharing.

Formally, an allocation $(\tilde{w}_1, \dots, \tilde{w}_H)$ is defined to be Pareto optimal if (i) it is feasible - i.e., $\sum_{h=1}^H \tilde{w}_h(\omega) = \tilde{w}_m(\omega)$ in each state of the world ω , and (ii) there does not exist any other feasible allocation $(\tilde{w}'_1, \dots, \tilde{w}'_H)$ such that

$$E[u_h(\tilde{w}'_h)] \geq E[u_h(\tilde{w}_h)]$$

for all h , with

$$E[u_h(\tilde{w}'_h)] > E[u_h(\tilde{w}_h)]$$

for some h . For the sake of brevity, the term “allocation” will mean “feasible allocation” in the remainder of the chapter.

A simple example of an allocation that does not involve efficient risk sharing (with homogeneous beliefs) and hence is not Pareto optimal is as follows. Suppose there are two risk-averse investors and two possible states of the world, with \tilde{w}_m being the same in both states, say, $\tilde{w}_m = 6$, and with the two states being equally likely. The allocation

$$\tilde{w}_1 = \begin{cases} 2 & \text{in state 1,} \\ 4 & \text{in state 2,} \end{cases}$$

$$\tilde{w}_2 = \begin{cases} 4 & \text{in state 1,} \\ 2 & \text{in state 2,} \end{cases}$$

is not Pareto optimal, because both investors would prefer to receive 3 in each state, which is also a feasible allocation.

In Section 3.3, we will establish a stronger property. Even if the states in this example have different probabilities, so it is not necessarily true that each investor would prefer to receive 3 in each state, the allocation specified above cannot be Pareto optimal. The reason is that the aggregate wealth is constant across the two states (6 in each state), and, as we will show, in any Pareto optimal allocation each investors’s wealth must be constant across states in which market wealth is constant. At a Pareto optimum there must be perfect insurance against everything except fluctuations in aggregate wealth, and insurance is imperfect in the above example.

3.2 Social Planner’s Problem

It is a standard result from microeconomics that a Pareto optimum maximizes a weighted average of utility functions. We can prove this as follows: Consider an allocation $(\tilde{w}'_1, \dots, \tilde{w}'_H)$. Define

$$U_h = E[u_h(\tilde{w}'_h)]$$

for each h . If the allocation is Pareto optimal, then it must solve

$$\text{maximize}_{\tilde{w}'_2, \dots, \tilde{w}'_H} E \left[u_1 \left(\tilde{w}_m - \sum_{h=2}^H \tilde{w}_h \right) \right] \quad (3.2)$$

subject to $E[u_h(\tilde{w}_h)] \geq U_h$ for $h = 2, \dots, H$.

The Lagrangean for this problem is

$$E \left[u_1 \left(\tilde{w}_m - \sum_{h=2}^H \tilde{w}_h \right) \right] + \sum_{h=2}^H \lambda_h E[u_h(\tilde{w}_h)] - \sum_{h=2}^H \lambda_h U_h.$$

Because of concavity, the optimum for (3.2) maximizes the Lagrangean. Taking $\lambda_1 = 1$ we conclude that a Pareto optimal allocation must solve:

$$\text{maximize } \sum_{h=1}^H \lambda_h E[u_h(\tilde{w}_h)] \quad (3.3)$$

$$\text{subject to } \sum_{h=1}^H \tilde{w}_h = \tilde{w}_m.$$

for some $(\lambda_1, \dots, \lambda_H)$. The problem (3.3) is called the “social planner’s problem.” Note that the constraint $\sum_{h=1}^H \tilde{w}_h = \tilde{w}_m$ is really a system of constraints: It should hold in each state of the world.

There are no constraints in the social planner’s problem that operate across different states of the world, so to achieve the maximum value of the objective function we simply need to maximize in each state of the world. In other words, the social planner’s problem is equivalent to

$$\text{maximize } \sum_{h=1}^H \lambda_h u_h(\tilde{w}_h(\omega)) \quad (3.4)$$

$$\text{subject to } \sum_{h=1}^H \tilde{w}_h(\omega) = \tilde{w}_m(\omega).$$

in each state of the world ω . The maximum value in (3.4) can be considered to be the social planner’s utility in state ω . It depends on market wealth $\tilde{w}_m(\omega)$ via the constraint in (3.4).

To define the social planner’s utility function more generally, given individual utility functions u_1, \dots, u_H and weights $\lambda_1, \dots, \lambda_H$, set

$$u_m(w) = \max \left\{ \sum_{h=1}^H \lambda_h u_h(w_h) \mid \sum_{h=1}^H w_h = w \right\} \quad (3.5)$$

for any constant wealth w . This means that the social planner’s utility is the maximum of the weighted sum of utilities that can be achieved by allocating aggregate wealth w across the investors. If the functions u_h are concave, then the social planner’s utility function is also concave (Problem 3.1). The maximum value in (3.3) is $E[u_m(\tilde{w}_m)]$.

The Lagrange multiplier for the constraint $\sum w_h = w$ in (3.5) depends on w , and we

will write it as $\eta(w)$. The Lagrangean for (3.5) is

$$\sum_{h=1}^H \lambda_h u_h(w_h) - \eta(w) \left(\sum_{h=1}^H w_h - w \right),$$

and the first-order condition is

$$(\forall h) \quad \lambda_h u'_h(w_h) = \eta(w). \quad (3.6)$$

The Lagrange multiplier $\eta(w)$ is also the marginal value of wealth for the social planner, so

$$(\forall h) \quad u'_m(w) = \lambda_h u'_h(w_h), \quad (3.7)$$

where (w_1, \dots, w_H) solves the optimization problem in (3.5).

A Pareto optimal allocation solves the social planner's problem (3.5) with $w = \tilde{w}_m(\omega)$ in each state of the world ω . Therefore, at a Pareto optimum, we have

$$(\forall h) \quad \lambda_h u'_h(\tilde{w}_h) = \eta(\tilde{w}_m) = u'_m(\tilde{w}_m). \quad (3.8)$$

3.3 Pareto Optima and Sharing Rules

If an allocation $(\tilde{w}_1, \dots, \tilde{w}_H)$ is Pareto optimal, then each individual must be allocated higher wealth in states in which market wealth is higher. This says nothing about which individuals get higher wealth than others, only that all individuals must share in market prosperity, and all must suffer (relatively speaking) when market wealth is low. As we will see, this is a simple consequence of the first-order condition (3.8) and risk aversion.

For any two investors j and h , the first-order condition (3.8) implies

$$\lambda_j u'_j(\tilde{w}_j(\omega)) = \lambda_h u'_h(\tilde{w}_h(\omega)). \quad (3.9)$$

Considering two different states ω_1 and ω_2 and dividing (3.9) in state ω_1 by (3.9) in state ω_2 , we have

$$\frac{u'_j(\tilde{w}_j(\omega_1))}{u'_j(\tilde{w}_j(\omega_2))} = \frac{u'_h(\tilde{w}_h(\omega_1))}{u'_h(\tilde{w}_h(\omega_2))}. \quad (3.10)$$

This is the familiar result from microeconomics that marginal rates of substitution must be equalized across individuals at a Pareto optimum. Here, wealth (consumption) in different states of the world plays the role of different commodities in the usual consumer choice problem.

Assuming strict risk aversion (strictly diminishing marginal utilities), the equality (3.10)

of marginal rates of substitutions produces the following chain of implications:

$$\begin{aligned}
\tilde{w}_j(\omega_1) > \tilde{w}_j(\omega_2) &\implies \frac{u'_j(\tilde{w}_j(\omega_1))}{u'_j(\tilde{w}_j(\omega_2))} < 1 \\
&\implies \frac{u'_h(\tilde{w}_h(\omega_1))}{u'_h(\tilde{w}_h(\omega_2))} < 1 \\
&\implies \tilde{w}_h(\omega_1) > \tilde{w}_h(\omega_2).
\end{aligned}$$

Because this is true for every pair of investors, it follows that all investors must have higher wealth in states in which market wealth is higher. Thus, each investor's wealth is related in a one-to-one fashion with market wealth; i.e., letting f_h denote the one-to-one relationship for investor h , we have $\tilde{w}_h(\omega) = f_h(\tilde{w}_m(\omega))$ in each state of the world ω . The functions f_h are called "sharing rules."

A particular consequence of these sharing rules is that if market wealth is the same in two different states of the world, then each investor's wealth must be constant across the two states of the world. This shows that the example in Section 3.1 is inconsistent with Pareto optimality when investors are strictly risk averse.

If investors have linear risk tolerance $\tau(w) = A + Bw$ with the same B coefficient, then the sharing rules f_h must be affine (linear plus a constant) functions. Specifically, if an allocation $(\tilde{w}_1, \dots, \tilde{w}_H)$ is Pareto optimal, then there exist constants a_h and $b_h > 0$ for each h such that

$$\tilde{w}_h(\omega) = a_h + b_h \tilde{w}_m(\omega). \quad (3.11)$$

for each ω . Thus, the wealths of different investors move together and do so in a linear way. We will establish (3.11) in Section 3.7.

3.4 Competitive Equilibria

A competitive equilibrium is characterized by two conditions: (i) markets clear, and (ii) each agent optimizes, taking prices as given. We will take production decisions as given and model the economy as an exchange economy. Thus, part (ii) means that each investor chooses an optimal portfolio. Hence, we can conclude from the last sentence of the previous section that at a Pareto optimal competitive equilibrium, the marginal utility of the representative investor evaluated at market wealth is proportional to a stochastic discount factor.

To define a competitive equilibrium formally, let $\bar{\theta}_{hi}$ denote the number of shares of asset i owned by investor h before trade at date 0. The value of the shares, which of course depends on the asset prices, is the investor's wealth at date 0. We assume investor h invests his date 0 wealth in a portfolio $\theta_h = (\theta_{h1}, \dots, \theta_{hn})$ of the n assets. Take the payoffs \tilde{x}_i of the assets as given and set $\bar{\theta}_i = \sum_{h=1}^H \bar{\theta}_{hi}$, which is the total supply of asset i . We can assume that investors have (possibly random) endowments at date 1, which they consume in addition to their portfolio values. Let \tilde{y}_h denote the endowment of investor h at date 1.

A competitive equilibrium is a set of prices (p_1, \dots, p_n) and a set of portfolios $(\theta_1, \dots, \theta_H)$ such that markets clear,¹ i.e.,

$$(\forall i) \quad \sum_{h=1}^H \theta_{hi} = \bar{\theta}_i,$$

and such that each investor's portfolio is optimal, i.e, for each h , θ_h solves

$$\text{maximize } E[u_h(\tilde{w}_h)] \tag{3.12}$$

$$\text{subject to } \sum_{i=1}^n \theta_{hi} p_i = \sum_{i=1}^n \bar{\theta}_{hi} p_i \text{ and } (\forall \omega) \quad \tilde{w}_h(\omega) = \tilde{y}_h(\omega) + \sum_{i=1}^n \theta_{hi} \tilde{x}_i(\omega).$$

3.5 Representative Investors

One says that a competitive equilibrium of an economy admits a representative investor if the equilibrium prices are also equilibrium prices of an economy consisting of a single investor who owns all the assets and endowments of the original economy. The wealth of this representative investor is the market wealth of the original economy, so his first-order condition is that his marginal utility at market wealth is proportional to a stochastic discount factor. As discussed in the introduction to this chapter, this is important because it produces a stochastic discount factor that depends only on aggregate variables (market wealth). The assumption that there is a representative investor is made frequently in finance to simplify valuation.

If a competitive equilibrium is Pareto optimal and investors are risk averse, then the social planner is a representative investor. To see this, it suffices to observe that, by (3.8), the social planner's marginal utility is proportional to each investor's marginal utility and hence proportional to a stochastic discount factor. This is the first-order condition for optimization over portfolios. Because the social planner's utility function is concave when investors' utility functions are concave (Problem 3.1), the first-order condition is sufficient for optimization. Therefore, market wealth - i.e., holding all of the assets - is optimal for an investor that has the social planner's utility function and owns all the assets and endowments in the economy. Thus, if conditions are satisfied to guarantee that a competitive equilibrium is Pareto optimal, then a representative investor exists.

The utility function of the social planner depends on the weights λ_h . These weights depend on the initial endowments. If endowments are redistributed, making some investors poorer and others richer, then the economy will reach a different competitive equilibrium.

¹As is standard in microeconomics, equilibrium prices can be scaled by a positive constant and remain equilibrium prices because the set of budget-feasible choices for each agent (in this model, the set of feasible portfolios θ_h) is unaffected by the scaling - technically, budget equations are "homogeneous of degree zero" in prices. In microeconomics, it is common to resolve this indeterminacy by requiring the price vector to lie in the unit simplex. As mentioned on p. 1, it is customary to resolve it in finance by setting the price of the consumption good equal to 1 (making the consumption good the "numeraire"). Without consumption at date 0, some other choice must be made. This choice will affect all of the returns \tilde{x}_i/p_i .

Assuming the equilibria are Pareto optimal, the new equilibrium maximizes a social planner's utility function where the now poorer individuals are given smaller weights than before and the now richer individuals are given higher weights than before. Equilibrium prices will typically be changed by the reallocation of endowments, and the representative investor's utility function will typically be changed also. However, if investors have linear risk tolerance with the same B coefficient, then neither equilibrium prices nor the representative investor's utility function will be affected by the redistribution. This is the case of Gorman aggregation and is discussed in Section 3.7.

3.6 Complete Markets

A securities market is said to be complete if, for any \tilde{w} , there exists a portfolio $(\theta_1, \dots, \theta_n)$ such that

$$(\forall \omega) \quad \sum_{i=1}^n \theta_i \tilde{x}_i(\omega) = \tilde{w}(\omega). \quad (3.13)$$

Thus, any desired distribution of wealth across states of the world can be achieved by choosing the appropriate portfolio.

It should be apparent that true completeness is a rare thing. For example, if there are infinitely many states of the world, then (3.13) is an infinite number of constraints, which we are supposed to satisfy by choosing a finite-dimensional vector $(\theta_1, \dots, \theta_n)$. This will be impossible. Note that there must be infinitely many states if we want the security payoffs \tilde{x}_i to be normally distributed, or to be log-normally distributed, or to have any other continuous distribution. Thus, single-period markets with finitely many continuously-distributed assets will not be complete. On the other hand, if significant gains are possible by improving risk sharing, then we would expect assets to be created to enable those gains to be realized. The history of financial markets and particularly the explosion of financial products in recent decades is a testament to the power of this process. Also, as will be shown later, dynamic trading can dramatically increase the “span” of securities markets. The real impediments to achieving at least approximately complete markets are moral hazard and adverse selection. For example, there are very limited opportunities for obtaining insurance against employment risk, due to moral hazard.

In any case, completeness is a useful benchmark against which to compare actual security markets. As remarked above, to have complete markets in a one-period model with a finite number of securities, there must be only finitely many possible states of the world. For the remainder of this section, suppose there are k possible states and index the states as ω_j , for $j = 1, \dots, k$. Set $x_{ij} = \tilde{x}_i(\omega_j)$ and $w_j = \tilde{w}(\omega_j)$. Then our definition of market completeness is equivalent to: For each $w \in \mathbb{R}^k$, there exists $\theta \in \mathbb{R}^n$ such that

$$(\forall j = 1, \dots, k) \quad \sum_{i=1}^n \theta_i x_{ij} = w_j. \quad (3.14)$$

This is a system of linear equations in the variables θ_i and it has a solution for every $w \in \mathbb{R}^k$ if and only if the $n \times k$ -matrix (x_{ij}) has rank k . Thus, in particular, market completeness implies $n \geq k$; i.e., there must be at least as many securities as states of the world.

Completeness means the existence of a solution $\theta \in \mathbb{R}^n$ to (3.14); it does not require that the solution be unique. However, if there are multiple solutions (for the same (w_1, \dots, w_k)) having different costs $\sum_{i=1}^n p_i \theta_i$, then an investor cannot have an optimum, because buying the cheaper solution and shorting the more expensive solution is an arbitrage. If the cost is unique, then we say that the “law of one price” holds. The law of one price and its relation to the existence of stochastic discount factors is discussed further in Section 4.3.

If the market is complete and the law of one price holds, then, for each state ω_j , there is a portfolio that generates a wealth of one when state ω_j occurs and zero if any other state occurs. The unique cost of this random wealth is some number q_j , and the unique stochastic discount factor is given by

$$\tilde{m}(\omega_j) = \frac{q_j}{\text{prob}_j},$$

where prob_j denotes the probability of state j . Moreover, the market is equivalent to a market for multiple goods under certainty: View consumption in state j as good j costing q_j per unit. Thus, the Pareto optimality of a competitive equilibrium follows (under the assumption of strictly monotone utilities) from the First Welfare Theorem of microeconomics. We will discuss this finite-state model further in the next chapter.

3.7 Linear Risk Tolerance

In this section, we assume that each investor h has linear risk tolerance as in (1.11) with coefficients A_h and B , the coefficient B being the same for each investor. Except in the first two subsections, which characterize Pareto optima, we assume there are no end-of-period endowments and we assume there is a risk-free asset. For convenience, we let n denote the number of other assets, there being $n + 1$ assets including the risk-free asset.

To simplify, but still address the most important cases, assume that utility functions are strictly monotone; that is, we exclude the shifted power case with $\rho < 0$. We can rewrite the shifted power utility function as

$$\frac{1}{1-\rho}(w-\zeta)^{1-\rho}$$

and view log and power utility as special cases of shifted log and shifted power respectively. Thus, we are assuming one of the following conditions holds:

- (a) Each investor has CARA utility, with possibly different absolute risk aversion coefficients α_h .
- (b) Each investor has shifted CRRA utility with the same coefficient $\rho > 0$ ($\rho = 1$ meaning

shifted log and $\rho \neq 1$ meaning shifted power) and with possibly different (and possibly zero) shifts ζ_h .

Sharing Rules with Linear Risk Tolerance

We will show in this subsection that any Pareto optimal allocation involves an affine sharing rule as in (3.11). Moreover, we will show that $\sum_{h=1}^H a_h = 0$ and $\sum_{h=1}^H b_h = 1$. The converse - that affine sharing rules produce Pareto optimal allocations - is left as an exercise (Problems. 3.8 and 3.9). Given a Pareto optimum $(\tilde{w}_1, \dots, \tilde{w}_H)$, let $\lambda_1, \dots, \lambda_H$ be weights such that the Pareto optimum solves the social planning problem (3.3). We can show:

(a) If each investor h has CARA utility with some absolute risk aversion coefficient α_h , then (3.11) holds with

$$a_h = \tau_h \left[\log(\lambda_h \alpha_h) - \sum_{j=1}^H \frac{\tau_j}{\tau} \log(\lambda_j \alpha_j) \right] \quad \text{and} \quad b_h = \frac{\tau_h}{\tau}, \quad (3.15)$$

where $\tau_h = 1/\alpha_h$ is the coefficient of risk tolerance, and $\tau = \sum_{j=1}^H \tau_j$.

(b) If each investor h has shifted CRRA utility with the same coefficient $\rho > 0$, then, setting $\zeta = \sum_{h=1}^H \zeta_h$, (3.11) holds with

$$a_h = \zeta_h - b_h \zeta \quad \text{and} \quad b_h = \frac{\lambda_h^{1/\rho}}{\sum_{j=1}^H \lambda_j^{1/\rho}}. \quad (3.16)$$

Note that the two cases are somewhat different, because the weights λ_h in the social planning problem affect only the intercepts a_h in the CARA case, whereas for shifted CRRA utility, an investor with a higher weight λ_h has a higher coefficient b_h , i.e., an allocation \tilde{w}_h with a greater sensitivity to market wealth \tilde{w}_m . Note also that (3.16) implies the sharing rule $\tilde{w}_h = a_h + b_h \tilde{w}_m$ in the shifted CRRA case can be written in the perhaps more transparent form:

$$\tilde{w}_h - \zeta_h = b_h (\tilde{w}_m - \zeta). \quad (3.17)$$

We will prove the CARA case. The shifted CRRA case, which is similar, is left as an exercise.

We need to solve the optimization problem in (3.5). As noted in Section 3.2, the social planning problem can be solved state by state. Specializing the first-order condition (3.8) to the case of CARA utility, it becomes

$$(\forall h) \quad \lambda_h \alpha_h e^{-\alpha_h \tilde{w}_h} = \tilde{\eta},$$

where we write $\tilde{\eta}$ for $\eta(\tilde{w}_m)$. We need to find $\tilde{\eta}$, which we can do by (i) solving for \tilde{w}_h :

$$\begin{aligned}\tilde{w}_h &= -\frac{1}{\alpha_h} \log \tilde{\eta} + \frac{\log(\lambda_h \alpha_h)}{\alpha_h} \\ &= -\tau_h \log \tilde{\eta} + \tau_h \log(\lambda_h \alpha_h),\end{aligned}\tag{3.18}$$

(ii) adding over investors to obtain

$$\tilde{w}_m = -\tau \log \tilde{\eta} + \sum_{j=1}^H \tau_j \log(\lambda_j \alpha_j),$$

and then (iii) solving for $\tilde{\eta}$ as

$$-\log \tilde{\eta} = \frac{1}{\tau} \tilde{w}_m - \frac{1}{\tau} \sum_{j=1}^H \tau_j \log(\lambda_j \alpha_j).$$

Substituting this back into (3.18) yields

$$\tilde{w}_h = \frac{\tau_h}{\tau} \tilde{w}_m - \frac{\tau_h}{\tau} \sum_{j=1}^H \tau_j \log(\lambda_j \alpha_j) + \tau_h \log(\lambda_h \alpha_h).$$

This establishes the affine sharing rule (3.15).

Social Planner with Linear Risk Tolerance

In this subsection, we will show that the utility function (3.5) of the social planner has linear risk tolerance with the same B coefficient as the individual investors and with intercept

$$A = \sum_{h=1}^H A_h.$$

This is equivalent to the following statements:

(a) If each investor h has CARA utility with some absolute risk aversion coefficient α_h , then the utility function (3.5) is of the CARA type with risk tolerance coefficient $\tau = \sum_{h=1}^H \tau_h$, where $\tau_h = 1/\alpha_h$.

(b) If each investor h has shifted CRRA utility with the same coefficient $\rho > 0$ ($\rho = 1$ meaning shifted log and $\rho \neq 1$ meaning shifted power) and some shift ζ_h , then the utility function (3.5) is of the shifted CRRA type with coefficient ρ and shift $\zeta = \sum_{h=1}^H \zeta_h$.

In case (a), the social planner's absolute risk aversion coefficient α is the aggregate absolute risk aversion defined in Section 1.3. The proofs of (a) and (b) are quite similar. We again give the proof for CARA utility, leaving the shifted CRRA case as an exercise.

Set $\alpha = 1/\tau$. Given the affine sharing rule (3.15), the social planner's utility is

$$\begin{aligned}
-\sum_{h=1}^H \lambda_h \exp\{-\alpha_h \tilde{w}_h\} &= -\sum_{h=1}^H \lambda_h \exp\{-\alpha_h a_h - \alpha_h b_h \tilde{w}_m\} \\
&= -\sum_{h=1}^H \lambda_h \exp\{-\alpha_h a_h - \alpha \tilde{w}_m\} \\
&= -\exp\{-\alpha \tilde{w}_m\} \sum_{h=1}^H \lambda_h \exp\{-\alpha_h a_h\}. \tag{3.19}
\end{aligned}$$

The sum

$$\sum_{h=1}^H \lambda_h \exp\{-\alpha_h a_h\}$$

is a positive constant (it does not depend on \tilde{w}_m or on the state of the world in any other way), so (3.19) is a monotone affine transform of $-e^{-\alpha \tilde{w}_m}$, as claimed.

Competitive Equilibria with Linear Risk Tolerance

By normalizing the number of shares of the risk-free asset outstanding, we can take its payoff to equal 1. Denote its price by p_0 , so the risk-free return is $R_f = 1/p_0$. Let $\bar{\theta}_{h0}$ denote investor h 's endowment of the risk-free asset, let $\bar{\theta}_0 = \sum_{h=1}^H \bar{\theta}_{h0}$ denote the aggregate endowment of the risk-free asset, and let θ_{h0} denote the optimal holding of the risk-free asset by investor h . We call a price vector (p_0, p_1, \dots, p_n) an "equilibrium price vector" if there exist portfolios $\theta_1, \dots, \theta_H$ such that the prices and portfolios form an equilibrium. We will analyze equilibrium price vectors in which $p_i \neq 0$ for each i .

Walras' Law implies that the market for the risk-free asset clears if the markets for the other n assets clear, so markets clear if and only if

$$\sum_{h=1}^H \phi_{hi} = p_i \bar{\theta}_i \tag{3.20}$$

for $i = 1, \dots, n$, where $\phi_{hi} = p_i \theta_{hi}$ denotes the investment in asset i by investor h .

In Section 2.4, it was shown that

$$\phi_{hi} = \xi_i A_h + \xi_i B R_f w_{h0}, \tag{3.21}$$

where ξ_i is independent of A_h and w_{h0} . In our current model, investors differ only with regard to A_h and w_{h0} , so ξ_i is the same for each investor h . Consequently, the aggregate investment in risky asset i is

$$\sum_{h=1}^H \phi_{hi} = \xi_i A + \xi_i B R_f w_0, \tag{3.22}$$

where $A = \sum_{h=1}^H A_h$ and B define the risk tolerance of the social planner and where $w_0 = \sum_{i=0}^n p_i \bar{\theta}_i$ is aggregate initial wealth.

Combining (3.20) and (3.22), markets clear if and only if

$$\xi_i A + \xi_i B R_f \sum_{j=0}^n p_j \bar{\theta}_j = p_i \bar{\theta}_i \quad (3.23)$$

for each i . The same relations (3.23) describe market clearing in an economy with a single investor, that investor being the social planner. The coefficients ξ_i depend on the returns and therefore on the prices. Hence, (3.23) is not an explicit solution for the equilibrium prices. However, what it does show is that a vector of nonzero prices is an equilibrium price vector if and only if it is an equilibrium price vector in the economy in which the social planner is the single investor. Thus, there is a representative investor at any competitive equilibrium.

The social planner's utility function does not depend on the initial wealth distribution across investors. Neither does the characterization (3.23) of equilibrium prices. Therefore, equilibrium prices do not depend on the wealth distribution. In microeconomics in general, this property is called "Gorman aggregation," and it relies on "Engel curves" (also called "income expansion paths") being linear and parallel. The linearity in the present model is the relationship (3.21) between optimal investments and initial wealth, which also shows that the Engel curves of different investors are parallel if the investors have the same B coefficient.

Implementing Affine Sharing Rules

When markets are incomplete, there are allocations that cannot be achieved via security trading. However, when investors have linear risk tolerance with the same B coefficient, Pareto optimal sharing rules are affine, as shown earlier in this section. This implies, as we will show here, that any Pareto optimal allocation can be implemented in the securities market. This property is some-times described as the market being "effectively complete."

As in the previous subsection, we take asset 0 to be risk free. We need the risk-free asset to generate the intercepts a_h in the sharing rules. We can assume without loss of generality that asset 0 is in zero net supply ($\bar{\theta}_0 = 0$). This is without loss of generality because we can also take one of the other assets to be risk-free, if there is actually a positive supply of the risk-free asset.

To implement the Pareto optimal allocation, include an investment of a_h/R_f in asset 0 in the portfolio of investor h . As noted before, $\sum_{h=1}^H a_h = 0$, so the total investment in asset 0 of all investors will be zero, equaling the supply. For each asset $i \leq n$, set the number of shares held by investor h to be $\theta_{hi} = b_h \bar{\theta}_i$. Because $\sum_{h=1}^H b_h = 1$, the total number of shares held by investors of asset i is $\bar{\theta}_i$. Thus, the proposed portfolios are feasible. Because

we have taken asset 0 to be in zero net supply, market wealth is

$$\tilde{w}_m = \sum_{i=1}^n \bar{\theta}_i \tilde{x}_i,$$

and we have for each investor h that

$$\tilde{w}_h = a_h + \sum_{i=1}^n b_h \bar{\theta}_i \tilde{x}_i = a_h + b_h \tilde{w}_m.$$

Thus, these portfolios implement the affine sharing rules.

Note that the relative investment in any two assets $i, j \leq n$ for any investor h is

$$\frac{b_h \bar{\theta}_i p_i}{b_h \bar{\theta}_j p_j} = \frac{\bar{\theta}_i p_i}{\bar{\theta}_j p_j}.$$

This means that the relative investment is equal to the relative “market capitalizations” of the two assets. Thus, we say that each investor holds the market portfolio of risky assets. It was shown in Section 2.4, using the result on linear Engle curves repeated above as (3.21), that each investor’s optimal portfolio must be the market portfolio. Thus, Pareto optimal portfolios and equilibrium portfolios seem to coincide, both equalling the market portfolio. This is shown more explicitly in the next subsection.

First Welfare Theorem with Linear Risk Tolerance

In this subsection, we will show that any competitive equilibrium in this economy must be Pareto optimal. The key fact is the effective completeness of markets established in the previous subsection. Furthermore, we will show that any allocation that is Pareto dominated is Pareto dominated by a Pareto optimum. Therefore, if we suppose that a competitive equilibrium is not Pareto optimal, it follows that it is Pareto dominated by an allocation that can be implemented in the securities market. But this dominant allocation cannot be budget feasible for each investor; because, if it were, it would have been chosen instead of the supposed competitive equilibrium allocation. Adding budget constraints across investors shows that the Pareto dominant allocation is not feasible, which is a contradiction. The remainder of this section provides the details of the proof.

We argue by contradiction. Consider a competitive equilibrium allocation

$$(\tilde{w}_1, \dots, \tilde{w}_H)$$

and suppose there is a feasible Pareto superior allocation. Without loss of generality, suppose that the first investor’s expected utility can be feasibly increased without reducing the expected utility of the other investors. Let $U_h = E[u_h(\tilde{w}_h)]$ for $h > 1$. We now define a Pareto optimum that increases the expected utility of the first investor without changing

the expected utilities of other investors.

(a) If each investor h has CARA utility with some absolute risk aversion coefficient α_h , define a_h for $h > 1$ by

$$E[u_h(a_h + b_h \tilde{w}_m)] = U_h,$$

where $b_h = \tau_h/\tau$ and $\tau = \sum_{h=1}^H \tau_h$. Set $a_1 = -\sum_{h=2}^H a_h$ and

$$(\forall h) \quad \tilde{w}'_h = a_h + b_h \tilde{w}_m.$$

(b) If each investor h has shifted CRRA utility with the same coefficient $\rho > 0$ ($\rho = 1$ meaning shifted log and $\rho \neq 1$ meaning shifted power) and some shift ζ_h , define b_h for $h > 1$ by

$$E[u_h(\zeta_h + b_h(\tilde{w}_m - \zeta))] = U_h,$$

where $\zeta = \sum_{h=1}^H \zeta_h$. Set $b_1 = 1 - \sum_{h=2}^H b_h$ and

$$(\forall h) \quad \tilde{w}'_h = \zeta_h + b_h(\tilde{w}_m - \zeta).$$

In case (a), each random wealth \tilde{w}'_h is feasible for each investor and $\sum_{h=1}^H \tilde{w}'_h = \tilde{w}_m$. The allocation $(\tilde{w}'_1, \dots, \tilde{w}'_H)$ is Pareto optimal (see Problem 3.8) and can be implemented in the securities market. Because we assumed the first investor's utility could be increased without decreasing the utility of other investors and because $(\tilde{w}'_1, \dots, \tilde{w}'_H)$ is a Pareto optimum that does not change the expected utility of investors 2, ..., H , we must have $E[u_1(\tilde{w}'_1)] > E[u_1(\tilde{w}_1)]$. From here, the proof of the contradiction follows the same lines as the usual proof of the first welfare theorem: because of strictly monotone utilities, the random wealth \tilde{w}'_h must cost at least as much as \tilde{w}_h for each h and cost strictly more for investor $h = 1$. Adding the investor's budget constraints shows that $\tilde{w}_m = \sum_{h=1}^H \tilde{w}'_h$ costs more than $\tilde{w}_m = \sum_{h=1}^H \tilde{w}_h$, which is a contradiction.

In case (b), if $\sum_{h=2}^H b_h < 1$, then \tilde{w}'_1 is feasible for investor 1, and the same reasoning as in the previous paragraph leads to a contradiction - see Problem 3.9 for the fact that the allocation in (b) is Pareto optimal. In the next paragraph, we show that $\sum_{h=2}^H b_h \geq 1$ also produces a contradiction.

Define $\tilde{w}_m^* = \tilde{w}_m - \zeta_1$. Because $\tilde{w}_1 \geq \zeta_1$, we have

$$\tilde{w}_m^* \geq \tilde{w}_m - \tilde{w}_1 = \sum_{h=2}^H \tilde{w}_h.$$

Note that for $h > 1$

$$\tilde{w}'_h = \zeta_h + b_h(\tilde{w}_m - \zeta) = \zeta_h + b_h \left(\tilde{w}_m^* - \sum_{h=2}^H \zeta_h \right).$$

If $\sum_{h=2}^H b_h = 1$, then the allocation $(\tilde{w}'_2, \dots, \tilde{w}'_H)$ is a Pareto optimal allocation of the wealth \tilde{w}_m^* among investors 2, ..., H . However, it leaves only $\tilde{w}_m - \tilde{w}_m^* = \zeta_1$ for investor 1, which is either infeasible or the worst possible level of wealth for investor 1. Hence, it is impossible to give each investor $h > 1$ the expected utility $E[u_h(\tilde{w}'_h)] = E[u_h(\tilde{w}_h)]$ while increasing the expected utility of the first investor above $E[u_1(\tilde{w}_1)]$, contradicting our maintained hypothesis. If $\sum_{h=2}^H b_h > 1$, then the allocation $(\tilde{w}'_2, \dots, \tilde{w}'_H)$ actually dominates (for investors 2, ..., H) the Pareto optimal allocation

$$\tilde{w}_h'' = \zeta_h + \frac{b_h}{\sum_{h=2}^H b_h} \left(\tilde{w}_h^* - \sum_{h=2}^H \zeta_h \right)$$

of the wealth \tilde{w}_m^* among investors 2, ..., H . Hence, the same reasoning leads to a contradiction.

3.8 Separating Distributions

The previous section showed that, with a risk-free asset and no end-of-period endowments, two-fund separation holds if all investors have linear risk tolerance $\tau(w) = A + Bw$ with the same B coefficient. This is independent of the distribution of risky asset payoffs. This section addresses a parallel question: For what distributions of risky asset payoffs, independent of investor preferences, does two-fund separation hold?

We assume again that there is a risk-free asset and no end-of-period endowments. Regarding investors, we assume only that they are risk averse. Assume, for $i = 1, \dots, n$, that the payoff \tilde{x}_i of asset i satisfies

$$\tilde{x}_i = a_i + b_i \tilde{y} + \tilde{\epsilon}_i \quad \text{and} \quad E[\tilde{\epsilon}_i | \tilde{y}] = 0, \quad (3.24)$$

for constants a_i and b_i and some random variable \tilde{y} . We take the risk-free asset to be one of the n assets. For that asset, we have $b_i = 0$ and $\tilde{\epsilon}_i = 0$. Assume

$$\sum_{i=1}^n \bar{\theta}_i b_i \neq 0 \quad \text{and} \quad \sum_{i=1}^n \bar{\theta}_i \tilde{\epsilon}_i = 0. \quad (3.25)$$

The model (3.24) is an example of a “factor model,” in which \tilde{y} is the common factor, and the random variables $\tilde{\epsilon}_i$ are the “residuals.” The assumption of mean-independence of the residuals from the common factor is stronger than the assumption of zero covariance that is usually made. Condition (3.25) means that the market payoff has no residual risk but is not risk-free (is exposed to the common factor), which is expressed as “the market portfolio is well diversified.” From (3.25), the payoff of the market portfolio is

$$\tilde{x}_m = a_m + b_m \tilde{y},$$

where

$$\tilde{x}_m = \sum_{i=1}^n \bar{\theta}_i \tilde{x}_i, \quad a_m = \sum_{i=1}^n \bar{\theta}_i a_i \quad \text{and} \quad b_m = \sum_{i=1}^n \bar{\theta}_i b_i \neq 0.$$

As remarked in footnote 1, there is one degree of indeterminacy in equilibrium prices. We can normalize prices by using the risk-free asset as the numeraire and setting $R_f = 1$. Define $p_i = a_i + b_i \lambda$ for a constant λ . We will show that these are equilibrium prices, for some constant λ . Set $\tilde{z} = \tilde{y} - \lambda$. Then we have

$$\tilde{x}_i = p_i + b_i \tilde{z} + \tilde{\epsilon}_i,$$

implying

$$\tilde{R}_i = R_f + \beta_i \tilde{z} + \tilde{\xi}_i \quad \text{and} \quad E[\tilde{\xi}_i | \tilde{z}] = 0, \quad (3.26)$$

where $\beta_i = b_i/p_i$ and $\tilde{\xi}_i = \tilde{\epsilon}_i/p_i$.

Condition (3.26) is a factor model for returns, with the “intercept” for each return being the risk-free return. Moreover, there exists a portfolio $\pi^* = (\pi_1^*, \dots, \pi_n^*)$ that is well diversified in the sense that

$$\sum_{i=1}^n \pi_i^* b_i \neq 0 \quad \text{and} \quad \sum_{i=1}^n \pi_i^* \tilde{\xi}_i = 0, \quad (3.27)$$

namely, $\pi_i^* = p_i \bar{\theta}_i$. Conditions (3.26) and (3.27) constitute a necessary and sufficient condition for two-fund separation when there is a risk-free asset. The condition seems very special, but actually it holds whenever the risky asset returns have a joint normal distribution. More generally, it holds whenever the risky asset returns have an elliptical distribution.

The following facts about this model will be established below:

1. Two-fund separation holds.
2. There exists a constant λ such that the p_i are equilibrium prices. Moreover, $\lambda \neq E[\tilde{y}]$, implying $E[\tilde{z}] \neq 0$.
3. The optimal portfolio of each investor has the minimum variance among all portfolios with the same mean payoff (it is on the “mean-variance frontier”).
4. The Capital Asset Pricing Model holds:

$$E[\tilde{R}_i] = R_f + \frac{\text{cov}(\tilde{R}_i, \tilde{R}_m)}{\text{var}(\tilde{R}_m)} (E[\tilde{R}_m] - R_f) \quad (3.28)$$

for each i , where \tilde{R}_m denotes the market return.

The last two facts anticipate Chaps. 5 and 6, respectively.

1. Two-fund separation follows from the factor model (3.26) and the existence (3.27) of a well-diversified portfolio. Letting π_i denote the fraction of wealth invested in asset i , the

end-of-period wealth of the investor is

$$w_0 \left(R_f + \sum_{i=1}^n \pi_i (\tilde{R}_i - R_f) \right) = w_0 \left(R_f + \sum_{i=1}^n \pi_i (\beta_i \tilde{z} + \tilde{\xi}_i) \right).$$

Notice that

$$\sum_{i=1}^n \pi_i^* \beta_i = \sum_{i=1}^n \bar{\theta}_i b_i = b_m \neq 0,$$

as asserted in (3.27). Define δ by

$$\delta b_m = \sum_{i=1}^n \pi_i \beta_i.$$

The portfolio $\delta\pi^*$ produces wealth

$$w_0 \left(R_f + \sum_{i=1}^n \delta \pi_i^* (\beta_i \tilde{z} + \tilde{\xi}_i) \right) = w_0 \left(R_f + \sum_{i=1}^n \pi_i \beta_i \tilde{z} \right).$$

Because the investor is averse to noise as described in Sect. 1.8, the portfolio $\delta\pi^*$ is preferred to π . This shows that there can be no better portfolio than $\delta\pi^*$ for some δ .

2. If an investor chooses a portfolio $\delta\pi^*$ for any δ , his end-of-period wealth is

$$w_0(R_f + \delta b_m \tilde{z}) = w_0(R_f + \delta b_m (\tilde{y} - \lambda)).$$

Let $\delta_h(\lambda)$ denote the optimal choice of δ for investor h . Clearly, $\delta_h(\lambda)$ is decreasing as λ increases. The amount invested in asset i by investor h is

$$w_{h0} \delta_h(\lambda) \pi_i^* = w_{h0} \delta_h(\lambda) p_i \bar{\theta}_i.$$

Market clearing requires that the aggregate amount invested equal $p_i \bar{\theta}_i$, which is equivalent to

$$\sum_{h=1}^H w_{h0} \delta_h(\lambda) = 1.$$

Choosing λ to satisfy this equation yields an equilibrium [some details still need to be supplied here]. Notice that if $\lambda = E[\tilde{y}]$, then expected end-of-period wealth is $w_0 E[R_f]$ for every portfolio. By risk aversion, the optimal portfolio for every investor in this circumstance would be the risk-free asset. Because this is inconsistent with market clearing, we conclude that $\lambda \neq E[\tilde{y}]$.

3. The expected return of a portfolio π is

$$R_f + \sum_{i=1}^n \pi_i \beta_i E[\tilde{z}],$$

and $E[\tilde{z}] \neq 0$. Consider a portfolio $\delta\pi^*$. Any other portfolio π with the same expected

return must satisfy

$$\sum_{i=1}^n \pi_i \beta_i = \sum_{i=1}^n \delta \pi_i^* \beta_i = \delta b_m.$$

Therefore, the variance of the portfolio return is

$$\delta^2 b_m^2 \text{var}(\tilde{z}) + \text{var} \left(\sum_{i=1}^n \pi_i \tilde{\xi}_i \right) \geq \delta^2 b_m^2 \text{var}(\tilde{z}),$$

and the right-hand side is the variance of the return of $\delta \pi^*$. Thus, each portfolio $\delta \pi^*$ has minimum variance among all portfolios with the same expected return.

4. We have

$$\begin{aligned} \tilde{x}_m &= a_m + b_m \lambda + b_m \tilde{z} \\ &= \sum_{i=1}^n \bar{\theta}_i p_i + b_m \tilde{z} \\ &= p_m + b_m \tilde{z}, \end{aligned}$$

where p_m denotes the cost of the market portfolio $\bar{\theta}$. Also, $R_f = 1$. Therefore,

$$\tilde{R}_m = R_f + \beta_m \tilde{z},$$

where $\beta_m = b_m/p_m$. This implies $\text{cov}(\tilde{R}_i, \tilde{R}_m) = \beta_i \beta_m \text{var}(\tilde{z})$, $\text{var}(\tilde{R}_m) = \beta_m^2 \text{var}(\tilde{z})$, and $E[\tilde{R}_m] - R_f = \beta_m E[\tilde{z}]$, yielding

$$R_f + \frac{\text{cov}(\tilde{R}_i, \tilde{R}_m)}{\text{var}(\tilde{R}_m)} (E[\tilde{R}_m] - R_f) = R_f + \beta_i E[\tilde{z}] = E[\tilde{R}_i].$$

3.9 Beginning-of-Period Consumption

Including beginning-of-period consumption does not change any of the results of this chapter. As mentioned before, in this model with no production and only one consumption good, Pareto optimality is about efficient risk sharing. Whether Pareto optimality can be achieved in competitive markets depends on the nature of asset markets and on investors' risk tolerances regarding date 1 consumption. Of course, the allocation of consumption at date 0 is also relevant for Pareto optimality, but any allocation of date 0 consumption can be achieved by trading the consumption good against assets at date 0, so competitive equilibria are Pareto optimal - and there exists a representative investor - under the same circumstances described earlier. This section provides a few details to support this claim.

Suppose investors have endowments y_{h0} at date 0 and choose consumption at date 0 as well as asset investments. Let y_{h1} denote the date 1 endowment of investor h . Assume time-additive utility as in Section 2.8. Then investor h chooses a portfolio θ_h and date 0

consumption c_{h0} to maximize

$$u_{h0}(c_{h0}) + E[u_{h1}(\tilde{c}_{h1})]$$

subject to

$$c_{h0} + \sum_{i=1}^n \theta_{hi} p_i = y_{h0} + \sum_{i=1}^n \bar{\theta}_{hi} p_i \quad \text{and} \quad (\forall \omega) \quad \tilde{c}_{h1}(\omega) = \tilde{y}_{h1}(\omega) + \sum_{i=1}^n \theta_{hi} \tilde{x}_i(\omega).$$

Note that we are taking the price of the consumption good to be one at both dates; i.e., we are using the consumption good as the numeraire. This resolves the indeterminacy of date 0 prices mentioned in footnote 1 in Section 3.4.

Denote aggregate consumption at dates 0 and 1 by

$$\bar{c}_0 = \sum_{h=1}^H y_{h0} \quad \text{and} \quad \bar{c}_1 = \sum_{h=1}^H \tilde{y}_{h1} + \sum_{i=1}^n \bar{\theta}_i \tilde{x}_i.$$

A Pareto optimal allocation $(c_{10}, \dots, c_{H0}, \tilde{c}_{11}, \dots, \tilde{c}_{H1})$ solves

$$\max \sum_{h=1}^H \lambda_h \{u_{h0}(c_{h0}) + E[u_{h1}(\tilde{c}_{h1})]\}$$

subject to

$$\sum_{h=1}^H c_{h0} = \bar{c}_0 \quad \text{and} \quad (\forall \omega) \quad \sum_{h=1}^H \tilde{c}_{h1}(\omega) = \bar{c}_1(\omega),$$

for some vector of weights $\lambda = (\lambda_1, \dots, \lambda_H)$. This social planning problem is separable across dates and states of the world. For any constant c , define

$$u_0(c) = \max \sum_{h=1}^H \lambda_h u_{h0}(c_h) \quad \text{subject to} \quad \sum_{h=1}^H c_h = c$$

and

$$u_1(c) = \max \sum_{h=1}^H \lambda_h u_{h1}(c_h) \quad \text{subject to} \quad \sum_{h=1}^H c_h = c.$$

Then the maximum value of the social planning problem is

$$u_0(\bar{c}_0) + E[u_1(\bar{c}_1)].$$

If a competitive equilibrium is Pareto optimal, then there is a representative investor with utility functions $u_0(c)$ and $u_1(c)$. Thus, in complete markets there is a representative investor.

When there is a representative investor, the marginal rate of substitution

$$\frac{\partial u_1(\bar{c}_1)/\partial c}{\partial u_0(\bar{c}_0)/\partial c}$$

is a stochastic discount factor. This is especially tractable when the utility functions u_{h0} and u_{h1} of each investor are the same except for a discount factor δ that is common across investors - i.e., $u_{h0} = u_h$ and $u_{h1} = \delta u_h$ - and when the functions u_h have linear risk tolerance with the same B coefficient for each h . Adopting this assumption and assuming there is a risk-free asset and no end-of-period endowments, there is a representative investor with utility functions as follows:

(a) If u_h is a CARA utility function with absolute risk aversion coefficient α_h , then

$$u_0(c) = -e^{-\alpha c} \quad \text{and} \quad u_1(c) = -\delta e^{-\alpha c},$$

where α is the aggregate absolute risk aversion defined in Section 1.3.

(b) If u_h is a shifted log utility function with shift ζ_h , then

$$u_0(c) = \log(c - \zeta) \quad \text{and} \quad u_1(c) = \delta \log(c - \zeta),$$

where $\zeta = \sum_{h=1}^H \zeta_h$.

(c) If u_h is a shifted power utility function with shift ζ_h and relative risk aversion coefficient $\rho > 0$ that is common across investors, then

$$u_0(c) = \frac{1}{1-\rho}(c - \zeta)^{1-\rho} \quad \text{and} \quad u_1(c) = \delta \frac{1}{1-\rho}(c - \zeta)^{1-\rho},$$

where $\zeta = \sum_{h=1}^H \zeta_h$.