Financial Economics - Asset Pricing and Portfolio Selection
Lecture 5

Hongtao Zhou *

5 Mean-Variance Analysis

In this chapter, we describe the portfolios of that are on the mean-variance frontier, meaning that their returns have minimum variance among all portfolios with the same expected return. The study of these portfolios can be motivated by the assumption that investors have mean-variance preferences, but understanding the mean-variance frontier has importance beyond this special case. This will become clear in the next chapter, where the relation between beta-pricing models and the mean-variance frontier is discussed.

We will use the following notation: There are n risky assets, $\tilde{R}_i$ is the return of asset i, $\tilde{R}^{\text{vec}}$ is the n-dimensional column vector with $\tilde{R}_i$ as its i-th element, μ is the column vector of expected returns (the n-vector with i-th element $E[\tilde{R}_i]$), Σ is the covariance matrix (the $n \times n$ matrix with $\text{cov}(\tilde{R}_i, \tilde{R}_j)$ as its (i, j)-th element), and $\mathbf{1}$ is an n-dimensional column vector of ones. We assume the covariance matrix is nonsingular, which is equivalent to no portfolio of the risky assets being risk free. In some parts of the chapter, we will assume there is a risk-free asset with return $R_f > 0$.

We define portfolios in terms of the fraction of wealth invested in each risky asset. Denote a portfolio as a column vector π. If the portfolio is fully invested in the risky assets, then the components π_i must sum to one, which can be represented as $\mathbf{1}'\pi = 1$, where $'$ denotes the transpose operator. The return of a portfolio fully invested in the risky assets is a weighted average of the risky asset returns, namely $\pi'\tilde{R}^{\text{vec}}$. It has mean $\pi'\mu$ and, as explained in Section 2.5, its variance is $\pi'\Sigma\pi$.

If all of the risky assets have the same expected return, then all portfolios of risky assets have the same expected return, and the only way to trade off mean and variance is by varying the proportion invested in the risk-free asset (if one is assumed to exist). This case is not very interesting, so we assume that at least two of the risky assets have different expected returns. This means that μ is not proportional to $\mathbf{1}$.

*These notes are based on *Asset Pricing and Portfolio Choice Theory* by Kerry Back. We thank Professor Back for his kindly and enormous support.

5.1 The Calculus Approach

The traditional approach to computing the mean-variance frontier is to solve the problems:

$$\min \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \mu'\pi = \mu_p \quad \text{and} \quad \mathbf{1}'\pi = 1.$$

Here μ_p is the mean return we want the portfolio to have. A portfolio solving this problem for some μ_p is said to be on the mean-variance frontier. The mean-variance frontier is traced out by varying μ_p. The factor 1/2 is included here only for convenience - obviously, minimizing one-half the variance is equivalent to minimizing the variance.

The Lagrangean for the problem is

$$\frac{1}{2}\pi'\Sigma\pi - \delta(\mu'\pi - \mu_p) - \gamma(\mathbf{1}'\pi - 1),$$

where δ and γ are the Lagrange multipliers, and the first-order condition is

$$\Sigma\pi = \delta\mu + \gamma\mathbf{1}.$$

Together with the constraints, the first-order condition is necessary and sufficient for a solution. Solving the first-order condition gives us

$$\pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\mathbf{1}, \tag{5.1}$$

so the constraints imply

$$\begin{aligned} \mu_p &= \delta\mu'\Sigma^{-1}\mu + \gamma\mu'\Sigma^{-1}\mathbf{1}, && (5.2a) \\ 1 &= \delta\mathbf{1}'\Sigma^{-1}\mu + \gamma\mathbf{1}'\Sigma^{-1}\mathbf{1}. && (5.2b) \end{aligned}$$

The system (5.2) is linear in the Lagrange multipliers δ and γ and can be solved for δ and γ to produce the solution π to the minimum-variance problem. However, a less direct route may be more instructive.

Define

$$\begin{aligned} \pi_\mu &= \frac{1}{\mathbf{1}'\Sigma^{-1}\mu}\Sigma^{-1}\mu, && (5.3a) \\ \pi_{\mathbf{1}} &= \frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}. && (5.3b) \end{aligned}$$

Because $\mathbf{1}'\pi_\mu = \mathbf{1}'\pi_{\mathbf{1}} = 1$, both π_μ and $\pi_{\mathbf{1}}$ are portfolios of risky assets. The solution (5.1) to the minimum-variance problem can be written as

$$\pi = (\delta\mathbf{1}'\Sigma^{-1}\mu)\pi_\mu + (\gamma\mathbf{1}'\Sigma^{-1}\mathbf{1})\pi_{\mathbf{1}}.$$

The constraint (5.2b) states that the coefficients multiplying π_μ and $\pi_{\mathbf{1}}$ in this equation add to one, so, defining

$$\lambda = \delta \mathbf{1}'\Sigma^{-1}\mu,$$

we have

$$\pi = \lambda\pi_\mu + (1-\lambda)\pi_{\mathbf{1}}. \tag{5.4}$$

Now we can compute λ in terms of μ_p from

$$\mu_p = \mu'\pi = \lambda\mu'\pi_\mu + (1-\lambda)\mu'\pi_{\mathbf{1}} \;\Rightarrow\; \lambda = \frac{\mu_p - \mu'\pi_{\mathbf{1}}}{\mu'(\pi_\mu - \pi_{\mathbf{1}})}. \tag{5.5}$$

5.2 Two-Fund Spanning

If investors have mean-variance preferences (liking mean and disliking variance) then they will choose portfolios on the mean-variance frontier. Equation (5.4) shows that all such investors will be content to allocate their wealth among two mutual funds, namely π_μ and $\pi_{\mathbf{1}}$. Both π_μ and $\pi_{\mathbf{1}}$ are on the frontier, and any other frontier portfolio is spanned by - is a weighted average of - these two portfolios.

However, these two funds are far from unique. In fact, any two portfolios on the mean-variance frontier span the entire frontier. We can see this by considering two arbitrary frontier portfolios

$$\begin{aligned}
\pi_a &= \lambda_a\pi_\mu + (1-\lambda_a)\pi_{\mathbf{1}},\\
\pi_b &= \lambda_b\pi_\mu + (1-\lambda_b)\pi_{\mathbf{1}},
\end{aligned}$$

or $\lambda_a \neq \lambda_b$ and considering any other frontier portfolio

$$\pi = \lambda\pi_\mu + (1-\lambda)\pi_{\mathbf{1}}.$$

We have

$$\pi = \phi_a\pi_a + \phi_b\pi_b$$

where ϕ_a and ϕ_b solve

$$\begin{aligned}
\phi_a\lambda_a + \phi_b\lambda_b &= \lambda,\\
\phi_a(1-\lambda_a) + \phi_b(1-\lambda_b) &= 1-\lambda;
\end{aligned}$$

i.e., $\phi_a = (\lambda - \lambda_b)/(\lambda_a - \lambda_b)$ and $\phi_b = 1 - \phi_a$. Thus, π_a and π_b also span the mean-variance frontier.

5.3 The Mean-Standard Deviation Trade-Off

The variance of any frontier portfolio is

$$[\lambda\pi_\mu + (1-\lambda)\pi_{\mathbf{1}}]'\Sigma[\lambda\pi_\mu + (1-\lambda)\pi_{\mathbf{1}}].$$

Calculation of the variance is facilitated by defining the constants

$$A = \mu'\Sigma^{-1}\mu, \quad B = \mu'\Sigma^{-1}\mathbf{1}, \quad C = \mathbf{1}'\Sigma^{-1}\mathbf{1}, \tag{5.6}$$

so (5.5) is equivalent to

$$\lambda = \frac{BC\mu_p - B^2}{AC - B^2}. \tag{5.7}$$

Making this substitution, some tedious algebra shows that the portfolio variance is

$$\sigma_p^2 = \frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}, \tag{5.8}$$

Thus, the variance is a quadratic function of the mean μ_p.

The mean-variance frontier is usually depicted graphically as the locus of (standard deviation, mean) pairs. Thus, it consists of the pairs

$$\left(\sqrt{\frac{A - 2B\mu_p + C\mu_p^2}{AC - B^2}}, \mu_p\right)$$

for real μ_p. This locus is a hyperbola, as shown in Figure 5.1.

5.4 Global Minimum Variance Portfolio and Mean-Variance Efficiency

The "global minimum variance portfolio" is the portfolio of risky assets that has minimum variance among all portfolios of risky assets. Minimizing the variance (5.8) in μ_p produces $\mu_p = B/C$ and substituting this in (5.7) produces $\lambda = 0$. Thus, $\pi_{\mathbf{1}}$ is the global minimum variance portfolio (this is also easy to show directly - see Problem 5.1).

If a frontier portfolio has an expected return less than the expected return B/C of the global minimum variance portfolio, then we say it is on the inefficient part of the mean-variance frontier. It is inefficient in the sense that moving to the global minimum variance portfolio will both increase expected return and reduce variance. On the other hand, there is a real trade-off between mean and variance for expected returns above B/C. Frontier portfolios with expected return above B/C are said to be mean-variance efficient or to be on the efficient part of the mean-variance frontier. They are efficient in the sense that it is impossible to increase expected return without simultaneously increasing variance.

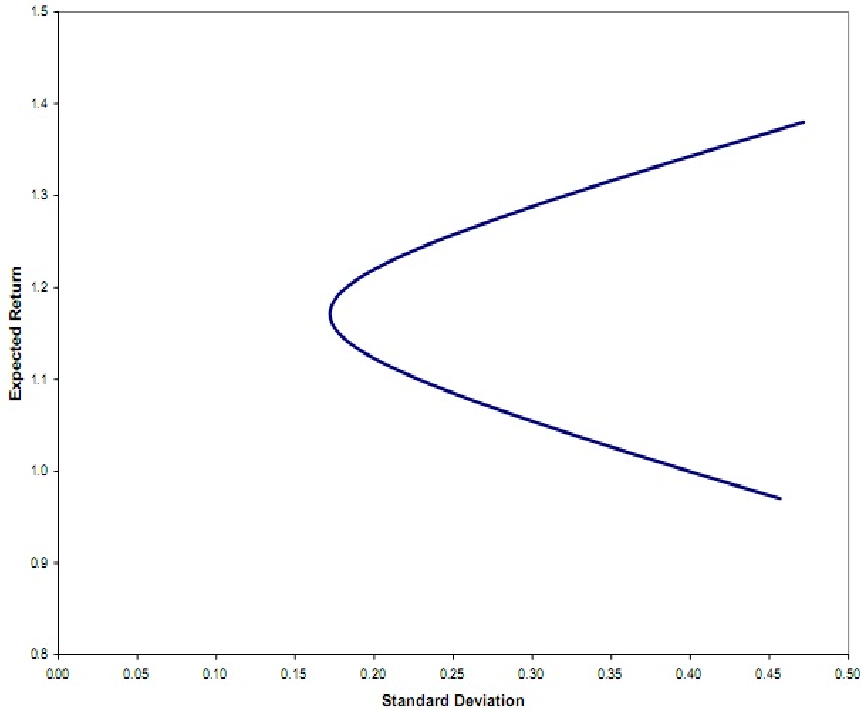

Figure 5.1. This depicts a locus of (standard deviation, mean) pairs corresponding to frontier portfolios of risky assets.

In terms of the formula (5.4), the mean-variance efficient portfolios are those with $\lambda > 0$ if the expected return of π_μ is larger than that of $\pi_{\mathbf{1}}$ and are those with $\lambda < 0$ in the opposite case. To say this another way, if π_μ is mean-variance efficient (has an expected return above that of the global minimum variance portfolio) then the portfolios (5.4) with $\lambda > 0$ are mean-variance efficient; otherwise, it is the portfolios (5.4) with $\lambda < 0$ that are mean-variance efficient.

5.5 Calculus Approach with a Risk-Free Asset

Suppose now that there is a risk-free asset with return R_f . Continue to let π denote a portfolio of the risky assets, μ the vector of expected returns of the risky assets, and Σ the covariance matrix of the risky assets. The fraction of wealth invested in the risk-free asset is $1 - \mathbf{1}'\pi$. The variance of a portfolio return is still $\pi'\Sigma\pi$, but the expected return is

$$(1 - \mathbf{1}'\pi)R_f + \mu'\pi = R_f + (\mu - R_f\mathbf{1})'\pi.$$

Thus, the risk premium is $(\mu - R_f\mathbf{1})'\pi$.

The minimum variance problem is

$$\min \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad (\mu - R_f\mathbf{1})'\pi = \mu_p - R_f,$$

with first-order condition $\Sigma\pi = \delta(\mu - R_f\mathbf{1})$ for some Lagrange multiplier δ. This implies

$$\pi = \delta\Sigma^{-1}(\mu - R_f\mathbf{1}). \tag{5.9}$$

Using the constraint $(\mu - R_f\mathbf{1})'\pi = \mu_p - R_f$, we compute

$$\delta = \frac{\mu_p - R_f}{(\mu - R_f\mathbf{1})'\Sigma^{-1}(\mu - R_f\mathbf{1})},$$

which implies the frontier portfolios are

$$\pi = \frac{\mu_p - R_f}{(\mu - R_f\mathbf{1})'\Sigma^{-1}(\mu - R_f\mathbf{1})}\Sigma^{-1}(\mu - R_f\mathbf{1}). \tag{5.10}$$

The standard deviation of a frontier portfolio is

$$\sqrt{\pi'\Sigma\pi} = \frac{|\mu_p - R_f|}{\sqrt{(\mu - R_f\mathbf{1})'\Sigma^{-1}(\mu - R_f\mathbf{1})}}.$$

The Sharpe ratio (ratio of risk premium to standard deviation) of a frontier portfolio is therefore

$$\sqrt{(\mu - R_f\mathbf{1})'\Sigma^{-1}(\mu - R_f\mathbf{1})} \tag{5.11}$$

when the risk premium is positive and minus (5.11) when the risk premium is negative. The maximum possible Sharpe ratio is (5.11).

The mean-variance efficient portfolios are the frontier portfolios with non-negative risk premia. The frontier portfolios with $\mu_p < R_f$ are on the inefficient part of the mean-variance frontier, because those portfolios have higher risk and lower expected return than the risk-free asset. In (standard deviation, mean) space, the frontier consists of two rays (forming a cone) emanating from $(0, R_f)$. The upper part of the cone is the efficient part of the frontier and has slope equal to (5.11). The lower part of the cone is the inefficient part; it has slope equal to minus (5.11). This is illustrated in Figures 5.2 and 5.3.

Assume $\mathbf{1}'\Sigma^{-1}(\mu - R_f\mathbf{1}) \neq 0$ and define

$$\pi_* = \frac{1}{\mathbf{1}'\Sigma^{-1}(\mu - R_f\mathbf{1})}\Sigma^{-1}(\mu - R_f\mathbf{1}). \tag{5.12}$$

Then $\mathbf{1}'\pi_* = 1$, so π_* is a portfolio fully invested in the risky assets. The portfolio π_* is of the form (5.10) with

$$\frac{\mu_p - R_f}{(\mu - R_f \mathbf{1})'\Sigma^{-1}(\mu - R_f \mathbf{1})} = \frac{1}{\mathbf{1}'\Sigma^{-1}(\mu - R_f \mathbf{1})}. \tag{5.13}$$

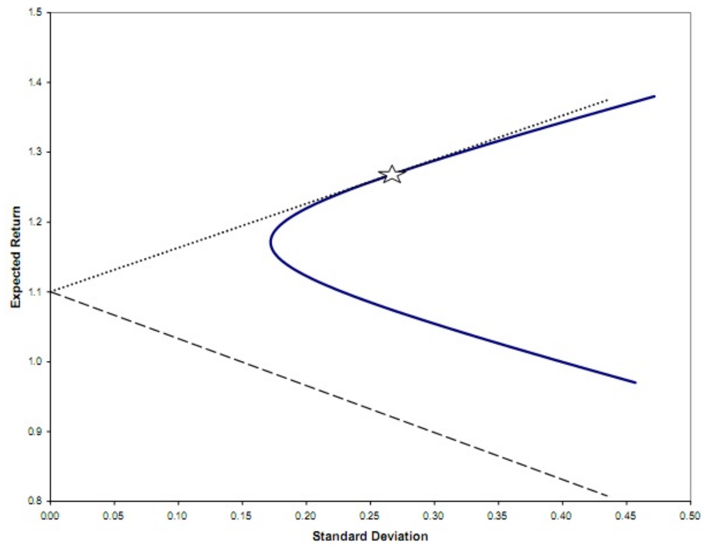

Figure 5.2. This illustrates the case $R_f < B/C$. The hyperbola is the locus of (standard deviation, mean) pairs corresponding to frontier portfolios of risky assets. The dotted line is the efficient part of the mean-variance frontier achievable by including the risk-free asset. The dashed line is the inefficient part of the frontier. The star is at the standard deviation and mean of the tangency portfolio π_*.

Therefore, it is on the mean-variance frontier defined by the risky and risk-free assets. Furthermore, because it consists entirely of risky assets and solves the minimum variance problem with all assets, it must solve the minimum-variance problem for only the risky assets; i.e., it is on the mean-variance frontier of the risky assets. The portfolio π_* is called the "tangency portfolio."

From (5.13), we can see that the tangency portfolio π_* is on the efficient part of the frontier (has a nonnegative risk premium) if and only if

$$\frac{1}{\mathbf{1}'\Sigma^{-1}(\mu - R_f \mathbf{1})} > 0.$$

This is equivalent to

$$\frac{\mathbf{1}'\Sigma^{-1}\mu}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} > R_f,$$

Recall that the ratio on the left-hand side is what was called B/C earlier, and it is the expected return of the global minimum variance portfolio. Thus, if the expected return of the global minimum variance portfolio is at least as large as the risk-free return, then the tangency portfolio is mean-variance efficient among all portfolios of the risky and risk-free assets (it plots on the upper part of the cone). This is illustrated in Figure 5.2. The opposite case $R_f > B/C$ is shown in Figure 5.3.

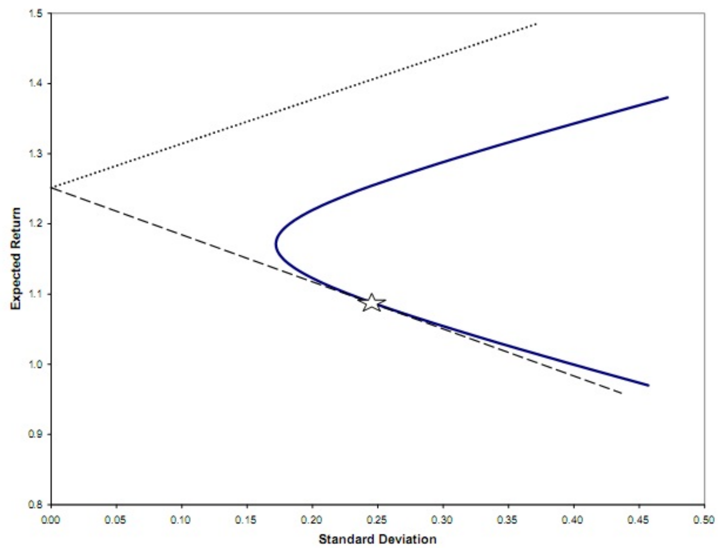

Figure 5.3. This illustrates the case $R_f > B/C$. The hyperbola is the locus of (standard deviation, mean) pairs corresponding to frontier portfolios of risky assets. The dotted line is the efficient part of the mean-variance frontier achievable by including the risk-free asset. The dashed line is the inefficient part of the frontier. The star is at the standard deviation and mean of the tangency portfolio π_*.

5.6 Two-Fund Spanning Again

With a risk-free asset, any portfolio proportional to the tangency portfolio is on the mean-variance frontier, because any such portfolio is of the form (5.10). This means we have two-fund spanning. Except for the tangency portfolio itself, all portfolios proportional to the tangency portfolio satisfy either $\mathbf{1}'\pi < 1$ or $\mathbf{1}'\pi > 1$. In the former case, the portfolio involves a positive (long) position in the risk-free asset, and in the latter a negative (short) position in the risk-free asset. Thus, the risk-free asset and the tangency portfolio span the mean-variance frontier. The tangency portfolio can be replaced in this statement by any other frontier portfolio (excluding the portfolio consisting entirely of the risk-free asset). The tangency portfolio is special only because it consists entirely of risky assets; thus, the two funds - tangency portfolio and risk-free asset - consist of different assets.

If the tangency portfolio is inefficient, then all efficient portfolios consist of a long position in the risk-free asset and a short position in the tangency portfolio. Because investors must in equilibrium be long risky assets, it is customary to assume the tangency portfolio is efficient. As explained in the previous section, this is equivalent to the risk-free return being below the expected return of the global minimum variance portfolio, as shown in Figure 5.2.

5.7 Orthogonal Projections and Frontier Returns

In the remainder of the chapter, we will analyze the mean-variance frontier via a different approach. This is equivalent to the previous analysis in a one-period model with a finite number of assets, which is what we are studying in this chapter. The real power of this different approach is in markets with infinitely many assets (an idealization considered in the study of the Arbitrage Pricing Theory, to be discussed in the next chapter) and in markets with dynamic trading. All that is needed is the existence of a stochastic discount factor (not necessarily strictly positive) and the existence of orthogonal projections on the asset (or return) space. The analysis in this section applies to markets with and without a risk-free asset (and to the risky assets only by excluding the risk-free asset if it exists). The return $\tilde{R}_p$ and excess return $\tilde{e}_p$ described below will vary depending on the market (and whether the risk-free asset is included or excluded).

As discussed in Section 4.3, in a one-period model with a finite number of assets having finite-variance returns, the law of one price implies that there is a (not necessarily strictly positive) stochastic discount factor. As before, let $\tilde{m}_p$ denote the unique orthogonal projection of any stochastic discount factor onto the span of the assets. Being in the span of the assets means it is the payoff of some portfolio. By the definition of a stochastic discount factor, the cost of the payoff $\tilde{m}_p$ is $E[\tilde{m}_p^2]$. Define the return $\tilde{R}_p = \tilde{m}_p/E[\tilde{m}_p^2]$.

The set of excess returns is the set of random variables (i) $\pi'\tilde{R}^{\text{vec}}$ with $\mathbf{1}'\pi = 0$ if there is no risk-free asset, and (ii) $\pi'\tilde{R}^{\text{vec}} - R_f\mathbf{1}'\pi$ if there is a risk-free asset. In either case, excess returns are the payoffs of zero-cost portfolios. For example, going long asset i and short an equal amount of asset j produces the excess return $a(\tilde{R}_i - \tilde{R}_j)$, where a denotes the amount invested long (=size of the short position). The set of excess returns is a finite-dimensional linear subspace of the space of finite-second-moment random variables; thus, as discussed in Section 4.9*, orthogonal projections onto this set are well defined.

Let $\tilde{e}_p$ denote the projection of the random variable that is identically equal to 1 onto the space of excess returns. This means that $\tilde{e}_p$ is an excess return and $\tilde{e}_p = 1 - \tilde{\xi}$, where $\tilde{\xi}$ is orthogonal to all excess returns. Thus, for each excess return $\tilde{x}$, we have $E[\tilde{e}_p\tilde{x}] = E[\tilde{x}] - E[\tilde{\xi}\tilde{x}] = E[\tilde{x}]$. Because $E[\tilde{e}_p\tilde{x}] = E[\tilde{x}]$, we say that $\tilde{e}_p$ "represents the expectation operator on the space of excess returns."

We will show that any return $\tilde{x}$ equals $\tilde{R}_p + b\tilde{e}_p + \tilde{\epsilon}$, for some constant b and some excess return $\tilde{\epsilon}$ that has zero mean and is uncorrelated with $\tilde{R}_p$ and $\tilde{e}_p$. Moreover, the frontier portfolios are the portfolios with returns $\tilde{R}_p + b\tilde{e}_p$ for some b.

Given any return $\tilde{x}$, we can write

$$\tilde{x} = a\tilde{R}_p + b\tilde{e}_p + \tilde{\epsilon},$$

where $a\tilde{R}_p + b\tilde{e}_p$ is the orthogonal projection of $\tilde{x}$ onto the span of $\tilde{R}_p$ and $\tilde{e}_p$. We will use the following facts:

1. $E[\tilde{R}_p\tilde{\epsilon}] = 0$.
2. $E[\tilde{e}_p\tilde{\epsilon}] = 0$.
3. $E[\tilde{m}_p\tilde{x}] = 1$.
4. $E[\tilde{m}_p\tilde{R}_p] = 1$.
5. $E[\tilde{m}_p\tilde{e}_p] = 0$.
6. $E[\tilde{m}_p\tilde{\epsilon}] = 0$.
7. $a = 1$ (so $\tilde{x} = \tilde{R}_p + b\tilde{e}_p + \tilde{\epsilon}$).
8. $E[\tilde{R}_p\tilde{e}_p] = 0$.
9. $\tilde{R}_p + b\tilde{e}_p$ is a return.
10. $\tilde{\epsilon}$ is an excess return.
11. $E[\tilde{\epsilon}] = 0$.
12. $\text{cov}(\tilde{R}_p, \tilde{\epsilon}) = 0$.
13. $\text{cov}(\tilde{e}_p, \tilde{\epsilon}) = 0$.
14. $E[\tilde{x}] = E[\tilde{R}_p + b\tilde{e}_p]$.
15. $\text{var}(\tilde{x}) = \text{var}(\tilde{R}_p + b\tilde{e}_p) + \text{var}(\tilde{\epsilon})$.

The proofs of these facts are as follows:

1. Definition of orthogonal projection.
2. Definition of orthogonal projection.
3. $\tilde{x}$ is a return.
4. $\tilde{R}_p$ is a return.
5. $\tilde{e}_p$ is an excess return.
6. Fact 1 and the proportionality of $\tilde{R}_p$ to $\tilde{m}_p$.
7. Facts 3-6, which imply

$$1 = E[\tilde{m}_p\tilde{x}] = aE[\tilde{m}_p\tilde{R}_p] + bE[\tilde{m}_p\tilde{e}_p] + E[\tilde{m}_p\tilde{\epsilon}] = a.$$

8. Fact 5 and the proportionality of $\tilde{R}_p$ to $\tilde{m}_p$.

9. $\tilde{R}_p$ is a return and $\tilde{e}_p$ is an excess return.

10. $\tilde{x}$ is a return, fact 9, and fact 7 (which implies $\tilde{\epsilon} = \tilde{x} - \tilde{R}_p - b\tilde{e}_p$).

11. Facts 2 and 10 and the fact that $\tilde{e}_p$ represents the expectation operator on the space of excess returns, implying $0 = E[\tilde{e}_p\tilde{\epsilon}] = E[\tilde{\epsilon}]$.

12. Facts 1 and 11.

13. Facts 2 and 11.

14. Facts 7 and 11.

15. Facts 12 and 13.

From facts 9, 14 and 15, we conclude that $\tilde{R}_p + b\tilde{e}_p$ is a return with the same mean as $\tilde{x}$ and a lower variance than $\tilde{x}$ (unless $\tilde{\epsilon} = 0$). Thus, the frontier portfolios are the portfolios with returns $\tilde{R}_p + b\tilde{e}_p$.

We have two-fund spanning: specifically, returns $\tilde{R}_p + b_1\tilde{e}_p$ and $\tilde{R}_p + b_2\tilde{e}_p$ span the mean-variance frontier, for any $b_1 \neq b_2$. This is easy to see: for any b, we can write

$$\tilde{R}_p + b\tilde{e}_p = \lambda(\tilde{R}_p + b_1\tilde{e}_p) + (1-\lambda)(\tilde{R}_p + b_2\tilde{e}_p)$$

for

$$\lambda = \frac{b - b_2}{b_1 - b_2}.$$

Thus any frontier return $\tilde{R}_p + b\tilde{e}_p$ is spanned by these two frontier returns. For example, the frontier is spanned by $\tilde{R}_p$ and $\tilde{R}_p + b\tilde{e}_p$ for any $b \neq 0$.

5.8 Risk-Free Return Proxies

There are three frontier returns that serve as analogues of the risk-free return in various contexts. If there is a risk-free asset, then all of these returns are equal to the risk-free return. The returns are

Minimum Variance Return:

$$\tilde{R}_p + b_m\tilde{e}_p \quad \text{where} \quad b_m = \frac{E[\tilde{R}_p]}{1 - E[\tilde{e}_p]}, \tag{5.14}$$

Zero Beta Return:

$$\tilde{R}_p + b_z\tilde{e}_p \quad \text{where} \quad b_z = \frac{\text{var}[\tilde{R}_p]}{E[\tilde{R}_p]E[\tilde{e}_p]}, \tag{5.15}$$

Constant Mimicking Return:

$$\tilde{R}_p + b_c\tilde{e}_p \quad \text{where} \quad b_c = \frac{E[\tilde{R}_p^2]}{E[\tilde{R}_p]}, \tag{5.16}$$

If there is no risk-free asset, then $\tilde{R}_p + b_m\tilde{e}_p$ is the return of the global minimum variance portfolio as described in Section 5.4. The return $\tilde{R}_p + b_z\tilde{e}_p$ is the frontier return that is uncorrelated with $\tilde{R}_p$. The return $\tilde{R}_p + b_c\tilde{e}_p$ is the projection of the random variable that is identically equal to b_c onto the space of returns. It has the property that

$$E[\tilde{R}(\tilde{R}_p + b_c\tilde{e}_p)] = b_c E[\tilde{R}] \tag{5.17}$$

for every return $\tilde{R}$. The verification of these facts is left for the exercises. If there is a risk-free asset, it is obvious that the risk-free return R_f has the minimum variance, is uncorrelated with $\tilde{R}_p$ and "mimics a constant" (i.e., $E[R_f\tilde{R}] = R_f E[\tilde{R}]$ for every return $\tilde{R}$). Thus,

$$\tilde{R}_p + b_m\tilde{e}_p = \tilde{R}_p + b_z\tilde{e}_p = \tilde{R}_p + b_c\tilde{e}_p = R_f. \tag{5.18}$$

One can also show (see Problem 5.4) that

$$\tilde{R}_p + R_f\tilde{e}_p = R_f, \tag{5.19}$$

which means that $b_m = b_z = b_c = R_f$. Note that (5.19) provides a formula for the excess return $\tilde{e}_p$ when there is a risk-free asset: $\tilde{e}_p = (R_f - \tilde{R}_p)/R_f$.

5.9 Inefficiency of $\tilde{R}_p$

If there is a risk-free asset, then $\tilde{R}_p$ is on the inefficient part of the mean-variance frontier (plots on the lower part of the cone). If there is no risk-free asset and $E[\tilde{R}_p] > 0$ - which is equivalent to the expected return of the global minimum variance portfolio being positive - then $\tilde{R}_p$ is on the inefficient part of the frontier (plots on the lower part of the hyperbola). This is the usual circumstance (with limited liability, all returns are nonnegative). Certainly, $E[\tilde{R}_p] > 0$ if $\tilde{m}_p$ is strictly positive.

To establish the inefficiency of $\tilde{R}_p$, it suffices to show that $E[\tilde{R}_p]$ is less than the expected return of the minimum-variance portfolio, which is $E[\tilde{R}_p] + b_m E[\tilde{e}_p]$. In other words, it suffices to show that $b_m E[\tilde{e}_p] > 0$.

Because $\tilde{e}_p$ represents the expectation operator on the space of excess returns, the following are true:

16. $E[\tilde{e}_p^2] = E[\tilde{e}_p]$.
17. $\text{var}(\tilde{e}_p) = E[\tilde{e}_p](1 - E[\tilde{e}_p])$.

Thus,

$$b_m E[\tilde{e}_p] = \frac{E[\tilde{R}_p]E[\tilde{e}_p]}{1 - E[\tilde{e}_p]} = \frac{E[\tilde{R}_p]E[\tilde{e}_p]^2}{\text{var}(\tilde{e}_p)},$$

which has the same sign as $E[\tilde{R}_p]$. This also shows that $E[\tilde{R}_p + b_m\tilde{e}_p]$ has the same sign as $E[\tilde{R}_p]$.

5.10 Hansen-Jagannathan Bound with a Risk-Free Asset

The Hansen-Jagannathan bound, with a risk-free asset, is that the ratio of standard deviation to mean of any stochastic discount factor must be at least as large as the maximum Sharpe ratio of all portfolios. Hansen and Jagannathan also showed that the orthogonal projection $\tilde{m}_p$ defined in Section 4.5 has the minimum standard deviation-to-mean ratio and that, for the stochastic discount factor $\tilde{m}_p$, the ratio equals the maximum Sharpe ratio.

We derived the Hansen-Jagannathan bound in Section 4.7 and showed there that $\tilde{m}_p$ has the minimum standard deviation-to-mean ratio. We can now show that this ratio $\text{stdev}(\tilde{m}_p)/E[\tilde{m}_p]$ equals the maximum Sharpe ratio.

We have already shown that $\tilde{R}_p$ and $\tilde{R}_p + b\tilde{e}_p$ span the mean-variance frontier for any $b \neq 0$, so $\tilde{R}_p$ and $\tilde{R}_p + b_m\tilde{e}_p = R_f$ span the frontier. Thus, any frontier return is of the form

$$\lambda\tilde{R}_p + (1-\lambda)R_f = R_f + \lambda(\tilde{R}_p - R_f).$$

Hence, the risk premium of any frontier portfolio is $(E[\tilde{R}_p] - R_f)$, and the standard deviation is $\text{stdev}(\tilde{R}_p)$. Given that $R_f > E[\tilde{R}_p]$, the maximum Sharpe ratio is

$$\frac{R_f - E[\tilde{R}_p]}{\text{stdev}(\tilde{R}_p)}.$$

Thus, we want to show that

$$\frac{\text{stdev}(\tilde{m}_p)}{E[\tilde{m}_p]} = \frac{R_f - E[\tilde{R}_p]}{\text{stdev}(\tilde{R}_p)}. \tag{5.20}$$

Because $\tilde{R}_p$ is proportional to $\tilde{m}_p$, this is equivalent to

$$\frac{\text{stdev}(\tilde{R}_p)}{E[\tilde{R}_p]} = \frac{R_f - E[\tilde{R}_p]}{\text{stdev}(\tilde{R}_p)}. \tag{5.21}$$

Equation (5.21) is a consequence of the following facts:

18. $E[\tilde{R}_p^2] = 1/E[\tilde{m}_p^2]$.
19. $R_f E[\tilde{R}_p] = 1/E[\tilde{m}_p^2]$.
20. $\text{var}(\tilde{R}_p) = R_f E[\tilde{R}_p] - E[\tilde{R}_p]^2$.

The proofs of these are as follows:

18. The definition $\tilde{R}_p = \tilde{m}_p / E[\tilde{m}_p^2]$, which implies $E[\tilde{R}_p^2] = E[\tilde{m}_p \tilde{R}_p]/E[\tilde{m}_p^2]$, and fact 4.
19. The definition $\tilde{R}_p = \tilde{m}_p/E[\tilde{m}_p^2]$, which implies $E[\tilde{R}_p R_f] = E[\tilde{m}_p R_f]/E[\tilde{m}_p^2]$, and $E[\tilde{m}_p R_f] = 1$.
20. Facts 18 and 19, which imply $E[\tilde{R}_p^2] = R_f E[\tilde{R}_p]$, and the definition of variance.
Fact 20 implies (5.21) directly.

5.11 Frontier Returns and Stochastic Discount Factors

The relation between stochastic discount factors and mean-variance analysis is as follows: There is a stochastic discount factor that is an affine function of a return if and only if the return is on the mean-variance frontier and not equal to the constant-mimicking return (so not equal to R_f if there is a risk-free asset). This is demonstrated below.

First, we will show that, for any frontier return $\tilde{R}_* = \tilde{R}_p + b\tilde{e}_p$ with $b \neq b_c$,

$$\tilde{x} = -\beta b + \beta \tilde{R}_*$$

is a stochastic discount factor for some constant β. We can write $\tilde{x}$ as

$$\tilde{x} = \beta \tilde{R}_p + \beta b(\tilde{e}_p - 1).$$

Both $\tilde{R}_p$ and $\tilde{e}_p - 1$ are orthogonal to excess returns. Therefore, $\tilde{x}$ is orthogonal to excess returns, implying, for any return $\tilde{R}$,

$$E[\tilde{x}\tilde{R}] = E[\tilde{x}\tilde{R}_p] + E[\tilde{x}(\tilde{R} - \tilde{R}_p)] = E[\tilde{x}\tilde{R}_p].$$

Moreover, using $E[\tilde{R}_p \tilde{e}_p] = 0$ (fact 8 from Section 5.7), we have

$$E[\tilde{x}\tilde{R}_p] = \beta E[\tilde{R}_p^2] - \beta b E[\tilde{R}_p].$$

Hence, setting

$$\beta = \frac{1}{E[\tilde{R}_p^2] - bE[\tilde{R}_p]},$$

we have, for every return $\tilde{R}$,

$$E[\tilde{x}\tilde{R}] = \beta E[\tilde{R}_p^2] - \beta b E[\tilde{R}_p] = 1.$$

This shows that $\tilde{x}$ is a stochastic discount factor.

Now, given any return $\tilde{R}_* = \tilde{R}_p + b\tilde{e}_p + \tilde{\epsilon}$, suppose

$$\tilde{m} = \gamma + \beta \tilde{R}_*$$

is a stochastic discount factor for some constants γ and β. We will show that $\tilde{\epsilon} = 0$ and therefore $\tilde{R}_*$ is a frontier return; moreover, $b \neq b_c$. Using the definition of a stochastic discount factor and the mutual orthogonality of $\tilde{R}_p$, $\tilde{e}_p$ and $\tilde{\epsilon}$, we can calculate

$$
\begin{aligned}
1 &= E[\tilde{m}\tilde{R}_*] = \gamma E[\tilde{R}_p] + \gamma b E[\tilde{e}_p] + \beta E[\tilde{R}_p^2] + \beta b^2 E[\tilde{e}_p^2] + \beta E[\tilde{\epsilon}^2], && (5.22)\\
1 &= E[\tilde{m}\tilde{R}_p] = \gamma E[\tilde{R}_p] + \beta E[\tilde{R}_p^2], && (5.23)\\
0 &= E[\tilde{m}\tilde{e}_p] = \gamma E[\tilde{e}_p] + \beta b E[\tilde{e}_p^2]. && (5.24)
\end{aligned}
$$

Subtracting (5.23) and b times (5.24) from (5.22) yields $E[\tilde{\epsilon}^2] = 0$. Thus, $\tilde{R}_*$ is a frontier return. Moreover, (5.24) and $E[\tilde{e}_p^2] = E[\tilde{e}_p]$ (fact 16) implies $\gamma = -\beta b$, so $\tilde{m} = -\beta b + \beta \tilde{R}_*$. This implies $E[\tilde{m}\tilde{R}_p] = 0$, contradicting (5.23), if $b = b_c$, so we conclude $b \neq b_c$.