Financial Economics - Asset Pricing and Portfolio Selection
Lecture 4

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4 Arbitrage and Stochastic Discount Factors

The previous chapter established that, under fairly restrictive conditions, there is a stochastic discount factor that is a function of market wealth - in fact, proportional to a representative investor's marginal utility of market wealth. In this chapter, it will be shown that under weak assumptions (absence of arbitrage opportunities) there exists some strictly positive stochastic discount factor. In fact, the existence of a strictly positive stochastic discount factor is equivalent to the absence of arbitrage opportunities. Dropping the "strictly positive" condition, the existence of a stochastic discount factor is equivalent to the law of one price, which is a weaker condition than absence of arbitrage opportunities.

This chapter also defines "risk neutral probabilities" and describes the structure of stochastic discount factors as equalling a stochastic discount factor spanned by the assets plus something orthogonal to the assets. Furthermore, it presents the Hansen-Jagannathan bound, which is a lower bound on the volatility of any stochastic discount factor.

In previous chapters, we implicitly assumed that asset payoffs and stochastic discount factors had finite variances whenever it was convenient to do so. We will be a bit more careful in this chapter to specify exactly when assumptions of this type are necessary.

4.1 A Binomial Example

Suppose there is a risk-free asset with return R_f and a single risky asset. For brevity, call the risky asset a "stock." Let S denote the date 0 price of the stock and suppose there are two possible states of the world that will be revealed by the end of the period. Call the states "up" and "down," and let S_u denote the date 1 price of the stock in the up state and S_d the date 1 price in the down state, with $S_u > S_d$.

Assume

$$\frac{S_u}{S} > R_f > \frac{S_d}{S}. \tag{4.1}$$

*These notes are based on *Asset Pricing and Portfolio Choice Theory* by Kerry Back. We thank Professor Back for his kindly and enormous support.

This means that the return on the stock in the up state is greater than the risk-free return, and the return on the stock in the down state is less than the risk-free return. If it were not true, there would be an arbitrage opportunity: if the return on the stock were at least as large as the risk-free return in both states, then one should buy an infinite amount of the stock on margin, and conversely if the return on the stock were no more than the risk-free return in both states, then one should short an infinite amount of stock and put the proceeds in the risk-free asset. So what we are assuming is that there are no arbitrage opportunities in the market for the stock and risk-free asset.

Consider any random variable, and let P_u denote its value in the up state and P_d its value in the down state. Some interesting examples are:

- Call option with strike K: $P_u = \max(S_u - K, 0)$ and $P_d = \max(S_d - K, 0)$.
- Put option with strike K: $P_u = \max(K - S_u, 0)$ and $P_d = \max(K - S_d, 0)$.
- Arrow security for up state: $P_u = 1$ and $P_d = 0$.
- Arrow security for down state: $P_u = 0$ and $P_d = 1$.

All four of these, and indeed any random variable in this two-state model, can be considered a "derivative security," because the payoff can be written as a function of the end-of-period stock price. The last two of the above are called "Arrow securities" in recognition of the seminal work of Kenneth Arrow (1964). They are the building blocks of all other securities. For example, the call option is equivalent to a portfolio consisting of $\max(S_u - K, 0)$ units of the first Arrow security and $\max(S_d - K, 0)$ units of the second Arrow security.

The "delta" of any derivative security is defined as $\delta = (P_u - P_d)/(S_u - S_d)$. Multiplying by $S_u - S_d$ gives us $\delta(S_u - S_d) = P_u - P_d$ and rearranging yields $\delta S_u - P_u = \delta S_d - P_d$, which is critical to what follows. Consider purchasing δ shares of the stock at date 0 and investing

$$\frac{P_u - \delta S_u}{R_f} = \frac{P_d - \delta S_d}{R_f}$$

in the risk-free asset at date 0. Then the position in the risk-free asset will be worth

$$P_u - \delta S_u = P_d - \delta S_d$$

at date 1, and hence the value of the portfolio at date 1 will be

$$\begin{aligned} \delta S_u + P_u - \delta S_u &= P_u \text{ in the up state,} \\ \delta S_d + P_d - \delta S_d &= P_d \text{ in the down state.} \end{aligned}$$

Thus, this portfolio of buying delta shares and investing in the risk-free asset "replicates" the derivative security. What we have shown is that this binomial securities market is complete: any payoff pattern (P_u, P_d) is the payoff of some portfolio.

Because the derivative security can be replicated, in the absence of arbitrage opportunities the value P of the derivative security at date 0 must be the date 0 cost of the replicating portfolio; i.e.,

$$P = \delta S + \frac{P_u - \delta S_u}{R_f}. \tag{4.2}$$

We will now rewrite the formula (4.2) in terms of "state prices." By substituting for δ in (4.2), we can rearrange it as

$$P = \left(\frac{S - S_d/R_f}{S_u - S_d}\right) P_u + \left(\frac{S_u/R_f - S}{S_u - S_d}\right) P_d. \tag{4.3a}$$

A little algebra also shows that

$$S = \left(\frac{S - S_d/R_f}{S_u - S_d}\right) S_u + \left(\frac{S_u/R_f - S}{S_u - S_d}\right) S_d, \tag{4.3b}$$

and

$$1 = \left(\frac{S - S_d/R_f}{S_u - S_d}\right) R_f + \left(\frac{S_u/R_f - S}{S_u - S_d}\right) R_f. \tag{4.3c}$$

Denote the factors appearing in these equations as

$$v_u = \frac{S - S_d/R_f}{S_u - S_d} \quad \text{and} \quad v_d = \frac{S_u/R_f - S}{S_u - S_d}. \tag{4.4}$$

With these definitions, we can write (4.3a)-(4.3c) as

$$P = v_u P_u + v_d P_d, \tag{4.5a}$$

$$S = v_u S_u + v_d S_d, \tag{4.5b}$$

$$1 = v_u R_f + v_d R_f. \tag{4.5c}$$

These equations have the following interpretation: the value of a security at date 0 is its value in the up state times v_u plus its value in the down state times v_d. This applies to (4.5c) by considering an investment of one in the risk-free asset - it has value one at date 0 and will have value R_f in both the up and down states at date 1.

If we consider the example of the first Arrow security, we have $P_u = 1$ and $P_d = 0$ and hence (4.5a) implies $P = v_u$. Hence, v_u is the price of the first Arrow security at date 0. Likewise, v_d is the price of the second Arrow security at date 0. The prices v_u and v_d of the Arrow securities are called "state prices," because v_u is the price at date 0 of receiving one unit of the consumption good in the up state at date 1, and v_d is the price at date 0 of receiving one unit of the consumption good in the down state at date 1. Any other security can be interpreted as a portfolio of the Arrow securities, as noted above - namely, a portfolio of P_u units of the first Arrow security and P_d units of the second. Thus, we can interpret (4.5a) for any other security as saying that its price P equals the cost of the equivalent portfolio of Arrow securities.

The state prices should be positive, and a little algebra shows that the conditions $v_u > 0$ and $v_d > 0$ are exactly equivalent to our "no-arbitrage" assumption (4.1). Thus, we conclude that in the absence of arbitrage opportunities, there exist positive state prices such that the price of any security is the sum across the states of the world of its payoff multiplied by the state price.

It is important to understand that we can compute the state prices without introducing the derivative security or the delta hedging (replication) argument. Equations (4.5b)-(4.5c) can be viewed as two equations in the two unknowns v_u and v_d. The unique solution of this system of equations is (4.4). Having solved (4.5b)-(4.5c), we can compute the value of any derivative security by (4.5a). To prepare for the more general model to be studied in the next section, note that we can write the system of equations (4.5b)-(4.5c) in matrix form as

$$\begin{pmatrix} S_u & S_d \\ R_f & R_f \end{pmatrix} \begin{pmatrix} v_u \\ v_d \end{pmatrix} = \begin{pmatrix} S \\ 1 \end{pmatrix} \tag{4.6}$$

The assumption that $S_u \neq S_d$ implies that the matrix on the left-hand side is invertible; hence the state prices v_u and v_d are unique (moreover, the no-arbitrage assumption (4.1) implies that the state prices are positive). Thus, the price of any derivative security is uniquely determined. The delta hedging argument shows why this should be the case: any derivative security can be replicated at a unique cost.

In the next section, we will consider a model in which there may be more states of the world than assets. If there are more states than assets, state prices will not be unique (there will be more unknowns than equations in the system analogous to (4.6)), so some derivative securities cannot be uniquely priced. These are derivative securities that cannot be replicated.

Finally, we want to convert from state prices to a stochastic discount factor. Let prob_u denote the probability of the up state and prob_d denote the probability of the down state. Define

$$m_u = \frac{v_u}{\text{prob}_u} \quad \text{and} \quad m_d = \frac{v_d}{\text{prob}_d}. \tag{4.7}$$

Then (4.5a)-(4.5c) can be written as

$$P = \text{prob}_u m_u P_u + \text{prob}_d m_d P_d, \tag{4.8a}$$

$$S = \text{prob}_u m_u S_u + \text{prob}_d m_d S_d, \tag{4.8b}$$

$$1 = \text{prob}_u m_u R_f + \text{prob}_d m_d R_f. \tag{4.8c}$$

The right-hand sides are expectations. For example, the right-hand side of equation (4.8a) is the expectation of the random variable that equals $m_u P_u$ in the up state and $m_d P_d$ in the down state. This shows that the random variable with value m_u in the up state and m_d in the down state is a stochastic discount factor. As noted previously, a stochastic discount factor is also called a "state price density," because it equals the state prices per unit probability.

4.2 Fundamental Theorem on Existence of SDF's

As in previous chapters, let n denote the number of assets, let p_i denote the date 0 price of asset i, and let $\tilde{x}_i$ denote the date 1 payoff of asset i. An arbitrage opportunity is defined to be a portfolio θ satisfying

(i) $\sum_{i=1}^n \theta_i p_i \leq 0$,
(ii) $\sum_{i=1}^n \theta_i \tilde{x}_i \geq 0$ with probability one, and
(iii) Either $p'\theta < 0$ or $\sum_{i=1}^n \theta_i \tilde{x}_i > 0$ with positive probability (or both).

Thus, an arbitrage opportunity is a portfolio that requires no investment at date 0, has a nonnegative value at date 1, and either produces income at date 0 or has a positive value with positive probability at date 1. If there is an arbitrage opportunity, then no investor with a strictly monotone utility function can have an optimal portfolio, because he will want to exploit the arbitrage opportunity at infinite scale.

We will show below that if there are only finitely many states of the world and no arbitrage opportunities, then there must be a strictly positive stochastic discount factor. The same thing is true if there are infinitely many states of the world; however, the proof with infinitely many states is more difficult and will not be given here. The most constructive proof with infinitely many states relies on the fact that a CARA investor has an optimal portfolio in any market in which there are no arbitrage opportunities. This is a nontrivial fact (and also will not be proven here), but given the existence of an optimal portfolio, the fact that there is a stochastic discount factor proportional to the investor's marginal utility follows as in Section 2.1.

For the remainder of this section, suppose there are k possible states of the world. Denote the payoff of asset i in state j as x_{ij}. A state-price vector is a vector $v = (v_1, ..., v_k)'$ satisfying

$$\begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \tag{4.9}$$

We will show that if there are no arbitrage opportunities, then there is a solution to (4.9) satisfying $v_j > 0$ for each j. Given strictly positive state prices, we can define a strictly positive stochastic discount factor as in the previous section: $m_j = v_j/\text{prob}_j$ in each state j, where prob_j denotes the probability of state j.

Note that we have made no assumption regarding the relative magnitudes of k and n, so there can be more states of the world than assets; i.e., markets can be incomplete. The proof below is similar to the standard proof of the second welfare theorem, involving a separating hyperplane argument. The converse (positive state prices implies no arbitrage) is easy to show and left as an exercise.

Let $p = (p_1, ..., p_n)'$, and let X denote the $n \times k$ matrix in (4.9). Consider the linear subspace of $\mathbb{R}^{k+1}$ consisting of all vectors $Y\theta$ for $\theta \in \mathbb{R}^n$ where Y is the $(k+1) \times n$ matrix defined as

$$Y = \begin{pmatrix} -p' \\ X' \end{pmatrix}.$$

The assumption of no arbitrage opportunities means that this subspace intersects the non-negative orthant of $\mathbb{R}^{k+1}$ only at the origin. A standard separating hyperplane theorem [need a reference here] implies the existence of a vector $\phi \in \mathbb{R}^{k+1}$ such that

$$\phi' z > 0 = \phi' Y \theta$$

for all $z \in \mathbb{R}^{k+1}$ such that $z \geq 0$ and $z \neq 0$ and all $\theta \in \mathbb{R}^n$. The inequality $\phi' z > 0$ for all z as described implies $\phi_j > 0$ for $j = 1, ..., k+1$. Define $v_j = \phi_{j+1}/\phi_1$ for $j = 1, ..., k$. The equality $0 = \phi' Y \theta$ for all $\theta \in \mathbb{R}^n$ implies

$$p'\theta = v' X' \theta$$

for all $\theta \in \mathbb{R}^n$, which implies $p = Xv$.

4.3 Law of One Price and Stochastic Discount Factors

The "law of one price" is a weaker condition than absence of arbitrage opportunities - it is implied by absence of arbitrage but does not imply the absence of arbitrage. The law of one price means that for any two portfolios θ and ψ, if

$$\sum_{i=1}^{n} \theta_i \tilde{x}_i = \sum_{i=1}^{n} \psi_i \tilde{x}_i$$

with probability one, then

$$\sum_{i=1}^{n} \theta_i p_i = \sum_{i=1}^{n} \psi_i p_i.$$

Thus, any two portfolios with the same payoff must have the same cost. If the law of one price fails, then there is a simple arbitrage opportunity: buy low and sell high by going long the cheaper portfolio and shorting the more expensive portfolio.

One says that a random variable $\tilde{x}$ is "marketed" if it is the payoff of some portfolio, i.e., if it equals $\sum_{i=1}^{n} \theta_i \tilde{x}_i$ for some portfolio θ. The set of marketed variables is also called the "linear span" of the asset payoffs. The law of one price states that every marketed payoff has a unique price. In complete markets, every random variable is marketed; however, in incomplete markets, the set of marketed payoffs is a linear subspace of the set of all random variables. The existence of a stochastic discount factor depends essentially on whether the given prices for marketed payoffs can be "extended" to form a linear valuation operator on the space of all random variables. In particular, if there is a stochastic discount factor $\tilde{m}$, then the function $\tilde{x} \mapsto E[\tilde{m}\tilde{x}]$ extends the prices of marketed payoffs to the class of all $\tilde{x}$ for which $E[\tilde{m}\tilde{x}]$ exists.

If there are only finitely many states of the world, or, more generally, if each asset payoff $\tilde{x}_i$ has a finite variance, then the law of one price implies the existence of a stochastic discount factor $\tilde{m}$. This stochastic discount factor has a finite variance but need not be strictly positive. Because $\tilde{m}$ has a finite variance, $E[\tilde{m}\tilde{x}]$ exists and is finite whenever $\tilde{x}$ has a finite variance. Thus, if each asset payoff $\tilde{x}_i$ has a finite variance and the law of one price holds, it is possible to extend the prices of marketed payoffs to a linear valuation operator $\tilde{x} \mapsto E[\tilde{m}\tilde{x}]$ defined for all $\tilde{x}$ having a finite variance. See Problem 4.6. The converse (the existence of a stochastic discount factor implies the law of one price) follows immediately from the linearity of the expectation operator, as is shown below.

The definition of a stochastic discount factor $\tilde{m}$ is that $p_i = E[\tilde{m}\tilde{x}_i]$ for each asset i. Consider two portfolios such that

$$\sum_{i=1}^{n} \theta_i \tilde{x}_i = \sum_{i=1}^{n} \psi_i \tilde{x}_i$$

and assume there is a stochastic discount factor $\tilde{m}$. Taking the expected product of each side of the above with $\tilde{m}$ and using the linearity of the expectation operator, we have

$$\sum_{i=1}^{n} \theta_i E[\tilde{m}\tilde{x}_i] = \sum_{i=1}^{n} \psi_i E[\tilde{m}\tilde{x}_i].$$

Now, using the definition of a stochastic discount factor produces

$$\sum_{i=1}^{n} \theta_i p_i = \sum_{i=1}^{n} \psi_i p_i,$$

showing that the law of one price holds.

4.4 Risk Neutral Probabilities

Another way to represent prices, which is equivalent to a stochastic discount factor, is via a "risk neutral probability." This concept will be defined here.

Let $\tilde{m}$ be a strictly positive stochastic discount factor. For any event A, let 1_A denote the indicator function of A, that is, $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ if $\omega \notin A$.

Suppose first that there is a risk-free asset. For any event A, define

$$\mathbb{P}^*(A) = R_f E[\tilde{m} 1_A]. \tag{4.10}$$

Then $\mathbb{P}^*$ is a probability measure: $\mathbb{P}^*(A) \geq 0$, $\mathbb{P}^*(\Omega) = 1$, and if $A_1, A_2, \ldots$ is a sequence of disjoint events, then $\mathbb{P}^*(\cup A_i) = \sum \mathbb{P}^*(A_i)$. As with any probability measure, there is an expectation operator associated with $\mathbb{P}^*$. Denote it by E^*. The definition of $\mathbb{P}^*$ implies that

$$E^*[\tilde{x}] = R_f E[\tilde{m}\tilde{x}]$$

for every $\tilde{x}$ for which the expectation $E[\tilde{m}\tilde{x}]$ exists. Thus, the price of a payoff $\tilde{x}$ is

$$E[\tilde{m}\tilde{x}] = \frac{1}{R_f} E^*[\tilde{x}]. \tag{4.11}$$

This shows that one can compute the price by taking the expectation relative to the probability measure $\mathbb{P}^*$ and then discounting at the risk-free rate. Because of this, the probability measure $\mathbb{P}^*$ is called a "risk neutral probability."

If there is no risk-free asset, then risk neutral probabilities are defined in the same way by substituting $1/E[\tilde{m}]$ for R_f in (4.10). In a complete market, there is a unique stochastic discount factor and hence a unique risk-neutral probability. In an incomplete market, different stochastic discount factors $\tilde{m}$ define different risk neutral probabilities $\mathbb{P}^*$.

In the binomial example in Section 4.1, the definition (4.10) of the risk-neutral probability and the formula (4.7) for the stochastic discount factor implies that the risk-neutral probabilites are

$$\begin{aligned}
\mathbb{P}^*(\{\text{UP}\}) &= R_f m_u \text{prob}_u = R_f v_u, \\
\mathbb{P}^*(\{\text{DOWN}\}) &= R_f m_d \text{prob}_d = R_f v_d,
\end{aligned}$$

where v_u and v_d are the state prices defined in (4.4). Thus, the formula (4.11) for the value of a payoff $\tilde{x}$ is equivalent to

$$\frac{1}{R_f} E^*[\tilde{x}] = \frac{1}{R_f}(R_f v_u x_u + R_f v_d x_d) = v_u x_u + v_d x_d,$$

where x_u denotes the realization of $\tilde{x}$ in the up state and x_d denotes the realization of $\tilde{x}$ in the down state. This illustrates the fact that valuing payoffs via a risk-neutral probability is equivalent to valuing via state prices.

4.5 Projecting SDF's onto the Asset Span

In the remainder of this chapter, we will assume that all asset payoffs have finite variances and the law of one price holds, so there is some stochastic discount factor with a finite variance. Every stochastic discount factor in the remainder of the chapter is implicitly assumed to have a finite variance. The projections and residuals that we define will automatically have finite variances.

In this section, we will show that any stochastic discount factor $\tilde{m}$ is equal to $\tilde{m}_p + \tilde{\varepsilon}$ where $\tilde{m}_p$ is the unique stochastic discount factor spanned by the assets and $\tilde{\varepsilon}$ is orthogonal to the assets. By "spanned by the assets," it is meant that $\tilde{m}_p$ is the payoff of some portfolio. By "orthogonal to the assets," it is meant that $E[\tilde{\varepsilon}\tilde{x}_i] = 0$ for each asset i. The stochastic discount factor $\tilde{m}_p$ is called the "orthogonal projection of $\tilde{m}$ onto the span of the assets."

To explain the orthogonal projection, we first digress to consider ordinary least-squares estimates of linear regression coefficients. This is an example of orthogonal projections with which the reader should be familiar, and it is helpful to understand that "orthogonal projection" (on a finite-dimensional space) means the same thing as "linear regression."

The usual multivariate linear regression model is written as

$$y = X\hat{\beta} + \epsilon$$

where y is a $T \times 1$ vector of observations of the dependent variable, X is a $T \times K$ matrix of observations on K independent variables (one of which may be a constant), $\hat{\beta}$ denotes the $K \times 1$ vector of estimated regression coefficients, and ϵ is the $K \times 1$ vector of residuals. The vector $X\hat{\beta}$ is the vector of "predicted" values of the dependent variable, given the observations of the independent variables, and we can denote it by y_p ("p" for "predicted" or "projected"). The fact that y_p is of the form $X\hat{\beta}$ for some $\hat{\beta}$ means that y_p is a linear combination of the columns of the X matrix, i.e., is "spanned" by the columns of X.

The vector $\hat{\beta}$ is chosen to minimize the sum of squared errors $(y - y_p)'(y - y_p)$, which is equivalent to choosing y_p as the closest point to y in the span of the columns of X. This is also equivalent to choosing y_p so that the error $\epsilon = y - y_p$ is orthogonal to the columns of X. In other words, $\hat{\beta}$ is defined by the equations $X'(y - X\hat{\beta}) = 0$. Assuming $X'X$ is invertible, this is equivalent to

$$\hat{\beta} = (X'X)^{-1}X'y \quad \text{and} \quad y_p = X(X'X)^{-1}X'y. \tag{4.12}$$

The formula for the orthogonal projection $\tilde{m}_p$ is analogous to that for y_p. An example of an orthogonal projection is presented in Fig. 4.1.

Let $\tilde{X}$ denote the column vector of dimension n that has $\tilde{x}_i$ as its i-th element. For $\tilde{m}_p$ to be in the span of the asset payoffs means that $\tilde{m}_p = \tilde{X}'\theta$ for some $\theta \in \mathbb{R}^n$. The projection $\tilde{m}_p$ is defined by the condition that

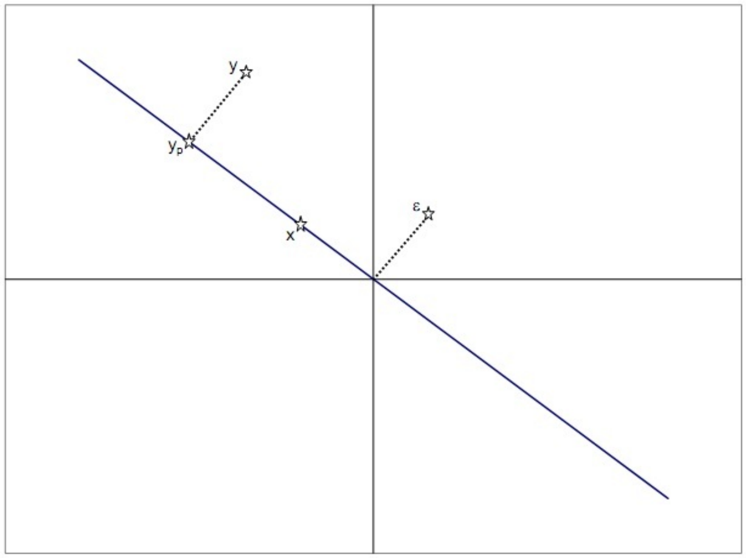

Figure 4.1. This illustrates the orthogonal projection of a vector in $\mathbb{R}^2$ on the linear span of another vector in $\mathbb{R}^2$, corresponding to $T = 2$ and $K = 1$ in the linear regression model. The solid line is the linear space spanned by x. The vector y_p is the orthogonal projection of y on the span of x. The residual is $\epsilon = y - y_p$.

it be in the span of the asset payoffs and the condition that the residual $\tilde{m} - \tilde{m}_p$ is orthogonal to each of the x_i. We are defining orthogonality in terms of the probability weighted inner product, i.e., the expectation. Thus, the orthogonality equations are

$$E[\tilde{X}(\tilde{m} - \tilde{X}'\theta)] = 0.$$

This can be solved as

$$E[\tilde{X}m] = E[\tilde{X}\tilde{X}']\theta \qquad \Rightarrow \qquad \theta = E[\tilde{X}\tilde{X}']^{-1}E[\tilde{X}m] \tag{4.13a}$$

$$\Rightarrow \qquad \tilde{m}_p = E[\tilde{X}m]'E[\tilde{X}\tilde{X}']^{-1}\tilde{X}. \tag{4.13b}$$

Here we have assumed that the matrix $E[\tilde{X}\tilde{X}']$ is invertible. If it is not invertible, then the portfolio θ is not uniquely defined, even though the projection $\tilde{m}_p$ is uniquely defined: There will be multiple portfolios θ satisfying $\tilde{X}'\theta = \tilde{m}_p$.

Note that $E[\tilde{X}\tilde{m}]$ in (4.13b) is the n-dimensional column vector with $E[\tilde{x}_i\tilde{m}]$ as its i-th element. By the definition of a stochastic discount factor, this i-th element is p_i. Thus, $E[\tilde{X}\tilde{m}] = p$, and the formula (4.13b) is equivalent to

$$\tilde{m}_p = p'E[\tilde{X}\tilde{X}']^{-1}\tilde{X}. \tag{4.14}$$

This shows that the projection $\tilde{m}_p$ is unique - the same for every stochastic discount factor $\tilde{m}$ - as claimed at the beginning of the section.

4.6 Projecting onto a Constant and the Asset Span

Another (perhaps identical) projection is often useful. Stack the payoffs of the risky assets in a column vector $\tilde{X}_0$. We want to project a stochastic discount factor $\tilde{m}$ onto the span of the risky payoffs and a constant. This means that we want to write

$$\tilde{m} = a + \theta_0' \tilde{X}_0 + \tilde{\epsilon}, \tag{4.15}$$

for some constant a and vector θ_0, where ϵ is orthogonal to a constant ($E[\tilde{\epsilon}] = 0$) and to each of the risky asset payoffs ($E[\tilde{\epsilon}\tilde{x}_i] = 0$ for each risky asset i). The orthogonal projection is $a + \theta_0' \tilde{X}_0$. If there is a risk-free asset, then projecting on the span of the risky assets and a constant is the same as projecting on the span of all of the assets, so $a + \theta_0' \tilde{X}_0$ must equal $\tilde{m}_p$ defined in (4.14). In this case, we will derive a different, but equivalent, formula for $\tilde{m}_p$. If there is no risk-free asset (and no linear combination of the risky assets is risk-free) then $a + \theta_0' \tilde{X}_0$ is the payoff of a portfolio and equal to $\tilde{m}_p$ only if $a = 0$.

Let $\tilde{m}$ be a stochastic discount factor, and define $\nu = E[\tilde{m}]$. We will show that the orthogonal projection of $\tilde{m}$ on the span of the risky assets and a constant is given by

$$\tilde{m}_{\nu p} = \nu + (p_0 - \nu E[\tilde{X}_0])' \Sigma_x^{-1} (\tilde{X}_0 - E[\tilde{X}_0]), \tag{4.16}$$

where p_0 is the vector of prices of the risky assets and Σ_x is the covariance matrix of the risky asset payoffs.

Note that $E[\tilde{m}_{\nu p}] = \nu$. Furthermore, for any value of ν, $\tilde{m}_{\nu p}$ defined in (4.16) satisfies $E[\tilde{m}_{\nu p} \tilde{X}_0'] = p_0'$. This defines $\tilde{m}_{\nu p}$ as a stochastic discount factor if there is no risk-free asset. Thus, in the absence of a risk-free asset, there is a stochastic discount factor with any given mean ν. In the presence of a risk-free asset, $\tilde{m}_{\nu p}$ defined in (4.16) is a stochastic discount factor if and only if $\nu = 1/R_f$, in which case, as noted before, $\tilde{m}_{\nu p} = \tilde{m}_p$.

The vector $p_0 - \nu E[\tilde{X}_0]$ in (4.16) equals

$$E[\tilde{m}\tilde{X}_0] - E[\tilde{m}]E[\tilde{X}_0] = \mathrm{Cov}(\tilde{X}_0, \tilde{m}),$$

where $Cov(\tilde{X}_0, \tilde{m})$ denotes the column vector with i-th element equal to $\mathrm{cov}(\tilde{x}_i, \tilde{m})$, so an equivalent formula for $\tilde{m}_{\nu p}$ is

$$\tilde{m}_{\nu p} = E[\tilde{m}] + \mathrm{Cov}(\tilde{X}_0, \tilde{m})' \Sigma_x^{-1} (\tilde{X}_0 - E[\tilde{X}_0]). \tag{4.17}$$

If all of the asset prices are positive, then we can define the vector of returns with i-th component $\tilde{R}_i = \tilde{x}_i / p_i$. In this case, the vector $\tilde{X}$ in the formula (4.13b) for $\tilde{m}_p$ can be replaced by the vector of returns, and the vector $\tilde{X}_0$ in the formula (4.17) for $\tilde{m}_{\nu p}$ can be replaced by the vector of risky asset returns, because the linear span of returns is the same as the linear span of payoffs. When there is a risk-free asset, the resulting formula for

$\tilde{m}_{\nu p}(=\tilde{m}_p)$ is shown explicitly in (5.25).

To project onto the span of a constant and the risky asset payoffs, it is convenient to first "de-mean" everything. We will have

$$\tilde{m} = a + \theta'\tilde{X} + \tilde{\epsilon},$$

for some constant, where the residual $\tilde{\epsilon}$ is orthogonal to the $\tilde{x}_i$ and to a constant. Being orthogonal to a constant means that $E[\tilde{\epsilon}] = 0$, so taking expectations throughout shows that

$$a = E[\tilde{m}] - \theta' E[\tilde{X}].$$

Thus,

$$\tilde{m} - E[\tilde{m}] = \theta'(\tilde{X} - E[\tilde{X}]) + \tilde{\epsilon}.$$

Furthermore, the residual $\tilde{\epsilon}$ being orthogonal to the $\tilde{x}_i$ and to a constant implies that it is orthogonal to the random variables $\tilde{x}_i - E[\tilde{x}_i]$. Thus, the vector θ can be obtained by projecting $\tilde{m} - E[\tilde{m}]$ onto the linear span of the $\tilde{x}_i - E[\tilde{x}_i]$. This means that we replace $\tilde{X}$ in (4.13a) with $\tilde{X} - E[\tilde{X}]$ and $\tilde{m}$ with $\tilde{m} - E[\tilde{m}]$, yielding

$$\begin{aligned} \theta &= E[(\tilde{X} - E[\tilde{X}])(\tilde{X} - E[\tilde{X}])']^{-1} E[(\tilde{X} - E[\tilde{X}])(\tilde{m} - E[\tilde{m}])] \\ &= \Sigma_x^{-1} \text{Cov}(\tilde{X}, \tilde{m}). \end{aligned} \tag{4.18}$$

Hence, the projection of $\tilde{m}$ is

$$\begin{aligned} a + \theta'\tilde{X} &= E[\tilde{m}] - \theta' E[\tilde{X}] + \theta'\tilde{X} \\ &= E[\tilde{m}] + \text{Cov}(\tilde{X}, \tilde{m})' \Sigma_x^{-1} (\tilde{X} - E[\tilde{X}]). \end{aligned}$$

This establishes (4.17) and therefore (4.16).

4.7 Hansen-Jagannathan Bound with a Risk-Free Asset

By the law of one price, if $\tilde{w}$ is in the span of the asset payoffs (i.e., it is the payoff of some portfolio), then there is unique cost at which it can be purchased. Denote this cost by $C[\tilde{w}]$. Consider any $\tilde{w}$ in the span of the asset payoffs with $C(\tilde{w}) > 0$, and define the return $\tilde{R} = \tilde{w}/C[\tilde{w}]$.

Assume there is a risk-free asset. Then (2.19) holds for any stochastic discount factor $\tilde{m}$, which we repeat here:

$$E[\tilde{R}] - R_f = -R_f \text{cov}(\tilde{m}, \tilde{R}).$$

Letting ρ denote the correlation of $\tilde{m}$ with $\tilde{R}$, we can write this as

$$\rho \times \text{stdev}(\tilde{m}) = -\frac{E[\tilde{R}] - R_f}{R_f \text{stdev}(\tilde{R})}.$$

Because the correlation ρ is between -1 and 1, this implies

$$\text{stdev}(\tilde{m}) \geq \frac{E[\tilde{R}] - R_f}{R_f \text{stdev}(\tilde{R})}. \tag{4.19}$$

The ratio $|E[\tilde{R}] - R_f|/\text{stdev}(\tilde{R})$ on the right-hand side of (4.19) is the absolute value of the "Sharpe ratio" of the portfolio with return $\tilde{R}$. Hence the standard deviation of any stochastic discount factor must be at least as large as the maximum absolute Sharpe ratio of all portfolios divided by the risk-free return. This is one version of the Hansen-Jagannathan (1991) bounds.

The Hansen-Jagannathan bound (4.19) has real economic significance. As discussed previously, an asset pricing model is a specification of a stochastic discount factor $\tilde{m}$. A model can be rejected by the Hansen-Jagannathan bound if it does not imply sufficient variability of $\tilde{m}$. For example, in a representative investor model with CRRA utility and log-normal consumption growth, the Hansen-Jagannathan bound implies a lower bound on the coefficient of relative risk aversion of the representative investor (see Problem 4.4).

Continuing to assume there is a risk-free asset, we can show that the stochastic discount factor with the minimum standard deviation is the unique stochastic discount factor $\tilde{m}_p$ in the span of the assets. Furthermore, it can be shown for this stochastic discount factor that $R_f \text{stdev}(\tilde{m})$ equals the Sharpe ratio of some portfolio; in other words, the inequality in (4.19) is an equality for the stochastic discount factor $\tilde{m}_p$ and the return $\tilde{R}$ with the maximum Sharpe ratio.

We will leave the calculation of the return $\tilde{R}$ with the maximum Sharpe ratio for the next chapter, but we will show here that $\tilde{m}_p$ has the minimum standard deviation. This is an easy calculation. For any stochastic discount factor $\tilde{m}$, we have $\tilde{m} = \tilde{m}_p + \tilde{\epsilon}$, where $\tilde{\epsilon}$ is orthogonal to the assets. The orthogonality implies $\tilde{\epsilon}$ is orthogonal to $\tilde{m}_p$; furthermore, given the existence of a risk-free asset, the orthogonality implies $E[\tilde{\epsilon}] = 0$. Therefore,

$$\begin{aligned}
\text{var}(\tilde{m}) &= \text{var}(\tilde{m}_p) + \text{var}(\tilde{\epsilon}) + 2\text{cov}(\tilde{m}_p, \tilde{\epsilon}) \\
&= \text{var}(\tilde{m}_p) + \text{var}(\tilde{\epsilon}) + 2(E[\tilde{m}_p\tilde{\epsilon}] - E[\tilde{m}_p]E[\tilde{\epsilon}]) \\
&= \text{var}(\tilde{m}_p) + \text{var}(\tilde{\epsilon}).
\end{aligned}$$

The standard deviation is therefore minimized by taking $\tilde{\epsilon} = 0$, i.e., $\tilde{m} = \tilde{m}_p$.

4.8 Hansen-Jagannathan Bound with No Risk-Free Asset

Now we drop the assumption that there is a risk-free asset. Consider again a w in the span of the assets with $C[\tilde{w}] > 0$ and set $\tilde{R} = \tilde{w}/C[\tilde{w}]$. For any stochastic discount factor $\tilde{m}$, (2.18) holds, namely:

$$1 = \mathrm{cov}(\tilde{m}, \tilde{R}) + E[\tilde{m}]E[\tilde{R}].$$

Therefore,

$$\rho \times \mathrm{stdev}(\tilde{m}) \times \mathrm{stdev}(\tilde{R}) = 1 - E[\tilde{m}]E[\tilde{R}],$$

implying

$$\mathrm{stdev}(\tilde{m}) \geq \frac{|E[\tilde{m}]E[\tilde{R}] - 1|}{\mathrm{stdev}(\tilde{R})}, \tag{4.20}$$

where ρ again denotes the correlation of $\tilde{m}$ with $\tilde{R}$. The maximum of the right-hand side of (4.20) over all returns $\tilde{R}$ defines a lower bound on the standard deviation of any stochastic discount factor, with the lower bound depending on the mean $E[\tilde{m}]$.