Financial Economics - Asset Pricing and Portfolio Selection
Lecture 2

Hongtao Zhou *

2 Portfolio Choice and Stochastic Discount Factors

Consider an investor in a one-period problem who can invest in a given set of assets. Let $\tilde{x}_i$ denote the payoff of asset i and p_i its price. We can assume $p_i \geq 0$ (if $p_i < 0$, just replace asset i with a short position in asset i, having price $-p_i$ and payoff $-\tilde{x}_i$). If the price of asset i is positive, then we can define the return on asset i as

$$\tilde{R}_i = \frac{\tilde{x}_i}{p_i}.$$

For each unit of the consumption good invested, the investor obtains $\tilde{R}_i$. The term "return" will be used in this book more generally for the payoff of a portfolio with a unit price. The rate of return is defined as

$$\tilde{r}_i = \tilde{R}_i - 1 = \frac{\tilde{x}_i - p_i}{p_i}.$$

If there is a risk-free asset, then we let R_f denote its return. The risk premium of a risky asset is defined to be $E[\tilde{R}_i] - R_f$. This extra average return is an investor's compensation for bearing the risk of the asset. Explaining why different assets have different risk premia is the main goal of asset pricing theory.

Except for Sections 2.7 and 2.8, we will focus on the optimal investment problem, assuming consumption at the beginning of the period is already determined. Let w_0 denote the amount invested at the beginning of the period, and let θ_i denote the number of shares the investor chooses to hold of asset i. Assume there are no short sale constraints (so $\theta_i < 0$ is feasible) and no margin requirements. For extra generality, we can assume the investor has some possibly random endowment $\tilde{y}$ at the end of the period (for example, labor income) which he consumes in addition to the end-of-period portfolio value. Letting n denote the number of assets, the investor's choice problem is:

$$\text{maximize } E[u(\tilde{w})] \tag{2.1}$$

*These notes are based on *Asset Pricing and Portfolio Choice Theory* by Kerry Back. We thank Professor Back for his kindly and enormous support.

$$\text{subject to } \sum_{i=1}^{n} \theta_i p_i = w_0 \quad \text{and} \quad \tilde{w} = \tilde{y} + \sum_{i=1}^{n} \theta_i \tilde{x}_i .$$

2.1 The First-Order Condition

Substituting in the second constraint, the Lagrangean for the choice problem (2.1) is

$$E\left[u\left(\tilde{y} + \sum_{i=1}^{n} \theta_i \tilde{x}_i\right)\right] - \gamma\left(\sum_{i=1}^{n} \theta_i p_i - w_0\right),$$

where γ is the Lagrange multiplier. The usual necessary condition for optimality is that the partial derivative with respect to each θ_i should be zero. Boundary constraints (e.g., $u(w)$ may only be defined for $w = 0$) can matter, as will be discussed further at the end of this section; however, these seem to be more of a technical issue than anything of real economic importance. Therefore, we will suppose that the partials are zero at the optimum. Furthermore, we will suppose that we can interchange differentiation and expectation in the Lagrangean; i.e., that

$$\frac{\partial}{\partial \theta_i} E[u(\cdot)] = E[\frac{\partial}{\partial \theta_i} u(\cdot)]. \tag{2.2}$$

Thus, we will suppose that the following holds at the optimum:

$$(\forall i) \quad E[u'(\tilde{w})\tilde{x}_i] - \gamma p_i = 0. \tag{2.3}$$

where

$$\tilde{w} = \tilde{y} + \sum_{i=1}^{n} \theta_i \tilde{x}_i$$

is the optimal end-of-period wealth (θ here being the optimal portfolio). We can rearrange (2.3) as

$$(\forall i) \quad E\left[\frac{u'(\tilde{w})}{\gamma}\tilde{x}_i\right] = p_i. \tag{2.4}$$

If $p_i \neq 0$, we can write (2.4) in terms of the return of asset i as

$$E\left[\frac{u'(\tilde{w})}{\gamma}\tilde{R}_i\right] = 1, \tag{2.5}$$

and if $p_j \neq 0$ also, then we have

$$E\left[u'(\tilde{w})\left(\tilde{R}_i - \tilde{R}_j\right)\right] = 0. \tag{2.6}$$

Each equation (2.4)-(2.6) is important in asset pricing theory, as will be explained in the next section.

The random variable $\tilde{R}_i - \tilde{R}_j$ in (2.6) is the payoff of the portfolio consisting of a unit of the consumption good invested in asset i and an equal short position in asset j. We call the payoff of a zero-cost portfolio such as this an "excess return." Moreover, two random variables $\tilde{y}$ and $\tilde{z}$ satisfying $E[\tilde{y}\tilde{z}] = 0$ are called "orthogonal." So, the first-order condition (2.6) can be expressed as: marginal utility evaluated at the optimal wealth is orthogonal to each excess return.

The simple intuition for (2.6) is that the expectation

$$E\left[u'(\tilde{w})\left(\tilde{R}_i - \tilde{R}_j\right)\right]$$

is the marginal value of adding the zero-cost portfolio to the optimal portfolio. If the expectation were positive, then adding a little of the zero-cost portfolio to the optimal portfolio θ would yield a portfolio even better than the optimal portfolio, which is of course impossible. If the expectation were negative, then shorting a little of the zero-cost portfolio (i.e., adding a little of the portfolio with payoff $\tilde{R}_j - \tilde{R}_i$) would lead to an improvement in utility. Because it is impossible to improve upon the optimum, the expectation must be zero (at the optimal wealth $\tilde{w}$). If there is a risk-free asset, then we can take $\tilde{R}_j = R_f$ in the first-order condition (2.6), leading to

$$E\left[u'(\tilde{w})\left(\tilde{R}_i - R_f\right)\right] = 0, \tag{2.7}$$

showing that the excess return of any risky asset over the risk-free rate must be orthogonal to marginal utility evaluated at the optimal wealth. We will now give a proof of the first-order condition. The key assumption that we need (see (2.8) and (2.9) below) is that it is actually feasible to add to and subtract from the optimal portfolio a little of the zero-cost portfolio with payoff $\tilde{R}_i - \tilde{R}_j$. A simple example in which it is infeasible to do this and the first-order conditions (2.4)-(2.7) may fail is if there is a risk-free asset, the utility function is only defined for nonnegative wealth (as with CRRA utility), there is no end-of-period endowment $\tilde{y}$, and the risky asset returns are normally distributed. In this case, regardless of the expected returns of the risky assets, the only feasible portfolio is to invest all of one's wealth in the risk-free asset. This produces a constant optimal wealth $\tilde{w}$, so

$$E\left[u'(\tilde{w})\left(\tilde{R}_i - \tilde{R}_j\right)\right] = u'(\tilde{w})E\left[\tilde{R}_i - \tilde{R}_j\right],$$

which need not be zero.

We will prove (2.6) when $p_i > 0$ and $p_j > 0$. Let θ denote the optimal portfolio, so

$$\tilde{w} = \tilde{y} + \sum_{i=1}^{n} \theta_i \tilde{x}_i$$

is the optimal wealth. Suppose the utility function is defined for all $w > \underline{w}$, where $\underline{w}$ is some constant, possibly equal to $-\infty$. Assume the utility function is concave and differentiable.

Assume there exists $\epsilon > 0$ such that

$$\tilde{w}(\omega) + \delta(\tilde{R}_i(\omega) - \tilde{R}_j(\omega)) \geq \underline{w} \tag{2.8}$$

in all states of the world ω and all δ such that $|\delta| \leq \epsilon$. Assume further that

$$E[u(\tilde{w} + \delta(\tilde{R}_i - \tilde{R}_j))] > -\infty \tag{2.9}$$

for all δ such that $|\delta| \leq \epsilon$. The optimality of $\tilde{w}$ implies

$$E\left[\frac{u(\tilde{w} + \delta(\tilde{R}_i - \tilde{R}_j)) - u(\tilde{w})}{\delta}\right] \leq 0 \tag{2.10}$$

for all $\delta > 0$.

We will use the following property of any concave function u: For any $w > \underline{w}$ and any real a,

$$\frac{u(w + \delta a) - u(w)}{\delta} \uparrow \quad \text{as } \delta \downarrow, \tag{2.11}$$

taking $\delta > 0$ and sufficiently small that $w + \delta a > \underline{w}$. To prove (2.11), consider $0 < \delta_2 < \delta_1$. Define $\lambda = \delta_2/\delta_1$. Apply the definition of concavity in footnote 1 of Chapter 1 with $w_2 = w$ and $w_1 = w + \delta_1 a$, noting that

$$\lambda w_1 + (1 - \lambda)w_2 = w_2 + \lambda(w_1 - w_2) = w + \delta_2 a.$$

This yields

$$u(w + \delta_2 a) \geq \frac{\delta_2}{\delta_1} u(w + \delta_1 a) + \left(1 - \frac{\delta_2}{\delta_1}\right) u(w),$$

so

$$\frac{u(w + \delta_2 a) - u(w)}{\delta_2} \geq \frac{u(w + \delta_1 a) - u(w)}{\delta_1},$$

as claimed.

We apply (2.11) in each state of the world with $w = \tilde{w}(\omega)$ and $a = \tilde{R}_j(\omega) - \tilde{R}_i(\omega)$. This shows that the expression inside the expectation in (2.10) is monotonically increasing as δ decreases. We can write this expression as

$$\frac{u(\tilde{w} + \delta(\tilde{R}_i - \tilde{R}_j)) - u(\tilde{w})}{\delta(\tilde{R}_i - \tilde{R}_j)} (\tilde{R}_i - \tilde{R}_j),$$

showing that it converges to

$$u'(\tilde{w})(\tilde{R}_i - \tilde{R}_j)$$

in each state of the world as $\delta \downarrow 0$. The monotone convergence theorem in conjunction with (2.10) therefore yields

$$E[u'(\tilde{w})(\tilde{R}_i - \tilde{R}_j)] \leq 0.$$

Repeating the argument with i and j reversed yields

$$E[u'(\tilde{w})(\tilde{R}_j - \tilde{R}_i)] \leq 0,$$

so (2.6) holds.

2.2 Stochastic Discount Factors

A stochastic discount factor is any random variable $\tilde{m}$ such that

$$(\forall i) \quad E[\tilde{m}\tilde{x}_i] = p_i. \tag{2.12}$$

This definition is of fundamental importance in asset pricing theory. The name"stochastic discount factor" reflects the fact that the price of an asset can be computed by "discounting" the future cash flow $\tilde{x}_i$ by the stochastic factor $\tilde{m}$ and then taking the expectation.

If there are only finitely many states of the world, say $\omega_1, ..., \omega_k$, then (2.12) can be written as

$$\sum_{j=1}^{k} \tilde{m}(\omega_j)\tilde{x}_i(\omega_j)\mathbf{P}(\omega_j) = p_i, \tag{2.12$'$}$$

where $\mathbf{P}(\omega_j)$ denotes the probability of the j-th state. Consider a security that pays one unit of the consumption good in a particular state ω and zero in all other states (sometimes called an "Arrow security"). The price p of such a security is called a "state price." From (2.12'), we have $\tilde{m}(\omega)\mathbf{P}(\omega) = p$, implying $\tilde{m}(\omega) = p/\mathbf{P}(\omega)$. Thus, the value of the stochastic discount factor in a particular state of the world is the ratio of the corresponding state price to the probability of the state. If there are infinitely many states of the world, then one can interpret $\tilde{m}$ similarly, though a little more care is obviously needed because individual states may have zero probabilities.

Because $\tilde{m}$ specifies the price of a unit of the consumption good in each state per unit probability, it is also called a "state price density." Another name for $\tilde{m}$ is "pricing kernel." The multiplicity of names (there are even others besides these) is one indicator of of the importance of the concept.

If each p_i is positive, the definition (2.12) of a stochastic discount factor is equivalent to

$$(\forall i) \quad E[\tilde{m}\tilde{R}_i] = 1, \tag{2.13}$$

and this implies

$$(\forall i, j) \quad E[\tilde{m}(\tilde{R}_i - \tilde{R}_j)] = 0. \tag{2.14}$$

Each of these is an important property of a stochastic discount factor. Moreover, if $\tilde{m}$ is a stochastic discount factor, then (2.12)-(2.14) hold for portfolios as well as individual assets. To see this, consider a portfolio consisting of θ_i shares of asset i for each i. Multiplying both sides of (2.12) by θ_i and adding over i implies

$$E[\tilde{m}\tilde{x}] = p, \tag{2.15}$$

where $\tilde{x} = \sum_{i=1}^n \theta_i \tilde{x}_i$ is the payoff of the portfolio and $p = \sum_{i=1}^n \theta_i p_i$ is the price of the portfolio. For every portfolio with a positive price (cost), we therefore have

$$E[\tilde{m}\tilde{R}] = 1, \tag{2.16}$$

where $\tilde{R} = \tilde{x}/p$ is the return of the portfolio.

An asset pricing theory is simply a set of hypotheses that implies some particular form for $\tilde{m}$. Already, we have one asset pricing theory: the first-order condition (2.4) states that

$$u'(\tilde{w}) = \gamma\tilde{m} \tag{2.17}$$

for a stochastic discount factor $\tilde{m}$ and constant γ. We will refine this (add more detail) throughout the book, as well as consider models that do not depend on individual investor optimization.

In the introduction to this chapter, it was noted that asset pricing theory is concerned with explaining the risk premia of different assets. It has now been said that asset pricing theory is about deriving a stochastic discount factor. It is important to understand that these two statements are consistent. Use the fact that the covariance of any two random variables is the expectation of their product minus the product of their expectations to write (2.16) as

$$1 = \text{cov}(\tilde{m}, \tilde{R}) + E[\tilde{m}]E[\tilde{R}]. \tag{2.18}$$

Suppose there is a risk-free asset. Then (2.16) with $\tilde{R} = R_f$ implies $E[\tilde{m}] = 1/R_f$. Substituting this in (2.18) and rearranging gives the following formula for the risk premium of any asset or portfolio with return $\tilde{R}$:

$$E[\tilde{R}] - R_f = -R_f \text{cov}(\tilde{m}, \tilde{R}). \tag{2.19}$$

This shows that risk premia are determined by covariances with any stochastic discount factor.

It is worthwhile to point out one additional implication of the first-order condition (2.4), equivalently (2.17). Concavity of utility implies marginal utility is a decreasing function of wealth. Therefore, the first-order condition (2.17) implies that optimal wealth must be inversely related to a stochastic discount factor $\tilde{m}$. This is intuitive: Investors consume less in states that are more expensive.

2.3 A Single Risky Asset

Returning to the derivation of optimal portfolios, this section will address the special case in which there is a risk-free asset, a single risky asset with return $\tilde{R}$, and no end-of-period endowment ($\tilde{y} = 0$). The investor chooses an amount ϕ to invest in the risky asset, leaving $w_0 - \phi$ to invest in the risk-free asset. This leads to wealth

$$\begin{aligned} \tilde{w} &= \phi\tilde{R} + (w_0 - \phi)R_f \\ &= w_0 R_f + \phi(\tilde{R} - R_f). \end{aligned} \tag{2.20}$$

The first-order condition is

$$E\left[u'(\tilde{w})\left(\tilde{R} - R_f\right)\right] = 0. \tag{2.21}$$

Investment is Positive if the Risk Premium is Positive

Suppose $E[\tilde{R}] \neq R_f$, i.e., the risk premium is nonzero, and the investor has strictly monotone utility. Then it cannot be optimal for him to invest 100% of his wealth in the risk-free asset. If it were, then $\tilde{w}$ would be nonrandom, which means that $u'(\tilde{w})$ could be taken out of the expectation in (2.21), leading to

$$u'(\tilde{w})E[\tilde{R} - R_f]$$

which is nonzero by assumption. Therefore, putting 100% of wealth in the risk-free asset contradicts the first-order condition (2.21).

In fact, if $E[\tilde{R}] > R_f$, then it must be optimal to invest a positive amount in the risky asset, and if $E[\tilde{R}] < R_f$, then it is optimal to short the risky asset. We can see this in either of two ways. Consider the case $E[\tilde{R}] > R_f$. Starting with zero investment in the risky asset and hence a nonrandom w we have

$$E\left[u'(w)\left(\tilde{R} - R_f\right)\right] = u'(w)E\left[\tilde{R} - R_f\right] > 0.$$

As discussed in the previous section, this means that adding some of the portfolio consisting of a unit long position in the risky asset and an equal short position in the risk-free asset will improve utility. This means reducing the investment in the risk-free asset to less than 100% of wealth.

A second way to see that the investor will invest a positive amount in the risky asset when it has a positive risk premium is to write the expectation in (2.21) as

$$E[u'(\tilde{w})]E[\tilde{R} - R_f] + \mathrm{cov}(u'(\tilde{w}), \tilde{R} - R_f). \tag{2.22}$$

The first term in this sum is positive because marginal utility is positive and the risk premium is assumed to be positive. Therefore, the covariance must be negative at the optimal wealth. Because marginal utility is decreasing in wealth (by risk aversion), the covariance is negative only if there is a positive association between $\tilde{w}$ and $\tilde{R}$, which means that the investor must be long the risky asset.

CARA/Normal Example

As an example, we will solve the portfolio choice problem when the risky asset return is normally distributed and the investor has CARA utility. Let μ denote the mean and σ^2 the variance of $\tilde{R}$. Given an amount ϕ invested in the risky asset, the utility is

$$-\exp\left\{-\alpha[w_0 R_f + \phi(\tilde{R} - R_f)]\right\}.$$

The random variable

$$-\alpha[w_0 R_f + \phi(\tilde{R} - R_f)]$$

is normally distributed with mean

$$-\alpha w_0 R_f - \alpha\phi(\mu - R_f)$$

and variance $\alpha^2\phi^2\sigma^2$. Therefore, using the fact that the expectation of the exponential of a normally distributed random variable is the exponential of the mean plus one-half the variance, the expected utility is

$$-\exp\left\{-\alpha w_0 R_f - \alpha\phi(\mu - R_f) + \frac{1}{2}\alpha^2\phi^2\sigma^2\right\}. \tag{2.23}$$

Maximizing (2.23) is equivalent to minimizing the negative of (2.23), which is equivalent to minimizing the exponent. Dividing the exponent by $-\alpha$, we conclude that maximizing the expected utility is equivalent to maximizing

$$w_0 R_f + \phi(\mu - R_f) - \frac{1}{2}\alpha\phi^2\sigma^2. \tag{2.24}$$

Thus, the investor's expected utility depends on his investment ϕ only through the risk premium $\mu - R_f$ and variance σ^2 of the risky asset. This is another example of mean-variance preferences.

Differentiating (2.24) with respect to ϕ and setting the derivative equal to zero yields

$$\phi = \frac{\mu - R_f}{\alpha\sigma^2}. \tag{2.25}$$

Thus, the optimal amount ϕ to invest is an increasing function of the risk premium $\mu - R_f$, a decreasing function of the variance σ^2, and a decreasing function of the investor's absolute risk aversion coefficient α. Note that $\phi > 0$ when the risk premium is positive, as shown more generally before. Also, note that ϕ does not depend on the initial wealth w_0. This is another illustration of the absence of wealth effects discussed in Section 1.5. An investor with CARA utility would invest the same amount in the risky asset whether his initial wealth were \$1,000 or \$1,000,000,000. Obviously, this depends on the assumption that the investor can buy on margin - i.e., short sell the risk-free asset - and there are no minimum margin requirements. However, as will be shown in Sect. 2.4, the absence of wealth effects does not depend on the return being normally distributed.

Decreasing Absolute Risk Aversion

Assuming decreasing absolute risk aversion (which, as noted before, includes CRRA utilities), we can show that a wealthier investor would invest more in the risky asset. We assume the asset has a positive risk premium, implying that the optimal investment is positive. We derive this comparative statics result by differentiating the first-order condition (2.21), where the random wealth $\tilde{w}$ is defined in (2.20), and we assume the optimal investment ϕ is a continuously differentiable function of w_0. Because the first-order condition holds for all w_0, the derivative of (2.21) with respect to w_0 must be zero; thus, using the formula (2.20) to compute the derivative, we have

$$E\left[(\tilde{R}-R_f)u''(\tilde{w})\left\{R_f+(\tilde{R}-R_f)\frac{\mathrm{d}\phi}{\mathrm{d}w_0}\right\}\right].$$

Therefore,

$$R_f E\left[(\tilde{R}-R_f)u''(\tilde{w})\right]+E\left[(\tilde{R}-R_f)^2u''(\tilde{w})\right]\frac{\mathrm{d}\phi}{\mathrm{d}w_0},$$

implying

$$\frac{\mathrm{d}\phi}{\mathrm{d}w_0}=\frac{-R_f E\left[(\tilde{R}-R_f)u''(\tilde{w})\right]}{E\left[(\tilde{R}-R_f)^2u''(\tilde{w})\right]}. \tag{2.26}$$

The denominator in (2.26) is negative, due to risk aversion. Our claim is that the numerator is also negative, leading to $\mathrm{d}\phi/\mathrm{d}w_0 > 0$. This is obviously equivalent to

$$E\left[(\tilde{R}-R_f)u''(\tilde{w})\right]>0, \tag{2.27}$$

which will be established below.

It may be surprising that (2.27) can be true. The second derivative is negative due to risk aversion and we are assuming the risk premium $E[\tilde{R}-R_f]$ is positive, so one might think the expectation in (2.27) should be negative. We can write the expectation as

$$E\left[\tilde{R}-R_f\right]E[u''(\tilde{w}]+\operatorname{cov}(\tilde{R}-R_f,u''(\tilde{w})).$$

As we have just explained, the first term is negative. Thus, for (2.27) to be true, the covariance must be positive. Keeping in mind that $u'' < 0$, this means that large values of $\tilde{R} - R_f$ must correspond to values of $u''(\tilde{w})$ that are close to zero, and small or negative values of $\tilde{R} - R_f$ must correspond to values of $u''(\tilde{w})$ that are large in absolute value. In other words, there must be less concavity ($u''(\tilde{w})$ closer to zero) when the return $\tilde{R}$, and hence the wealth $\tilde{w}$, is larger. This is precisely what we are assuming - decreasing absolute risk aversion.

To prove (2.27), define $w_f = w_0 R_f$ (the wealth level when $\tilde{R} = R_f$) and substitute

$$\begin{aligned} u''(\tilde{w}) &= -\alpha(\tilde{w})u'(\tilde{w}) \\ &= -\alpha(w_f)u'(\tilde{w}) + [\alpha(w_f) - \alpha(\tilde{w})]u'(\tilde{w}) \end{aligned}$$

in the left-hand side of (2.27) to obtain

$$-\alpha(w_f)E\left[(\tilde{R} - R_f)u'(\tilde{w})\right] + E\left[[\alpha(w_f) - \alpha(\tilde{w})](\tilde{R} - R_f)u'(\tilde{w})\right].$$

The first term in this expression is zero, due to the first-order condition (2.21). The second term is positive because

$$[\alpha(w_f) - \alpha(\tilde{w})](\tilde{R} - R_f)$$

is everywhere positive, due to $\tilde{w}$ being greater than w_f whenever $\tilde{R} > R_f$ and the assumption that absolute risk aversion is a decreasing function of wealth. Therefore, (2.27) holds.

2.4 Linear Risk Tolerance

We showed in the previous section that the amount a CARA investor will invest in a single normally-distributed asset is independent of his initial wealth. We will consider here the more general case of multiple risky assets with general returns (i.e., not necessarily normally distributed) and preferences with linear risk tolerance. We want to see how the optimal investments depend on the initial wealth. Suppose that there is a risk-free asset and that the utility function has risk tolerance $\tau(w) = A + Bw$. Continue to assume there is no end-of-period endowment ($\tilde{y} = 0$).

Let ϕ_i denote the optimal investment in risky asset i, and let ϕ denote the total investment in risky assets, so $w_0 - \phi$ is the amount invested in the risk-free asset. We will show that ϕ_i/ϕ is independent of w_0 and independent of A. Thus, if the economy were populated by investors with linear risk tolerance and with the same B coefficient, then all investors would hold the same portfolio of risky assets: If someone invests twice as much in stock i as in stock j, then all other investors do the same. In this circumstance, the market value of stock i must be twice as much as that of stock j. More generally, all investors must hold the market portfolio of risky assets. This is an example of "two-fund separation," which means that all investors allocate their wealth across two funds, in this case the risk-free asset and the market portfolio of risky assets.

It is slightly more convenient here to let n denote the number of risky assets, so there are $n+1$ assets including the risk-free asset, and $\phi = \sum_{i=1}^n \phi_i$. Set

$$\xi_i = \frac{\phi_i}{A + BR_f w_0} \tag{2.28}$$

and $\xi = \sum_{i=1}^n \xi_i$. Clearly, $\phi_i/\phi = \xi_i/\xi$. We will show that ξ_i is independent of w_0 and A, which implies the same of ξ and therefore the same of ϕ_i/ϕ.

We can write (2.28) as

$$\phi_i = \xi_i A + \xi_i BR_f w_0, \tag{2.29}$$

showing that the optimal investment in each risky asset is an affine (constant plus linear) function of initial wealth w_0. Special cases of (2.29) are

$$\text{CARA utility:} \quad B = 0, \text{ so } \phi_i = \xi_i A, \tag{2.29a}$$

$$\text{CRRA utility:} \quad A = 0, \text{ so } \frac{\phi_i}{w_0} = \xi_i BR_f. \tag{2.29b}$$

The case of CARA utility is of course the case considered in the previous section, and, as in the previous section, (2.29a) shows that the optimal amount to invest in each risky asset is independent of initial wealth. However, we allow for multiple risky assets and non-normal return distributions. For CRRA utility, (2.29b) asserts that the optimal fraction of initial wealth to invest in each risky asset is independent of initial wealth.

The remainder of this section consists of the proof that ξ_i defined in (2.28) is independent of w_0 and A.

The wealth achieved by the investor is

$$\begin{aligned} \tilde{w} &= \left(w_0 - \sum_{i=1}^n \phi_i\right) R_f + \sum_{i=1}^n \phi_i \tilde{R}_i \\ &= w_0 R_f + \sum_{i=1}^n \phi_i (\tilde{R}_i - R_f). \end{aligned} \tag{2.30}$$

For negative exponential utility, we can write the expected utility as

$$-e^{-\alpha w_0 R_f} E\left[\exp\left\{-\alpha \sum_{i=1}^n \phi_i (\tilde{R}_i - R_f)\right\}\right].$$

To maximize this expected utility is equivalent to maximizing

$$-E\left[\exp\left\{-\alpha \sum_{i=1}^n \phi_i (\tilde{R}_i - R_f)\right\}\right].$$

Substituting $\xi_i = \phi_i/A = \alpha\phi_i$, the optimization problem is to maximize

$$-E\left[\exp\left\{-\sum_{i=1}^n \xi_i(\tilde{R}_i - R_f)\right\}\right],$$

and this does not depend on w_0 or A; hence, ξ_i is independent of w_0 and A.

For CRRA utility, define $\pi_i = \phi_i/w_0 = BR_f\xi_i$, and write the wealth (2.30) as

$$\tilde{w} = w_0\left[R_f + \sum_{i=1}^n \pi_i(\tilde{R}_i - R_f)\right]. \tag{2.31}$$

For logarithmic utility, the expected utility equals

$$\log w_0 + \log\left[R_f + \sum_{i=1}^n \pi_i(\tilde{R}_i - R_f)\right],$$

and maximizing this is equivalent to maximizing

$$\log\left[R_f + \sum_{i=1}^n \pi_i(\tilde{R}_i - R_f)\right].$$

This optimization problem does not depend on w_0, so the optimal π_i and hence ξ_i do not depend on w_0. For power utility, the expected utility equals

$$w_0^{1-\rho}\frac{1}{1-\rho}\left[R_f + \sum_{i=1}^n \pi_i(\tilde{R}_i - R_f)\right]^{1-\rho}.$$

Maximizing this is equivalent to maximizing the same thing without the constant factor $w_0^{1-\rho}$, an optimization problem that does not depend on w_0, so we conclude that ξ_i is independent of w_0 for power utility also.

Now consider shifted logarithmic and shifted power utility, recalling that the risk tolerance is

$$\tau(w) = \frac{w-\zeta}{\rho},$$

so $A = -\zeta/\rho$ and $B = 1/\rho$, with $\rho = 1$ for log utility. It is convenient to solve these portfolio choice problems in two steps: first invest ζ/R_f in the risk-free asset, and then invest $w_0 - \zeta/R_f$ optimally in the risk-free and risky assets. This is without loss of generality, because the amount invested in the risk-free asset in the first step can be disinvested in the second step if this is optimal. The first investment produces ζ, so the total wealth achieved is

$$\tilde{w} = \zeta + \left(w_0 - \frac{\zeta}{R_f}\right)\tilde{R},$$

where $\tilde{R}$ denotes the return on the second investment. We can write this as

$$\tilde{w} = \zeta + \left(w_0 - \frac{\zeta}{R_f}\right)\left[R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f)\right],$$

where we define

$$\pi_i = \frac{\phi_i}{w_0 - \zeta/R_f} = BR_f\xi_i. \tag{2.32}$$

This implies that the utility achieved is, for shifted log,

$$\log\left[\left(w_0 - \frac{\zeta}{R_f}\right)\left[R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f)\right]\right].$$

or, for shifted power,

$$\frac{1}{1-\rho}\left[\left(w_0 - \frac{\zeta}{R_f}\right)\left[R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f)\right]\right]^{1-\rho}.$$

In either case, the logic of the previous paragraph leads to the conclusion that the optimal π_i and hence ξ_i are independent of w_0 and A.

2.5 Multiple Asset CARA-Normal Example

To illustrate the result of the previous section, consider a CARA investor who chooses among multiple normally distributed assets. We continue to assume there is no end-of-period endowment. Suppose there is a risk-free asset with return R_f and n risky assets with returns $\tilde{R}_i$ that are joint normally distributed. Let $\tilde{R}^{\text{vec}}$ denote the n-dimensional column vector with $\tilde{R}_i$ as its i-th element, μ the vector of expected returns (the n-dimensional column vector with i-th element $E[\tilde{R}_i]$), and $\mathbf{1}$ the n-dimensional column vector of ones. Let ϕ_f denote the investment in the risk-free asset, let ϕ_i denote the investment in risky asset i, and let ϕ denote the n-dimensional column vector with ϕ_i as its i-th element.

The budget constraint of the investor is

$$\phi_f + \sum_{i=1}^{n} \phi_i = w_0,$$

where w_0 is the given initial wealth. We can also write this as

$$\phi_f = w_0 - \mathbf{1}'\phi,$$

where ' denotes the transpose operator. The end-of-period wealth is

$$\begin{aligned} \phi_f R_f + \sum_{i=1}^n \phi_i \tilde{R}_i &= \phi_f R_f + \phi' \tilde{R}^{\text{vec}} \\ &= w_0 R_f + \phi'(\tilde{R}^{\text{vec}} - R_f \mathbf{1}), \end{aligned}$$

and the expected end-of-period wealth is

$$w_0 R_f + \phi'(\mu - R_f \mathbf{1}).$$

Let Σ denote the covariance matrix of the risky asset returns. The (i, j)-th element of Σ is $\text{cov}(\tilde{R}_i, \tilde{R}_j)$. Of course, the diagonal elements are variances. In matrix notation, Σ is given by $E[(\tilde{R}^{\text{vec}} - \mu)(\tilde{R}^{\text{vec}} - \mu)']$. The variance of end-of-period wealth is

$$\text{var}\left(\sum_{i=1}^n \phi_i \tilde{R}_i\right) = \phi' \Sigma \phi.$$

To see this, note that, because the square of a scalar equals the scalar multiplied by its transpose, the variance is

$$\begin{aligned} E\left[(\phi'(\tilde{R}^{\text{vec}} - \mu))^2\right] &= E\left[\phi'(\tilde{R}^{\text{vec}} - \mu)(\tilde{R}^{\text{vec}} - \mu)'\phi\right] \\ &= \phi' E\left[(\tilde{R}^{\text{vec}} - \mu)(\tilde{R}^{\text{vec}} - \mu)'\right]\phi \\ &= \phi' \Sigma \phi. \end{aligned}$$

Assume Σ is nonsingular, which is equivalent to there being "no redundant asset."

As was the case with a single normally-distributed risky asset, maximizing expected CARA utility with multiple normally distributed assets is equivalent to solving a mean-variance problem: choose ϕ to maximize

$$(u - R_f \mathbf{1})'\phi - \frac{1}{2}\alpha \phi' \Sigma \phi.$$

Differentiating with respect to ϕ and equating the derivative to zero produces:

$$u - R_f \mathbf{1} - \alpha \Sigma \phi = 0$$

with solution

$$\phi = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f \mathbf{1}). \tag{2.33}$$

This is a straightforward generalization of the formula (2.25) for the optimal portfolio of a CARA investor with a single normally-distributed asset. As asserted in (2.29a), the optimal investments are independent of initial wealth. As will be seen in Section 5.5, the portfolio $\Sigma^{-1}(\mu - R_f \mathbf{1})$ has a special significance in mean-variance analysis even when asset returns are not normally distributed.

The formula (2.33) implies the single-asset formula (2.25) if the return of the asset is independent of all other asset returns. For such an asset i, (2.33) implies

$$\phi_i = \frac{\mu_i - R_f}{\alpha\sigma_i^2}, \tag{2.34}$$

where σ_i^2 is the variance of $\tilde{R}_i$. In general, (2.33) states that the demand for each asset i depends on the entire vector of risk premia and the covariances between asset i and the other assets.

2.6 Mean-Variance Preferences

We have seen two examples of mean-variance preferences: normal returns with CARA utility in the previous section, and quadratic utility in Section 1.7. In this section, we ask: What conditions imply that investors rank portfolios based on the means and variances of their payoffs?

Assume there are no end-of-period endowments, so the end-of-period wealth of an investor is the payoff of his portfolio. Regardless of the distribution of the asset payoffs, an investor with quadratic utility $u(w) = -\frac{1}{2}(w-\zeta)^2$ will choose portfolios based on mean and variance to maximize

$$\zeta E[\tilde{w}] - \frac{1}{2}E[\tilde{w}]^2 - \frac{1}{2}\text{var}(\tilde{w}),$$

as discussed in Section 1.7. An alternative question is: For what payoff distributions will all investors, regardless of their utility functions, choose portfolios based on mean and variance?

Let now $\tilde{x}_1, ..., \tilde{x}_n$ denote the payoffs of the risky assets (meaning all asset payoffs if there is no risk-free asset, and the payoffs of the assets other than the risk-free asset if there is a risk-free asset) and let $\tilde{x}$ denote the column vector with $\tilde{x}_i$ as its i-th component. A sufficient condition for portfolios to be chosen based on mean and variance is that $\tilde{x}$ have a multivariate normal distribution. In this circumstance, each portfolio payoff is a linear combination of joint normally distributed variables and therefore has a normal distribution. Moreover, a normal distribution is entirely characterized by its mean and variance. Thus, if portfolios θ and ψ have payoffs with the same mean and variance, then the payoffs of θ and ψ have the exact same distribution, and all investors must be indifferent between them. Only mean and variance can matter if payoffs are normally distributed.

In general, all investors will choose portfolios based on mean and variance if and only if the distributions of portfolio payoffs are completely characterized by their means and variances, as with normal distributions. If there is a risk-free asset, a necessary and sufficient condition for this to be the case is that $\tilde{x}$ have an "elliptical distribution." If $\tilde{x}$ has a density function, then it is said to be elliptically distributed if there is a positive definite matrix Σ

and a vector μ such that the density function is constant on each set

$$\{x : (x-\mu)'\Sigma^{-1}(x-\mu) = a\}$$

for $a > 0$. These sets are ellipses. If the $\tilde{x}_i$ have finite variances, then μ is the vector of means and Σ is the covariance matrix of $\tilde{x}$. The class of elliptical distributions includes distributions that are bounded (and hence can satisfy limited liability) and distributions with "fat tails" (and hence may match empirical returns better than normal distributions do). It also includes distributions that do not have finite means and variances, in which case μ is interpreted as a location parameter and Σ as a scale parameter, and investors have "location-scale preferences."

If there are end-of-period endowments, then investors will typically care about how portfolios hedge or exacerbate their endowment risk. For example an investor with quadratic utility will choose a portfolio with return $\tilde{R}$ that maximizes

$$\zeta E[w_0\tilde{R}+\tilde{y}] - \frac{1}{2}E[w_0\tilde{R}+\tilde{y}]^2 - \frac{1}{2}\text{var}(w_0\tilde{R}+\tilde{y}).$$

The variance here equals

$$w_0^2\text{var}(\tilde{R}) + \text{var}(\tilde{y}) + 2w_0\text{cov}(\tilde{R},\tilde{y}).$$

Therefore, the covariance between the portfolio return and the endowment will affect the investor's expected utility, implying that the investor cares about more than just the mean and variance of the portfolio return. A similar example with normal returns and CARA utility is in Problem 2.5.

2.7 Beginning-of-Period Consumption

Consider now the problem of choosing consumption optimally at the beginning of the period in addition to choosing an optimal portfolio. Call the beginning of the period "date 0" and the end of the period "date 1." Now let w_0 denote the beginning-of-period wealth before consuming. This includes the value of any shares held plus any date 0 endowment. Letting $v(c_0, c_1)$ denote the utility function, the choice problem is:

$$\text{maximize} \quad E[v(c_0,\tilde{c}_1)] \tag{2.35}$$

$$\text{subject to } c_0 + \sum_{i=1}^n \theta_i p_i = w_0 \text{ and } (\forall\omega)\ \tilde{c}_1(\omega) = \tilde{y}(\omega) + \sum_{i=1}^n \theta_i\tilde{x}_i(\omega).$$

Substituting in the second constraint, the Lagrangean for this problem is

$$E\left[v\left(c_0, \tilde{y} + \sum_{i=1}^n \theta_i\tilde{x}_i\right)\right] - \gamma\left(c_0 + \sum_{i=1}^n \theta_i p_i - w_0\right),$$

and the first-order conditions are:

$$E\left[\frac{\partial}{\partial c_0}v(c_0,\tilde{c}_1)\right] = \gamma, \tag{2.36}$$

$$(\forall i) \quad E\left[\frac{\partial}{\partial c_1}v(c_0,\tilde{c}_1)\tilde{x}_i\right] = \gamma p_i. \tag{2.37}$$

These are equivalent to:

$$(\forall i) \quad E\left[\frac{\partial}{\partial c_1}v(c_0,\tilde{c}_1)\tilde{x}_i\right] = p_i E\left[\frac{\partial}{\partial c_0}v(c_0,\tilde{c}_1)\right]. \tag{2.38}$$

As before, these are necessary conditions for optimality provided it is feasible, starting from the optimal portfolio, to add a little or subtract a little of asset i.

The new feature, relative to the problem considered previously in this chapter, is that we have an explicit formula for the Lagrange multiplier γ. By substituting (2.36) into (2.37), we obtain

$$(\forall i) \quad E[\tilde{m}\tilde{x}_i] = p_i. \tag{2.39}$$

where

$$\tilde{m} = \frac{\partial v(c_0,\tilde{c}_1)/\partial c_1}{E[\partial v(c_0,\tilde{c}_1)/\partial c_0]}. \tag{2.40}$$

This looks more complicated than the corresponding formula (2.17), but it says the same thing: The marginal utility of end-of-period wealth is proportional to a stochastic discount factor. The only difference is that here we have an explicit formula for the constant of proportionality.

2.8 Time-Additive Utility

To obtain strong results in the model with optimal beginning-of-period consumption, we can assume the investor has time-additive utility, meaning that there are functions u_0 and u_1 such that $v(c_0,c_1) = u_0(c_0) + u_1(c_1)$. In this circumstance, we have

$$\frac{\partial}{\partial c_0}v(c_0,c_1) = u_0'(c_0) \quad \text{and} \quad \frac{\partial}{\partial c_1}v(c_0,c_1) = u_1'(c_1).$$

Therefore, the stochastic discount factor $\tilde{m}$ in (2.40) is

$$\tilde{m} = \frac{u_1'(\tilde{c}_1)}{u_0'(c_0)}. \tag{2.41}$$

Thus, with time-additive utility, the investor's marginal rate of substitution between date 0 consumption and date 1 consumption is a stochastic discount factor. The first-order conditions (2.38) with time-additive utility can be expressed as:

$$(\forall i) \quad E[u_1'(\tilde{c}_1)\tilde{x}_i] = p_i u_0'(c_0). \tag{2.42}$$

This equation is called the "Euler equation."

A leading special case is when the functions u_0 and u_1 are the same except for a discounting of future utility u_1. So suppose there is a function u and discount factor $0 < \delta < 1$ such that $u_0 = u$ and $u_1 = \delta u$. Then the stochastic discount factor $\tilde{m}$ in (2.40) is

$$\tilde{m} = \frac{\delta u'(\tilde{c}_1)}{u'(c_0)}. \tag{2.43}$$

As was remarked, time-additive utility will lead to strong results. For example, in continuous time it will produce the Consumption-based Capital Asset Pricing Model (CCAPM) of Breeden (1979). However, it is also a strong assumption. In particular, it links the way an investor trades of consumption at different dates with the investor's tolerance for risk, which one might think should be distinct aspects of an investor's preferences. The precise statement of the link is that the "elasticity of intertemporal substitution" equals the reciprocal of the coefficient of relative risk aversion.