AERO3600 — Embedded Control Systems

Practice problems: Continuous-time state-feedback control design[1]

Important

This document proposes several practice problem on state-feedback control desing for continuous time systems.

Problem 1

Consider the system

$$\dot{x}_1 = -3x_1 + x_2 + u$$
$$\dot{x}_2 = -2x_1 + 1.5x_2 + 4u.$$

a) Is the system completely controllable? Justify your answer.

Check your solutions with MATLAB.

Problem 2

Consider four different LTI system written in the form

$$\dot{x} = A_i\, x + B_i\, u$$

where $i = \{1, 2, 3, 4\}$ indicate the index of the system i. The corresponding controllability matrices of the systems are as follows

$$Co_1 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \quad Co_2 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad Co_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Co_4 = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}.$$

a) Determine the dimension of the matrices A_i and B_i that corresponde to system i.

b) Analyse the controllability of each system. Justify your answer.

Check your solutions with MATLAB.

Problem 3

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

a) Is the system internally stable?

b) Design a zero-state regulator controller with desired closed-loop eigenvalues at $\lambda_{1,2} = -1.8 \pm j2.4$.

c) Draw the block-diagram of the control system.

Check your solutions with MATLAB.

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[1]Updated: 5 May 2021.

Problem 4

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

a) Is the system controllable? Justify your answer

b) Find the controller that results in closed-loop eigenvalues $\lambda_1 = \lambda_2 = -1$.

Check your solutions with MATLAB.

Problem 5

Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

a) Find the state transformation T that takes the system into the companion form.

b) Find the controller that results in closed-loop eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

c) Find the set-point regulator that takes the output y to a desired constant output $\bar{y}$, that is

$$\lim_{t \to \infty} y = \bar{y}$$

Check your solutions with MATLAB.

Problem 6

For each of the following systems, use linearisation to design a state-feedback controller to stabilise the origin. Select the desired closed-loop eigenvalues to achieve a particular dynamic response, for example certain overshoot and time response.

1.

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= 3x_1^2 x_2 + x_1 + u \\ y &= -x_1^3 + x_2 \end{aligned}$$

2.

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_1 x_2^2 - x_1 + x_3 \\ \dot{x}_3 &= u \\ y &= -x_1^3 + x_2 \end{aligned}$$

3.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_1 x_3 + u \\ \dot{x}_3 &= x_1 + x_1 x_2 - 2x_3 \\ y &= x_1 \end{aligned}$$

Check your solutions with MATLAB.

Problem 7

A simplified model of a low-frequency motion of a ship can be written as follows

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10x_2 + 0.1u\\ y &= x_1\end{aligned}$$

where x_1 is the heading angle (in radians), x_2 is the heading angular rate and u is the rudder angle.

1. Find the equilibrium point and input that correspond to $\bar{y} = \frac{\pi}{4}$.
2. Use linearisation to design a state feedback controller to ensure that the output y tracks the reference values $\bar{y}$ and such that the closed loop eigenvalues are -1 and -1.
3. Design an integral action that ensure that the output tracks the reference $\bar{y}$. The desired eigenvalues of the closed loop are -1 -1 -15.

Check your solutions with MATLAB.

Problem 8

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u.$$

a) Determine if the system is controllable. Justify your answer.

b) Find the gain of the controller $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x$, such that the closed loop eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$.

c) Determine if it possible to arbitrarily assign the closed loop eigenvalues, that is, can we assign the values λ_1 and λ_2 arbitrarily?

d) Consider now the controller $u = -\begin{bmatrix} k_3 & 3k_3 \end{bmatrix} x$. Determine the gain k_3 such that one of the closed loop eigenvalues $\lambda_1 = -1$. Determine if it is possible to arbitrarily assign the second closed loop eigenvalue λ_2.

e) Consider the controller in d). Determine if the closed loop is stable. Justify your answer.

Check your solutions with MATLAB.

Problem 9

Consider the dynamics of the magnetic levitation system[2]

$$\begin{aligned}35\ddot{\phi} + 4\ddot{\theta} - 100\sin(\phi) &= -\tau\\ 4\ddot{\phi} + \ddot{\theta} - 4\sin(\phi)\dot{\phi}^2 &= \tau\\ y_1 &= \phi\\ y_2 &= \theta\end{aligned}$$

where ϕ is the absolute angle of the body and θ is angle of the ball.

1. Write the system in state space form with $x_1 = \phi$, $x_2 = \theta$, $x_3 = \dot{\phi}$ and $x_4 = \dot{\theta}$.
2. Find the equilibrium point and input that correspond to $\bar{x}_1 = 0$ and $\bar{x}_2 = 0$.
3. Use linearisation to design a state feedback controller to ensure that the output y tracks the reference values $\bar{y}$ and such that the closed loop eigenvalues are -10, -10, -15 and -15.
4. Design an integral action that ensures that the outputs track the reference values $\bar{y}_1 = 0$ and $\bar{y}_2 = 4$. If possible, the desired eigenvalues of the closed loop should be -10, -10, -15, -15, -5 and -5.

Solve numerically using MATLAB and simulate the closed loop.

[2] See for example the article *Balancing and Transferring Control of a Ball Segway Using a Double-Loop Approach* in IEEE Control Systems Magazine, vol. 38, no. 2, pp. 15-37, 2018, D. B. Pham, H. Kim, J. Kim and S. Lee.

Problem 10

Consider the dynamics of the robot known as Ballbot system[3]

$$\begin{aligned} 35\ddot{\phi} + 4\ddot{\theta} - 100\sin(\phi) &= -\tau \\ 4\ddot{\phi} + \ddot{\theta} - 4\sin(\phi)\dot{\phi}^2 &= \tau \\ y_1 &= \phi \\ y_2 &= \theta \end{aligned}$$

where ϕ is the absolute angle of the body and θ is angle of the ball.

1. Write the system in state space form with $x_1 = \phi$, $x_2 = \theta$, $x_3 = \dot{\phi}$ and $x_4 = \dot{\theta}$.
2. Find the equilibrium point and input that correspond to $\bar{x}_1 = 0$ and $\bar{x}_2 = 0$.
3. Use linearisation to design a state feedback controller to ensure that the output y tracks the reference values $\bar{y}$ and such that the closed loop eigenvalues are -10, -10, -15 and -15.
4. Design an integral action that ensures that the outputs track the reference values $\bar{y}_1 = 0$ and $\bar{y}_2 = 4$. If possible, the desired eigenvalues of the closed loop should be -10, -10, -15, -15, -5 and -5.

Solve numerically using MATLAB and simulate the closed loop.

Additional exercises

You can find additional exercises in the chapter 11—the Design of State variable feedback systems—of the book *Modern Control Systems* by R. C. Dorf and R. H. Bishop. The book is available at UON Library.

[3]See for example the article *Balancing and Transferring Control of a Ball Segway Using a Double-Loop Approach* in IEEE Control Systems Magazine, vol. 38, no. 2, pp. 15-37, 2018, D. B. Pham, H. Kim, J. Kim and S. Lee.

Answer to a some practice problems

(state-space feedback Control design)

Problem 1

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \; ; \; C_{AB} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$$

$\mathrm{rank}(C_{AB}) = 1$ $\longrightarrow$ The matrix C_{AB} is square (2-by-2) and its determinant is zero $\Rightarrow \mathrm{rank}(C_{AB}) = 1$

Also notice that the columns of C_{AB} are linearly dependent.

(see the note on mathematical review for the definition of rank of a matrix)

Since $\mathrm{rank}(C_{AB})$ is smaller than the number of the states $\Rightarrow$ the system is not completely controllable

Problem 2

First notice the that for a system $\dot{x} = Ax + Bu$, with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $u \in \mathbb{R}^m$, where n is the number of states and m is number of inputs, the controllability matrix is.

$$C_o = [\underbrace{B}_{n\times m} \quad \underbrace{AB}_{n\times m} \quad \underbrace{A^2B}_{n\times m} \cdots \underbrace{A^{n-1}B}_{n\times m}] \Rightarrow C_o \in \mathbb{R}^{n\times nm}$$

, that is the matrix C_o is a n-by-nm matrix

then for.

- System 1: $C_{o1} \in \mathbb{R}^{2\times 2} \Rightarrow A_1 \in \mathbb{R}^{2\times 2}$; $B_1 \in \mathbb{R}^{2\times 1}$; $\left.\begin{matrix}\mathrm{rank}(C_{o1}) = 2 \\ \text{n° of states} = 2\end{matrix}\right\} \Rightarrow$ System 1 is completely controllable.

- System 2. $C_{o2} \in \mathbb{R}^{2\times 2} \Rightarrow A_2 \in \mathbb{R}^{2\times 2}$; $B_2 \in \mathbb{R}^{2\times 1}$; $\left.\begin{matrix}\mathrm{rank}(C_{o2}) = 1 \\ \text{n° of states} = 2\end{matrix}\right\} \Rightarrow$ System 2 is not completely controllable

- System 3: $C_{o3} \in \mathbb{R}^{3\times3} \Rightarrow A_3 \in \mathbb{R}^{3\times3}$; $B_3 \in \mathbb{R}^{3\times1}$; $\left.\begin{array}{l}\text{rank}(C_{o3}) = 3 \\ \text{n° of states} = 3\end{array}\right\} \Rightarrow$ System 3 is Completelly Controllable.

- System 4: $C_{o3} \in \mathbb{R}^{2\times4} \Rightarrow A_4 \in \mathbb{R}^{2\times2}$, $B_4 \in \mathbb{R}^{2\times2}$; $\left.\begin{array}{l}\text{rank}(C_{o4}) = 2 \\ \text{n° of states} = 2\end{array}\right\} \Rightarrow$ System 4 is Completelly Controllable.

$\boxed{\text{Problem 3}}$

$$A = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \; ; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

a) Eigenvalues of A $\rightarrow$ $\det[\lambda I - A] = \det\begin{bmatrix} \lambda & -1 \\ -20.6 & \lambda \end{bmatrix} = \lambda^2 - 20.6 \rightarrow \lambda_{1,2} = \pm\sqrt{20.6}$

the system is internally unstable because A has a positive eigenvalue.

b) Desired Characteristic polynomial

$$(s+1.8+j2.4)(s+1.8-j2.4) = s^2 + 1.8s - \cancel{j2.4s} + 1.8s + 1.8^2 - \cancel{j2.4\cdot1.8} + \cancel{j2.4s}$$
$$\cancel{+j2.4\cdot1.8} + 2.4^2$$
$$\boxed{= s^2 + 3.6s + 9} \quad (*)$$

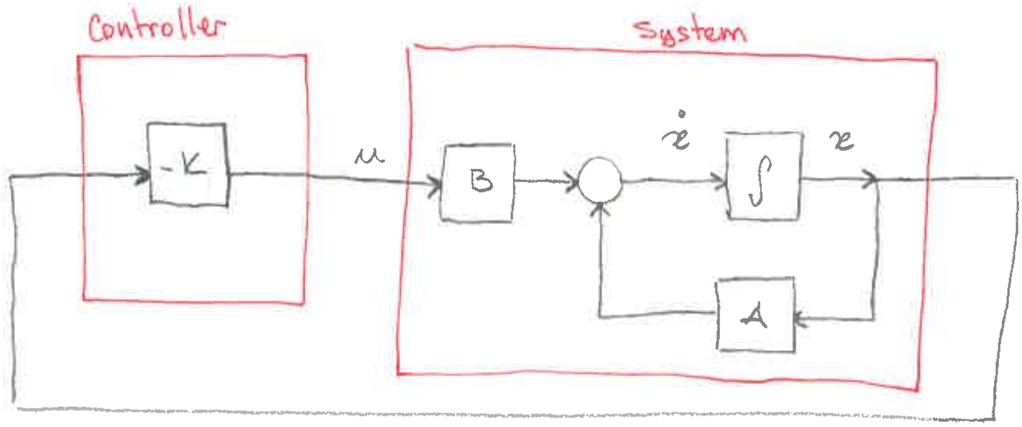
$$K = [k_0 \quad k_1] \Rightarrow (A - BK) = \begin{bmatrix} 0 & 1 \\ -(-20.6+k_0) & -k_1 \end{bmatrix}$$

$$\text{Char}(A-BK) = \boxed{s^2 + k_1 s + (-20.6+k_0)} \quad (**)$$

From $(*)$ and $(**)$ $\Rightarrow$ $\boxed{k_1 = 3.6}$ $\boxed{k_0 = 29.6}$

$$\boxed{K = [29.6 \quad 3.6]} \qquad \boxed{u = -K \cdot x}$$

c)



Problem 4

a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $\left.\begin{matrix} \text{rank}(C_{AB}) = 2 \\ n^\circ \text{ of states} = 2 \end{matrix}\right\} \Rightarrow$ the system is Completely Controllable

b) . Desired characteristic polynomial

$$(s+1)(s+1) = s^2 + 2s + 1$$

the system is in companion form, then with $K = [k_0 \quad k_1]$

$$(A - BK) = \begin{bmatrix} 0 & 1 \\ -k_0 & -k_1 \end{bmatrix} \rightarrow \text{Char}(A-BK) = s^2 + k_1 s + k_0$$

Matching the desired char. polynomial and char(A-BK) $\Rightarrow$ $\boxed{k_0 = 1 \; ; \; k_1 = 2}$

$$\boxed{u = -[1 \quad 2]\, x}$$

Problem 5

a) $A = \begin{bmatrix} 1 & 1 \\ -4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $C_{AB} = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix}$; $\left.\begin{matrix} \text{rank}(C_{AB}) = 2 \\ n^\circ \text{ of states} = 2 \end{matrix}\right\} \Rightarrow$ the system is Completely Controllable

$$\text{Char}(A) = \det(sI - A) = \det\begin{bmatrix} s-1 & -1 \\ 4 & s-3 \end{bmatrix} = s^2 - 4s + 7$$

then, the companion form is

$$A_c = \begin{bmatrix} 0 & 1 \\ -7 & 4 \end{bmatrix} \; ; \; B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \qquad C_{A_cB_c} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}$$

$\exists \; T \; / \; z = T^{-1}x \rightarrow \dot{z} = A_c z + B_c u$, with $A_c = T^{-1}AT$
$B_c = T^{-1}B$

$$C_{A_cB_c} = \left[B_c \quad A_cB_c \right] = \left[T^{-1}B \quad T^{-1}ATT^{-1}B \right] = T^{-1}\left[B \quad AB \right] = T^{-1}C_{AB} \Rightarrow$$

$$\Rightarrow \boxed{T = C_{AB}\, C_{A_cB_c}^{-1}} \quad \text{and} \quad \boxed{T^{-1} = C_{A_cB_c} \cdot C_{AB}^{-1}}$$

$$C_{A_cB_c}^{-1} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}}$$

b)

Desired characteristic polynomial $\lambda_1 = -1 \; ; \; \lambda_2 = -2$

$$(s+1)(s+2) = s^2 + 3s + 2$$

Desig using the companion form for the state space model $\dot{z} = A_c z + B_c u \Rightarrow$

$u = -K_z z \;$; with $K_z = \left[k_{z0} \quad k_{z1} \right]$.

$$(A_c - B_c K_z) = \begin{bmatrix} 0 & 1 \\ -(7+k_{z0}) & -(-4+k_{z1}) \end{bmatrix} \; ; \; \text{Char}(A_c - B_c K_z) = s^2 + (-4+k_{z1})s + (7+k_{z0})$$

$$\Rightarrow \left. \begin{aligned} 7 + k_{z0} &= 2 \\ -4 + k_{z1} &= 3 \end{aligned} \right\} \Rightarrow \begin{aligned} k_{z0} &= -5 \\ k_{z1} &= 7 \end{aligned} \Rightarrow \boxed{K_z = [-5 \quad 7]} \rightarrow \boxed{u = -K_z z}$$

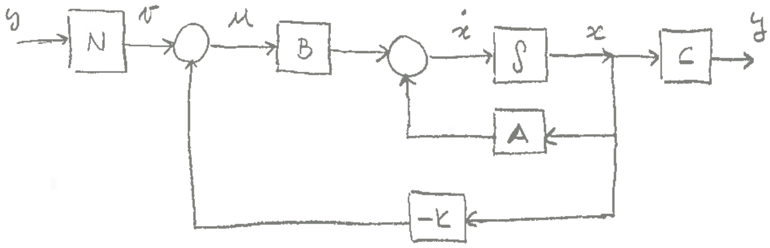
However, the controller has to be expressed in the original states

$$\boxed{u = -K_z T^{-1} x = -[-5 \quad 7] \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} x = -\left[1 \quad \tfrac{7}{2} \right] x}$$

$$\boxed{K = \left[1 \quad \tfrac{7}{2} \right]}$$

c)

Alternative solution



$\dot{x} = (A - BK)x + BN\bar{y}$
$y = Cx$
$\longrightarrow$ new states
$z \triangleq x + (A-BK)^{-1}BN\bar{y}$
$y \triangleq C\left[z - (A-BK)^{-1}BN\bar{y}\right]$

$\dot{z} = \dot{x} = (A-BK)x + BN\bar{y} = (A-BK)z$

The system in states z is

$$\begin{cases} \dot{z} = (A-BK)z \\ y = Cz - C(A-BK)^{-1}BN\bar{y} \end{cases}$$

Since $\text{eig}(A-BK) \begin{cases} -1 \\ -2 \end{cases}$, then the system is stable and thus $\lim_{t\to\infty} z = 0$, which implies that

$\lim_{t\to\infty} y = -C(A-BK)^{-1}BN\bar{y}$. On the other side, it is required that $\lim_{t\to\infty} y = \bar{y}$.

therefore $C(A-BK)^{-1}BN = -1 \Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -6 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} N = -1 \Rightarrow$

$\Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & -\frac{1}{2} \\ 3 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} N = -1 \Rightarrow -1\,N = -1 \Rightarrow \boxed{N = 1}$