AERO3600 — Embedded Control Systems

Practice problems: Continuous-time output-feedback control design[1]

Important

This document proposes several practice problem on observer design and output-feedback control design for continuous-time systems.

Problem 1

Consider the system

$$\begin{aligned}\dot{x}_1 &= -3x_1 + x_2 + u\\ \dot{x}_2 &= -2x_1 + 1.5x_2 + 4u.\\ y &= -x_1 + 2x_2\end{aligned}$$

a) Is the system completely observable? Justify your answer.

Check your solutions with MATLAB.

Problem 2

Consider four different LTI system written in the form

$$\begin{aligned}\dot{x} &= A_i\, x + B_i\, u\\ y &= C_i\, x\end{aligned}$$

where $i = \{1, 2, 3, 4\}$ indicate the index of the system i. The corresponding observability matrices of the systems are as follows

$$Ob_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad Ob_2 = \begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}, \quad Ob_3 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}, \quad Ob_4 = \begin{bmatrix}1 & 0\\ 0 & 1\\ -2 & 0\\ 0 & -2\end{bmatrix}.$$

a) Determine the dimension of the matrices A_i and C_i that correspond to system i.

b) Analyse the observability of each system. Justify your answer.

Check your solutions with MATLAB.

Problem 3

Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix}0 & 1\\ 20.6 & 0\end{bmatrix} x + \begin{bmatrix}0\\ 1\end{bmatrix} u\\ y &= \begin{bmatrix}1 & 0\end{bmatrix} x\end{aligned}$$

and a zero-state regulator controller $u = -Kx$ such that the closed-loop eigenvalues are $\lambda_{1,2} = -1.8 \pm j2.4$. Assume now that only the output y is available.

[1]Updated: 20 Feb 2021.

a) Is the system completely observable?

b) Compute the gain of an observer such that the eigenvalues of the observer error dynamics are $\lambda_1 = -20$ and $\lambda_2 = -25$.

c) Draw the block-diagram of the control system (i.e. plant, controller and observer).

Check your solutions with MATLAB.

Problem 4

Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \\
y_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} x
\end{aligned}
$$

and a zero-state regulator controller $u = -Kx$ such that the closed-loop eigenvalues are $\lambda_{1,2} = -1$. Assume now that only one of the outputs y_1 and y_2 is available.

a) Analyse the observability of the system with the output y_1 and the output y_2. Choose one output to design an state observer. Justify your choice.

b) Use the output selected in the previous item and compute the gain of a state observer such that the location of the eigenvalues of the observer error dynamics is 10 time faster than the control closed-loop dynamics.

Check your solutions with MATLAB.

Problem 5

Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x.
\end{aligned}
$$

and a zero-state regulator controller $u = -Kx$ such that the closed-loop eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. Assume now that only the output y is available.

a) Is the system completely observable? Justify your answer.

b) Design a state observer and compute the gain of the observer. Choose the values of the eigenvalues of the observer error dynamics.

Check your solutions with MATLAB.

Problem 6

Consider the following nonlinear systems:

1.

$$
\begin{aligned}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= 3x_1^2 x_2 + x_1 + u \\
y &= -x_1^3 + x_2
\end{aligned}
$$

2.

$$
\begin{aligned}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= x_1 x_2^2 - x_1 + x_3 \\
\dot{x}_3 &= u \\
y &= -x_1^3 + x_2
\end{aligned}
$$

3.

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = x_1 - x_2 - x_1 x_3 + u$$
$$\dot{x}_3 = x_1 + x_1 x_2 - 2x_3$$
$$y = x_1$$

a) Use linearisation to design a state-feedback controller to stabilise the origin. Select the desired closed-loop eigenvalues to achieve a particular dynamic response, for example certain overshoot and time response. (same as Problem 6 of SFC practice problem)

b) Consider that only measurements are the input and output signals. For each system, design an observer to estimate the states, if possible. Propose the desired eigenvalues.

Check your solutions with MATLAB.

Problem 8

A model of a field-controlled DC motor can be described by the second order model

$$\dot{x}_1 = -60\, x_1 - 0.5\, x_2\, u + 40$$
$$\dot{x}_2 = -6\, x_2 + 4 \cdot 10^4\, x_1\, u$$
$$y = x_2$$

where x_1 is the armature current, x_2 is the mechanical speed and u is the field current.

1. Find the equilibrium point and input that correspond to $\bar{y} = 200$.
2. Use linearisation to design a state feedback controller to ensure that the output y tracks the reference values $\bar{y}$ and such that the closed loop eigenvalues are -10 and -15.
3. Design an observer assuming that the input and output are the signals available to use in the controller. The desired eigenvalues of the estimation error dynamics are -50 and -60.

Check your solutions with MATLAB.

Problem 9

Consider the dynamics of the robot known as Ballbot system[2]

$$35\ddot{\phi} + 4\ddot{\theta} - 100\sin(\phi) = -\tau$$
$$4\ddot{\phi} + \ddot{\theta} - 4\sin(\phi)\dot{\phi}^2 = \tau$$
$$y_1 = \phi$$
$$y_2 = \theta$$

where ϕ is the absolute angle of the body and θ is angle of the ball.

1. Write the system in state space form with $x_1 = \phi$, $x_2 = \theta$, $x_3 = \dot{\phi}$ and $x_4 = \dot{\theta}$.
2. Find the equilibrium point and input that correspond to $\bar{x}_1 = 0$ and $\bar{x}_2 = 0$.
3. Use linearisation to design a state feedback controller with integral action that ensures that the outputs track the reference values $\bar{y}_1 = 0$ and $\bar{y}_2 = 4$. If possible, the desired eigenvalues of the closed loop should be -10, -10, -15, -15, -5 and -5. (This is Problem 10 of `AERO3600_CT_SFC_Practice_Problems.pdf`).
4. Design an observer to estimate the Ballbot states assuming that the input and output are the signals available to use in the controller. The desired eigenvalues of the estimation error dynamics are -50, -55, -60 and -65.

Solve numerically using MATLAB and simulate the closed loop.

[2]See for example the article *Balancing and Transferring Control of a Ball Segway Using a Double-Loop Approach* in IEEE Control Systems Magazine, vol. 38, no. 2, pp. 15-37, 2018, D. B. Pham, H. Kim, J. Kim and S. Lee.

Answer to Practice Problems

(Observer Design)

Problem 1

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \; ; \; C = \begin{bmatrix} -1 & 2 \end{bmatrix}$$

$$O_{AC} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \; ; \; \left.\begin{array}{l} \text{rank}(O_{AC}) = 1 \\ \text{n° states} = 2 \end{array}\right\} \Rightarrow \text{the system is not completely observable}$$

Problem 2

a)

$$O_{AC} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \begin{array}{l} C \in \mathbb{R}^{m\times n} \\ \\ A \in \mathbb{R}^{n\times n} \end{array} \qquad \begin{array}{l} m: \text{n° of output} \\ n: \text{n° of states} \end{array} \Rightarrow O_{AC} \in \mathbb{R}^{(nm)\times n}$$

$$\begin{array}{llll} A_1 \in \mathbb{R}^{2\times 2} & A_2 \in \mathbb{R}^{2\times 2} & A_3 \in \mathbb{R}^{3\times 3} & A_4 \in \mathbb{R}^{2\times 2} \\ C_1 \in \mathbb{R}^{1\times 2} & C_1 \in \mathbb{R}^{1\times 2} & C_3 \in \mathbb{R}^{1\times 3} & C_4 \in \mathbb{R}^{2\times 2} \end{array}$$

b)

System 1: $\left.\begin{array}{l} \text{rank}(Ob_1) = 1 \\ \text{n° states} = 2 \end{array}\right\} \Rightarrow$ the system is not completely observable

System 2: $\left.\begin{array}{l} \text{rank}(Ob_2) = 1 \\ \text{n° states} = 2 \end{array}\right\} \Rightarrow$ the system is not completely observable

System 3: $\left.\begin{array}{l} \text{rank}(Ob_3) = 3 \\ \text{n° states} = 3 \end{array}\right\} \Rightarrow$ the system is completely observable

System 4. $\left.\begin{array}{l} \text{rank}(Ob_4) = 2 \\ \text{n° states} = 2 \end{array}\right\} \Rightarrow$ the system is completely observable

Problem 3

$$A = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \; ; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a)

$$O_{AC} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; \left.\begin{array}{l} \text{rank}(O_{AC}) = 2 \\ \text{n° of states} = 2 \end{array}\right\} \Rightarrow \text{the system is completely observable.}$$

b) Desired characteristic polynomial $(s+20)(s+25) = s^2 + 45\,s + 500$

Observer $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$; $(A-LC) = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ 20.6-l_2 & 0 \end{bmatrix}$

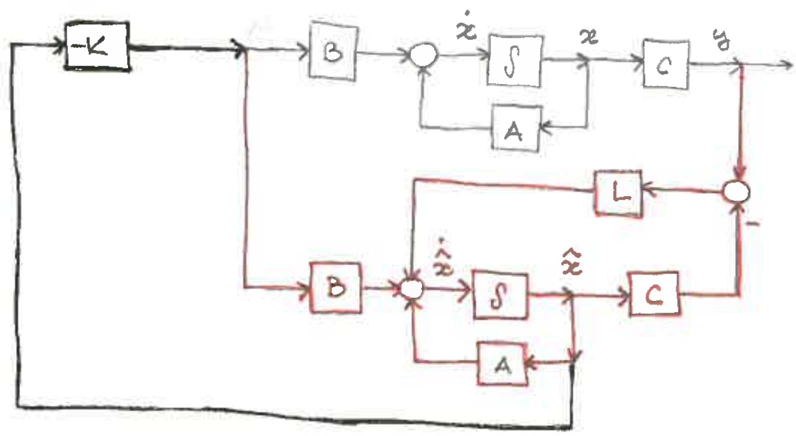
$$\mathrm{Char}(A-LC) = \det \begin{bmatrix} s+l_1 & -1 \\ -(20.6-l_2) & s \end{bmatrix} = s^2 + l_1 s - (20.6 - l_2)$$

$l_1 = 45$

$-20.6 + l_2 = 500 \rightarrow l_2 = 520.6$

$$L = \begin{bmatrix} 45 \\ 520.6 \end{bmatrix}$$

c)



Problem 4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$; $C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$

a) $O_{AC_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\left.\begin{array}{l} \mathrm{rank}(O_{AC_1}) = 2 \\ n^\circ \text{ states} = 2 \end{array}\right\}$ $\Rightarrow$ the system is completely Observable with output y_1

$O_{AC_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $\left.\begin{array}{l} \mathrm{rank}(O_{AC_2}) = 1 \\ n^\circ \text{ states} = 2 \end{array}\right\}$ $\Rightarrow$ the system is not completely observable with output y_2

We choose y_1 because the system is completely observable with this output
$\Rightarrow$ we can assign the eigenvalues of the observer error dynamics

b) Desired characteristic polynomial $(s+10)^2 = s^2+20s+100$

the rest of the problem is similar to Problem 3. Check the values with Matlab.

Problem 5

$$A = \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

a) $O_{AC} = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}$; $\left.\begin{matrix} \text{rank}(O_{AC}) = 2 \\ \text{n° states} = 2 \end{matrix}\right\} \Rightarrow$ completely observable.

b) Desired characteristic polynomial, for example :

$(s+20)(s+30) = s^2+50s+600$

The rest of the problem follows the same procedure as in problem 3

$\boxed{\text{PROBLEM 8}}$

PROBLEM SOLVED IN CONSULTATION. CHECK NUMERICAL SOLUTION

$$\begin{cases} \dot{x}_1 = -60x_1 - 0.5x_2 u + 40 \\ \dot{x}_2 = -6x_2 + 4\cdot 10^4 x_1 u \\ y = x_2 \end{cases}$$

1. FIND E.P. $\bar{x}$ AND $\bar{u}$ / $\bar{y} = 200$

$$\left.\begin{array}{l} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 = -60\bar{x}_1 - 0.5\bar{x}_2\bar{u} + 40 \\ 0 = -6\bar{x}_2 + 4\cdot 10^4 \bar{x}_1 \bar{u} \\ \\ \bar{y} = \bar{x}_2 = 200 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad 0 = -60\bar{x}_1 - 0.5 * 200\,\bar{u} + 40 \rightarrow 60\bar{x}_1 = -100\bar{u} + 40$$

$$0 = -6 * 200 + 4.10^4 \bar{x}_1 \bar{u}$$

$$\Rightarrow \quad 0 = -1200 + 4.10^4\left(\frac{-100\bar{u}+40}{60}\right)\bar{u}$$

$$0 = -1200 + 40000\left[-\frac{5}{3}\bar{u} + \frac{2}{3}\right]\bar{u}$$

$$0 = -1200 - 6.6 \times 10^4\,\bar{u}^2 + 2.6 \times 10^4\,\bar{u} \quad \begin{array}{l} \nearrow 0.3483 \\ \searrow 0.0517 \end{array}$$

$$\bar{x}_1 = -\frac{100}{60}\bar{u} + \frac{40}{60} \quad \begin{array}{l} \nearrow 0.0861 \\ \searrow 0.5805 \end{array}$$

EP$_1$: $\bar{x}_1 = 0.0861$, $\bar{x}_2 = 200$, $\bar{u} = 0.3483$

EP$_2$: $\bar{x}_1 = 0.5805$, $\bar{x}_2 = 200$, $\bar{u} = 0.0517$

2- SOLUTION USING EP$_1$

$$\dot{x}_1 = -60x_1 - 0.5x_2 u + 40$$
$$\dot{x}_2 = -6x_2 + 4\cdot 10^4 x_1 u$$
$$y = x_2$$

$$A = \left.\frac{\partial f}{\partial x}^T\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} ; \quad B = \left.\frac{\partial f}{\partial u}^T\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} ; \quad C = \left.\frac{\partial g}{\partial x}^T\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} ; \quad D = \left.\frac{\partial g}{\partial u}^T\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

$$\tilde{x} = x - \bar{x} \quad , \quad \tilde{u} = u - \tilde{u} \quad ; \quad \tilde{y} = y - \bar{y}$$

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \\ \tilde{y} = C\tilde{x} + D\tilde{u} \end{cases}$$

$$A = \left.\begin{bmatrix} -60 & -0.5u \\ 4.10^4 u & -6 \end{bmatrix}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} -60 & -0.1742 \\ 1.393\,10^4 & -6 \end{bmatrix}$$

$$B = \left.\begin{bmatrix} -0.5x_2 \\ 4\cdot 10^4 x_1 \end{bmatrix}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} = \begin{bmatrix} -100 \\ 3.445\times 10^3 \end{bmatrix} \quad ; \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} ; \quad D = 0$$

$$C_{AB} = \begin{bmatrix} -100 & 5400 \\ 3445 & -1.41\times10^6 \end{bmatrix} \rightarrow \text{RANK}(C_{AB}) = 2 \Rightarrow \text{Syst. Compl. Contr.}$$

* DESIRED CHAR. POLY. $(s+10)(s+15) = s^2 + 25s + 150$

* CHAR. POLY. of CLOSED LOOP: $\text{CHAR}(A - BK)$

$$\text{CHAR}\left(\begin{bmatrix} -60 & -0.1742 \\ 1.39\times10^4 & -6 \end{bmatrix} - \begin{bmatrix} -100 \\ 3445 \end{bmatrix} \begin{bmatrix} k_0 & k_1 \end{bmatrix} \right) =$$

$$\text{CHAR}\left(\begin{bmatrix} -60 + 100k_0 & -0.1742 + 100k_1 \\ 13900 - 3445k_0 & -6 - 3445k_1 \end{bmatrix} \right)$$

$$\text{DET}\left(s - [A - BK]\right) = \text{DET} \begin{bmatrix} s + 60 - 100k_0 & 0.1742 - 100k_1 \\ -13900 + 3445k_0 & s + 6 + 3445k_1 \end{bmatrix}$$

$$= (s + 60 - 100k_0)(s + 6 + 3445k_1) - (0.1742 - 100k_1)(-13900 + 3445k_0)$$

$$= s^2 + (60 - 100k_0 + 6 + 3445k_1)s + (60 - 100k_0)(6 + 3445k_1)$$

$$- (0.1742 - 100k_1)(-13900 + 3445k_0)$$

MATCH COEFF.

$$60 - 100k_0 + 6 + 3445k_1 = 25$$

$$(60 - 100k_0)(6 + 3445k_1) - (0.1742 - 100k_1)(-13900 + 3445k_0) = 150$$

$$\Rightarrow \begin{matrix} k_0 = 0.47 \\ k_1 = 0.0017 \end{matrix} \qquad \Rightarrow \tilde{u} = -[0.47 \quad 0.0017]\, \tilde{x}$$

$$u - \bar{u} = -[0.47 \quad 0.0017]\,(x - \bar{x}) \Rightarrow$$

$$\Rightarrow \boxed{u = -[0.47 \quad 0.0017]\,(x - \bar{x}) + \bar{u}}$$

3.

$$O_{AC} = \begin{bmatrix} 0 & 1 \\ 1.393\,10^4 & -6 \end{bmatrix} \Rightarrow \dots \Rightarrow \text{Syst. Compl. Observable.}$$

* DESIRED CHAR. ERROR $(s+50)(s+60) = s^2 + 110\, s + 3000$

* CHAR. $(A - LC)$

$$\begin{bmatrix} 0 & l_0 \\ 0 & l_1 \end{bmatrix}$$

$$\text{CHAR}\left(\begin{bmatrix} -60 & -0.1742 \\ 1.39 \times 10^4 & -6 \end{bmatrix} - \begin{bmatrix} l_0 \\ l_1 \end{bmatrix} [0 \quad 1] \right)$$

$$\text{CHAR}\left(\begin{bmatrix} -60 & -0.1742 - l_0 \\ 13900 & -6 - l_1 \end{bmatrix} \right) = \text{DET} \begin{bmatrix} s+60 & 0.1742 + l_0 \\ -13900 & s+6+l_1 \end{bmatrix} =$$

$$= s^2 + (66 + l_1)\, s + 60(6 + l_1) + 13900\,(0.1742 + l_0)$$

*MATCH COEFF.

$$\begin{aligned} 66 + l_1 &= 110 \\ 60(6 + l_1) + 13900\,(0.1742 + l_0) &= 3000 \end{aligned} \Rightarrow \begin{matrix} l_1 = 44 \\ l_0 = -0.1742 \end{matrix}$$

$$\Rightarrow L = \begin{bmatrix} -0.1742 \\ 44 \end{bmatrix}$$