AERO3600 — Embedded Control Systems

Practice problems: Continuous-time optimal control design[1]

Important

This document proposes several practice problem on control design usign linear quadratic regulator for continuous-time systems.

Problem 1

Find the optimal feedback controller for the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u.\end{aligned}$$

with cost functional

$$J(x_0, u) = \int_0^\infty (x_1^2 + x_2^2 + 4u^2)dt.$$

Check your solutions with MATLAB using the command `[K,S]=lqr(A,B,Q,R)`.

Problem 2

Consider the system

$$\ddot{x} + \frac{1}{2}\dot{x} + x = u$$

with cost functional

$$J(x_0, u) = \int_0^\infty (3x^2 + u^2)dt.$$

Find the optimal control $u = -Kx$.

Check your solutions with MATLAB using the command `[K,S]=lqr(A,B,Q,R)`.

Problem 3

Find the optimal feedback controller for the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 + 2u.\end{aligned}$$

with cost functional

$$J(x_0, u) = \int_0^\infty (3x_1^2 + 3x_2^2 + u^2)dt.$$

Check your solutions with MATLAB using the command `[K,S]=lqr(A,B,Q,R)`.

[1] Updated: 18 Mar 2021.

Problem 4

Consider the system

$$\dot{x} = 3x + 4u$$

and the optimal controller $u = -2x$ that minimises the cost functional

$$J(x_0, u) = \int_0^{\infty} (x^2 + r\, u^2) dt.$$

Find the value of r that characterises the cost functional.

Additional exercises

Solve the examples in the lecture slides *"Introduction to Optimal Control"*.

Answer to Practice Problems

(LQR Design)

Problem 1

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad R = 4.$$

Riccati equation: $A^T S + SA - (SB) R^{-1} (B^T S) + Q = 0$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \qquad S.A = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & S_{11} - S_{12} \\ 0 & S_{12} - S_{22} \end{bmatrix}$$

$$SB = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ S_{11} - S_{12} & S_{12} - S_{22} \end{bmatrix} + \begin{bmatrix} 0 & S_{11} - S_{12} \\ 0 & S_{12} - S_{22} \end{bmatrix} - \frac{1}{4} \begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix} \begin{bmatrix} S_{12} & S_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & S_{11} - S_{12} \\ S_{11} - S_{12} & 2(S_{12} - S_{22}) \end{bmatrix} - \frac{1}{4} \begin{bmatrix} S_{12}^2 & S_{12} S_{22} \\ S_{12} S_{22} & S_{22}^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_{12}^2 = 4 \longrightarrow \boxed{S_{12} = 2}$$

$$4(S_{11} - S_{12}) = S_{12} S_{22} \longrightarrow S_{11} = \frac{S_{12}}{4}(S_{22} + 4) \longrightarrow \boxed{S_{11} = 3}$$

$$4(S_{11} - S_{12}) = S_{12} S_{22}$$

$$2(S_{12} - S_{22}) - \frac{S_{22}^2}{4} = -1 \longrightarrow S_{22}^2 + 8 S_{22} - 20 = 0 \quad \begin{cases} S_{22} = -10 \\ \boxed{S_{22} = 2} \end{cases}$$

$$K = R^{-1} B^T S = \frac{1}{4} \begin{bmatrix} 0 & 1 \end{bmatrix} . \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow u = -\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} x$$

Problem 2

$$z_1 \triangleq x ; \quad z_2 \triangleq \dot{x} \qquad \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(the system is controllable because is in companion form)

$$Q = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} ; \quad R = 1 \qquad SA = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -S_{12} & S_{11} - \frac{S_{12}}{2} \\ -S_{22} & S_{12} - \frac{S_{22}}{2} \end{bmatrix}$$

$$SB = \begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$$

Riccati equation:

$$\begin{bmatrix} -2S_{12} & S_{11} - \frac{S_{12}}{2} - S_{22} \\ S_{11} - \frac{S_{12}}{2} - S_{22} & 2S_{12} - S_{22} \end{bmatrix} - \begin{bmatrix} S_{12}^2 & S_{12} S_{22} \\ S_{12} S_{22} & S_{22}^2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} &-2S_{12} - S_{12}^2 = -3 \longrightarrow S_{12} = 1 \\ &S_{11} - \frac{S_{12}}{2} - S_{22} = 0 \longrightarrow S_{11} = 2.5 \\ &2S_{12} - S_{22} - S_{22}^2 = 0 \longrightarrow S_{22} = 1 \end{aligned}$$

$$K = R^{-1} B^T S = \begin{bmatrix} 1 & 1 \end{bmatrix} ; \quad u = -\begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Problem 3

$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$; $R = 1$

$C_{AB} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$; $\left.\begin{matrix} \text{rank}(C_{AB}) = 2 \\ \text{n° states} = 2 \end{matrix}\right\} \Rightarrow$ Completely Controllable

Similar procedure as in problem 1 and 2 $\longrightarrow S = \begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \longrightarrow K = [1 \;\; 2]$

Problem 4

$A = 3$; $B = 4$; $Q = 1$; $R = r$

$K = R^{-1} B^T S = \frac{1}{r} 4 S = 2 \Rightarrow S = \frac{r}{2}$

Riccati equation $\Rightarrow A^T S + S A - (SB) R^{-1} (B^T S) + Q = 0$

$6S - \frac{16}{r} S^2 + Q \Rightarrow 3r - 4r + 1 = 0 \Rightarrow \boxed{r = 1}$