STAT2110 PASS Worksheet 2 Solutions

Saturday, 10 February 2024 12:02 pm

1. The probability that a vehicle entering the Luray Caverns has Canadian license plates is 0.12; the probability that it is a camper is 0.28; and the probability that it is a camper with Canadian license plates is 0.09. What is the probability that

   (a) a camper entering the Luray Caverns has Canadian license plates?

   (b) a vehicle with Canadian license plates entering the Luray Caverns is a camper?

   (c) a vehicle entering the Luray Caverns does not have Canadian plates or is not a camper?

Let A be the event that the vehicle has Canadian license plates
Let B be the event that it is a camper.
Let C be the event that it is a camper with Canadian license plates.

Now we may write some statements in terms of our letter variables:

$P(A) = 0.12$, $P(B) = 0.28$, $P(C) = P(A \cap B) = 0.09$

a) We may write this problem as a conditional probability statement:

$$
\begin{aligned}
P(\text{Canadian license plates} \mid \text{camper}) &= P(A \mid B) \\
&= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.09}{0.28} \\
&= \frac{9}{28}
\end{aligned}
$$

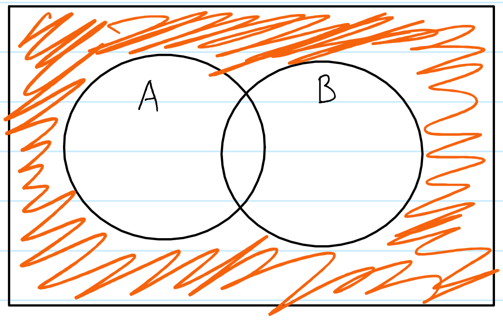
b)
$$
\begin{aligned}
P(\text{camper} \mid \text{Canadian license plates}) &= P(B \mid A) \\
&= \frac{P(B \cap A)}{P(A)} \\
&= \frac{0.09}{0.12} \\
&= \frac{3}{4}
\end{aligned}
$$

c) The probability statement can also be represented by a Venn diagram.
Where events A and B intersect is known as $A \cap B$ or A AND B.
Since we are looking at the complement of two individual events being

where events A and B intersect is known as $A \cap B$ or A AND B.
Since we are looking at the complement of two individual events being OR'd together (i.e. $\bar{A} \cup \bar{B}$), we may use De Morgan's theorem to say that the $\bar{A} \cap \bar{B} = \overline{A \cup B}$

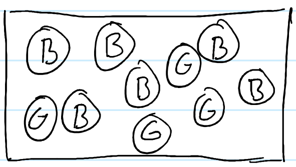
CHECK THIS.

$$P(A \cap B) = P(A \cup B) - P(A) - P(B)$$



$$\Rightarrow P(\overline{A \cup B}) \qquad 1 - P(A \cap B)$$
$$= 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)$$

2.



6 B, 4 G.

There are two events corresponding to all 3 balls being the same colour.

- BBB $\longrightarrow$ $n(BBB) = \binom{6}{3} = 20$
- GGG $\longrightarrow$ $n(GGG) = \binom{4}{3} = 4$

So $n(BBB) + n(GGG) = 280$.

The number of ways to draw any ball: $N_{total} = \binom{10}{3} = 120$

$$\therefore P(\text{All 3 balls are the same colour}) = \frac{n(BBB) + n(GGG)}{N_{total}}$$
$$= \frac{24}{120}$$
$$= \frac{1}{5}$$

3. The different ways that x defective TVs can be purchased is:

1. 0 defective TV sets purchased, i.e. $x = 0$:

$\binom{5}{3}\binom{2}{0}$

no. of ways to select ... three TVs ...

$(3)(0)$

2. 1 defective :

$$\binom{5}{2}\binom{2}{1}$$

3. 2 defective:

$$\binom{5}{1}\binom{2}{2}$$

PDF

$$f(x) = \frac{\binom{5}{3-x}\binom{2}{x}}{\binom{7}{3}}$$

no. of ways to select any three TVs.

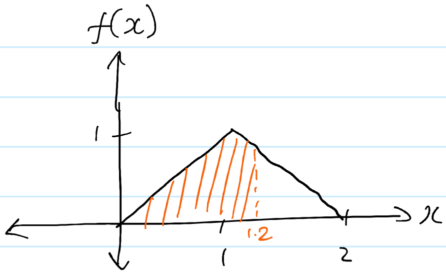
Evaluating the PDF for $x = 0, 1, 2$:

x	0	1	2
$f(x)$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

This shows the probability distribution of x since $f(x)$ is discrete.

4.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$



a) Can be calculated geometrically or through the use of integrals.

much quicker here.

We want to find: $P(X \le 1.2) = \int_{-\infty}^{1.2} f(x)\,dx$

$1.2 \times 100 = 120$hrs

$$= \int_0^1 x\,dx + \int_1^{1.2} (2-x)\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1 + \left[2x - \frac{1}{2}x^2\right]_1^{1.2}$$

$$= 0.5 + 0.18$$

$$= 0.68$$

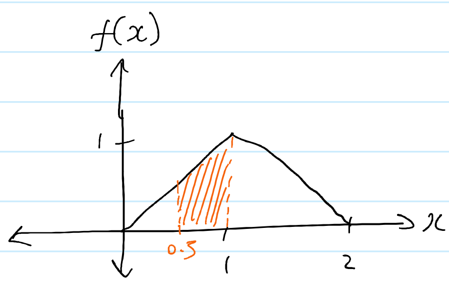
b) $P(\underset{(50)}{0.5} < x < \underset{(100)}{1})$ hours

$$= \int_{0.5}^{1} f(x)\,dx$$

$$= \int_{0.5}^{1} x\,dx$$

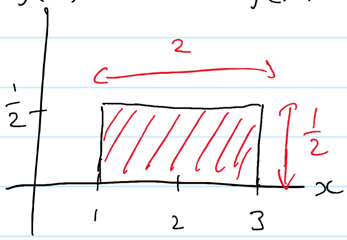
$$= \left[\frac{1}{2}x^2\right]_{0.5}^{1}$$

$$= 0.5 - 0.125$$



$= 0.375$

5. $f(x) = \frac{1}{2}$ for $1 \le x \le 3$



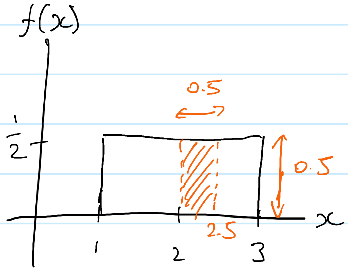
a) Much quicker to solve graphically:

Area under the curve = lb (area of rectangle)

$= 2 \times \frac{1}{2}$

$= 1$.

b) $P(2 < X < 2.5)$

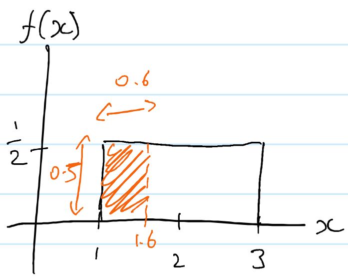


Area under the curve corresponds to probability for a given pdf.

$P(2 < X < 2.5) = 0.5 \times 0.5$

$= 0.25$.

c) $P(X \le 1.6)$



$P(X \le 1.6) = 0.6 \times 0.5$

$= 0.3$

6. $f(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0 < x < 2, \quad 2 < y < 4 \\ 0, & \text{elsewhere.} \end{cases}$

$P(1 < Y < 3 \mid X = 1)$

$$f(y \mid x) = \frac{f(x,y)}{g(x)} \longrightarrow g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_2^4 \frac{6-x-y}{8}\,dy$$

$$= \frac{1}{8}\left[6y - xy - \frac{1}{2}y^2\right]_2^4$$

$$= \frac{1}{8}\left[24 - 4x - 8 - (12 - 2x - 2)\right]$$

$$= \frac{\frac{6-x-y}{8}}{\frac{6-2x}{8}}$$

$$f(y \mid x) = \frac{6-x-y}{6-2x}$$

$$f(y|x) = \frac{6-x-y}{6-2x}$$

$$= \frac{1}{8}\left[24-4x-8-(12-2x-2)\right]$$

$$= \frac{1}{8}\left[6-2x\right]$$

$$= \frac{6-2x}{8}$$

So $P(1<Y<3|X=1) = \int_1^3 f(y|x=1)\,dy$

y only defined for $2<y<4$

$$= \int_2^3 \frac{5-y}{4}\,dy$$

$$= \frac{1}{4}\left[5y-\frac{1}{2}y^2\right]_2^3$$

$$= \frac{1}{4}\left[15-4.5-(10-2)\right]$$

$$= 0.625.$$

7. $f(x,y) = \begin{cases} \frac{3x-y}{9}, & 1<x<3,\ 1<y<2 \\ 0, & \text{elsewhere.} \end{cases}$

a) Marginal distributions are found as shown below:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_1^2 \frac{3x-y}{9}\,dy$$

$$= \frac{1}{9}\left[3xy-\frac{1}{2}y^2\right]_1^2$$

$$= \frac{1}{9}\left[6x-2-\left(3x-\frac{1}{2}\right)\right]$$

$$g(x) = \frac{3x-1.5}{9}$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_1^3 \frac{3x-y}{9}\,dx$$

$$= \frac{1}{9}\left[\frac{3x^2}{2}-xy\right]_1^3$$

$$= \frac{1}{9}\left[\frac{27}{2}-3y-\left(\frac{3}{2}-y\right)\right]$$

$$= \frac{1}{9}\left[12-2y\right]$$

$$h(y) = \frac{12-2y}{}$$

$$g(x) = \frac{3x - 1.5}{9}$$

$$h(y) = \frac{12 - 2y}{9}$$

b) marginal distributions are independent if $f(x,y) = g(x)h(y)$.

$$g(x)h(y) = \frac{3x - 1.5}{9} \times \frac{12 - 2y}{9}$$

$$= \frac{36x - 18 - 6xy + 3y}{81}$$

$$\neq f(x,y)$$

$\therefore$ X and Y are not independent.

c) $$P(X > 2) = \int_2^\infty g(x)\,dx$$

$$= \int_2^3 \frac{3x - 1.5}{9}\,dx$$

$$= \frac{1}{9}\left[\frac{3x^2}{2} - 1.5x\right]_2^3$$

$$= \frac{1}{9}\left[\frac{27}{2} - 4.5 - (6 - 3)\right]$$

$$= \frac{6}{9}$$

$$= \frac{2}{3}$$