STAT2110 PASS Worksheet 3 Solutions

Monday, 18 March 2024 8:08 pm

1.

x	\$4000	−\$1000
$f(x)$	0.3	0.7

\$1000 Loss = −\$1000

$$E(X) = \sum_x x f(x)$$
$$= 4000 \times 0.3 + (-1000) \times 0.7$$
$$= 500$$

$\therefore$ Expected gain is \$500.

2.

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$
$$= \int_0^1 x \cdot \frac{4}{\pi(1+x^2)}\, dx$$
$$= \frac{2}{\pi} \int_0^1 \frac{2x}{1+x^2}\, dx$$
$$= \frac{2}{\pi} \left[\ln(1+x^2) \right]_0^1$$
$$= \frac{2}{\pi} \ln(2) - \cancel{\frac{2}{\pi} \ln(1)}$$
$$= \frac{2}{\pi} \ln(2)$$

$E(X) \approx 0.44$

3.

$f(x,y)$		x = 2	x = 4	$h(y)$
y	1	0.10	0.15	0.25
	3	0.20	0.30	0.5
	5	0.10	0.15	0.25

$g(X,Y) = XY^2$

y	3	0.20	0.30	0.5
	5	0.10	0.15	0.25
	$g(x)$	0.4	0.6	1

$$g(X,Y) = XY^2$$

a) $$E[g(x,y)] = \sum_x \sum_y g(x,y) f(x,y)$$

$$= \sum_x \sum_y x y^2 f(x,y)$$

$$= (2)(1)^2(0.1) + (2)(3)^2(0.2) + (2)(5)^2(0.1)$$
$$+ (4)(1)^2(0.15) + (4)(3)^2(0.3) + (4)(5)^2(0.15)$$

$$= 35.2$$

b) Using the added rows/columns in the table above:

$$\mu_X = \sum_x x\, g(x) \qquad\qquad \mu_Y = \sum_y y\, h(y)$$

$$= 2 \times 0.4 + 4 \times 0.6 \qquad\qquad = 1 \times 0.25 + 3 \times 0.5 + 5 \times 0.25$$

$$= 3.2 \qquad\qquad = 3$$

4. $Y = 3X - 2$, $\quad f(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$

$$\int_a^b v' u \, dx = [uv]_a^b - \int_a^b u' v \, dx$$

Remember $\lim_{x \to \infty} e^{-ax} = 0$.

mean of Y

$$\mu_Y = E(Y) = E(3X - 2)$$
$$= 3E(X) - 2$$
$$= 3 \times 4 - 2$$
$$\mu_Y = 10$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad u = x \quad v' = e^{-x/4} \qquad u' = 1 \quad v = -4e^{-x/4}$$

$$= \int_0^{\infty} x \frac{1}{4} e^{-x/4}\, dx$$

$$= \frac{1}{4}\left\{ \left[-4x e^{-x/4}\right]_0^{\infty} - \int_0^{\infty} -4 e^{-x/4}\, dx \right\}$$

$$= \frac{1}{\cancel{4}}\left\{ [0 - 0] + \cancel{4} \int_0^{\infty} e^{-x/4}\, dx \right\}$$

$$= \left[-4 e^{-x/4}\right]^{\infty}$$

variance of Y

$$\sigma_Y^2 = E[(Y - \mu_Y)^2]$$

$$= E(Y^2) - \mu_Y^2 \quad \text{(alternate formula)}$$

$= E(Y^2) - \mu_Y^2$ (alternate formula)

$= E(9X^2 - 12X + 4) - 10^2$

$= 9E(X^2) - 12E(X) - 96$

$= 9E(X^2) - 12 \times 4 - 96$

$\sigma_Y^2 = 9E(X^2) - 144$

$= \left[-4e^{-x/4}\right]_0^\infty$

$= 0 - (-4)$

$E(X) = 4$

$E(X^2)$ requires two iterations of integration by parts, which will be omitted as it is similar to finding $E(X)$.

$E(X^2) = \frac{1}{4}\int_0^\infty x^2 e^{-x/4} dx$

$= \ldots$

$E(X^2) = 32.$

Hence, $\sigma_Y^2 = 9 \times 32 - 144$

$\sigma_Y^2 = 144.$

5.

x	-2	3	5
$f(x)$	0.3	0.2	0.5

$\mu_X = \sum_x x f(x)$

$= -2 \times 0.3 + 3 \times 0.2 + 5 \times 0.5$

$\mu_X = 2.5$

$\sigma_X^2 = \sum_x (x - \mu_X)^2 f(x)$

$= (-2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0.2 + (5 - 2.5)^2 \times 0.5$

$\sigma_X^2 = 9.25$ — variance of X

$\therefore \sigma_X = \sqrt{9.25} \approx 3.04$ — standard deviation of X

6.

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

$$E(X) = \sum_x x f(x)$$

$$= -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3}$$

$$= 5.5$$

$$E(X^2) = \sum_x x^2 f(x)$$

$$= (-3)^2 \times \frac{1}{6} + 6^2 \times \frac{1}{2} + 9^2 \times \frac{1}{3}$$

$$E(X^2) = 46.5$$

$$E[(2X+1)^2] = E[4X^2 + 4X + 1]$$

$$= 4E(X^2) + 4E(X) + 1$$

$$= 4 \times 46.5 + 4 \times 5.5 + 1$$

$$= 209$$