

# 1 Trajectory Generation

The material for these notes has been adopted from:  
John J. Craig: Robotics: Mechanics and Control.

This example assumes that we have a starting position and goal pose of the end effector and we are asked to move the joint angles to move the end effector from one pose to another. Here we describe a strategy how to do so by designing a trajectory in joint space from one end point to another. Assuming that we know the inverse kinematics of the system, we can compute the desired joint angle for goal position of the end effector. This example shows how to design a trajectory of a single joint  $\theta(t)$  as function of time. Suppose that we have following constraints of our trajectory: we have desired position at the beginning and end of the trajectory and we the velocity at the beginning and end has to be zero. Hence our desired trajectory has to satisfy the following constraints:

$$\theta(0) = \theta_0; \theta(t_f) = \theta_d \quad \dot{\theta}(0) = 0 \quad \dot{\theta}(t_f) = 0 \quad (1)$$

**Cubic polynomials** In order to satisfy the above constraints, our trajectory has to be at least polynomial of the 3<sup>rd</sup> order, which has four coefficients, and hence can satisfy the above 4 constraints. This can be achieved by third order cubic polynomial which has the following form

$$\theta(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Given the above form the joint velocity and acceleration will have the following forms

$$\dot{\theta} = a_1 + 2a_2t + 3a_3t^2 \quad (2)$$

$$\ddot{\theta} = 2a_2 + 6a_3t \quad (3)$$

Using the above equations and instantiating the constraints we can solve for the coefficients of the cubic polynomial and obtain

$$a_0 = \theta_0 \quad (4)$$

$$a_1 = 0 \quad (5)$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) \quad (6)$$

$$a_3 = \frac{2}{t_f^3}(\theta_f - \theta_0) \quad (7)$$

Now given a particular instance of the problem, we can substitute to the above equations the desired parameters  $\theta_0, \theta_f, t_f$  and obtain different trajectories.

**Linear functions with parabolic blends** If we were to simply just connect the desired position with a linear function, it would cause the velocity to be discontinuous at the beginning and end of the motion. Also note that of the shape of the part in the joint space is linear, that does not mean that the shape of the path in the end effector space is linear. Hence what can be done is to take a linear path in the end effector space and interpolate it linearly. We would like to do it in a way that the velocities at the would not be discontinuous at the places where the pieces meet. One way to achieve this is to add a parabolic blend region, such that the we will create a smooth and continuous path. During the blend portion of the trajectory the acceleration will be constant (i.e. we will assume that it will not be changing in time and that we can instantaneously generate the constant acceleration profile). To construct a one such single segment, we will assume that parabolic blend at the beginning and the end have the same duration and the same constant acceleration (with opposite signs) will be used during those blends. If the blends at the beginning and the end will have the same duration, the final solution will be always symmetric around the half way point  $t_h$  and  $\theta_h$ . To guarantee smoothness the velocity at the end of the blend has to be the same as the velocity of the linear section

$$\dot{\theta}_{t_b} = \frac{\theta_h - \theta_b}{t_h - t_b}$$

where  $\theta_b$  is the value of  $\theta$  at the end of the blend region. Since the blend is parabolic the value of  $\theta_b$  is given by

$$\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta}t_b^2$$

Combining the above two equations and denoting  $t = 2t_h$ , we get

$$\ddot{\theta}t_b^2 - \ddot{\theta}tt_b + (\theta_f - \theta_0) = 0$$

where  $t$  is the desired time of motion. Given the desired  $\theta_f, \theta_0$  and  $t$ , the above equation gives constraints on between  $\ddot{\theta}$  and  $t_b$  which the trajectory has to satisfy. Hence typically  $\ddot{\theta}$  is chosen and then we can use the equation to solve for  $t_b$  to obtain

$$t_b = \frac{t}{2} - \frac{\sqrt{\ddot{\theta}t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

Notice that depending on acceleration the time of the blend region will vary. Depending on the acceleration, the path will be composed from two parabolic blends which will meet in the middle with the same slope and the linear portion of the blend will go to zero. If the acceleration is high the blend region will be shorter. In the limit when acceleration is infinite, we will reach the simple linear interpolation case.

## 2 Control of Second-Order Systems

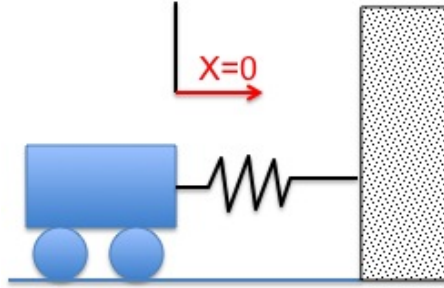


Figure 1: Block with mass  $m$  attached to the wall with spring with stiffness  $k$ .

Before we start considering the trajectory tracking problem, let's consider a simpler problem. Consider a block with the mass  $m$  sliding along a surface and attached with the spring to the wall. The equation of motion of the block is

$$m\ddot{x} + b\dot{x} + kx = 0$$

where  $x$  is the position the block as measured from the origin of the coordinate system, the  $b\dot{x}$  is the frictional force proportional to the velocity and  $kx$  is the related

to the position and stiffness of the spring. We would like to study the behavior of the system by understanding the trajectories  $x(t)$ . From the study of differential equations the form of the solution depends on the roots of its characteristic equation

$$ms^2 + bs + k = 0$$

with the roots

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m} \text{ and } s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}$$

It can be easily shown by substitution that the solution  $x(t)$  has the following form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where  $c_1$  and  $c_2$  are constants which can be determined from the initial conditions. We will now show 3 different cases of qualitatively different solutions which depend of the values  $s_1$  and  $s_2$  and consequently of the parameters of the system  $m, b, k$ .

1. The first case we consider is  $s_1 = -2$  and  $s_2 = -3$ , where two roots are real and have negative parts. In case the initial conditions,  $x(0) = -1$  and  $\dot{x}(0) = 0$ , substituting to the differential equation

$$c_1 + c_2 = -1 \tag{8}$$

$$-2c_1 - 3c_2 = 0 \tag{9}$$

which is satisfied by  $c_1 = -3$  and  $c_2 = 2$ . The motion of the system is then

$$x(t) = -3e^{-2t} + 2e^{-3t}$$

2. The second case we consider is when the two roots have complex roots and the solution has the form

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

where  $s_1 = \lambda + i\mu$  and  $s_2 = \lambda - i\mu$ . Using the well known **Euler formula**

$$e^{ix} = \cos x + i \sin x$$

we can rewrite the trajectory in the following form

$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

where the coefficients  $c_1$  and  $c_2$  can be computed from initial conditions. If we rewrite them in the following way

$$c_1 = r \cos \delta \quad (10)$$

$$c_2 = r \sin \delta \quad (11)$$

then using the formula for  $\cos(\alpha + \beta)$  we can write the trajectories in the following way

$$x(t) = re^{\lambda t} \cos(\mu t - \delta)$$

where

$$r = \sqrt{c_1^2 + c_2^2} \text{ and } \delta = \arctan(c_2, c_1)$$

In the above form it is easier to see that the resulting trajectories will be oscillations, with the amplitude exponentially decreasing to zero. This type of oscillatory system is also often described in terms of following parameters, which are the functions of the terms already defined above. First it is the natural frequency of the system  $\omega_n$ , the damping ratio  $\zeta$

$$\lambda = -\zeta \omega_n \quad (12)$$

$$\mu = \omega_n \sqrt{1 - \zeta^2} \quad (13)$$

These symbols are related to the canonical form of the characteristic equation of the second order system

$$s^2 + \zeta \omega_n s + \omega_n^2 = 0$$

3. Another interesting case is the case when, the solutions to the characteristic equations are two real repeated roots, i.e.

$$s_1 = s_2 = -\frac{b}{2m}$$

In this case the trajectory will have the following form

$$x(t) = (c_1 + c_2)e^{\frac{-b}{2m}t}$$

When the roots of the characteristics equations (also called poles of the second order system) are real and equal, the system is **critically damped** and exhibits the fastest possible non-oscillatory response.

## 2.1 Control of second order systems

We saw in the previous section that the behavior of the second order system (involving second derivatives of the position) depends on the coefficient of the system. The previous equations characterized the behavior of the system in the absence of any external forces. Support now we want to modify the behavior.

If we want to achieve a desired behavior we need to modified these coefficients by means of control. Suppose for example that we are going to apply some external force to the system, which will yield the following equation of motion

$$m\ddot{x} + b\dot{x} + kx = f$$

Assuming that we have at our disposal sensors which can measure the position  $x$  and the velocity  $\dot{x}$ , we would like to make the force proportional to the sensed feedback. Hence suppose the control of the following form

$$f = -k_p x - k_v \dot{x}$$

where  $k_p$  and  $k_v$  are some constants, also referred to as gains determining how big the force will be as proportion of velocity and position. This particular control law will strive to keep the position of the block at zero and stationary, i.e. when both  $x = 0$  and  $\dot{x} = 0$ , the applied force will be 0. If we now bring the equation of motion to the canonical form above (right hand side is zero), we will have an equation of motion of closed feedback loop system

$$m\ddot{x} + (b + k_v)\dot{x} + (k + k_p)x = 0$$

or

$$m\ddot{x} + b'\dot{x} + k'x = 0$$

Notice now that we can now chance the control gains  $k_p$  and  $k_v$  so as to obtain the coefficients of the second order system which would generate the desired behavior.

## 2.2 Trajectory following

So how is this related to the trajectory following?

Prior proceeding with the analysis we will make a simplification to the control formulation, which would enable us to separate the components which are related to model parameters and those which are related to the actual control law. Instead of writing the differential equation in the following form:

$$m\ddot{x} + b\dot{x} + kx = f$$

We assume that  $f$  has the form  $f = \alpha f' + \beta$ . In this case the equation above becomes

$$m\ddot{x} + b\dot{x} + kx = \alpha f' + \beta$$

If we choose  $\alpha = m$  and  $\beta = b\dot{x} + kx$  we will get

$$\ddot{x} = f'$$

which will make the system appear as unit mass. We can then proceed to control this system by setting

$$f' = -k_v\dot{x} - k_px$$

making the closed loop dynamics as follows

$$\ddot{x} + k_v\dot{x} + k_px = 0.$$

Now all we have to do it to worry about selecting parameters  $k_v$  and  $k_p$  to get the desired behavior and we do not have to worry about parameters of the system.

Now instead of designing a control to maintain the block a a particular position, we can design a control which will make the block to follow particular trajectory. Suppose now that the trajectory is given to us as  $x_d(t)$  which specifies the desired position of the block. We also assume that our trajectory is smooth (i.e. first two derivatives exist) and that our trajectory generator provides us with  $x_d, \dot{x}_d$  and  $\ddot{x}_d$  at all times. We now define the servo error as  $e = x_d - x$ . A servo control law which we will then use for trajectory following will have the following form

$$f = \ddot{x}_d + k_v\dot{e} + k_pe$$

If we combine the above equation with the simplified canonical equation of motion (see explanation above)  $\ddot{x} = f$ <sup>1</sup> we will obtain the following equation:

$$\ddot{x} = \ddot{x}_d + k_v\dot{e} + k_pe$$

or

$$\ddot{e} + k_v\dot{e} + k_pe = 0.$$

Notice that this again second order differential equation, hence we can determine the behavior of the error trajectories  $e(t)$  by setting the coefficients of the equation, based on the cases outlined at the beginning of this handout.

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<sup>1</sup>Any second order system can be rewritten to this form by simply grouping the other parameters of the system into  $f$ .

**Disturbance rejection** The control law in the previous example had the form

$$f = \ddot{x}_d + k_v \dot{e} + k_p e.$$

Now suppose that system is affected by some unknown disturbance  $f_{dist}$ . The closed loop dynamics of will then become:

$$\ddot{e} + k_v \dot{e} + k_p e = f_{dist}.$$

If  $f_{dist}$  is bounded then the solution to differential equation  $e(t)$  is also bounded. Lets consider a steady state behavior. If we set the derivatives to zero we get

$$k_p e = f_{dist}$$

and the so called steady-state error  $e = f_{dist}/k_p$  will be non-zero. The higher  $k_p$  the smaller the error will be. To eliminate the steady state error we can modify the control law to:

$$f = \ddot{x}_d + k_v \dot{e} + k_p e + \int e dt.$$

This would assure that the system has no steady-state error in case of constant disturbances. The closed loop dynamics would be

$$\ddot{e} + k_v \dot{e} + k_p e + k_i \int e dt = f_{dist}.$$

If the disturbance is constraint, we can conclude (by taking a derivative of both sides) that:

$$\ddot{e} + k_v \dot{e} + k_p \dot{e} + k_i e = \dot{f}_{dist}.$$

where the steady state error becomes

$$k_i e = 0.$$

The integral term accumulates the error over time.