Digital Control Systems
MAE/ECEN 5473

State space stability analysis: Lyapunov second method

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Continuous time systems

General nonlinear systems:

$$\dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n \tag{1}$$

Special cases: Linear time-invariant (LTI) systems

$$\dot{x} = Ax + Bu \tag{2}$$

Linear time-varying systems

$$\dot{x} = A(t)x + B(t)u \tag{3}$$

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By slight abuse of notation, both cases can be summarized as

$$\dot{x} = f(x, t)$$

Equilibrium (equilibria)

As the name suggests, at an equilibrium, the motion of a dynamical system vanishes, i.e., $\dot{x} = 0$ at an equilibrium x_e.

Find an equilibrium

Given $\dot{x} = f(x, t)$, a point x_e is an equilibrium if and only if it satisfies $f(x_e, t) = 0$, $\forall t$.

Observe: if $x(0) = x_e$, $x(t) = x_e$, $\forall t \geq 0$.

Linear and nonlinear examples

Stability definitions

Consider $\dot{x} = f(x,t)$, $x(t_0) = x_0$, $x \in \mathbb{R}^n$. Its solution is denoted by $\phi(t; x_0, t_0)$. Let x_e be an equilibrium of $\dot{x} = f(x,t)$.

Stable in the sense of Lyapunov

For every $\epsilon > 0$, let $S(\epsilon)$ be the set $\{x \mid \|x - x_e\| < \epsilon\}$. If there exists a $\delta > 0$ such that the trajectories $\phi(t; x_0, t_0)$ starting in $\|x_0 - x_e\| \leq \delta$ remain in $S(\epsilon)$, then x_e is stable in the sense of Lyapunov.

Asymptotic stability

Asymptotic stability (a.s.)

Stable in the sense of Lyapunov + convergence to the equilibrium x_e: Every solution starting inside $\|x_0 - x_e\| < \delta$ remains in $S(\epsilon)$ and converges to x_e as $t \to \infty$.

Global asymptotic stability (g.a.s.)

Stable in the sense of Lyapunov + global convergence to the equilibrium x_e: Every solution starting inside $\|x_0 - x_e\| < \delta$ remains in $S(\epsilon)$ and every solution converges to x_e as $t \to \infty$.

- G.A.S. is possible only if there is a single equilibrium.

Exponential stability

Exponential stability (e.s.)

a.s. + exponential convergence to the equilibrium x_e:
$\|\phi(t;x_0,t_0) - x_e\| \leq \|x_0 - x_e\| e^{-\lambda(t-t_0)}$, $\lambda > 0$.

Global exponential stability (g.e.s.)

g.a.s + exponential convergence

Instability

For some $\epsilon > 0$, any $\delta > 0$, there always exists a x_0 in $\|x_0 - x_e\| < \delta$ such that $\phi(t;x_0,t_0)$ leaves $S(\epsilon)$.

For LTI

- All stability is uniform, i.e., δ is independent of t_0.
- g.a.s. is equivalent to g.e.s.
- Either a single equilibrium or infinite many equilibria
- instability is equivalent to unbounded trajectories.

Without Loss of Generality (WLOG)

We always assume $x_e = 0$, i.e., the origin of x is an equilibrium.
Why?

Positive definite functions

Consider a scalar function $V(x) \in \mathbb{R}$ where $x \in \mathbb{R}^n$.

- $V(x)$ is positive definite (P.D.) in a region Ω where $0 \in \Omega$ if i) $V(x) > 0$, $\forall x \in \Omega, x \neq 0$, ii) $V(x) = 0$ when $x = 0$.
- $V(x)$ is positive semidefinite (PSD), if i) $V(x) \geq 0$, $\forall x \in \Omega$, ii) $V(x) = 0$ when $x = 0$.
- $V(x)$ is negative definite (N.D.) if $-V(x)$ is P.D.
- $V(x)$ is negative semidefinite (NSD) if $-V(x)$ is PSD

Examples

A special case

The scalar function $V(x) = x^T P x$ is PD if and only if (iff) the matrix $P \in \mathbb{R}^{n \times n}$ is a PD matrix.

PD matrix

A square matrix P is PD iff 1) $P \in \mathbb{R}^{n \times n}$ is symmetric, 2) all the eigenvalues of P must be positive.

Quiz Verify: i) $V(x) = x^T P x > 0$, $\forall x \neq 0$. ii) $x^T P x = 0$ when $x = 0$.

- P is P.S.D. iff 1) $P \in \mathbb{R}^{n \times n}$ is symmetric 2) all the eigenvalues of P are non-negative.
- $V(x) = x^T P x$ is PSD if P is PSD.
- $V(x) = x^T P x$ is ND (NSD) if $-P$ is PD (NSD).
- P can be sign-indefinite, as P can have both positive and negative eigenvalues.

Condition 2 in the PD matrix definition can be replaced with
2') its leading principal minors are positive.

The asymmetric case

Suppose that $V(x) = x^T Qx$, where Q is not a symmetric matrix.

Lyapunov stability theorem (second method)

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. Suppose that there exists a P.D. scalar function $V(x)$ such that

1. $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x)$ is N.S.D. Then the origin $x = 0$ is stable in the sense of Lyapunov.
2. $\dot{V}(x)$ is N.D. Then the origin $x = 0$ is a.s.
3. $\dot{V}(x)$ is N.D. and $V(x) \to \infty$ as $\|x\| \to \infty$. Then the origin is g.a.s.
4. $\dot{V}(x)$ is N.S.D. and $\dot{V}(x)$ doesn't vanish identically. Then the origin is a.s.

Example 1 (nonlinear)

Example 2 (linear)

Comments on the theorem

- Applies **only to** time-invariant systems!
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- Applies **only to** time-invariant systems!
- Condition 1-4 are only sufficient:
  If you can't find a $V(x)$ to satisfy either condition, it doesn't mean the origin is unstable.
  Finding $V(x)$ such that any condition is satisfied for $x \in \Omega$ doesn't mean that the system is unstable outside of Ω.
- $V(x)$: Lyapunov function, not unique
- Condition 4 is referred to as "LaSalle Invariance Principle".

How to find Lyapunov functions

Good news: For linear systems $\dot{x} = Ax$, the Lyapunov method is both sufficient and necessary.

Consider a *candidate* Lyapunov function $V(x) = x^T P x$, where P is PD.

Compute $\dot{V}(x)$.

- If $\dot{V}(x)$ is ND
- If $\dot{V}(x)$ is NSD
- If $\dot{V}(x)$ is NSD and

Necessary and Sufficient Condition for CT LTI

Given $\dot{x} = Ax$, a necessary and sufficient condition for $x = 0$ to be g.a.s. if that for any PD Q, there exists a PD P such that $A^T P + PA = -Q$. Then $V(x) = x^T Px$ is a Lyapunov function.

Example

Example

Instability theorem

Suppose that $W(x)$ is PD in a region Ω around the origin.
Suppose that $\dot{W}(x) = \frac{\partial W(x)}{\partial x} \cdot f(x)$ is PD in the same region Ω.
Then $x = 0$ is unstable.

Example

Extend to Discrete time systems: $x(k+1) = f(x(k))$

- Equilibrium

- Example

Lyapunov stability theorem in DT

Consider $x(k+1) = f(x(k))$. Assume that the origin is the equilibrium of interest. Suppose that there exists a scalar PD function $V(x)$ such that

1. $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = V(f(x(k))) - V(x(k))$ is NSD, then $x(k) = 0$ is stable in the sense of Lyapunov.
2. $\Delta V(x(k))$ is ND, then $x(k) = 0$ is a.s.
3. $\Delta V(x(k))$ is ND and $V(x(k)) \to \infty$ as $\|x(k)\| \to \infty$, $x(k) = 0$ is g.a.s.
4. $\Delta V(x(k))$ is NSD and $\Delta V(x(k))$ does not vanish identically for any solution satisfying $x(k+1) = f(x(k))$, $x(k) = 0$ is a.s.

For LTI systems in DT

Consider $x(k+1) = Gx(k)$, $x(k) \in \mathbb{R}^n$. Define a candidate Lyapunov function $V(x(k)) = x(k)^T P x(k)$, where $P \in \mathbb{R}^{n \times n}$ is PD. Calculate $\Delta V(x(k))$.

1. If
2. If
3. If

Necessary and Sufficient Condition for DT LTI

Given $x(k+1) = Gx(k)$, a necessary and sufficient condition for $x(k) = 0$ to be g.a.s. if that for any PD Q, there exists a PD P such that $G^T PG - P = -Q$. Then $V(x) = x^T Px$ is a Lyapunov function.

Example

Final comments on stability

Eigenvalue approach

Lyapunov approach

Final comments on stability

- Discretization does not change stability of LTI systems.

- Relationship to stability of transfer function.