Reduction from 4-sphere model to 3-sphere model as $r_4 \to r_3$ and $\sigma_4 \to \sigma_3$

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Here we show that the corrected analytical formulas for the four-sphere model fulfill the limiting case requiring the four-sphere model to reduce to the three-sphere model, when the radii and electrical conductivities are equal for the two outer spheres.

$$\Phi^4(r,\theta) = \Phi^3(r,\theta) \quad \forall \;\; r_4 = r_3$$

$$\Phi^3(r) = \frac{p}{4\pi\sigma_1 r_z^2} \sum_{n=1}^{\infty} \left[A_3^s\left(\frac{r}{r_3}\right)^n + B_n^3\left(\frac{r_3}{r}\right)^{n+1}\right] nP_n(\cos\theta) \tag{1}$$

$$\Phi^4(r) = \frac{p}{4\pi\sigma_1 r_z^2} \sum_{n=1}^{\infty} \left[A_4^s\left(\frac{r}{r_4}\right)^n + B_n^4\left(\frac{r_4}{r}\right)^{n+1}\right] nP_n(\cos\theta) \tag{2}$$

Inserting $r_3 = r_4$ into the expression for Φ^4, we can see that the requirement is fulfilled when $A_n^4 = A_n^3$ and $B_n^4 = B_n^3$.

As $r_4 \to r_3$ and $\sigma_4 \to \sigma_3$, we get the following expressions for A_n^4

$$\begin{aligned} A_n^4 &= \frac{n+1}{n} \frac{A_n^3 + B_n^3}{\frac{n+1}{n} r_{34}^n + r_{43}^{n+1}} \\ &= (A_n^3 + B_n^3) \frac{\frac{n+1}{n}}{\frac{n+1}{n} + 1}, \end{aligned}$$

and B_n^3

$$\begin{aligned} B_n^3 &= V_n A_n^3 \\ &= A_n^3 \frac{\frac{n}{n+1}\sigma_{34} - \frac{r_{34}^n - r_{43}^{n+1}}{\frac{n+1}{n} r_{34}^n + r_{43}^{n+1}}}{\sigma_{34} + \frac{r_{34}^n - r_{43}^{n+1}}{\frac{n+1}{n} r_{34}^n + r_{43}^{n+1}}} \\ &= \frac{n}{n+1} A_n^3. \end{aligned}$$

From here we can combine the two relations above, such that

$$A_n^4 = A_n^3 \frac{\frac{n+1}{n}\left(1 + \frac{n}{n+1}\right)}{\frac{n+1}{n} + 1} = A_n^3,$$

and find that $A_n^4 = A_n^3$ for $r_4 = r_3$ and $\sigma_4 = \sigma_3$.

Next we insert this into the expression for B_n^4, and make use of the relation $A_n^3 = \frac{n+1}{n} B_n^3$ from above

$$\begin{aligned} B_n^4 &= \frac{n}{n+1} A_n^4 \\ &= \frac{n}{n+1} A_n^3 \\ &= \frac{n}{n+1} \frac{n+1}{n} B_n^3 \\ &= B_n^3, \end{aligned}$$

and obtain that $B_n^4 = B_n^3$ for $r_4 = r_3$ and $\sigma_4 = \sigma_3$.