

Solutions to Problems in Introduction to Stochastic Processes (2nd Edition) by Gregory F. Lawler

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0. Preliminaries

0.1 Solution

The coefficient matrix (A) of this ODE is:

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \quad (0.1.1)$$

It is easy to see the diagonalization of matrix (A) being:

$$A = Q\Lambda Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & \\ & -4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \quad (0.1.2)$$

Therefore, the general solution to this ODE should be in the form of Eq. 0.1.3.

$$\vec{y}_c = e^{At}\vec{c} = Qe^{\Lambda t}Q^{-1}\vec{c} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & \\ & e^{-4t} \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \vec{c} = \frac{1}{4} \begin{bmatrix} 3 + e^{-4t} & 1 - e^{-4t} \\ 3 - 3e^{-4t} & 1 + 3e^{-4t} \end{bmatrix} \vec{c} \quad (0.1.3)$$

The particular solution for the given initial condition is:

$$\begin{aligned} \therefore \frac{1}{4} \begin{bmatrix} 3 + 1 & 1 - 1 \\ 3 - 3 & 1 + 3 \end{bmatrix} \vec{c} &= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \\ \therefore \vec{c} &= \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \end{aligned} \quad (0.1.4)$$

$$\therefore \vec{y}_p = \vec{c} = \frac{1}{4} \begin{bmatrix} 3 + e^{-4t} & 1 - e^{-4t} \\ 3 - 3e^{-4t} & 1 + 3e^{-4t} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad (0.1.5)$$

0.2 Solution

The characteristic equation for this problem is:

$$\lambda = \frac{1}{4} + \frac{3}{4}\lambda^2$$

$$\therefore \lambda_1 = 1, \lambda_2 = 1/3 \quad (0.2.1)$$

Its general solution is:

$$y_c = C_1 + C_2\left(\frac{1}{3}\right)^n \quad (0.2.2)$$

Under its given initial conditions, we have:

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + C_2/3 = 1 \end{cases}$$

$$\therefore C_1 = \frac{3}{2}, C_2 = -\frac{3}{2}$$

$$\therefore y_p = \frac{3}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^n \quad (0.2.3)$$

0.3 Solution

The characteristic equation for this problem is:

$$\lambda^2 - \lambda - 1 = 0$$

$$\therefore \lambda_1 = \frac{1 + \sqrt{5}}{2}, \lambda_2 = \frac{1 - \sqrt{5}}{2} \quad (0.3.1)$$

Its general solution is:

$$y_c = C_1\left(\frac{1 + \sqrt{5}}{2}\right)^n + C_2\left(\frac{1 - \sqrt{5}}{2}\right)^n \quad (0.3.2)$$

Under its given initial conditions, we have:

$$\begin{cases} C_1\left(\frac{1 + \sqrt{5}}{2}\right) + C_2\left(\frac{1 - \sqrt{5}}{2}\right) = 1 \\ C_1\left(\frac{1 + \sqrt{5}}{2}\right)^2 + C_2\left(\frac{1 - \sqrt{5}}{2}\right)^2 = 1 \end{cases}$$

$$\therefore C_1 = \frac{1}{\sqrt{5}}, C_2 = -\frac{1}{\sqrt{5}}$$

$$\therefore y_p = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n \quad (0.3.3)$$

0.4 Solution

The characteristic equation for this problem is:

$$3\lambda = 1 + \lambda^2 + \lambda^3$$

$$\therefore (\lambda - 1)(\lambda^2 + 2\lambda - 1) = 0$$

$$\therefore \lambda_1 = 1, \lambda_2 = -1 + \sqrt{2}, \lambda_3 = -1 - \sqrt{2}$$

Its general solution is:

$$\therefore y_c = C_1 + C_2(-1 + \sqrt{2})^n + C_3(-1 - \sqrt{2})^n \quad (0.4.1)$$

Under its initial conditions:

$$\therefore \lim_{n \rightarrow +\infty} y_c \text{ exists and } = 1$$

$$\therefore C_1 = 1, C_3 = 0 \quad (0.4.2)$$

$$\therefore y_p(n = 0) = C_1 + C_2 = 0$$

$$\therefore C_2 = -1$$

$$\therefore y_p = 1 - (\sqrt{2} - 1)^n \quad (0.4.2)$$

0.5 Solution

First, one needs to solve its corresponding homogeneous equation:

$$f(n) = \frac{1}{2}f(n+1) + \frac{1}{2}f(n-1)$$

Its general solution is:

$$\tilde{y}_c = C_1 + C_2 n \quad (0.5.1)$$

Then let's try and get a particular solution for this heterogeneous equation using **undetermined coefficient method** by assuming a particular solution in the form of Eq. 0.5.2. (In fact, B and C in Eq. 0.5.2 are not necessary, since they are already implied in its general solution.)

$$y_p = An^2 + Bn + C \quad (0.5.2)$$

By taking y_p into the original equation, we will see that $A = 1$. Hence the general solution to this heterogeneous equation is:

$$y_c = C_1 + C_2 n + n^2 \quad (0.5.3)$$

0.6 Solution

Both (a) and (b) have the identical characteristic equation:

$$\lambda^2 + \lambda + 1 = 0$$

$$\therefore \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Let's denote $\lambda = a \pm bi$ for simplicity.

(a) ODE

$$\begin{aligned} y_c &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} \\ &= e^{at} (C_1 (\cos bt + i \sin bt) + C_2 (\cos bt - i \sin bt)) \\ &= e^{at} (D_1 \cos bt + D_2 \sin bt) \\ \therefore y_c &= e^{-\frac{1}{2}t} (D_1 \cos \frac{\sqrt{3}}{2}t + D_2 \sin \frac{\sqrt{3}}{2}t) \end{aligned} \tag{0.6.1}$$

(b) Difference equation

$$y_c = C_1 \lambda_1^n + C_2 \lambda_2^n$$

On the other hand, we have:

$$\begin{aligned} \lambda &= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i = e^{\pm \frac{2}{3}i\pi} \\ \therefore \lambda^n &= e^{\pm \frac{2}{3}in\pi} = \cos \frac{2}{3}n\pi \pm i \sin \frac{2}{3}n\pi \end{aligned}$$

Ultimately, we have:

$$y_c = D_1 \cos \frac{2}{3}n\pi + D_2 \sin \frac{2}{3}n\pi$$