Solutions to Problems in Introduction to Stochastic Processes (2nd Edition) by Gregory F. Lawler

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0. Preliminaries

0.1 Solution

The coefficient matrix (A) of this ODE is:

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \tag{0.1.1}$$

It is easy to see the diagonalization of matrix (A) being:

$$A = Q\Lambda Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & \\ & -4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \tag{0.1.2}$$

Therefore, the general solution to this ODE should be in the form of Eq. 0.1.3.

$$\vec{y_c} = e^{At}\vec{c} = Qe^{\Lambda t}Q^{-1}\vec{c} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & \\ & e^{-4t} \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{bmatrix} \vec{c} = \frac{1}{4} \begin{bmatrix} 3+e^{-4t} & 1-e^{-4t} \\ 3-3e^{-4t} & 1+3e^{-4t} \end{bmatrix} \vec{c} \tag{0.1.3}$$

The particular solution for the given initial condition is:

$$\because \frac{1}{4} \begin{bmatrix} 3+1 & 1-1 \\ 3-3 & 1+3 \end{bmatrix} \vec{c} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\therefore \vec{c} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \tag{0.1.4}$$

$$\therefore \vec{y_p} = \vec{c} = \frac{1}{4} \begin{bmatrix} 3+e^{-4t} & 1-e^{-4t} \\ 3-3e^{-4t} & 1+3e^{-4t} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \tag{0.1.5}$$

0.2 Solution

The characteristic equation for this problem is:

$$\lambda = \frac{1}{4} + \frac{3}{4}\lambda^2$$

$$\therefore \lambda_1 = 1, \lambda_2 = 1/3 \tag{0.2.1}$$

Its general solution is:

$$y_c = C_1 + C_2(\frac{1}{3})^n \tag{0.2.2}$$

Under its given initial conditions, we have:

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 + C_2/3 = 1 \end{cases}$$

$$\therefore C_1 = \frac{3}{2}, C_2 = -\frac{3}{2}$$

$$\therefore y_p = \frac{3}{2} - \frac{3}{2}(\frac{1}{3})^n \tag{0.2.3}$$

0.3 Solution

The characteristic equation for this problem is:

$$\lambda^2 - \lambda - 1 = 0$$

$$\therefore \lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2} \tag{0.3.1}$$

Its general solution is:

$$y_c = C_1(\frac{1+\sqrt{5}}{2})^n + C_2(\frac{1-\sqrt{5}}{2})^n \tag{0.3.2}$$

Under its given initial conditions, we have:

$$\begin{cases} C_1(\frac{1+\sqrt{5}}{2}) + C_2(\frac{1-\sqrt{5}}{2}) = 1 \\ C_1(\frac{1+\sqrt{5}}{2})^2 + C_2(\frac{1-\sqrt{5}}{2})^2 = 1 \end{cases}$$

$$\therefore C_1 = \frac{1}{\sqrt{5}}, C_2 = -\frac{1}{\sqrt{5}}$$

$$\therefore y_p = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n \tag{0.3.3}$$

0.4 Solution

The characteristic equation for this problem is:

$$3\lambda = 1 + \lambda^2 + \lambda^3$$

$$\therefore (\lambda - 1)(\lambda^2 + 2\lambda - 1) = 0$$

$$\therefore \lambda_1 = 1, \lambda_2 = -1 + \sqrt{2}, \lambda_3 = -1 - \sqrt{2}$$

Its general solution is:

$$\therefore y_c = C_1 + C_2(-1 + \sqrt{2})^n + C_3(-1 - \sqrt{2})^n \tag{0.4.1}$$

Under its initial condistions:

$$\because \lim_{n \to +\infty} y_c \ exists\ and = 1$$

$$\therefore C_1 = 1, C_3 = 0 \tag{0.4.2}$$

$$\because y_p(n = 0) = C_1 + C_2 = 0$$

$$\therefore C_2 = -1$$
$$\therefore y_p = 1 - (\sqrt{2} - 1)^n \tag{0.4.2}$$

0.5 Solution

First, one needs to solve its corresponding homogeneous equation:

$$f(n) = \frac{1}{2} f(n+1) + \frac{1}{2} f(n-1)$$

Its general solution is:

$$\tilde{y_c} = C_1 + C_2 n \tag{0.5.1}$$

Then let's try and get a particular solution for this heterogeneous equation using **undetermined coefficient method** by assuming a particular solution in the form of Eq. 0.5.2. (In fact, B and C in Eq. 0.5.2 are not necessary, since they are already implied in its general solution.)

$$y_p = An^2 + Bn + C \tag{0.5.2}$$

By taking y_p into the original equation, we will see that $A = 1$. Hence the general solution to this heterogeneous equation is:

$$y_c = C_1 + C_2 n + n^2 \tag{0.5.3}$$

0.6 Solution

Both (a) and (b) have the identical characteristic equation:

$$\lambda^2 + \lambda + 1 = 0$$

$$\therefore \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Let's denote $\lambda = a \pm bi$ for simplicity.

(a) ODE

$$\begin{aligned} y_c &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t} \\ &= e^{at}(C_1(\cos bt + i \sin bt) + C_2(\cos bt - i \sin bt)) \\ &= e^{at}(D_1 \cos bt + D_2 \sin bt) \end{aligned}$$

$$\therefore y_c = e^{-\frac{1}{2}t}(D_1 \cos \frac{\sqrt{3}}{2} t + D_2 \sin \frac{\sqrt{3}}{2} t) \tag{0.6.1}$$

(b) Difference equation

$$y_c = C_1 \lambda_1^n + C_2 \lambda_2^n$$

On the other hand, we have:

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{\pm \frac{2}{3} i \pi}$$

$$\therefore \lambda^n = e^{\pm \frac{2}{3} i n \pi} = \cos \frac{2}{3} n\pi \pm i \sin \frac{2}{3} n\pi$$

Ultimately, we have:

$$y_c = D_1 \cos \frac{2}{3} n\pi + D_2 sin \frac{2}{3} n\pi$$

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Reference

Lawler, G. F. (2018). Introduction to stochastic processes. Chapman and Hall/CRC. https://doi.org/10.1201/9781315273600