Exercise 1.

As a first step for solving the Linear Program, we define the dual LP:

$$\begin{aligned} \textbf{min} \quad & \beta y_0 + \sum_{j=1}^{n} y_j \\ \textbf{s.t.} \quad & y_0 q_j + y_j \geq p_j \quad \forall j = 1, \dots, n \\ & y_j \geq 0 \quad \forall j = 1, \dots, n \end{aligned}$$

As a next step, we modify the dual problem, so that it is in standard form and we divide the constraints by q_j.

$$\begin{aligned} \textbf{max} \quad & -\beta y_0 - \sum_{j=1}^{n} y_j \\ \textbf{s.t.} \quad & -y_0 - \frac{y_j}{q_j} \leq -\frac{p_j}{q_j} \quad \forall j = 1, \dots, n \\ & y_j \geq 0 \quad \forall j = 1, \dots, n \end{aligned}$$

Now we can formulate this LP in slack form as a dictionary.

$$\begin{aligned} z &= -\beta y_0 - \sum_{j=1}^{n} y_j \\ y_{n+j} &= -\frac{p_j}{q_j} + y_0 + \frac{y_j}{q_j} \quad \forall j = 1, \dots, n \end{aligned}$$

Since the initial solution of the simplex method for this LP is not feasible, we look at the auxiliary LP:

$$\begin{aligned} z &= -y_{\text{aux}} \\ y_{n+j} &= -\frac{p_j}{q_j} + y_0 + \frac{y_j}{q_j} + y_{\text{aux}} \quad \forall j = 1, \dots, n \end{aligned}$$

We choose y_{aux} as the entering variable. Since $-\frac{p_n}{q_n}$ is the "most infeasible", we choose y_{2n} as the leaving variable.

$$\Rightarrow \quad y_{\text{aux}} = y_{2n} + \frac{p_n}{q_n} - y_0 - \frac{y_n}{q_n}$$

$$\begin{aligned} z &= -\frac{p_n}{q_n} + y_0 + \frac{y_n}{q_n} - y_{2n} \\ y_{n+j} &= \frac{p_n}{q_n} - \frac{p_j}{q_j} - \frac{y_n}{q_n} + \frac{y_j}{q_j} + y_{2n} \quad \forall j = 1, \dots, n-1 \\ y_{\text{aux}} &= \frac{p_n}{q_n} - y_0 - \frac{y_n}{q_n} + y_{2n} \end{aligned}$$

Because $\sum_{j=1}^{n} q_j = 1$ and $q_j > 0 \, \forall j = 1, \dots, n$, we know that $q_j \leq 1 \, \forall j = 1, \dots, n$. Therefore y_n is chosen as the entering variable, because $\frac{1}{q_n} \geq 1$. Since y_n has the same coefficient in all constraints, we choose the constraint with the smallest constant as the leaving variable. This is y_{2n-1}.

$$\Rightarrow \quad y_n = q_n \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} + \frac{y_{n-1}}{q_{n-1}} - y_{2n-1} + y_{2n} \right)$$

$$\begin{aligned}
z &= -\frac{p_{n-1}}{q_{n-1}} + y_0 + \frac{y_{n-1}}{q_{n-1}} - y_{2n-1} \\
y_{n+j} &= \frac{p_{n-1}}{q_{n-1}} - \frac{p_j}{q_j} - \frac{y_{n-1}}{q_{n-1}} + \frac{y_j}{q_j} + y_{2n-1} && \forall j = 1, \dots, n-2 \\
y_{\text{aux}} &= \frac{p_{n-1}}{q_{n-1}} - y_0 - \frac{y_{n-1}}{q_{n-1}} + y_{2n-1} \\
y_n &= q_n \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} + \frac{y_{n-1}}{q_{n-1}} - y_{2n-1} + y_{2n} \right)
\end{aligned}$$

This step is repeated for $i = 1, \dots, n-1$ times where always y_{n-i+1} is chosen as the entering and y_{2n-i} as the leaving variable. After step i we therefore have the following dictionary:

$$\begin{aligned}
z &= -\frac{p_{n-i}}{q_{n-i}} + y_0 + \frac{y_{n-i}}{q_{n-i}} - y_{2n-i} \\
y_{n+j} &= \frac{p_{n-i}}{q_{n-i}} - \frac{p_j}{q_j} - \frac{y_{n-i}}{q_{n-i}} + \frac{y_j}{q_j} + y_{2n-i} && \forall j = 1, \dots, n-i-1 \\
y_{\text{aux}} &= \frac{p_{n-i}}{q_{n-i}} - y_0 - \frac{y_{n-i}}{q_{n-i}} + y_{2n-i} \\
y_{n-j} &= q_{n-j} \left(\frac{p_{n-j}}{q_{n-j}} - \frac{p_{n-i}}{q_{n-i}} + \frac{y_{n-i}}{q_{n-i}} - y_{2n-i} + y_{2n-j} \right) && \forall j = 0, \dots, i-1
\end{aligned}$$

After the last step ($i = n-1$), we therefore get the following dictionary:

$$\begin{aligned}
z &= -\frac{p_1}{q_1} + y_0 + \frac{y_1}{q_1} - y_{n+1} \\
y_{\text{aux}} &= \frac{p_1}{q_1} - y_0 - \frac{y_1}{q_1} + y_{n+1} \\
y_{n-j} &= q_{n-j} \left(\frac{p_{n-j}}{q_{n-j}} - \frac{p_1}{q_1} + \frac{y_1}{q_1} - y_{n+1} + y_{2n-j} \right) && \forall j = 0, \dots, n-2
\end{aligned}$$

Now we can do one final step where y_1 is the entering and y_{aux} is the leaving variable:

$$\Rightarrow \quad y_1 = q_1 \left(\frac{p_1}{q_1} - y_0 + y_{n+1} - y_{\text{aux}} \right)$$

$$\begin{aligned}
z &= -y_{\text{aux}} \\
y_j &= q_j \left(\frac{p_j}{q_j} - y_0 - y_{\text{aux}} + y_{n+j} \right) && \forall j = 1, \dots, n
\end{aligned}$$

This tells us our initial solution for the modified dual LP:

$$\hat{y}_j = \begin{cases} 0 & \text{for } j = 0 \\ p_j & \text{for } j = 1, \dots, n \\ 0 & \text{for } j = n+1, \dots, 2n \end{cases}$$

Now we go back to the modified dual problem by dropping y_{aux} and changing the objective function.

$$z = -\beta y_0 - \sum_{j=1}^{n} y_j = -\beta y_0 - \sum_{j=1}^{n} p_j + \sum_{j=1}^{n} q_j y_0 - \sum_{j=1}^{n} q_j y_{n+j} = (1-\beta) y_0 - 1 - \sum_{j=1}^{n} q_j y_{n+j}$$

$$\begin{aligned} z &= -1 + (1-\beta)y_0 - \sum_{j=1}^{n} q_j y_{n+j} \\ y_j &= q_j \left(\frac{p_j}{q_j} - y_0 + y_{n+j} \right) && \forall j = 1, \dots, n \end{aligned}$$

From $\frac{\beta}{q_j} < 1 \, \forall j = 1, \dots, n$, we know that $\beta < q_j \, \forall j = 1, \dots, n$ and therefore $1-\beta > 0$. Therefore we choose y_0 as the entering variable. From each of the constraints, we find that we can increase y_0 to a maximum of $\frac{p_j}{q_j}$. This is smallest for the constraint for y_1, which is therefore chosen as the leaving variable.

$$y_0 = \frac{p_1}{q_1} - \frac{y_1}{q_1} + y_{n+1}$$

$$\begin{aligned} z &= (1-\beta)\frac{p_1}{q_1} - 1 - (1-\beta)\frac{y_1}{q_1} + (1-\beta-q_1)y_{n+1} - \sum_{j=2}^{n} q_j y_{n+j} \\ y_j &= q_j \left(\frac{p_j}{q_j} - \frac{p_1}{q_1} + \frac{y_1}{q_1} + y_{n+j} - y_{n+1} \right) && \forall j = 2, \dots, n \\ y_0 &= \frac{p_1}{q_1} - \frac{y_1}{q_1} + y_{n+1} \end{aligned}$$

We know that $\sum_{j=1}^{n} q_j = 1$ and that $\beta < q_j \, \forall j = 1 \dots, n$, therefore we know that a sum over not all q_j is smaller then $1-\beta$. Because of this, we know that $1-\beta-q_1 > 0$ and we choose y_{n+1} as the leaving variable. For each constraint we can increase y_{n+1} to a maximum of $\frac{p_j}{q_j}$, we therefore choose y_2 as the leaving variable.

$$y_{n+1} = \frac{p_2}{q_2} - \frac{p_1}{q_1} + \frac{y_1}{q_1} - \frac{y_2}{q_2} + y_{n+2}$$

$$\begin{aligned} z &= -1 + p_1 - y_1 + (1-\beta-q_1)(\frac{p_2}{q_2} - \frac{y_2}{q_2}) + (1-\beta-q_1-q_2)y_{n+2} - \sum_{j=3}^{n} q_j y_{n+j} \\ y_j &= q_j \left(\frac{p_j}{q_j} - \frac{p_2}{q_2} + \frac{y_2}{q_2} + y_{n+j} - y_{n+2} \right) \quad \forall j = 3, \dots, n \\ y_0 &= \frac{p_2}{q_2} - \frac{y_2}{q_2} + y_{n+2} \\ y_{n+1} &= \frac{p_2}{q_2} - \frac{p_1}{q_1} + \frac{y_1}{q_1} - \frac{y_2}{q_2} + y_{n+2} \end{aligned}$$

This step does now again repeat for $i = 2, \dots, n$ and we can formulate the dictionary after the ith step as follows:

$$\begin{aligned} z &= -1 + \sum_{j=1}^{i-1} (p_j - y_j) + (1-\beta-\sum_{j=1}^{i-1} q_j)(\frac{p_i}{q_i} - \frac{y_i}{q_i}) + (1-\beta-\sum_{j=1}^{i} q_j)y_{n+i} - \sum_{j=i+1}^{n} q_j y_{n+j} \\ y_j &= q_j \left(\frac{p_j}{q_j} - \frac{p_i}{q_i} + \frac{y_i}{q_i} + y_{n+j} - y_{n+i} \right) \quad \forall j = i+1, \dots, n \\ y_0 &= \frac{p_i}{q_i} - \frac{y_i}{q_i} + y_{n+i} \\ y_{n+j} &= \frac{p_i}{q_i} - \frac{p_j}{q_j} + \frac{y_j}{q_j} - \frac{y_i}{q_i} + y_{n+i} \quad \forall j = 1, \dots, i-1 \end{aligned}$$

After the last step ($i = n$), we therefore get the following dictionary:

$$
\begin{aligned}
z &= -1 + \sum_{j=1}^{n-1} (p_j - y_j) + (1 - \beta - \sum_{j=1}^{n-1} q_j)(\frac{p_n}{q_n} - \frac{y_n}{q_n}) - \beta y_{2n} \\
y_0 &= \frac{p_n}{q_n} - \frac{y_n}{q_n} + y_{2n} \\
y_{n+j} &= \frac{p_n}{q_n} - \frac{p_j}{q_j} + \frac{y_j}{q_j} - \frac{y_n}{q_n} + y_{2n} \quad \forall j = 1, \dots, n-1
\end{aligned}
$$

Here, all coefficients in the objective function are negative and the simplex method does therefore terminate. We get the following optimal solution for the (modified) dual problem:

$$
y_j^* = \begin{cases} \frac{p_n}{q_n} & \text{for } j = 0 \\ 0 & \forall j = 1, \dots, n \\ \frac{p_n}{q_n} - \frac{p_{n-j}}{q_{n-j}} & \forall j = n, \dots, 2n-1 \\ 0 & \text{for } j = 2n \end{cases}
$$

We can now find the solution for the primal problem with the help of the complementary slackness theorem. We know that where the dual solution is $\neq 0$, the primal constraint needs to be tight, therefore since $y_0 \neq 0$ we know that

$$
\sum_{j=1}^{n} q_j x_j = \beta \tag{1}
$$

In addition to that, we know that where the constraints of the dual are slack, the primal value needs to be zero. In the constraints of the dual we see the following:

$$
\begin{aligned}
-y_0 + \frac{y_j}{q_j} &\leq -\frac{p_j}{q_j} \forall j = 1, \dots, n \\
\Rightarrow \quad -y_0 &\leq -\frac{p_j}{q_j} \forall j = 1, \dots, n \\
\Rightarrow \quad -\frac{p_n}{q_n} &\leq -\frac{p_j}{q_j} \forall j = 1, \dots, n
\end{aligned}
$$

We can see that all constraints are slack, except for the one for $j = n$. Therefore we know that $x_j = 0 \, \forall j = 1, \dots, n-1$.

From that and (1), we can now compute x_n:

$$
\sum_{j=1}^{n} q_j x_j = \beta = q_n x_n \quad \Rightarrow \quad x_n = \frac{\beta}{q_n}
$$

The final (optimal) solution to the primal problem is therefore:

$$
x_j^* = \begin{cases} 0 & \forall j = 1, \dots, n \\ \frac{\beta}{q_n} & \text{for } j = n \end{cases}
$$

Exercise 2.

The Manhattan metrics between two points out of $\mathbb{R}^2$ is defined as:

$$d(p_i, p_j) = |x_i - x_j| + |y_i - y_j|$$

a) In the following figure, the rectangles along with the points of minimum distances are illustrated:

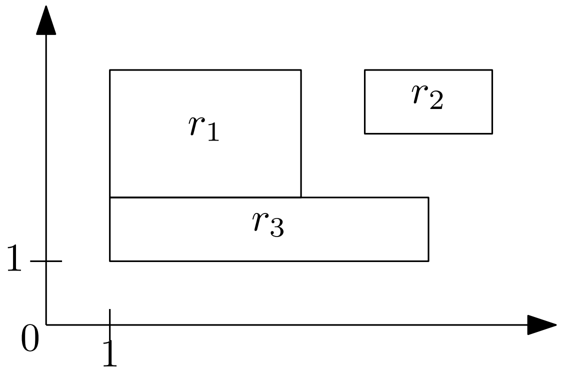
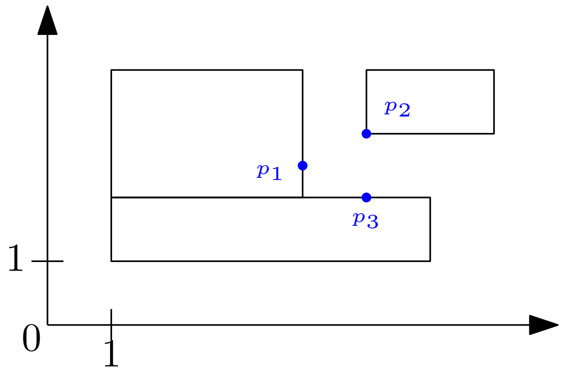


Illustration of the given rectangles



Solution to the given MMDPS problem

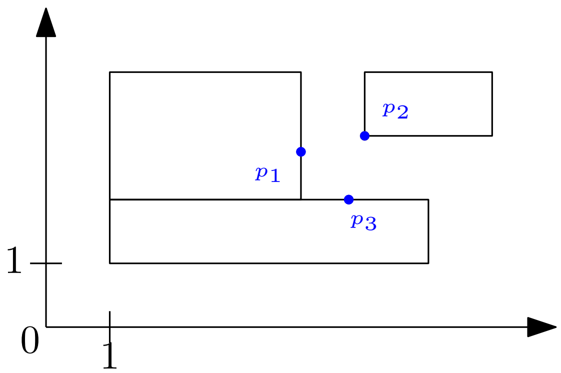
An optimal solution is:

$$p_1 = (4, 2.5), p_2 = (5, 3), p_3 = (5, 2)$$

with a diameter of 1.5.

Proof. The minimum distance from r_2 to r_1 or r_3 is 1 due to the gap. To keep the distance to r_3 in y direction and to r_1 in x direction minimal, p_2 is set to the bottom left corner. Then, with two distances of 1, there is a diameter of 2 since the point set would span a square at its corners. So, the lower bound of the diameter is 1, the upper bound 2 with this approach.
Next, it is to show that there is no point set with a lower diameter of 1.5. Suppose, there exists a point set p'_1, p'_2, p'_3 with fixed $p'_2 = p_2$ and a diameter of less than 1.5. Then, $d(p'_i, p'_j) < 1.5$. Look at w.l.o.g. fixed p'_2, p'_1. Then, every point p'_3 fulfilling the diameter constraint is not in r_3, for all fixed points, contradicting the Minimum Manhattan Diameter Point Set. □

Note that this is not the only optimal solution, consider this second solution in the figure below:



Alternative solution

b) The following LP formulates the MMDPS.

$$\mathbf{min} z$$

$$
\begin{array}{llrrrrl}
\textbf{s.t.} & x_i - & x_l^i & & & \geq 0 & (2)\\
& x_r^i - & x_i & & & \geq 0 & (3)\\
& y_t^i - & y_i & & & \geq 0 & (4)\\
& y_i - & y_b^i & & & \geq 0 & (5)\\
& x_i - & x_j + & y_i - & y_j & \leq z & (6)\\
& -x_i + & x_j + & y_i - & y_j & \leq z & (7)\\
& x_i - & x_j + & -y_i + & y_j & \leq z & (8)\\
& -x_i + & x_j + & -y_i + & y_j & \leq z & (9)\\
& & & & & \forall i,j \in \{1,...,n\}, i \neq j & (10)
\end{array}
$$

Constraints 1 to 4 state that the points shall be in the rectangles. Constraints 5 to 8 formulate the Manhattan distance by resolving a non-linear inequality with two linear inequalitites. z will be the diameter to minimize.

Exercise 3.

In order to solve the given LP with the primal-dual method, we first formulate the dual problem (D):

$$\begin{array}{rr}
 & (D) \\
\textbf{max} & 2y_1 + y_2 \\
\textbf{s.t.} & y_1 - y_2 \leq 2 \\
 & -2y_1 + 3y_2 \leq 1
\end{array}$$

As the initial solution for (D), we use $y^{(0)} = (0,0)^T$ and based on that formulate the first iteration of the set I:

$$I\left(y^{(0)}\right) = \left\{ j \mid 1 \leq j \leq 2 \vee \sum_i a_{ij} y_i^{(0)} = c_j \right\} = \emptyset$$

Based on that we can formulate the first restricted problem:

$$\begin{array}{rr}
 & (R^{(0)}) \\
\textbf{min} & s_1 + s_2 \\
\textbf{s.t.} & s_1 = 2 \\
 & s_2 = 1 \\
 & s \geq 0
\end{array}$$

The solution for that is obviously $s^{*(0)} = (2,1)^T$.

Now we consider the dual of this problem:

$$\begin{array}{rr}
 & (DR^{(0)}) \\
\textbf{max} & 2\pi_1 + \pi_2 \\
\textbf{s.t.} & \pi \leq 1
\end{array}$$

which has the solution $\pi^{*(0)} = (1,1)^T$.

Now we want to improve the solution to the dual problem. For that consider all $j \in \bar{I}$ and find that for $j = 2$, $\sum_i a_{ij} \pi_i^{*(0)} > 0$. Therefore we get the following t:

$$t^{(0)} = \min_{j \in \bar{I}} \frac{c_j - \sum_i a_{ij} y_i^{(0)}}{\sum_i a_{ij} \pi_i^{*(0)}} = \frac{c_2 - \sum_i a_{i2} y_i^{(0)}}{\sum_i a_{i2} \pi_i^{*(0)}} = 1$$

Now we improve y

$$y^{(1)} = y^{(0)} + t^{(0)} \pi^{*(0)} = (1,1)^T$$

and get a new set I:

$$I\left(y^{(1)}\right) = \{2\}$$

We formulate the restricted problem:

$$\begin{array}{rr}
 & (R^{(1)}) \\
\textbf{min} & s_1 + s_2 \\
\textbf{s.t.} & -2x_2 + s_1 = 2 \\
 & 3x_2 + s_2 = 1 \\
 & x_2 \geq 0 \\
 & s \geq 0
\end{array}$$

and find the solution with the help of an LP solver[1]: $s^{*(1)} = (\frac{2}{3}, 0)^T$.

We formulate the dual restricted problem:

$$
\begin{aligned}
&(DR^{(1)}) \\
&\textbf{max} \quad 2\pi_1 + \pi_2 \\
&\textbf{s.t.} \quad -2\pi_1 + 3\pi_2 \leq 0 \\
&\qquad\qquad \pi \leq 1
\end{aligned}
$$

which has the solution $\pi^{*(1)} = (1, \frac{2}{3})^T$.

We look at all $j \in \bar{I}$ and find that for $j = 2$, $\sum_i a_{ij}\pi_i^{*(0)} > 0$. Therefore we get the following t:

$$t^{(1)} = \frac{c_1 - \sum_i a_{i1} y_i^{(0)}}{\sum_i a_{i1}\pi_i^{*(0)}} = 6$$

We improve y again:

$$y^{(2)} = y^{(1)} + t^{(1)}\pi^{*(1)} = (7, 5)^T$$

and get a new set I:

$$I\left(y^{(2)}\right) = \{1, 2\}$$

We formulate the restricted problem:

$$
\begin{aligned}
&(R^{(2)}) \\
&\textbf{min} \quad s_1 + s_2 \\
&\textbf{s.t.} \quad x_1 + -2x_2 + s_1 = 2 \\
&\qquad -x_1 + 3x_2 + s_2 = 1 \\
&\qquad\qquad x \geq 0 \\
&\qquad\qquad s \geq 0
\end{aligned}
$$

and find the solution $s^{*(2)} = (0, 0)^T$. Since the objective function of the restricted problem is 0, we know that $y^{(2)}$ is optimal for (D) and that $x^* = (8, 3)^T$ as given by the solution of the restricted problem is the optimal solution for the primal problem.

[1] `https://online-optimizer.appspot.com`