On Smooth Orthogonal and Octilinear Drawings: Relations, Complexity and Kandinsky Drawings

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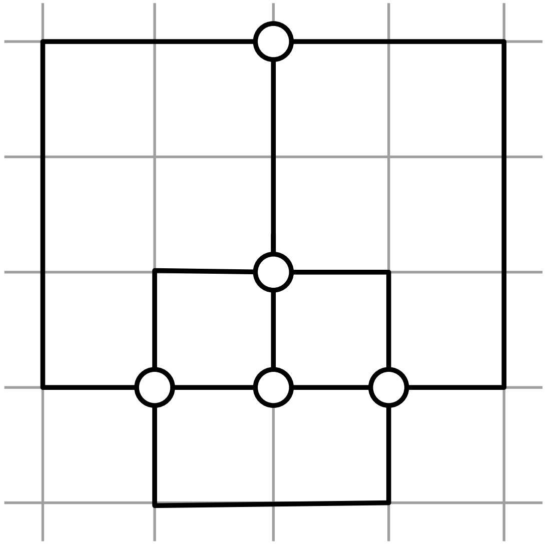
Wilhelm-Schickard-Institut für Informatik
Universität Tübingen, Germany

Smooth Orthogonal and Octilinear Drawings

- Extensions of the orthogonal graph drawing model



Orthogonal
Complexity 3

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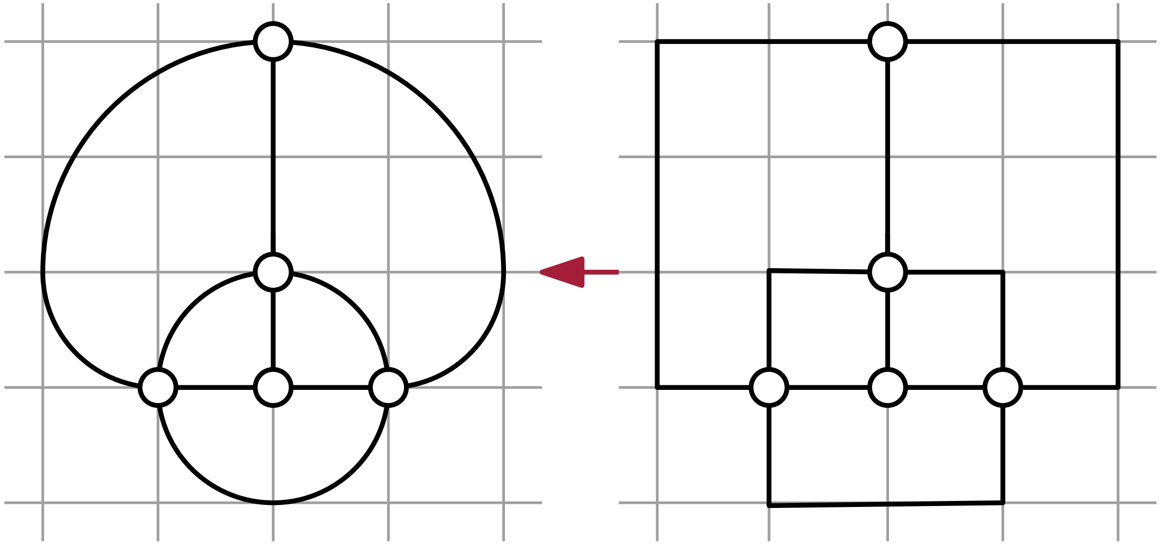

Smooth Orthogonal
Complexity 2

Orthogonal
Complexity 3

- Smooth orthogonal: Clarity of orthogonal layouts
  + Aesthetics of Lombardi drawings

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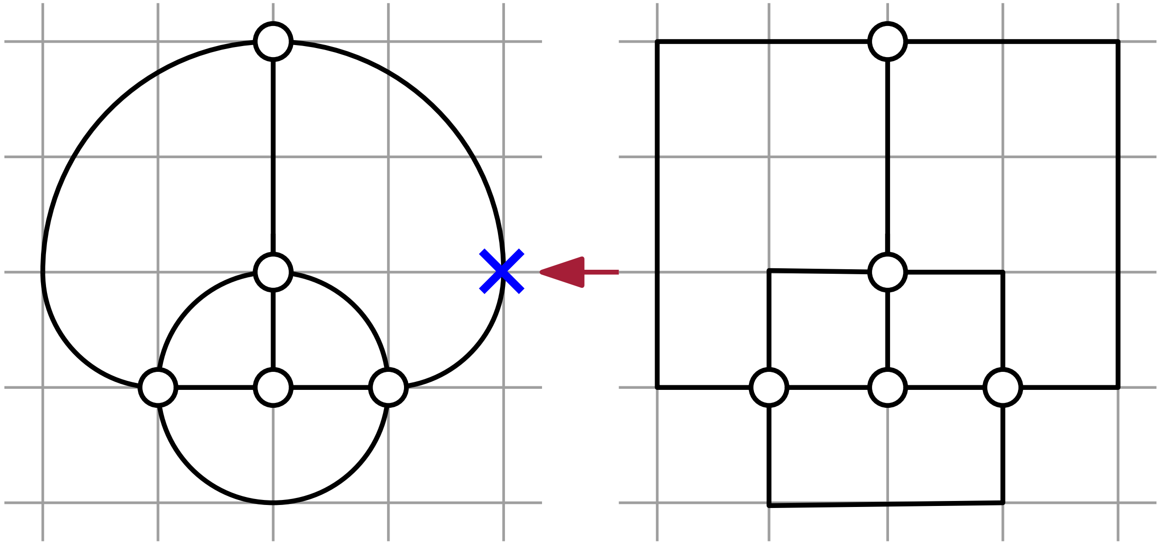

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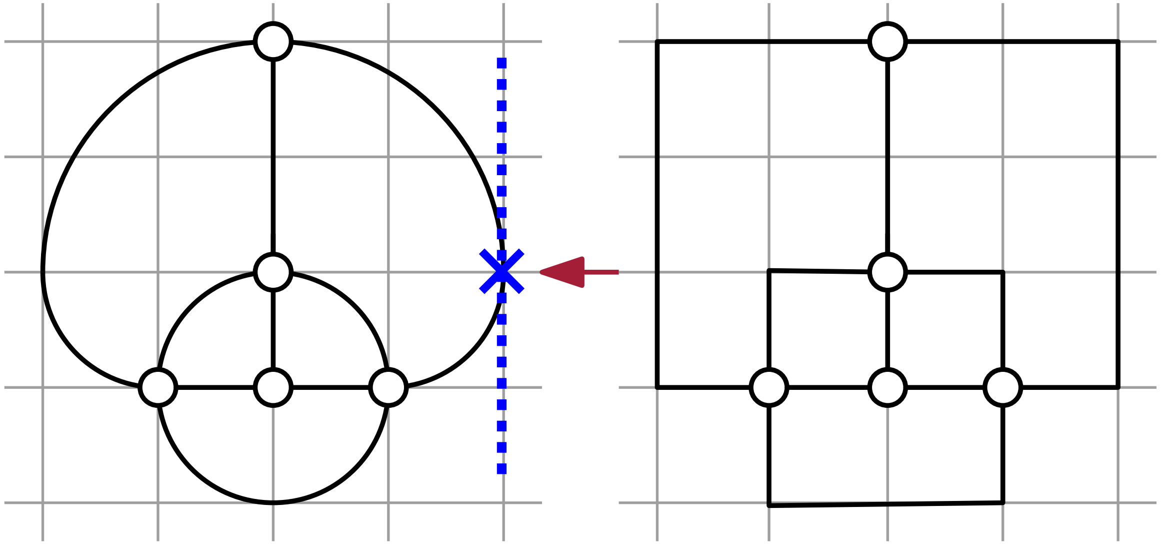

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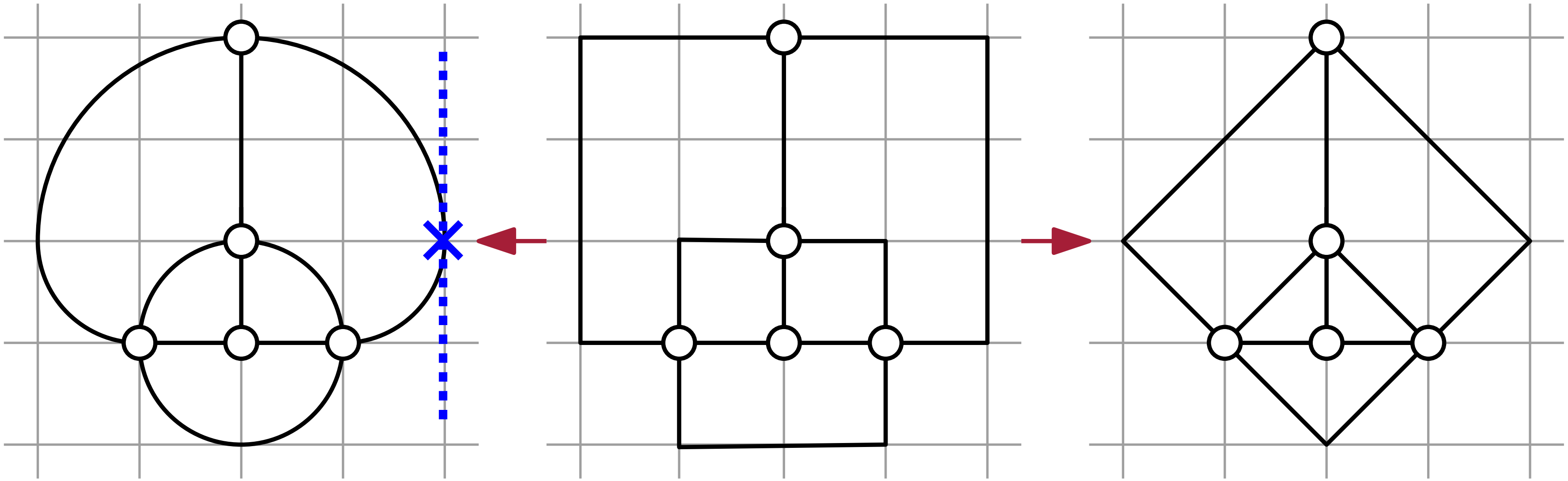

Smooth Orthogonal
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Orthogonal
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Octilinear
Complexity 2

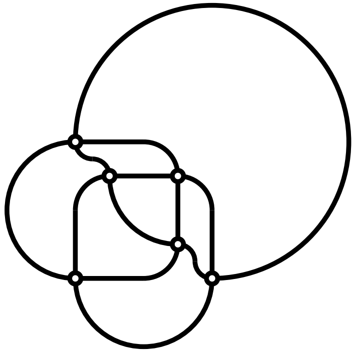
- Smooth orthogonal: Clarity of orthogonal layouts
  + Aesthetics of Lombardi drawings
- Octilinear: Generalization to max-degree 8
  + Metromap applications

Known Results

- **Relations**
- **Complexity**
- **Kandinsky Drawings**

Known Results

- **Relations**
  - Not all max-degree 4 graphs admit bendless smooth orthogonal/octilinear drawings [Bekos et al. 2013, Bekos et al. 2017]



  - 1 bend per edge suffices for max-degree 4 graphs in both models [Alam et al. 2014, Bekos et al. 2015]
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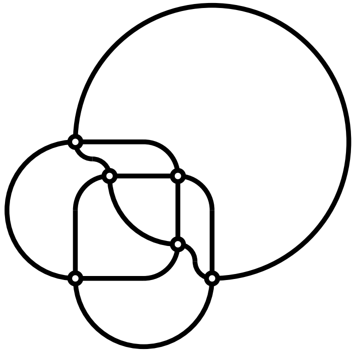

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- **Complexity**
  - Bendless octilinear drawing problem $\mathcal{NP}$-hard on max-degree 8 graphs
    [Nöllenburg 2005]
- **Kandinsky Drawings**

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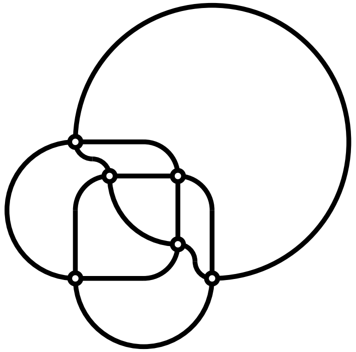

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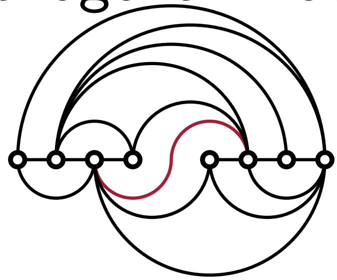
- **Kandinsky Drawings**
  - Book embedding inspired approach for smooth orthogonal model ($< n$ edges with edges of complexity 2)
    [Bekos et al. 2013, Cardinal et al. 2015]



Our Contribution

- **Relations**

- **Complexity**

- **Kandinsky Drawings**

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- **Relations**
  - Classes of bendless smooth orthogonal drawable (SC_1) and octilinear drawable ($8C_1$) graphs are incomparable
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Our Contribution

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  - Deciding if a smooth orthogonal or octilinear representation is realizable is $\mathcal{NP}$-hard on max-degree 4 graphs
- **Kandinsky Drawings**

Our Contribution

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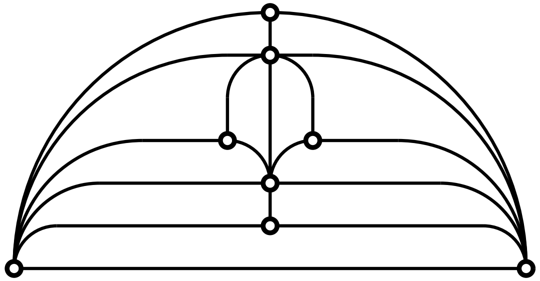
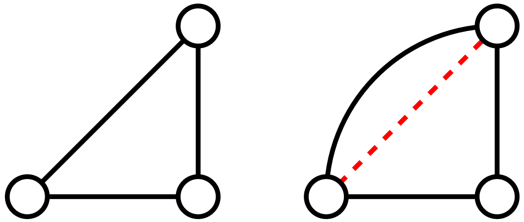
  - Classes of bendless smooth orthogonal drawable (SC_1) and octilinear drawable ($8C_1$) graphs are incomparable
- **Complexity**
  - Deciding if a smooth orthogonal or octilinear representation is realizable is $\mathcal{NP}$-hard on max-degree 4 graphs
- **Kandinsky Drawings**
  - Smooth orthogonal: Alternative approach producing aesthetically more pleasing drawings
  - Octilinear: First results



Relations

- Bendless smooth orthogonal and octilinear drawings require same endpoint positioning



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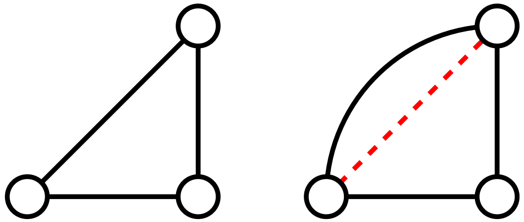
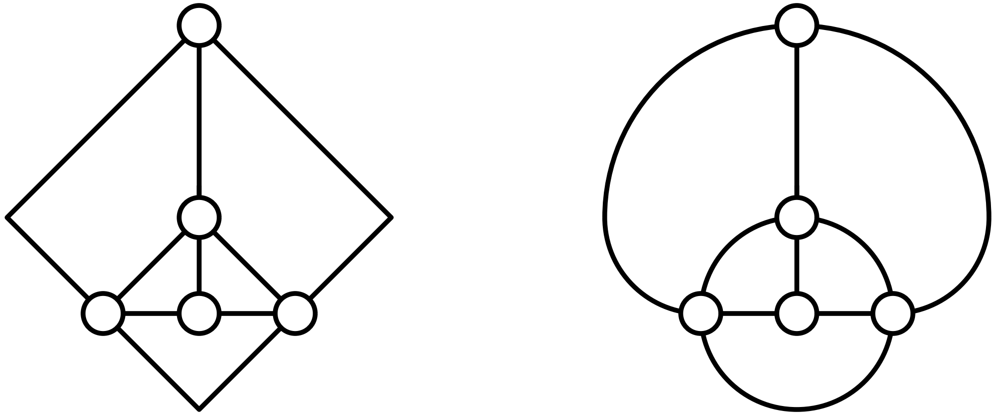


- Idea: Replace arcs with diagonals and vice versa



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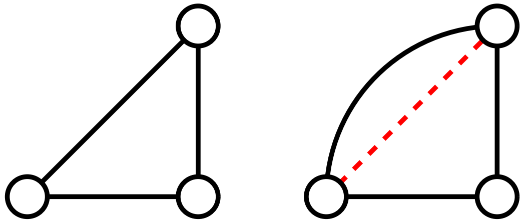

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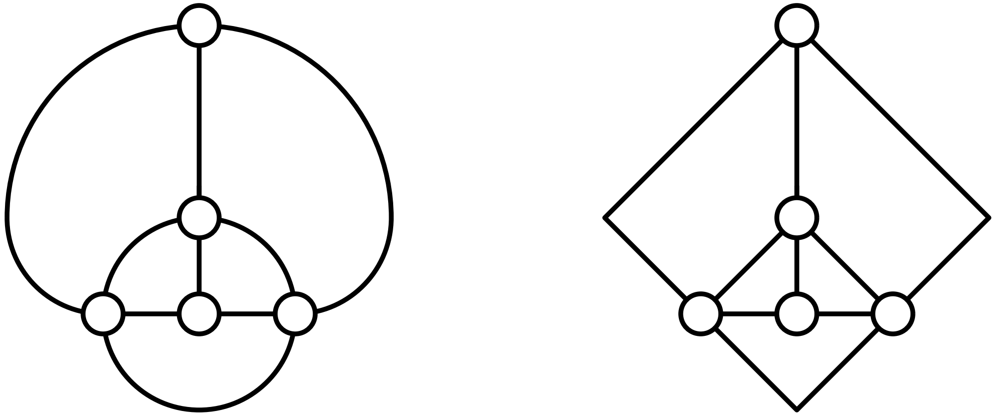

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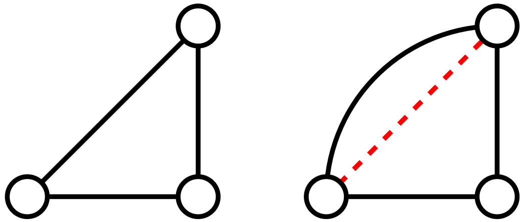

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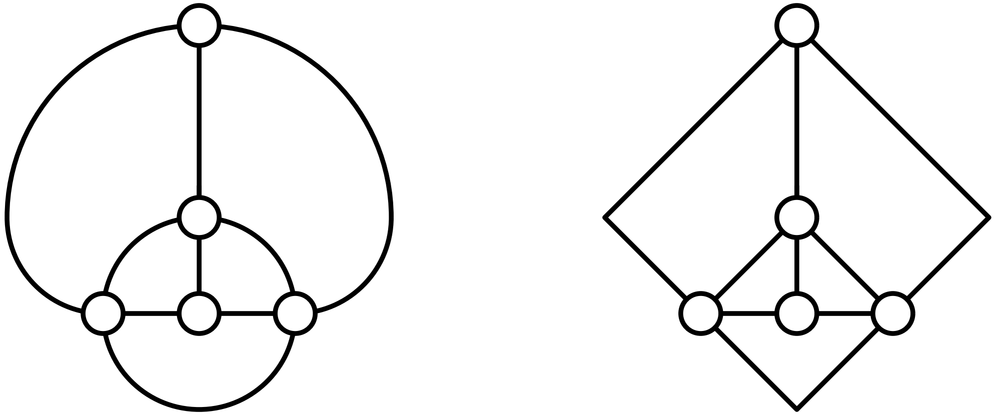

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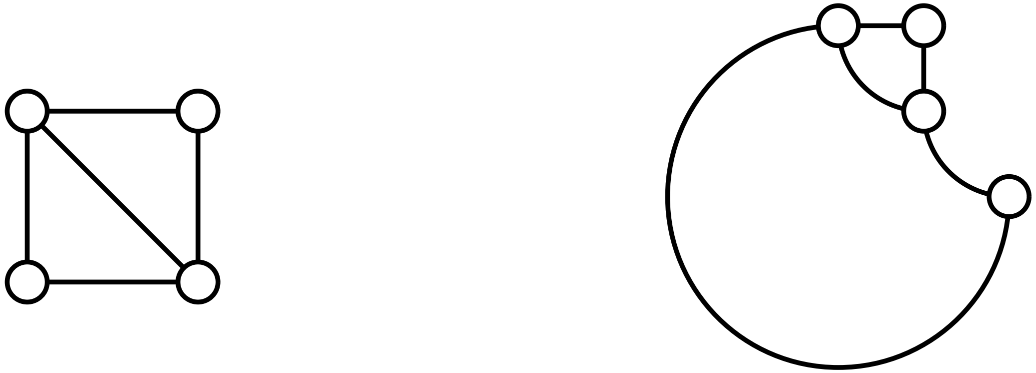

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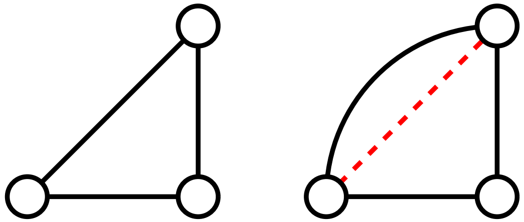

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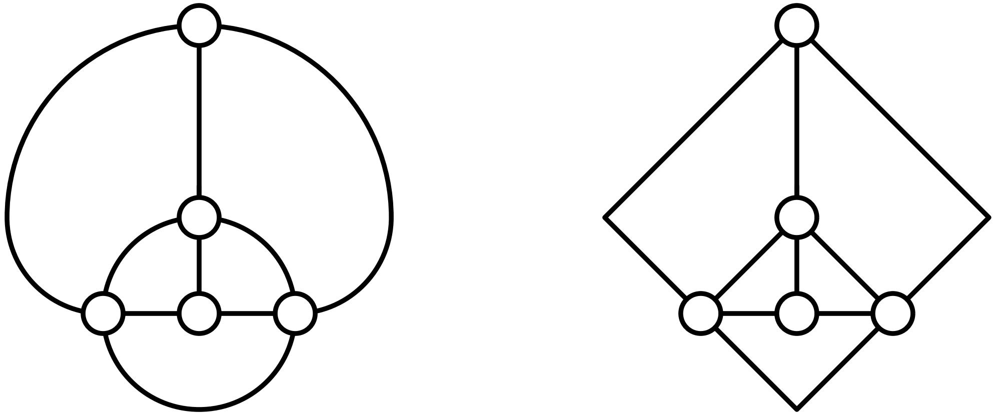

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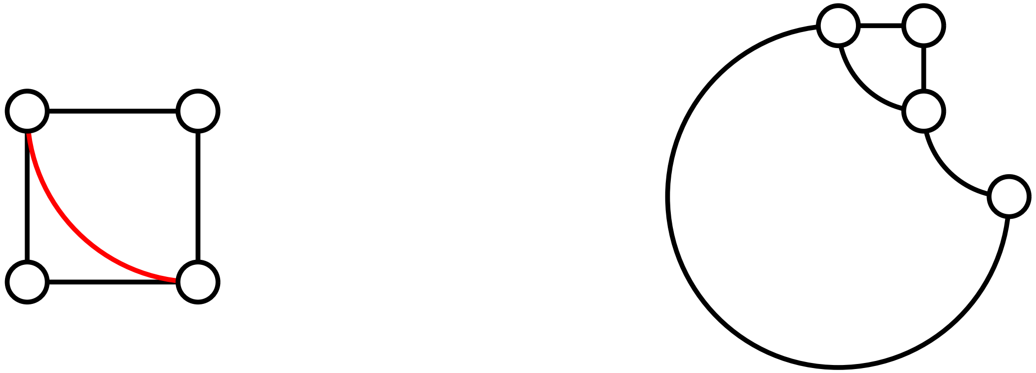

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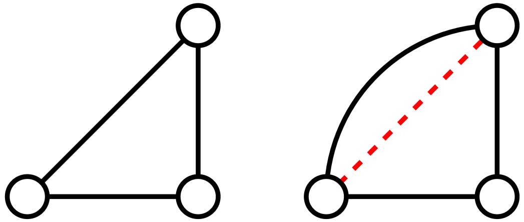

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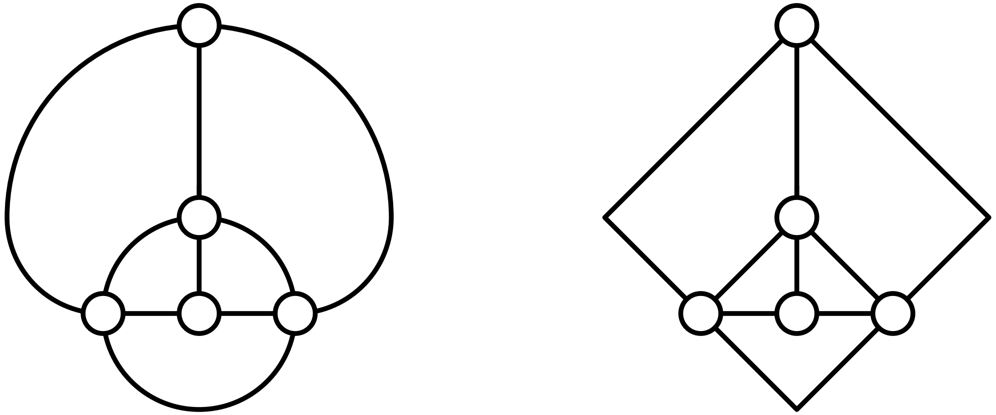

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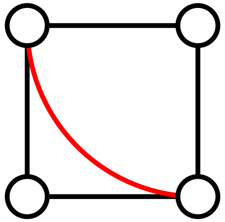

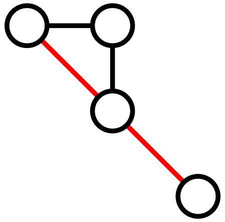

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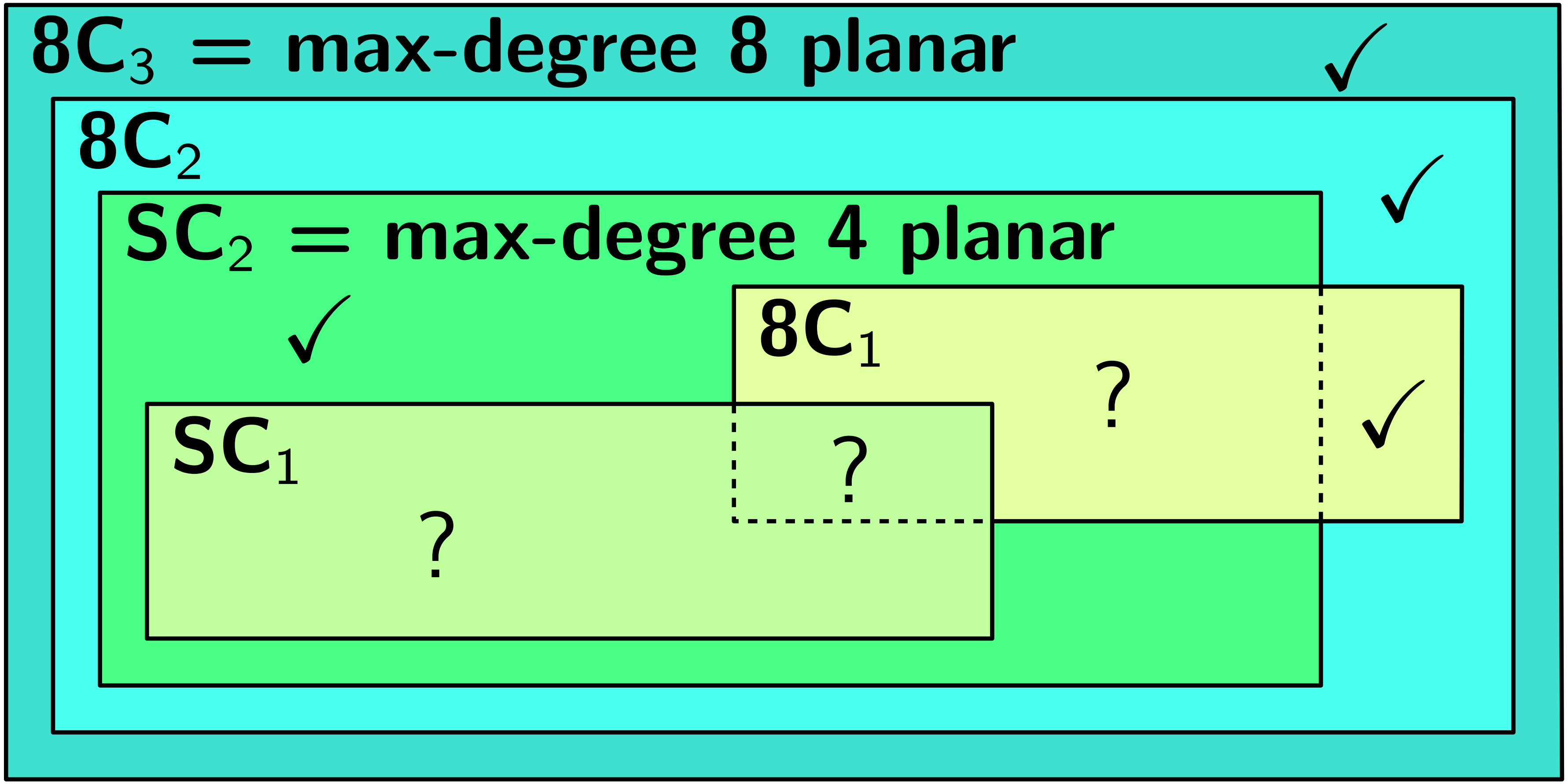
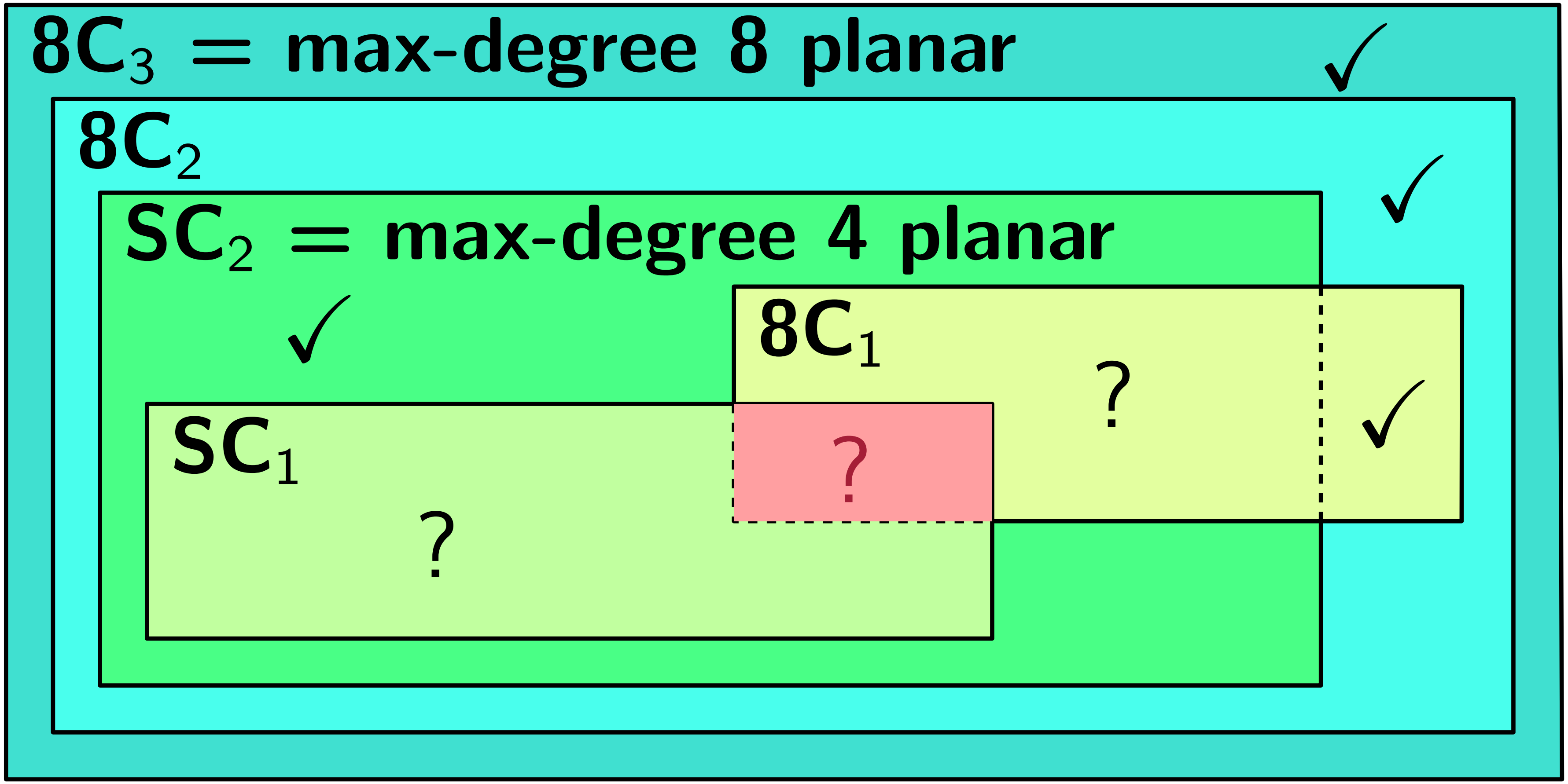


$8C_k$ = Graphs drawable with octilinear complexity k

SC_k = Graphs drawable with smooth orthogonal complexity k

Relations



$\mathbf{8C_3}$ = max-degree 8 planar
✓
$\mathbf{8C_2}$
✓
$\mathbf{SC_2}$ = max-degree 4 planar
✓
$\mathbf{8C_1}$
?
✓
$\mathbf{SC_1}$
?
?

$8C_k$ = Graphs drawable with octilinear complexity k

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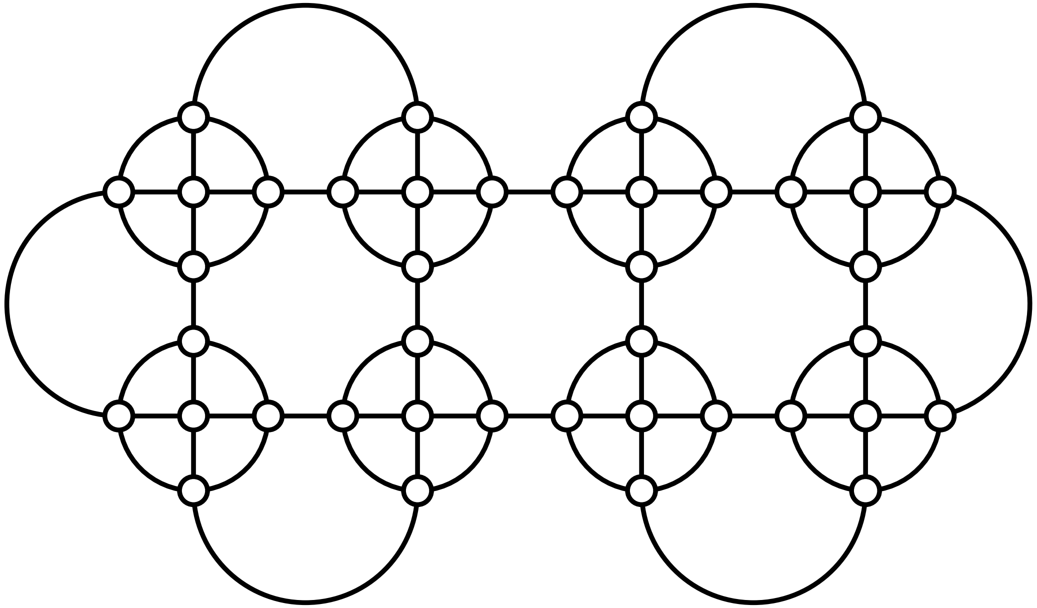
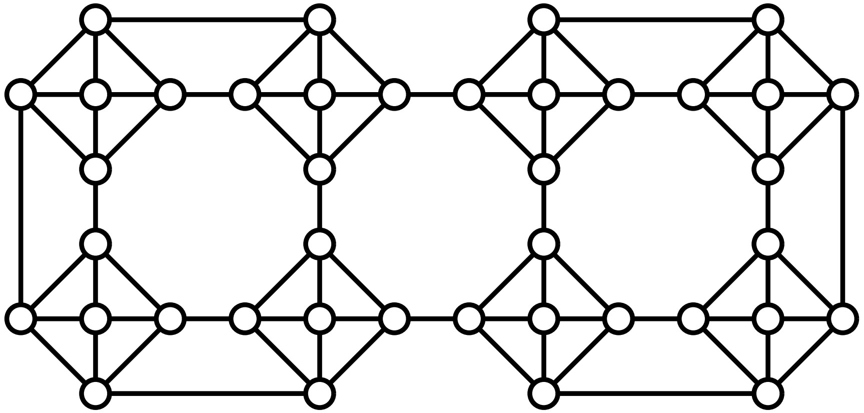
Intersection of SC_1 and $8C_1$

- Infinitely large graph family drawable with both styles:





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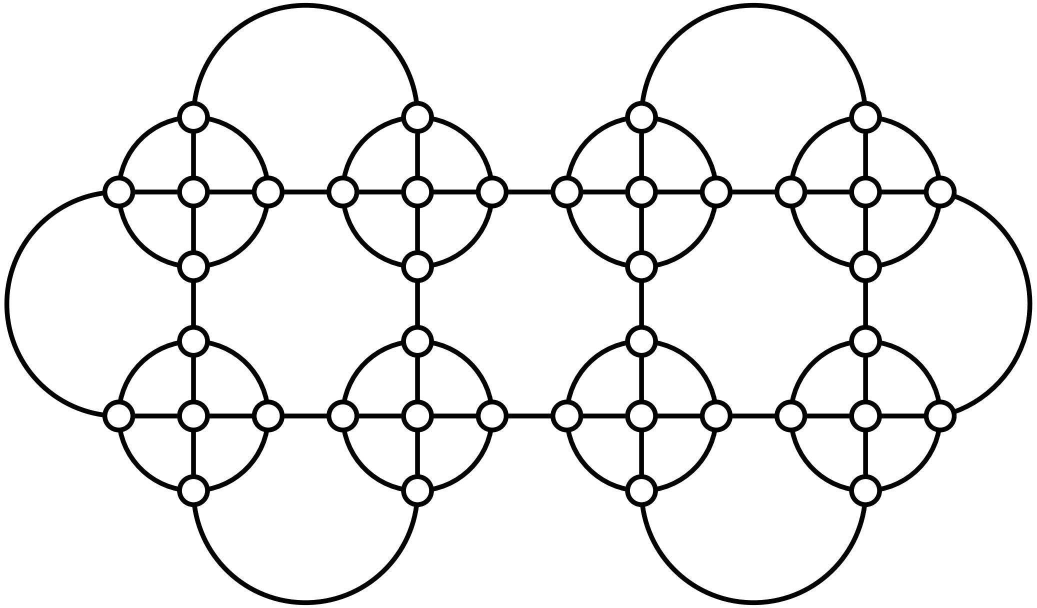
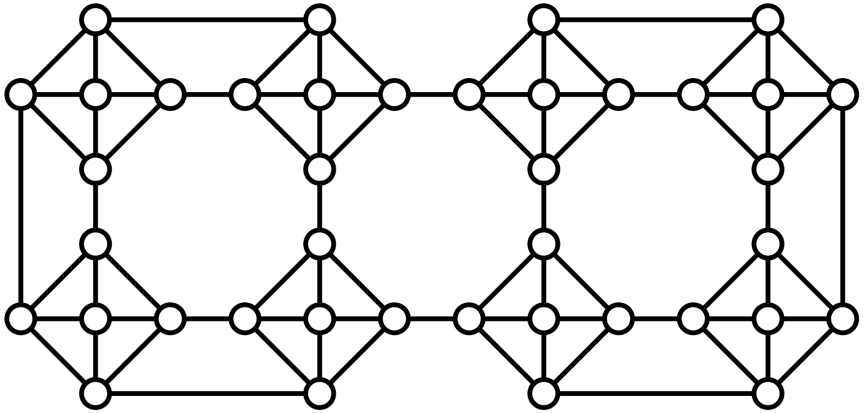
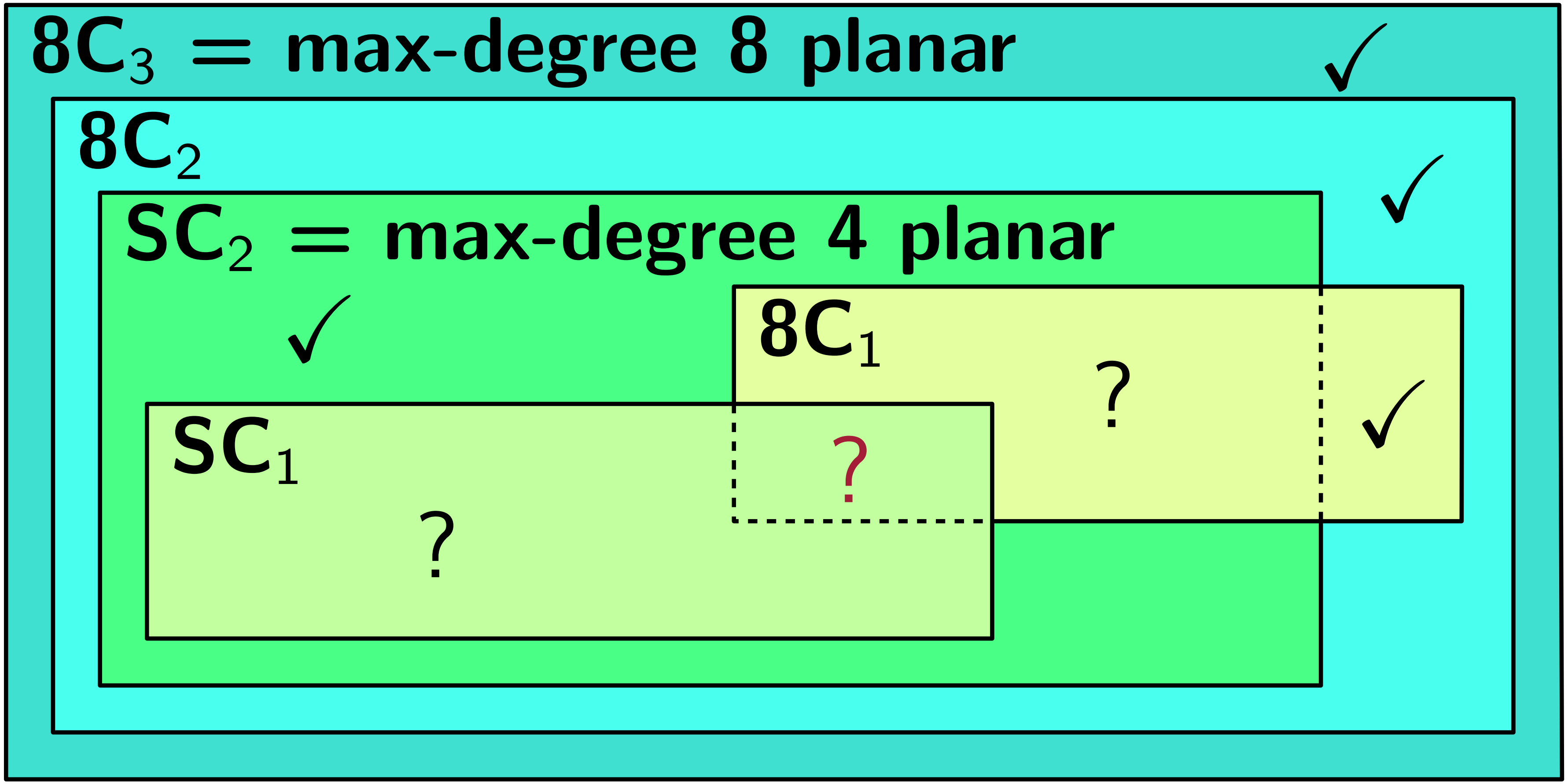




- Family is 4-regular → density does not divide classes

Relations



$8C_3$ = max-degree 8 planar ✓

$8C_2$ ✓

SC_2 = max-degree 4 planar ✓

$8C_1$? ✓

SC_1 ?

?

$8C_k$ = Graphs drawable with octilinear complexity k

SC_k = Graphs drawable with smooth orthogonal complexity k

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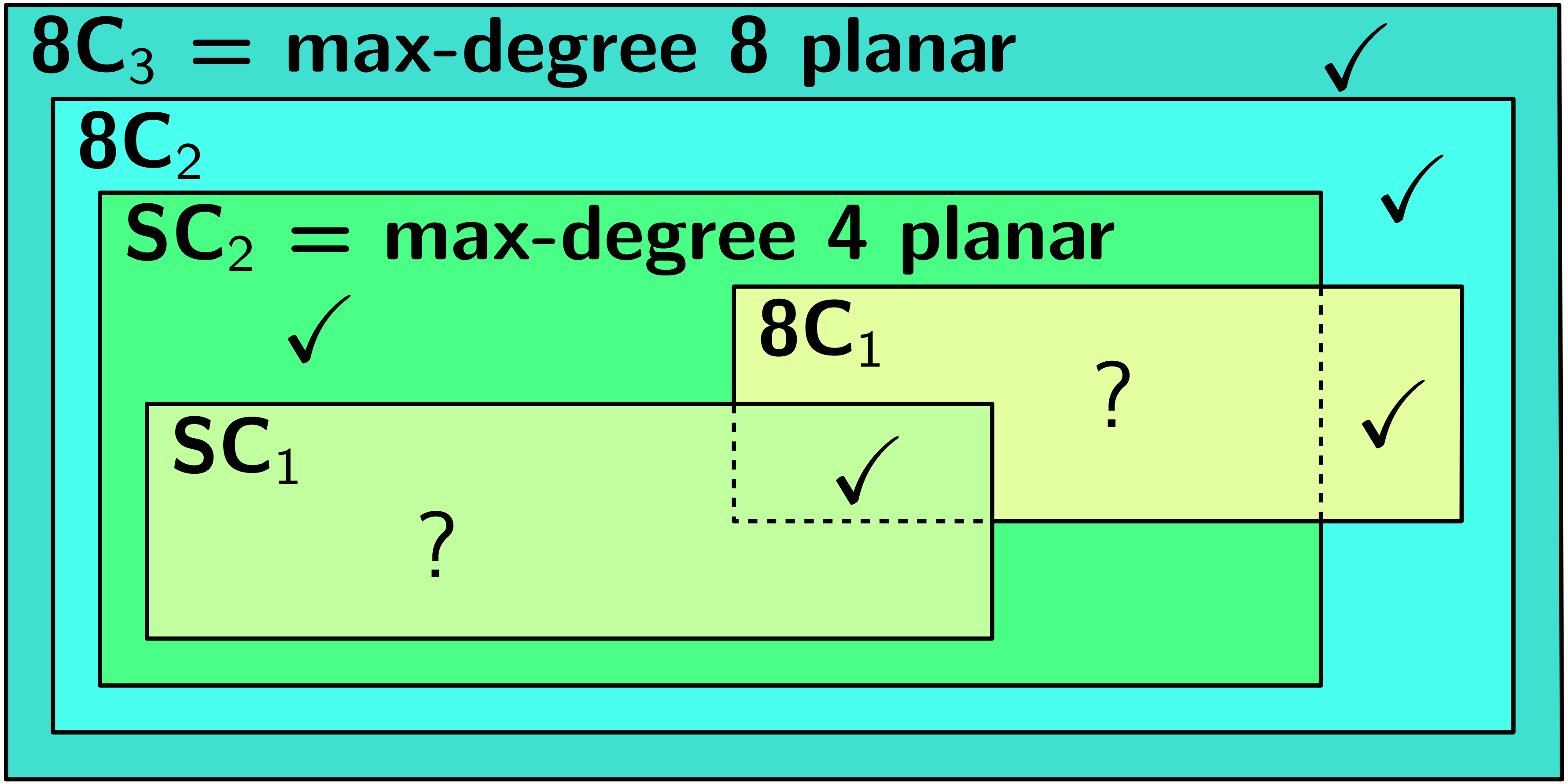

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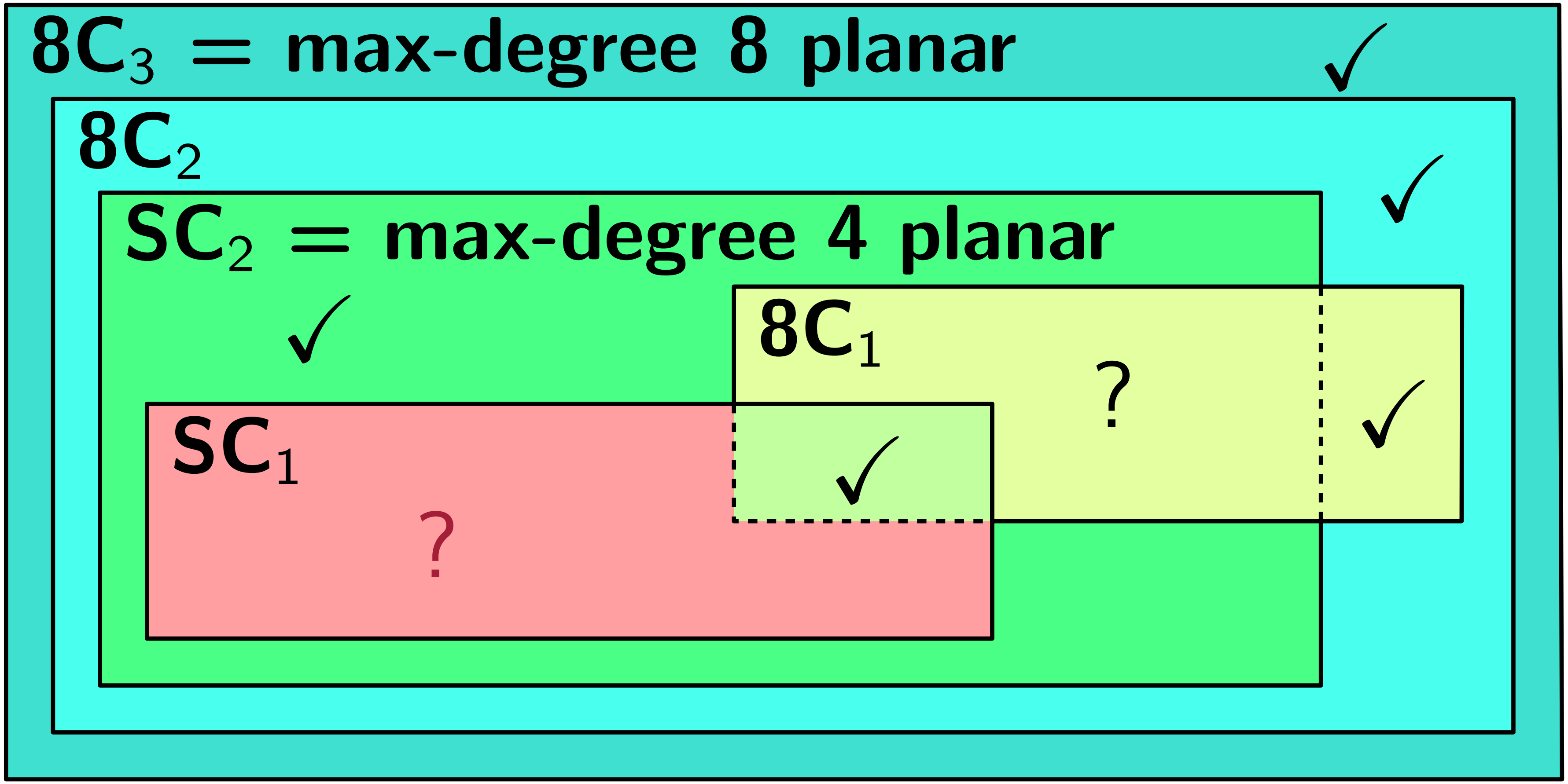

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$S C_1$ but not $8 C_1$

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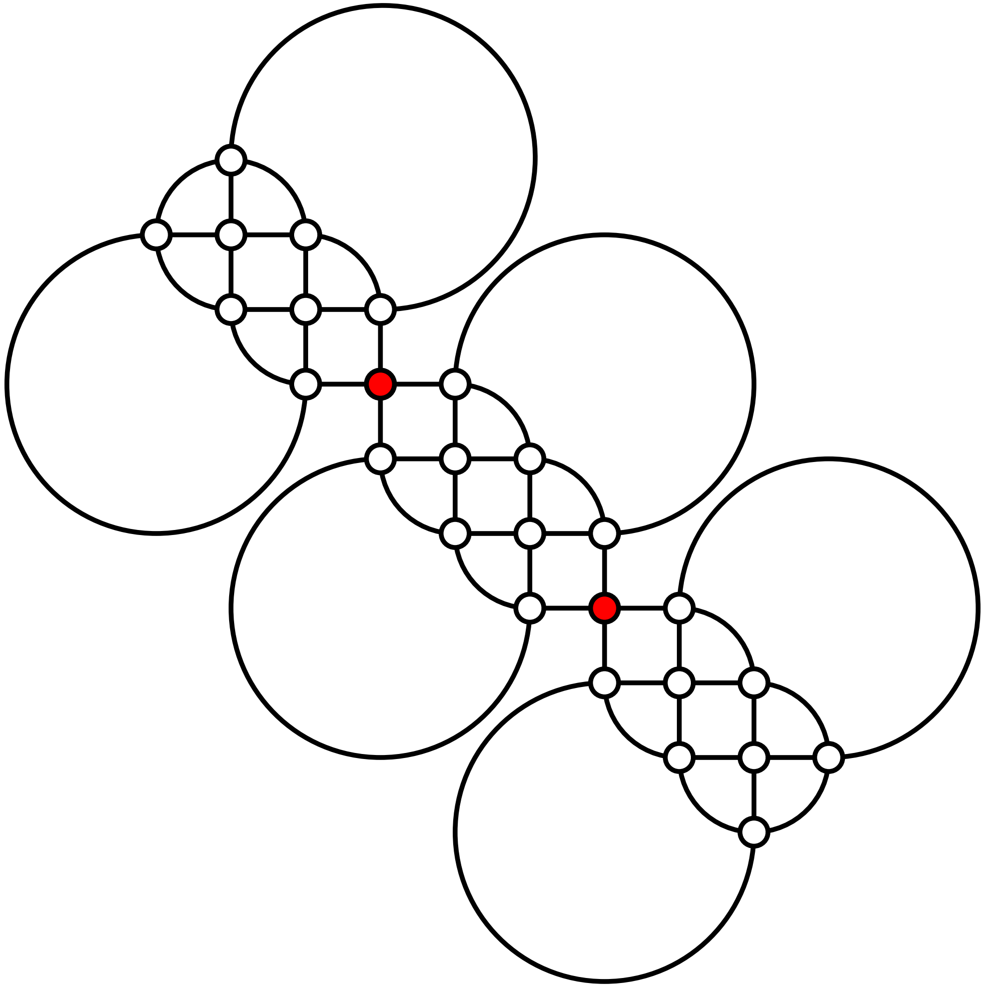

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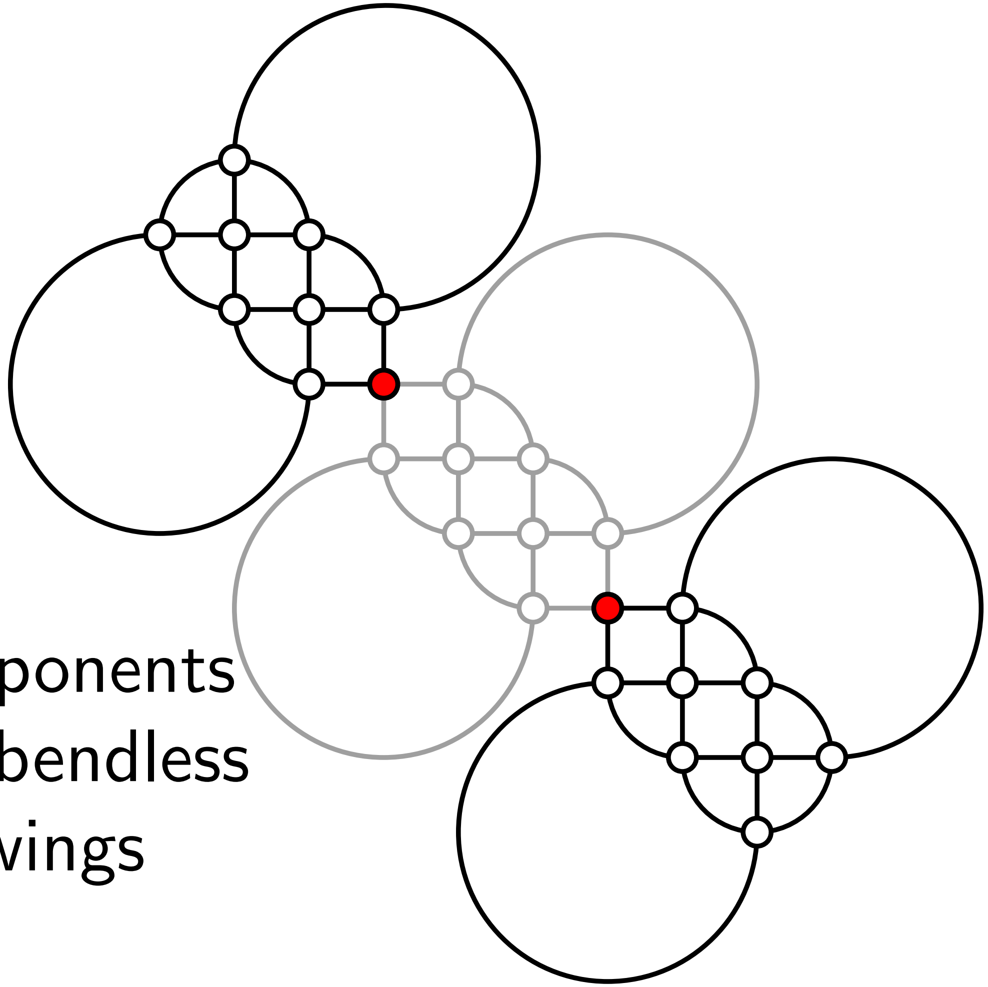
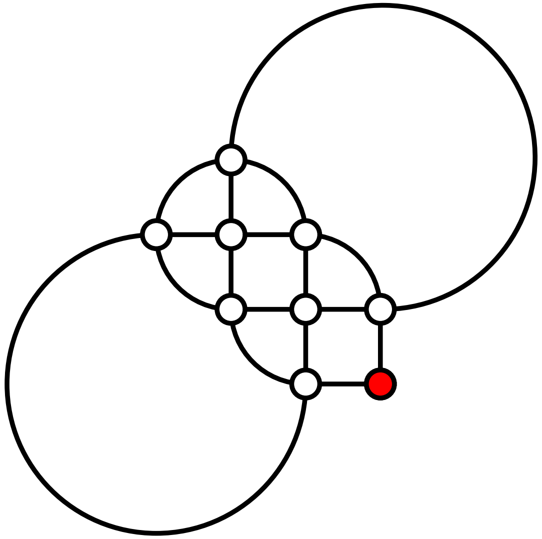
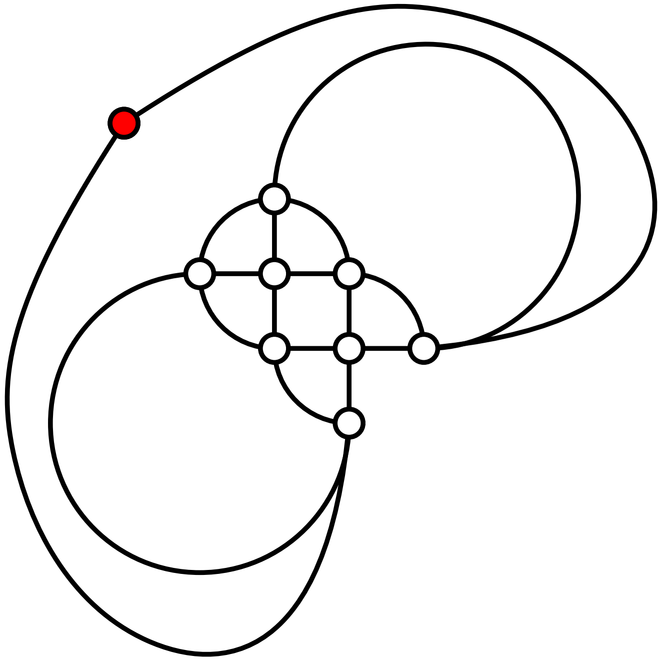


- Two end components do not admit bendless octilinear drawings

SC_1 but not $8C_1$

- End components only have one embedding





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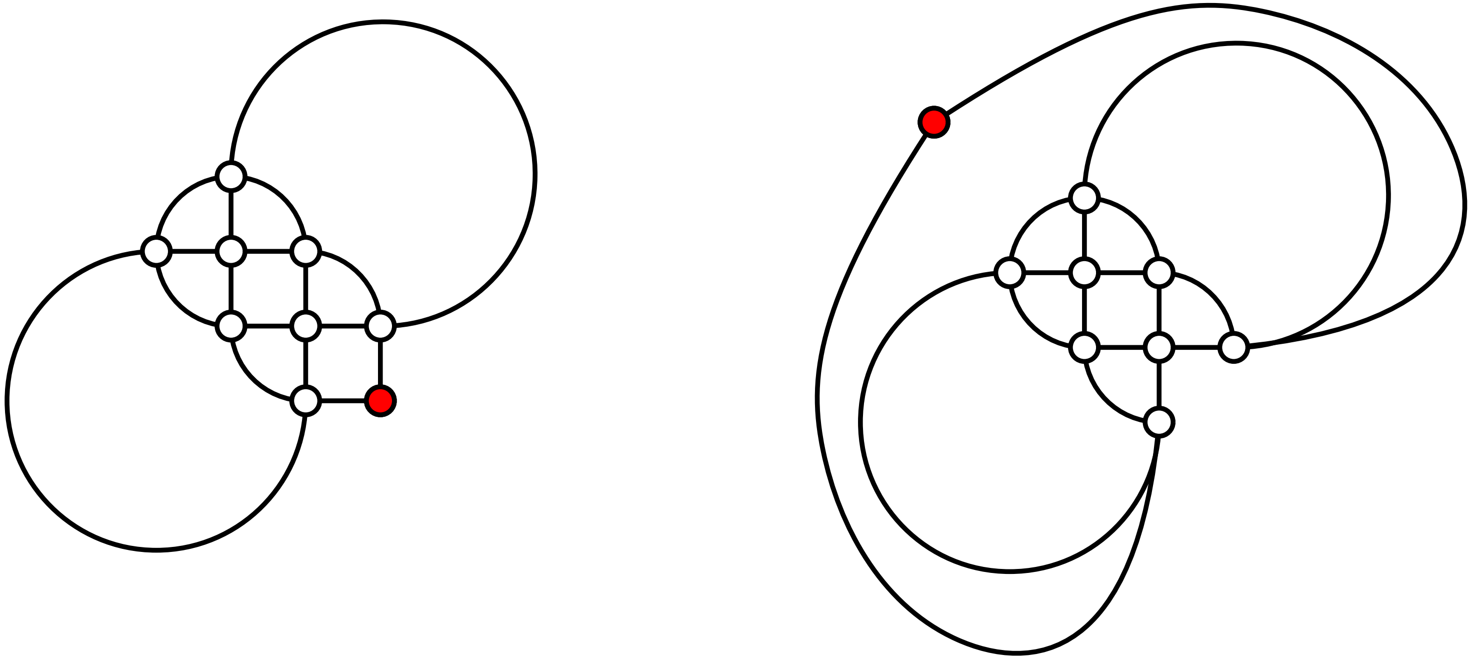

- Properties of this embedding:

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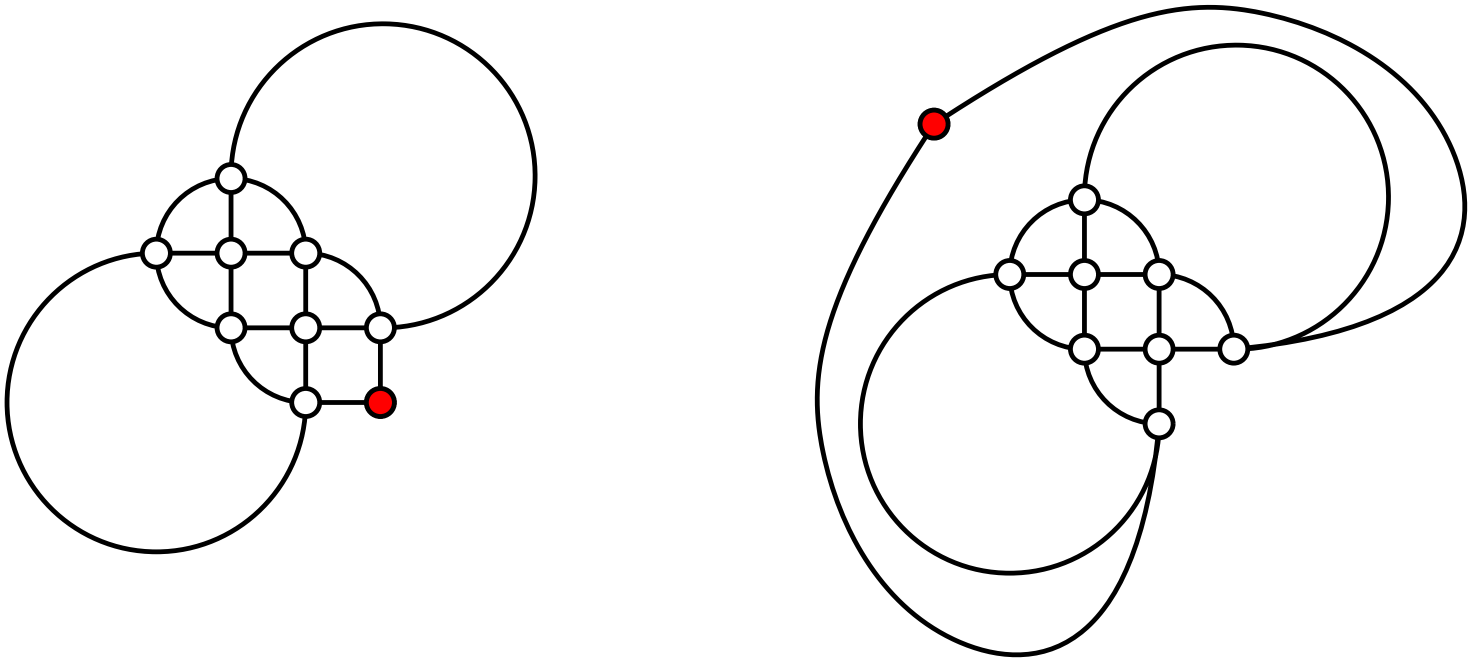

- Properties of this embedding:
  - Each face has length at most 5

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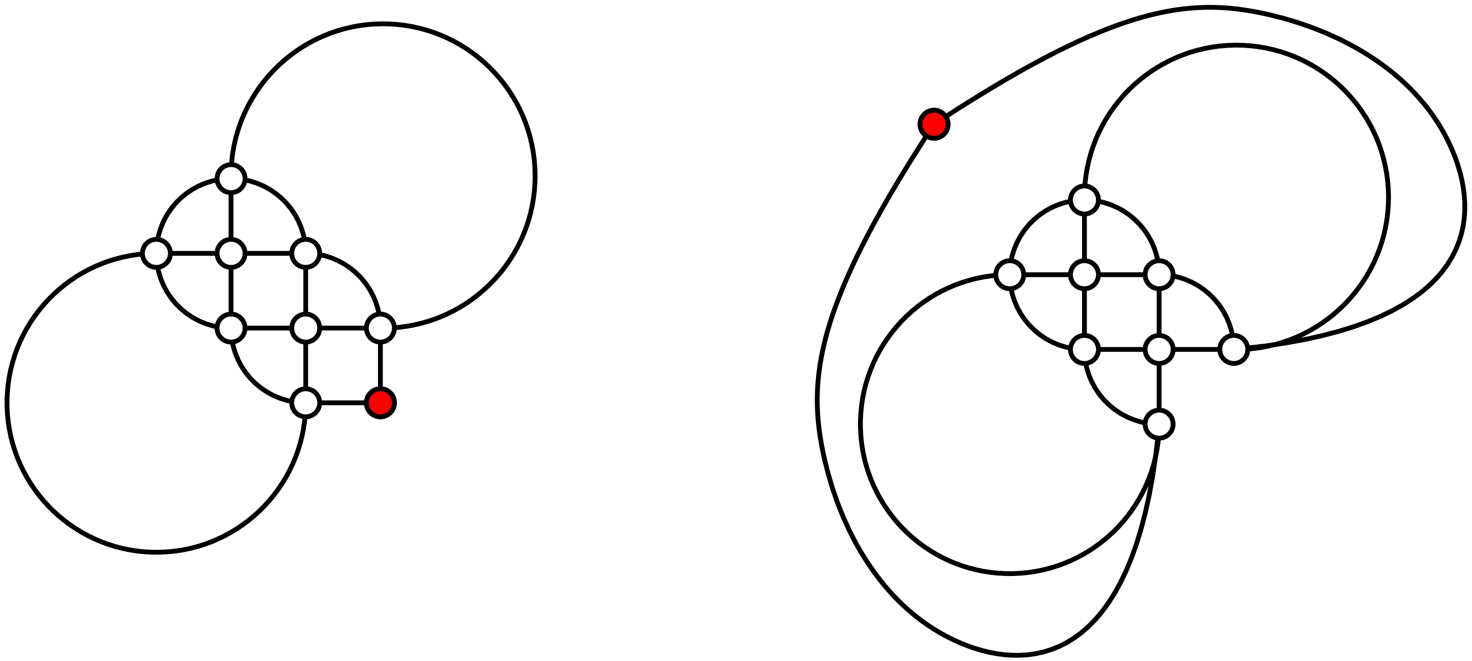
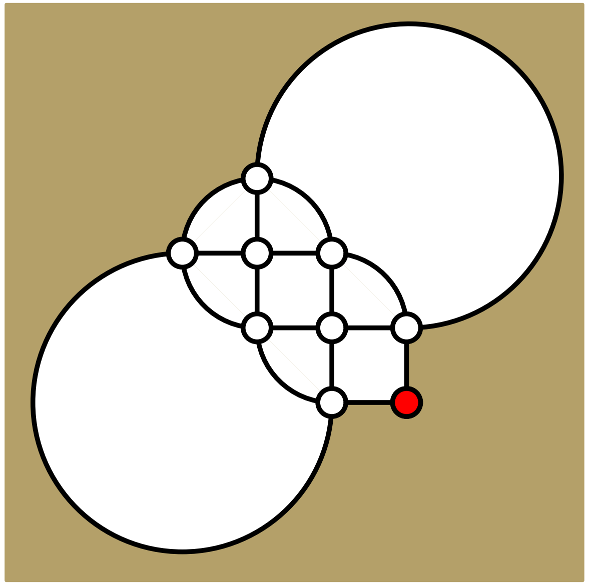
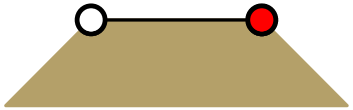


- Properties of this embedding:
  - Each face has length at most 5
  - All but one vertex on the outerface must support two ports to the interior of the drawing

SC_1 but not $8C_1$

- If we try to realize such a drawing, we find, that it is not possible to close the outerface





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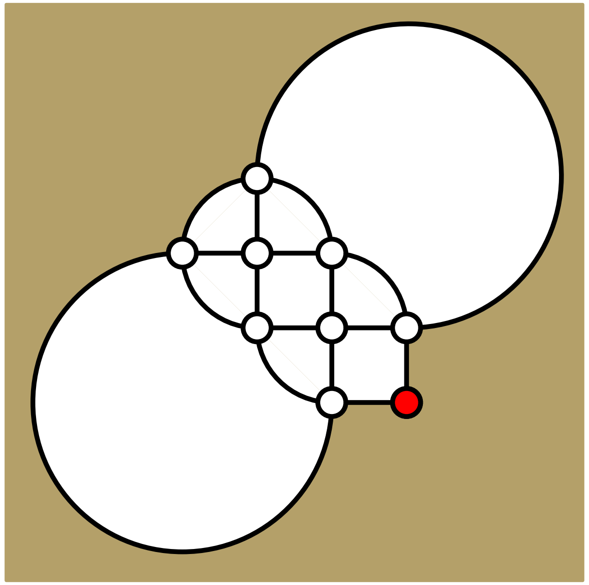

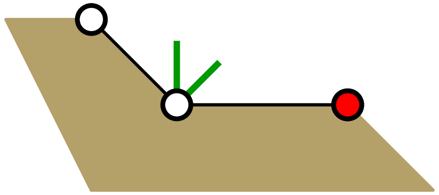

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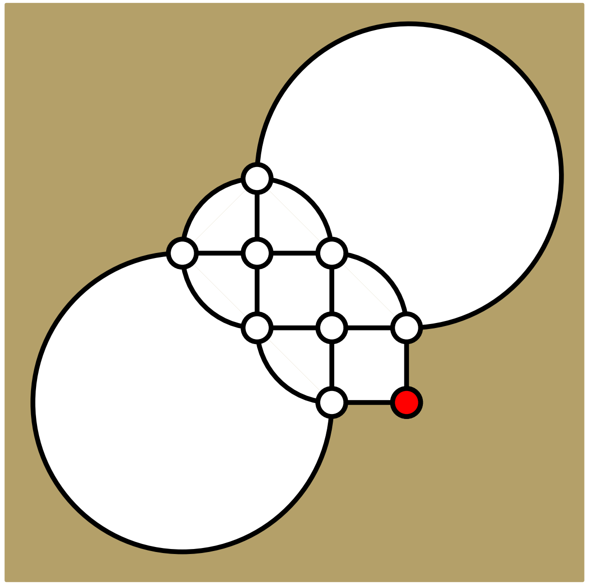

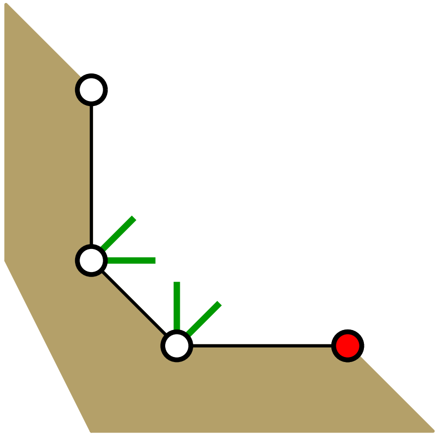

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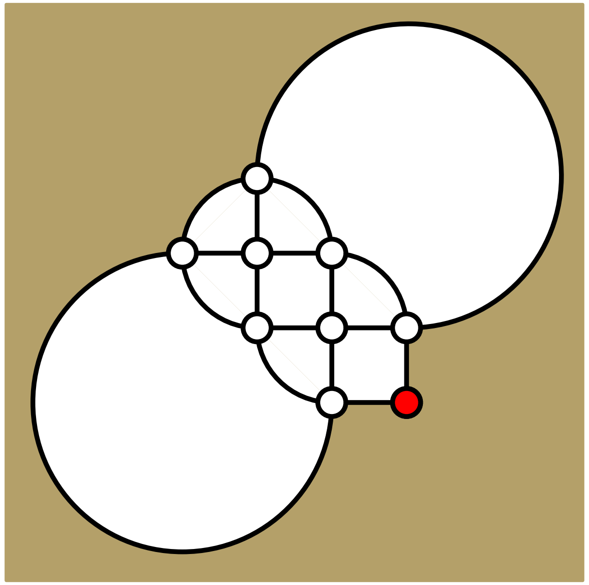

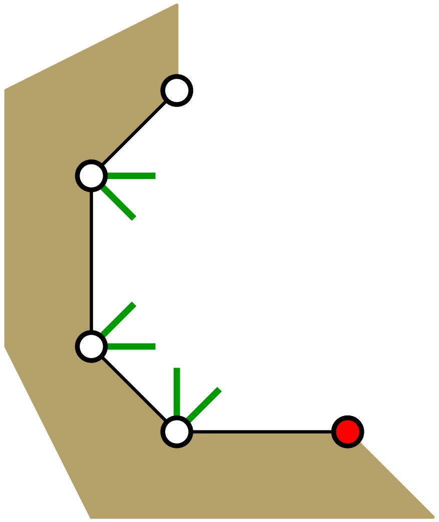

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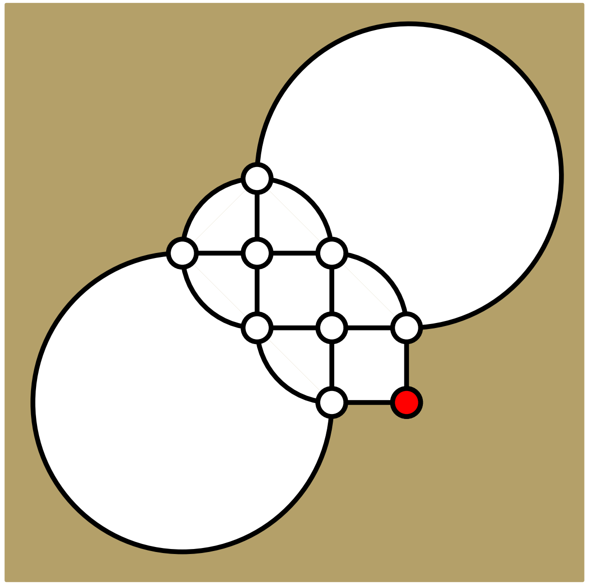

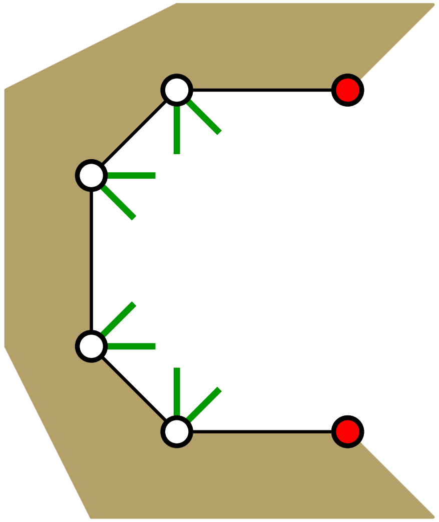

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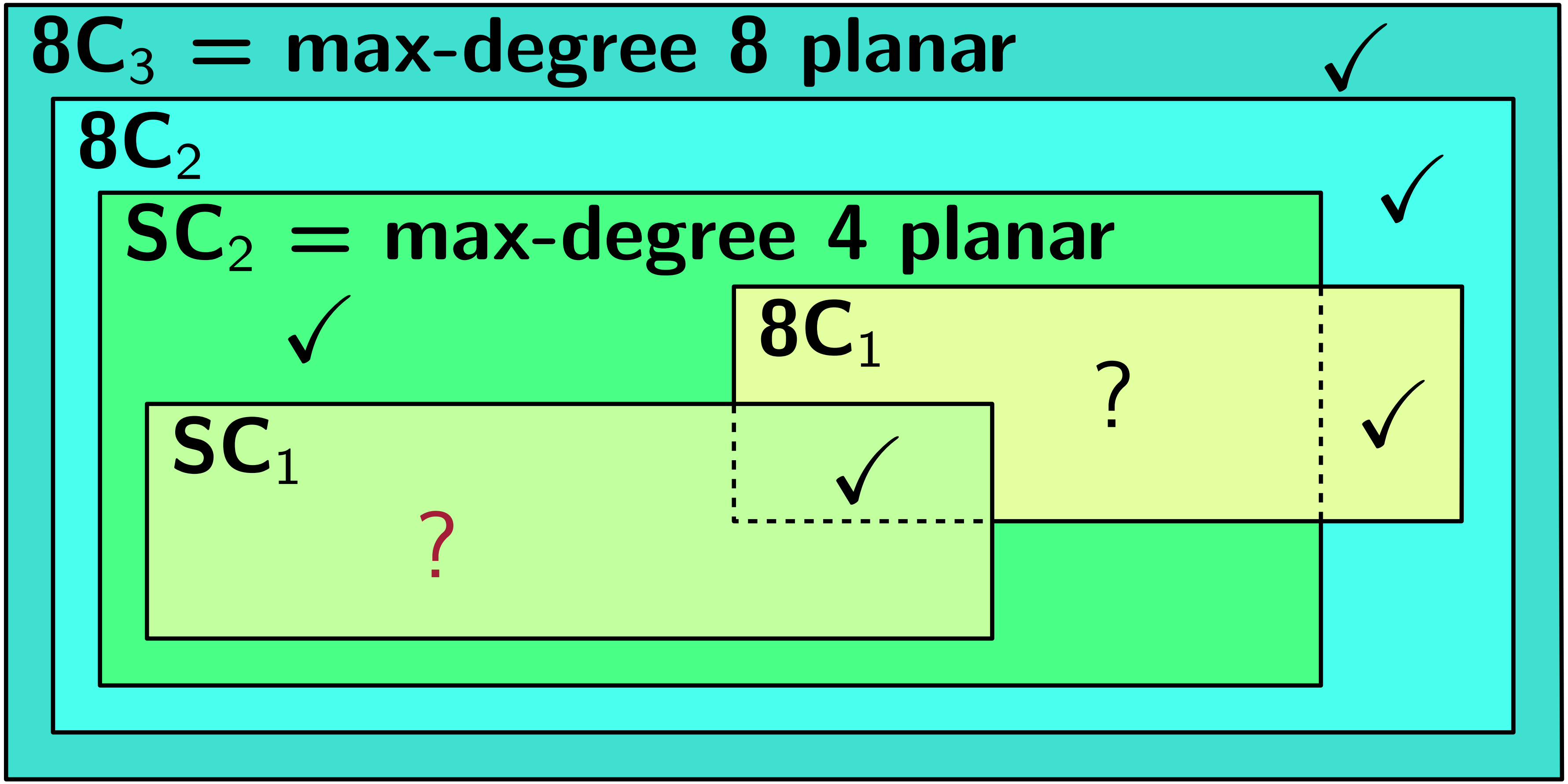

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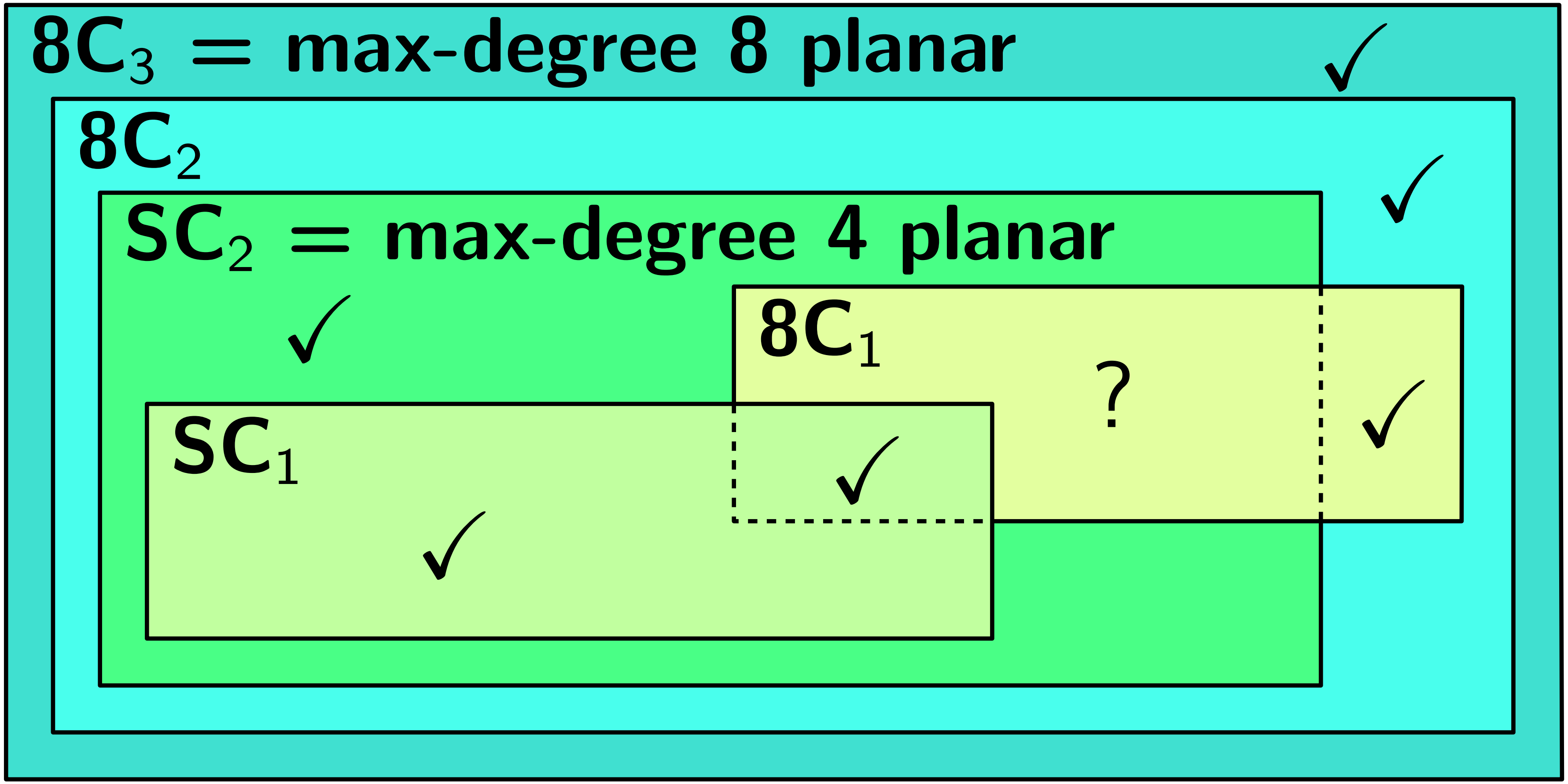

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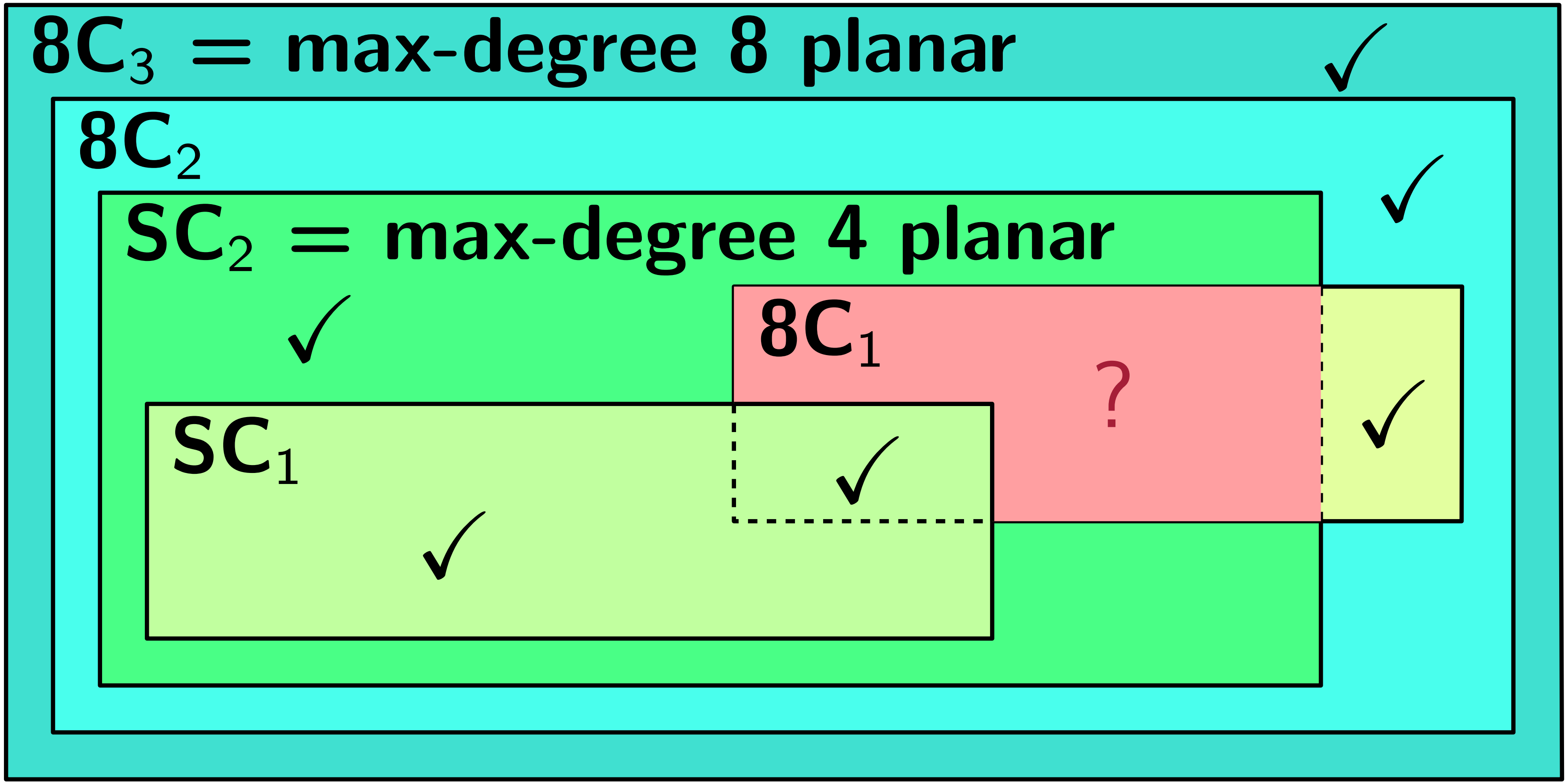

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$8C_1$ but not SC_1

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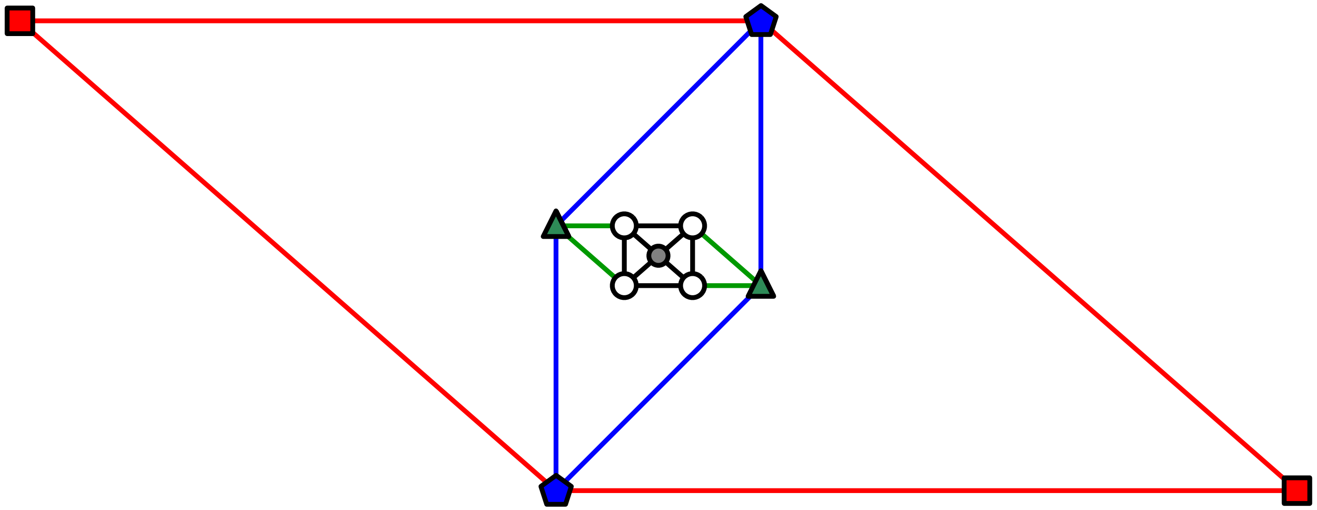

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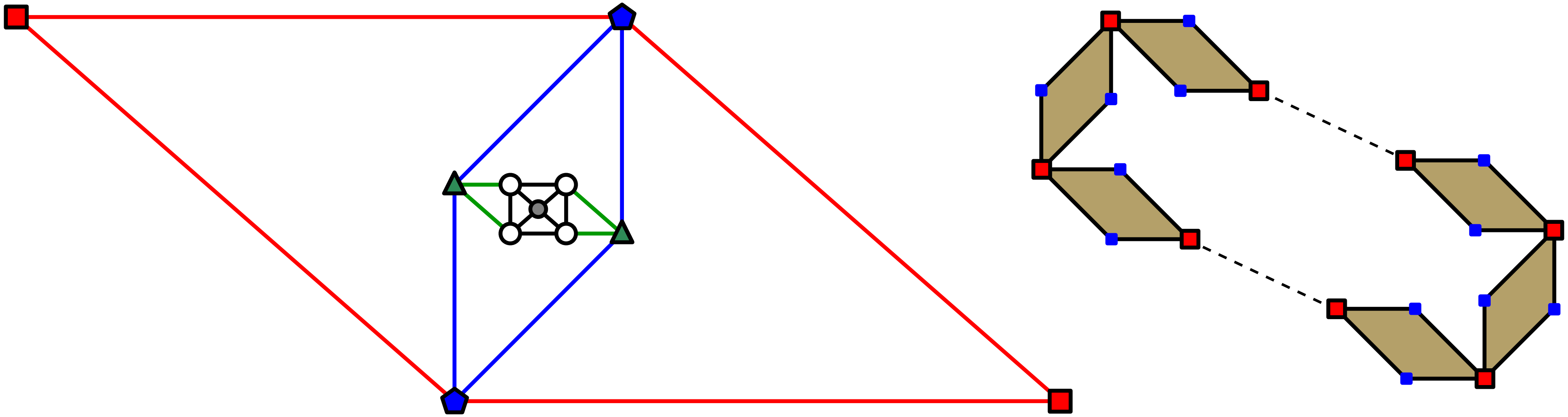

- Multiple copies of a basic component in a cycle

$8C_1$ but not SC_1

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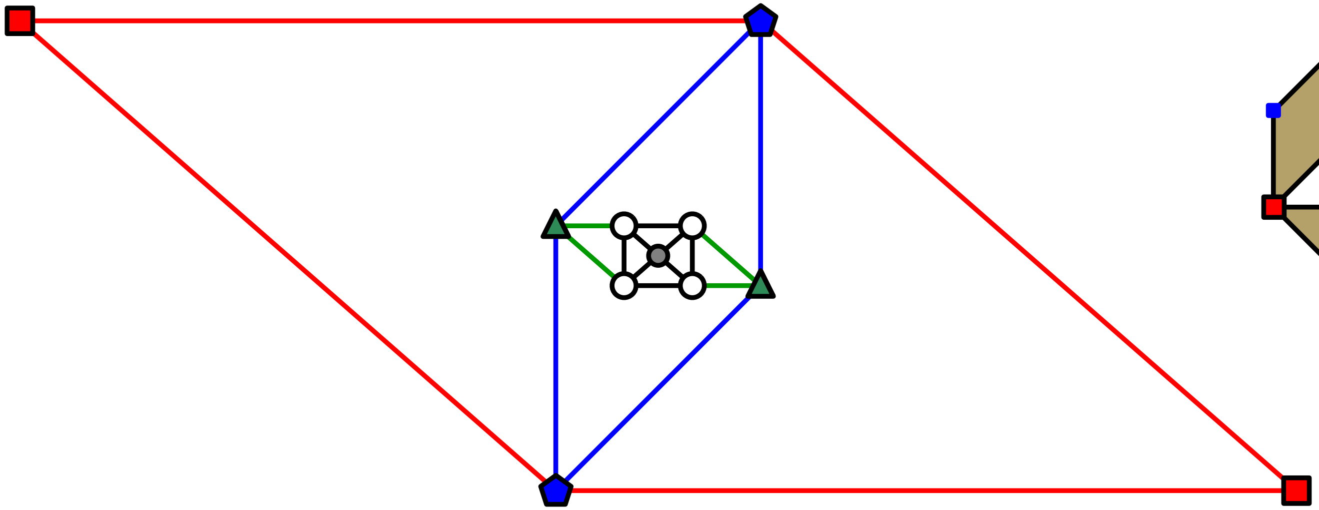

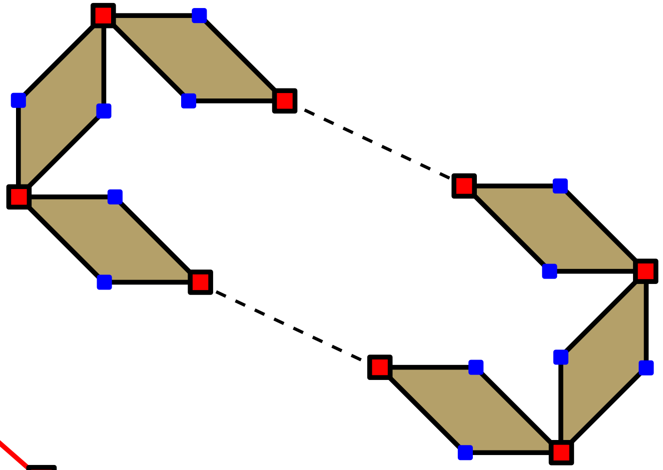

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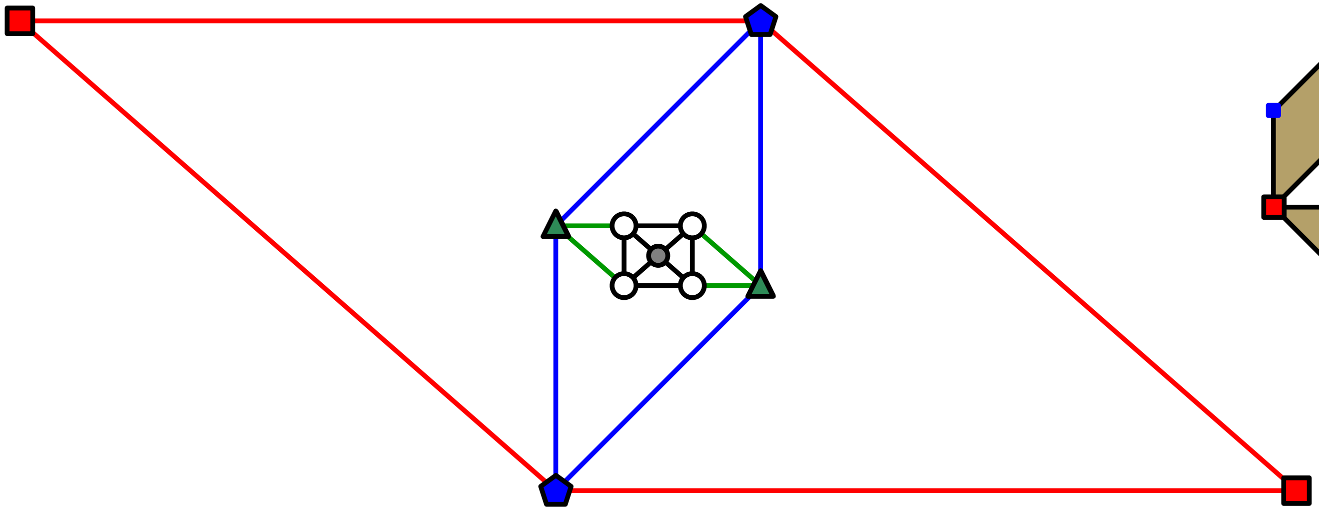

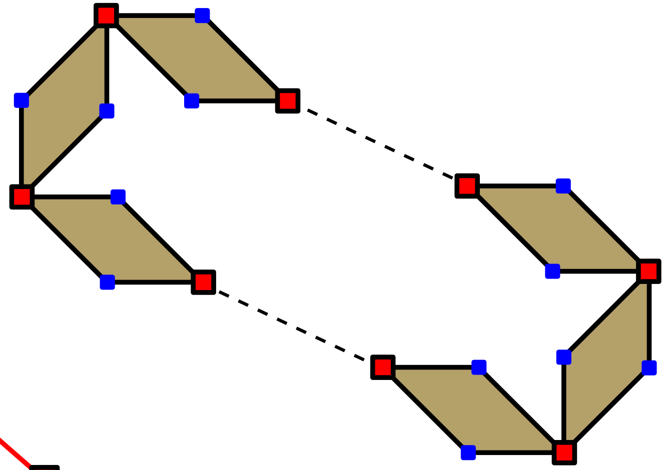

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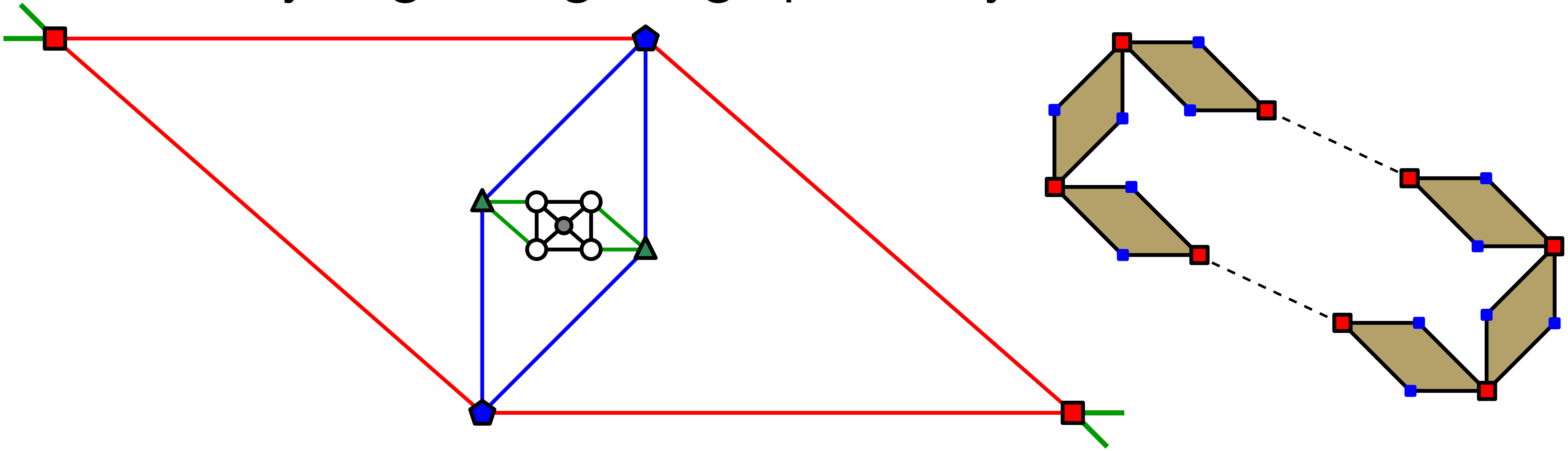

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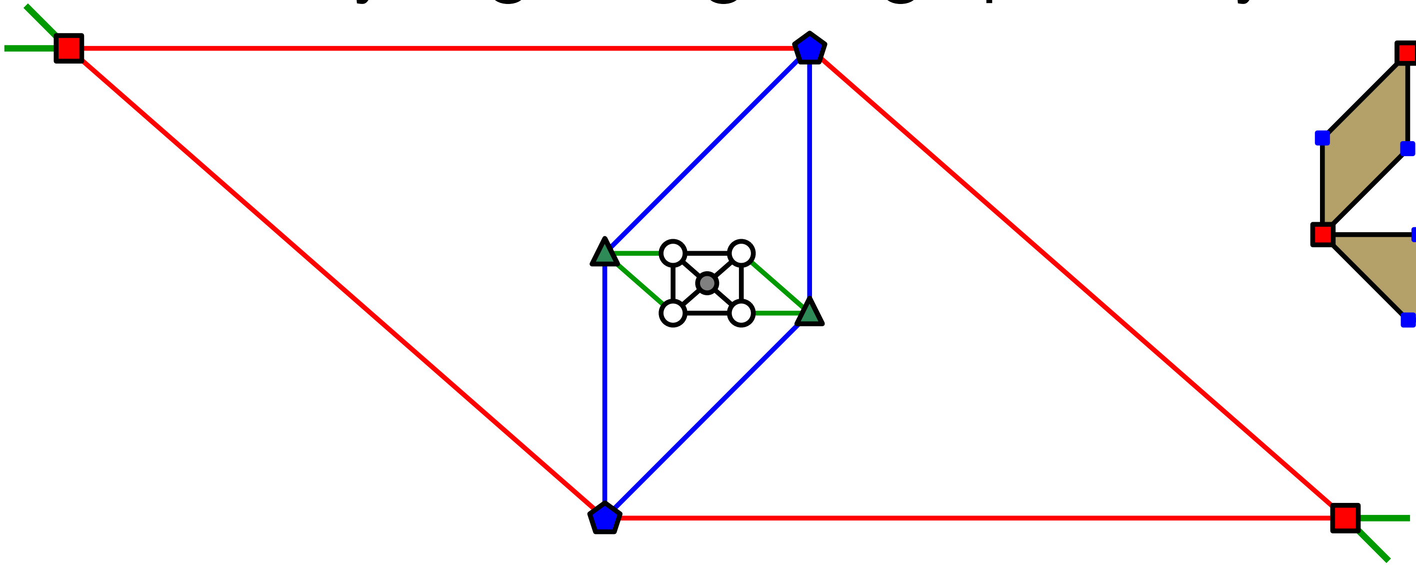

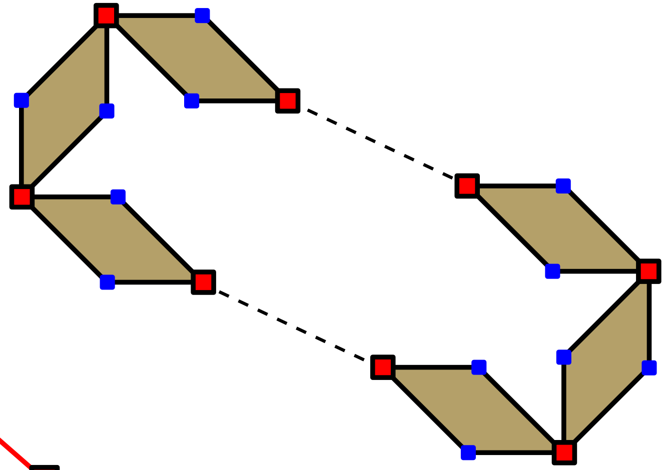

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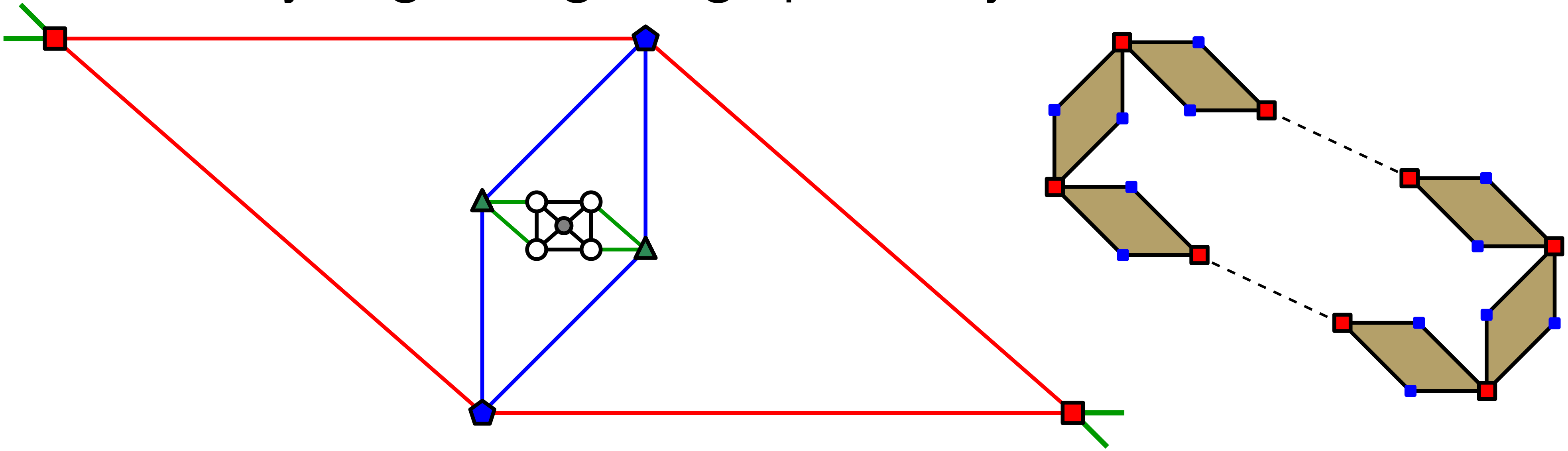

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- Smooth Orthogonal Representation Realizability
  - Input: angles between edges and edge segments along edges

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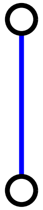

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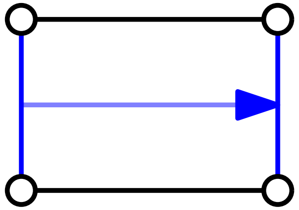

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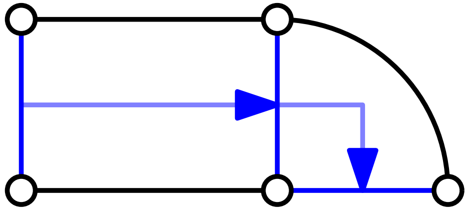

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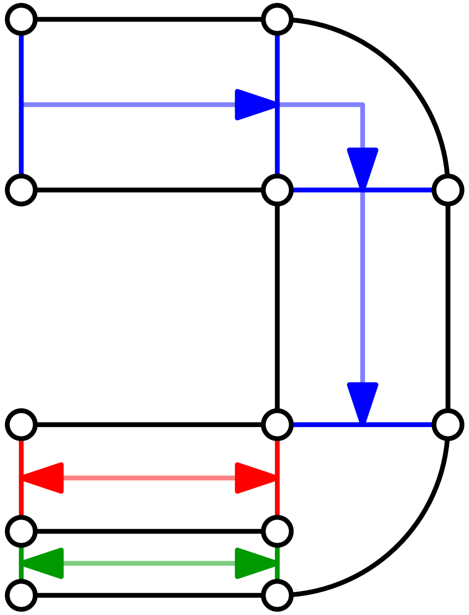

Auxiliary Gadgets

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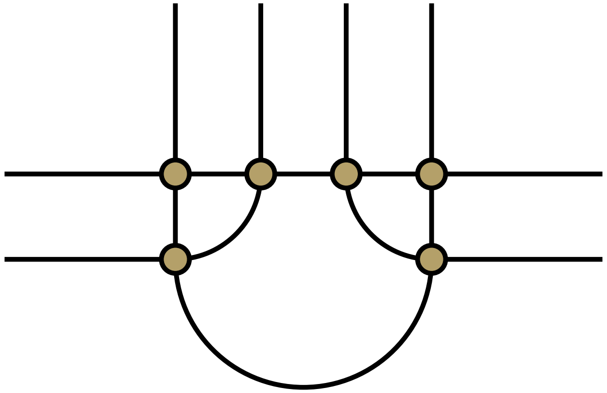

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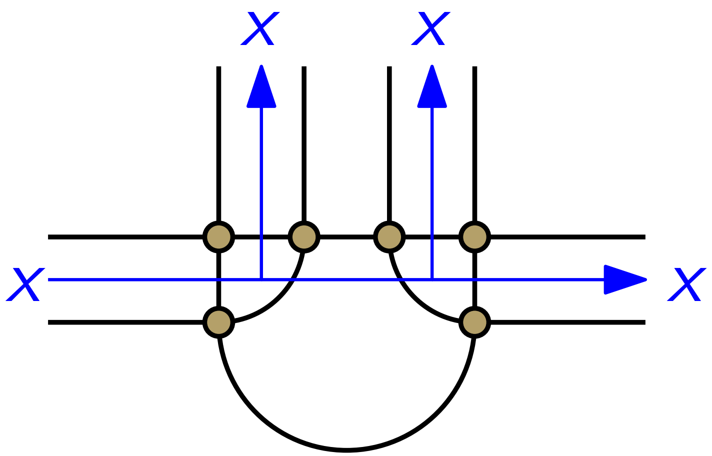

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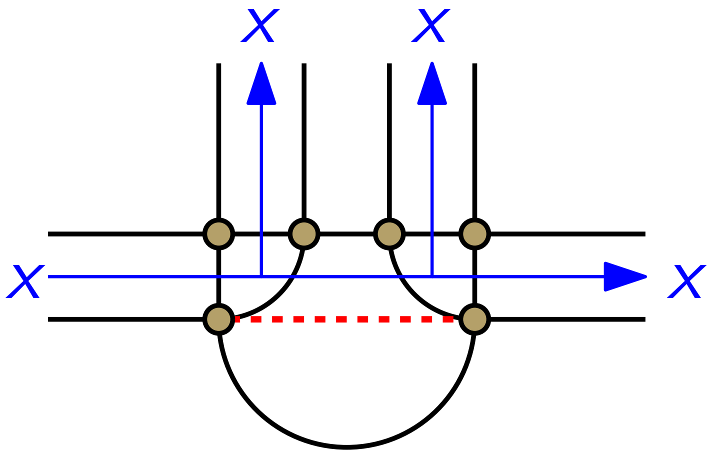

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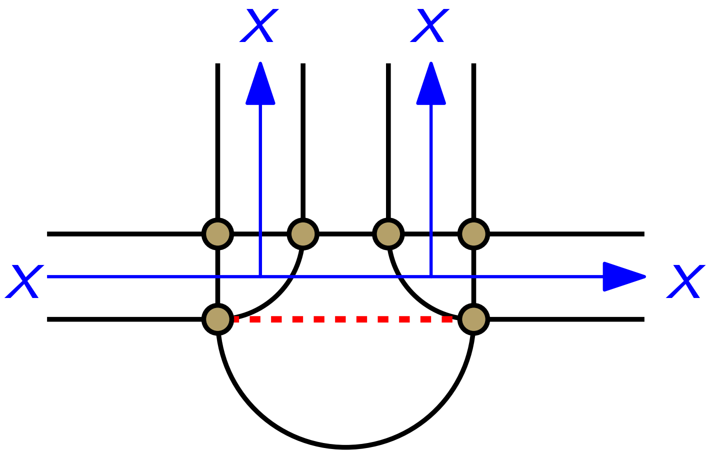

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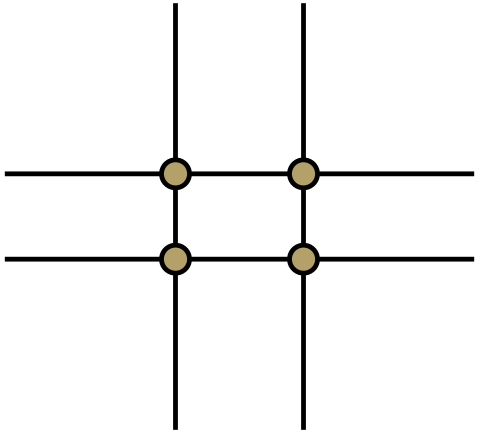

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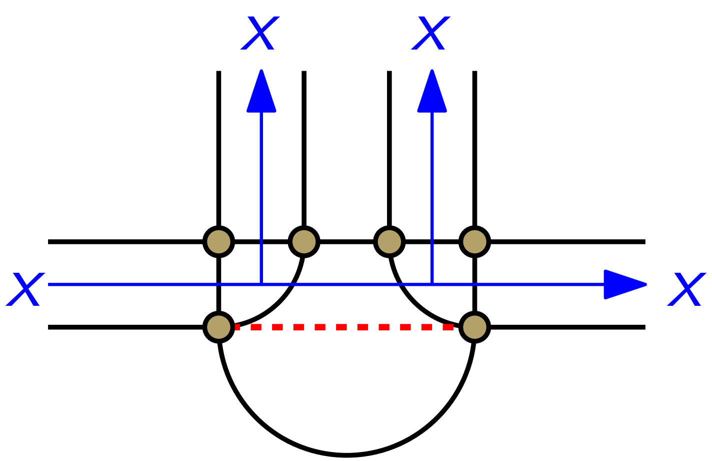

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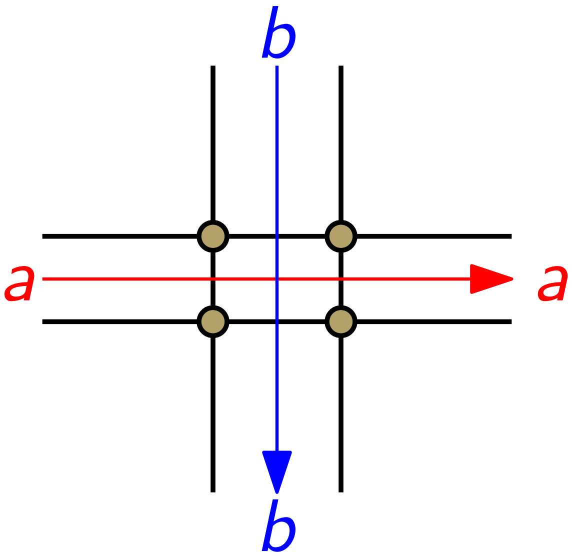

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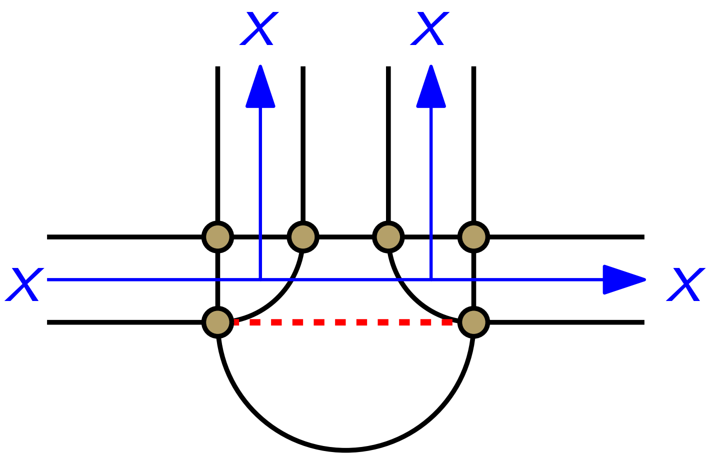

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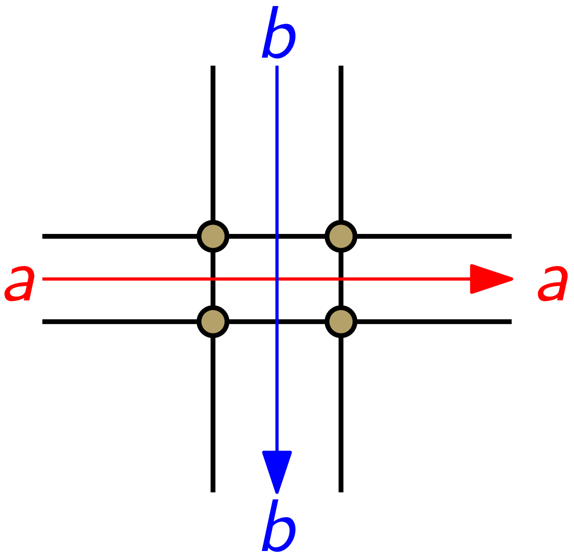
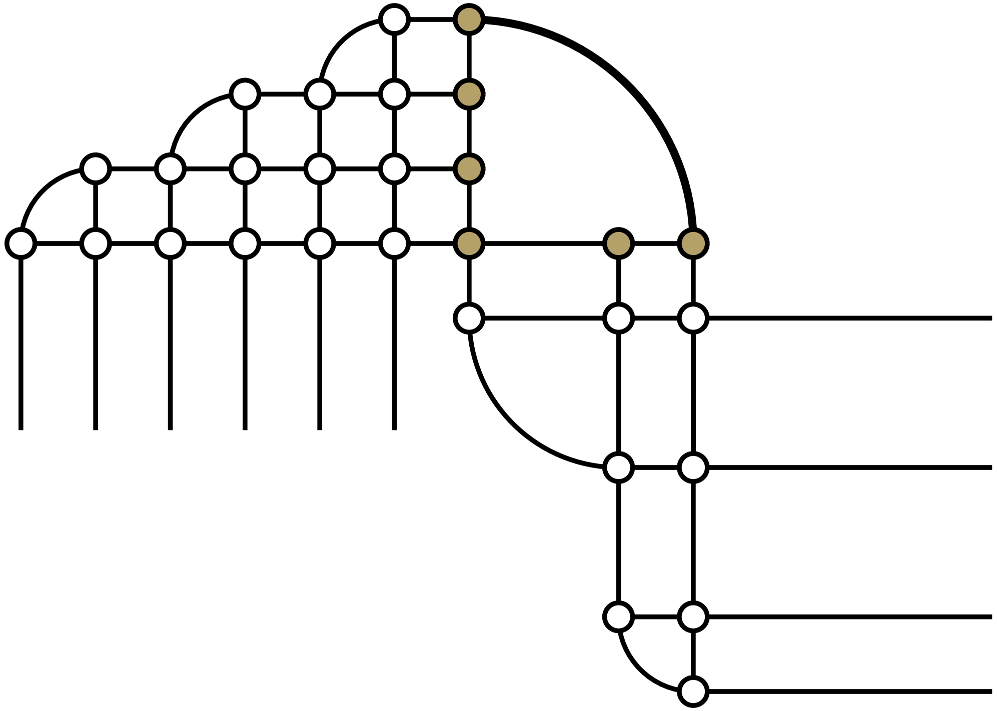


- We can connect literals and clauses properly

Variable Gadget



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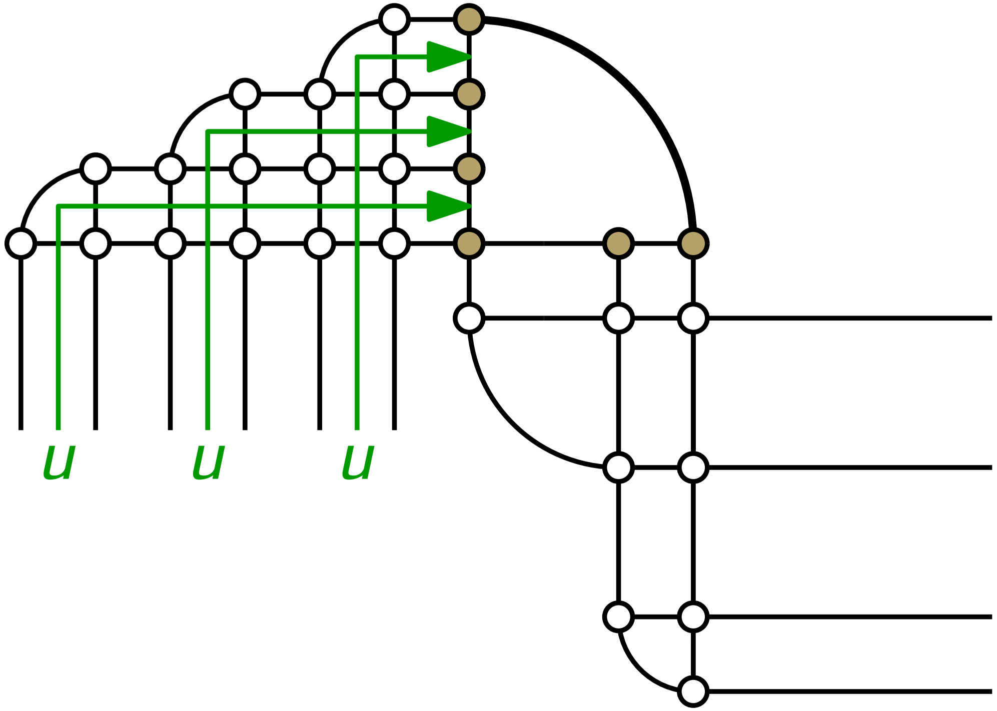

- Take 3 units of “flow” as input

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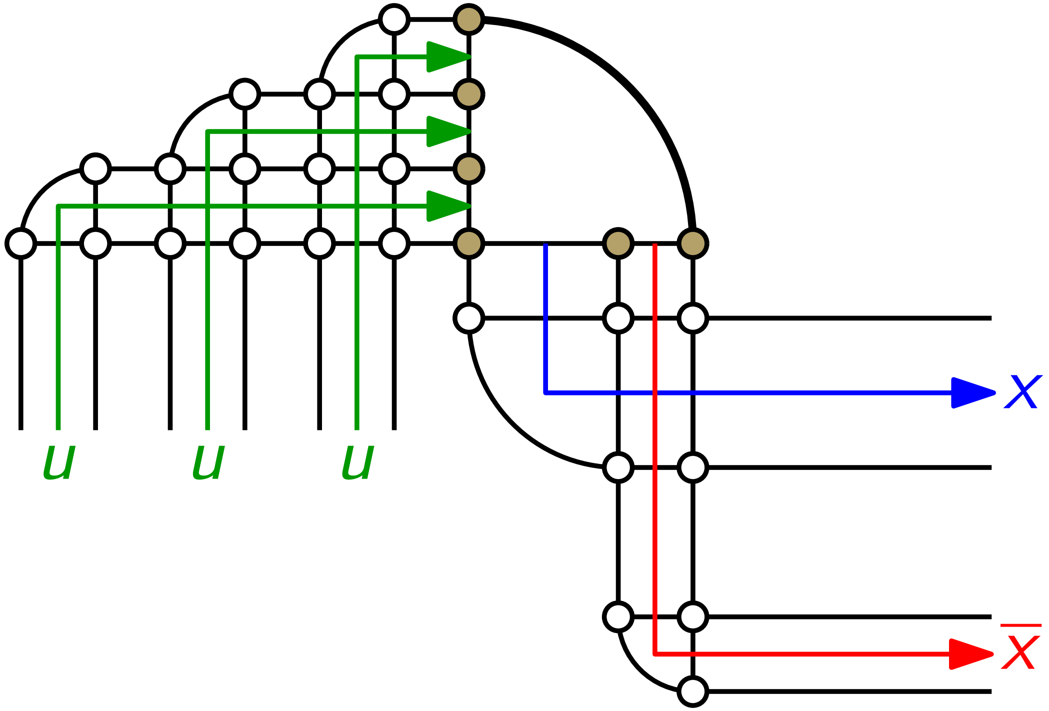

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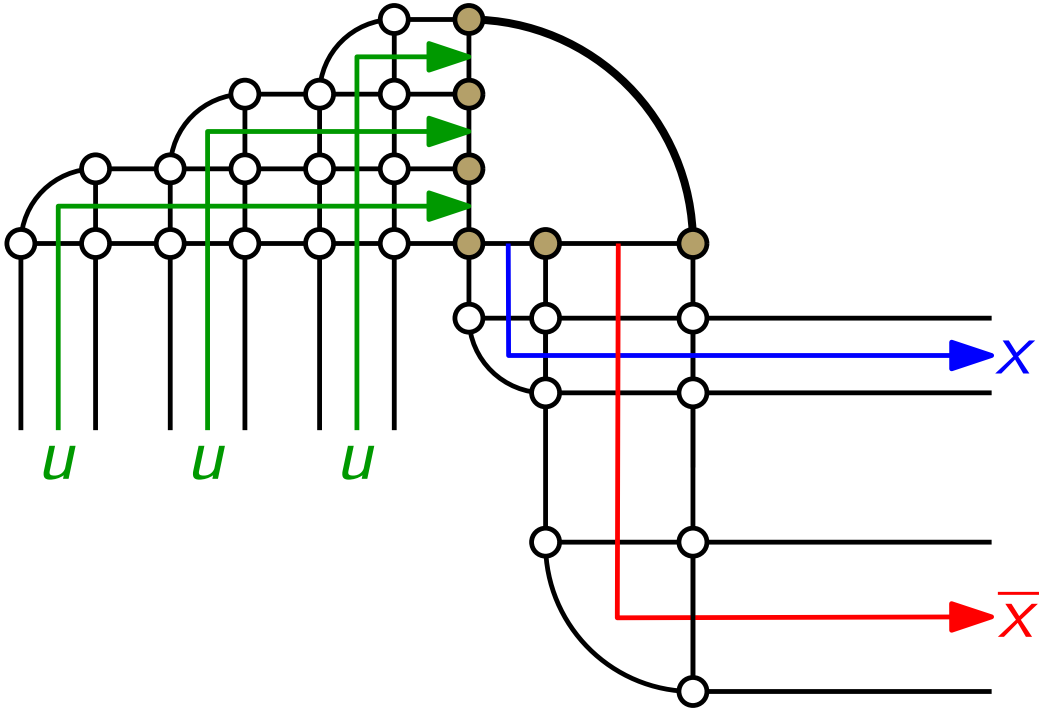

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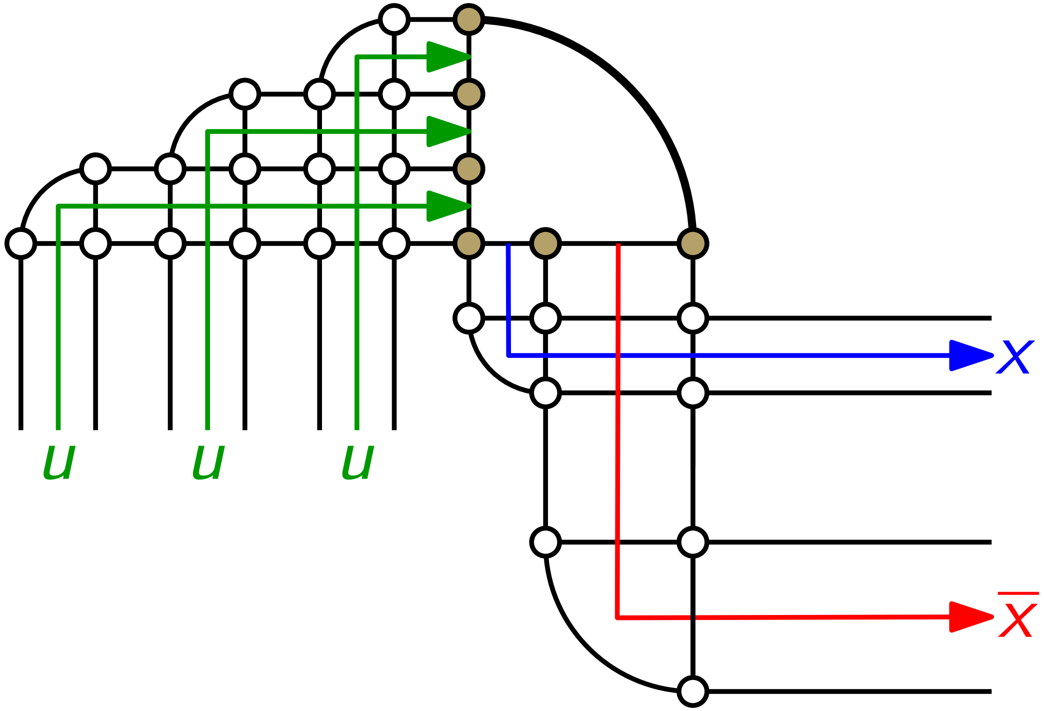

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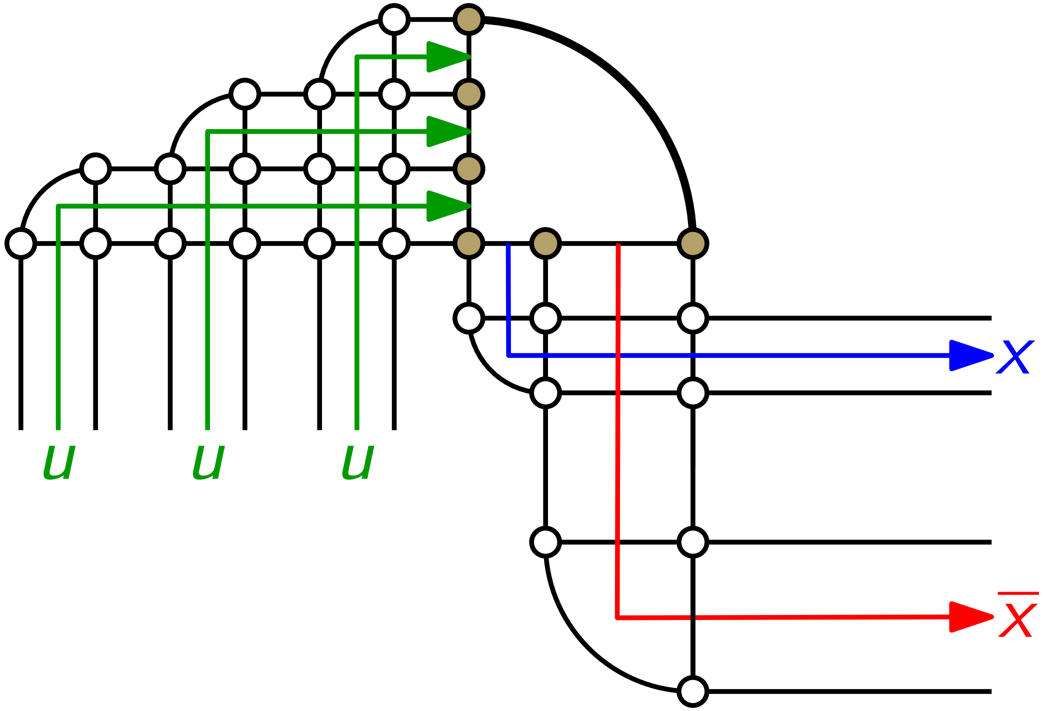

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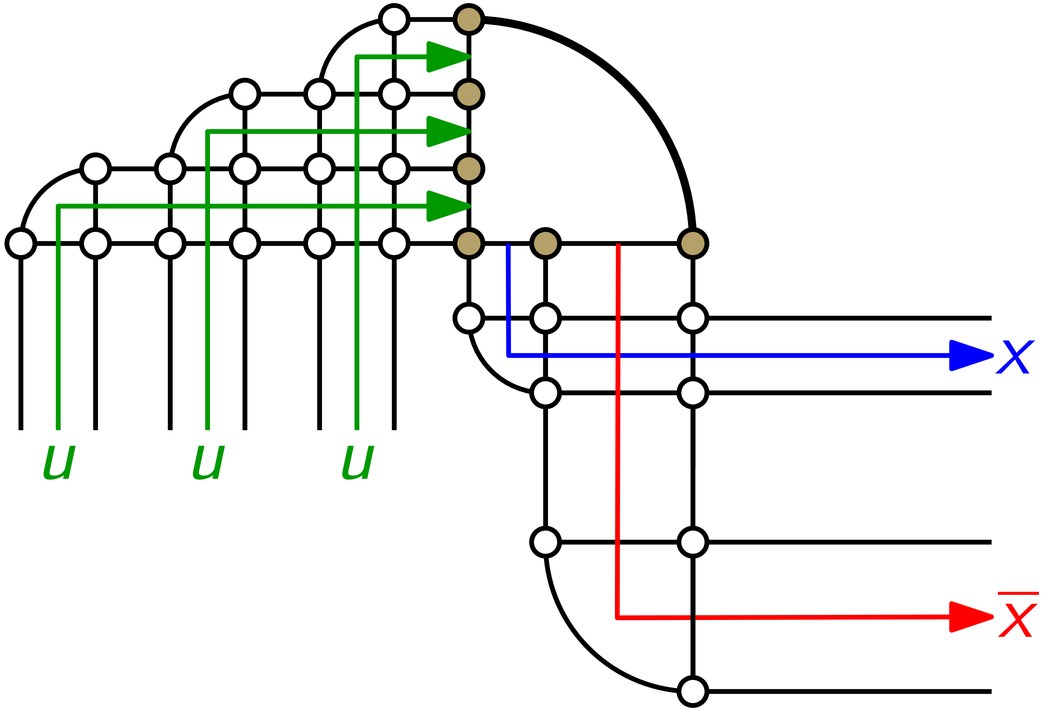

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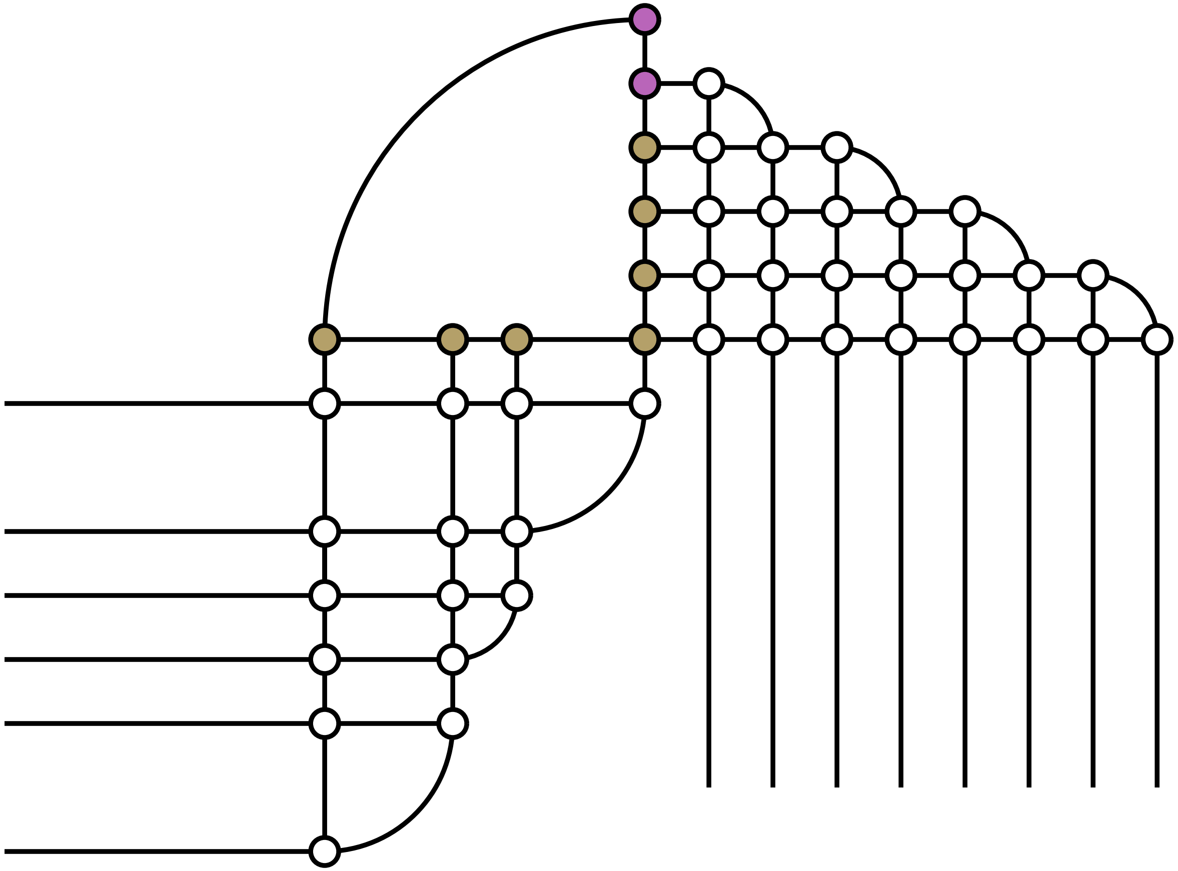
Clause Gadget



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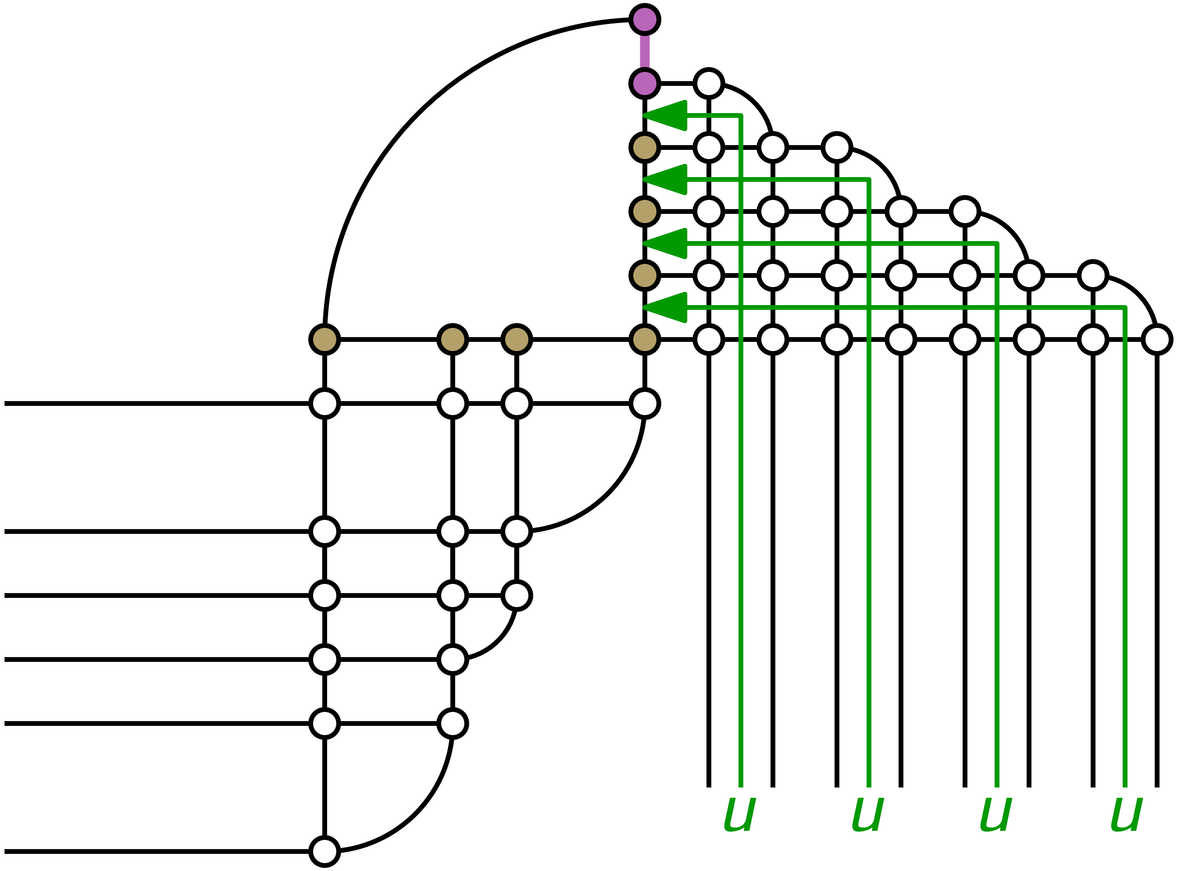

- One side of the arc is 4 units + a free edge's length long

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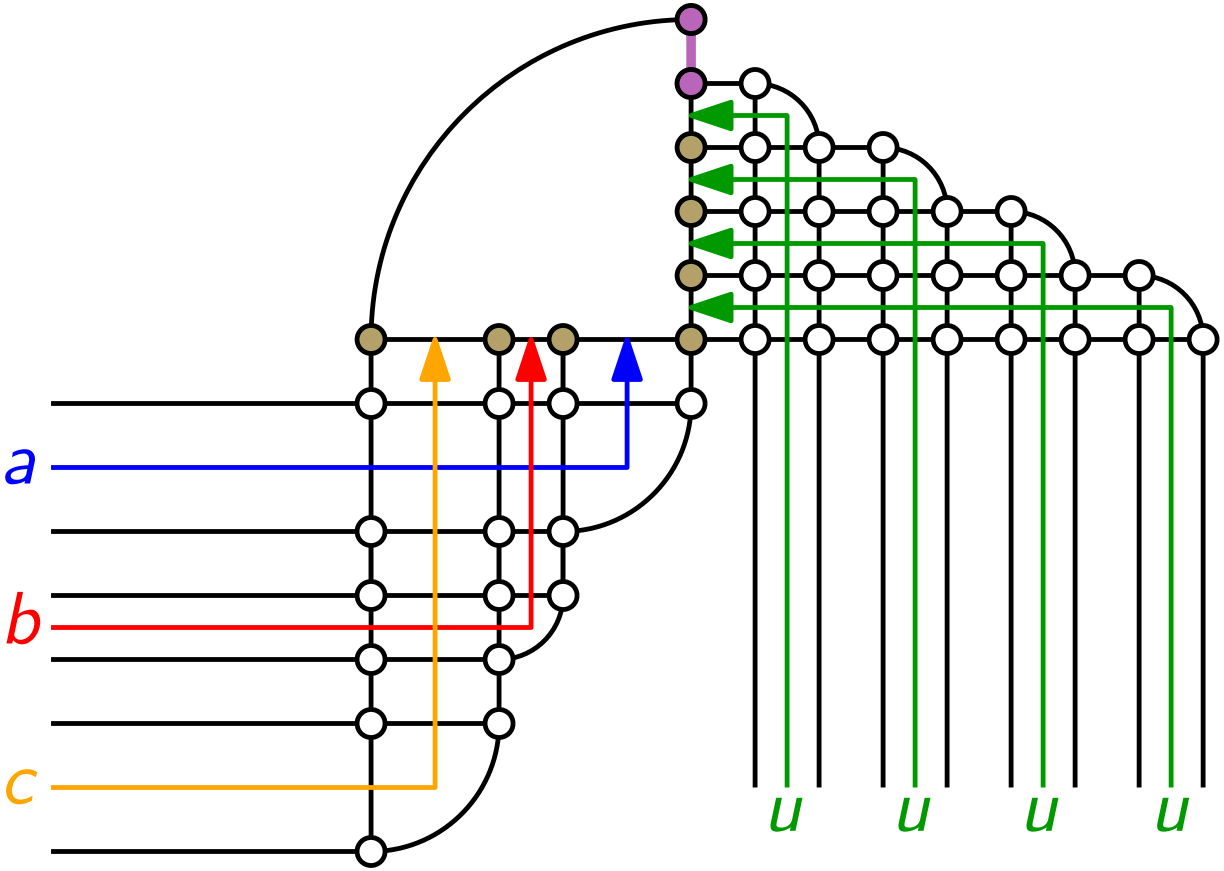

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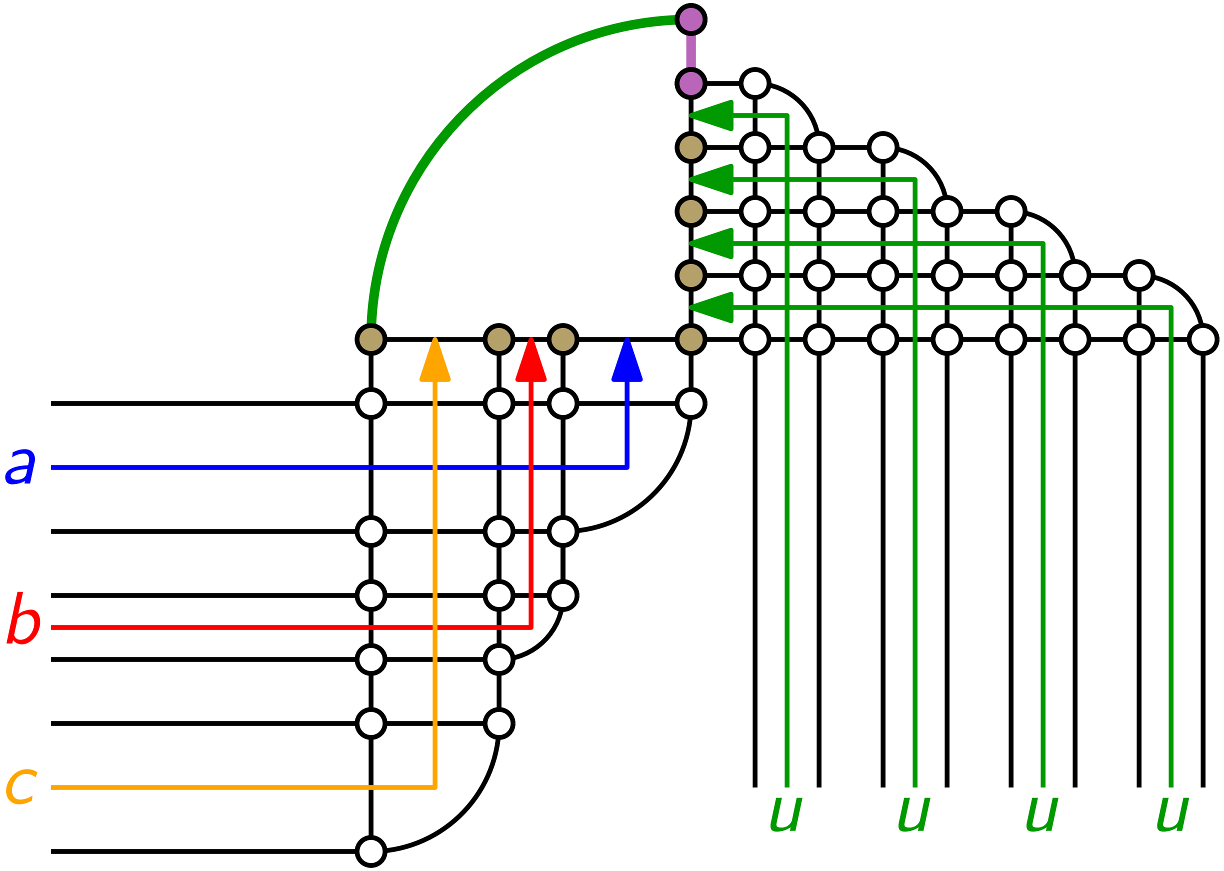

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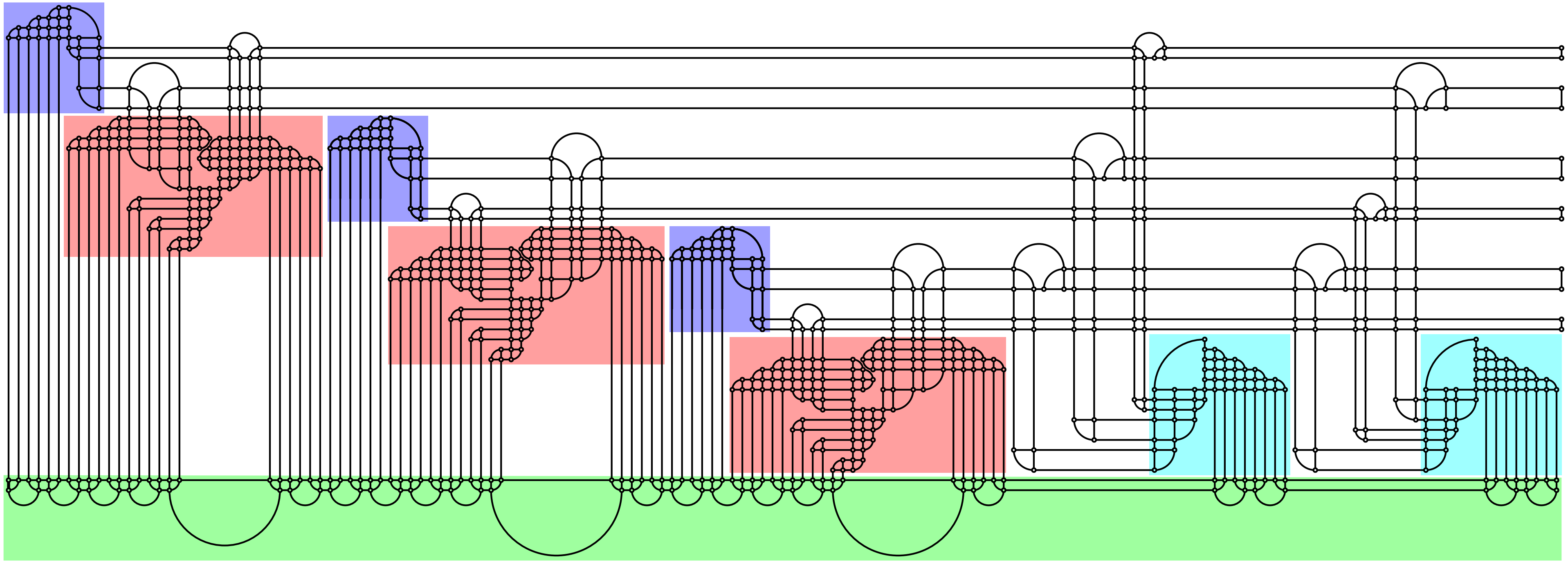
  $\Rightarrow$ at least one literal must be true

A "Small" Example

$(a \vee b \vee c) \wedge (\overline{a} \vee \overline{b} \vee c)$ with $a = \texttt{false}$ and $b = c = \texttt{true}$



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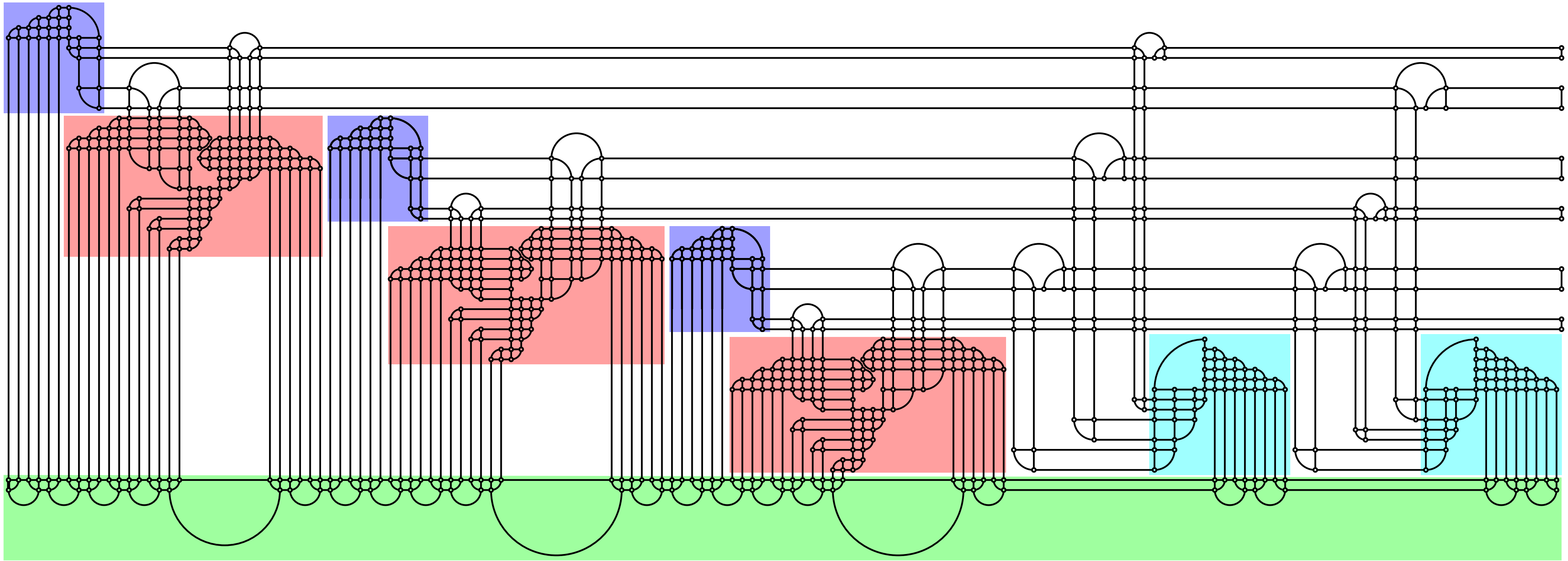

Copies of unit length

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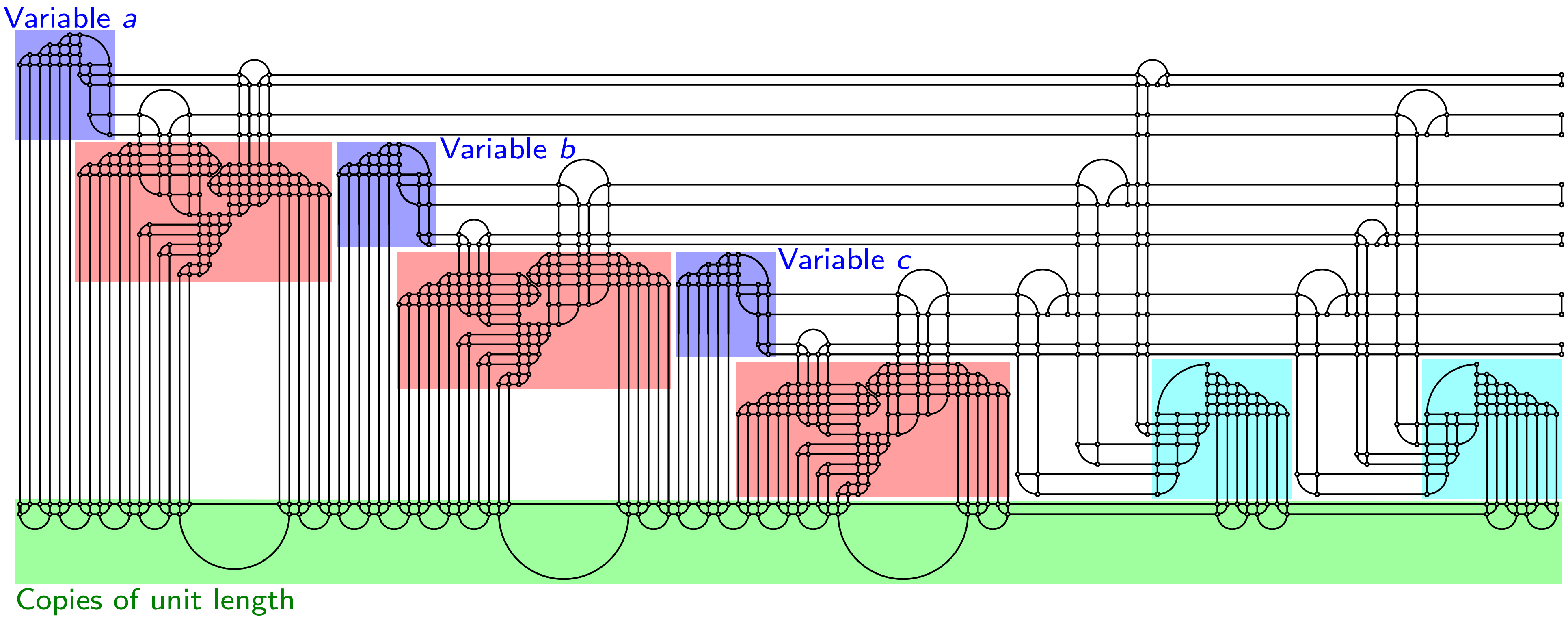

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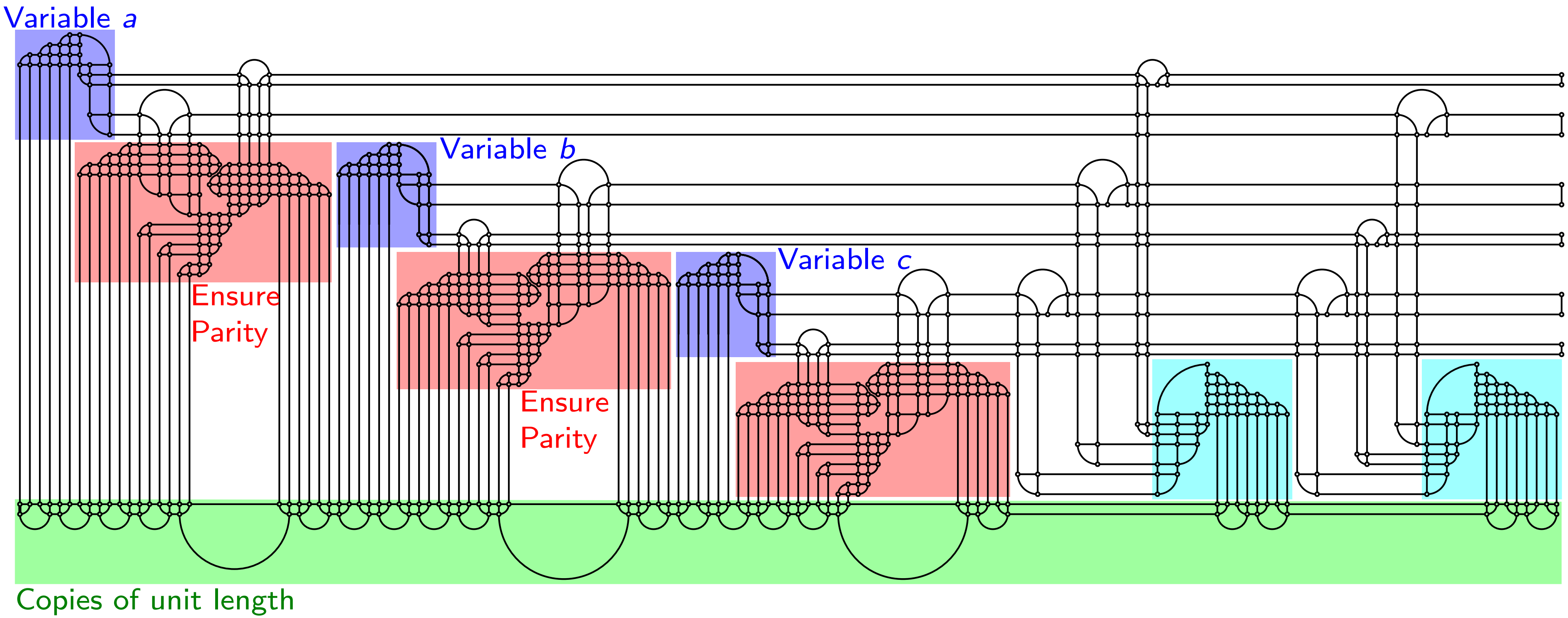

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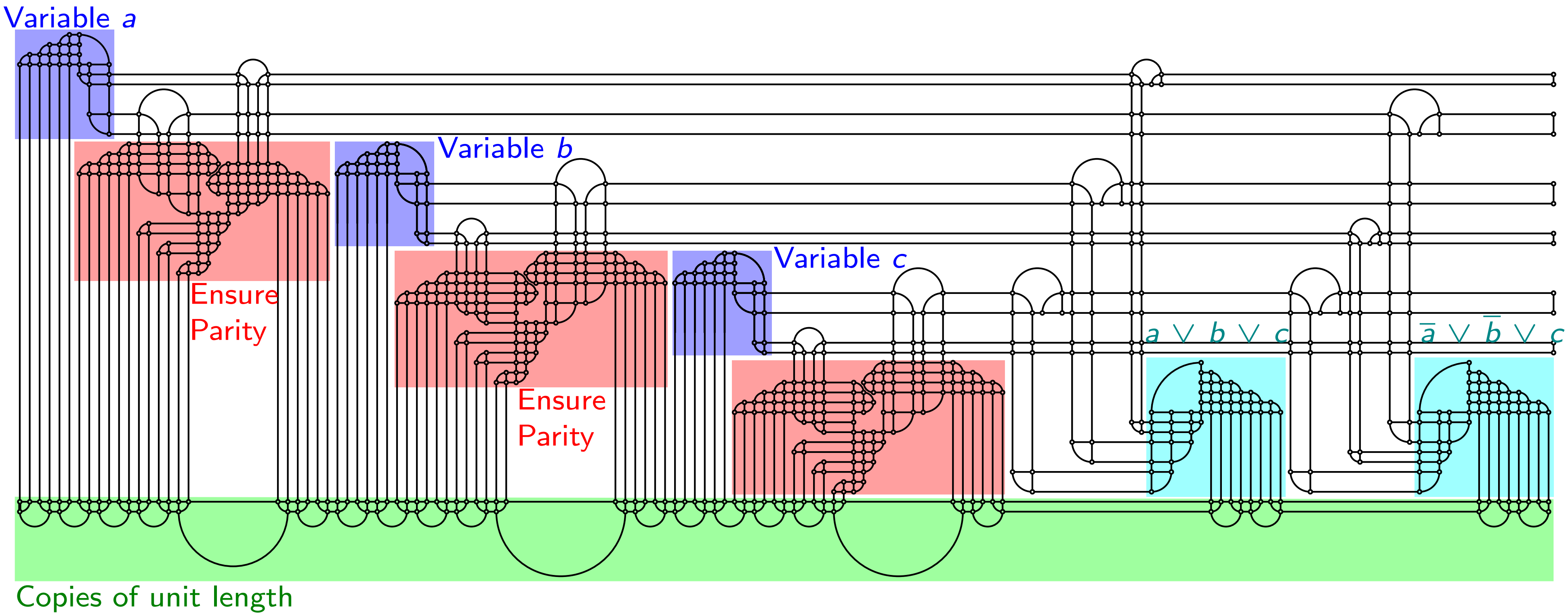

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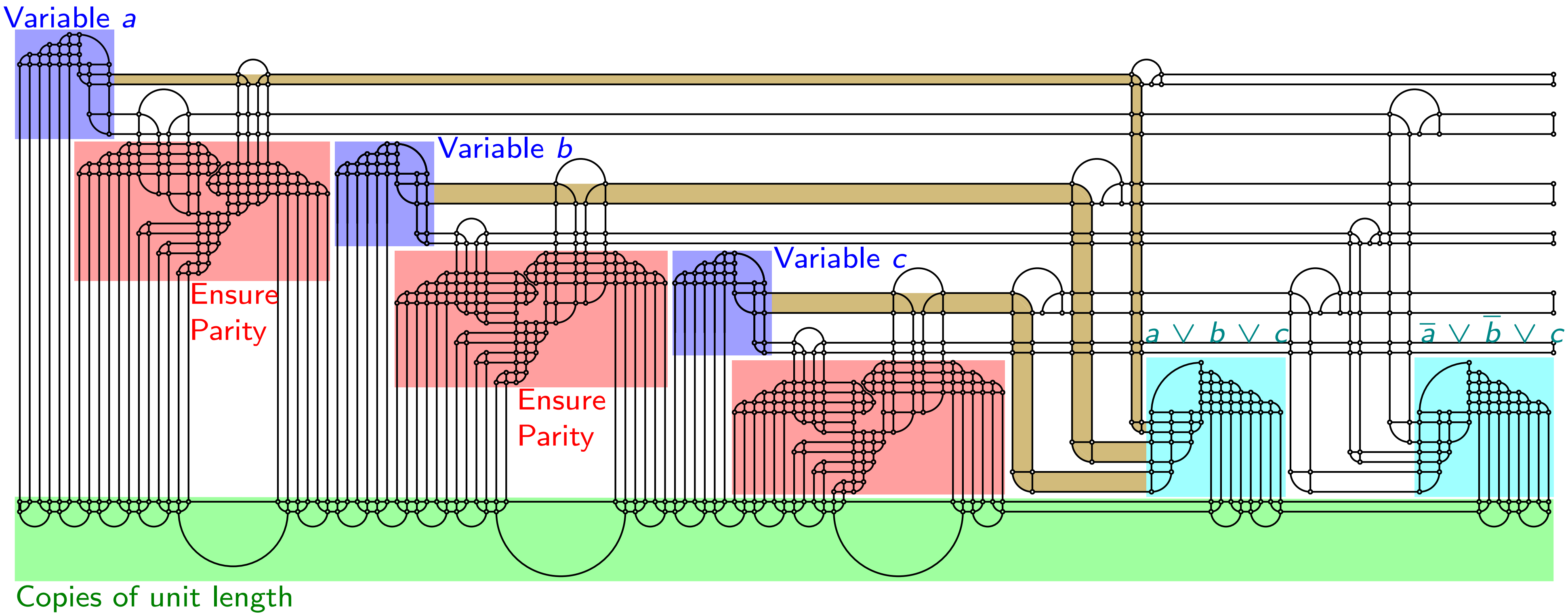

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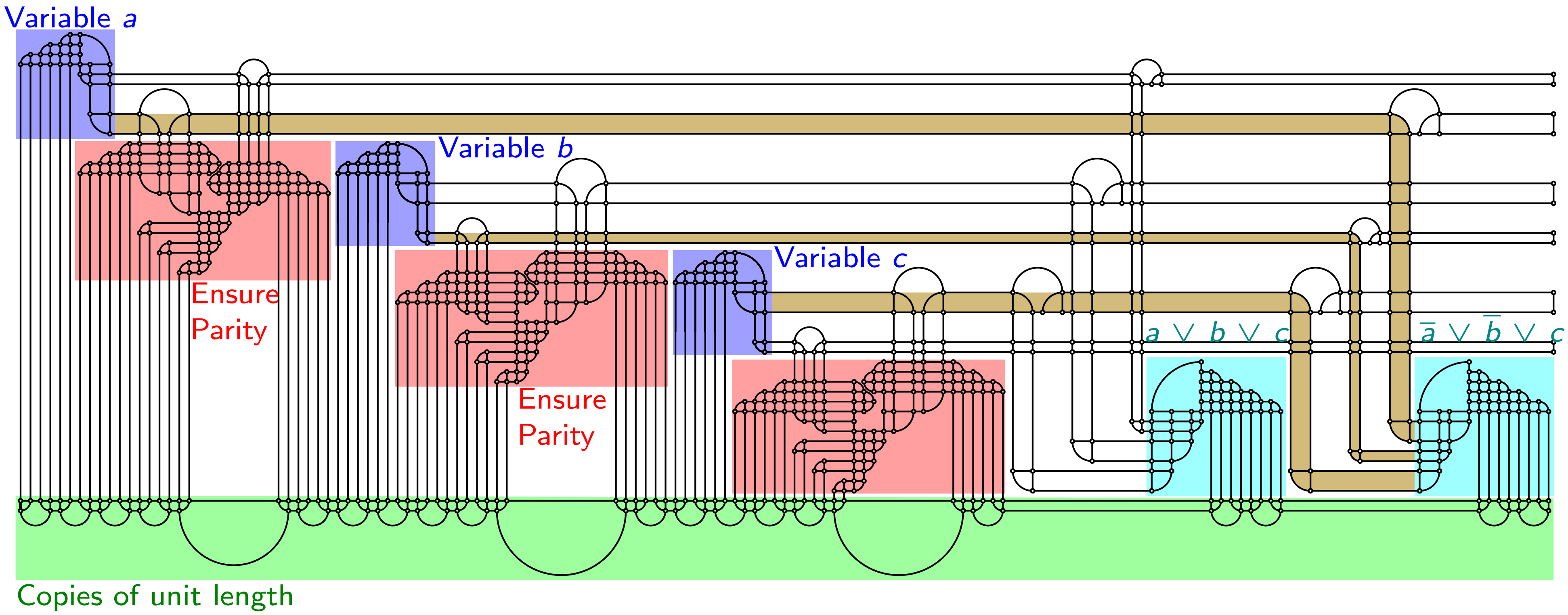

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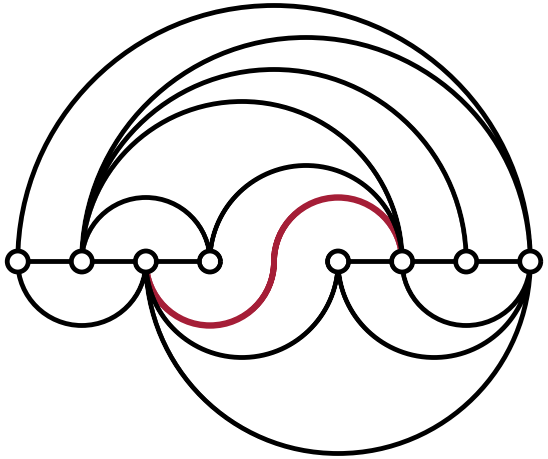
  - Same reduction scheme, most gagdets easy to transform
- TSM approach not suitable for smooth orthogonal and octilinear drawings

Kandinsky Drawings

- Kandinsky model in smooth orthogonal setting so far: Book Embedding Inspired

[Bekos et al. 2013]



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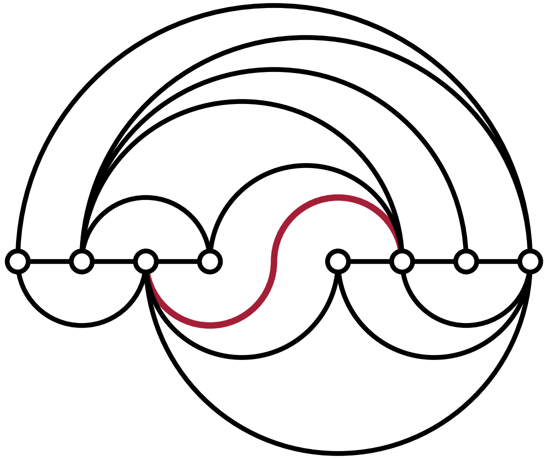

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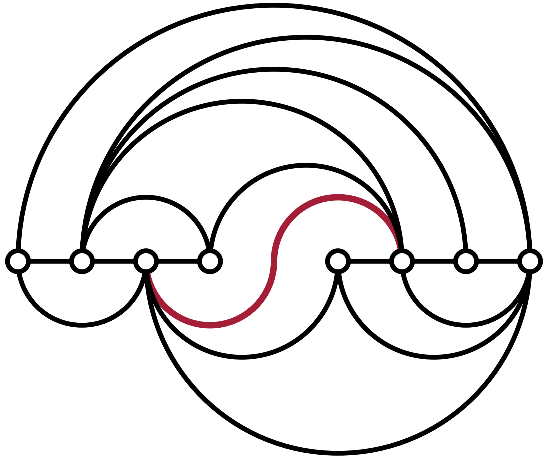
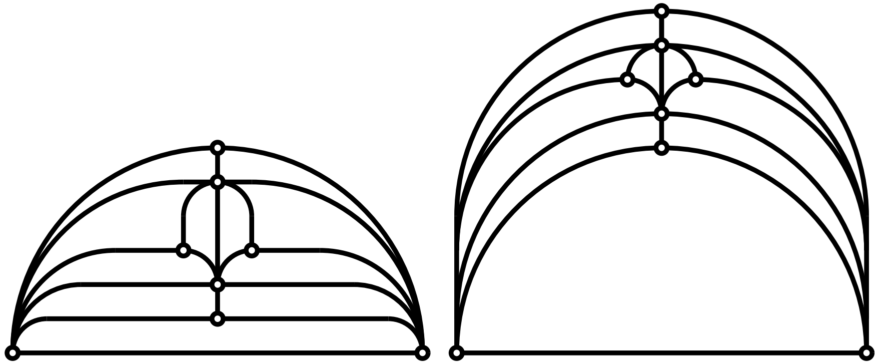


- $O(n)$ time, $O(n^2)$ area, $\leq n-2$ edges of complexity 2... But is it readable?
- Possible improvements:
  - Distribute vertices more evenly
  - Draw edges x, y-monotone



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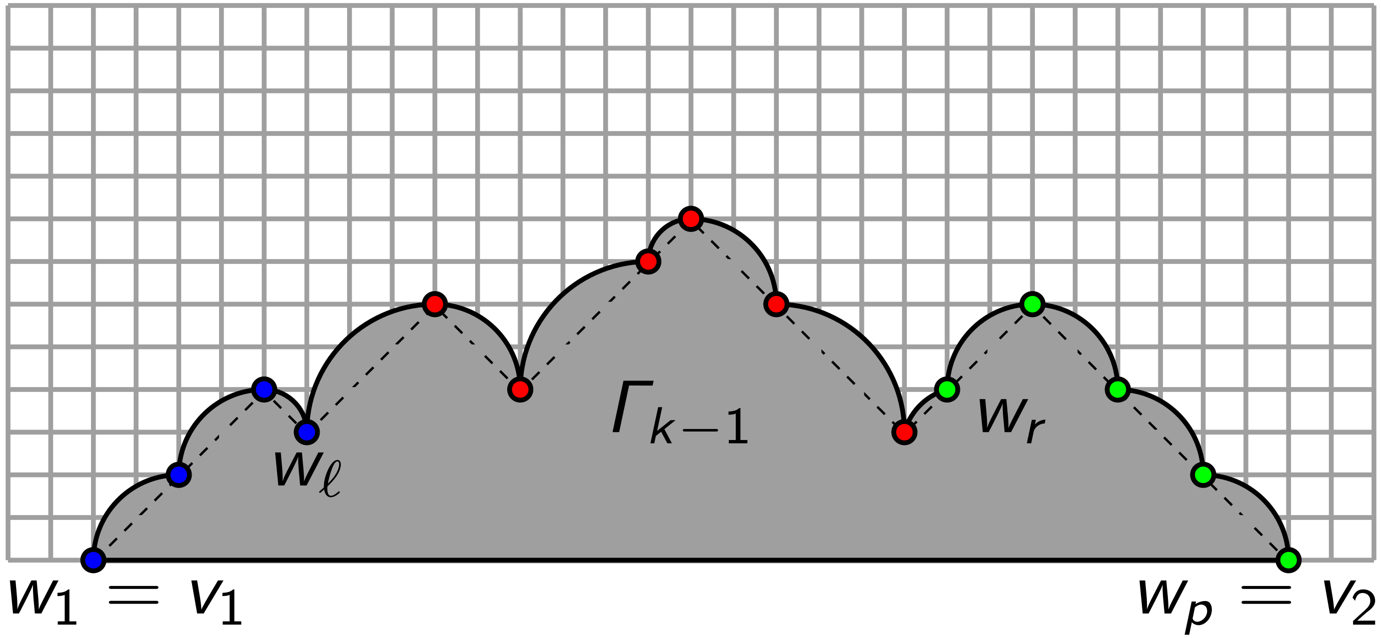

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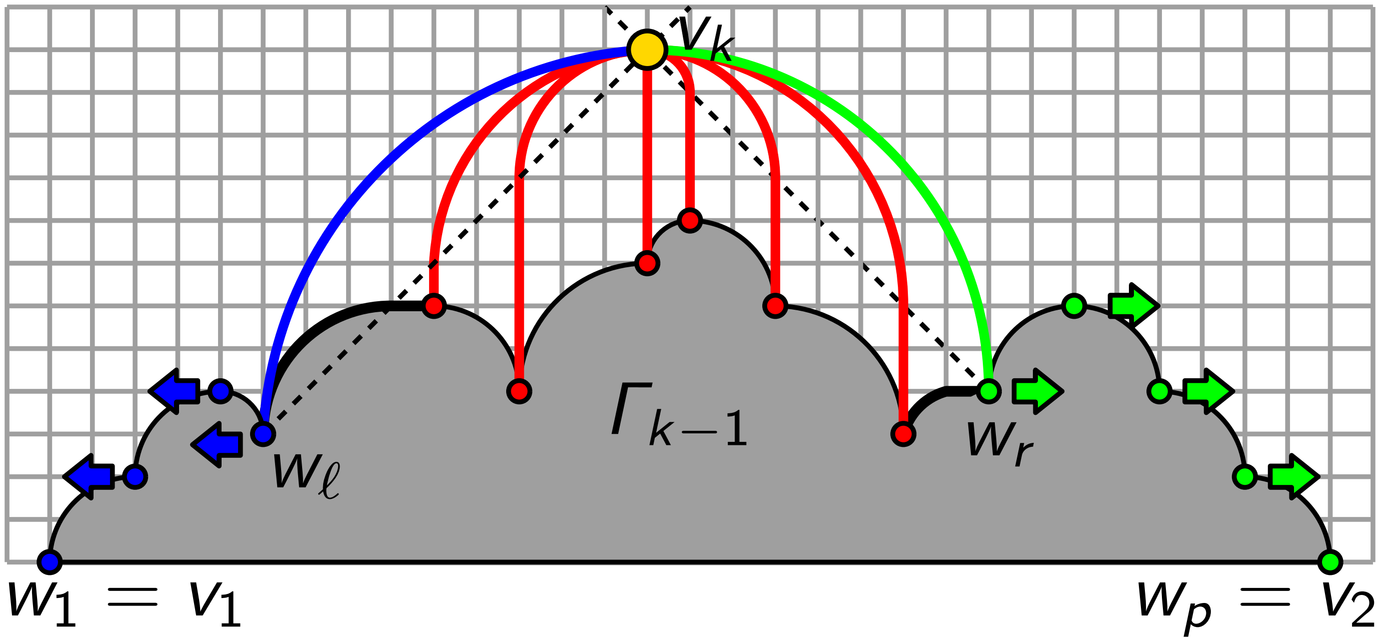

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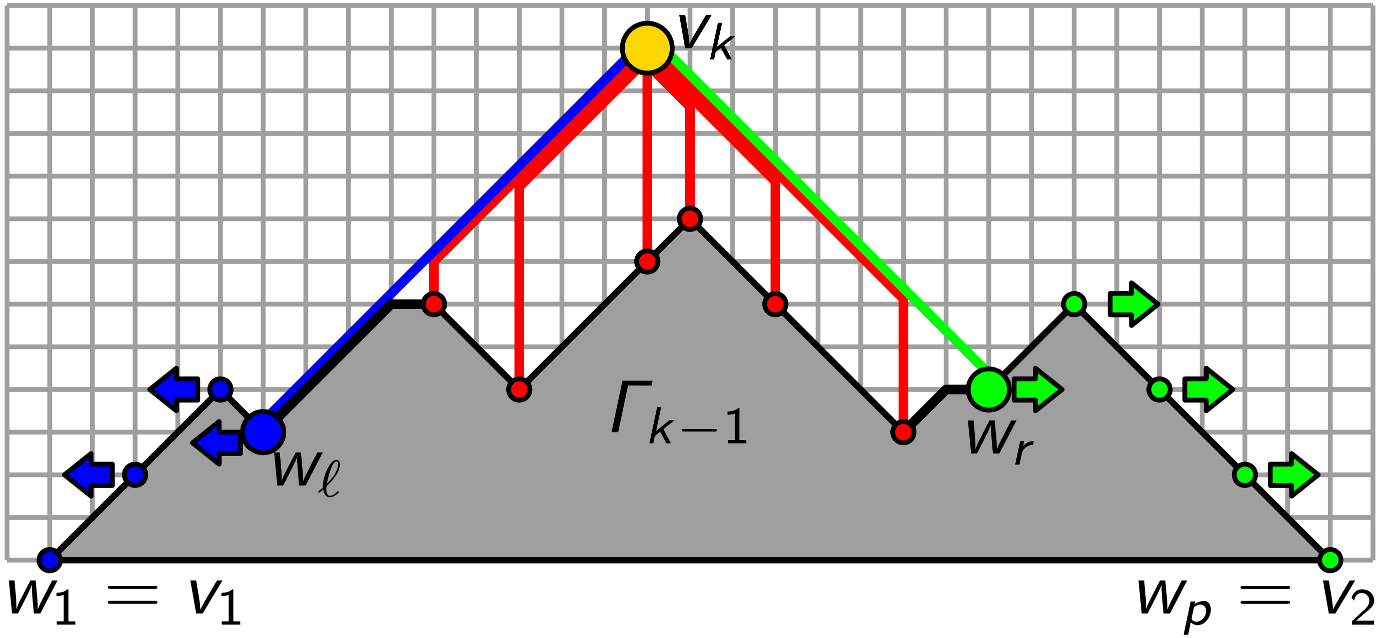

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  - We can use this approach for octilinear Kandinsky drawings too!

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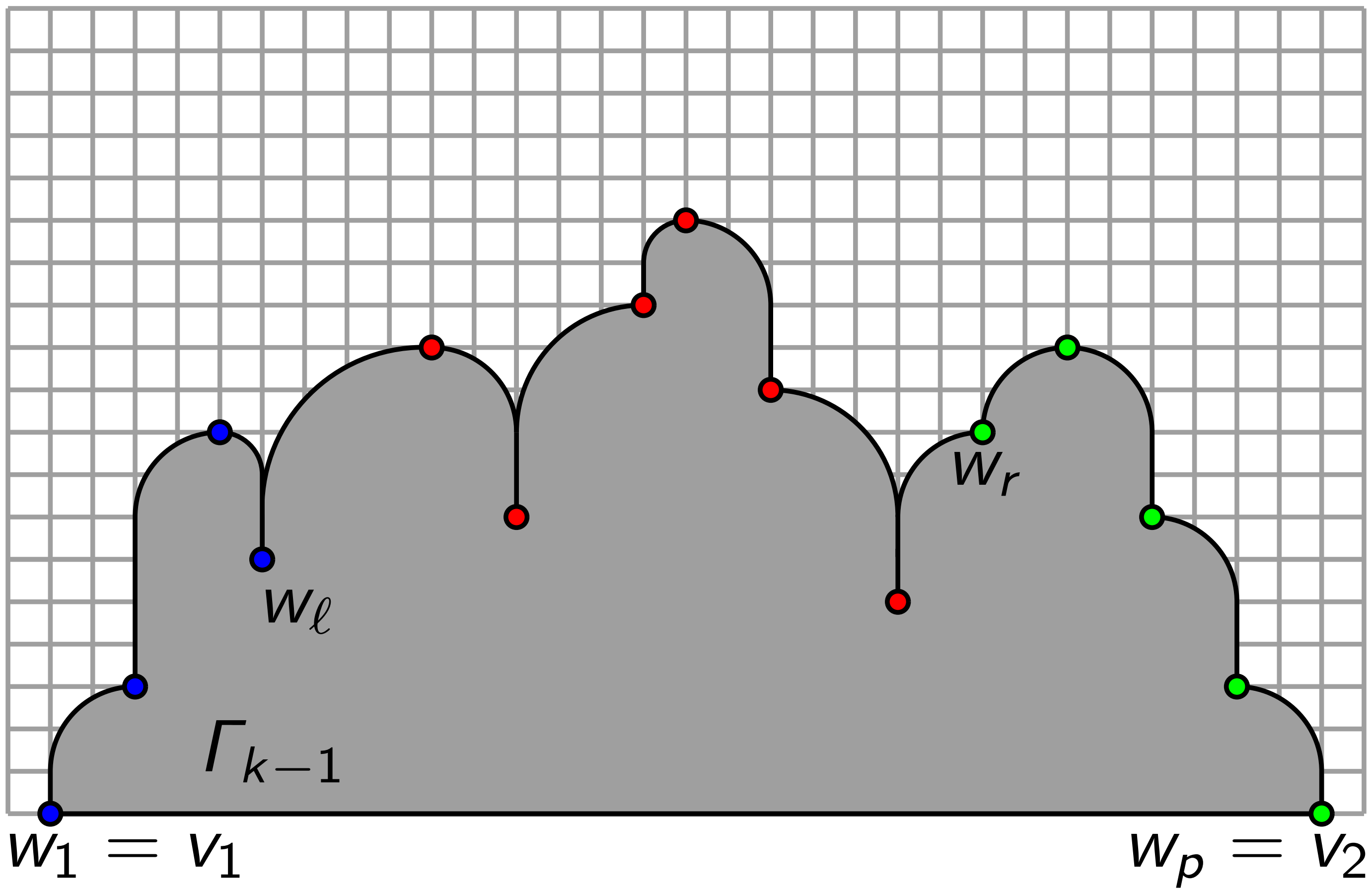
Thanks to the anonymous reviewers!

Drawings with Less Bends

- Now: steep mountains as contour



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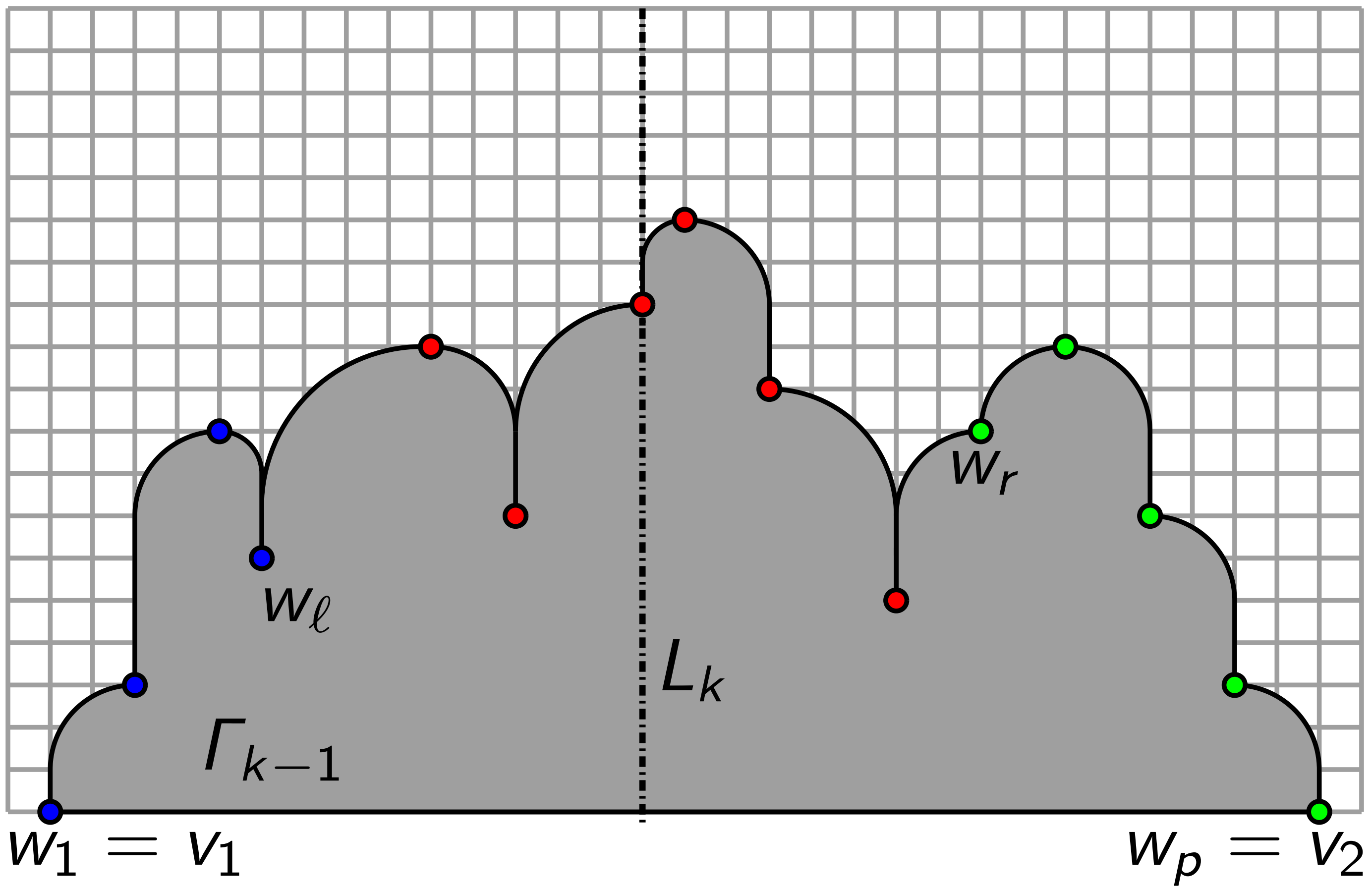

Drawings with Less Bends

- Now: steep mountains as contour
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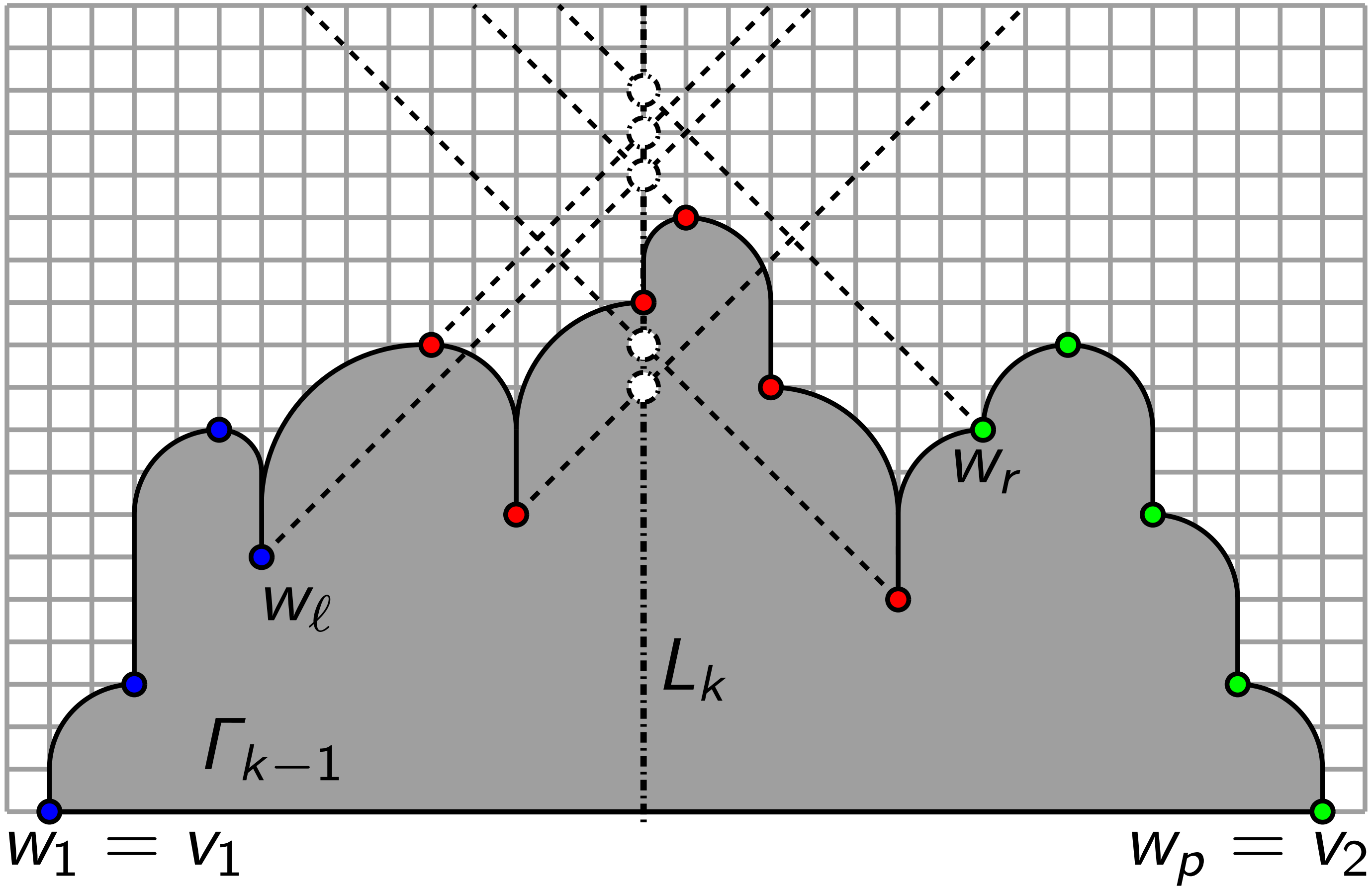

Drawings with Less Bends

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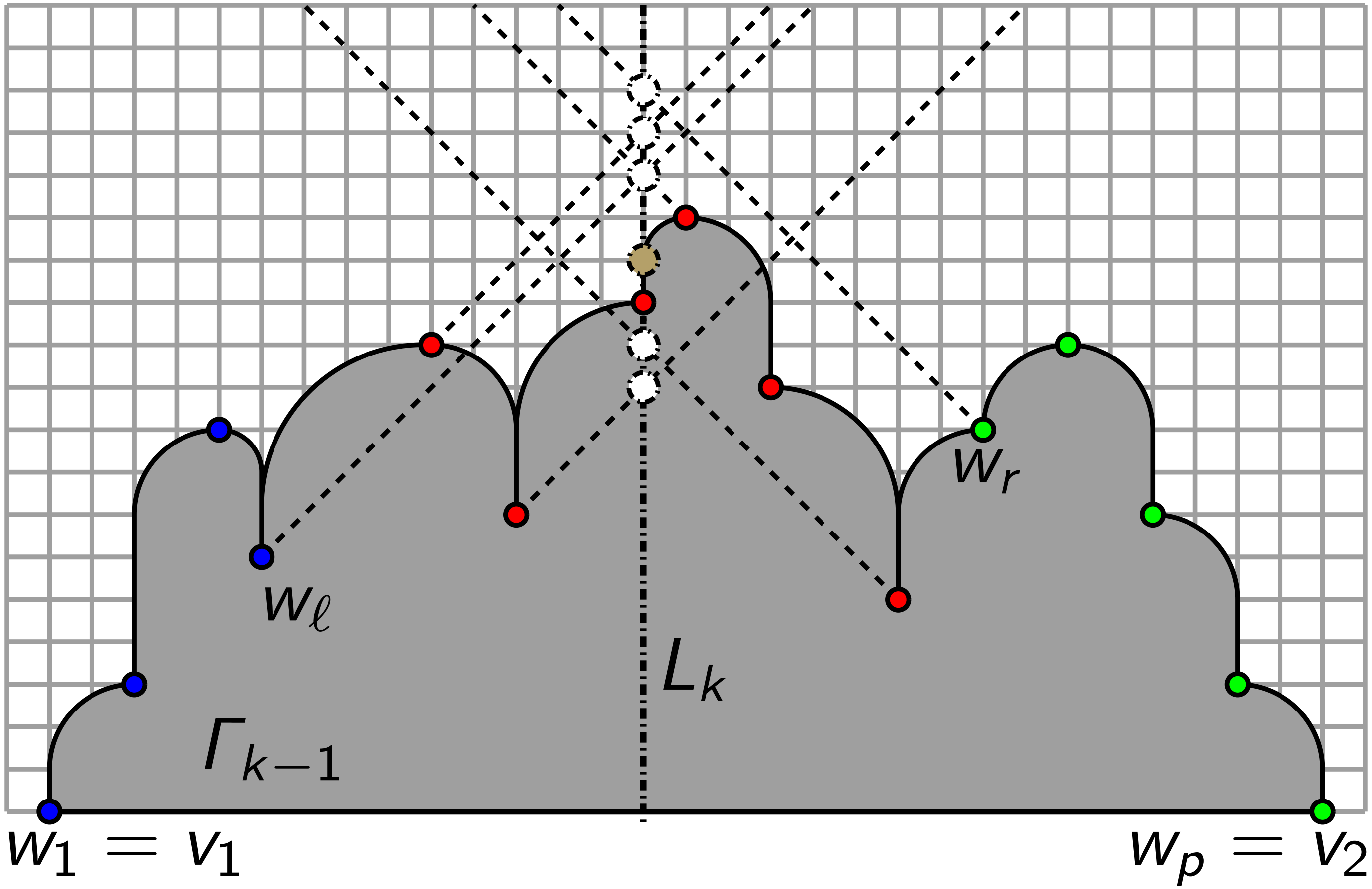

Drawings with Less Bends

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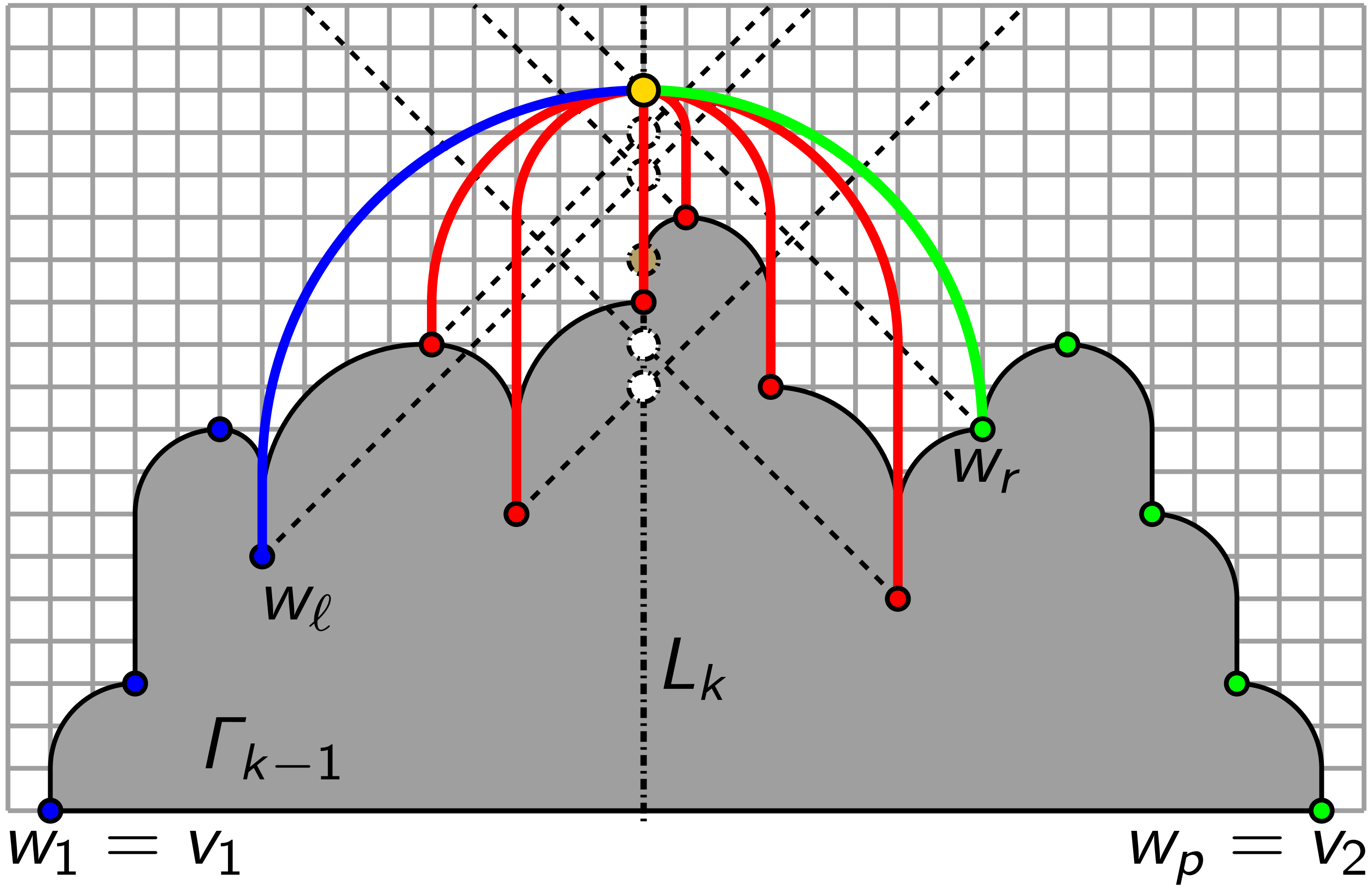
- Use highest candidate position to ensure planarity and contour condition



Open Problems

- **Relations**

- **Complexity**

- **Kandinsky Drawings**

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- **Relations**
  - Higher degree: (smooth) d-linear drawings
  - Relations of smooth d-linear and $2d$-linear on max-degree d
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Thanks for your attention!