# 1 Preliminaries

There are a variety of results regarding a special class of graphs - In the following section, let $G$ be a planar graph with maximum degree 4 and a port constraint. Every vertex is connected to at most one polyedge on each cardinal direction (North, South, East, West).

## 1.1 Fixed Layout Model

Considering a drawing of an orthogonal graph $\Gamma_G$ in the Fixed Layout Model, the position of the vertices cannot be altered. This is a rather strict constraint. The implementation of circular arcs in a polyline lead to an increase of the edge complexity. Every bend is substituted with a quarter arc, resulting in two new bends in the worst case.

**Theorem 1.1.** *In the Fixed Layout Model, the edge complexity of a given orthogonal graph $G$ might increase from $k$ to $2k - 1$.*

The reason for the edge complexity increase is the fixation of the vertices and following counterexample regarding so-called "Staircase Edges".

TODO: Staircase of 4-planar graph in Fixed Layout

It is already clear, that the Fixed Layout Model is rather too restrictive to process orthogonal graphs to SMOGs. A different approach is the Fixed Shape Model, where the embedding is contained regarding the circular ordering of the polyedges around a vertex.

## 1.2 Fixed Shape Model

In the Fixed Shape Model, the orientation of the vertices (North, East, West, South) and the embedding (circular ordering of the edges connected tp 1a vertex) is preserved. The vertices are of uniform size but can be repositioned on the course integer grid. Due to the horizontal stretching technique by the factor of $l$ (longest vertical segment), there is new space left and right from every vertical line segment. To be more precise, there is an empty box left and right from every vertical line with size $l' \times l'$, while $l'$ is the length of the regarding vertical line.
In practice, the SMOG Model is derived from the Kandinsky Model using basically two plane sweeps: The first plane sweep stretches the Kandinsky drawing horizontally by the factor of the longest vertical line segment. The second plane sweep erases 90 degree bends by circular arc substitution, making the drawing *smooth*. The resulting drawing is in $\mathcal{O}(n^2) \times \mathcal{O}(n)$ area due to the horizontal stretching.

**Theorem 1.2.** *If the bends of a polyline are purely uniform (in the same direction), then there is a SMOG representation of that polyline without an increase of complexity. Similarily, if the polyline is purely alternating, the edge complexity raises from $k$ to $\lceil \frac{3}{2}k \rceil$.*

*Proof (Sketch):* If a polyline is purely uniform, consider the vertical segments; If the a vertical segment lies between two horizonal segments, substitute that vertical segment with a semicircle arc. If a vertical segment is connected to a vertex and a segment, substitute the vertical segment with a quadrant arc. If the polyedge consists of one vertical segment, then two vertices are at both ends. The edge is not altered. The space around a vertical segment, necessary for the circle arc substitution, is guaranteed by stretching the entire drawing horizontally by a factor of the longest vertical segment. □

**Theorem 1.3.** *Let $G$ be an orthogonal graph. If any polyline is alternating at some point, $G$ can be minimized regarding the number of bends.*

*Proof (Sketch):* We show the theorem by flow minimization over the dual graph the following way: for each alternation we send one unit of flow from one face to another. Determining the unit of the minimized flow and the direction between both faces determines the minimal direction changes of an edge resulting in a lower number of bends. □

**Theorem 1.4.** *In the Fixed Shape Model, an orthogonal graph $G$ - with minimal number of bends and an edge complexity of $k$ - can be transferred to a SMOG without an edge complexity increase under $\mathcal{O}(n^2) \times \mathcal{O}(n)$ area.*

*Proof (Sketch):* If $G$ has got a minimal number of bends, then there is no alternation in any polyline by contraposition of Theorem 1.3. The polyedges are purely uniform and every vertical segment is replaced by either a quarter circle arc or a semicircle arc or it stays the same. As we already saw, uniform bends do not lead to an edge complexity increase. Planarity is preserved due to the stretching technique. □

# 2 Guarantee for the number of bends of a polyline in SMOG

Let $G$ be a planar graph and a Kandinsky drawing $\Gamma_G$ and an edge complexity of $k$. The main goal of this section is to examine lower and upper bounds for the edge complexity in SMOG.

## 2.1 Examining "good" and "bad" parts of a polyline

The general idea is to distinguish between two major cases: Line segments with alternating turns (staircase, zig-zags) and line segments with uniform turns. As we already saw, the SMOG representation of uniform turns does not increase the edge complexity. Therefore we examine polyedges regarding its properties of turns from one vertex to another. Try to maximize the uniform part and simultaneously minimize the alternating part of a polyedge. We achieve this by so-called *fragmentation* of a polyedge.

**Definition 2.1.** *An* ***edge fragment*** *is a non-empty sequence of line segments. A* ***fragmentation*** *of a polyedge is a sequence of fragments.*

**Lemma 2.1.** *A fragment is either purely uniform or purely alternating in its turns.*

The property of turns is crucial for the edge complexity thus the main criterion for the fragmentation.

**Algorithm 1:** Algorithm to determine the progress of turns

**Input:** Polyedge $e = e_1 e_2 e_3 \ldots e_{k+1}, k \geq 2$
**Result:** $e'$, a first fragmentation of $e$

$e'_1 \leftarrow (e_1, e_2, e_3)$
**if** $e'_1$ *uniform* **then**
  $e'_1$.`uniform` = `True`
**else**
  $e'_1$.`uniform` = `False`
**for** *all remaining segments* **do**
  **if** *the current segment fits into the turn property of* $e'_1$ **then**
    $e'_1$.`append`(current segment)
  **else**
    create a new fragment with $e'_m$.`uniform` $\leftarrow \neg e'_{m-1}$.`uniform` and continue
**return** $(e'_1, e'_2, \ldots, e'_n)$

**Lemma 2.2.** *The fragmentation of a given polyedge $e$ is not unique.*

*Proof (Sketch):* Consider following drawing:

`TODO: Non-unique fragmentation`

□

**Lemma 2.3.** *Let $e$ be a polyedge with $k$ bends. Then a fragmentation contains minimal one fragment, maximal $\lfloor \frac{k}{2} \rfloor$ fragments.*

**Definition 2.2.** *Let $e'$ and $f'$ be two fragmentations. Then*

$$e' \sim_R f' \Leftrightarrow \Gamma_{e'} = \Gamma_{f'}$$

1 This means that two fragmentations are relative iff they describe the same polyedge in a given drawing.

**Lemma 2.4.** *The relation from Definition 2.2 is an equivalence relation.*

*Proof (Sketch):* Describing the same image is trivially reflexive, symmetical and transitive. □

**Lemma 2.5.** *The amount of all possible fragmentations of a polyedge $e$ is finite. To be more precise:*

$$|[e]_{\sim_R}| < \infty$$

*Proof (Sketch):* Given a polyedge $e$, the amount of segments describing it is finite. Therefore, the amount of all possible fragmentations is finite and bounded by `TODO`. □ Now, given a mathematical backbone to a given polyedge $e$, we want to describe the "best way" to determine the number of bends in SMOG. So we will pick the "best"fragmentation, which is minimal in its number of alternating fragments and minimal in its total number of fragments.

**Definition 2.3.** *Let $e', f'$ be two fragmentations of the same polyedge $e$ ($e' \sim_R f'$). Then we define a relation:*

$$e' \leq f' \Leftrightarrow (\#_{altFrag}(e') \leq \#_{altFrag}(f')) \wedge (\#_{totalFrag}(e') \leq \#_{totalFrag}(f'))$$

**Lemma 2.6.** *The relation from Defintion 2.3 is sufficient to determine a minimum of $[e]_{\sim_R}$.*

*Proof (Sketch):* Obviously, the relation $\leq$ is reflexiv and transitive. Therefore the relation is a preorder and we are able to determine a minimum of $[e]_{\sim_R}$ because in a finite set there is a unique minimum regarding a preorder. □ Now we achieved the best representation of a polyedge in form of a fragmentation. This representation goes along with the already achieved results. The results of purely uniform or purely alternating polyedges also apply for purely uniform or purely alternating fragments because every fragment can be considered as a seperate polyedge simulated with dummy vertices. This means that a fragment with edge complexity $k'$ can be transferred to a SMOG fragment with edge complexity $\leq \lceil \frac{3}{2} k' \rceil$.

**Lemma 2.7.** *Let $e$ be a polyedge and $e'$ its best fragmentation regarding the number of alternating fragments and total fragment. To be more precise: $e' = \min_{\leq} [e]_{\sim_R}$. If $e'$ contains more than one fragment, the polyedge $e$ has not got a minimal number of bends.*

*Proof (Sketch):* Let $e' = (e'_1, e'_2)$ without loss of generality. Then $e'$ cannot be further reduced to a single fragment, hence it is the minimum of the equivalence class. If $e'_1$ or $e'_2$ were alternating, then the number of bends can be minimized. If $e'_1$ and $e'_2$ were uniform in its turns, the reason for the inability of a fragment merge must be a sort of alternation. Therefore, the polyedge hasn't got a minimal number of bends. □