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Conference Paper · September 2011

DOI: 10.1063/1.3636990

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Abstract. We consider maximum series-parallel graphs, propose their recursive description, and prove that every maximal series-parallel graph is maximum as well.

Keywords: series-parallel graph, directed acyclic graph, source, sink
PACS: 89.20.Ff

INTRODUCTION

A *graph* $G = (V,E)$ consists of a *vertex set* V and an *edge set* E, where each edge corresponds to a pair of vertices. If the edges are ordered pairs of vertices, then we call the graph *directed* or *digraph.* A vertex in a digraph is a *source* if no edges enter it, and a *sink* if no edges leave it. A two-terminal directed acyclic graph (*st-dag*) has only one source s and only one sink t. A *simple graph* is a graph without multiple edges.

A *series-parallel* (*SP*) *graph* is an st-dag defined recursively as follows:
(i) A single edge (u,v) is a series-parallel graph with source u and sink v.
(ii) If G_1 and G_2 are series-parallel graphs, so is the graph obtained by either of the following operations:
(a) Parallel composition: identify the source of G_1 with the source of G_2 and the sink of G_1 with the sink of G_2.
(b) Series composition: identify the sink of G_1 with the source of G_2.
Figure 1 shows an example of a non-simple series-parallel graph.

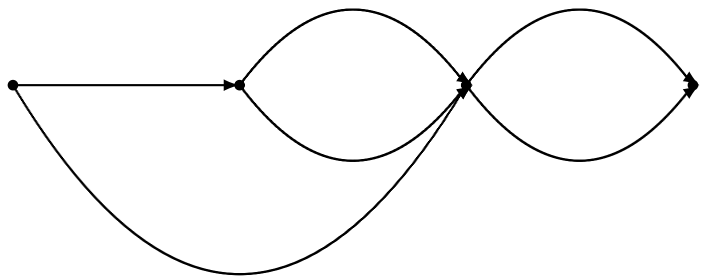

FIGURE 1. A series-parallel graph.

An n-vertex simple SP graph is a *maximum series-parallel graph* if it contains the maximum possible number of edges which can exist in an n-vertex simple SP graph. A simple SP graph G is called a *maximal series-parallel graph* if the addition of an edge between any two vertices of G which are not connected by an edge, turns G into a non-SP graph.

Many problems, which may be intractable for arbitrary st-dags, have efficient solutions for series-parallel graphs. The problems related to series-parallel graphs are considered in [1]–[5] and other works.

In this paper we investigate the structure of maximum series-parallel graphs.

RESULTS

Because the special case of an SP graph taking the form of a simple graph is of interest, and that is our focus in this paper, for the sake of brevity, we are able to omit the designation, "simple" before the term, "SP graph".

The following lemma and theorem were proved in [3].

Lemma 1 *An n-vertex maximum SP graph ($n > 1$) has an edge connecting its source and sink.*

Theorem 2 *The number m of edges in an n-vertex SP graph is bounded as*

$$n-1 \leq m \leq 2n-3 \quad (n>1).$$

An example of a 5-vertex maximum SP graph is presented in Figure 2. It contains $2 \cdot 5 - 3 = 7$ edges. Such a graph is a *centered series-parallel graph*. An n-vertex centered SP graph is obtained by the parallel composition of a single edge and an SP graph which is constructed by the series composition of an $n-1$-vertex centered SP graph and a single edge. A 2-vertex SP graph is an initial centered SP graph.

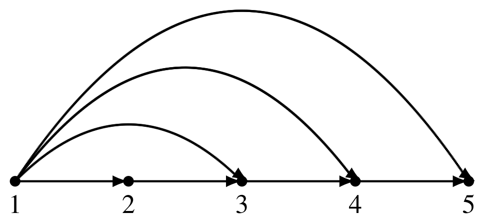

FIGURE 2. A centered series-parallel graph.

Proposition 3 *An n-vertex maximum SP graph has exactly* $2n-3$ *edges for any* $n > 1$.

Proof. The proof is based on induction on n. Consider a centered SP graph in Figure 2. A single-edge SP graph is a 2-vertex maximum SP graph, and it has $2 \cdot 2 - 3$ edges. Suppose, this proposition holds for an n-vertex centered SP graph. An increment of an additional vertex to such a graph leads to an increase in its edge number by two. Hence, the number of edges in an $n+1$-vertex centered SP graph is

$$2n-3+2 = 2(n+1)-3.$$

Thus, for any $n > 1$, there exists an n-vertex SP graph that has $2n-3$ edges. On the other hand, by Theorem 2, an n-vertex SP graph contains not more than $2n-3$ edges. The proof is complete. ■

An example of a 5-vertex maximum SP graph that is not centered is illustrated in Figure 3.

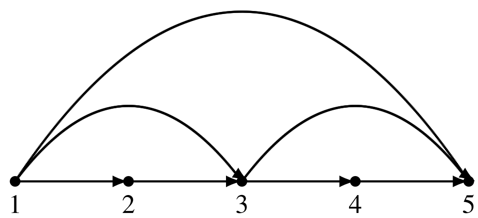

FIGURE 3. A maximum series-parallel graph.

We define an SP graph that is not maximum only since it does not have an edge connecting its source and sink as an *almost maximum series-parallel graph*.

Theorem 4 *A maximum SP graph may be defined recursively as follows:*

(i) A single edge is a 2*-vertex maximum SP graph.*

(ii) Suppose G_1 *is an* n_1*-vertex maximum SP graph and* G_2 *is an* n_2*-vertex almost maximum SP graph. In such a case, graph G which is constructed by the parallel composition of* G_1 *and* G_2 *is an* n_1+n_2-2*-vertex maximum SP graph.*

Proof. By Proposition 3, graph G_1 has $2n_1 - 3$ edges and graph G_2 has $2n_2 - 4$ edges. Therefore, the number of edges in an n-vertex graph G ($n = n_1 + n_2 - 2$) is

$$2n_1 - 3 + 2n_2 - 4 = 2(n_1 + n_2) - 7 = 2(n+2) - 7 = 2n - 3. \tag{1}$$

Thus, the graph constructed in accordance with our recursive definition is a maximum SP graph.

Now, we will show that any maximum SP graph can be constructed in accordance with our recursive definition. It is clear that if G_1 or G_2 is a non-simple graph, then G is a non-simple graph (for both series and parallel compositions of G_1 and G_2). Hence, G_1 and G_2 are simple graphs. By Lemma 1, an SP graph constructed by a series composition of two SP graphs cannot be a maximum SP graph. Hence, the last composition in generating a maximum SP graph is a parallel one. As follows from (1) an n-vertex SP graph G, constructed by the parallel composition of an n_1-vertex SP graph G_1 and an n_2-vertex SP graph G_2, is maximum and, by Proposition 3, has $2n-3$ edges only if G_1 contains not fewer than $2n_1 - 3$ edges, or G_2 contains not fewer than $2n_2 - 3$ edges. On the other hand, by Theorem 2, G_1 contains not more than $2n_1 - 3$ edges, and G_2 contains not more than $2n_2 - 3$ edges. Therefore, G_1 has exactly $2n_1 - 3$ edges, or G_2 has exactly $2n_2 - 3$ edges, that is, one of these graphs is a maximum SP graph. Without loss of generality, suppose that just G_1 has $2n_1 - 3$ edges. If G_2 has fewer than $2n_2 - 4$ edges, then, according to (1), G has fewer than $2n - 3$ edges, and it is not a maximum SP graph. If G_2 has $2n_2 - 3$ edges, then, by Lemma 1, G is a non-simple graph. If G_2 has $2n_2 - 4$ edges, including an edge connecting its source and sink, then G is also a non-simple graph. Thus, G_2 has to be an almost maximum SP graph. Thus, all possible maximum SP graphs can be constructed in accordance with our recursive definition, and the proof is complete. ■

Theorem 5 *An almost maximum SP graph may be defined recursively as follows:*

(i) The series composition of two edges is a 3*-vertex almost maximum SP graph.*

(ii) Suppose G_1 and G_2 are an n_1-vertex and an n_2-vertex maximum SP graphs, respectively. In such a case, graph G which is constructed by the series composition of G_1 and G_2 is an $n_1 + n_2 - 1$-vertex almost maximum SP graph.

(iii) Suppose G_1 and G_2 are an n_1-vertex and an n_2-vertex almost maximum SP graphs, respectively. In such a case, graph G which is constructed by the parallel composition of G_1 and G_2 is an $n_1 + n_2 - 2$-vertex almost maximum SP graph.

Proof. An n-vertex graph G constructed by the series composition of an n_1-vertex and an n_2-vertex maximum SP graphs (they have $2n_1 - 3$ and $2n_2 - 3$ edges, respectively, by Proposition 3) has $2n - 4$ edges ($n = n_1 + n_2 - 1$). Indeed, the number of edges in G in this case is

$$2n_1 - 3 + 2n_2 - 3 = 2(n_1 + n_2) - 6 = 2(n+1) - 6 = 2n - 4. \tag{2}$$

An n-vertex graph G constructed by the parallel composition of an n_1-vertex and an n_2-vertex almost maximum SP graphs (they have $2n_1 - 4$ and $2n_2 - 4$ edges, respectively) has $2n - 4$ edges ($n = n_1 + n_2 - 2$) as well. Indeed, the number of edges in G in this case is

$$2n_1 - 4 + 2n_2 - 4 = 2(n_1 + n_2) - 8 = 2(n+2) - 8 = 2n - 4. \tag{3}$$

Hence, in both cases G is not a maximum SP graph only due to the absence of an edge connecting its source and sink. Thus, the graph constructed in accordance with our recursive definition is an almost maximum SP graph.

Now, we will show that any almost maximum SP graph can be constructed in accordance with our definition. It is clear that G_1 and G_2 are simple graphs (see the proof of Theorem 4). An n-vertex SP graph G that does not have an edge connecting its source and sink can be obtained in two ways. The first way is the series composition of an n_1-vertex SP graph G_1 and an n_2-vertex SP graph G_2. The second way is the parallel composition of an n_1-vertex SP graph G_1 and an n_2-vertex SP graph G_2, on the condition that neither G_1 nor G_2 has an edge connecting its source and sink. In the case of the first way, as shown in (2), the number of edges in G can reach $2n - 4$ only if G_1 has $2n_1 - 3$ edges and G_2 has $2n_2 - 3$ edges, that is, both G_1 and G_2 are maximum SP graphs. In the case of the second way, as follows from (3) the number of edges in G can reach $2n - 4$ only if G_1 has $2n_1 - 4$ edges and G_2 has $2n_2 - 4$ edges, that is, both G_1 and G_2 are almost maximum SP graphs. Thus, all possible almost maximum SP graphs can be constructed in accordance with our recursive definition, and the proof is complete. ■

Lemma 6 *It is possible to add an edge to any n-vertex non-maximum SP graph G so that the new graph $G^{'}$ will be an n-vertex SP graph.*

Proof. Suppose G is obtained by a series composition of two SP graphs. In such a case, G is not a maximum SP graph and G' can be constructed by the parallel composition of G and a single edge. For a parallel composition, we prove the correctness of the lemma by induction on n. A 2-vertex SP graph is a maximum SP graph. A 3-vertex non-maximum SP graph is obtained by a series composition of two single-edge SP graphs. A graph obtained by the parallel composition of such a graph and a single edge is a 3-vertex SP graph, hence the statement holds. In the general case, G is constructed by the parallel composition of an n_1-vertex SP graph G_1 and an n_2-vertex SP graph G_2. In such a case, G_1 or G_2, or both, are non-maximum SP graphs. Let G_1 be a non-maximum SP graph and the lemma be correct for it. In such a case, a new n_1-vertex SP graph G_1' can be constructed by adding a new edge to G_1. Therefore, we obtained the new n-vertex SP graph G' constructed by the parallel composition of the n_1-vertex SP graph G_1' and the n_2-vertex SP graph G_2. On the other hand, G' is obtained by adding a new edge to G. Hence, the proof is complete. ■

Theorem 7 *The following are equivalent.*
1. G is an n-vertex maximum SP graph.
2. G is an n-vertex maximal SP graph.

Proof. $(1 \Rightarrow 2)$ The proof follows directly from the definitions of maximum and maximal SP graphs.
$(2 \Rightarrow 1)$ It is impossible to add an edge to G so that the new graph will be an SP graph. Hence, by Lemma 6, G is an n-vertex maximum SP graph. ■

CONCLUSION

The paper investigates the structure of maximum series-parallel graphs. We have proposed a recursive description of these graphs and have shown that an n-vertex maximum series-parallel graph has exactly $2n-3$ edges. Besides, we have proven that maximum and maximal series-parallel graphs are exactly the same.

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