Discrete Comput Geom (2011) 45: 141–160
DOI 10.1007/s00454-010-9310-z

Small Drawings of Outerplanar Graphs, Series-Parallel Graphs, and Other Planar Graphs

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Received: 18 August 2009 / Revised: 25 September 2010 / Accepted: 11 October 2010 /
Published online: 6 November 2010

Abstract In this paper, we study small planar drawings of planar graphs. For arbitrary planar graphs, $\Theta(n^2)$ is the established upper and lower bound on the worst-case area. A long-standing open problem is to determine for what graphs a smaller area can be achieved. We show here that series-parallel graphs can be drawn in $O(n^{3/2})$ area, and outerplanar graphs can be drawn in $O(n \log n)$ area, but 2-outerplanar graphs and planar graphs of proper pathwidth 3 require $\Omega(n^2)$ area. Our drawings are visibility representations, which can be converted to polyline drawings of asymptotically the same area.

Keywords Graph drawing · Planar graphs · Visibility representations · Series-parallel graphs · Outerplanar graphs · Partial 3-trees · 2-outerplanar graphs

1 Introduction

A planar graph is a graph that can be drawn without crossing. Fáry, Stein, and Wagner [12, 26, 29] proved independently that every planar graph has a planar drawing such that all edges are drawn as straight-line segments. It was established 20 years ago [18, 25] that every n-vertex planar graph has a straight-line drawing with $O(n^2)$ area where vertices are placed at grid points. This is asymptotically optimal, since there are planar graphs that need $\Omega(n^2)$ area [17].

A number of other graph drawing models (e.g., polyline drawings, orthogonal drawings, visibility representations; see Sect. 2 for formal definitions) exist for planar

Research supported by NSERC. Part of the work was done while the author was on sabbatical leave at the University of Passau. Some of the results in this paper appeared in preliminary versions at *Graph Drawing '02* and *Graph Drawing '09*.

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graphs. In all these models, $O(n^2)$ area can be achieved for planar graphs; see for example [22, 30]. On the other hand, $\Omega(n^2)$ area is needed, in all models, for the graph in [17]. This raises the natural question [5] of whether $o(n^2)$ area is possible for subclasses of planar graphs.

1.1 Known Results

Every *tree* has a straight-line drawing in $O(n \log n)$ area and in $O(n)$ area if the maximum degree is asymptotically smaller than n. See [8] for references and many other upper and lower bounds regarding drawings of trees under special restrictions.

It is easy to create straight-line drawings of *outerplanar graphs* that have area $O(nd)$, where d is the diameter of the dual tree of the graph. In another result, any n points in general position can be used for a straight-line drawing of any outerplanar graph [6, 20]. Neither result yields $o(n^2)$ area for all outerplanar graphs. In a preliminary version of the present paper [3], we showed that any outerplanar graph has a visibility representation (and hence a polyline drawing) of area $O(n \log n)$. Since then, some work has been done on improving the bounds for straight-line drawings, first to $O(\Delta n^{1.48})$ [19], and then to $O(n^{1.48})$ [9] and $O(\Delta n \log n)$ [14].

Many drawing results are known for *series-parallel graphs*; see, e.g., [1, 7, 21, 28]. However, the emphasis here was on displaying the series-parallel structure of the graph, and/or on allowing additional constraints. All known algorithms bound the area by $O(n^2)$ or worse. Frati proved lower bounds on the area of polyline drawings of series-parallel graphs of $\Omega(n \log n)$ [15] and—very recently—$\Omega(n2^{\sqrt{\log n}})$ [16].

No graph drawing results specifically tailored to *k-outerplanar graphs* (for $k \geq 2$), or planar graphs with *small treewidth/pathwidth* appear to be known for two-dimensional drawings. The pathwidth of a graph is an asymptotic lower bound on the height of a planar drawing [11]. Any tree has a drawing where the height is proportional to the pathwidth [27], but no such results appear to be known for other graph classes.

While higher-dimensional drawings are not the focus of our paper, it is worth mentioning that all graph classes considered in this paper can be drawn with linear area in three dimensions because they have constant treewidth [10]; see also [13] for some earlier three-dimensional results for outerplanar graphs. Graphs of constant treewidth also have nonplanar two-dimensional orthogonal point-drawings in $O(n)$ area if the maximum degree is at most 4 [23].

1.2 Our Results

In this paper, we provide the following results.

- Every series-parallel graph has a visibility representation with $O(n^{3/2})$ area.
- Every series-parallel graph has a visibility representation with $O(\Delta n \log n)$ area.
- Every outerplanar graph has a visibility representation with $O(n \log n)$ area.
- For outerplanar graphs, we can maintain the "standard" planar embedding with the same area bound: Every outerplanar graph has an orthogonal box-drawing with area $O(n \log n)$ in which all vertices are drawn on the outer face.

As we briefly recall in Sect. 2, visibility representations and orthogonal drawings can be converted to polyline drawings with asymptotically the same area. Hence, all upper bounds (given in Sects. 3 and 4) also hold for polyline drawings.

We also provide the following lower bounds for polyline drawings.

- There are series-parallel graphs that require $\Omega(n^2)$ area in any polyline drawing that respects the planar embedding.
- There are outerplanar graphs that require $\Omega(n^2)$ area in any polyline drawing with the vertices on the bounding box.
- There are graphs of constant proper pathwidth and constant maximum degree that require $\Omega(n^2)$ area.
- There are 2-outerplanar graphs that require $\Omega(n^2)$ area.

2 Background

2.1 Graphs and Graph Classes

Let $G = (V, E)$ be a graph with $n = n(G) = |V|$ vertices and $m = m(G) = |E|$ edges. Throughout this paper we assume that G is *simple* (has no loops or multiple edges) and *planar* (can be drawn in the plane without crossing). A planar drawing of G splits the plane into connected pieces; the unbounded piece is called the *outer face* and all other pieces are called *interior faces*.

An *outerplanar graph* is a planar graph that can be drawn such that all vertices are on the outer face. A *maximal outerplanar graph* is an outerplanar graph to which we cannot add an edge without destroying simplicity or outerplanarity. Such a graph consists of an n-cycle with chords such that every interior face is a triangle. We will only study how to draw maximal outerplanar graphs because we can make any outerplanar graph maximal by adding edges, draw the resulting graph, and then delete added edges from the drawing.

A *2-terminal series-parallel graph with terminals s, t* is a graph defined recursively with one of the following three rules. (a) An edge (s, t) is a 2-terminal series-parallel graph. (b) If G_i, $i = 1, 2$, is a 2-terminal series-parallel graph with terminals s_i and t_i, then in a *series composition* we identify t_1 with s_2 to obtain a 2-terminal series-parallel graph with terminals s_1 and t_2. (c) If G_i, $i = 1, \dots, k$, is a 2-terminal series-parallel graph with terminals s_i and t_i, then in a *parallel composition* we identify $s_1, s_2, \dots, s_k$ into one terminal s and $t_1, t_2, \dots, t_k$ into one terminal t to obtain a 2-terminal series-parallel graph with terminals s and t. Here k is as large as possible, i.e., none of the graphs G_i is itself obtained via a parallel composition. The *fan-out* of a series-parallel graph is the maximum number of subgraphs k used in a parallel composition.

Given a 2-terminal series-parallel graph G, a *subgraph from the composition* is any of the subgraphs $G_1, \dots, G_k$ that was used to create G, or recursively any subgraph from the compositions of $G_1, \dots, G_k$. Since we never consider any other subgraphs, we will say "subgraphs" instead of "subgraphs from the composition."

A *series-parallel graph*, or *SP-graph* for short, is a graph for which every biconnected component is a 2-terminal series-parallel graph. It is *maximal* if no edge can

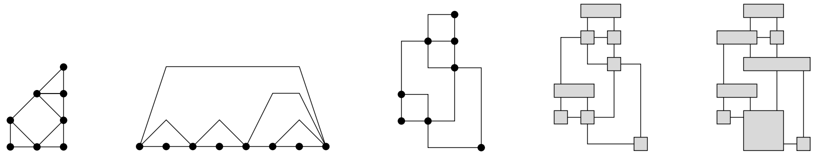


Fig. 1 The same graph in a straight-line drawing, a polyline drawing, an orthogonal point-drawing, an orthogonal box-drawing, and a visibility representation

be added while maintaining a simple SP-graph. Any maximal series-parallel graph is a 2-terminal series-parallel graph. Furthermore, in any parallel composition there exists an edge between the terminals, and in any series composition each subgraph is an edge or obtained from a parallel composition. As for outerplanar graphs, we will only study how to draw maximal series-parallel graphs.

The concept of *treewidth* has gained much attention in recent years, due to beautiful uses in the Graph Minor Theorem and the development of fixed-parameter tractable algorithms for some NP-hard graph problems. See Bodlaender's overview [4] for an exact definition of treewidth and applications of these graph classes. Graphs of treewidth 1 are exactly the forests, and graphs of treewidth 2 are exactly the series-parallel graphs. A graph has *pathwidth* k if it has a vertex order $v_1, \dots, v_n$ such that for $i > k$ at most k vertices in $\{v_1, \dots, v_i\}$ have a neighbor in $\{v_{i+1}, \dots, v_n\}$. A graph has *proper pathwidth* k if it has a vertex order $v_1, \dots, v_n$ such that for any edge (v_i, v_j) we have $|j - i| \leq k$. Graphs of proper pathwidth k are a subset of graphs of pathwidth k, which in turn are a subset of graphs of treewidth k.

A generalization of outerplanar graphs are k-outerplanar graphs defined as follows. Let G be a graph with a fixed planar embedding and outer face. G is 1*-outer-plane* if all vertices of G are on the outer-face. G is k*-outer-plane* if the graph that results from removing all vertices from the outer-face of G is $(k-1)$-outer-plane in the induced embedding and outer-face. A graph G is k*-outer-planar* if it is k-outer-plane for some choice of planar embedding and outer-face.

2.2 Drawing Models and Their Relationships

We consider the following drawing models (see also Fig. 1).

- *Straight-line drawings*: Vertices are points, edges are straight-line segments.
- *Polyline drawings*: Vertices are points, edges are sequences of contiguous straight-line segments. The transition point between two edge segments is called a *bend*.
- *Orthogonal point-drawings*: Vertices are points, edges are sequences of contiguous horizontal or vertical line segments. Such drawings exist only if the maximum degree is at most 4.
- *Orthogonal box-drawings*: Vertices are axis-aligned boxes (possibly degenerated to a line segment or a point), edges are sequences of contiguous horizontal or vertical line segments. In a *flat orthogonal box-drawing* all vertices are horizontal line segments.

  From now on, "orthogonal drawing" will always mean "orthogonal box-drawing." Also, "box" always refers to an axis-parallel box.

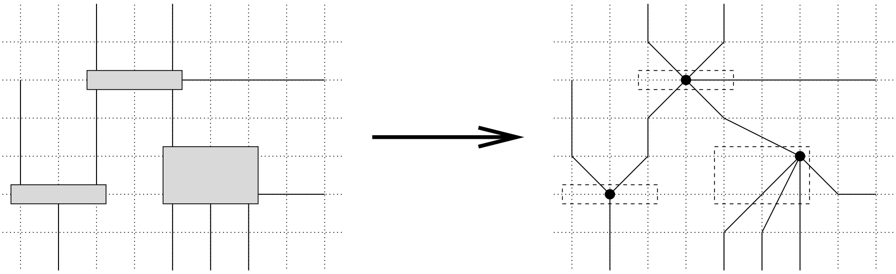


Fig. 2 Converting an orthogonal box-drawing to a polyline drawing. In this and other drawings, vertex boxes are slightly thickened to ease readability

- *Visibility representations*: Vertices are boxes, edges are horizontal or vertical line segments. In a *1-directional visibility representation* all edges are vertical line segments. In a *flat visibility representation* all vertices are horizontal line segments.

Whenever we speak of a drawing of a planar graph, we assume that the drawing has no crossing. We also assume that all defining features have integer coordinates; in particular, bends and vertices (or corners of boxes assigned to vertices) are at grid points.

The *width* of a box is the number of vertical grid lines (*columns*) that are overlapped by it. The *height* of a box is the number of horizontal grid lines (*rows*) that are overlapped by it. Observe that we count the number of overlapped grid lines for the width/height, rather than the more customary distance between the first and last grid line. This makes our bounds one unit larger than the bounds elsewhere, but eases summations of heights of drawings of subgraphs.

A drawing whose minimum enclosing box has width w and height h is called a $w \times h$-drawing; it has *area* $w \cdot h$ and *aspect ratio* $\max\{w/h, h/w\}$.

Note that by definition every visibility representation is an orthogonal box-drawing, and every straight-line drawing is a polyline drawing. Every 1-directional visibility representation can be assumed to be a flat visibility representation, since no vertex has incident horizontal edges. Other relationships between graph drawing models can be obtained by modifying the drawings, as we explain now.

2.2.1 *Orthogonal Box-Drawings to Polyline Drawings*

Any orthogonal box-drawing can be converted to a polyline drawing with asymptotically the same area as follows. Add empty grid lines until every segment of every edge has length at least 2; this will at most double the width and height. For any vertex v, replace the box of v by an arbitrary grid point inside the box, and reroute the incident edges of v locally as illustrated in Fig. 2.

2.2.2 *Flat Visibility Representations to 1-Directional Visibility Representations*

To convert a flat visibility representation to a 1-directional visibility representation, replace every grid line in it by two new grid lines; this will at most double the width and height. If $v_1, \ldots, v_k$ are the vertices in row r from left to right, then place

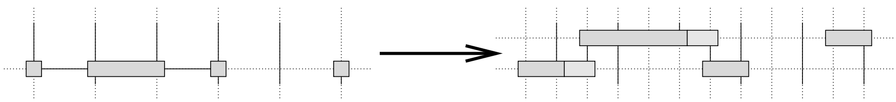


Fig. 3 Converting a flat visibility representation to a 1-directional visibility representation

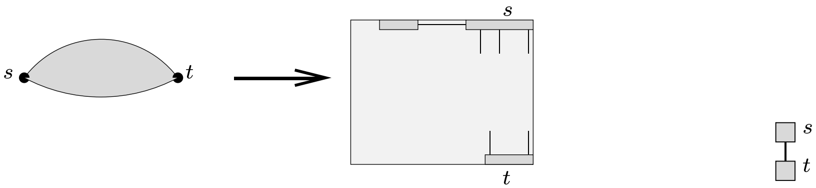
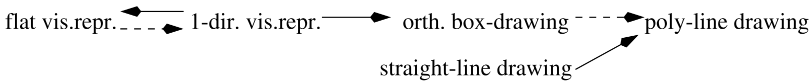


Fig. 4 Illustration of the invariant and the base case $m = 1$

$v_1, \dots, v_k$ alternatingly in the two rows r_1 and r_2 that replaced r. If a vertex intersected column c before, then let it now intersect both columns that replaced c.

Replace edges as illustrated in Fig. 3. If e was routed vertically in column c, then place it in the right of the two columns that replaced c. If e was routed horizontally, then it connected two vertices v_i and v_{i+1} that were consecutive in the same row r. Vertex v_i is now in (say) row r_1 and v_{i+1} in row r_2 that replaced r. Extend vertex v_i to the right until it overlaps the leftmost column of v_{i+1}, and route e vertically in this column.

We can summarizes all relationships between drawing models as follows:



Here, solid arrows mean that drawings in one model are automatically drawings in the other model, whereas dashed arrows mean that drawings in one model can be transformed to drawings in the other model with asymptotically the same area. It follows that flat visibility drawings and straight-line drawings are the most stringent model for upper bounds, while poly-line drawings are the most stringent model for lower bounds.

3 Visibility Representations of Series-Parallel Graphs

3.1 Invariant and Base Case

Let G be a maximal series-parallel graph (SP-graph). We now give a recursive algorithm to create a flat orthogonal drawing of G and all its subgraphs. To ease putting the drawings of such subgraphs together, we restrict the position of the terminals in the constructed drawings (see also Fig. 4).

- Vertex s contains the upper right corner of the bounding box.
- Vertex t contains the lower right corner of the bounding box.

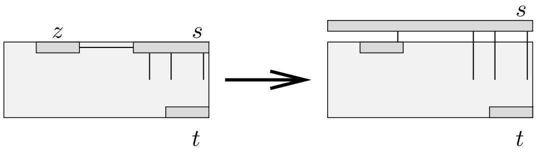


Fig. 5 Moving s so that it spans the top row

We also develop a recursive formula for the height: $h(m)$ is the maximum height of a drawing obtained with our algorithm over all maximal SP-graphs with m edges. (We have $m = 2n - 3$, but we use m to simplify the computations.)

In the base case ($m = 1$), simply place s atop t; see Fig. 4. The conditions are clearly satisfied, and $h(1) = 2$.

3.2 Modifying Drawings

If $m \geq 2$, then we obtain the drawing of G by merging the drawings of its subgraphs together suitably. Before doing this, we sometimes modify the drawings.

Lemma 3.1 *Let H be a maximal SP-graph with a flat orthogonal drawing $\Gamma(H)$ that has height $h \geq 2$ and satisfies the invariant. Then for any $h' \geq h$, there exists a flat orthogonal drawing $\Gamma'(H)$ of H of height h' that satisfies the invariant.*

Proof Simply insert $h' - h$ empty rows between the top row and the bottom row. □

A vertex *spans the top (bottom) row* of a drawing if it contains both the top (bottom) left point and the top (bottom) right point of the bounding box of the drawing.

Lemma 3.2 *Let H be a maximal SP-graph with terminals s and t, and let $\Gamma(H)$ be a flat orthogonal drawing of H that has height h and satisfies the invariant. Then there exists a flat orthogonal drawing $\Gamma'(H)$ of H that has height $h + 1$, satisfies the invariant, and in which vertex s spans the top row.*

Proof Add a new row above $\Gamma(H)$, and move s into this row, extending it all the way. Any edge that attached vertically at s is extended to the new row. There can be at most one edge that attached horizontally at s. If there is one, then let z be the other endpoint of its horizontal segment in the top row (z could be a vertex or a bend). Reroute the edge by continuing vertically upward from z towards the new position of s. See also Fig. 5. □

We can hence always achieve that terminal s spans the top row after adding a row; we call this *releasing terminal* s. Similarly, we can also release terminal t after adding a row.

3.3 The Inductive Step

If G has $m \geq 2$ edges, then we create a drawing of G by combining drawings of G's subgraphs. We distinguish cases depending on how G was obtained and some

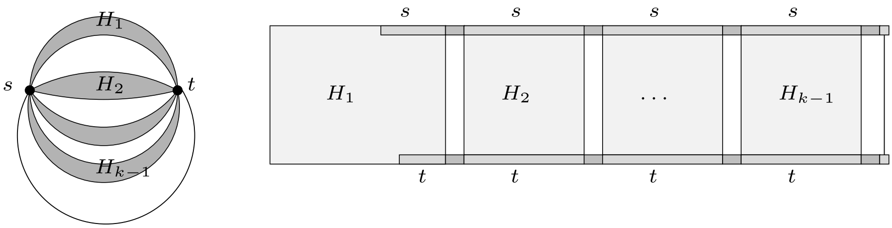


Fig. 6 Combining subgraphs in parallel

properties of its subgraphs. For each case α, we bound $h_\alpha(m)$, which is the maximum possible height that could be obtained in case α. The overall bound on the height is then $h(m) = \max_\alpha\{h_\alpha(m)\}$. We obtain recursive formulas for $h_\alpha(m)$ in this subsection, and derive upper hounds on $h(m)$ in the next subsection.

Parallel compositions (Case (P)): Assume that G is a parallel composition of subgraphs $H_1, \dots, H_k$, $k \geq 2$. After possible renaming, assume that $m_i = m(H_i)$ satisfies $m_1 \geq m_2 \geq \cdots \geq m_k$. Recursively obtain drawings of $H_1, \dots, H_k$; the drawing of H_i has height at most $h(m_i)$. Release both terminals in each of $H_2, \dots, H_k$. Then add rows so that all drawings have the same height (Lemma 3.1). Place H_1 leftmost, and $H_2, \dots, H_k$ to the right of it; this gives a flat orthogonal drawing of G that satisfies the invariant. See Fig. 6.

Since $m_1 \geq m_2 \geq m_3 \geq \cdots \geq m_k$, we have $m_i \leq \frac{m}{2}$ for $i \geq 2$. Therefore, the height of this drawing is

$$\begin{aligned} h_{(\mathrm{P})}(m) &\leq \max\bigl\{h(m_1), h(m_2) + 2, \dots, h(m_k) + 2\bigr\} \\ &\leq \max\left\{h(m-1), h\left(\frac{m}{2}\right) + 2\right\} \end{aligned} \tag{1}$$

Series compositions: Now assume that G (with terminals s, t) is a series composition of graphs H_a and H_b with terminals s, x and x, t, respectively. Since G is a maximal SP-graph, each of H_a and H_b is an edge or obtained from a parallel composition. We distinguish cases.

Case (S1): One subgraph, say H_b, is an edge. Draw H_a recursively, extend terminal s to the right, place t in the bottom row, and connect edge (x, t) horizontally. See Fig. 7. The case that H_a is an edge is symmetric. We have $h_{(\mathrm{S1})}(m) = h(m-1)$.

Case (S2): Both subgraphs have at least two edges. Assume that $m(H_b) \leq m(H_a)$; the other case is symmetric. Since H_b is not an edge, it is a parallel composition of subgraphs $H_1, \dots, H_k$, $k \geq 2$. After renaming, assume that $m(H_1) \geq \cdots \geq m(H_k)$. Note that H_k is the edge (x, t), which exists since G is a maximal SP-graph.

Let $L \geq 2$ be an integer; we will discuss later how to choose L. For all $i < \min\{L, k\}$, graph H_i is a series composition of two subgraphs H_i^a and H_i^b with terminals x, y_i and y_i, t, respectively. See also Fig. 8. Set $m_\alpha^\beta = m(H_\alpha^\beta)$ for any strings α and β.

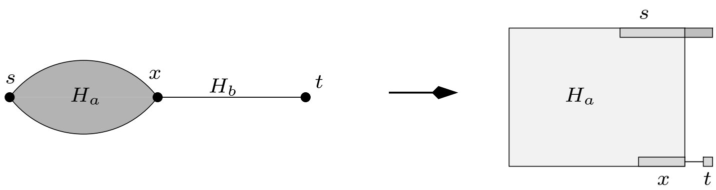


Fig. 7 A series composition if one subgraph is an edge

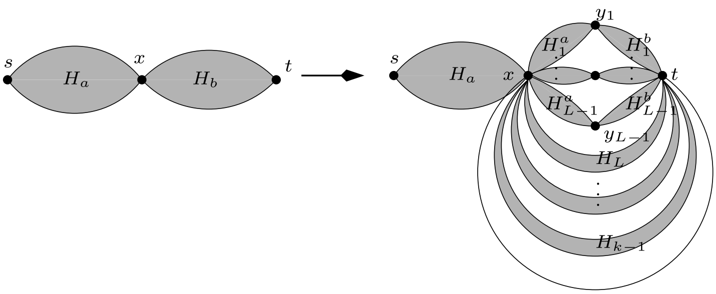


Fig. 8 Breaking down subgraph H_b

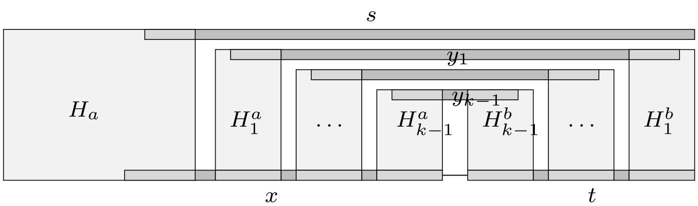


Fig. 9 Combining the subgraphs for a series composition. The case $k \le L$

Recursively draw each of the subgraphs H_a, H_i^a, H_i^b (for $i = 1, \dots, \min\{k, L\} - 1$) and H_i (for $i = L, \dots, k - 1$). Before we combine these drawings, we release terminals as follows.

- The drawing of H_a is unchanged and has height at most $h(m_a)$.
- For $i = 1, \dots, \min\{L, k\} - 1$, release terminal x in the drawing of H_i^a, and terminal t in the drawing of H_i^b. The drawings hence have height at most $h(m_i^a) + 1$ and $h(m_i^b) + 1$.
- For $i = L, \dots, k - 1$, release both terminals in the drawing of H_i.

To explain how to put these drawings together, we distinguish two subcases.

Case (S2a): Assume first that $k \le L$, and consider Fig. 9. Place H_a on the left, followed by $H_1^a, H_2^a, \dots, H_{k-1}^a$. All these graphs share terminal x and (after flipping drawings along a horizontal line, if necessary) x spans the bottom row for $H_1^a, H_2^a, \dots, H_{k-1}^a$. So this draws x as a horizontal segment.

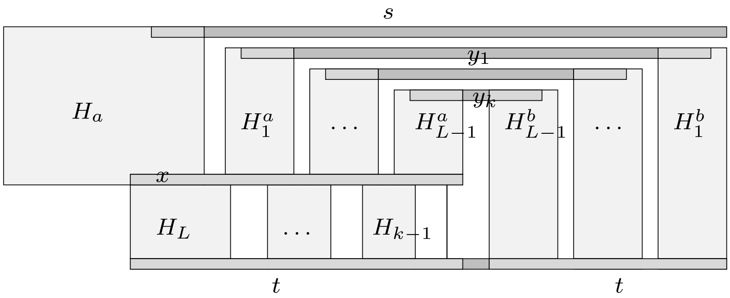


Fig. 10 Combining the subgraphs for a series composition. The case $k > L$

For $i = 1, \ldots, k-1$, flip the drawing of H_i^b so that terminal t spans the bottom row and terminal y_i occupies the top left corner. Add these drawings on the right in order $H_{k-1}^b, H_{k-2}^b, \ldots, H_1^b$; then t is in the bottom row and can be connected to x with a horizontal segment.

Increase the heights of these drawings (cf. Lemma 3.1) so that the two representations of y_i are in the same row, y_i is above the drawing of y_{i+1} (for $i < k-1$), and s is above all y_i's. Then all y_i's can be represented as line segments and s can be extended to the top right corner, so the invariant holds.

Let h_i be the height of the drawing of H_i^a and H_i^b together in the final drawing. Then $h_{k-1} \le \max\{h(m_{k-1}^a)+1, h(m_{k-1}^b)+1\} \le h(m_{k-1})+1$. For $i < k-1$, the height of H_i^a and H_i^b has been increased further to keep y_i above y_{i+1}, hence $h_i \le \max\{h(m_i)+1, h_{i+1}+1\}$. Therefore, the height of H_1^a and H_1^b satisfies $h_1 \le \max\{h(m_1)+1, h(m_2)+2, \ldots, h(m_{k-1})+k-1\}$. Vertex s is at least one unit higher, so the height of the drawing of G is bounded by

$$h_{(\mathrm{S2a})} \le \max\bigl\{h(m_a), h(m_1)+2, h(m_2)+3, \ldots, h(m_{k-1})+k\bigr\}$$

We bound this in two different ways. First, notice that for $i \ge 2$ we have $m_i \le m_1$ and $m_i \le m_b - m_1$, hence $m_i \le m_b/2 \le m_a/2$. We also know $m_1 \le m_b \le m_a$. By $k \le L$ we hence get

$$h_{(\mathrm{S2a})}(m) \le \max\bigl\{h(m_a)+2, h(m_a/2)+L\bigr\} \tag{2}$$

A simpler bound (which suffices in some cases) can be obtained by using $m_i \le m_b \le m/2$ for all $i \ge 1$:

$$h_{(\mathrm{S2a})}(m) \le \max\bigl\{h(m-1), h(m/2)+L\bigr\} \tag{3}$$

Case (S2b): Now assume $k > L$. Place $H_a, H_1^a, \ldots, H_{L-1}^a, H_{L-1}^b, \ldots, H_1^b$ exactly as in case (S2a). Graphs $H_L, \ldots, H_{k-1}$ are treated differently: Add rows until $H_L, \ldots, H_{k-1}$ all have the same height and place them *below* the segment of x, adding columns to x if it is not wide enough for the subgraphs. Increase the height of the drawings of $H_{L-1}^b, \ldots, H_1^b$ until the two occurrences of t match up, and draw edge (x,t) vertically. See Fig. 10.

The height for $H_a, H_1^a, \ldots, H_{L-1}^a, H_{L-1}^b, \ldots, H_1^b$ is no more than $h_{(\mathrm{S2a})}(m)$. Let h_d be the maximum height amount $H_L, \ldots, H_{k-1}$; since $m_L \geq \cdots \geq m_k$, we have $h_d \leq h(m_L) + 2$ (recall that both terminals were released for $H_L, \ldots, H_{k-1}$). To bound the height, we hence need to add $h_d - 1$ to inequality (2) and get

$$h_{(\mathrm{S2b})}(m) \leq \max\left\{h(m_a) + 2, h\left(\frac{m_a}{2}\right) + L\right\} + h(m_L) + 1 \tag{4}$$

This finishes the description of the recursive construction. Notice that it naturally implies a recursive algorithm to obtain the drawing. This algorithm has run time $O(m+n)$, since every edge is handled only once.

3.4 Analysis

Lemma 3.3 *For a suitable choice of L, we have $h(m) \leq 10\sqrt{m}$.*

Proof The proof is by induction on m, and the claim clearly holds for $m = 1$. In the inductive step, we bound $h(m) = \max\{h_{(\mathrm{P})}(m), h_{(\mathrm{S1})}(m), h_{(\mathrm{S2a})}(m), h_{(\mathrm{S2b})}(m)\}$ by proving that $h_\alpha(m) \leq 10\sqrt{m}$ for all cases α.

- In case (P), by inequality (1) and $m \geq 2$,

$$\begin{aligned} h_{(\mathrm{P})}(m) &\leq \max\left\{h(m-1), h\left(\frac{m}{2}\right) + 2\right\} \\ &\leq \max\left\{10\sqrt{m-1}, 10\sqrt{\frac{m}{2}} + 2\right\} \leq 10\sqrt{m}. \end{aligned}$$

- In case (S1), we have $h_{(\mathrm{S1})}(m) = h(m-1) \leq 10\sqrt{m-1} \leq 10\sqrt{m}$.
- In case (S2), we assumed $m_a \geq m_b$. Also, $m_b \geq 3$ (because H_1^a and H_1^b have each an edge, and (x,t) exists), and hence $m_a \geq 3$ and $m \geq 6$. We choose $L = \lceil 2.1\sqrt{m_a} + 1 \rceil$. (We would like to note that this L was chosen because the analysis works—many thanks to Jason Schattman for helping with MAPLE to find small constants. In practice, the best method would be to try all $L = 2, \ldots, k$ and choose the one that minimizes the height, although this would increase the running time of the algorithm.)
  – In case (S2a), by inequality (3),

$$\begin{aligned} h_{(\mathrm{S2a})}(m) &\leq \max\left\{h(m-1), h\left(\frac{m}{2}\right) + L\right\} \\ &\leq \max\left\{10\sqrt{m-1}, 10\sqrt{\frac{m}{2}} + 2.1\sqrt{m} + 2\right\} \quad \text{by definition of } L \\ &\leq \max\left\{10, \left(\frac{10}{\sqrt{2}} + 2.1 + \frac{2}{\sqrt{6}}\right)\right\}\sqrt{m} \quad \text{by } m \geq 6 \\ &\leq 10\sqrt{m} \end{aligned}$$

– In case (S2b), we have $k > L$, so $G_L \neq G_k$ contains at least two edges and $m_L \geq 2$. By inequality (4),

$$\begin{aligned} h_{(\mathrm{S2b})}(m) &\leq \max\bigl\{h(m_a), h(m_a/2) + L - 2\bigr\} + h(m_L) + 3 \\ &\leq \max\bigl\{10\sqrt{m_a}, 10\sqrt{m_a/2} + 2.1\sqrt{m_a}\bigr\} + 10\sqrt{m_L} + 3 \\ &\leq 10\sqrt{m_a} + 10\sqrt{m_L} + 3 \end{aligned}$$

Observe that

$$\begin{aligned} &\left(\sqrt{m_a} + \sqrt{m_L} + \frac{3}{10}\right)^2 \\ &\quad = m_a + m_L + \frac{9}{100} + 2\sqrt{m_a}\sqrt{m_L} + \frac{3}{5}\sqrt{m_a} + \frac{3}{5}\sqrt{m_L} \\ &\quad \leq m_a + m_L + \frac{0.09}{\sqrt{3}\cdot 2}\sqrt{m_a}m_L + \frac{2}{\sqrt{2}}\sqrt{m_a}m_L + \frac{0.6}{2}\sqrt{m_a}m_L \\ &\qquad + \frac{0.6}{\sqrt{3}\sqrt{2}}\sqrt{m_a}m_L \quad \text{by } m_a \geq 3 \text{ and } m_L \geq 2 \\ &\quad \leq m_a + m_L + (L-1)m_L \quad \text{by } L \geq 2.1\sqrt{m_a} + 1 \\ &\quad \leq m_a + m_L + m_1 + m_2 + \cdots + m_{L-1} \quad \text{by } m_i \geq m_L \text{ for } i < L \\ &\quad \leq m \end{aligned}$$

Hence $h_{(\mathrm{S2b})}(m) \leq 10(\sqrt{m_a} + \sqrt{m_L} + \frac{3}{10}) \leq 10\sqrt{m}$.

So $h_\alpha(m) \leq 10\sqrt{m}$ for all cases α, which proves that $h(m) \leq 10\sqrt{m}$ as desired. □

Theorem 1 *Any series-parallel graph has a flat visibility representation with area $O(n^{3/2})$.*

Proof Convert the graph into a maximal series-parallel graph (by adding edges if needed) and apply the above algorithm. By the previous lemma, the height is $O(\sqrt{m}) = O(\sqrt{n})$ since $m = 2n - 3$. To analyze the width, notice that we use at most one column for each edge. No additional columns are needed for vertices, since every vertex obtains an incident vertical edge in the base case. Hence, the width is at most $m = 2n - 3$, and the total area is $O(n^{3/2})$. No bend was ever created, so the flat orthogonal drawing is in fact a flat visibility representation. □

We get better bounds if case (S2b) does not occur, i.e., if the series-parallel graph has small fan-out.

Lemma 3.4 *If the fan-out is at most f, then with a suitable choice of L, we have $h(m) \leq f \log m + 2$.*

Proof We proceed by induction on the number of edges. In the base case $h(1) = 2 \le 2 + f \log m$. In case of a parallel composition, by inequality (1),

$$\begin{aligned} h_{(\mathrm{P})}(m) &\le \max\left\{h(m-1), h\left(\frac{m}{2}\right) + 2\right\} \\ &\le \max\left\{2 + f\log(m-1), 2 + f\log\left(\frac{m}{2}\right) + 2\right\} \le 2 + f\log m \end{aligned}$$

since $f \ge 2$. In case (S1), the height is $h_{(\mathrm{S1})}(m) = h(m-1) \le 2 + f\log(m-1) \le 2 + f\log m$. In case (S2), we choose $L = f$; hence $k \le f \le L$ and we are in case (S2a). Here, by inequality (3) the height is

$$\begin{aligned} h_{(\mathrm{S2a})}(m) &\le \max\left\{h(m-1), h\left(\frac{m}{2}\right) + f\right\} \\ &\le \max\left\{2 + f\log(m-1), 2 + f\log\left(\frac{m}{2}\right) + f\right\} \le 2 + f\log m. \end{aligned}$$

So $h(m) \le \max\{h_{(\mathrm{P})}(m), h_{(\mathrm{S1})}(m), h_{(\mathrm{S2a})}(m)\} \le 2 + f\log m$. □

Theorem 2 *Any series-parallel graph with fan-out f has a visibility representation of area $O(fn\log n)$.*

Proof If the graph is maximal, then by the previous lemma the height is $O(f\log m)$ and the claim is proved as in Theorem 1. If the graph is not maximal, then it can be made into a maximal SP-graph by adding edges; this adds at most one to the fan-out f and hence the drawing of the super-graph has area $O(fn\log n)$. □

Corollary 1 *Any series-parallel graph with maximum degree Δ has a visibility representation of area $O(\Delta n\log n)$.*

Our theorems only hold for simple series-parallel graphs. However, the same asymptotic area bounds can be shown for multi-graphs as follows. (1) Delete all but one copy of a multiple edge. (2) Create a flat visibility representation with our algorithm. (3) Convert it into a 1-directional visibility representation of asymptotically the same width and height. (4) Add back the multiple edges by inserting grid lines next to the one copy that was kept. Since we had a 1-directional visibility representation, this adds only width but not height. So the height is $O(\sqrt{m})$ (or $O(f\log m)$ or $O(\Delta\log m)$) as before, the width is again $O(m)$, and we obtain asymptotically the same area bounds.

4 Drawing Outerplanar Graphs

Every outerplanar graph is a series-parallel graph and hence has a visibility representation with $O(fn\log n)$ area. We now show that outerplanar graphs have fan-out $f \le 2$. Next we show that with a modified construction we can obtain orthogonal box-drawings of area $O(n\log n)$ that have all vertices of the graph drawn on the outer face.

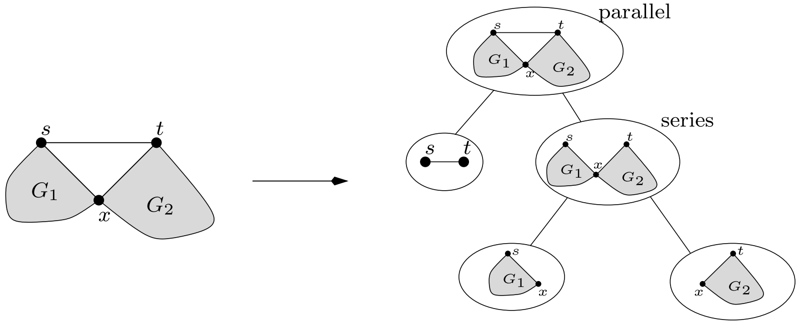


Fig. 11 A series-parallel composition for an outerplanar graph

4.1 Visibility Representation

We first prove the claim on the fan-out of an outerplanar graph by reproving the well-known fact that any outerplanar graph is series-parallel.

Lemma 4.1 *Let G be a maximal outerplanar graph embedded so that all vertices are on the outer face, and let (s,t) be an edge of G on the outer face. Then G is a series-parallel graph with terminals s,t with fan-out at most 2. Moreover, for every parallel composition, one subgraph is an edge.*

Proof We proceed by induction on the number of vertices; the claim is trivial for $n=2$ since G is an edge. For $n \geq 3$ consider the interior face F incident to edge (s,t). Since G is maximal outerplanar, F is a triangle; let x be the third vertex on the triangle. Let G_s be the subgraph induced by all vertices between s and x on the outer face of G and let G_t be the subgraph induced by all vertices between x and t on the outer face of G. See also Fig. 11.

G_s is outerplanar with edge (s,x) on the outer face and G_t is outerplanar with edge (t,x) on the outer face. So by induction they have a series-parallel composition with terminals s,x and x,t, respectively. Combine these two subgraphs in series and then combine the result in parallel with an edge (s,t). This gives a series-parallel composition for G that satisfies all conditions. □

Theorem 2 hence implies the following.

Corollary 2 *Every outerplanar graph has a flat visibility representation of area $O(n \log n)$.*

4.2 Orthogonal Box-Drawings

The drawing algorithm of Sect. 3 does not keep all vertices on the outer face for an outerplanar graph: in case (S2a) (see Fig. 12) edge (x,t) cuts off y_1 from the outer face. Also, the clockwise order of edges is not correct: At vertex x, we encounter (in clockwise order) the edges in H_a, then the edges in H_1^a ending with (x,y_1), then

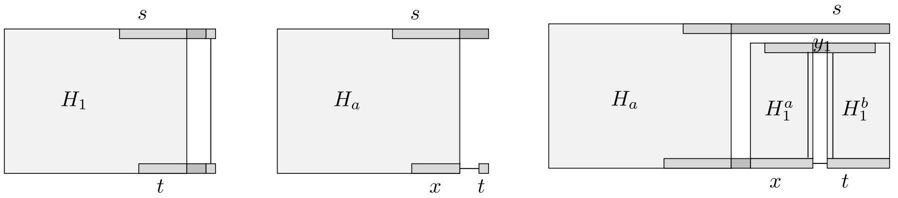


Fig. 12 The drawing algorithm of Sect. 3 applied to maximal outerplanar graph. Cases (P), (S1), and (S2a)

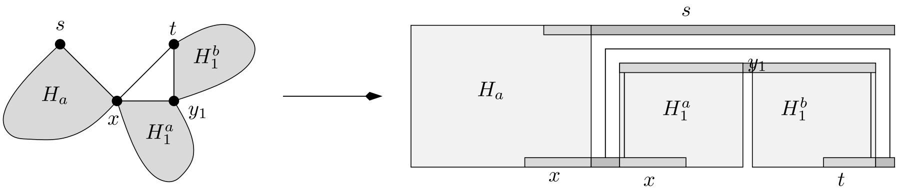


Fig. 13 Rerouting edge (x,t) in case (S2a) to keep all vertices on the outer face

(x,t). But the correct order of edges incident to x is "edges in H_a, then (x,t), then the edges in H_1^a beginning with edge (x,y_1)." (See also the left picture of Fig. 13.)

We can solve both problems by changing the algorithm in case (S2a). Release terminal y_1 (*not* x or t as done previously) in the drawings of H_1^a and H_1^b. Flip the drawing of H_1^a so that y_1 spans the top row and x covers the bottom left corner, i.e., so that edge (x,y_1) is in the leftmost column of H_1^a. Similarly, flip the drawing of H_1^b so that y_1 spans the top row and t covers the bottom right corner, i.e., edge (y_1,t) is in the rightmost column of H_1^b. Now place the drawings from left to right in order H_a, H_1^a, H_1^b. Route the edge (x,t) with two bends "around" H_1^a and H_1^b; see Fig. 13.

By induction one shows easily that all vertices are on the outer face. We use one additional row to route the edge (x,t), so the height formula in case (S2a) becomes

$$h_{(\mathrm{S2a})}(m) \le \max\{h(m-1), h(m/2)+3\}.$$

An inductive argument similar to that in the proof of Lemma 3.4 shows that the height of this construction is at most $3\log m + 2 \in O(\log m)$. The width increases (every edge with two bends uses two additional columns), but still remains $O(m)$.

Theorem 3 *Let G be an outerplanar graph. Then G has a flat orthogonal box-drawing of area $O(n\log n)$ with all vertices on the outer face. Every edge has at most two bends.*

5 Lower Bounds

Theorem 4 *The following lower bounds exist for polyline drawings.*

1. *There exists a series-parallel graph that requires $\Omega(n^2)$ area in any polyline drawing that respects the planar embedding.*

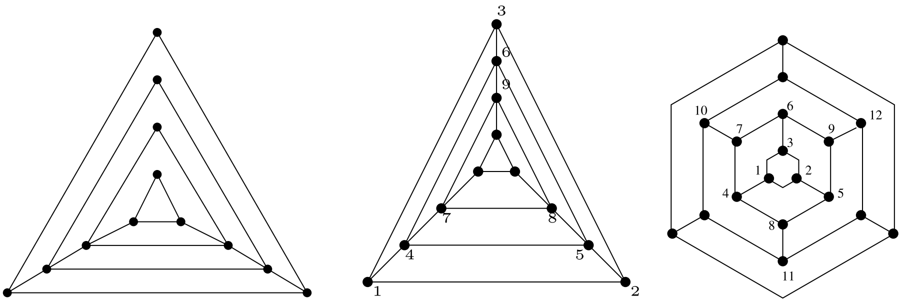


Fig. 14 Two graphs with $n/3$ stacked cycles, and a graph with $n/6 + 1$ stacked cycles

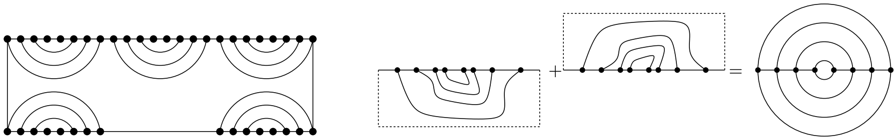


Fig. 15 A graph that requires $\Omega(n^2)$ area if vertices are drawn on the bounding box

2. *There exists an outerplanar graph that requires* $\Omega(n^2)$ *area in any polyline drawing with all vertices on the bounding box.*
3. *There exists a* 3*-connected graph of proper pathwidth* 3 *with maximum degree* 4 *that requires* $\Omega(n^2)$ *area in any polyline drawing.*
4. *There exists a* 3*-connected graph of proper pathwidth* 4 *with maximum degree* 3 *that requires* $\Omega(n^2)$ *area in any polyline drawing.*

Proof These proofs rely on the "stacked cycle argument" used previously (see, e.g., [2, 17, 24]): Assume we have a planar graph G with a fixed planar embedding and outer face. We say that disjoint cycles $C_1, \dots, C_k$ of G are *stacked cycles* if C_i is outside the region defined by C_{i-1} for all $i > 1$. If G has k stacked cycles, then it requires a $2k \times 2k$-grid in any planar polyline drawing that reflects the planar embedding and outer face. If we are allowed to choose the outer face of G (but not the planar embedding, or if the planar embedding is fixed since the graph is 3-connected), then there are at least $k/2$ stacked cycles, and the lower bound becomes $\Omega(k^2)$ area.

We now show how to construct graphs that fit into the specified graph classes and that have $\Theta(n)$ stacked cycles.

1. Observe that the left graph in Fig. 14 is series-parallel. Since it has $n/3$ stacked cycles, it requires a $\frac{2}{3}n \times \frac{2}{3}n$-grid in any drawing that respects the planar embedding and outer face.
2. Let G be the outerplanar graph in Fig. 15, which exists for n a multiple of 10. G consists of five groups of $n/5$ vertices with a matching within each group. Assume we have a drawing of G with all vertices on the bounding box $\mathcal{B}$. Of the five groups, at least one must be entirely on one side of $\mathcal{B}$, say the top. Duplicate the drawing and flip it upside down. The resulting drawing has asymptotically

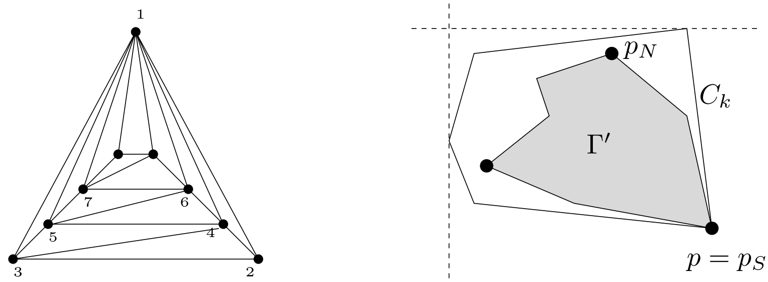


Fig. 16 A 2-outerplanar graph with $(n-1)/2$ 1-fused stacked cycles, and an illustration of adding a 1-fused cycle around a drawing

the same area and contains a multi-graph with $n/10$ stacked cycles. So its area is $\Omega(n^2)$.

3. Let G be the middle graph in Fig. 14. G has maximum degree 4 and (as indicated by the numbering) proper pathwidth 3. Since G is 3-connected, its planar embedding is unique. G has $n/3$ stacked cycles, so it requires a $\frac{2}{3}n \times \frac{2}{3}n$-grid in any drawing that respects the outer face.
4. Let G be the right graph in Fig. 14. G has maximum degree 3 and (as indicated by the numbering) proper pathwidth 4. Since G is 3-connected, its planar embedding is unique. G has $n/6+1$ stacked cycles, so it requires a $(\frac{n}{3}+2) \times (\frac{n}{3}+2)$-grid in any drawing that respects the outer face. □

Theorem 5 *There exists a triangulated* 2*-outerplanar graph of pathwidth* 3 *that requires* $\Omega(n^2)$ *area in any polyline drawing.*

Proof The proof relies on a modification of the stacked cycle argument where consecutive cycles are "fused" at one vertex. Let G be the graph shown in Fig. 16. G is 2-outerplanar and has pathwidth 3. Since G is triangulated, its planar embedding is unique. Observe that G consists of $k := (n-1)/2$ cycles $C_1, \dots, C_k$ that are edge-disjoint, the edges of C_i are outside the region defined by C_{i-1} for $i = 2, \dots, k$, and any two cycles have at most one vertex in common. We call $C_1, \dots, C_k$ *1-fused stacked cycles*.

By induction on k we can show that any graph with k 1-fused stacked cycles $C_1, \dots, C_k$ requires width and height at least $k+1$ in any polyline drawing that respects the planar embedding and outer face. Clearly, cycle C_1 needs width and height 2. For $k > 1$, let G' be the subgraph formed by the 1-fused stacked cycles $C_1, \dots, C_{k-1}$. Consider an arbitrary polyline drawing Γ of G, and let Γ' be the induced drawing of G', which has width and height at least k by induction. As illustrated in Fig. 16, the drawing of C_k in Γ must stay outside Γ', except at the point p where C_k and C_{k-1} have a vertex in common (if any). Let p_N and p_S be points at a vertex or bend in the topmost and bottommost row of Γ'; by $k \geq 2$ they are distinct. So $p \neq p_N$ or $p \neq p_S$; assume the former. To go around p_N, the drawing of C_k in Γ must reach a point strictly higher than p_N, and hence uses at least one more row above Γ'. Similarly, one shows that Γ has at least one more column than Γ'.

Therefore, any graph with k 1-fused stacked cycles requires width and height $k+1$, and our specific graph G requires width and height $(n+1)/2$ in any polyline drawing that respects the outer face. For any other choice of outer face, graph G has $\Theta(n)$ 1-fused stacked cycles and hence requires $\Omega(n^2)$ area. □

We would like to offer a few comments.

- Our lower bound for series-parallel graphs (Theorem 4(1)) crucially requires that the planar embedding must be respected. In a different embedding this graph can easily be drawn in $O(n)$ area. Frati [15] showed that another series-parallel graph (consisting of $K_{2,n}$ and a complete ternary tree) needs $\Omega(n \log n)$ in any polyline drawing. (Very recently, he improved the lower bound to $\Omega(n2^{\sqrt{\log n}})$ [16].) Closing the gap between this lower bound and our $O(n^{3/2})$ upper bound remains open.
- One possible criticism of our drawings is that the aspect ratio is very large: The width is $\Theta(m)$ while the height is $O(\sqrt{m})$ or $O(f \log m)$. However, one can argue that a smaller aspect ratio is possible only at the expense of more area: Assume we have a polyline drawing of $K_{2,n}$ with width W and height H. Frati [15] showed that $\max\{W, H\} \in \Omega(n)$. Assume $W \geq H$ and the aspect ratio is at least k. Then $W \in \Omega(n)$ and $W/H \leq k$, so $H \in \Omega(n/k)$ and the area is $\Omega(n^2/k)$.
- Our lower bound for outerplanar graphs (Theorem 4(2)) only holds if vertices are forced to be on the bounding box. It can be extended to other models of "being near the boundary," such as "having a horizontal or vertical segment that reaches to the boundary of the enclosing rectangle," or even "having an escape hatch" (see [23]).

  It remains open whether some outerplanar graphs require $\Omega(n \log n)$ area in any polyline drawing. An $\Omega(n \log n)$ area lower bound is trivial for visibility representations, even for trees: Combine $K_{1,\Theta(n)}$ with a complete ternary tree on $\Theta(n)$ nodes: This requires width and height $\Omega(\log n)$ for the ternary tree, and width or height $\Omega(n)$ to accommodate the box of the node of degree $\Theta(n)$.
- One open question is to ask for other subclasses of planar graphs that have drawings with $o(n^2)$ area. Two natural subclasses to study would be planar graphs of treewidth 3 (which generalize series-parallel graphs) and 2-outerplanar graphs (which generalize outerplanar graphs). Our lower bounds in Theorems 4(3, 4), and 5 show that these graph classes require $\Omega(n^2)$ area in general, so this avenue of extending our results is closed.

6 Conclusion

In this paper, we studied planar polyline drawings of some subclasses of planar graphs. Using a recursive algorithm, we achieved $O(n^{3/2})$ area for series-parallel graphs, and $O(fn \log n)$ area for series-parallel graphs with fan-out f. In particular, this implies $O(\Delta n \log n)$ area for series-parallel graphs with maximum degree Δ, and $O(n \log n)$ area for outerplanar graphs. We created flat visibility representations, from which polyline drawings of the same asymptotic area are easily obtained. For outerplanar graphs, a variant achieved orthogonal box-drawings of area $O(n \log n)$ with all vertices on the outer face.

We also gave some lower bounds, and showed that $\Omega(n^2)$ area is sometimes required for graphs of proper pathwidth 3 and for 2-outerplanar graphs. Many open problems remain.

- What subclasses of planar graphs have *straight-line drawings* of area $o(n^2)$? Can we achieve $o(n^2)$ straight-line drawings of series-parallel graphs, at least under some conditions on other graph parameters? Recall that much progress has been made for outerplanar graphs [9, 14, 19], although whether $O(n \log n)$ area straight-line drawings exist still remains open for outerplanar graphs.
- For outerplanar graphs, can we achieve visibility representations of area $O(n \log n)$ that keep all vertices on the outer face? Or even better, that respect any given planar embedding?
- The emphasis in this paper was on achieving area $o(n^2)$, not on obtaining a particularly small constant. Following the steps of the proof, one can show that the area is $20\sqrt{2}n^{3/2} + o(n^{3/2})$ for flat visibility drawings of series-parallel graphs. How much can this be improved with a more careful analysis, and/or by considering larger subgraphs for base cases? What is the best constant for polyline drawings of series-parallel graphs?

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