On Maximizing the Euclidian Distance between Adjacent Vertices in Planar Drawings of Small Area

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Universität Tübingen, Germany

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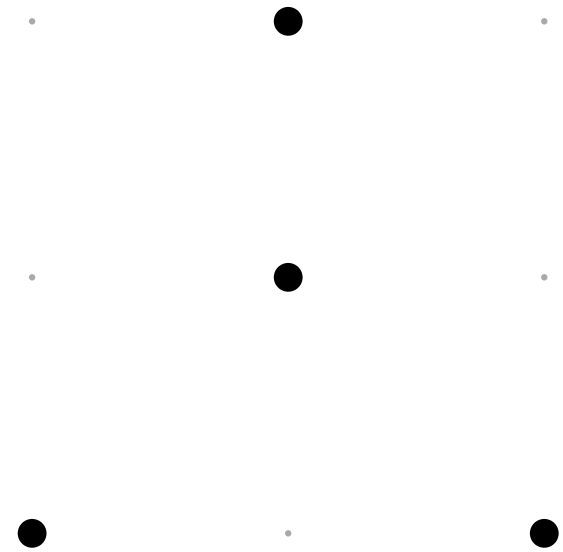

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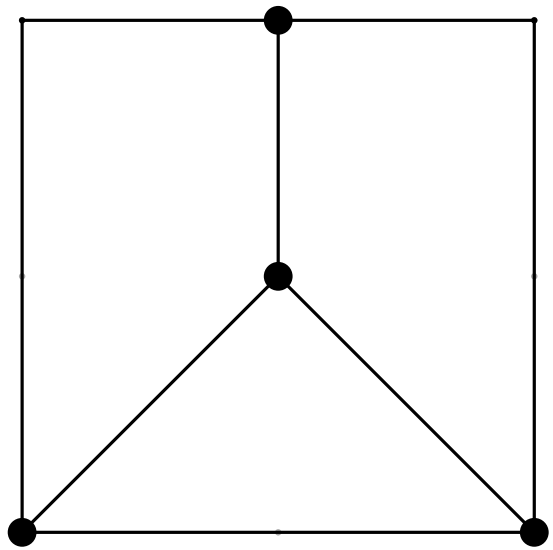

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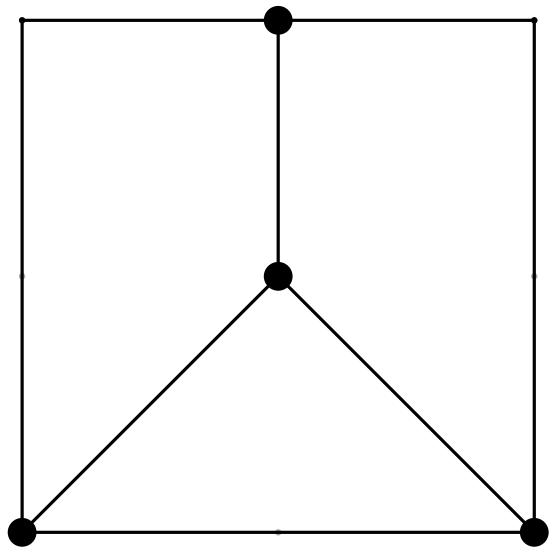

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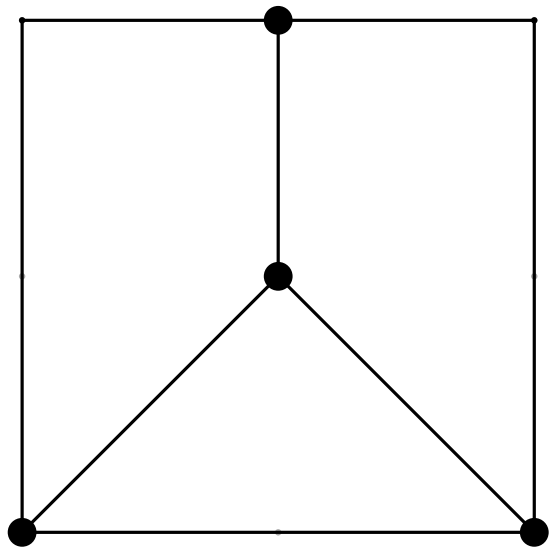

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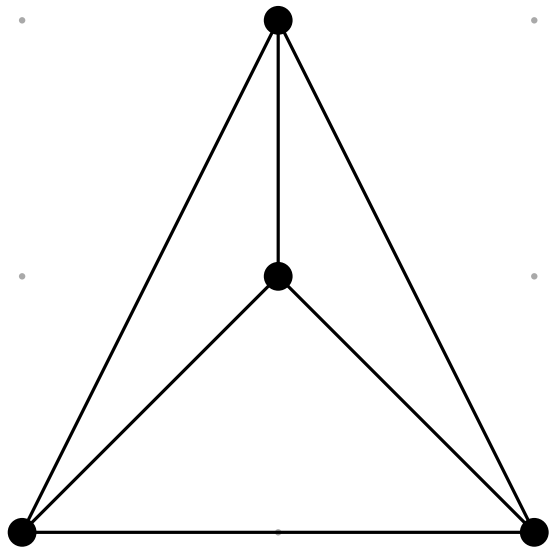

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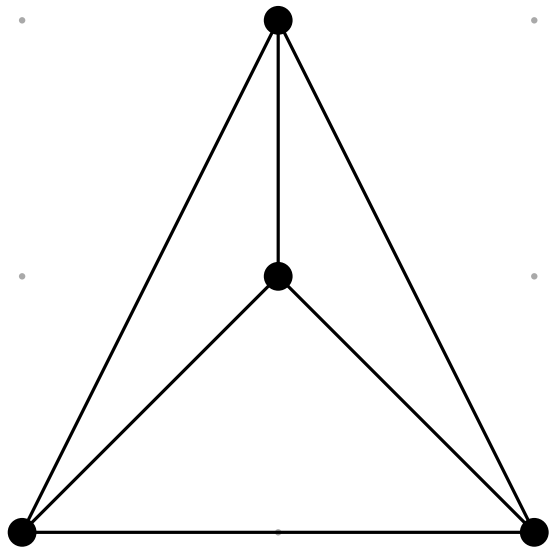

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- This work addresses producing drawing algorithms for specific graph classes which will approach the geometrical problems with results described asymptotically

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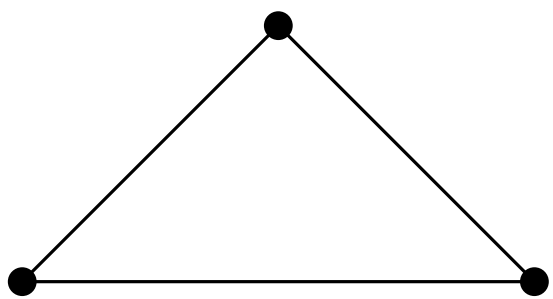

$r_{2021} \neq 1$

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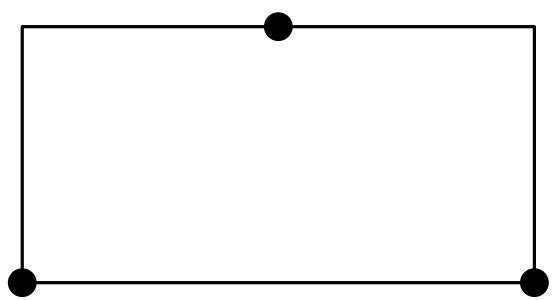

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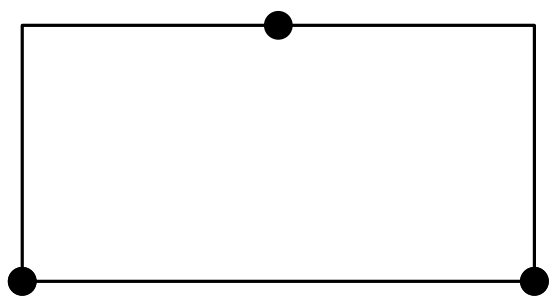

$$r = \sqrt{2}$$

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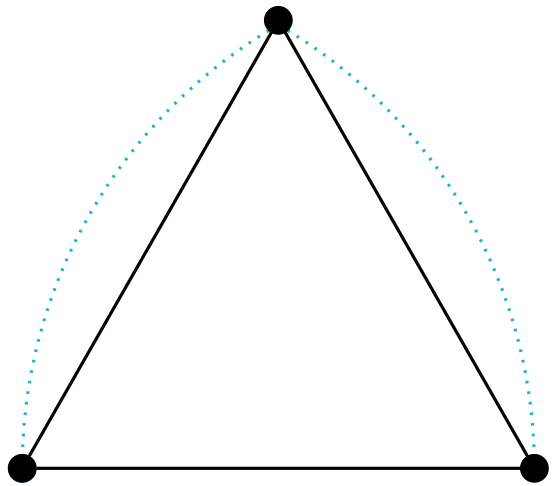
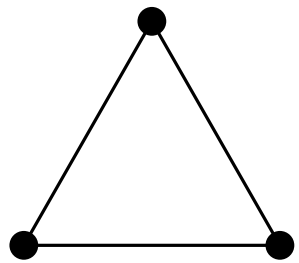


$r = 1 + \varepsilon$, ε depends on grid size

Adjacent and Nested Triangles

- $r = 1 + \varepsilon$



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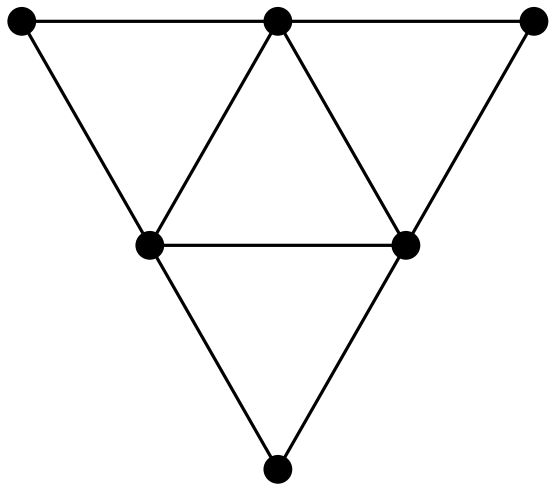

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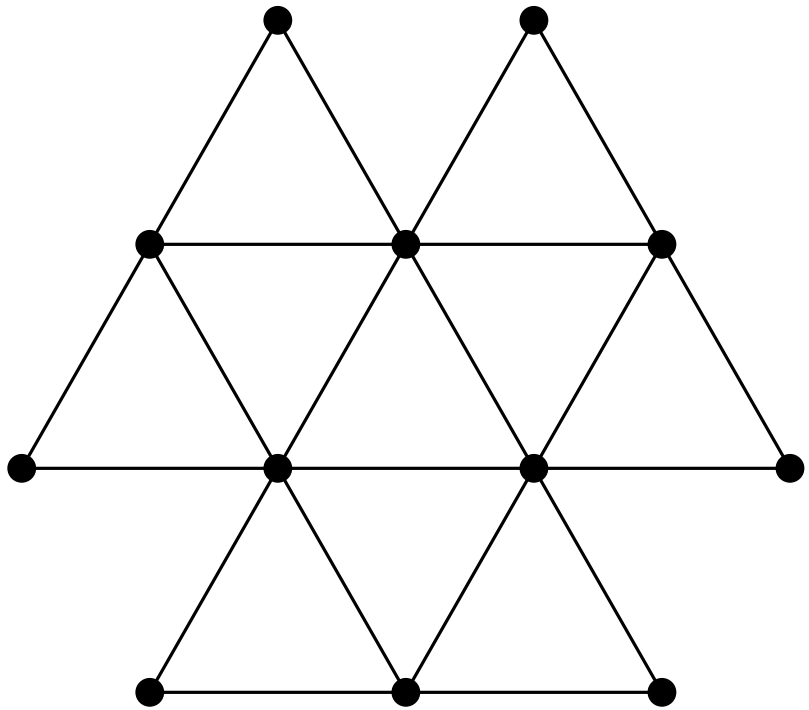

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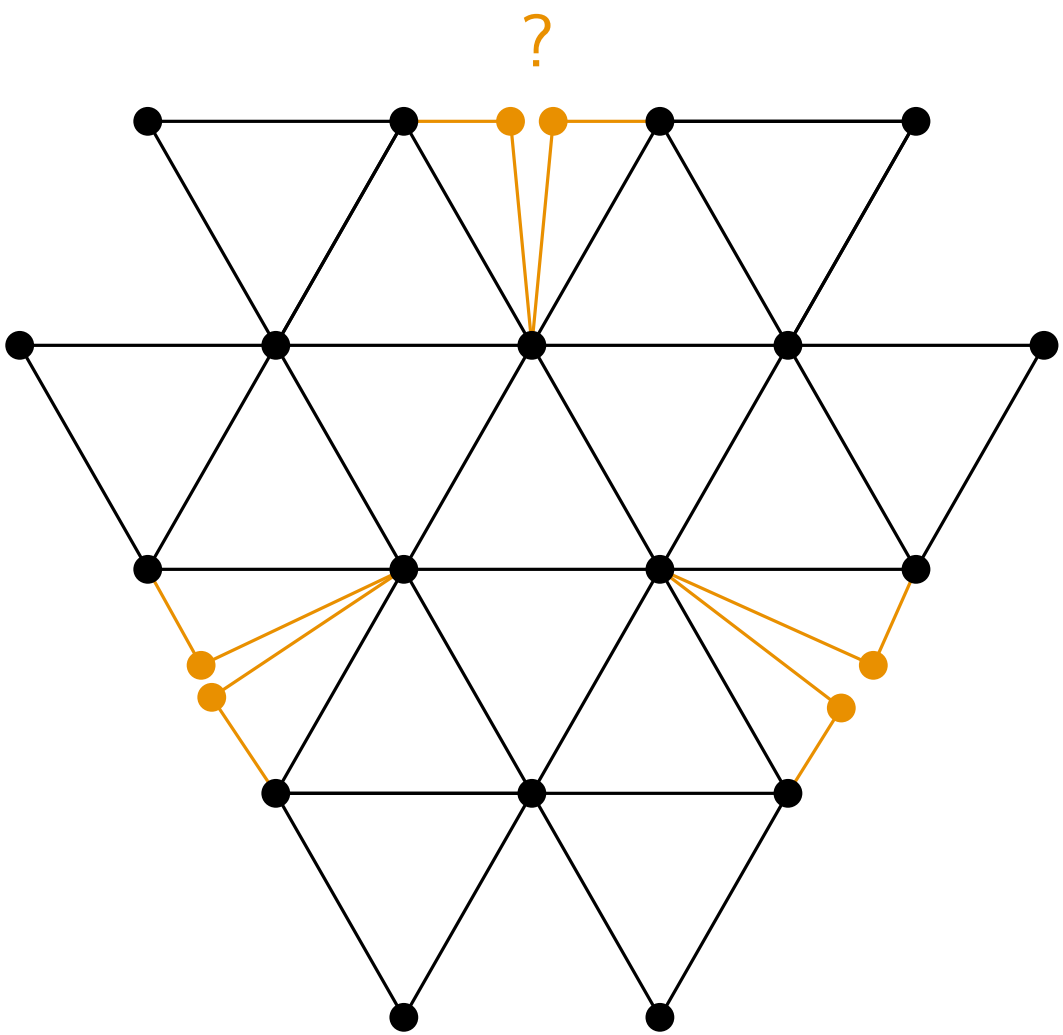

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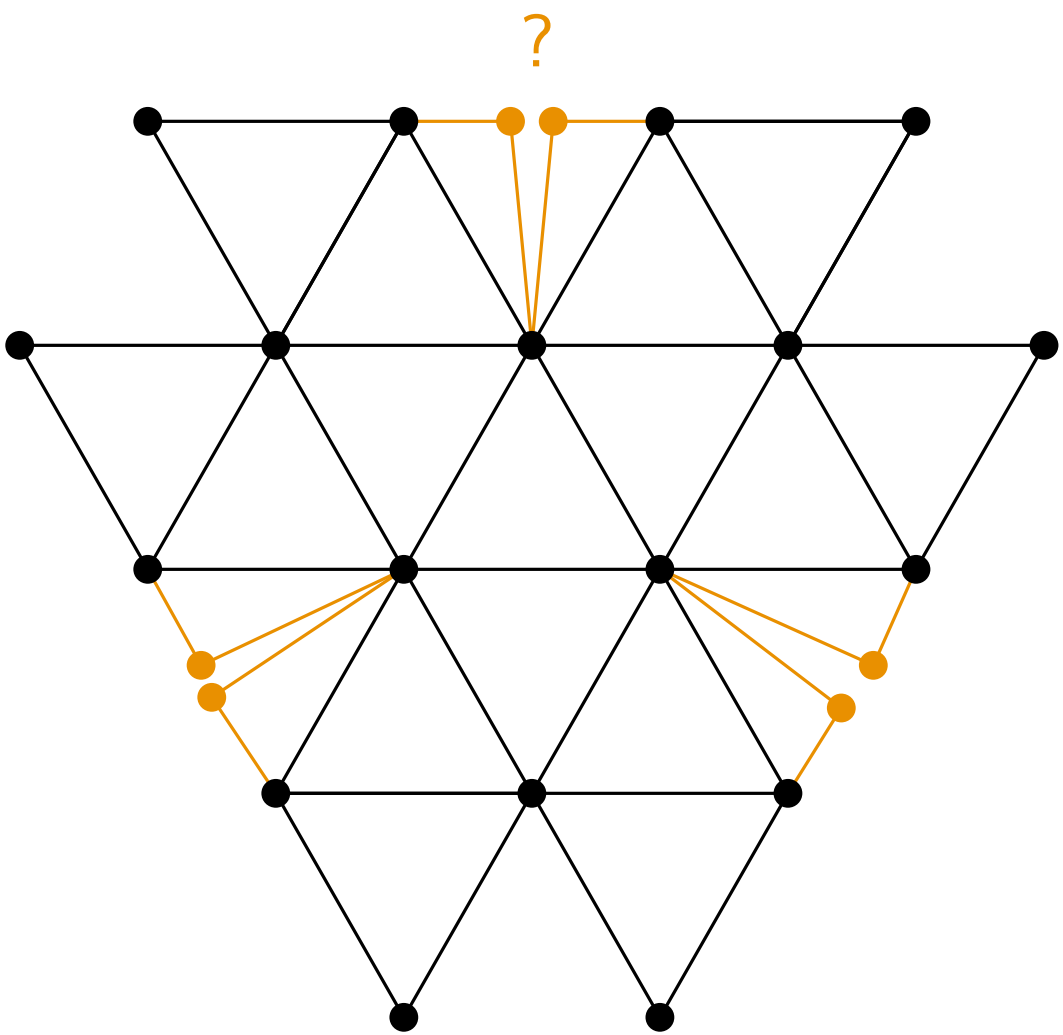

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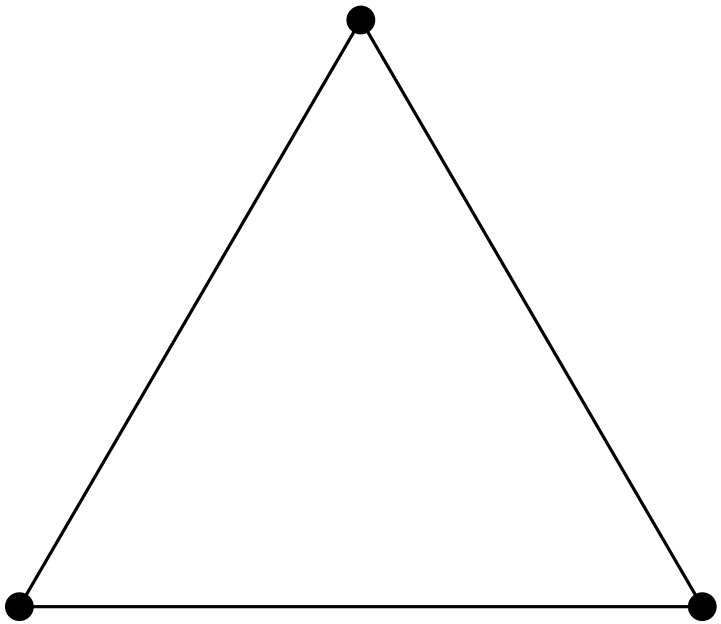

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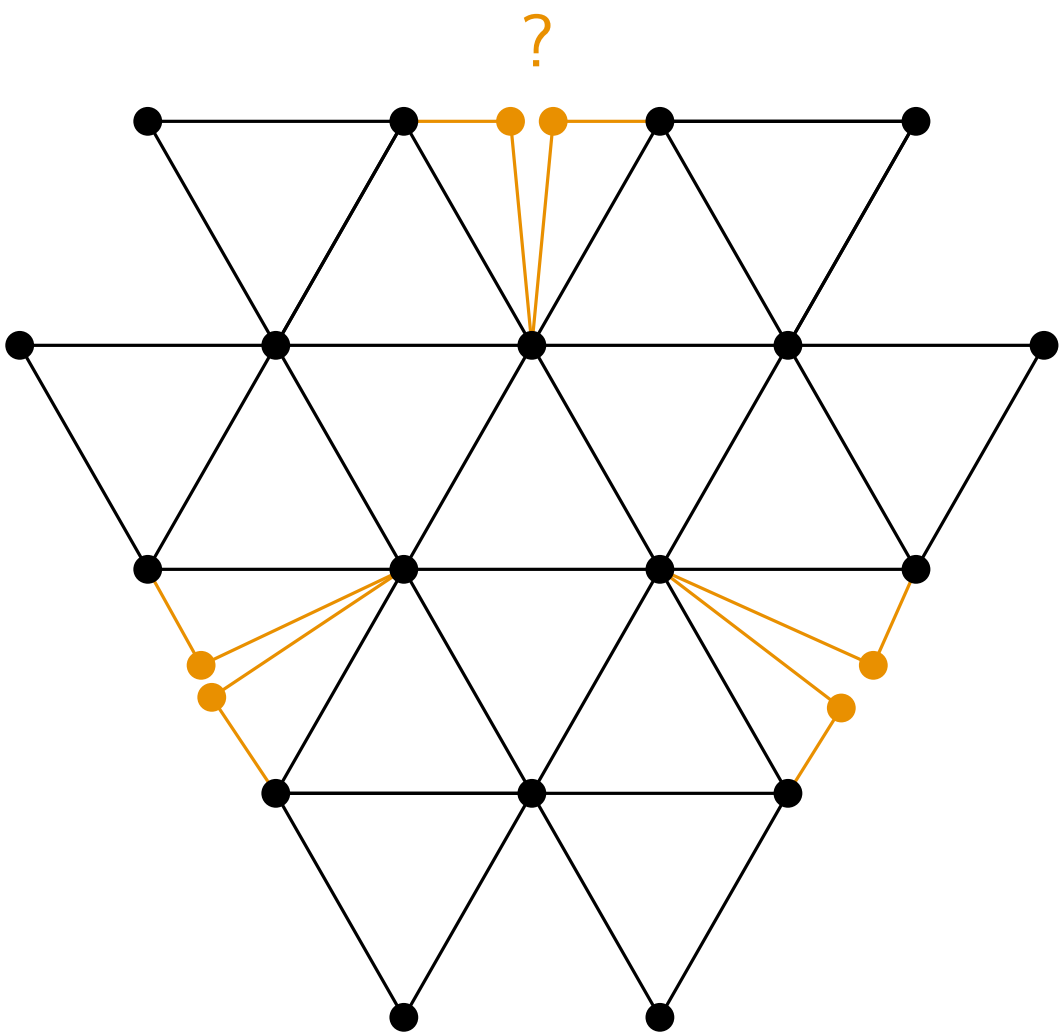

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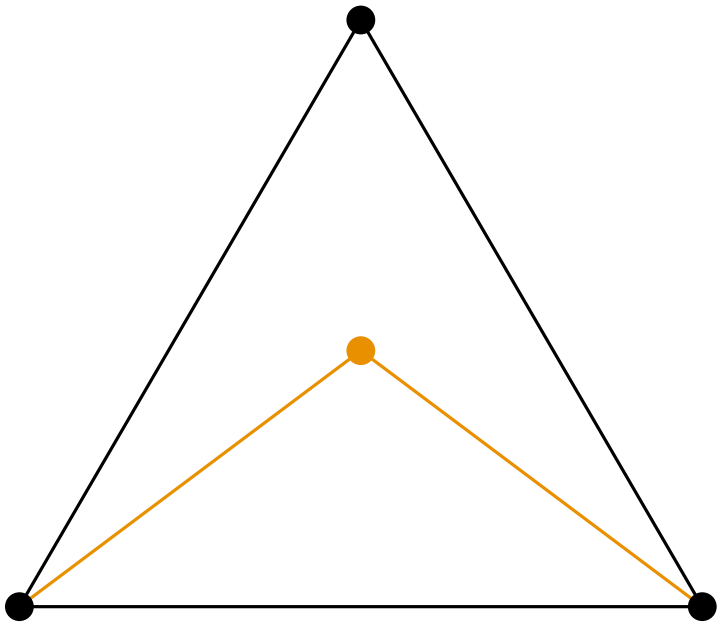

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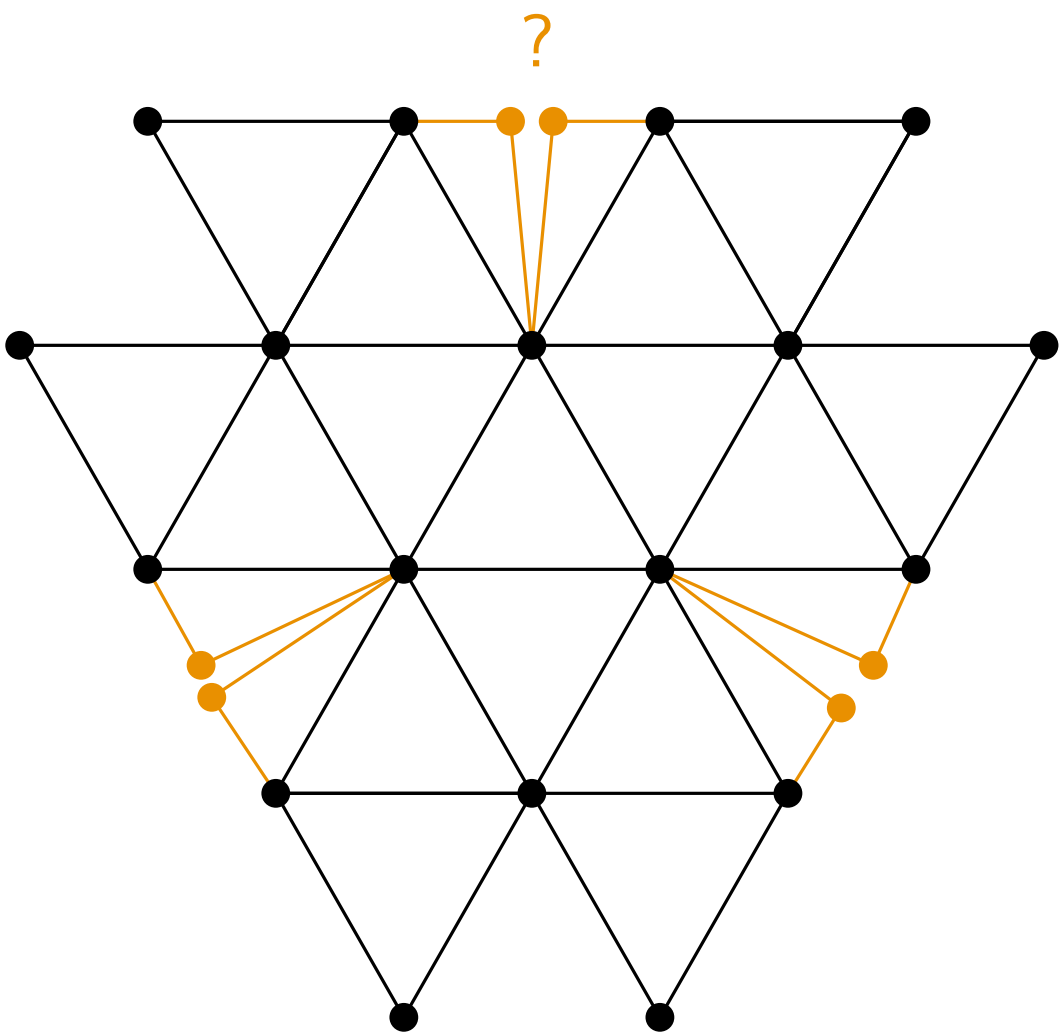
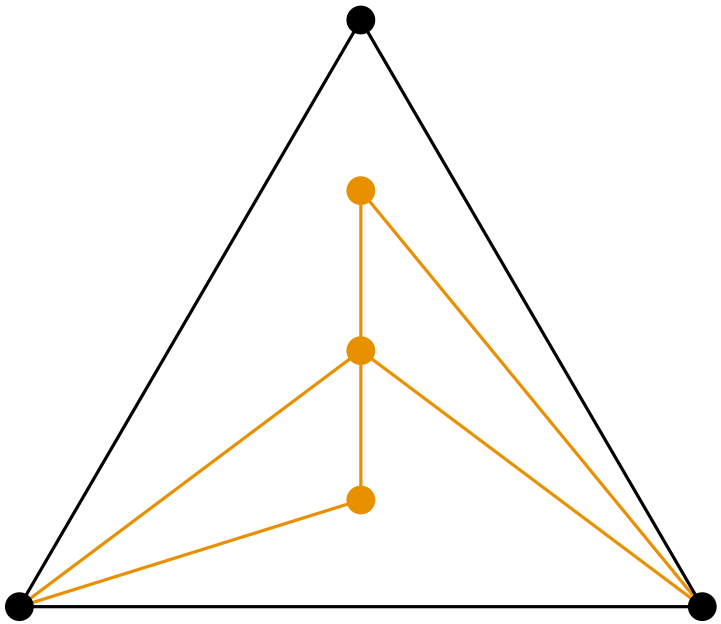


- $r = 4 + \varepsilon$



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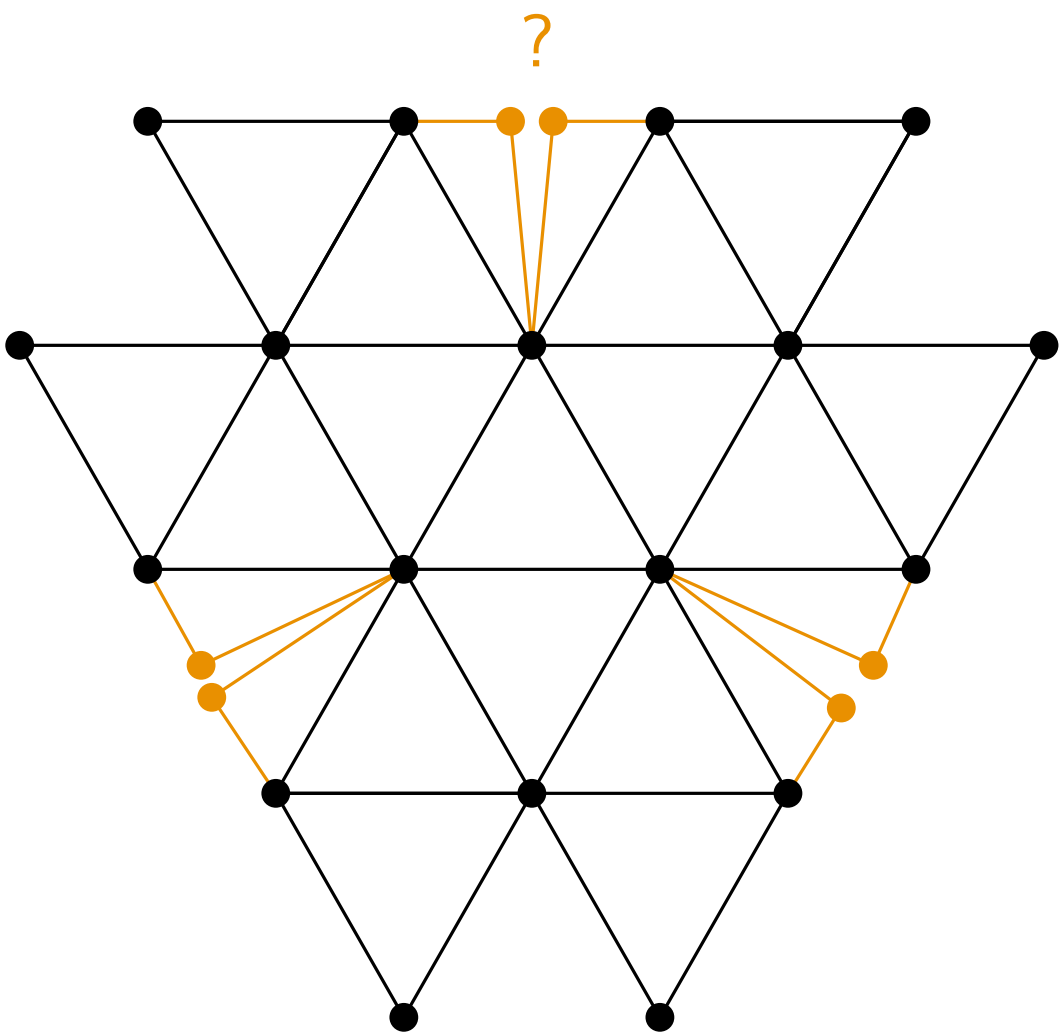

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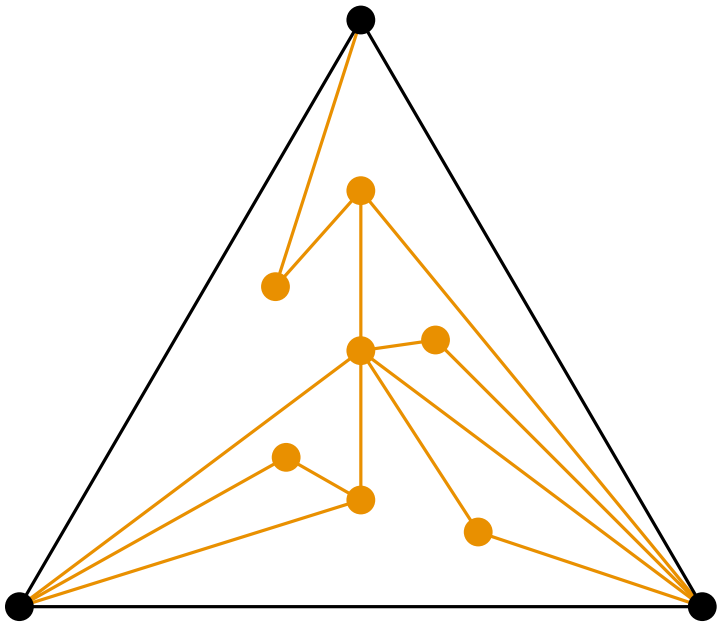

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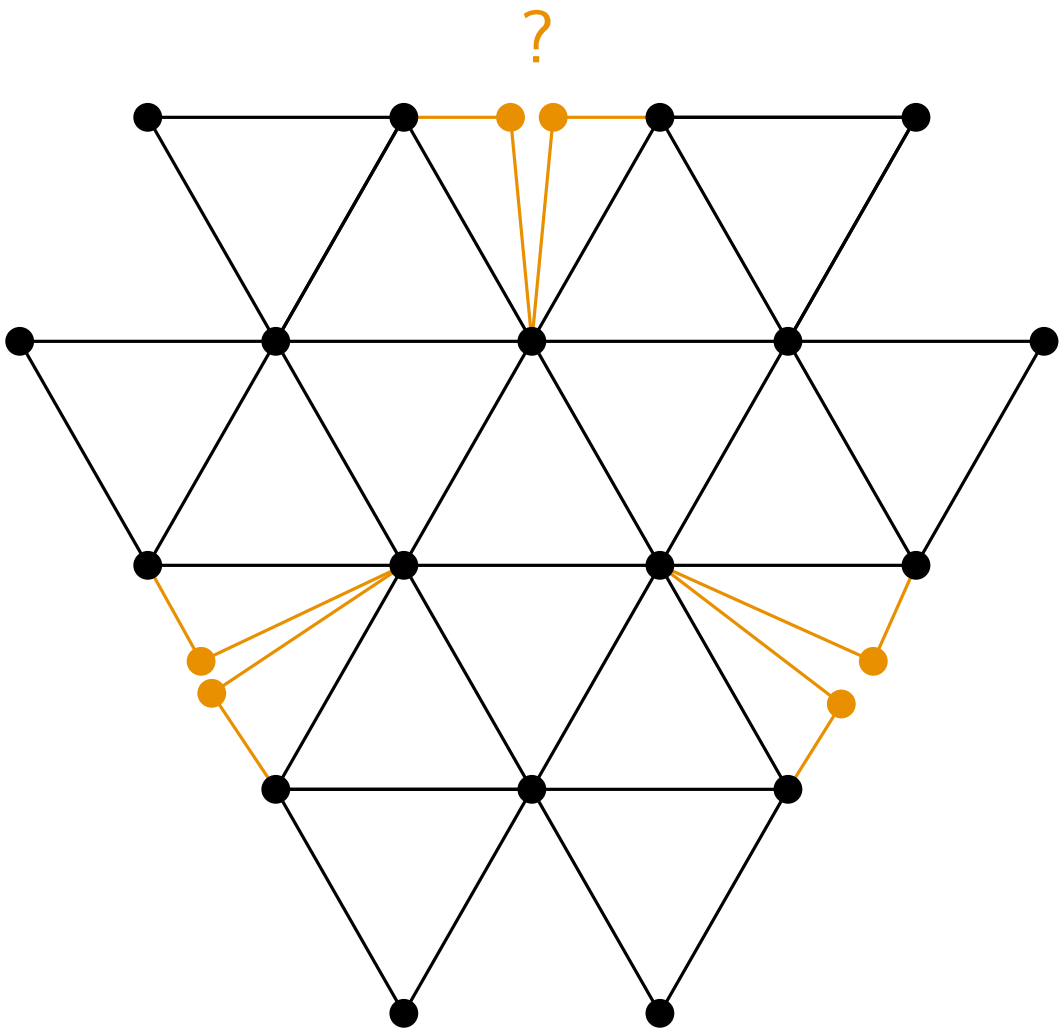
- *Outerplanar graphs*



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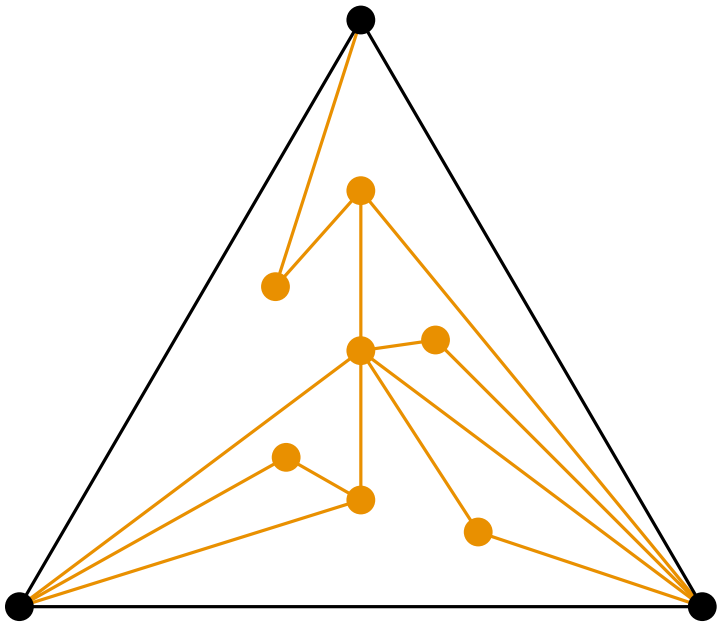

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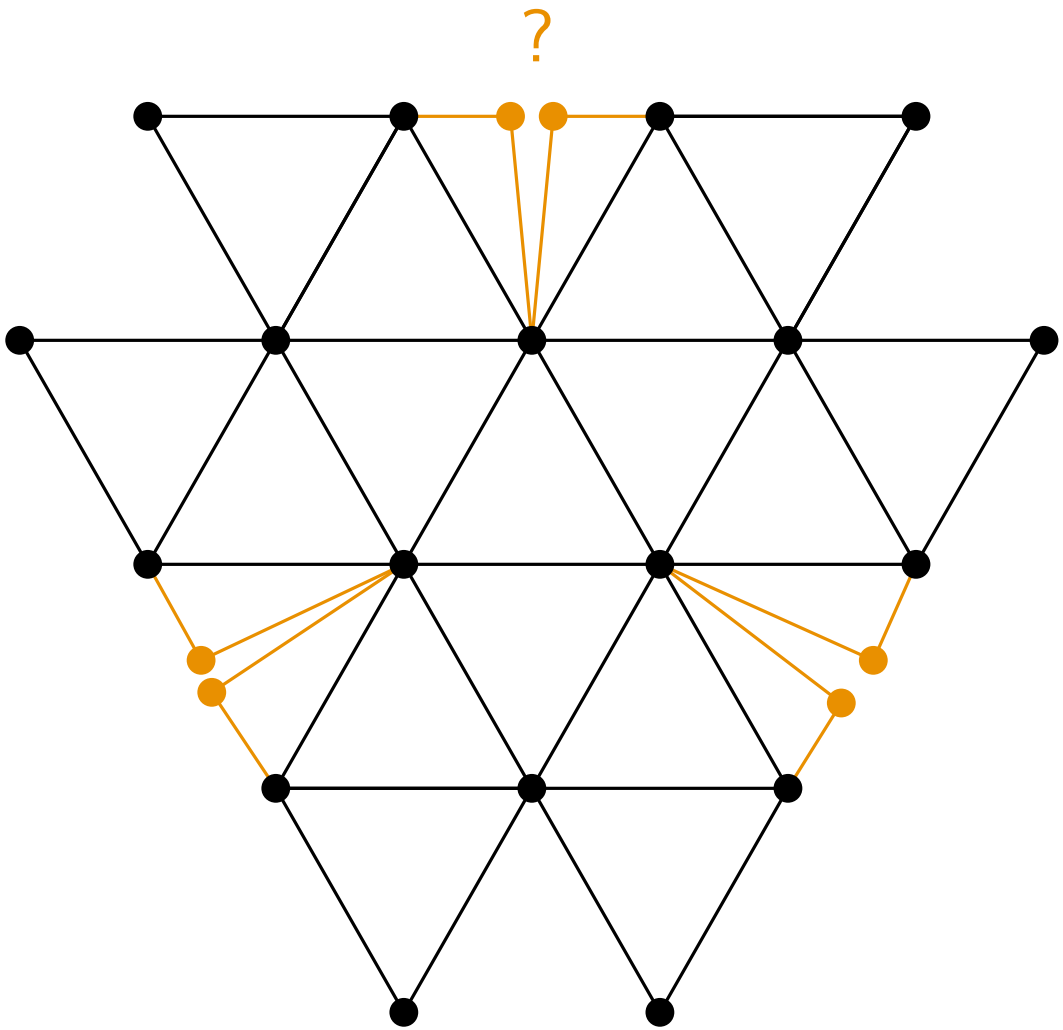

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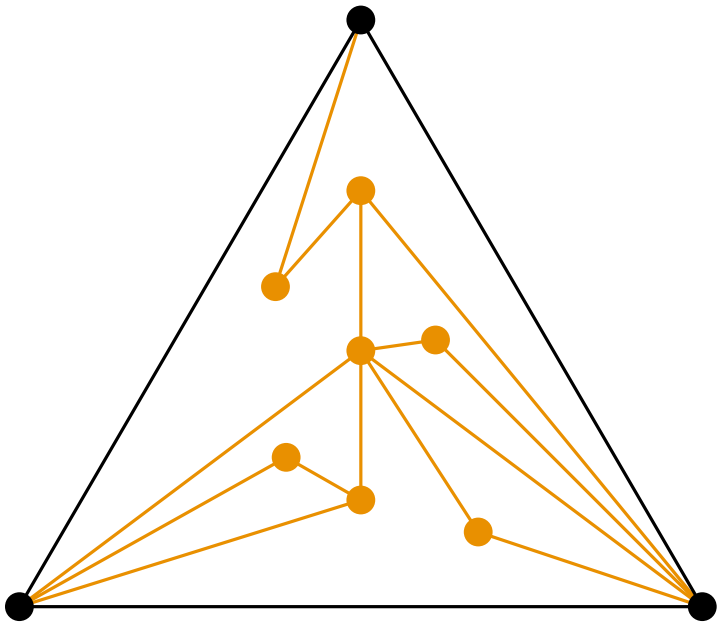

...logarithmic?

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- Appendix
  - Example Drawing of a Series-Parallel Graph
  - General Trees
  - Planar 3-trees
  - Implementation of Complete k-ary Tree Drawer

Complete k-ary Trees

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- k-ary tree is a rooted tree with at most k children per inner node
- k-ary tree is *complete* if every inner node has k children and all leaves are on the same height

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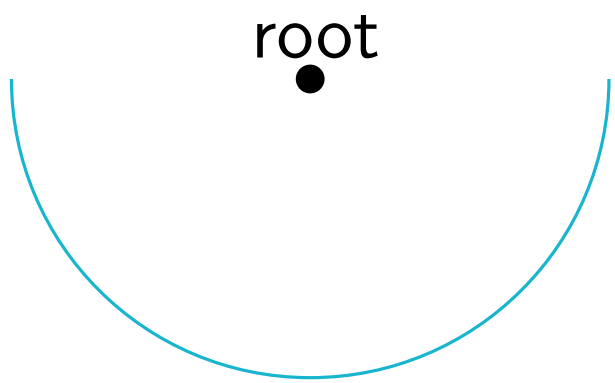

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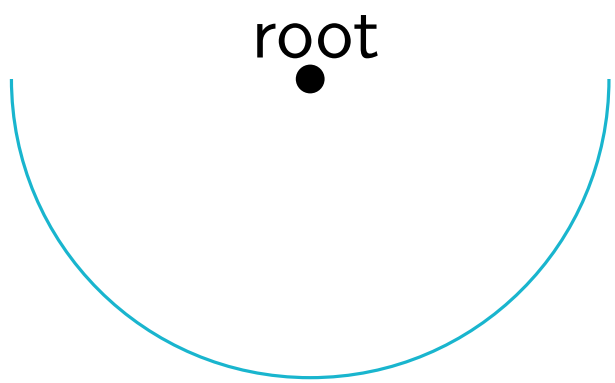

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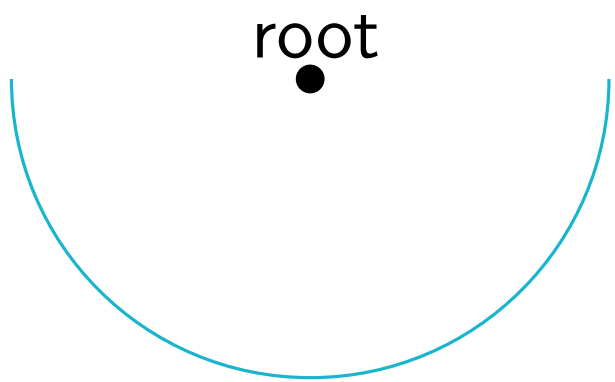

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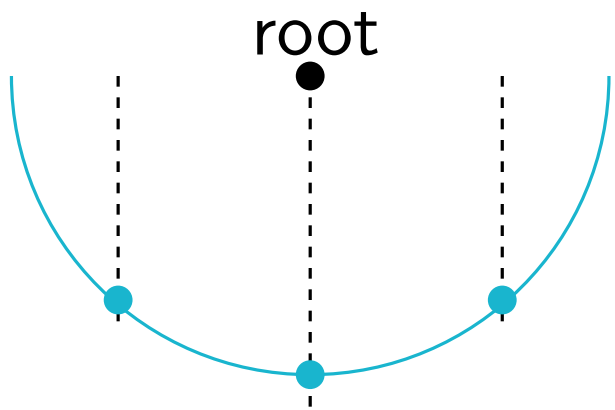

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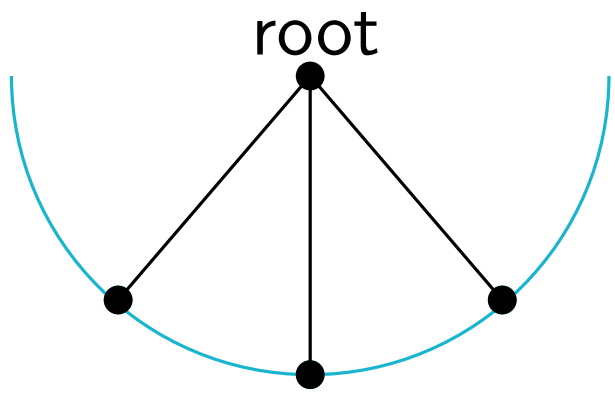

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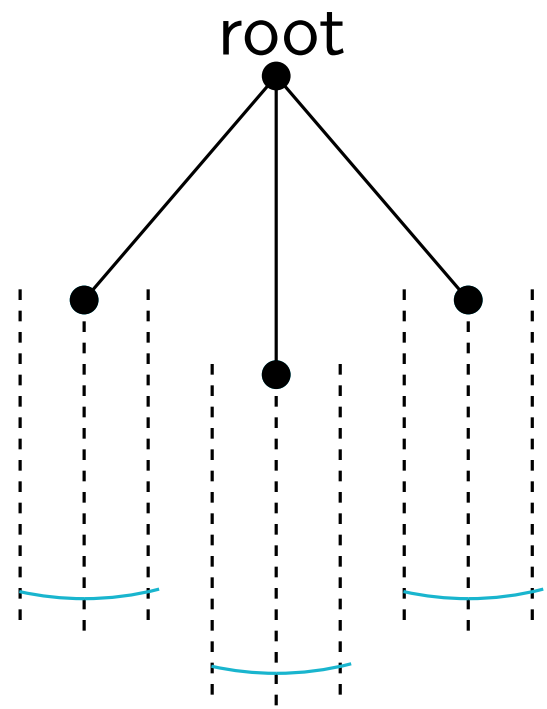

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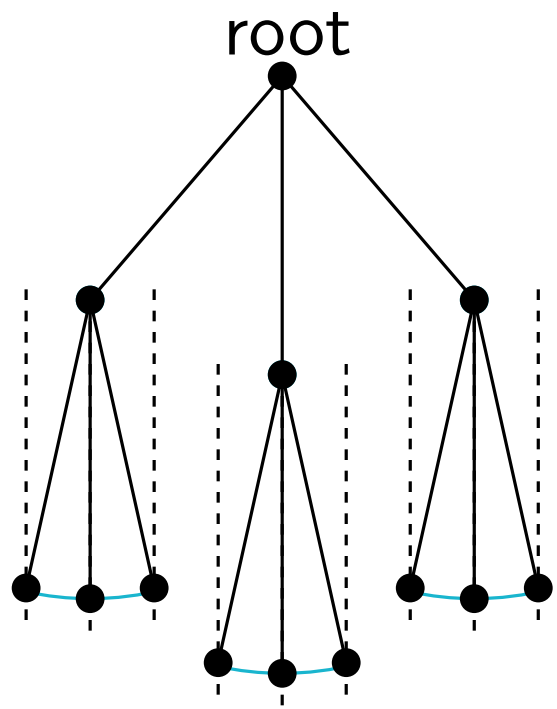

Drawing A Complete k-ary Tree T

- Unidirectional from top to bottom
- Place the root of T on the grid
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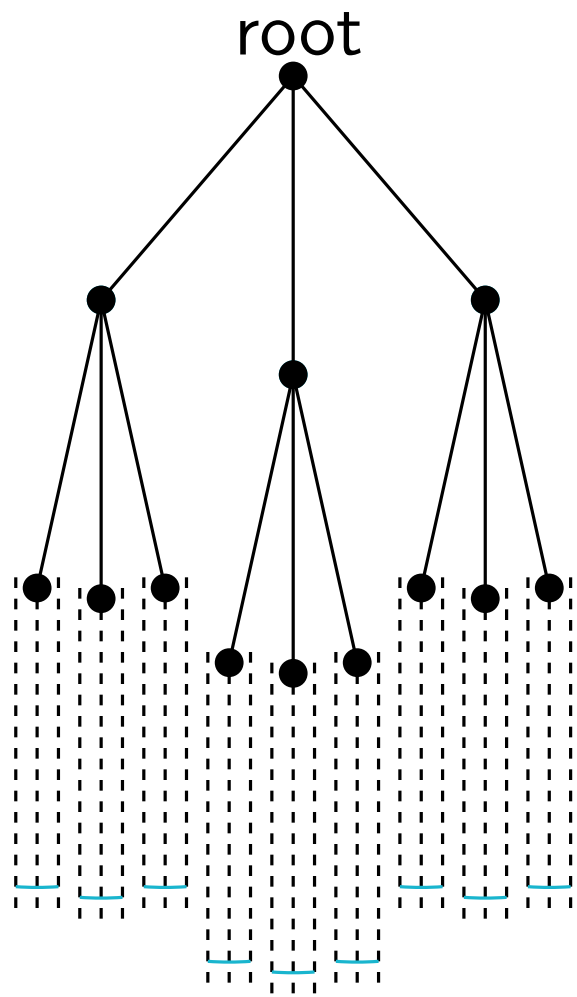

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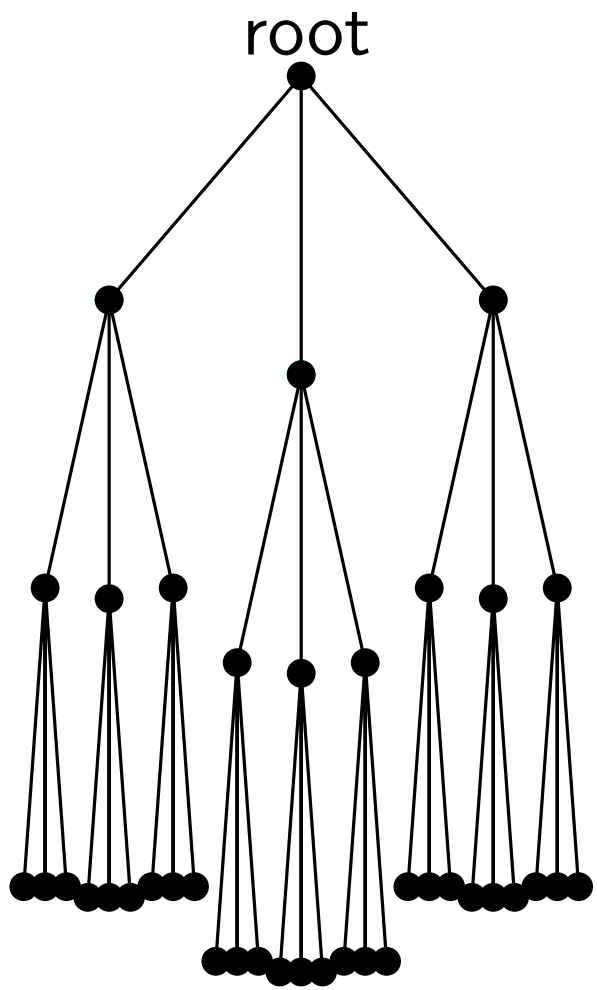

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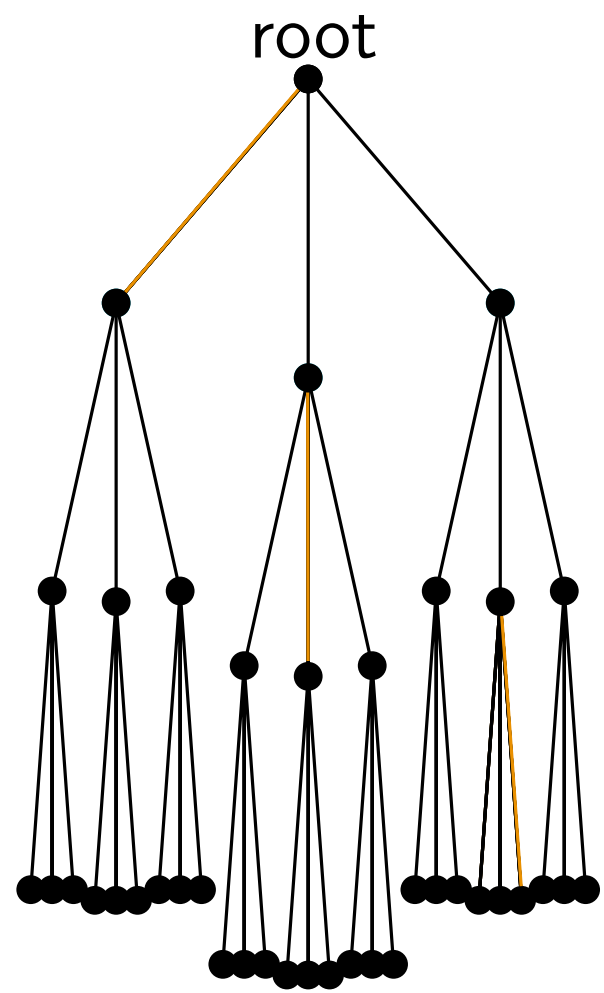

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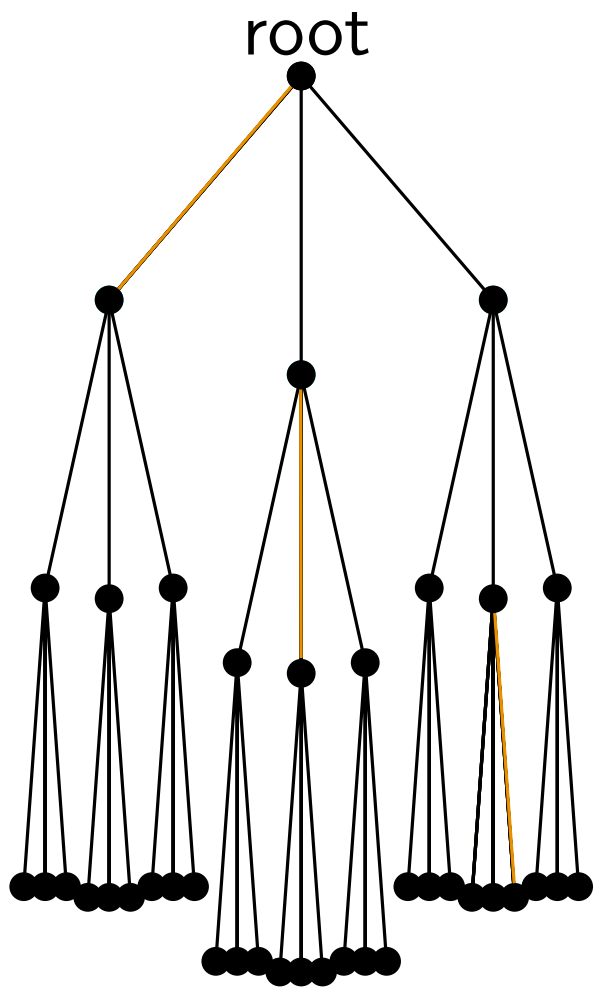

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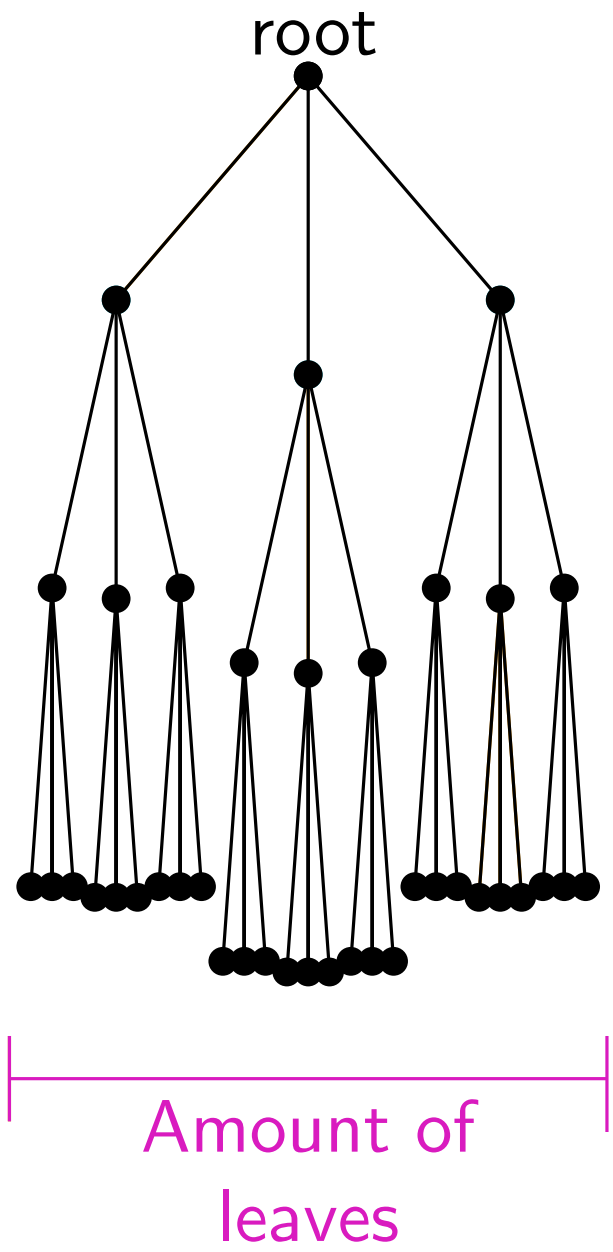

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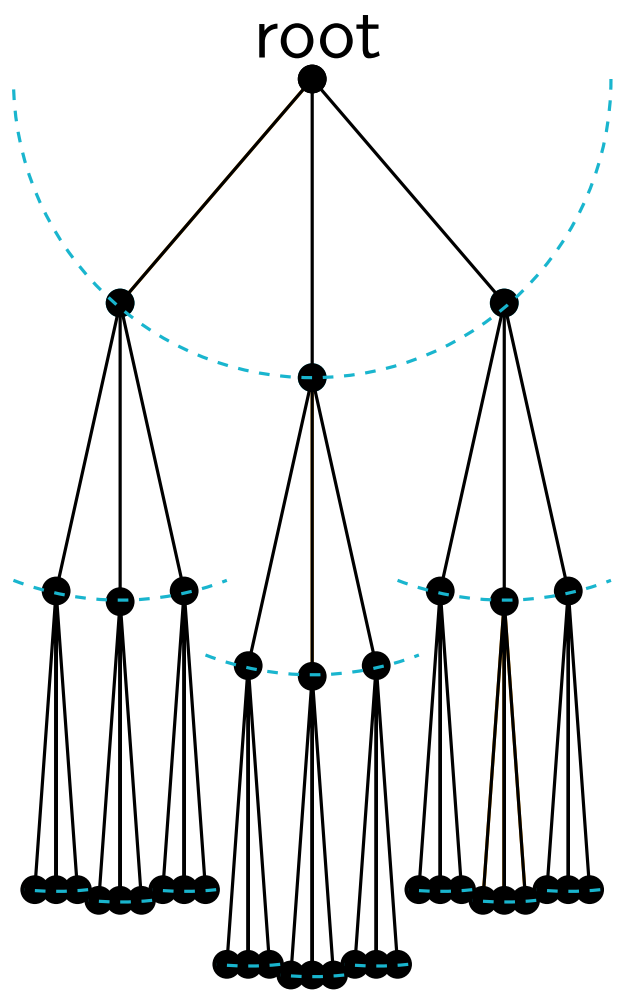

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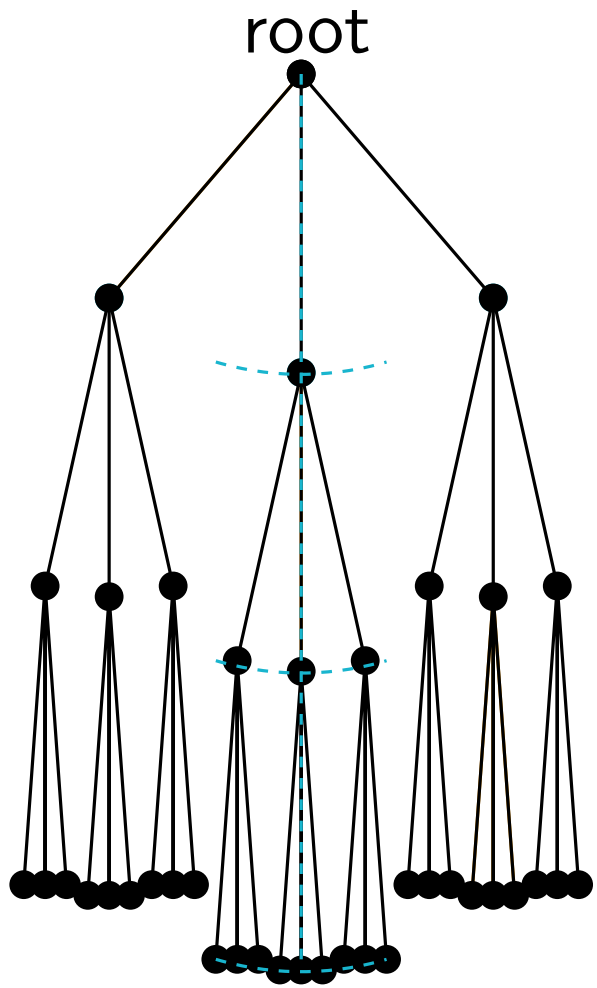

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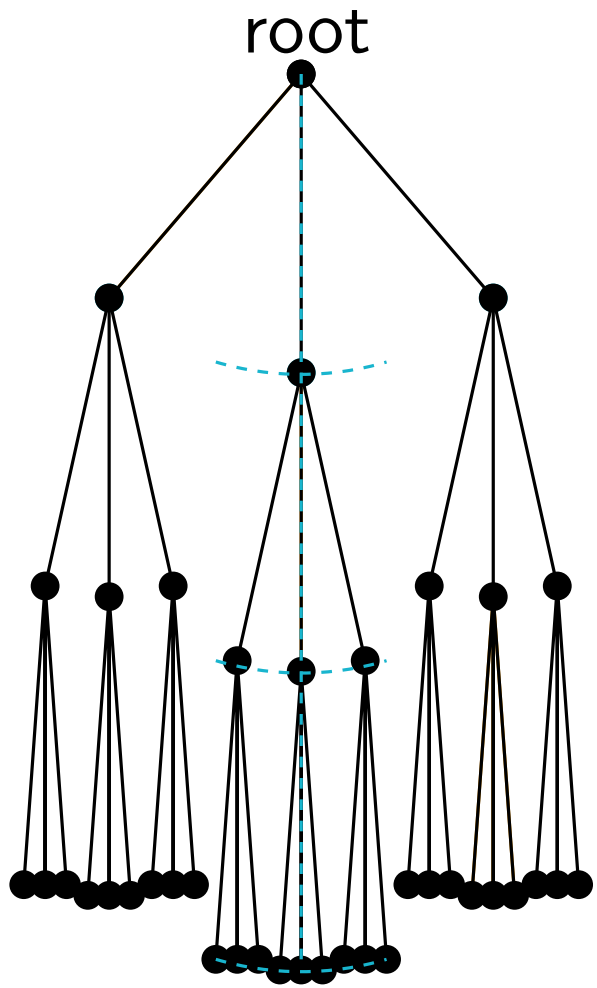

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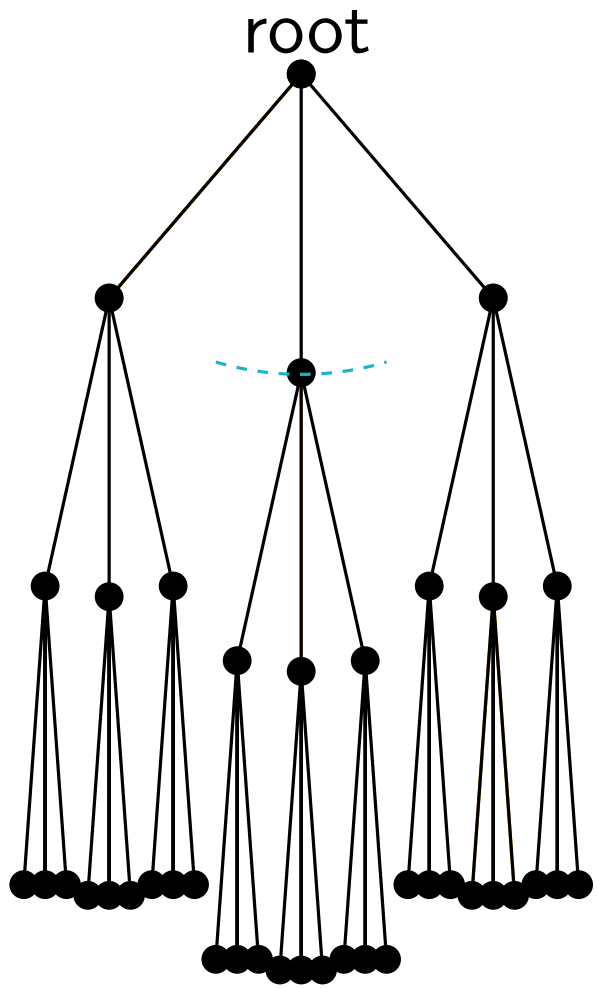

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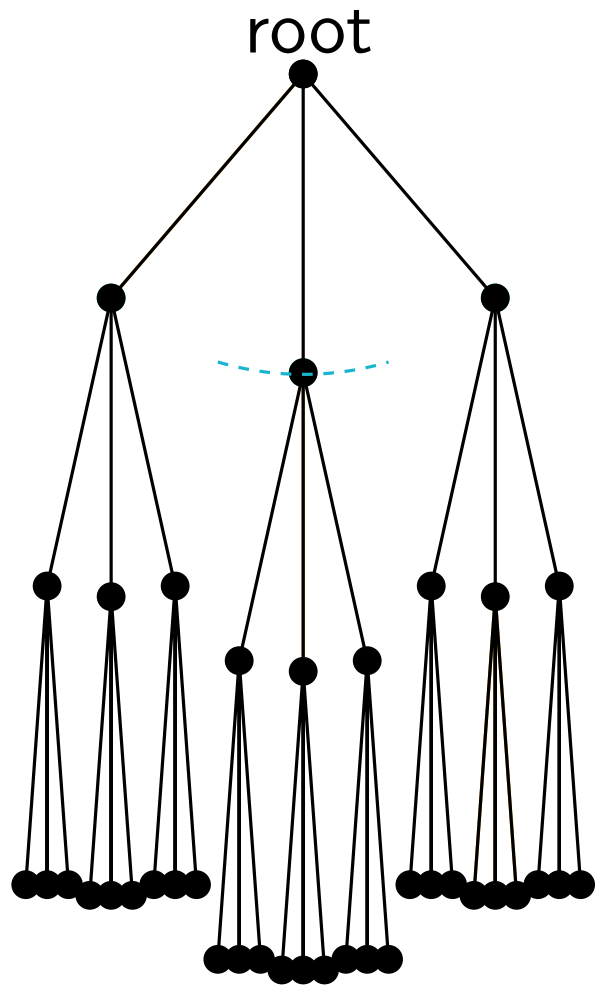

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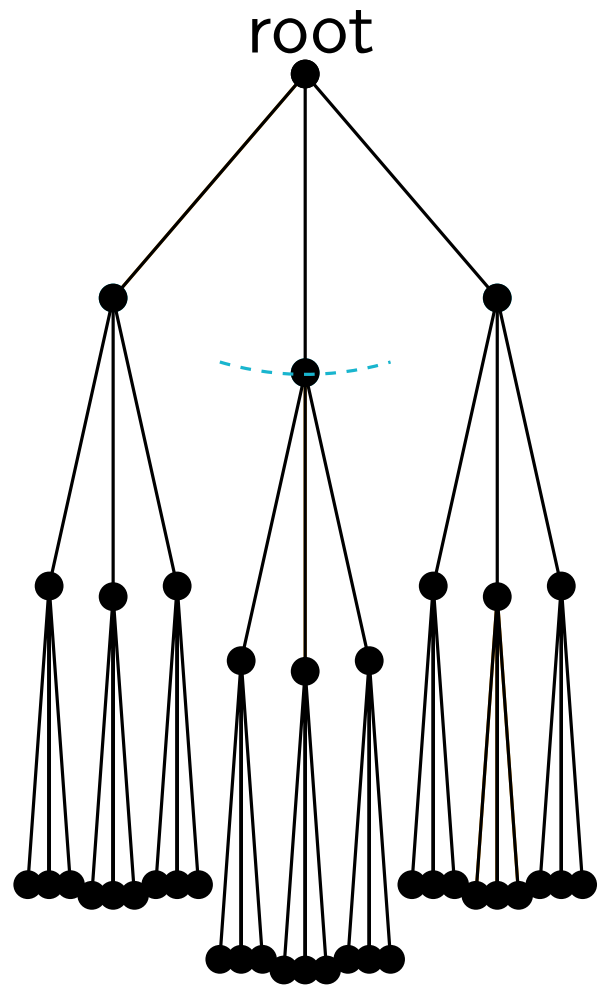

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Outerplanar Graphs and Series-Parallel Graphs

Outerplanar Graphs

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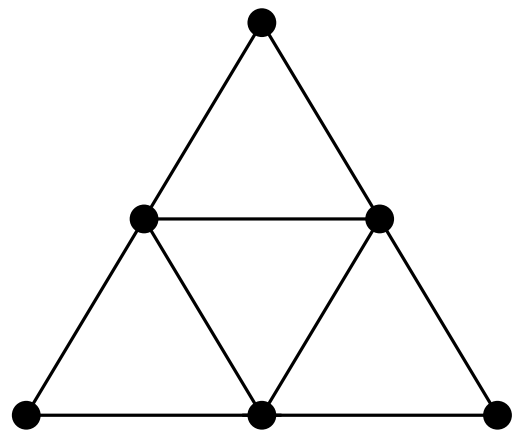
- G admitting a drawing where every vertex lies on the outer face is *outerplanar*



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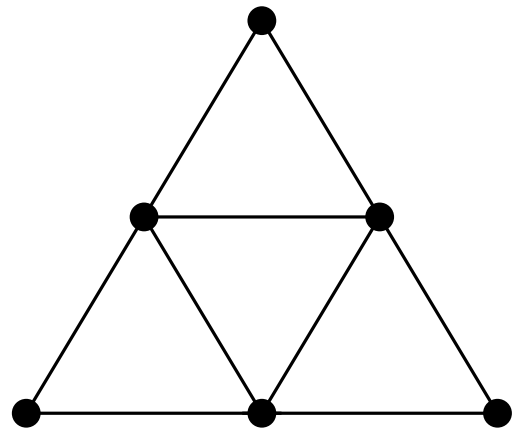

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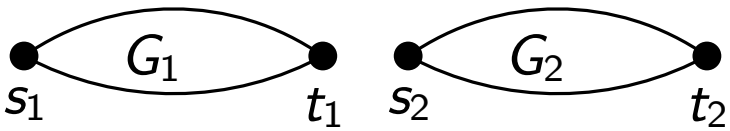

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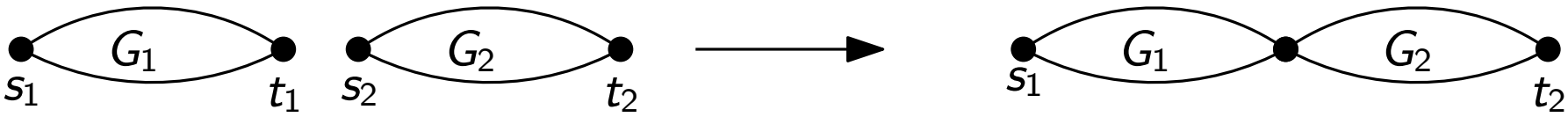

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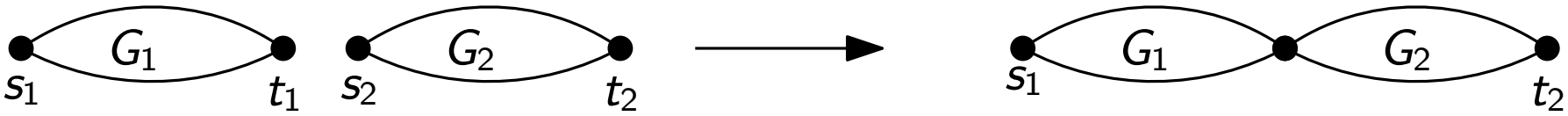

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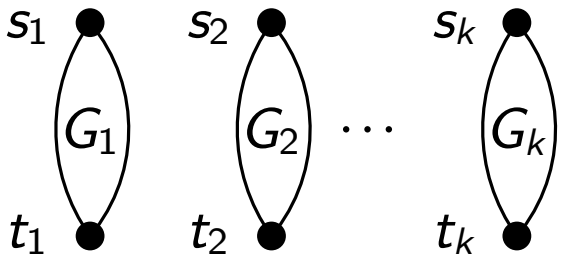

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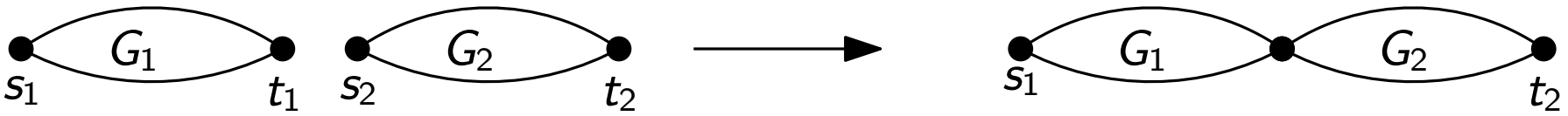

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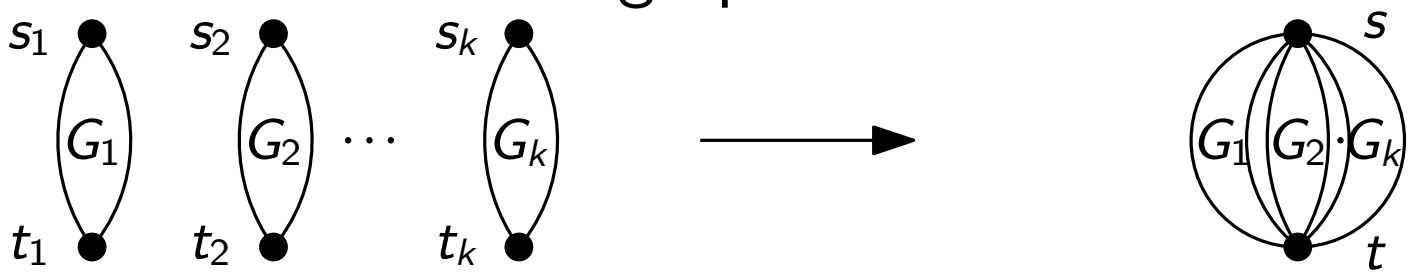

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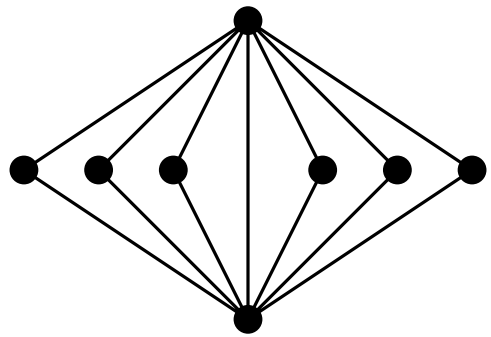

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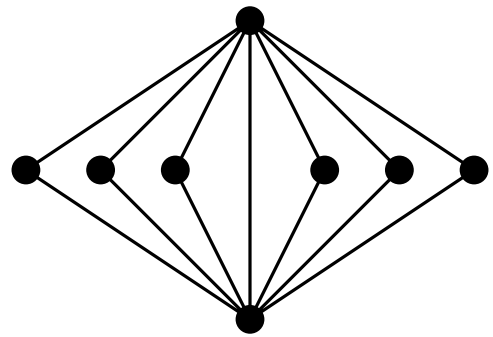

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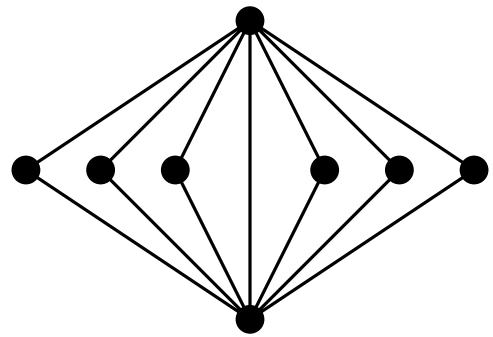

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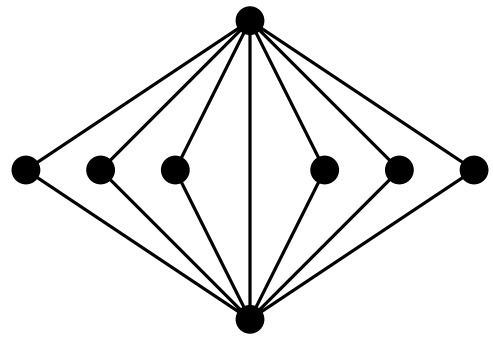
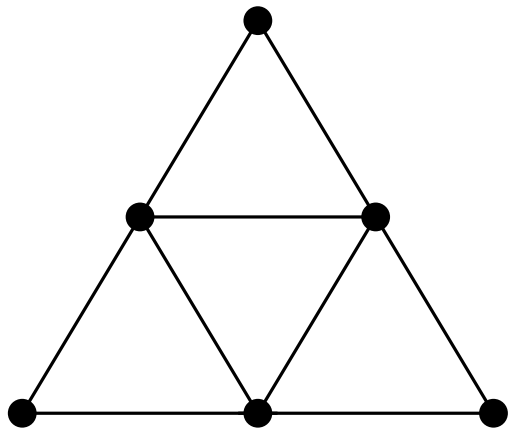


- a series-parallel graph is *maximal* if no edges can be inserted while maintaining a series-parallel graph
- maximal outerplanar graphs are maximal SP-graphs
  - Drawing approaches deal with outerplanar graphs first
  - Extended to SP-graphs, if possible

Approach I: Using Weak Dual Graph G^*

- outerplanar graph *maximal* if no edges can be inserted without destroying outerplanarity



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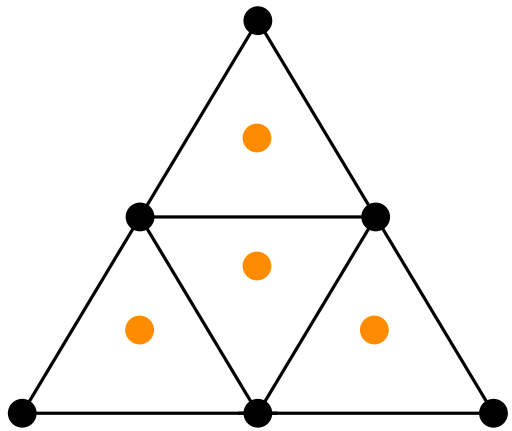

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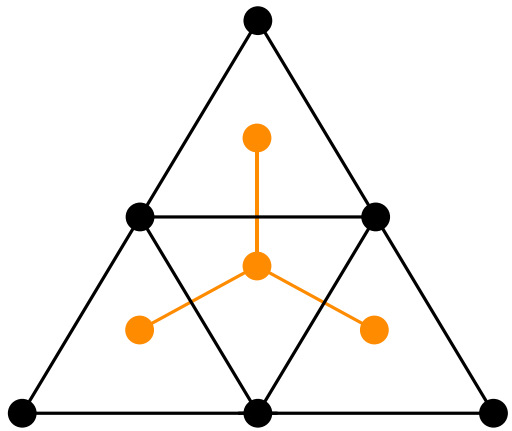

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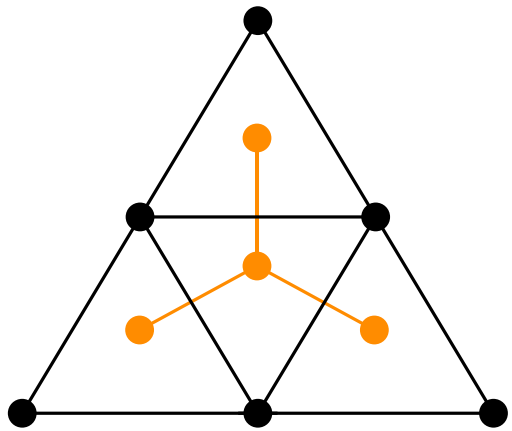

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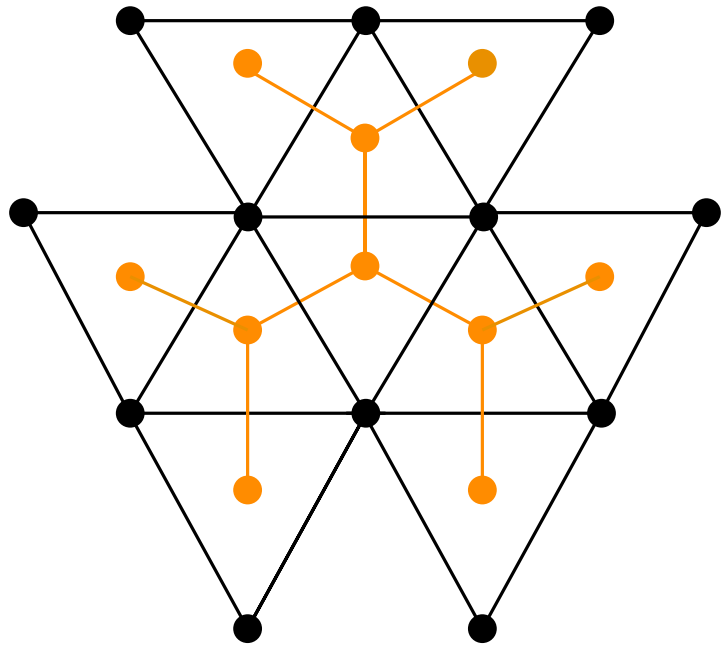

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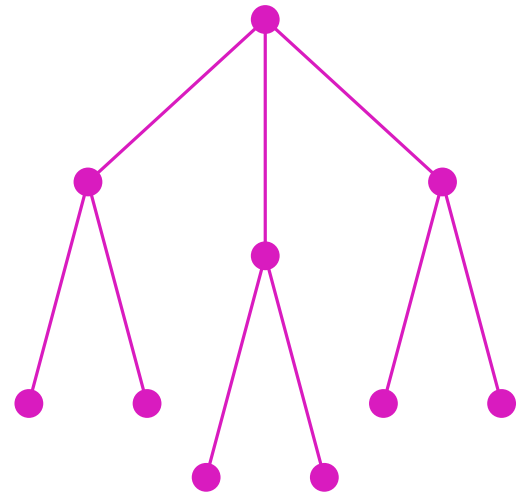

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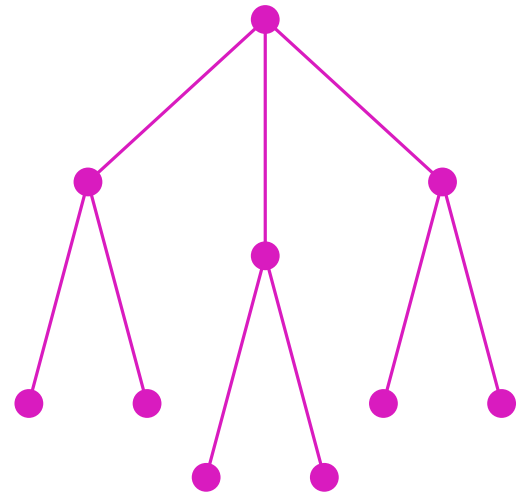

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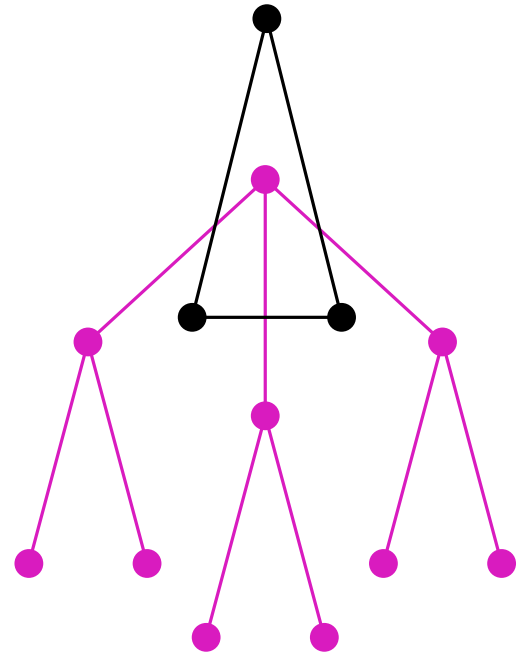

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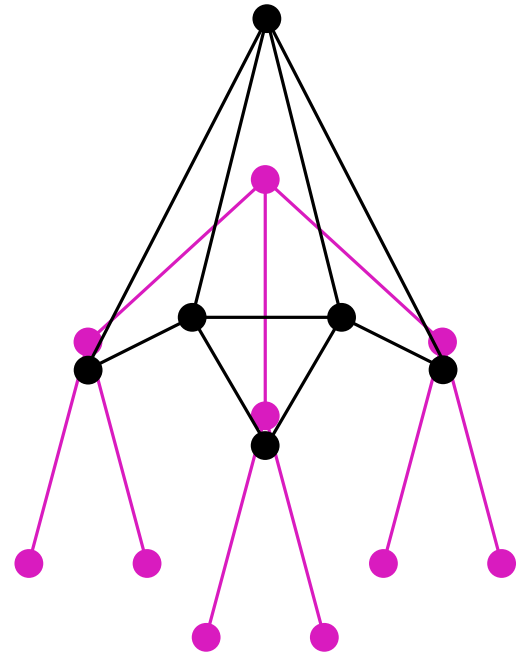

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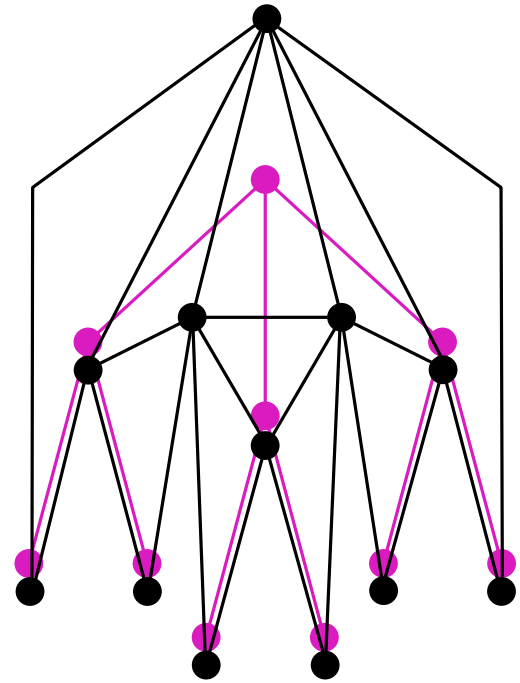

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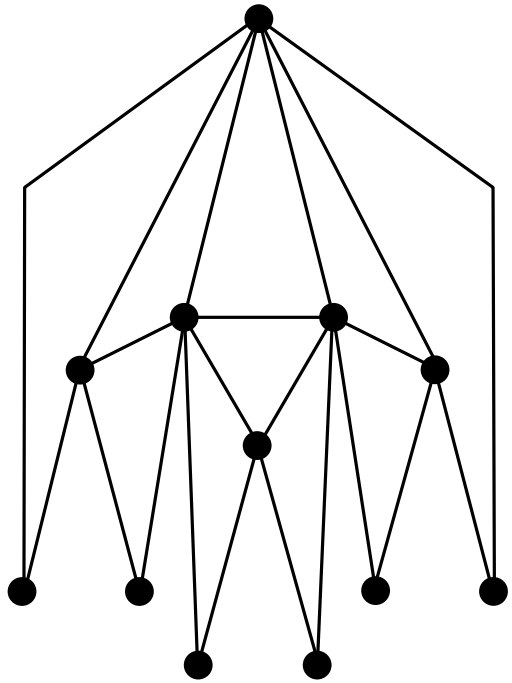

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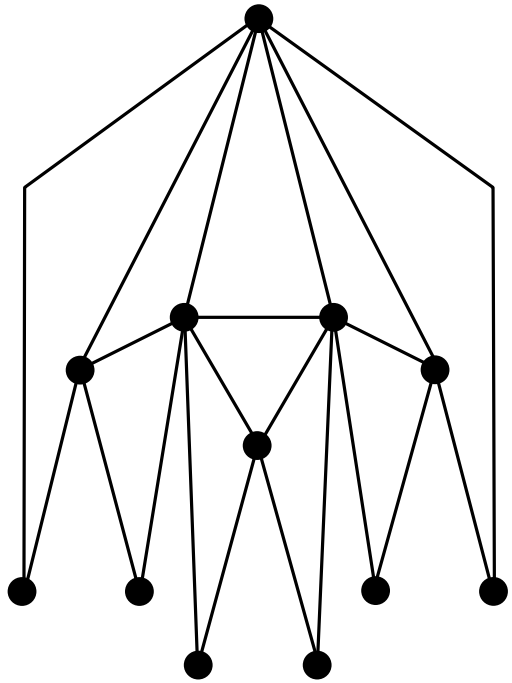

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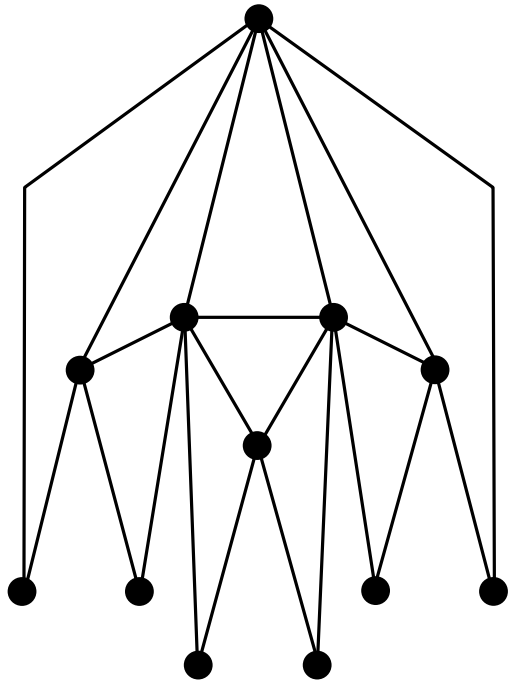

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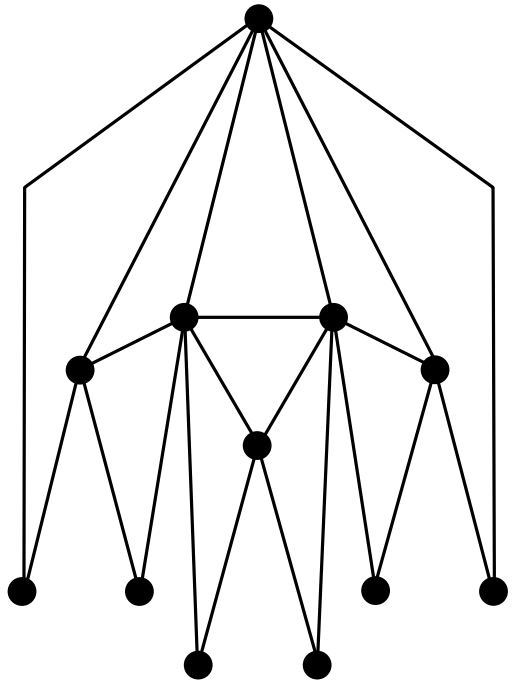

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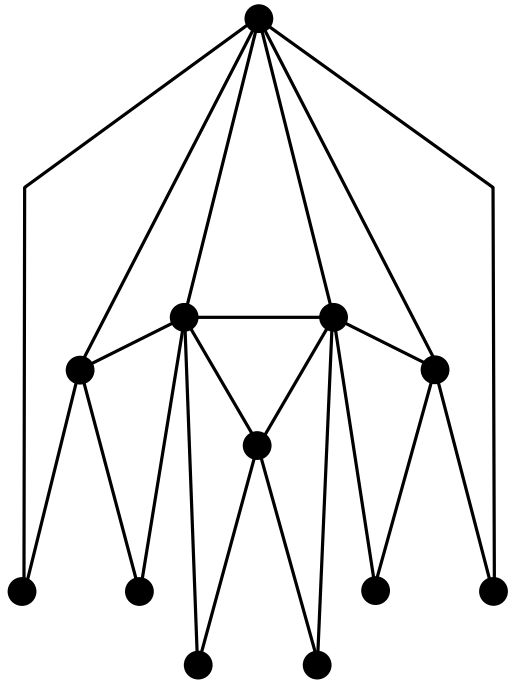

- If height of G^* in $\mathcal{O}(\log n)$
  - Radius in $\Theta(n)$, drawing height in $\mathcal{O}(n \log n)$
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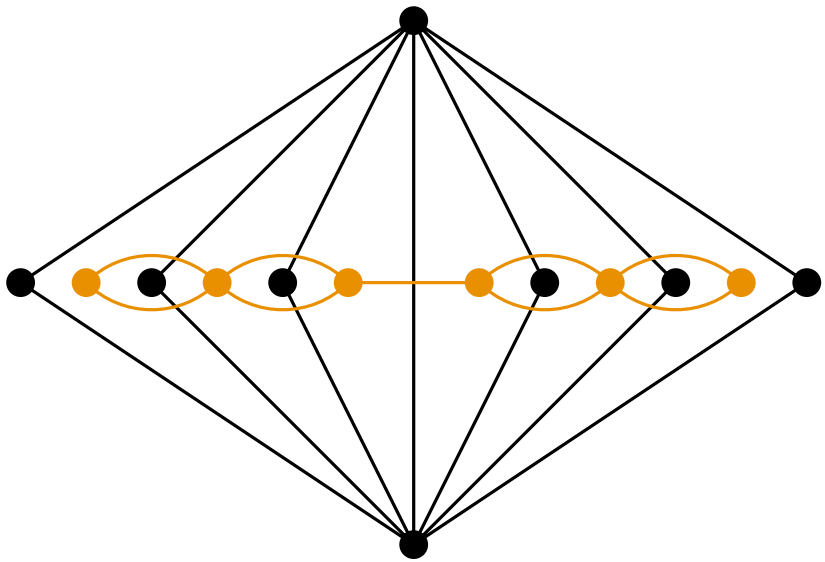

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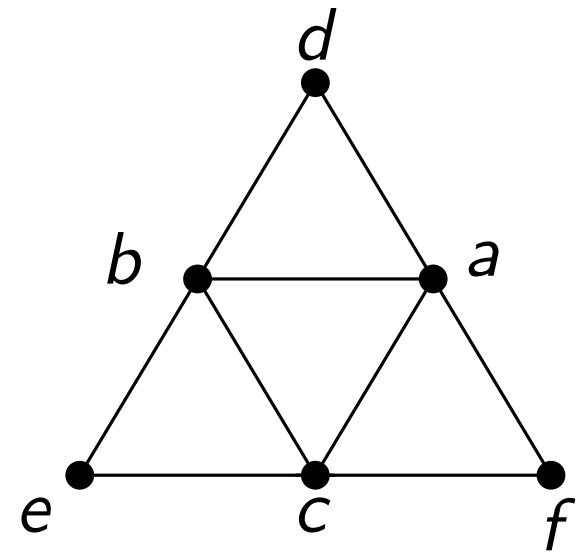

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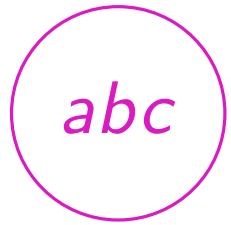

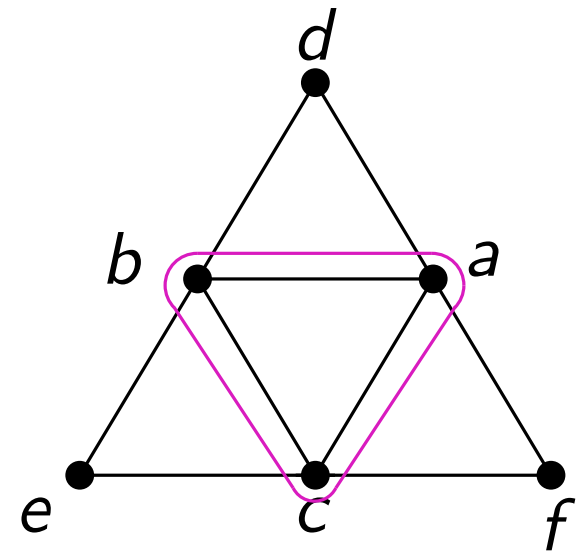

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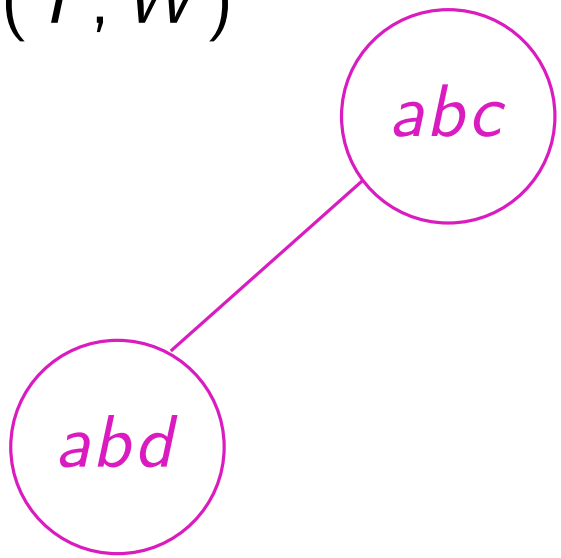

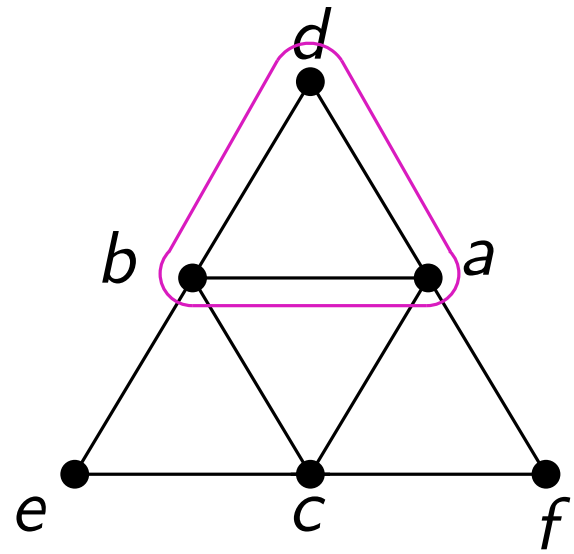

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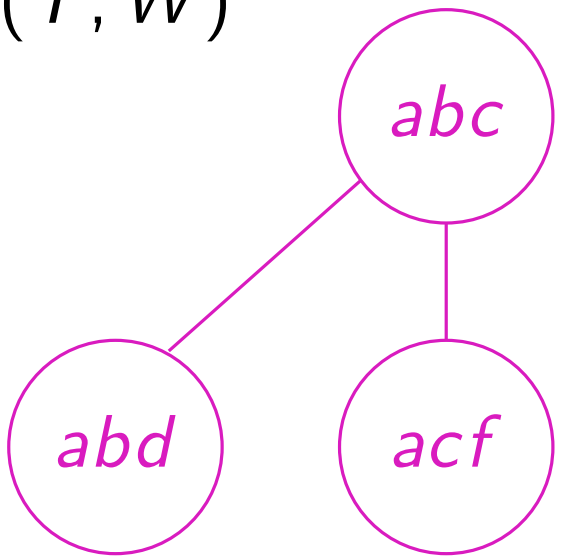

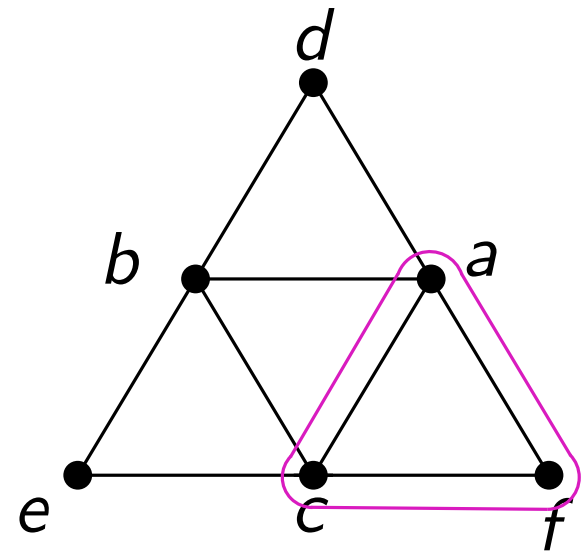

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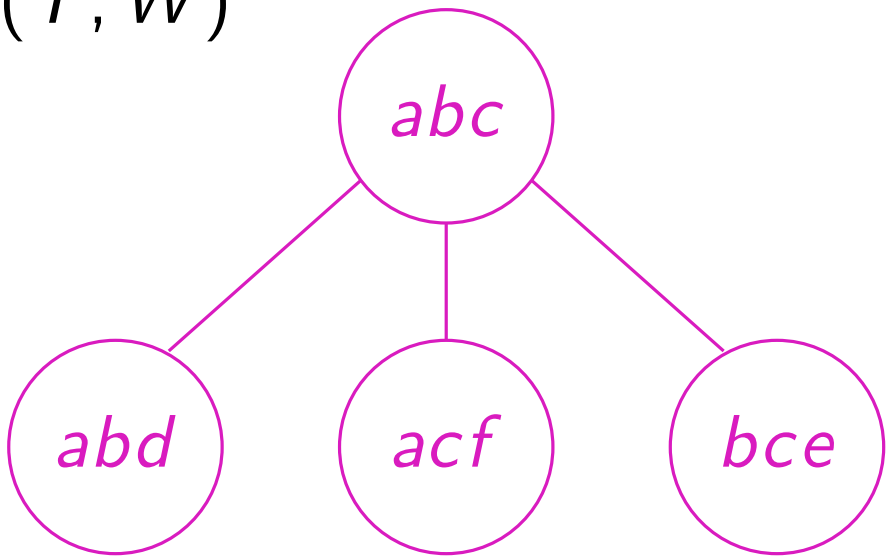

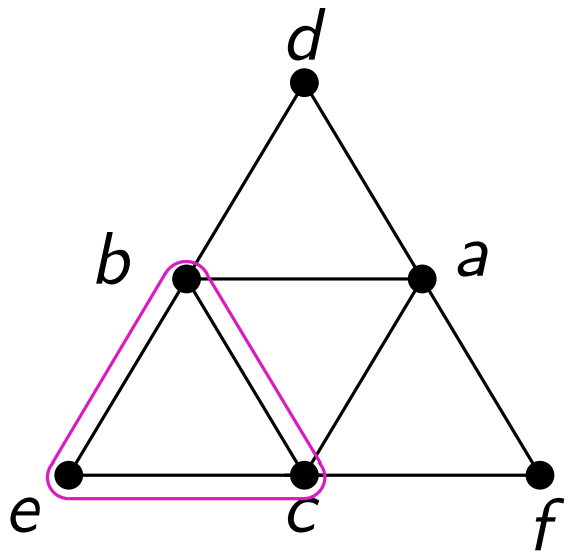

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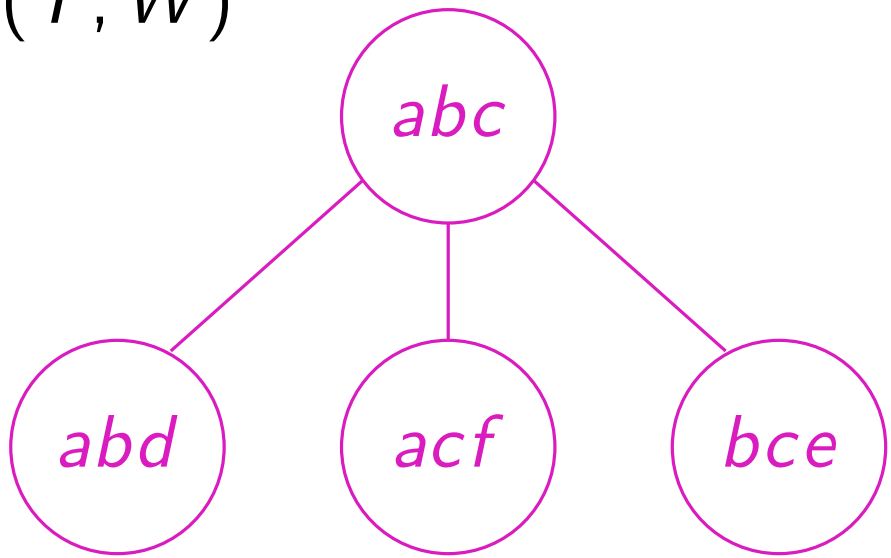

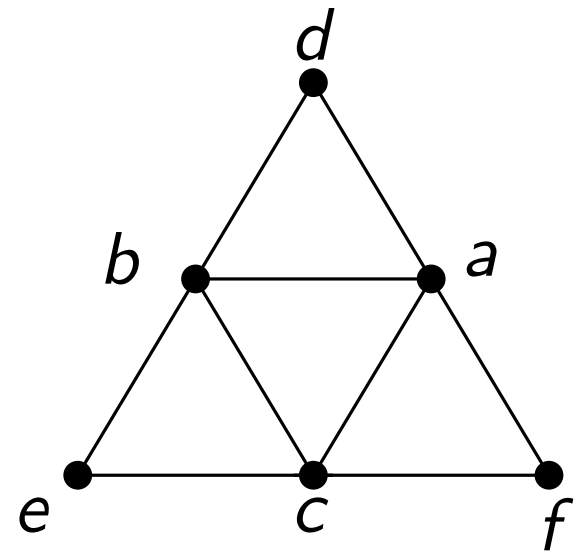

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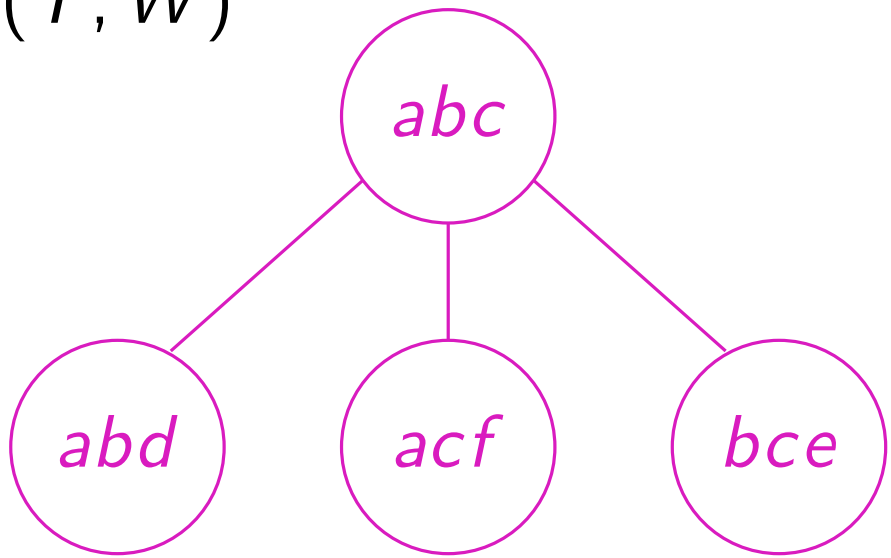

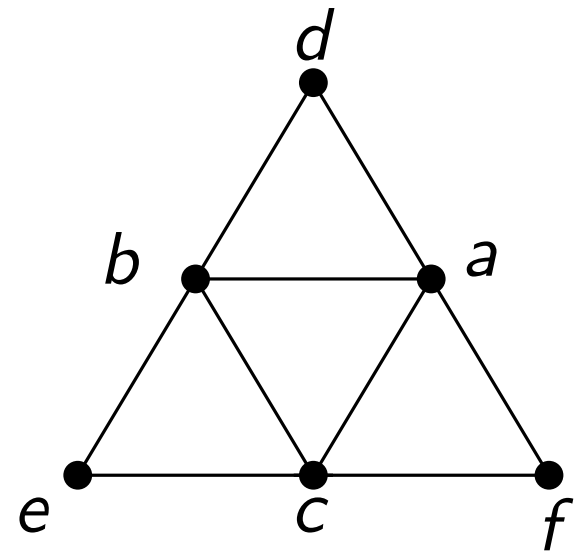

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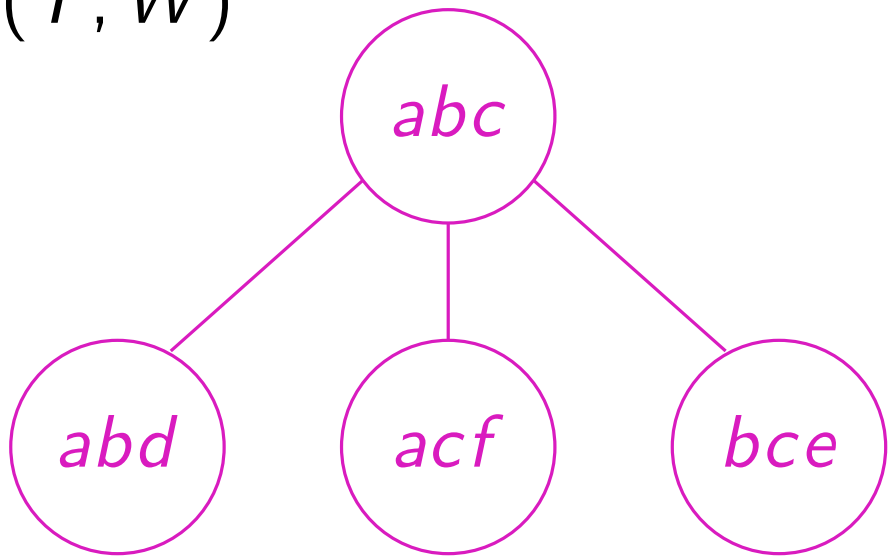

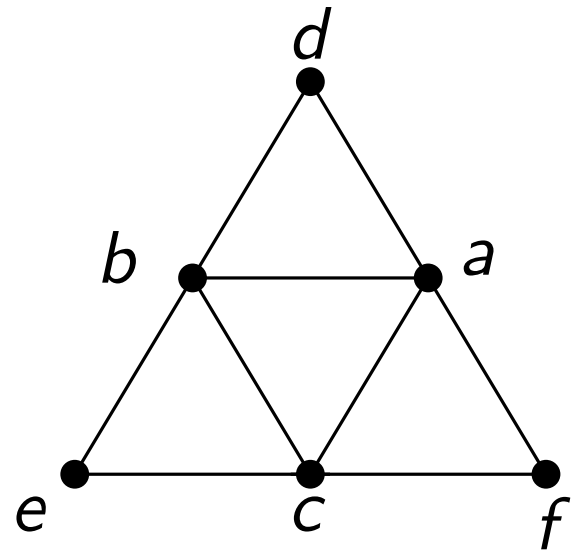

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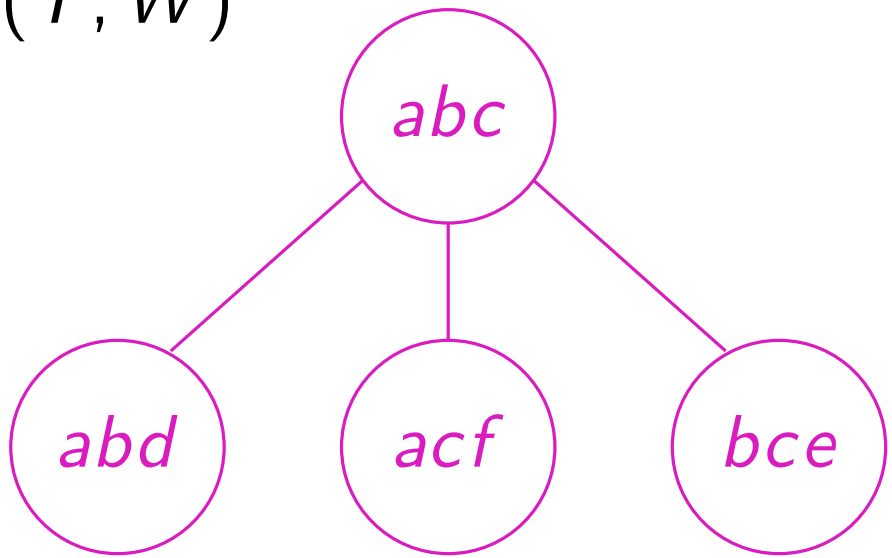

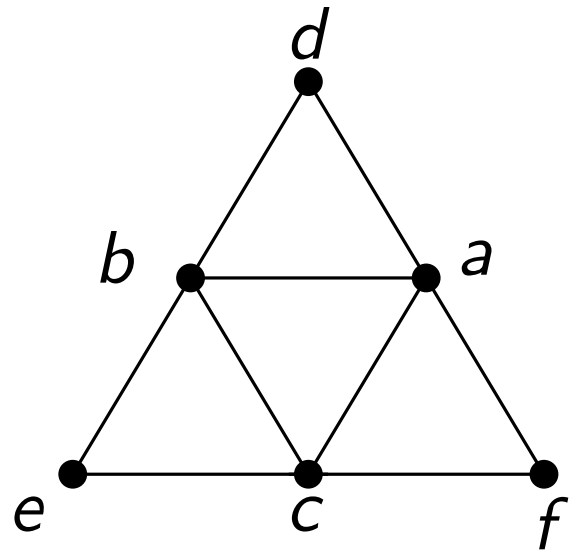

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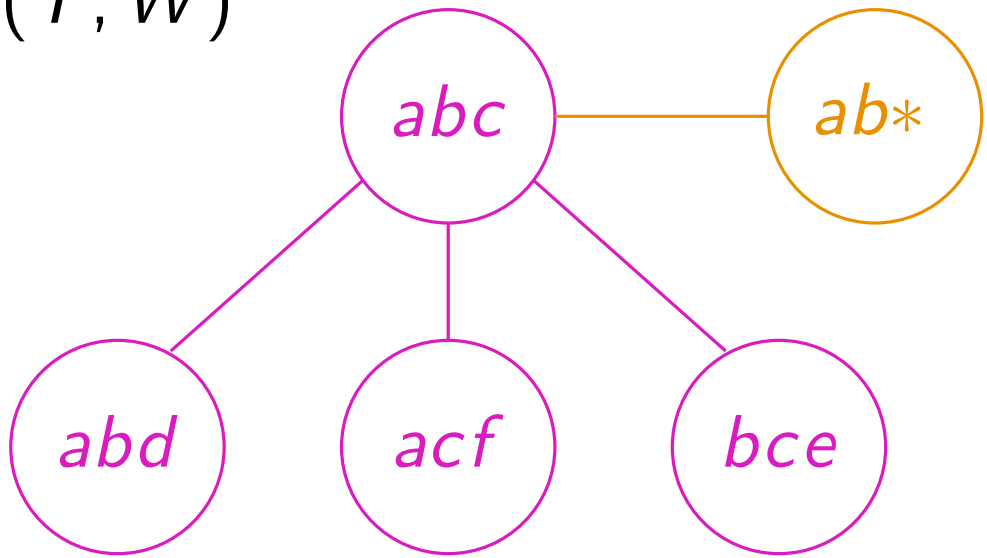

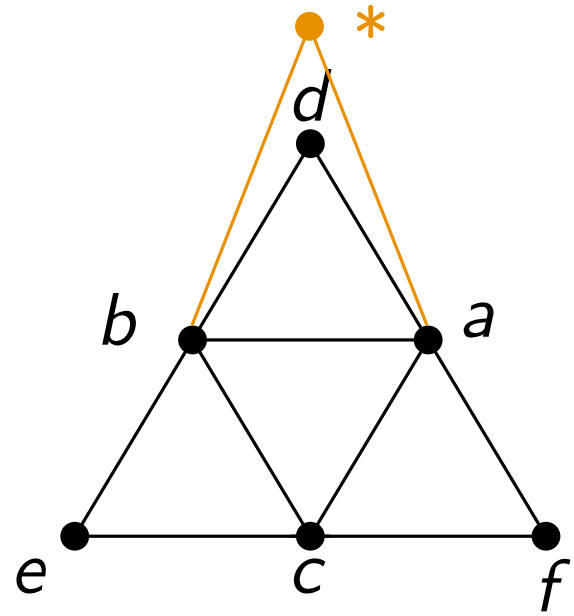

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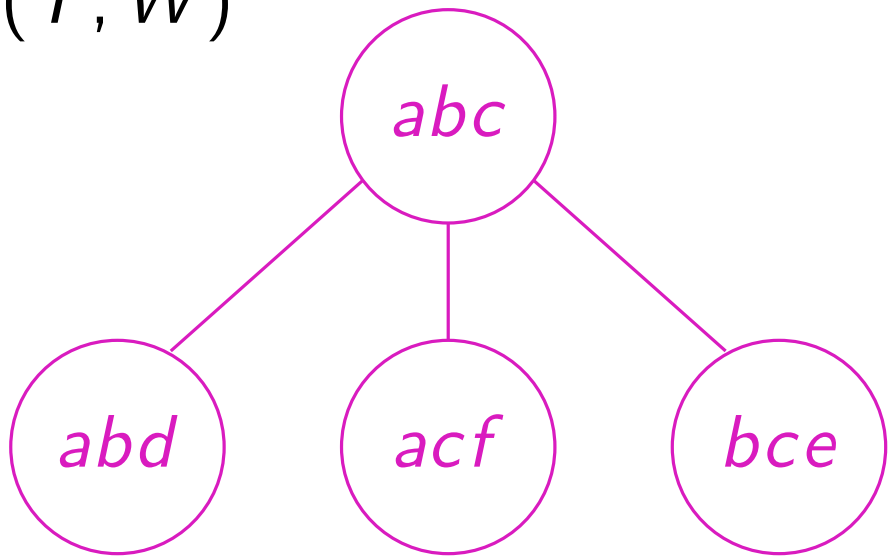

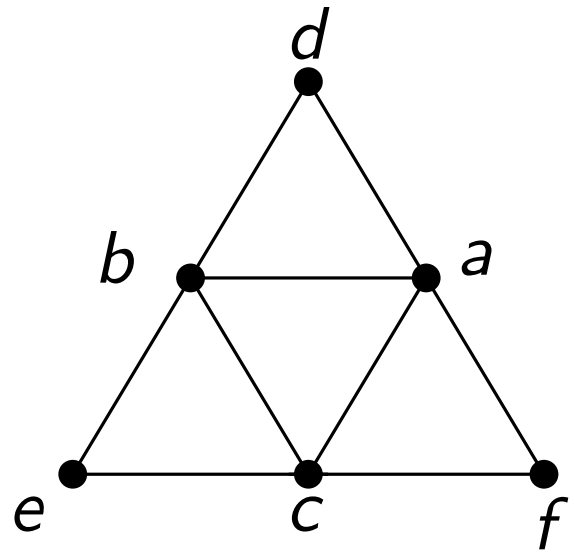

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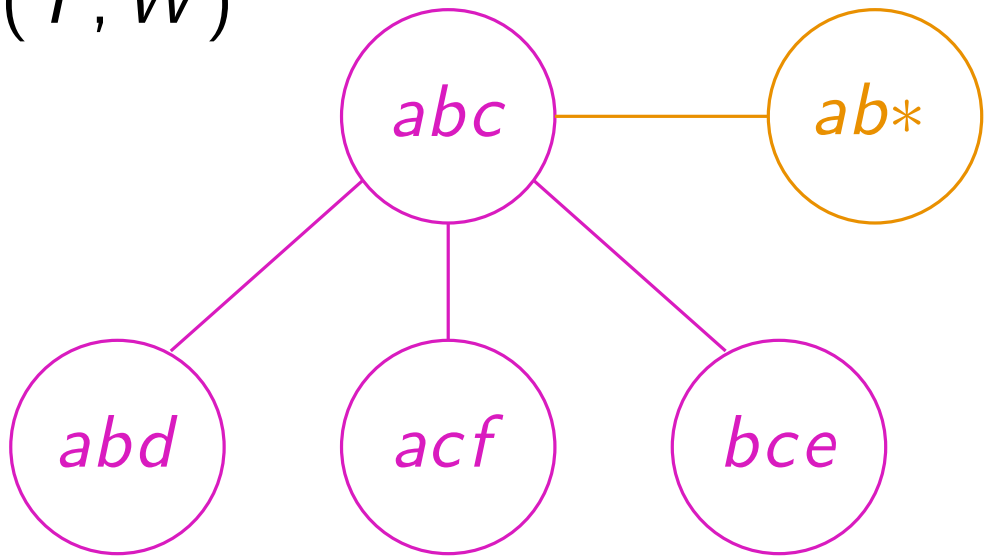

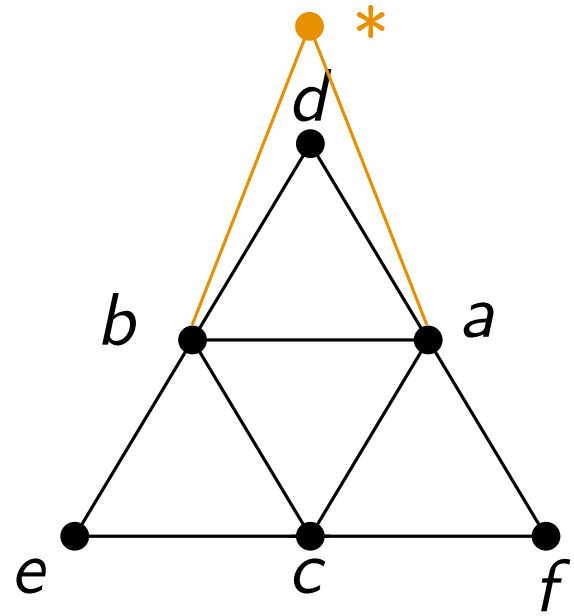

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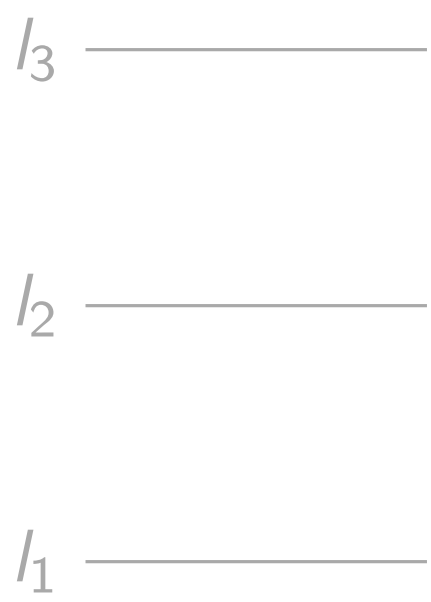

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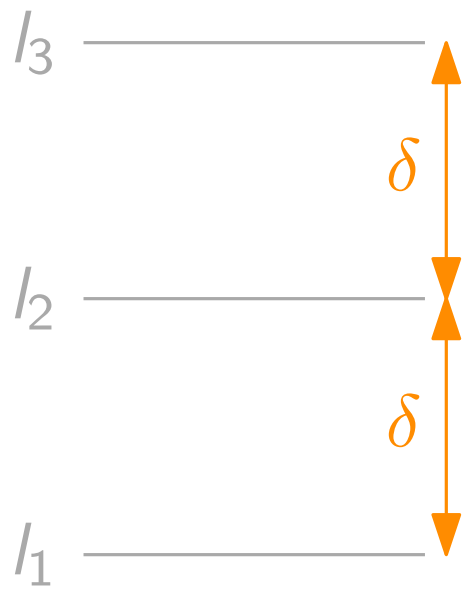

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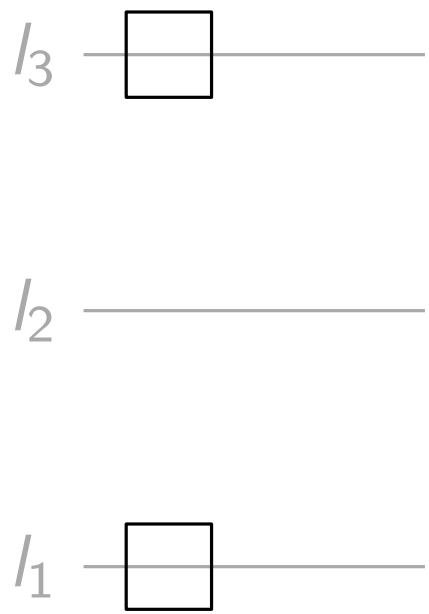

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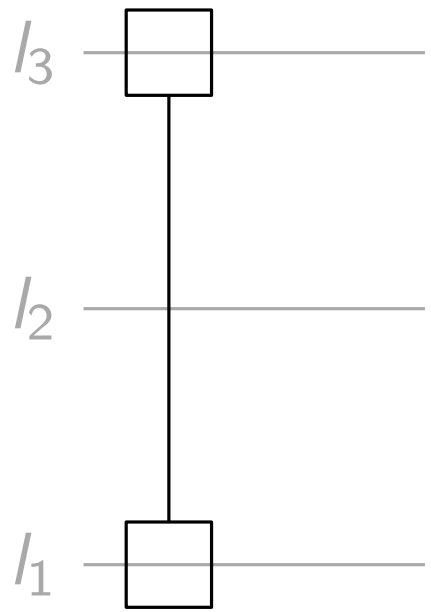

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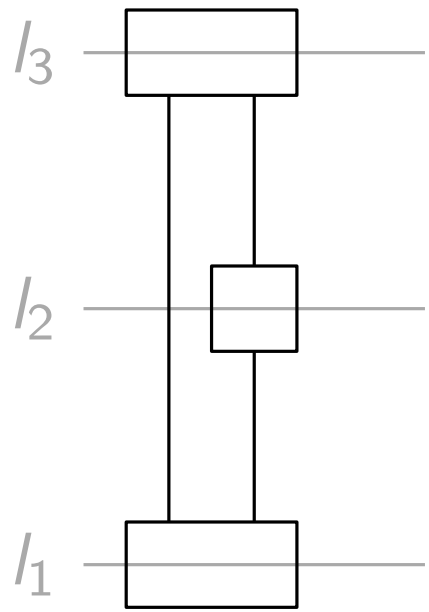

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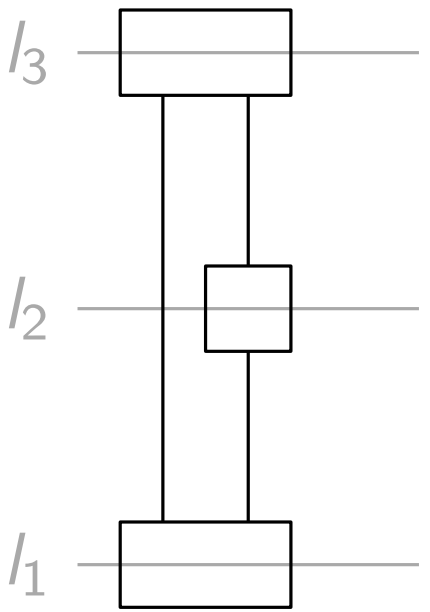

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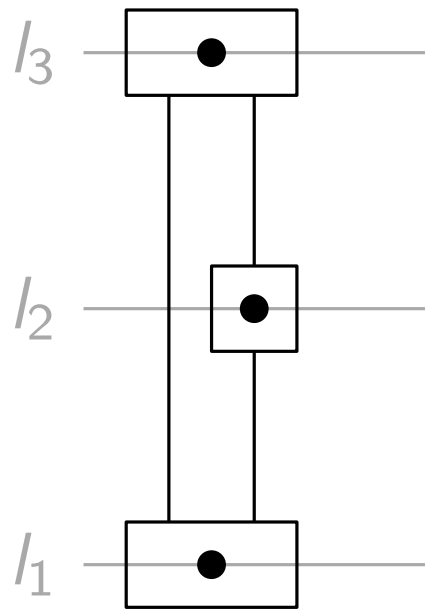
- Traverse T and draw G in a *layered box drawing* B
  - Layering property: If $(v, w) \in E(G)$, then v and w do not share the same y value in drawing
  - Minimal distance δ between layers induces minimal Euclidian distance
  - *Box drawing*: vertices as axis-aligned boxes with minimal width and height of 2
  - Edges only vertically
  - Vertex insertions in free, *reachable* layers
  - Box drawings transferable to polyline drawings with same area bounds, 2 bends



Approach II: Scheme

- Make outerplanar graph G maximal, if necessary
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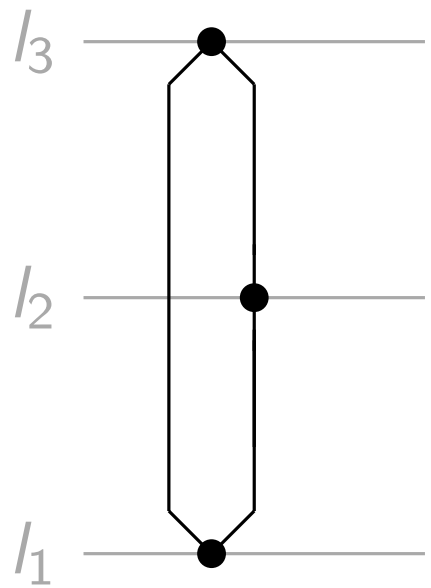

- Transfer B into a polyline drawing, remove edges

Vertex Insertions In Box Drawing

- Vertices inserted on layers as extendable boxes

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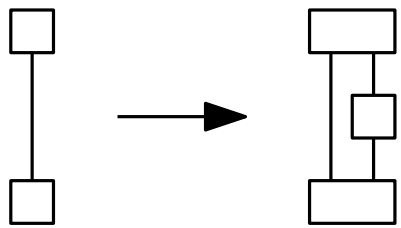

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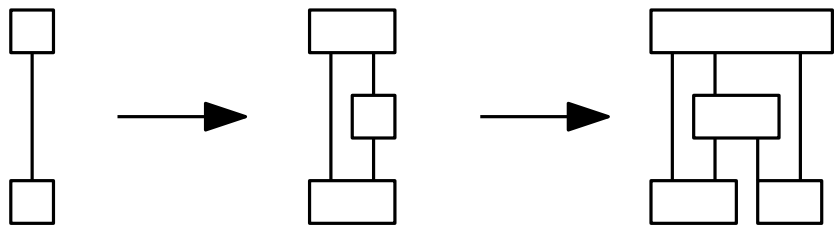

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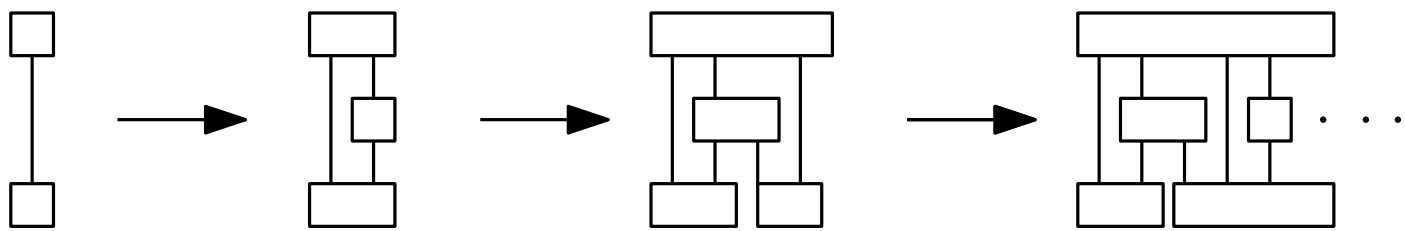

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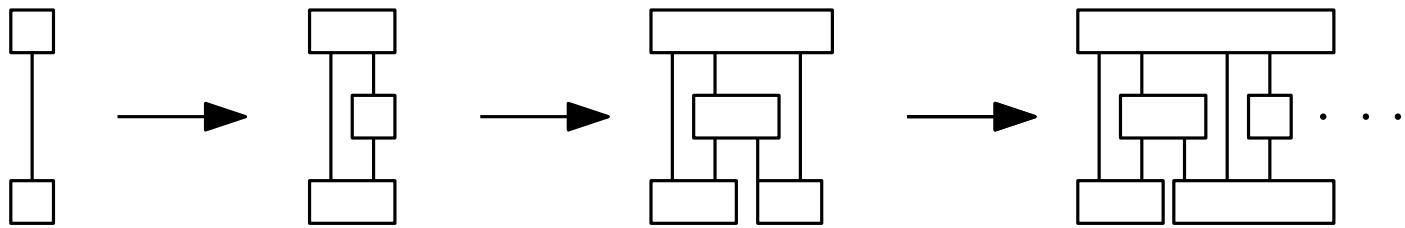

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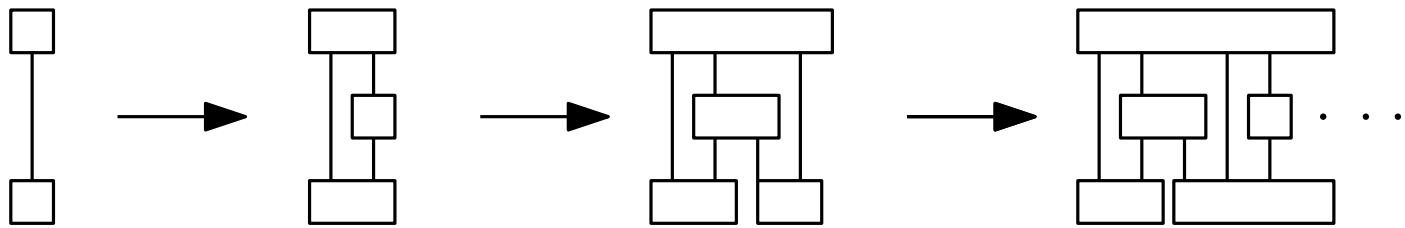

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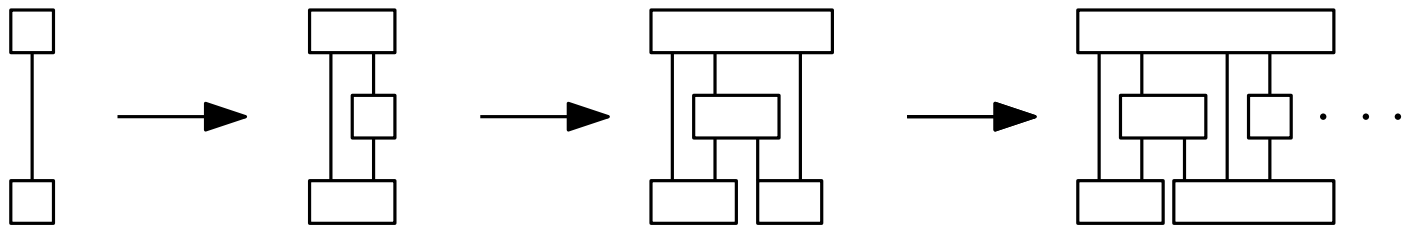

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  - Amount of layers determines the height and the ratio of the drawing
- A layer is *reachable* if it is free, v can be placed on it and edges to a, b can be drawn without destroying planarity

Reachability of Layers

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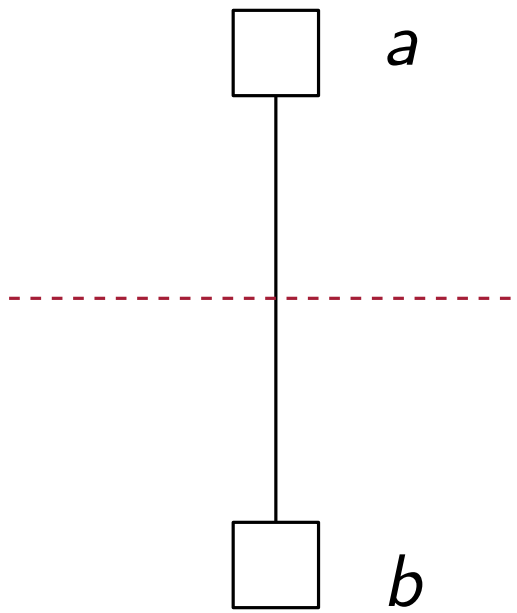

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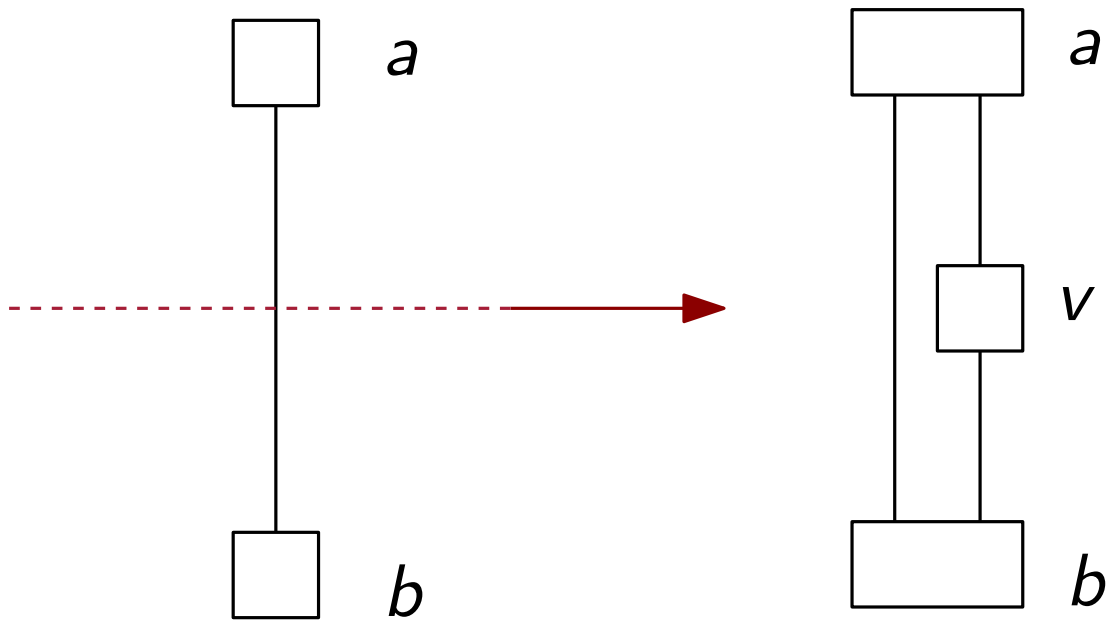

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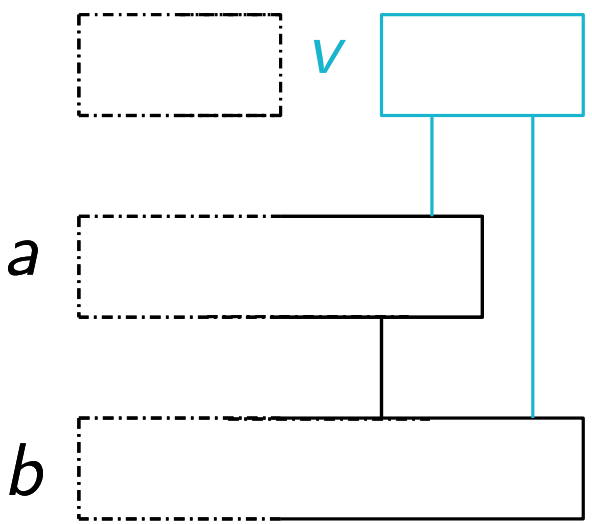

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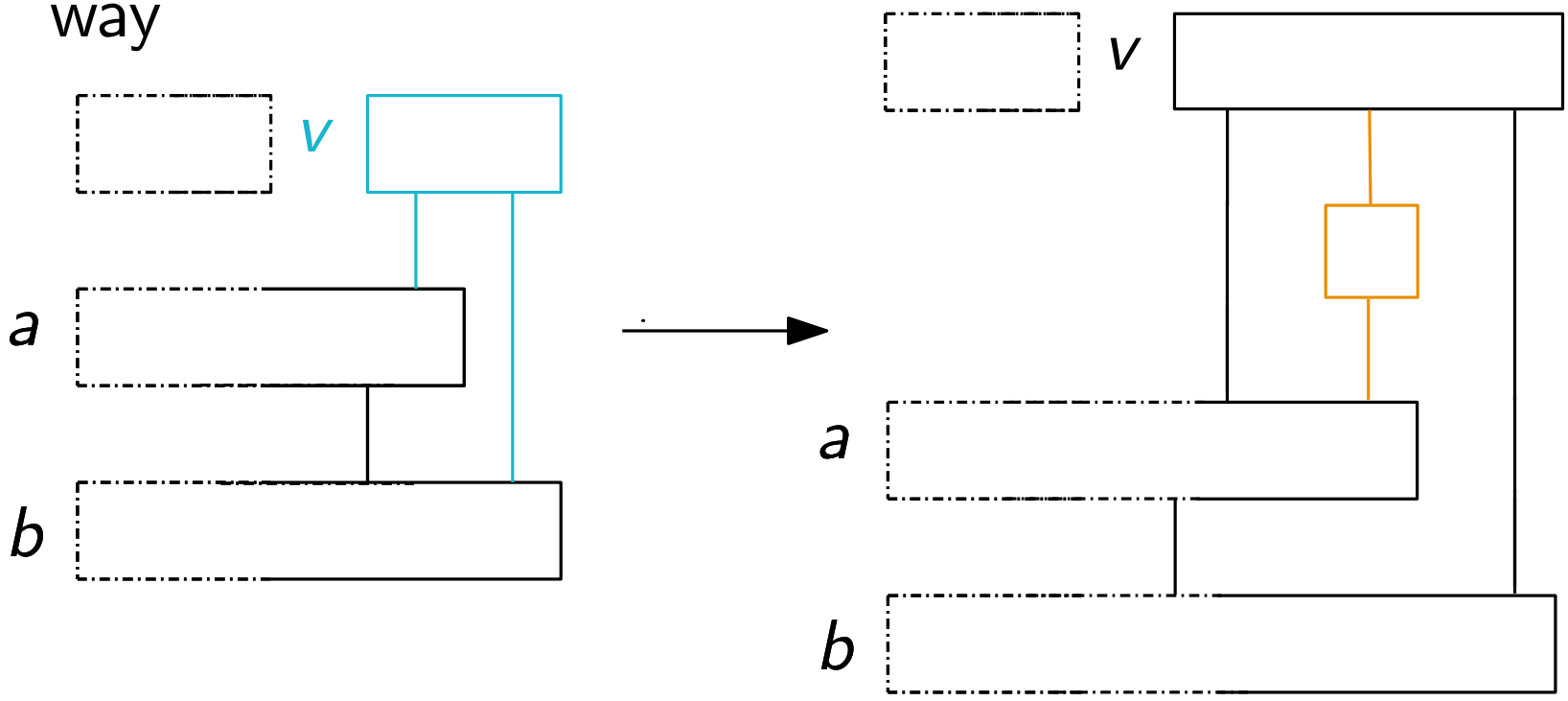

  - Subsequent vertex insertions between v and a
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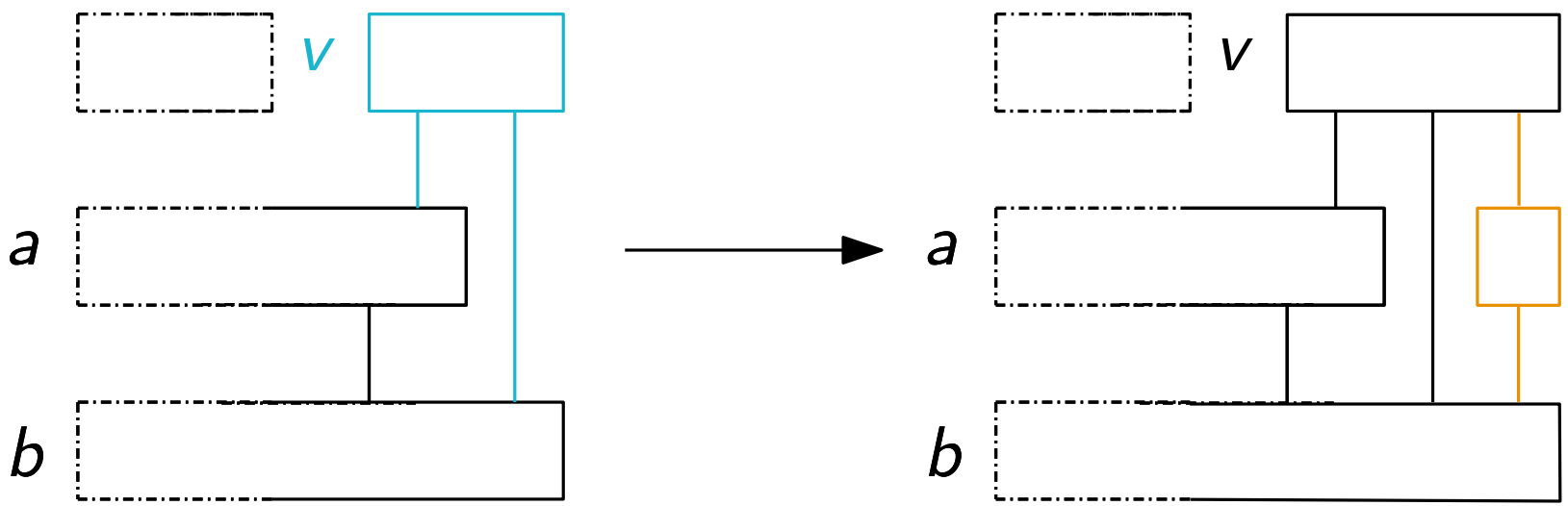

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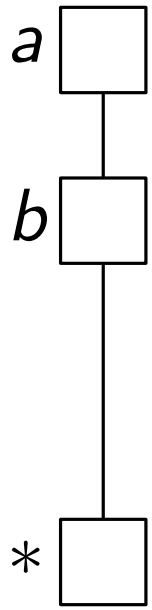

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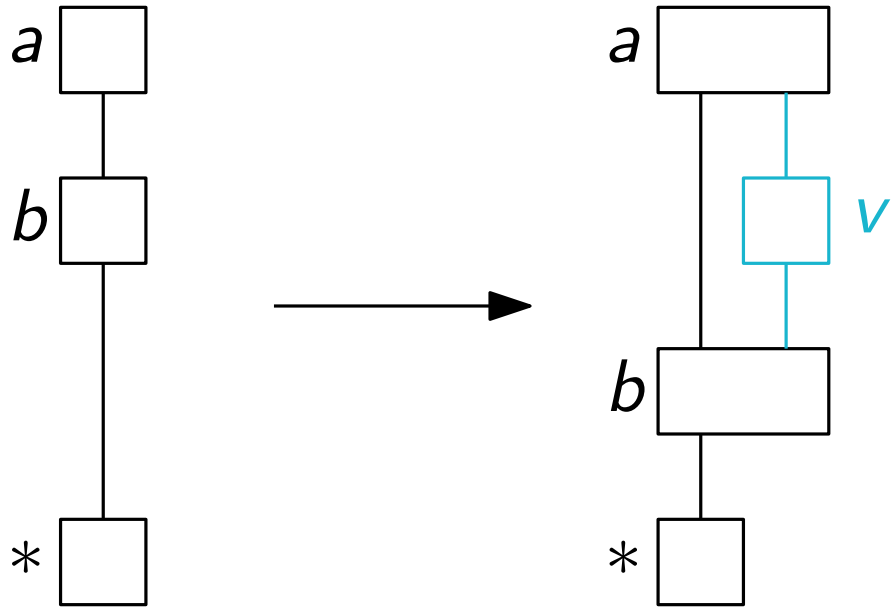

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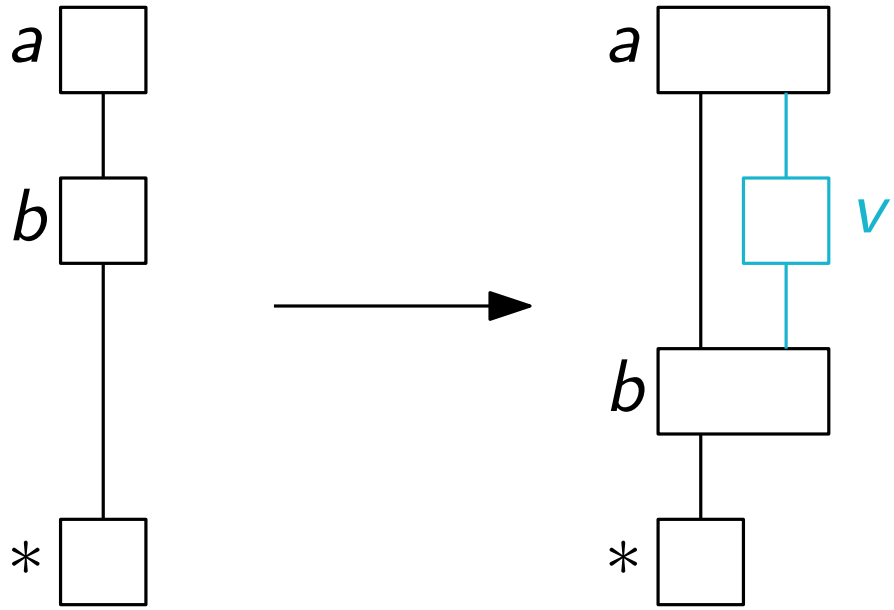

- Then, either case 1 or case 2 will apply
- If neither of those cases possible, insert a new layer between a and b

Traversal Of Tree Decomposition

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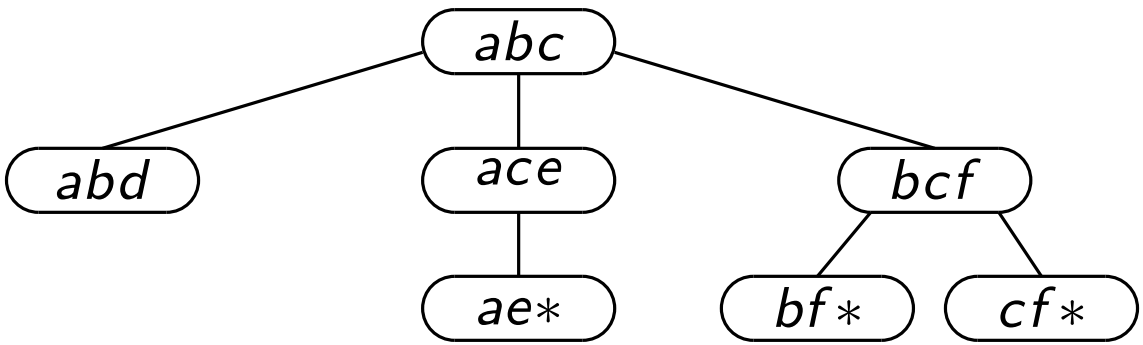

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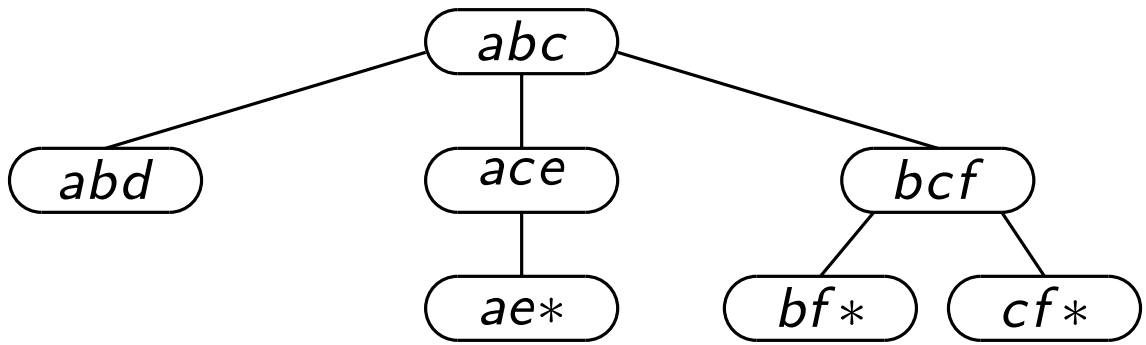

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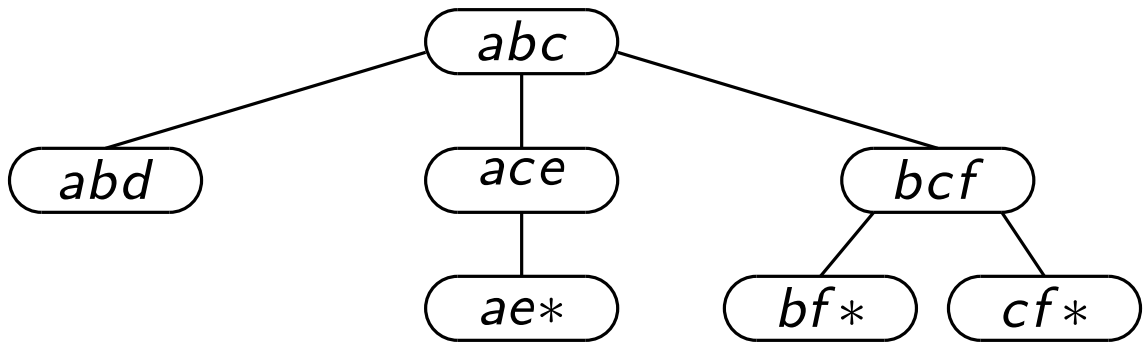

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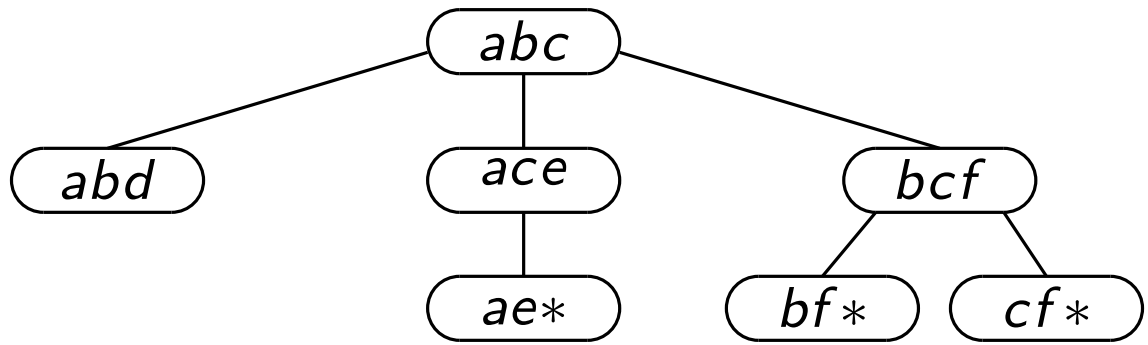

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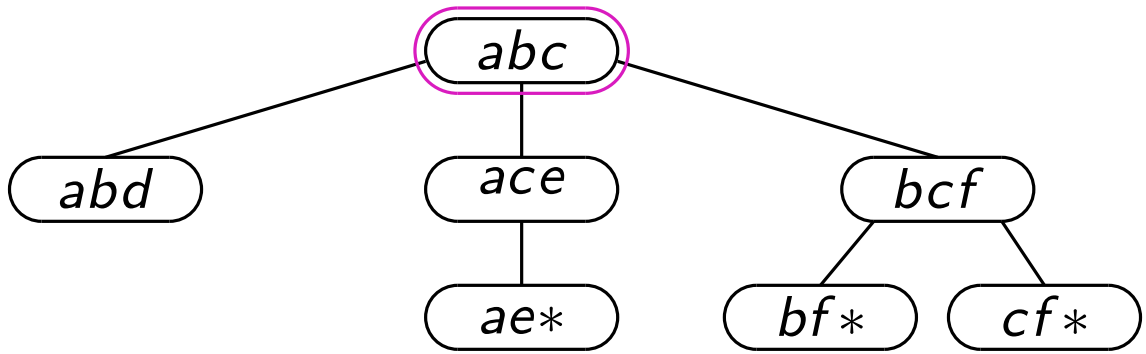

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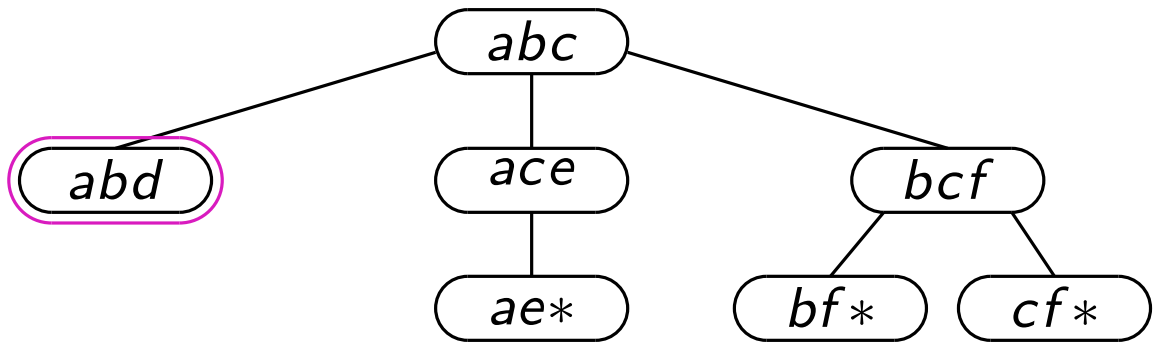

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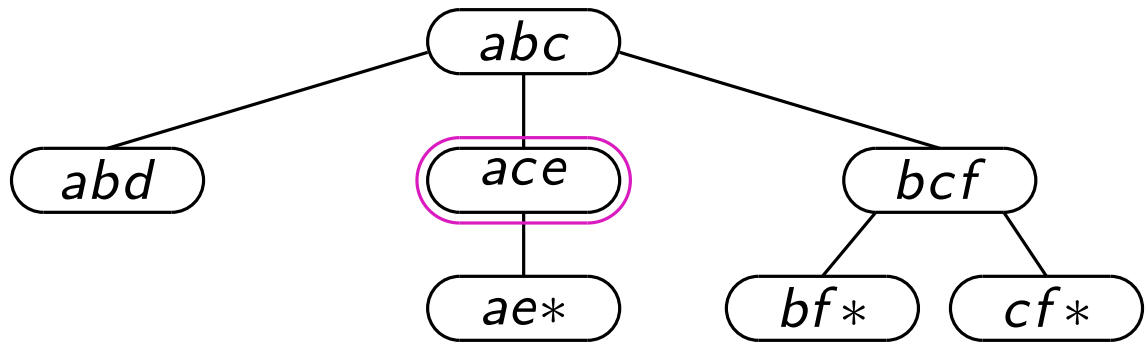

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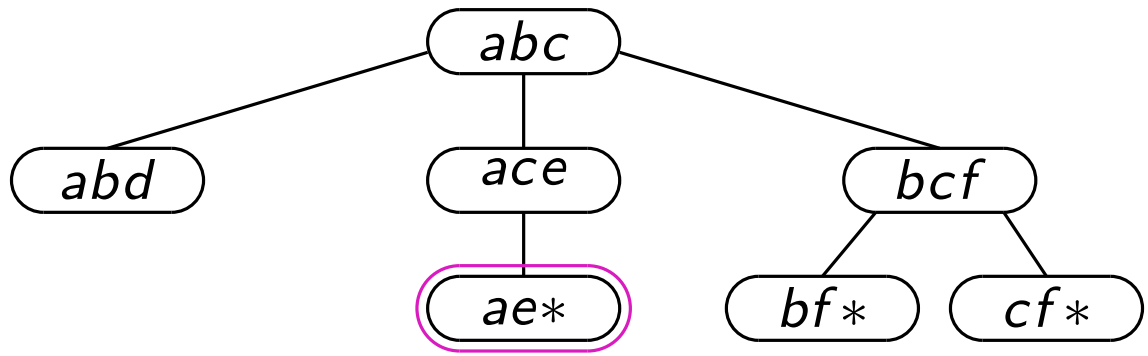

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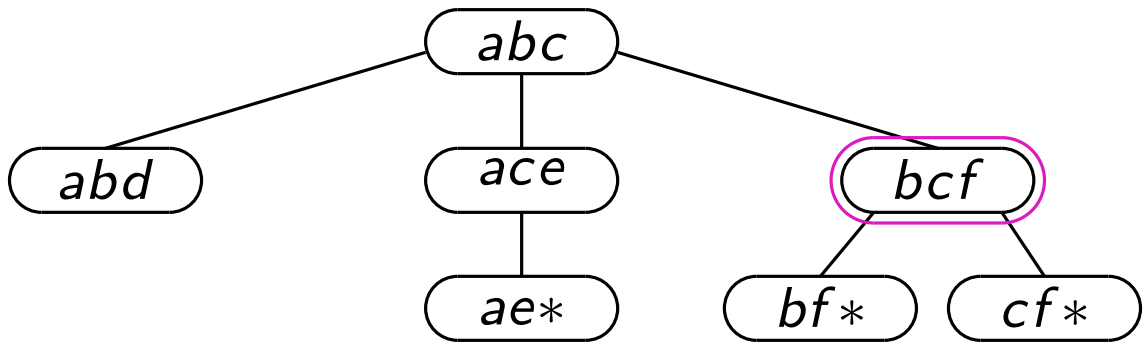

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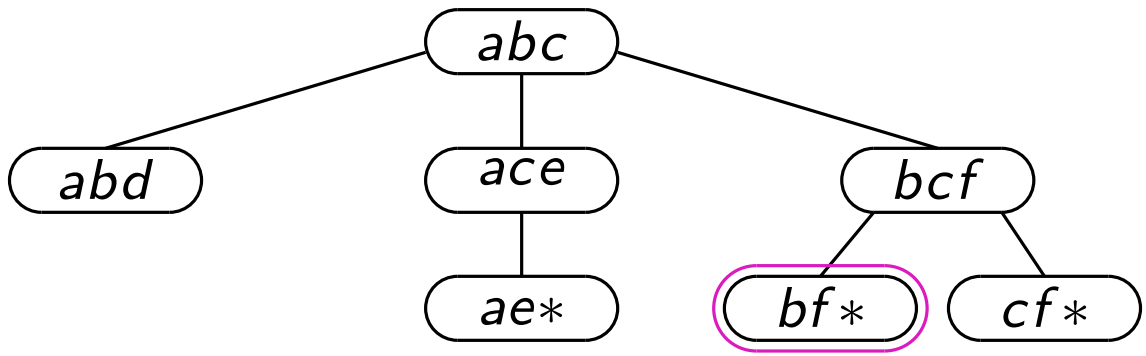

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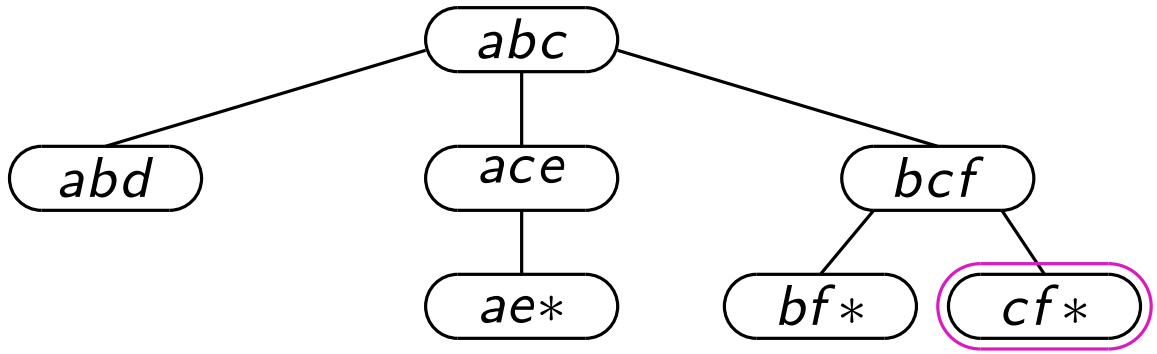

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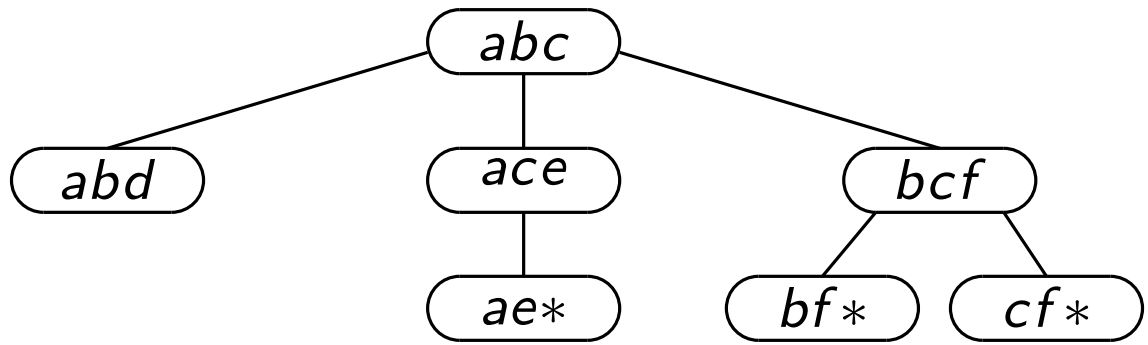

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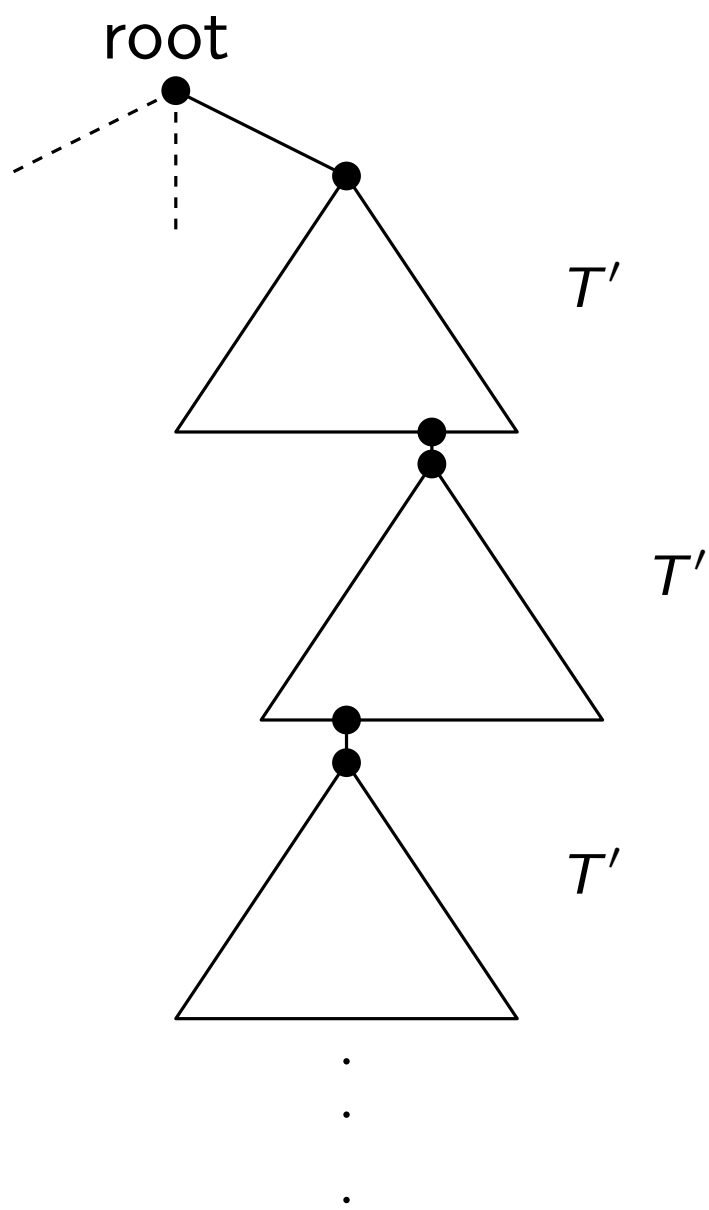

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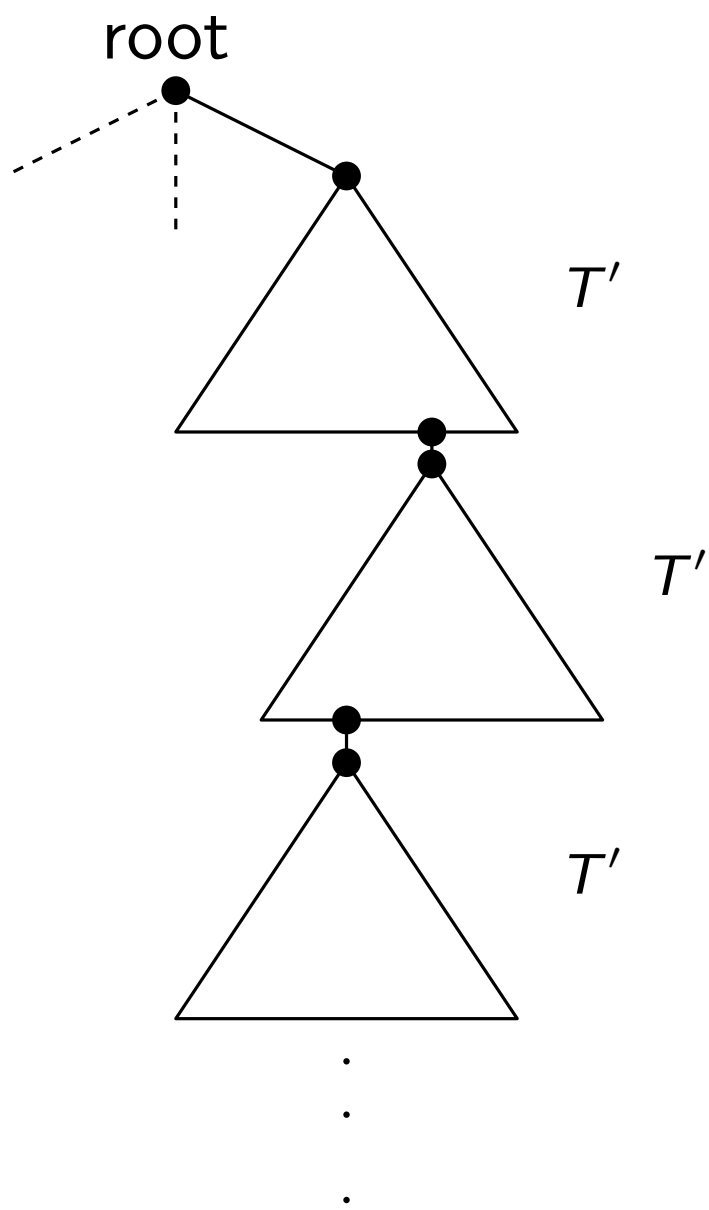

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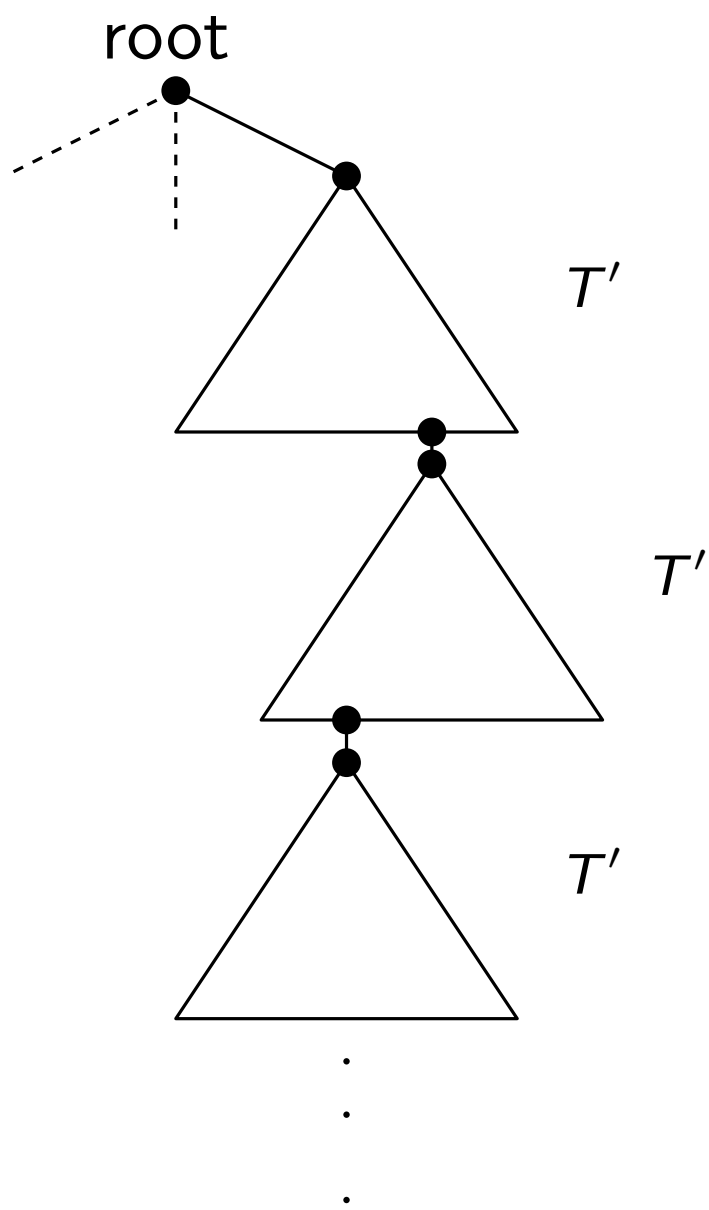

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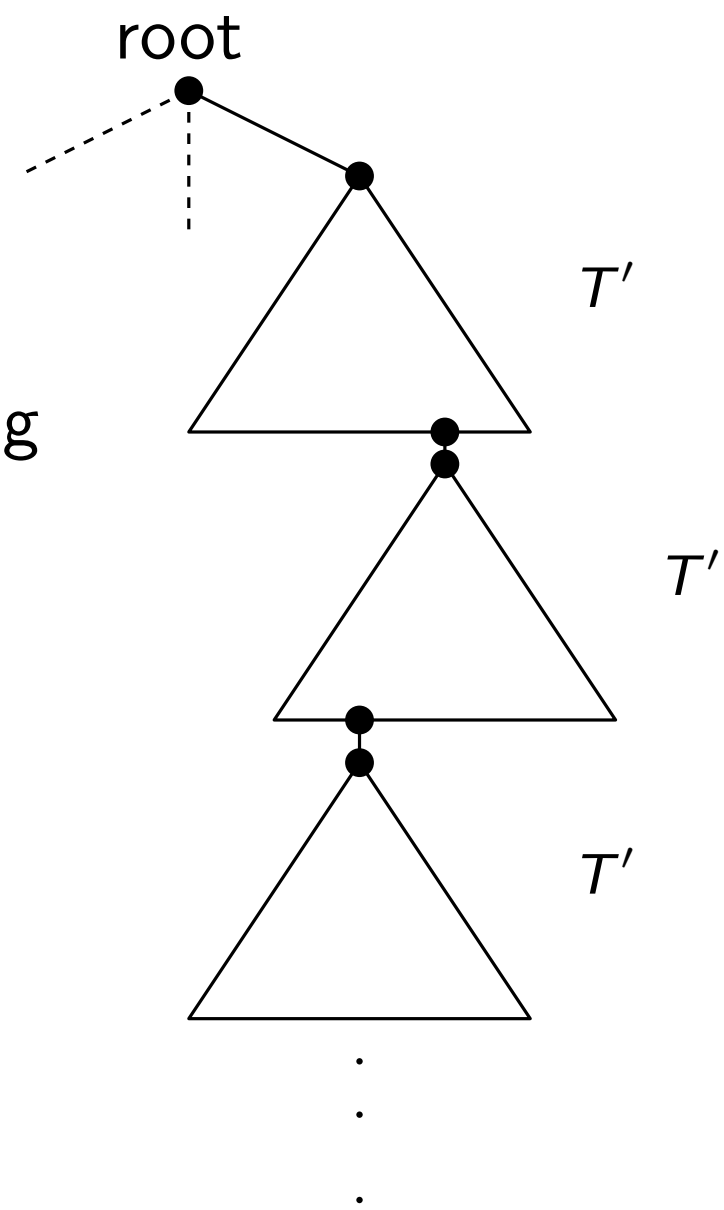

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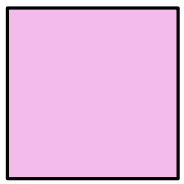

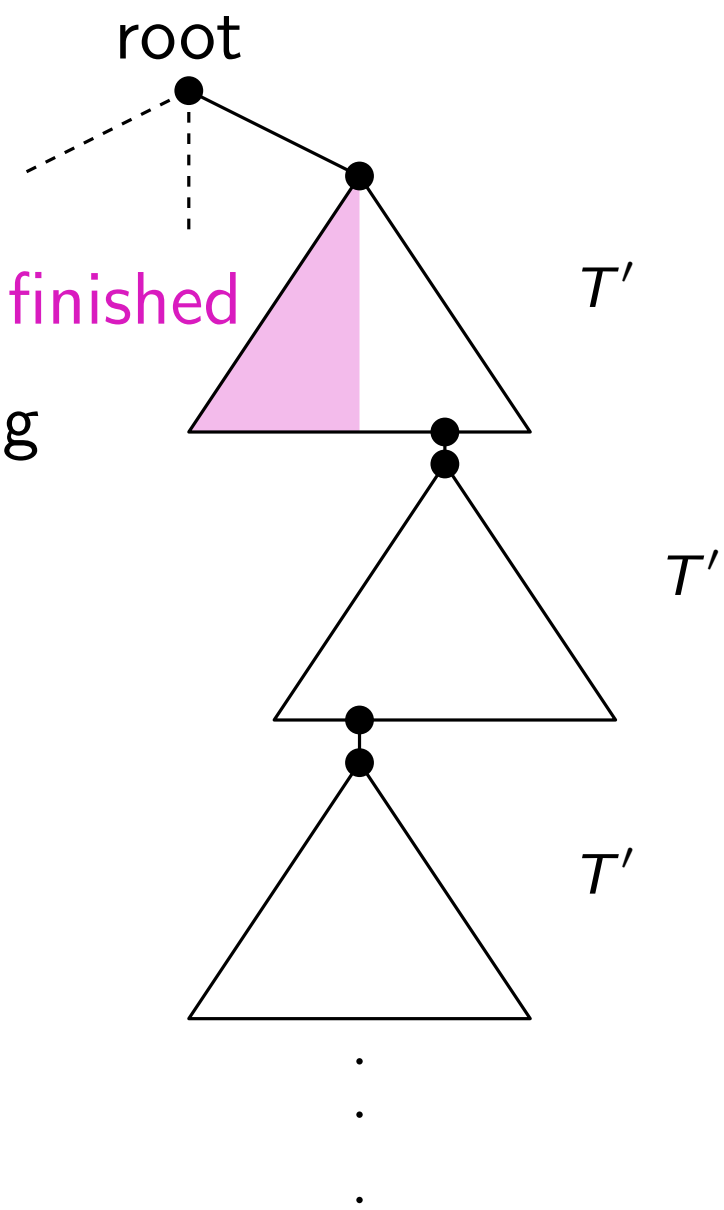

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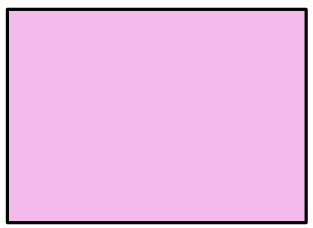

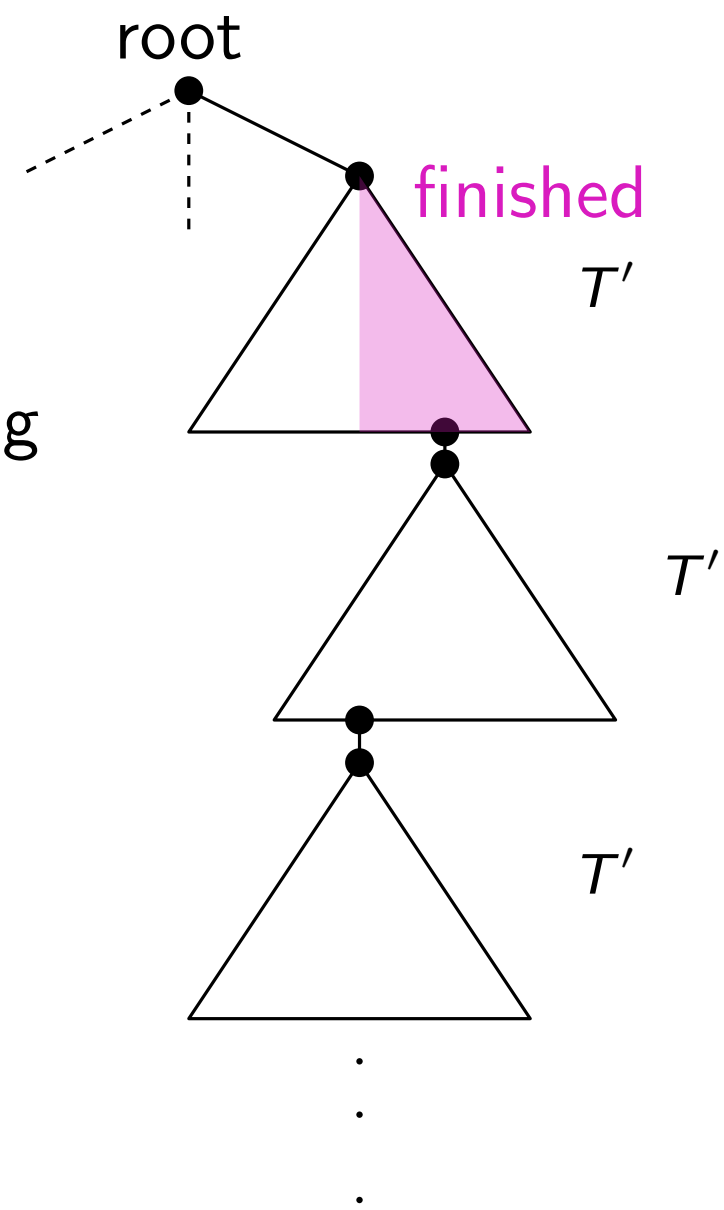

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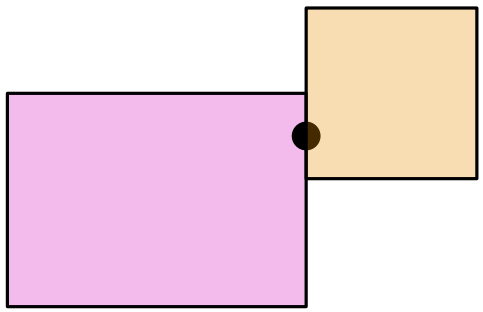

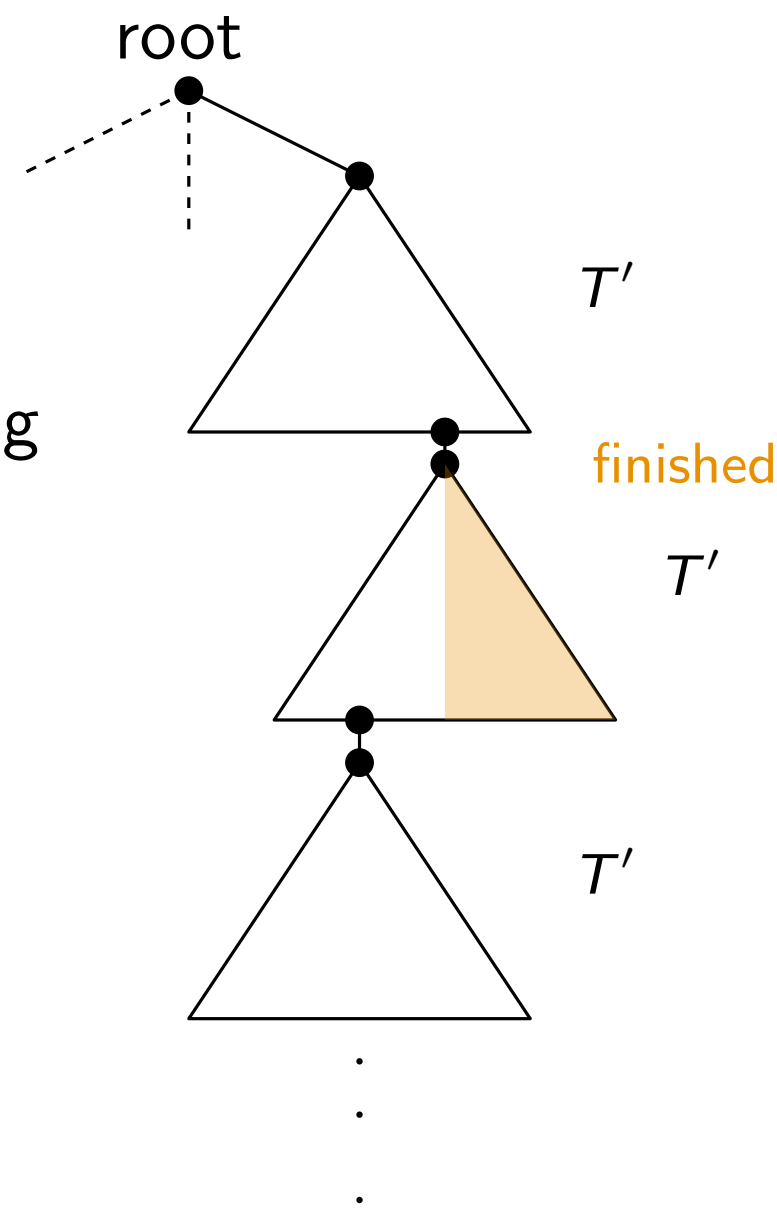

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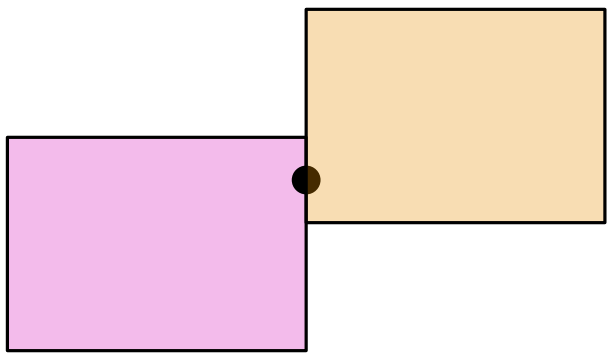

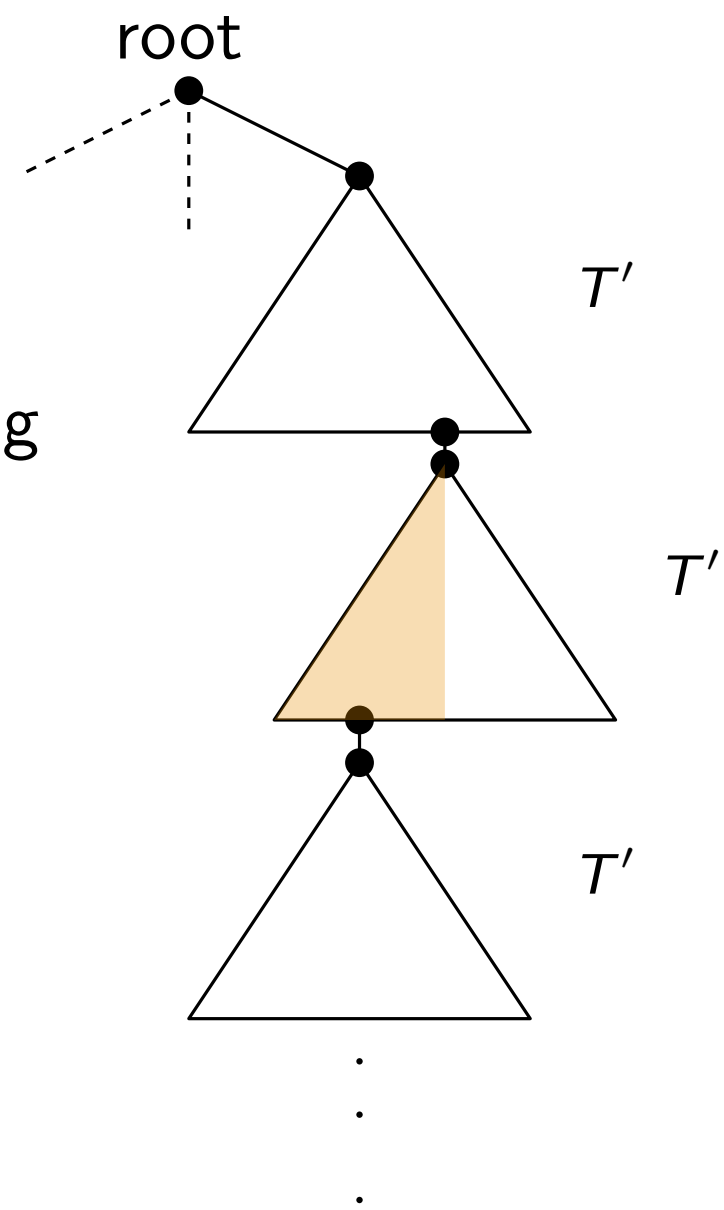

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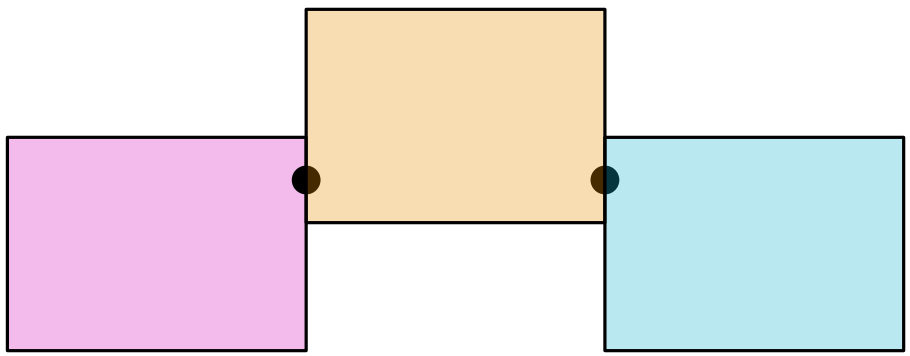

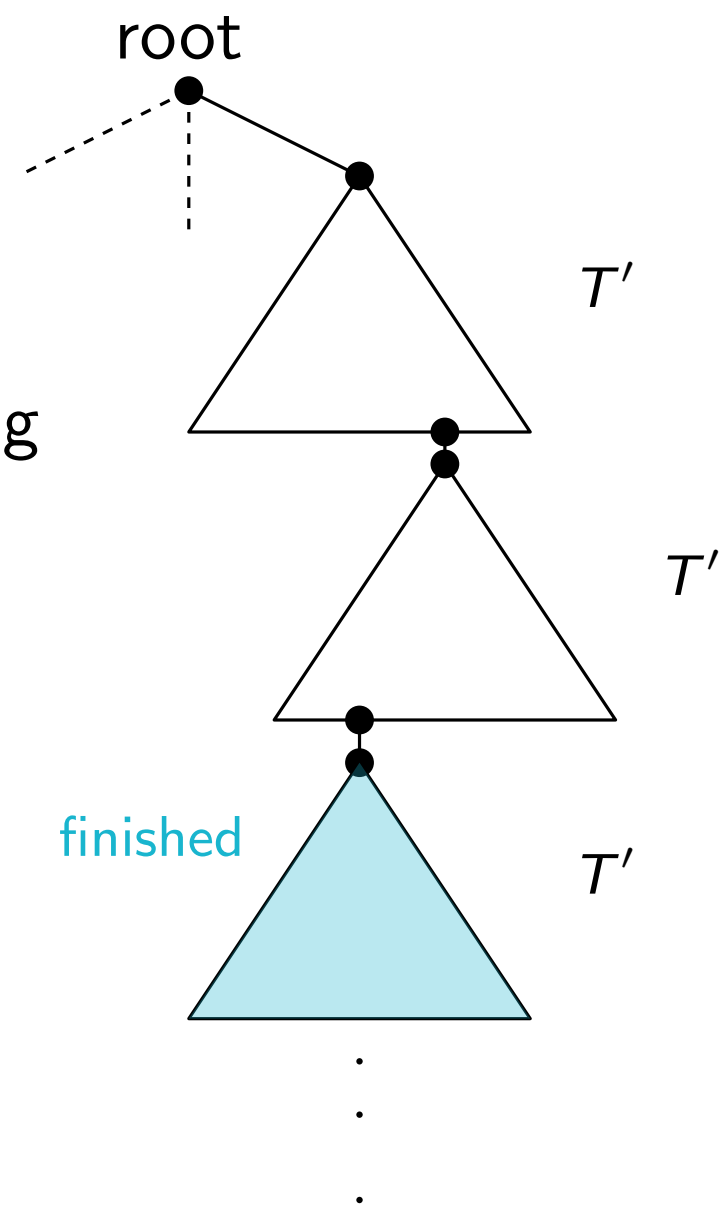

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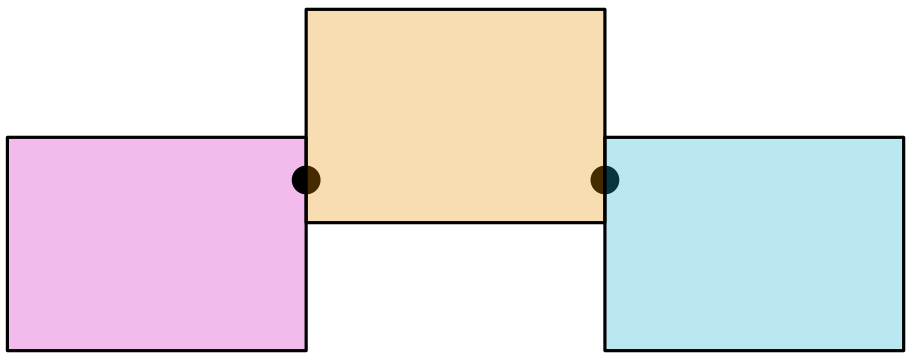

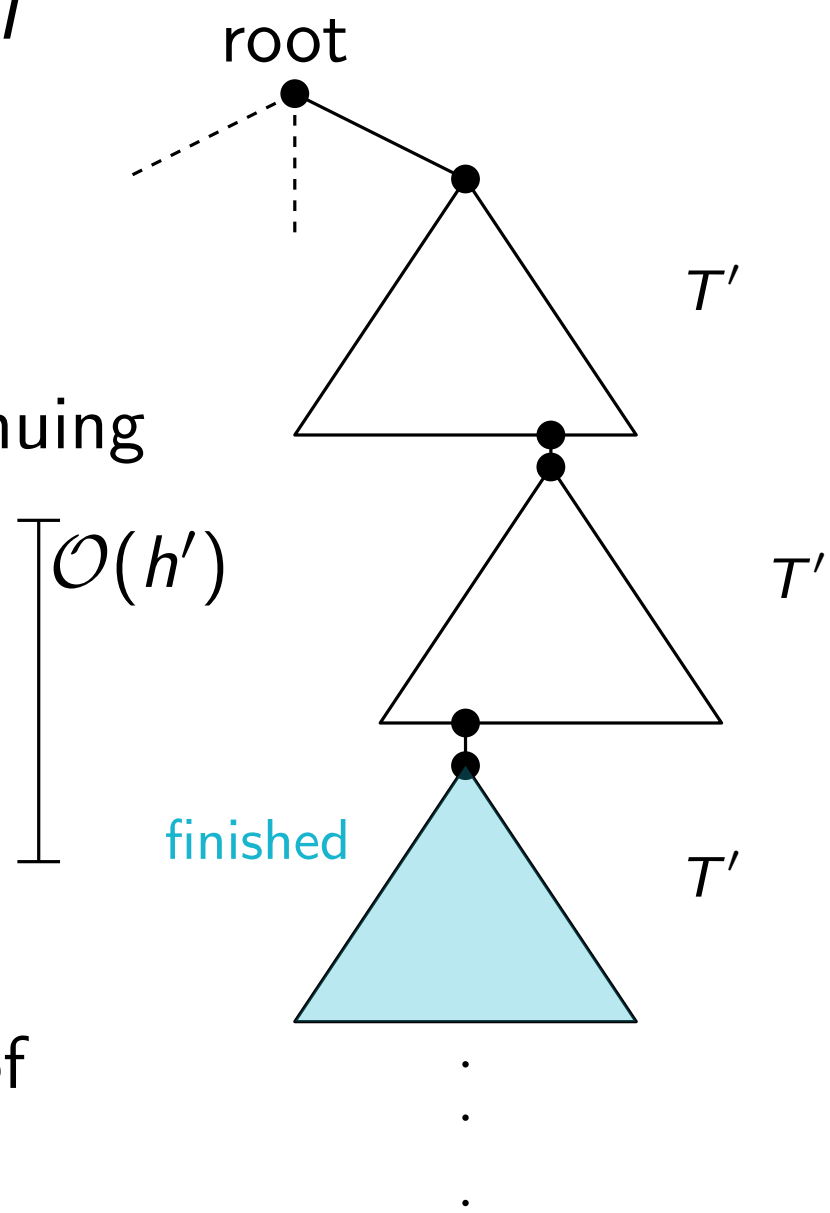

- Total amount of layers equals constant multiple of amount of layers induced by drawing T'

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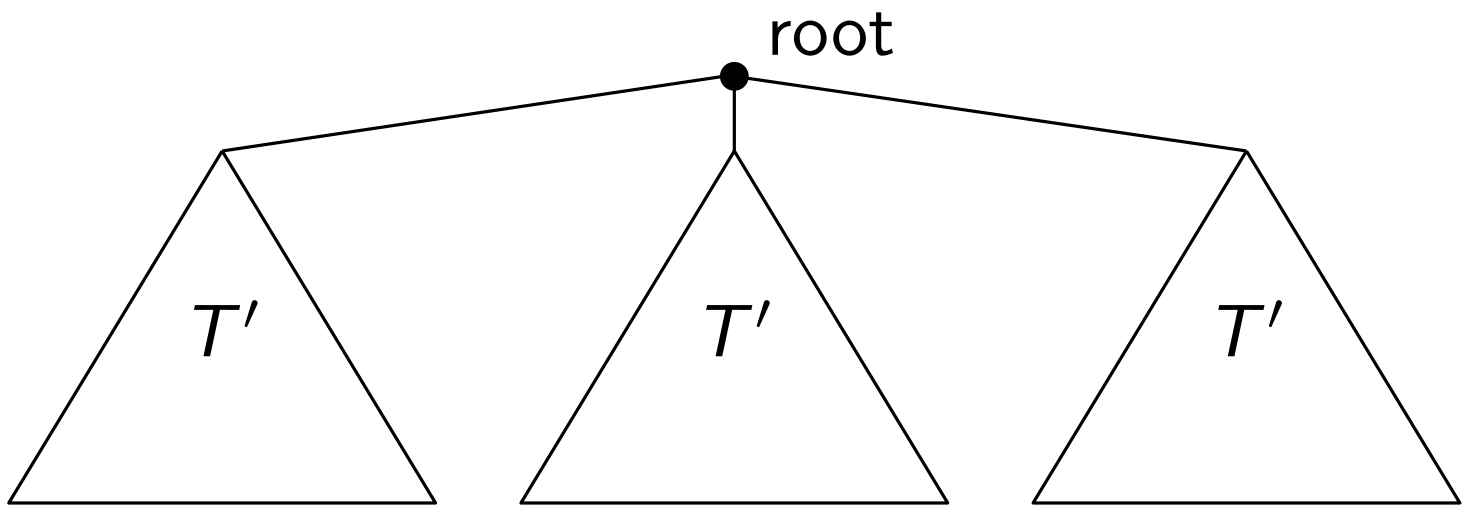

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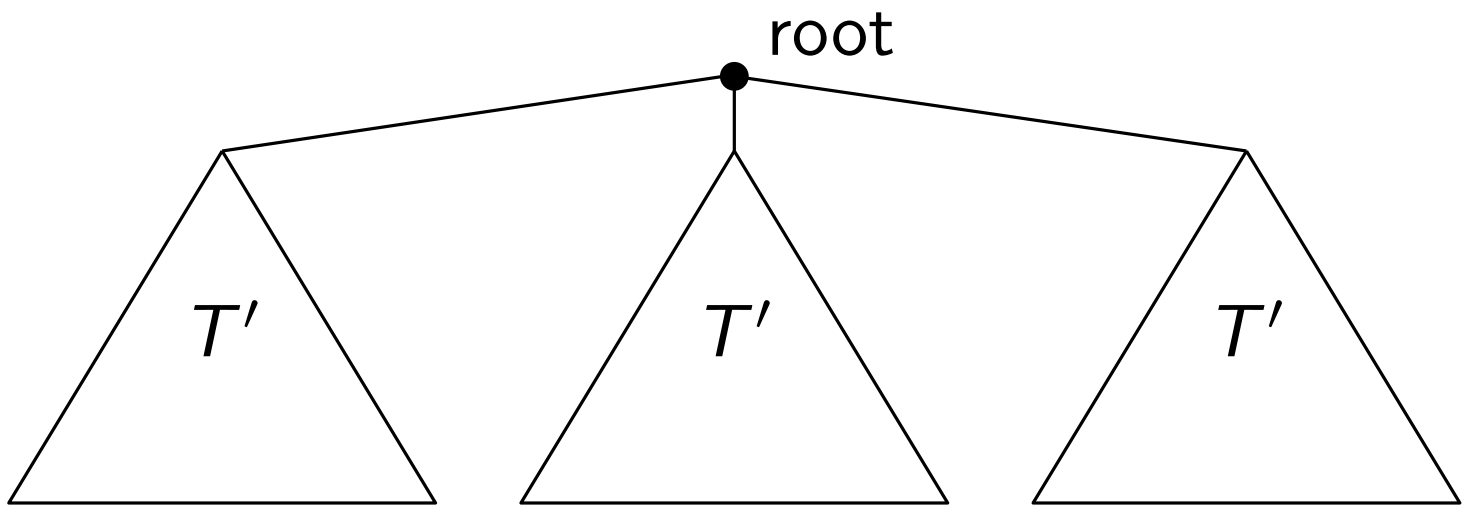

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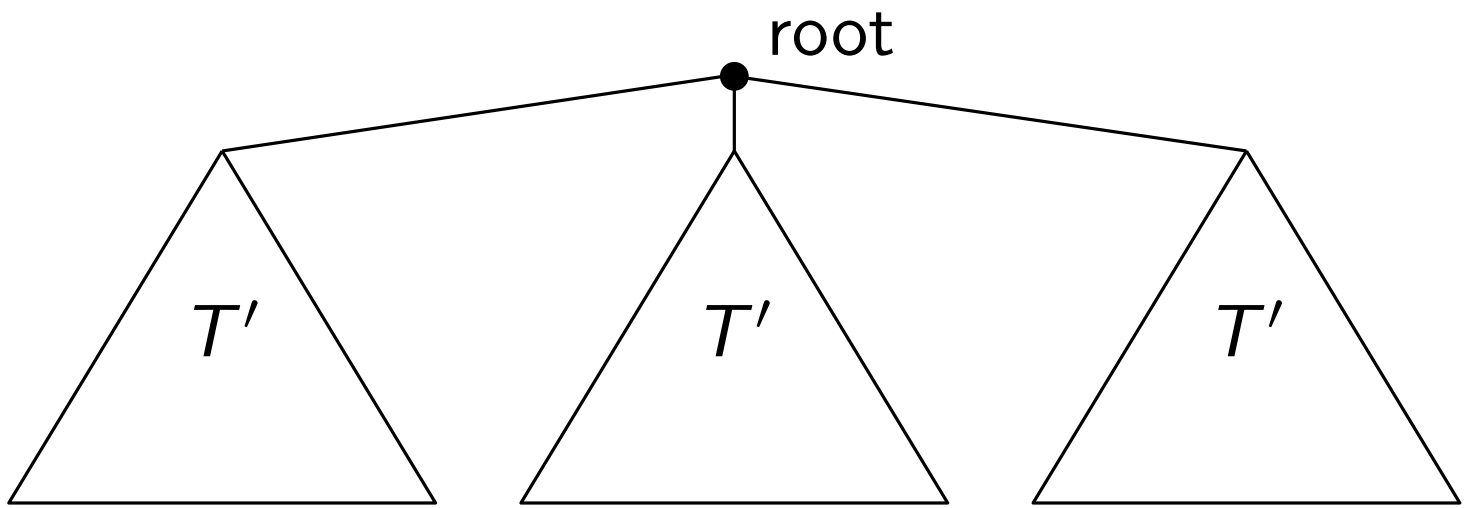

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Layer Amount For Complete Binary Subtree

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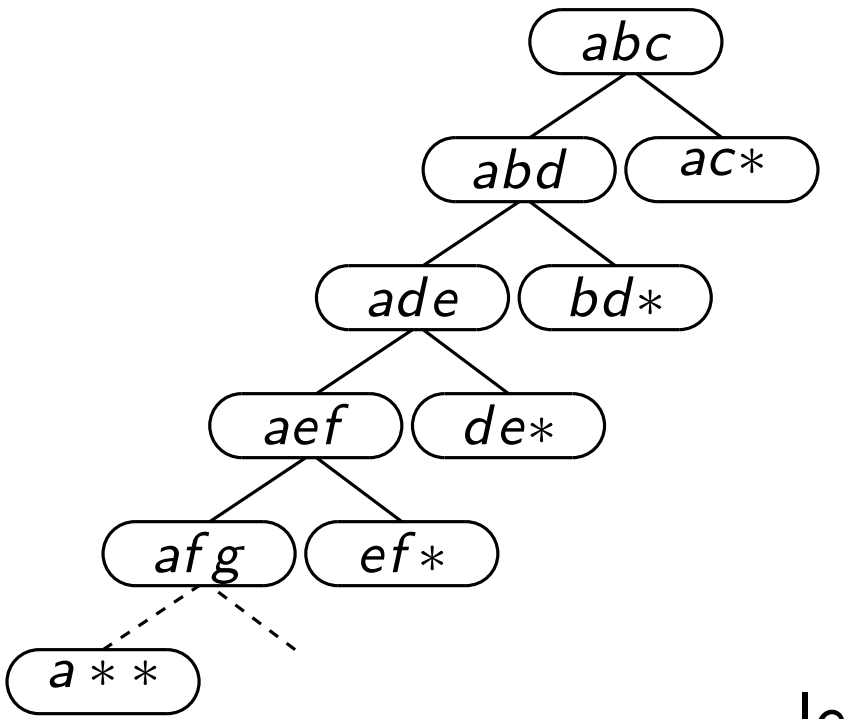

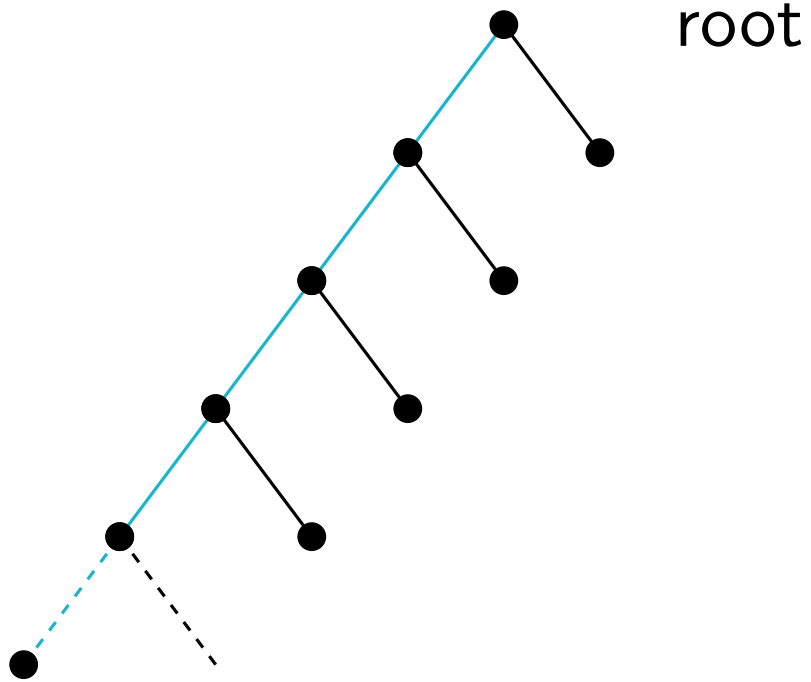

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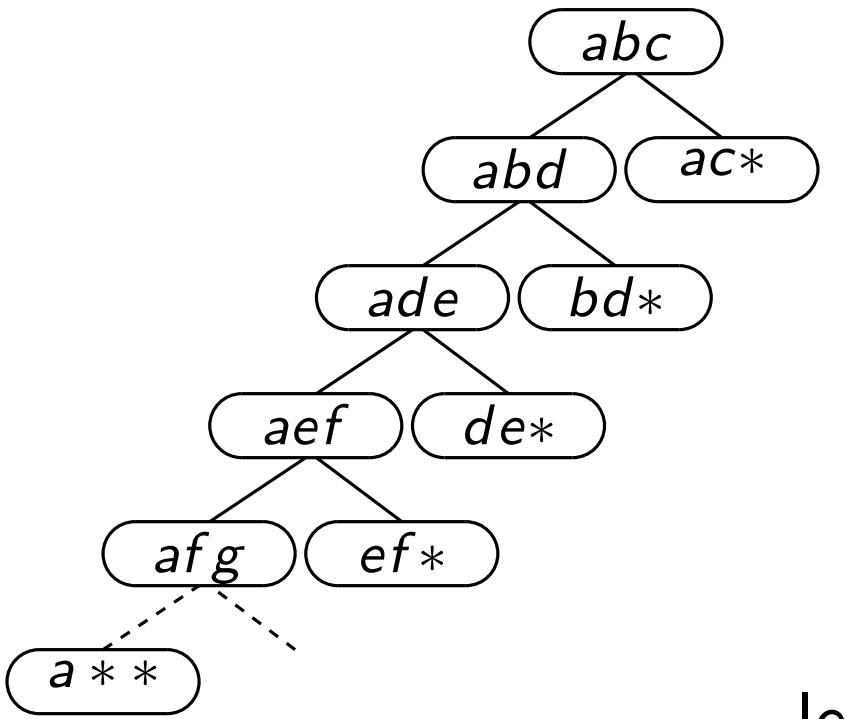

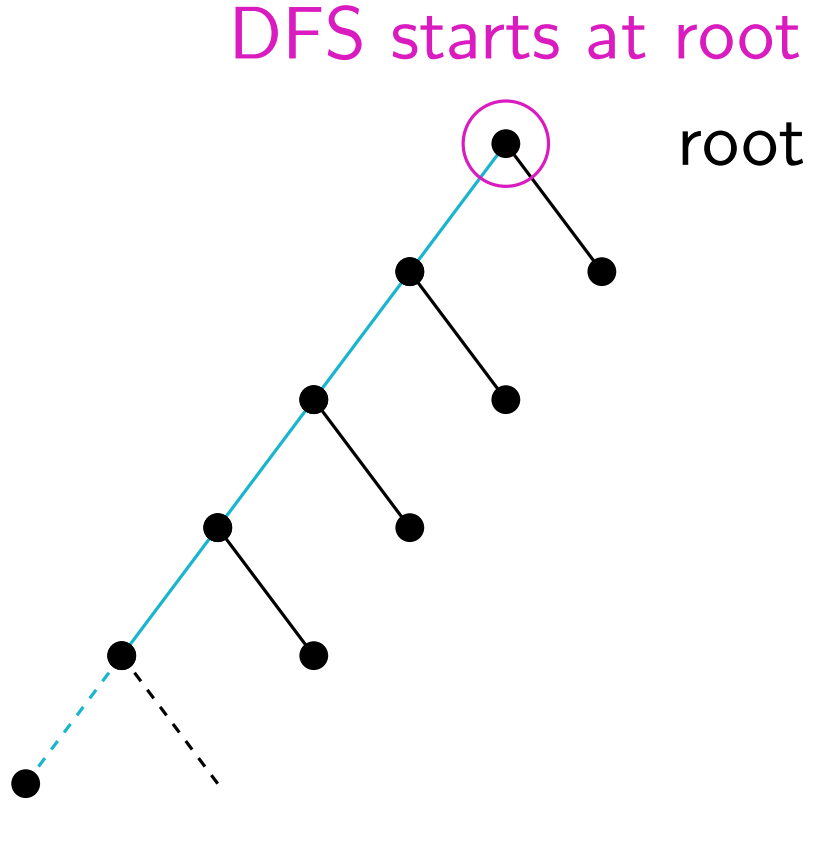

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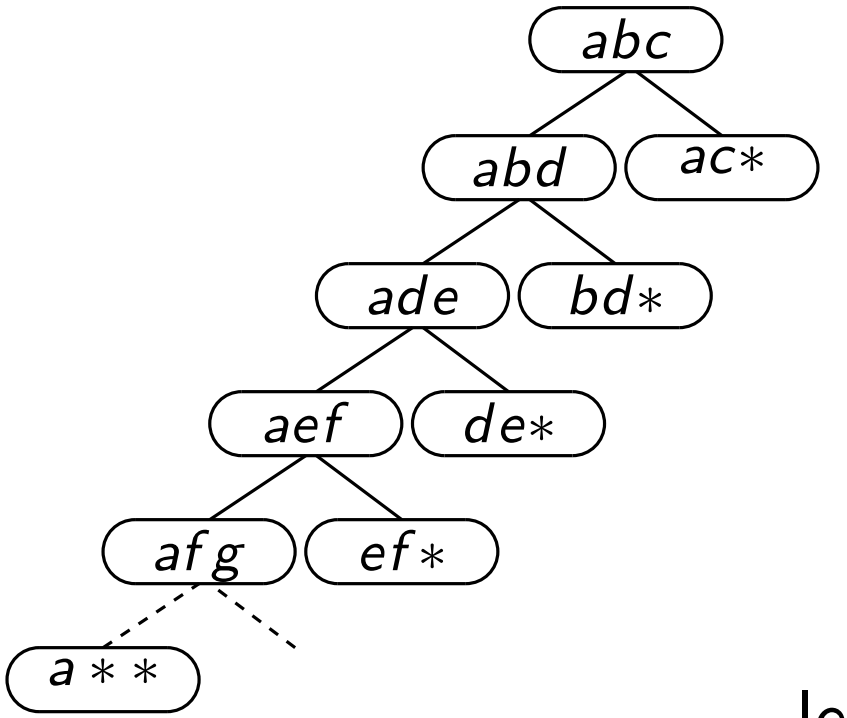

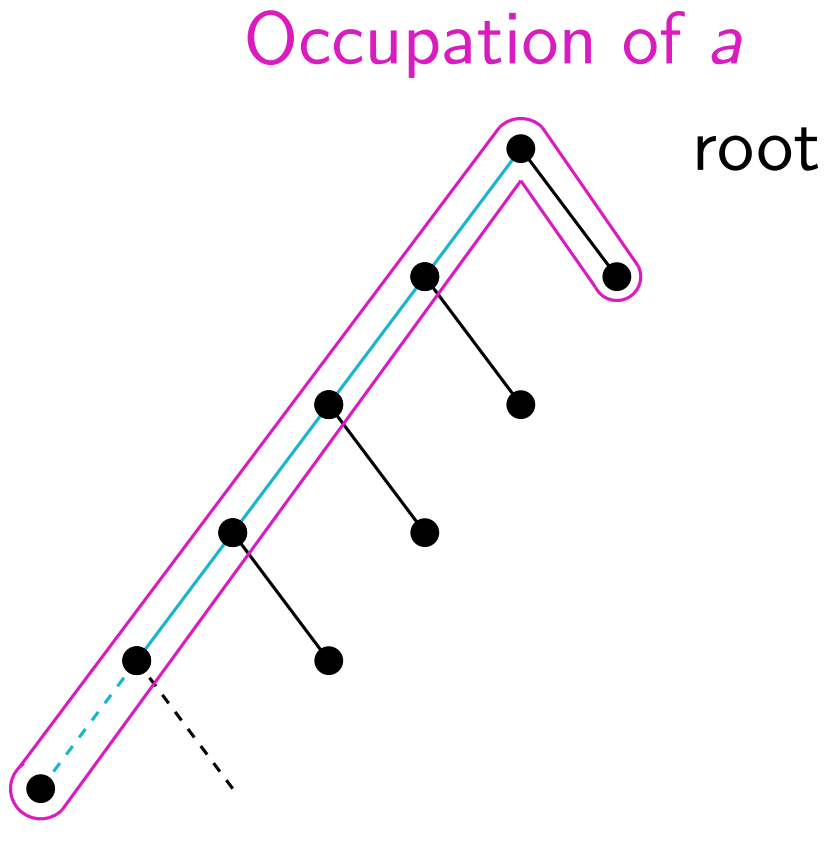

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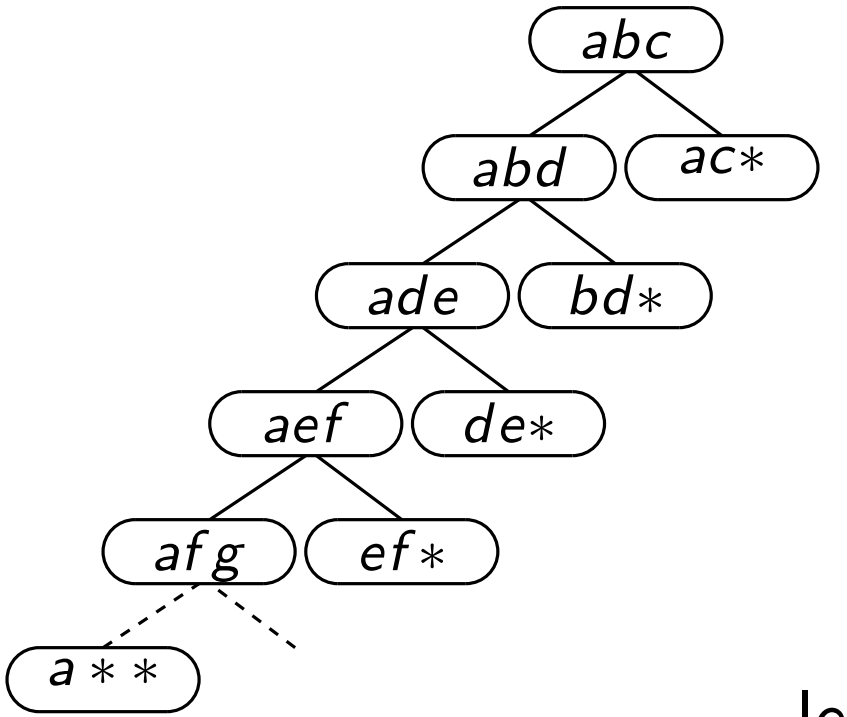

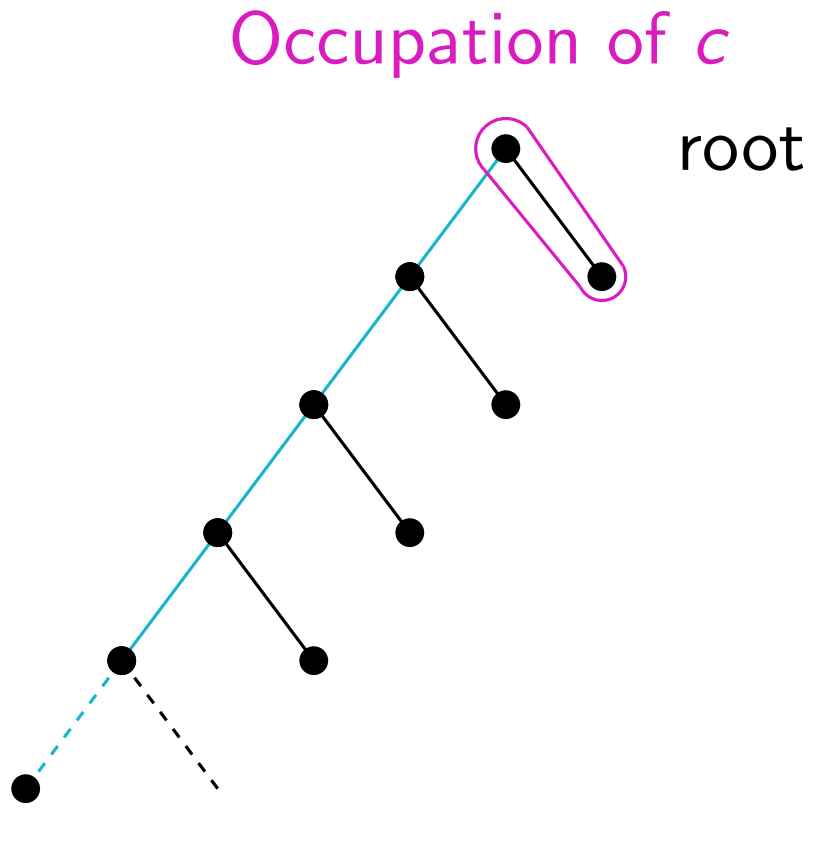

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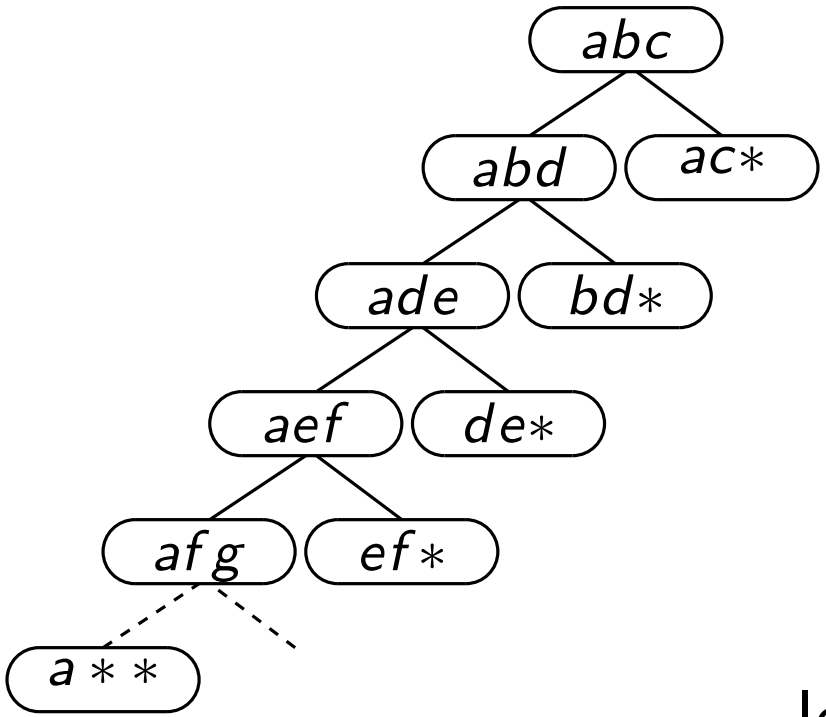
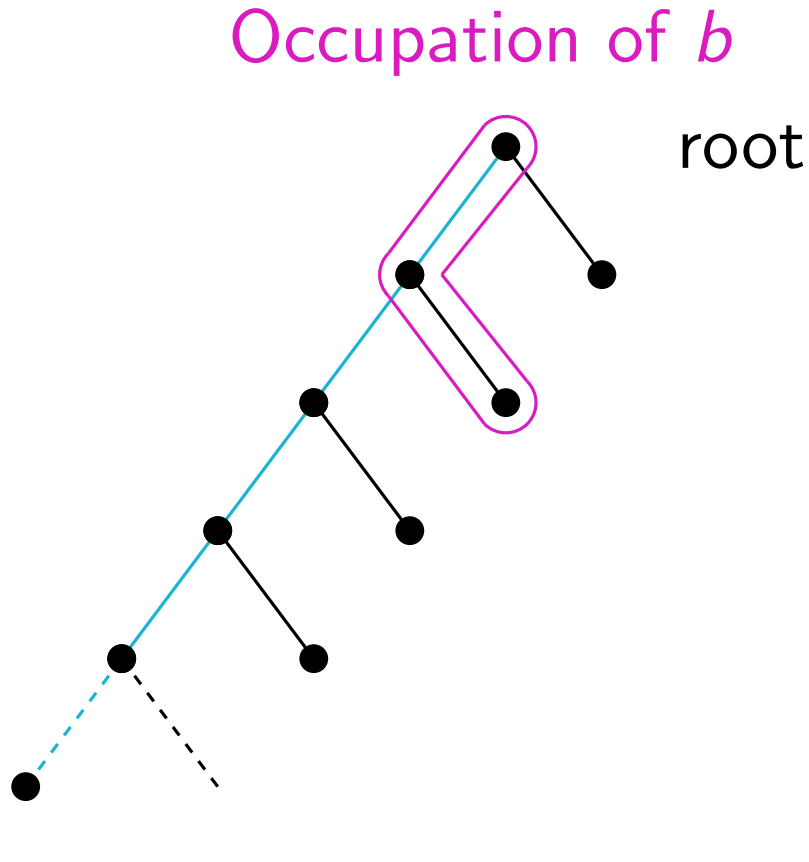
- In complete binary tree T', consider path from root to a leaf

`Amount of layers occupied` ≥ 3





Layer Amount For Complete Binary Subtree

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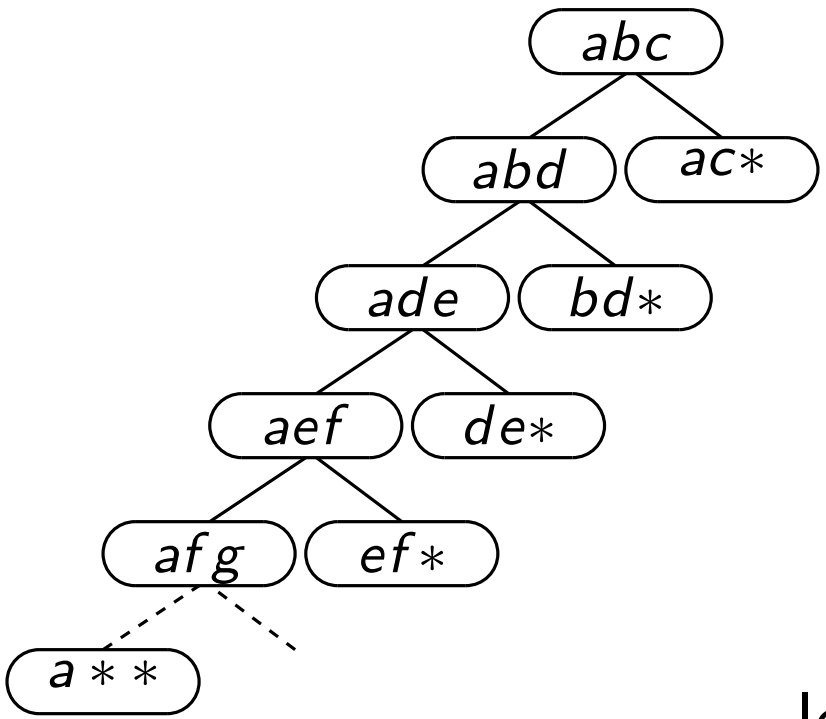
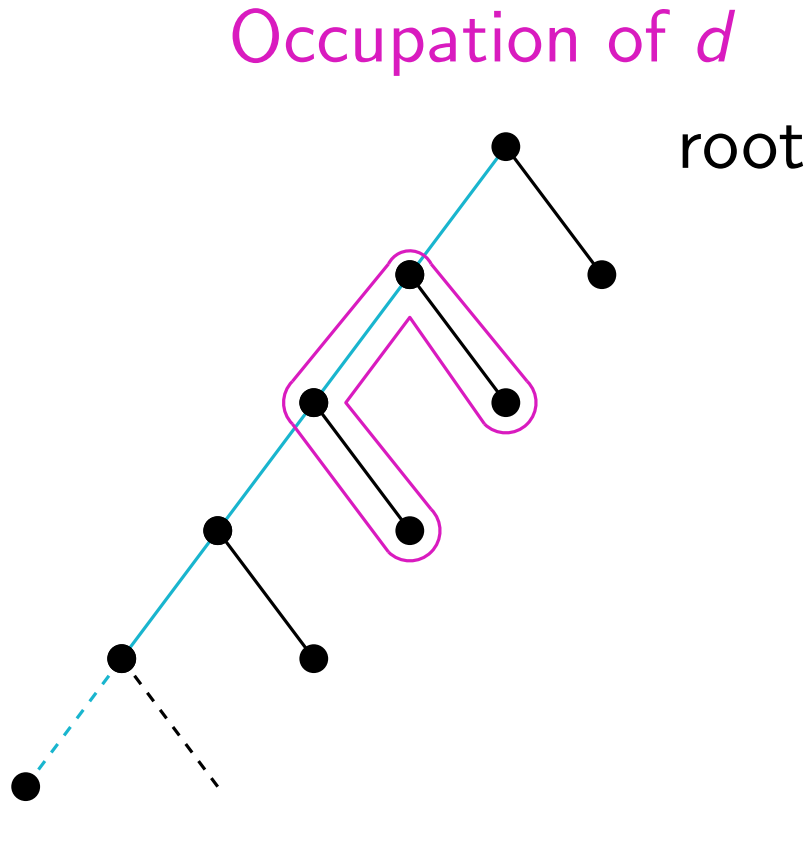
`Amount of layers occupied` ≥ 4





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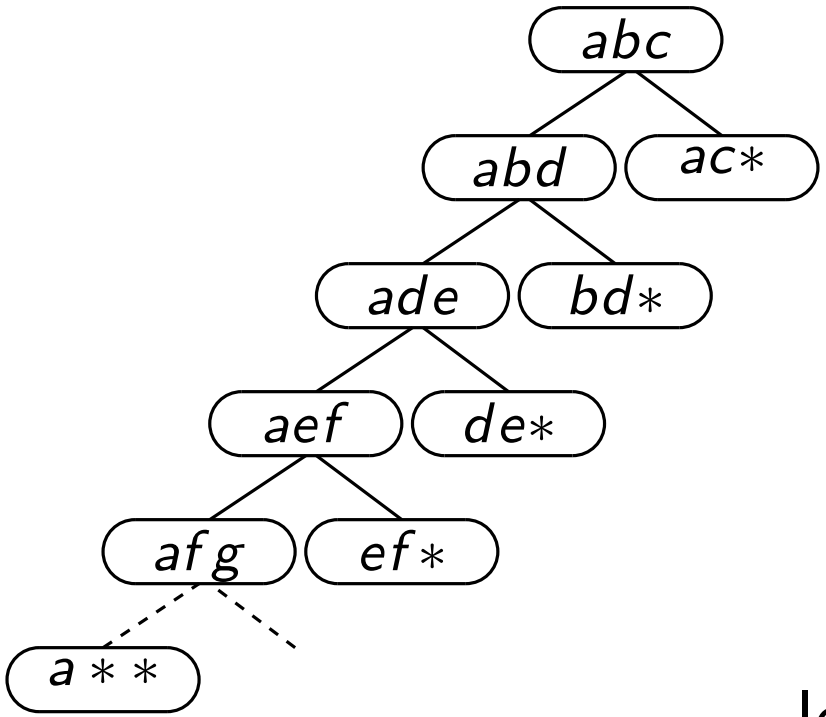
- In complete binary tree T', consider path from root to a leaf

`Amount of layers occupied` ≥ 5



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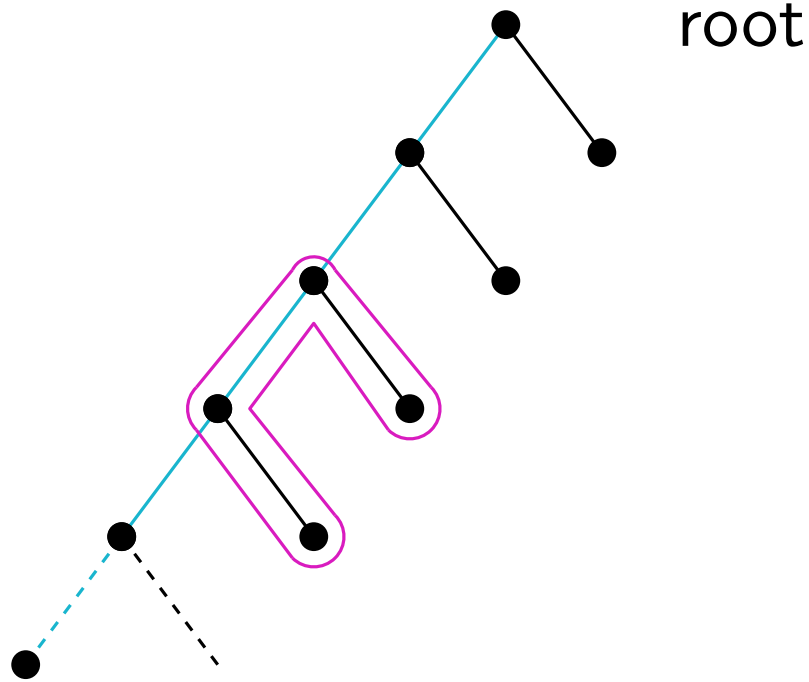

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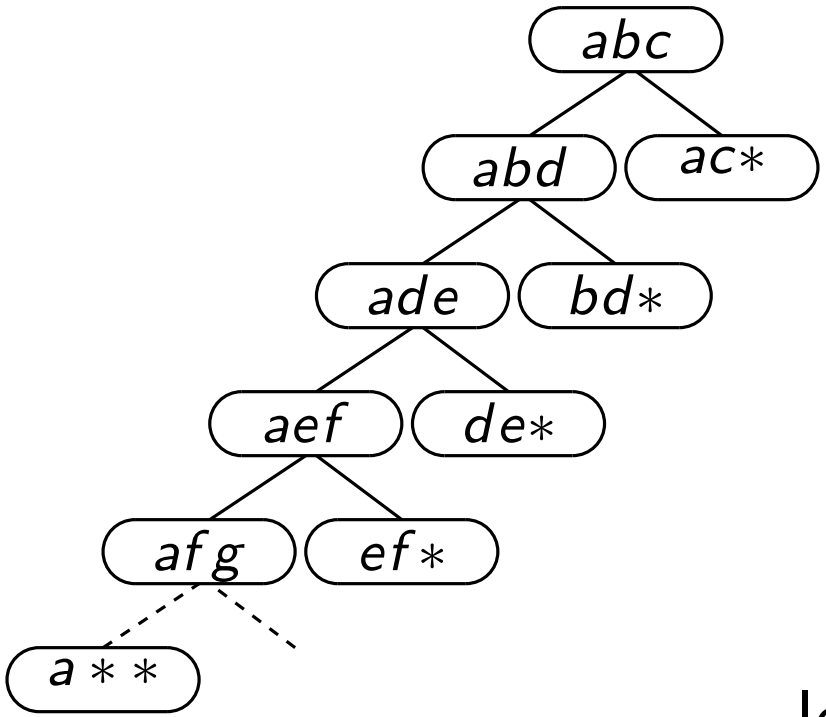

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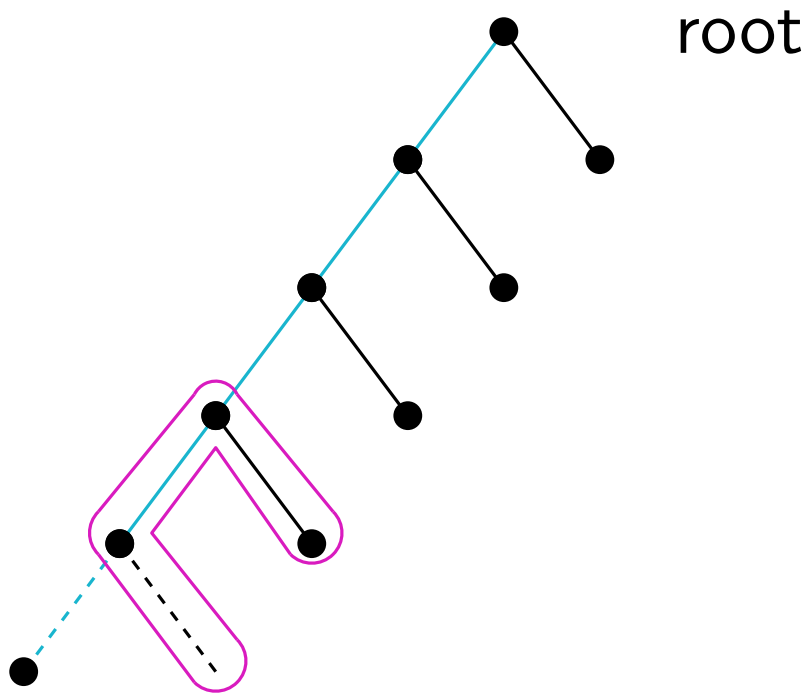

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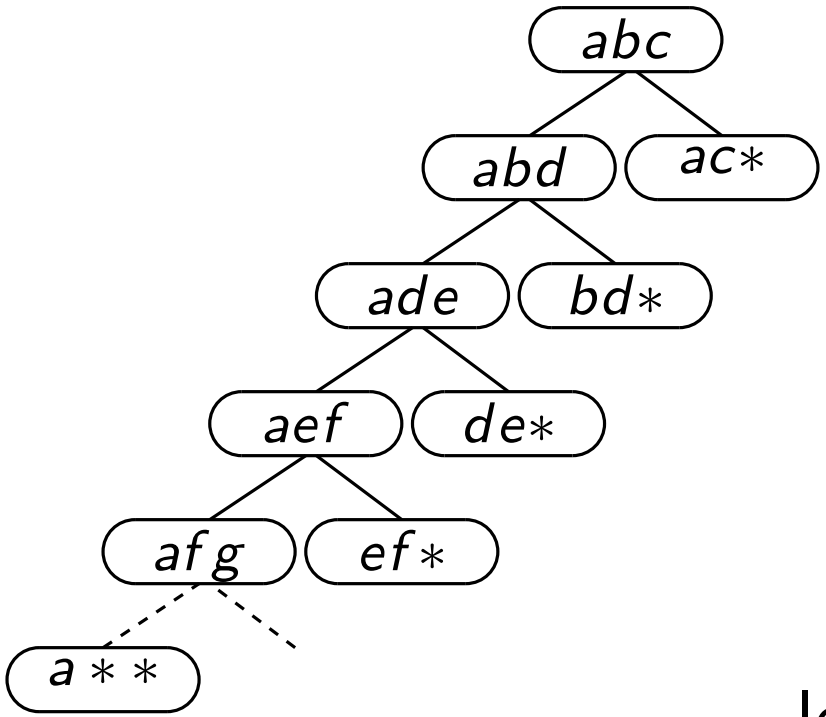
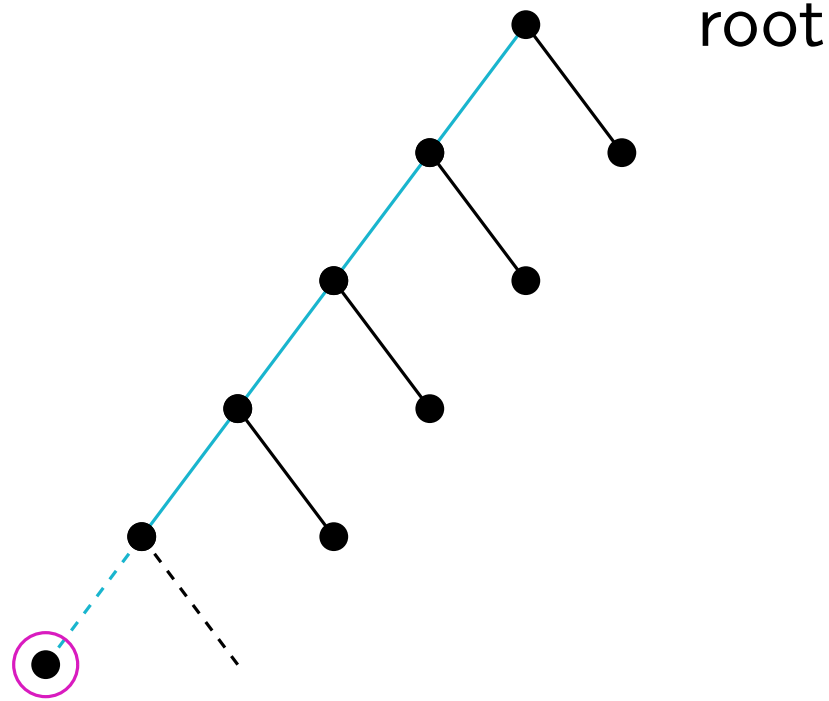
`Amount of layers occupied` $\mathcal{O}(\log n')$

DFS ends at leaf





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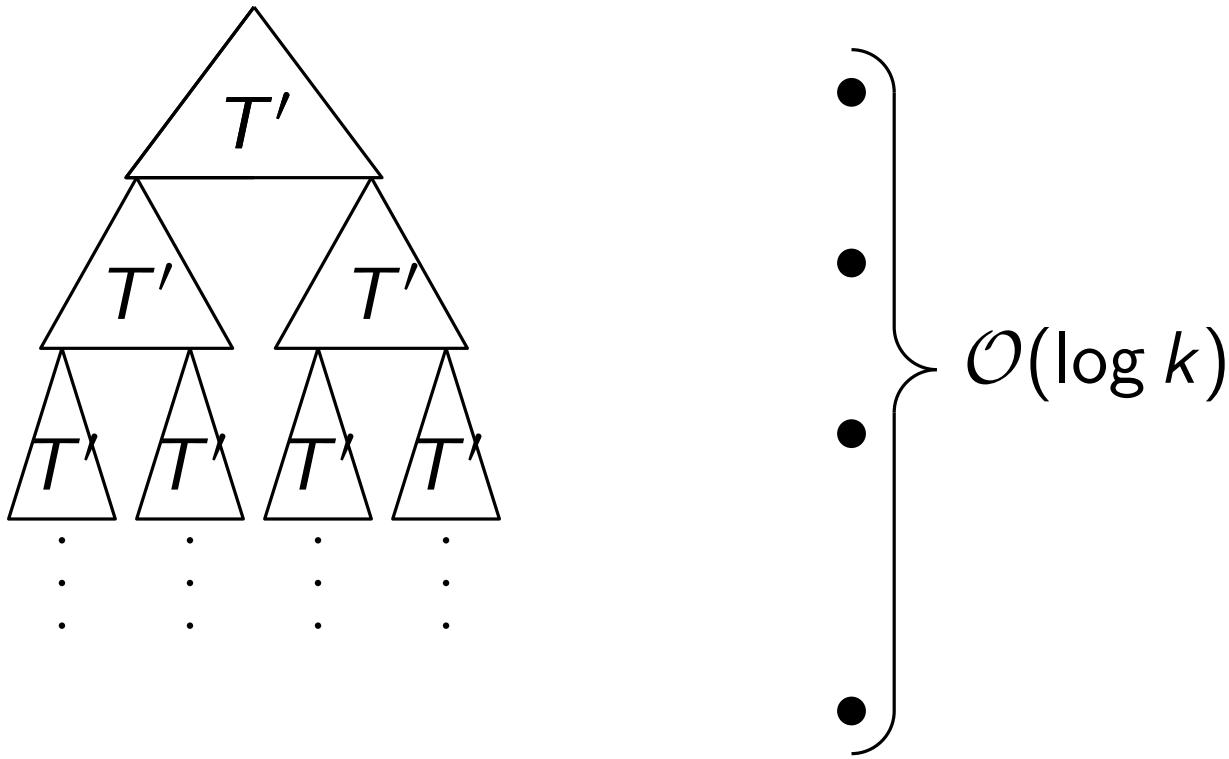

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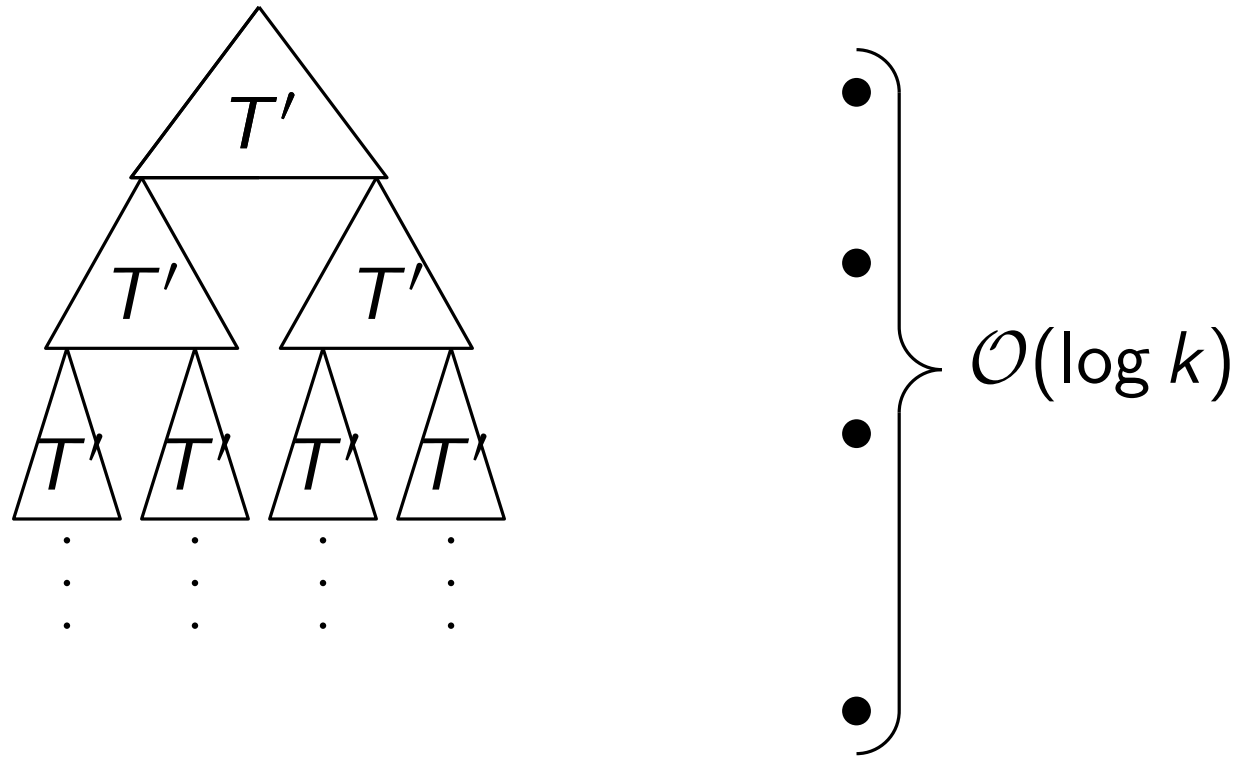

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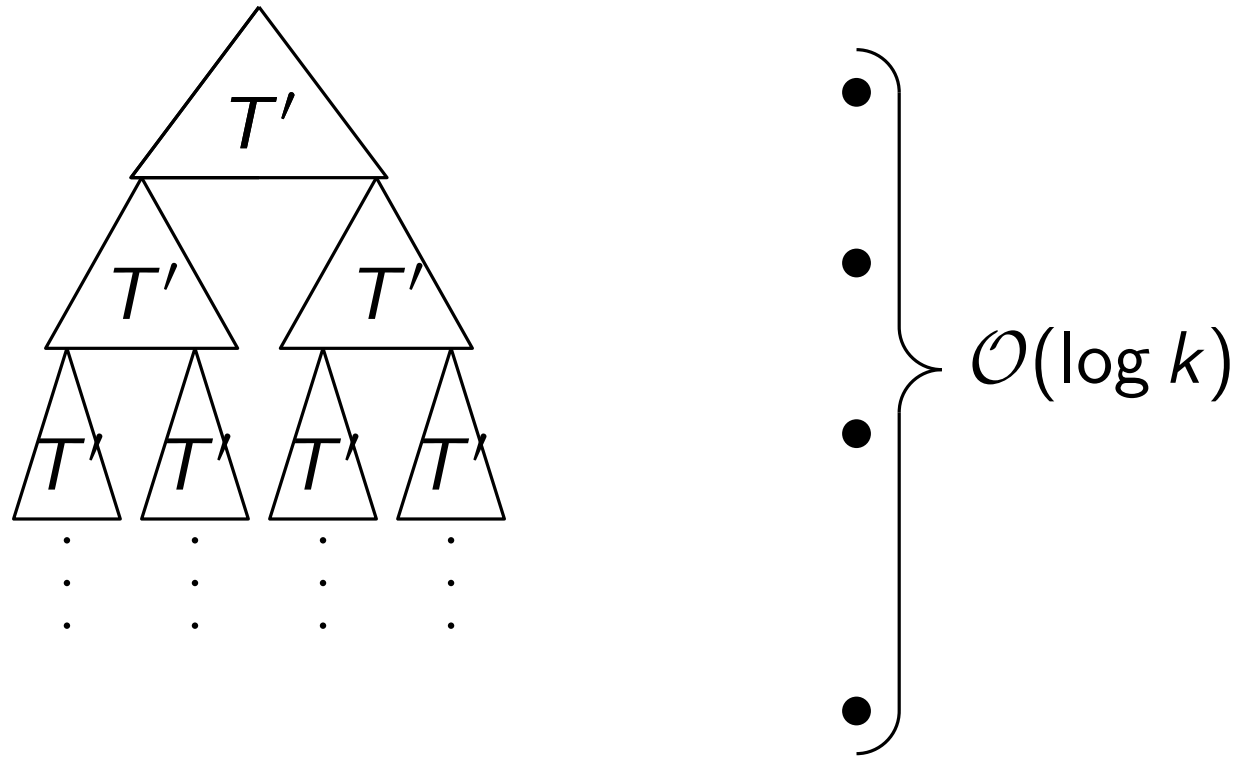

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- Any outerplanar graph G admits a polyline drawing in $\mathcal{O}(n^2 \log^2 n)$ with $r \in \mathcal{O}(\log^2 n)$ and two bends per edge

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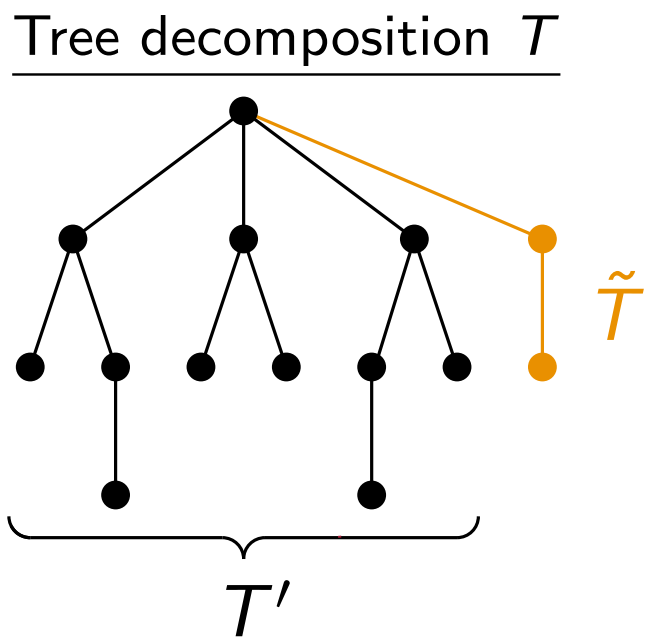

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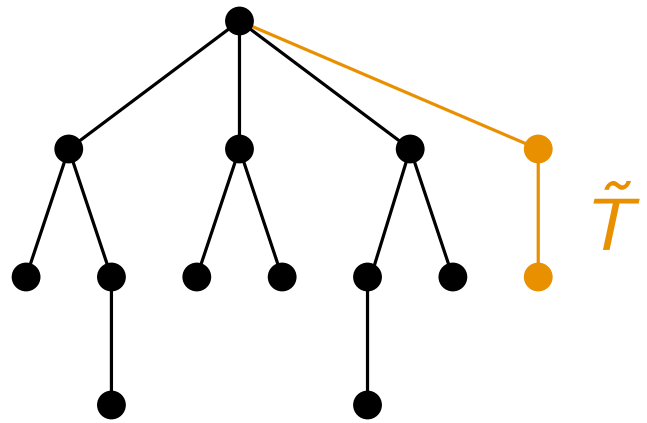

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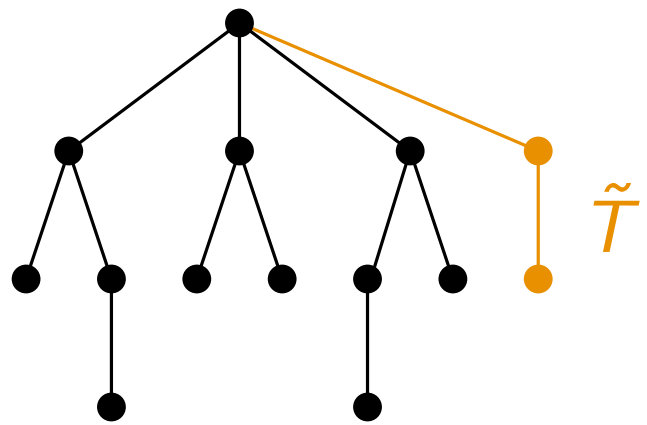

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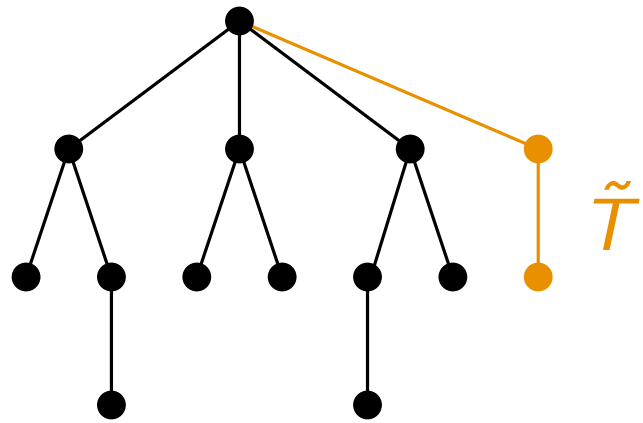

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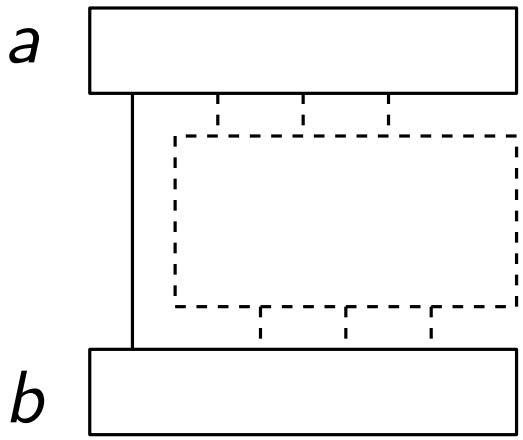

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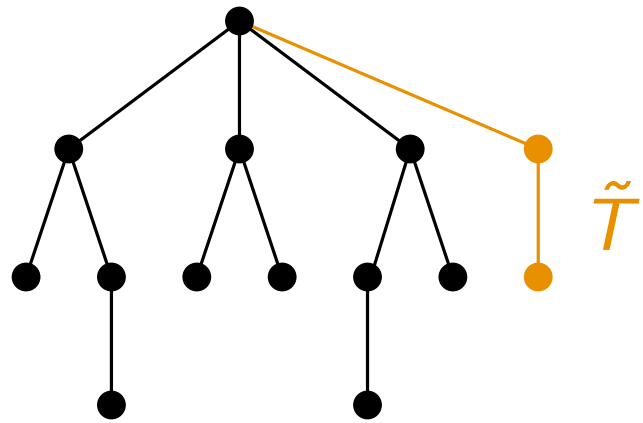

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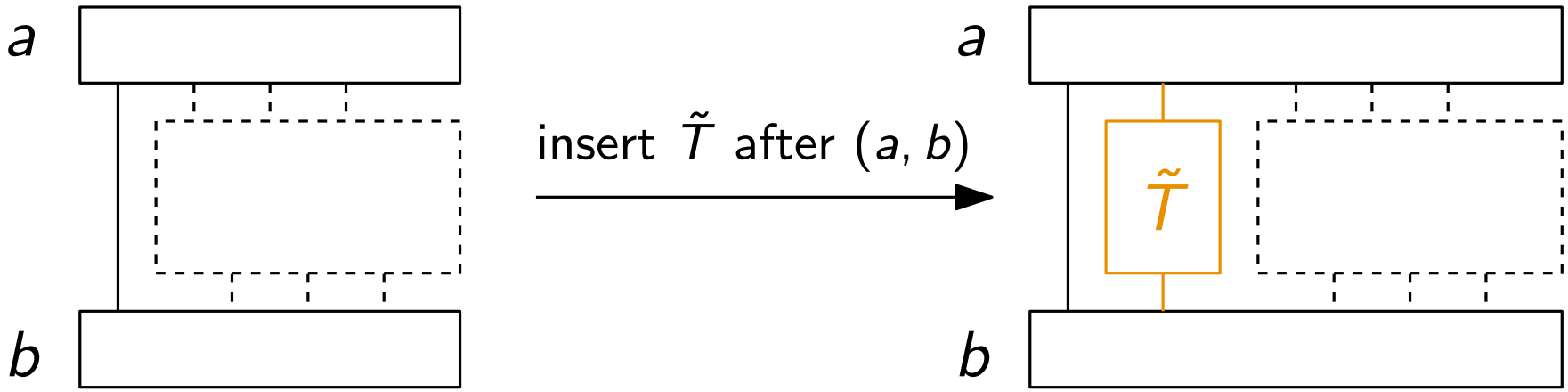

- Does not alter the area bounds of $B_{G'}$ asymptotically!

Results for Series-Parallel Graphs

- Since the area bounds stay the same, the results of outerplanar graphs also hold for series-parallel graphs

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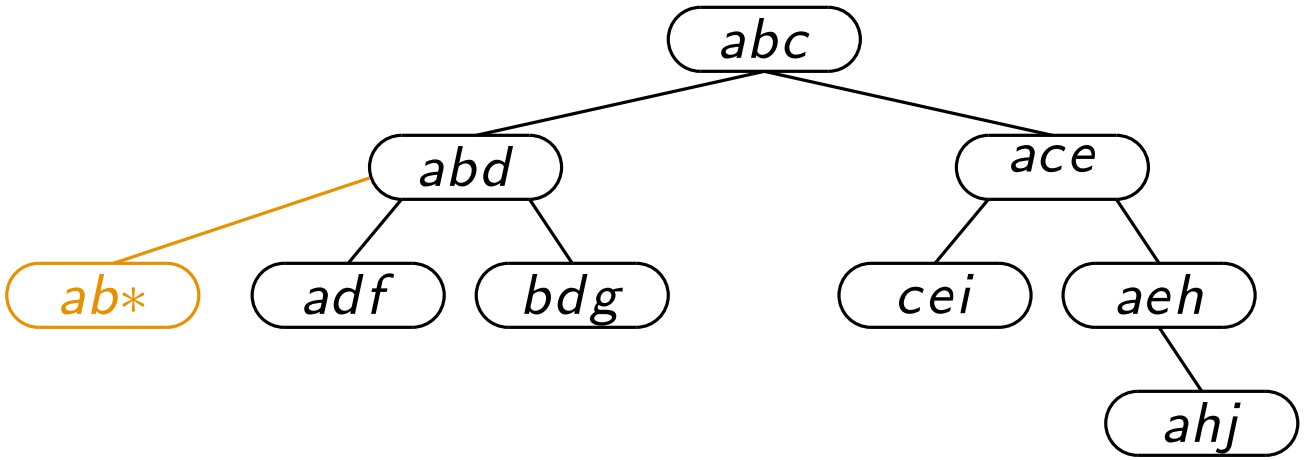

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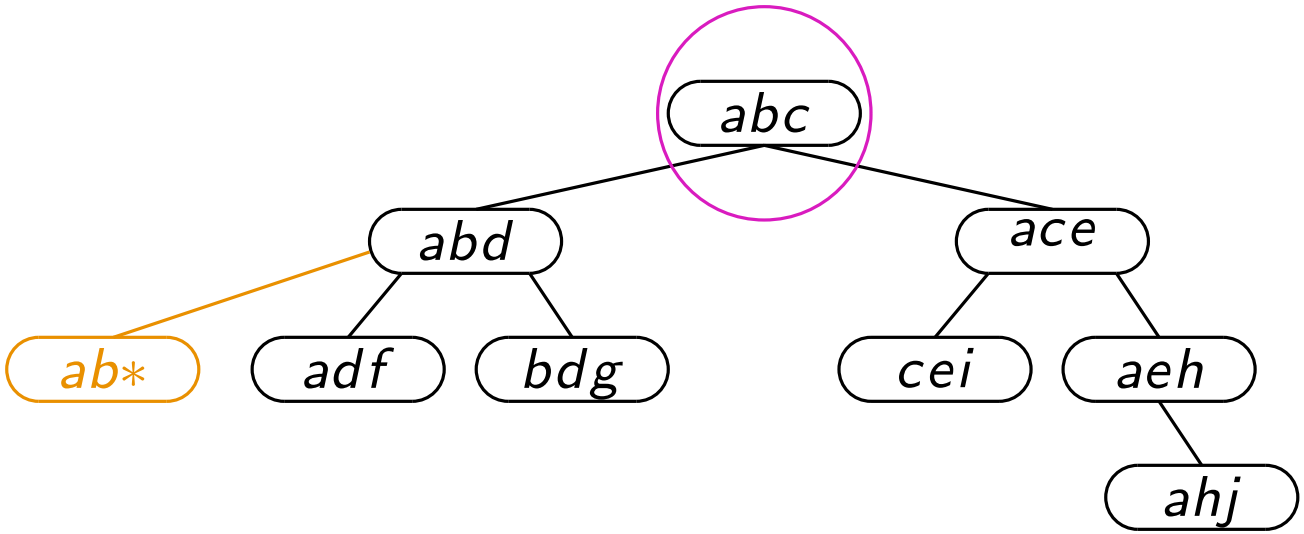

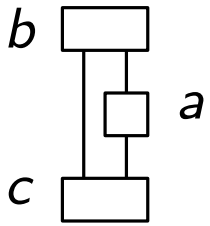

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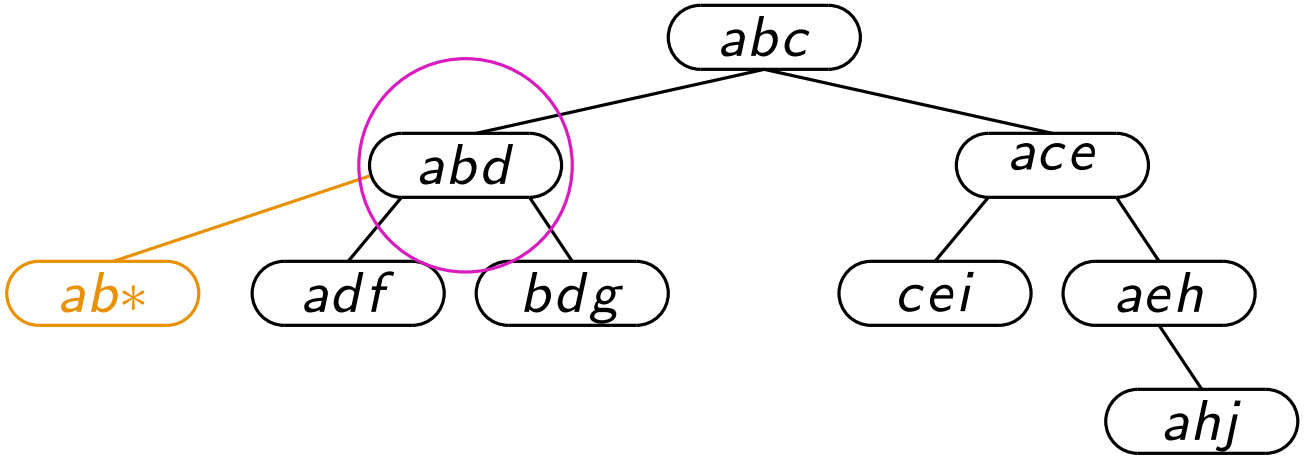

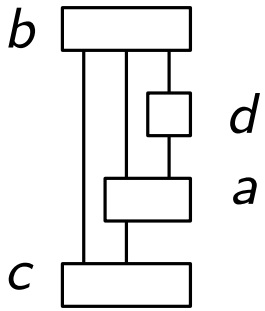

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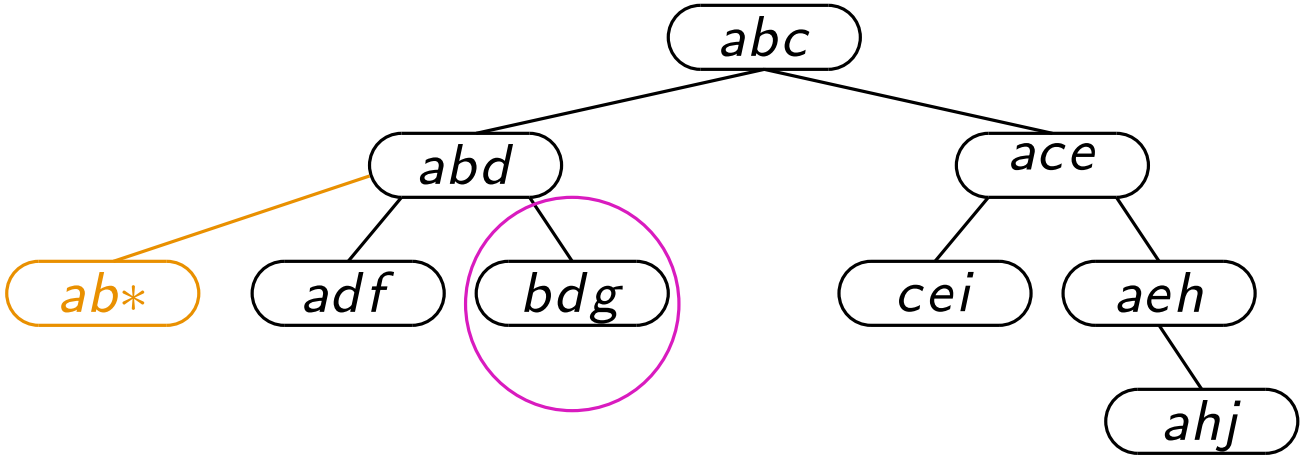
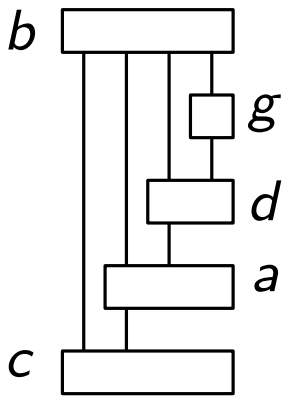


layer of b becomes free



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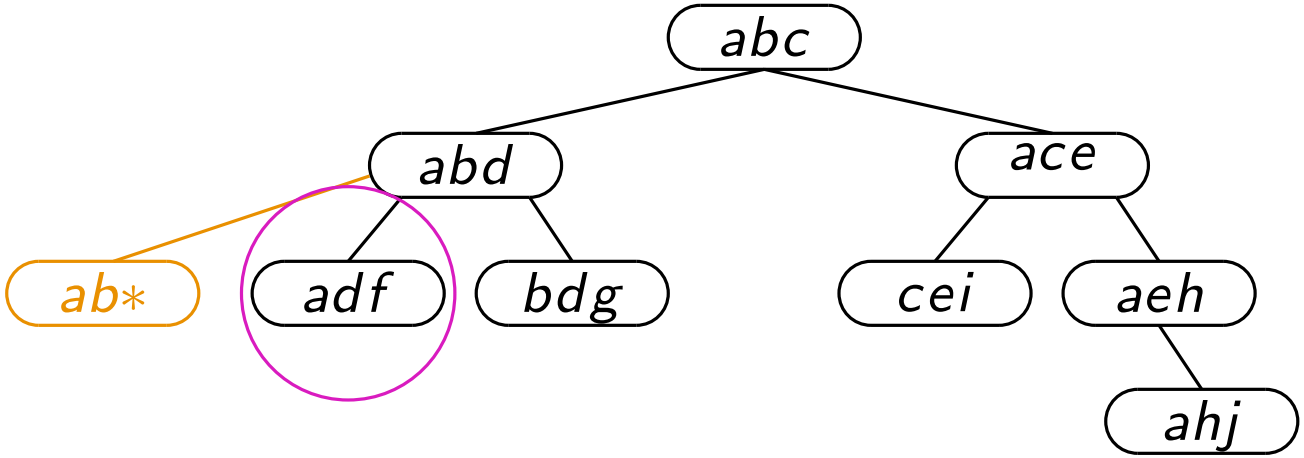

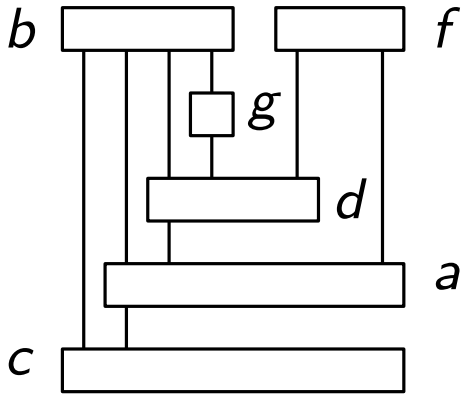

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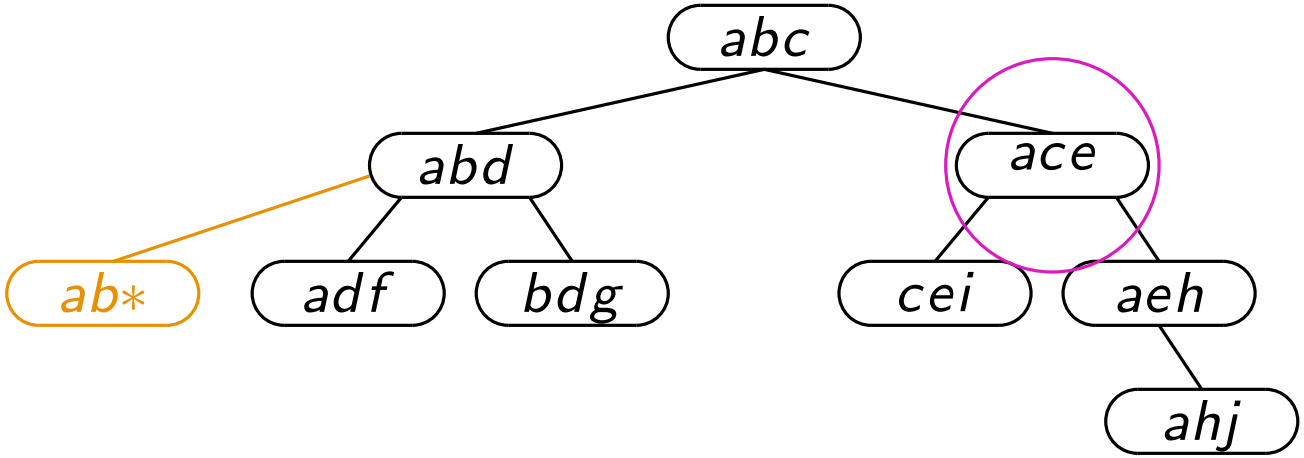

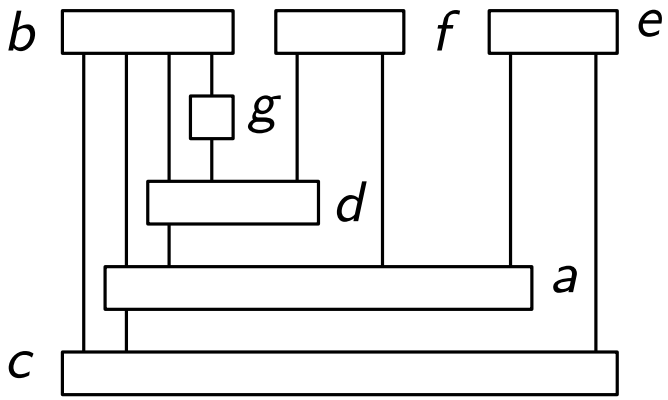

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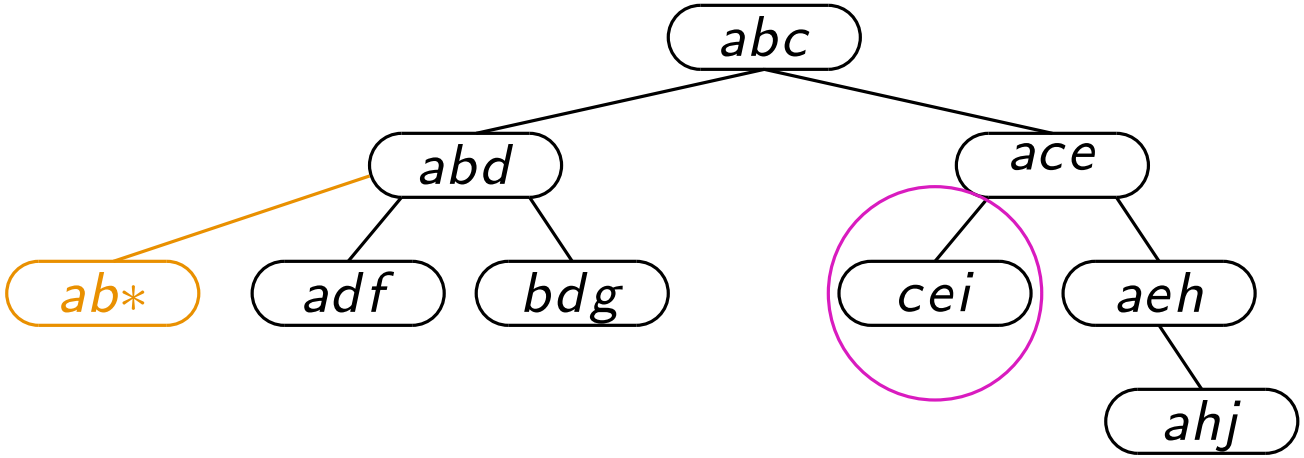

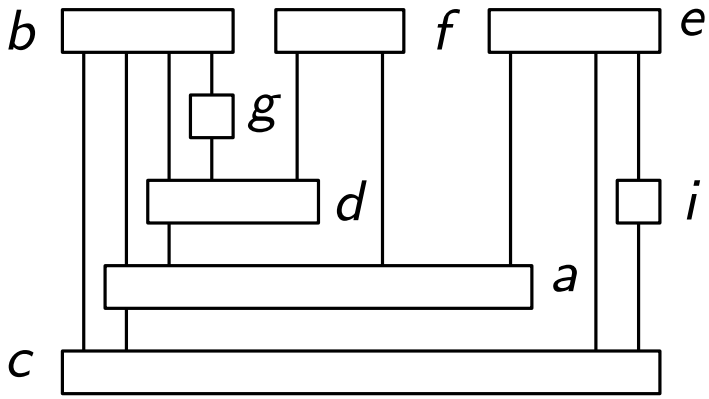

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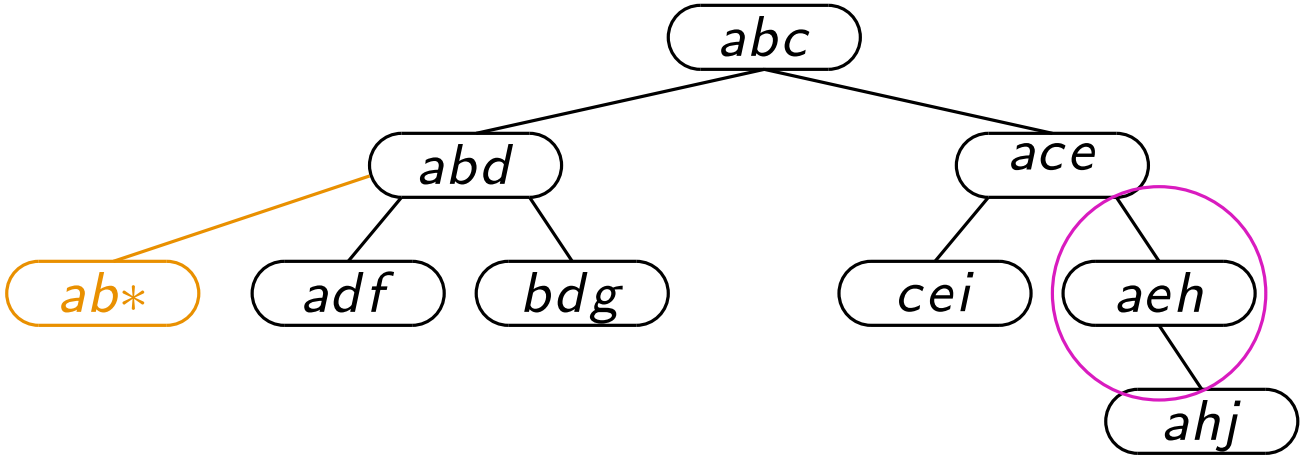

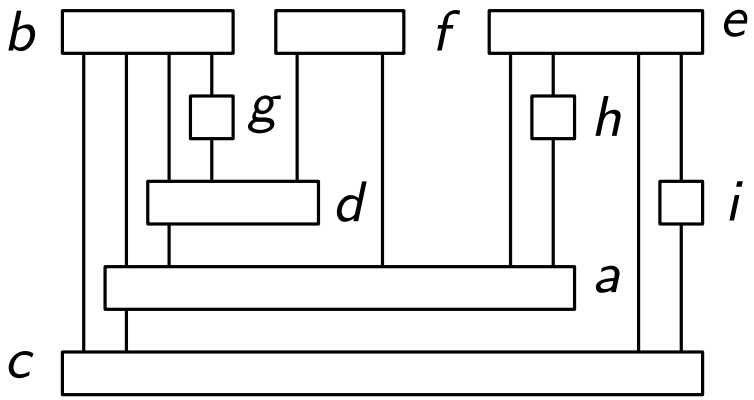

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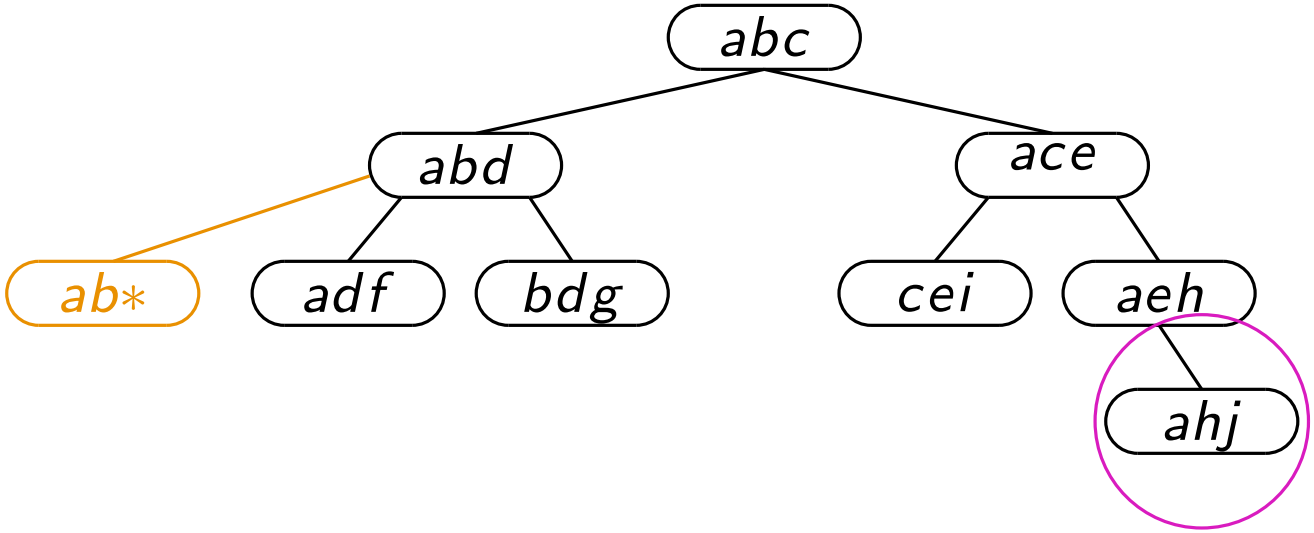

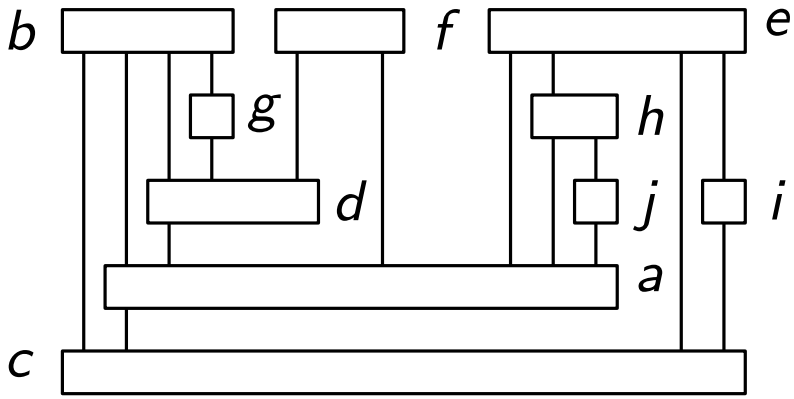

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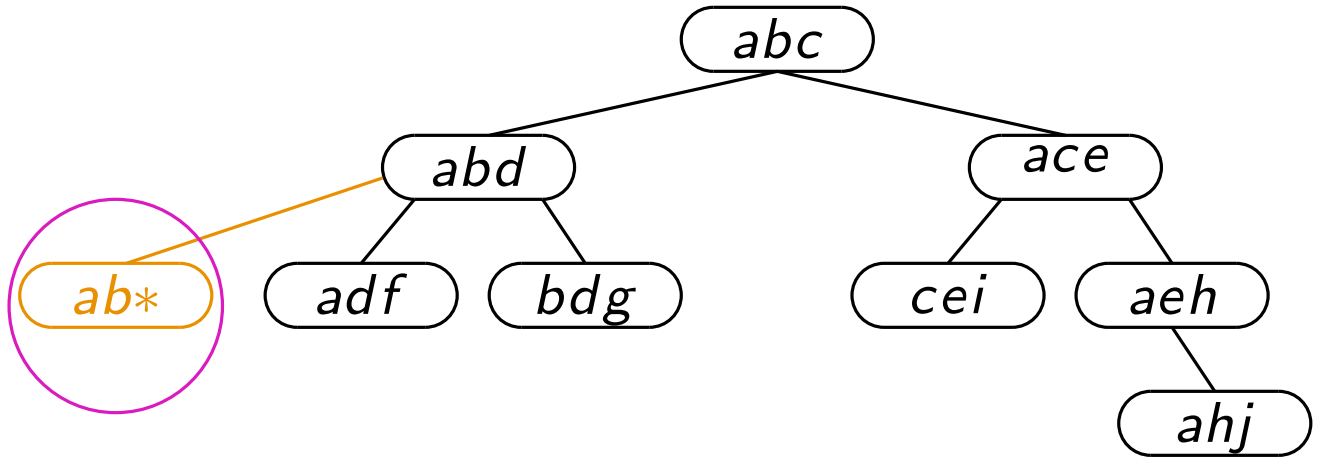

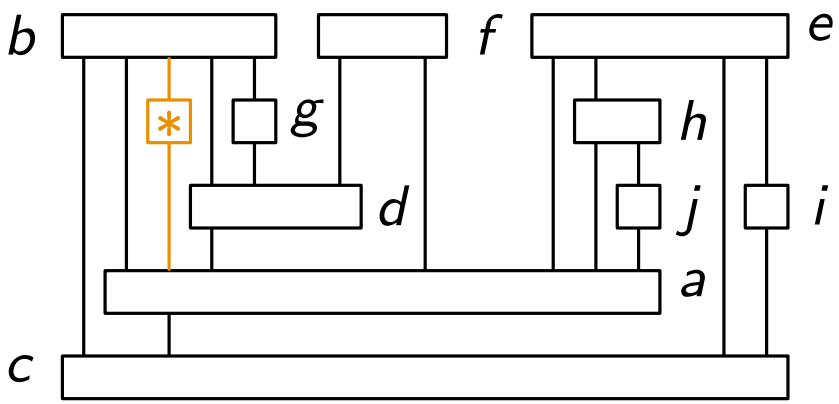

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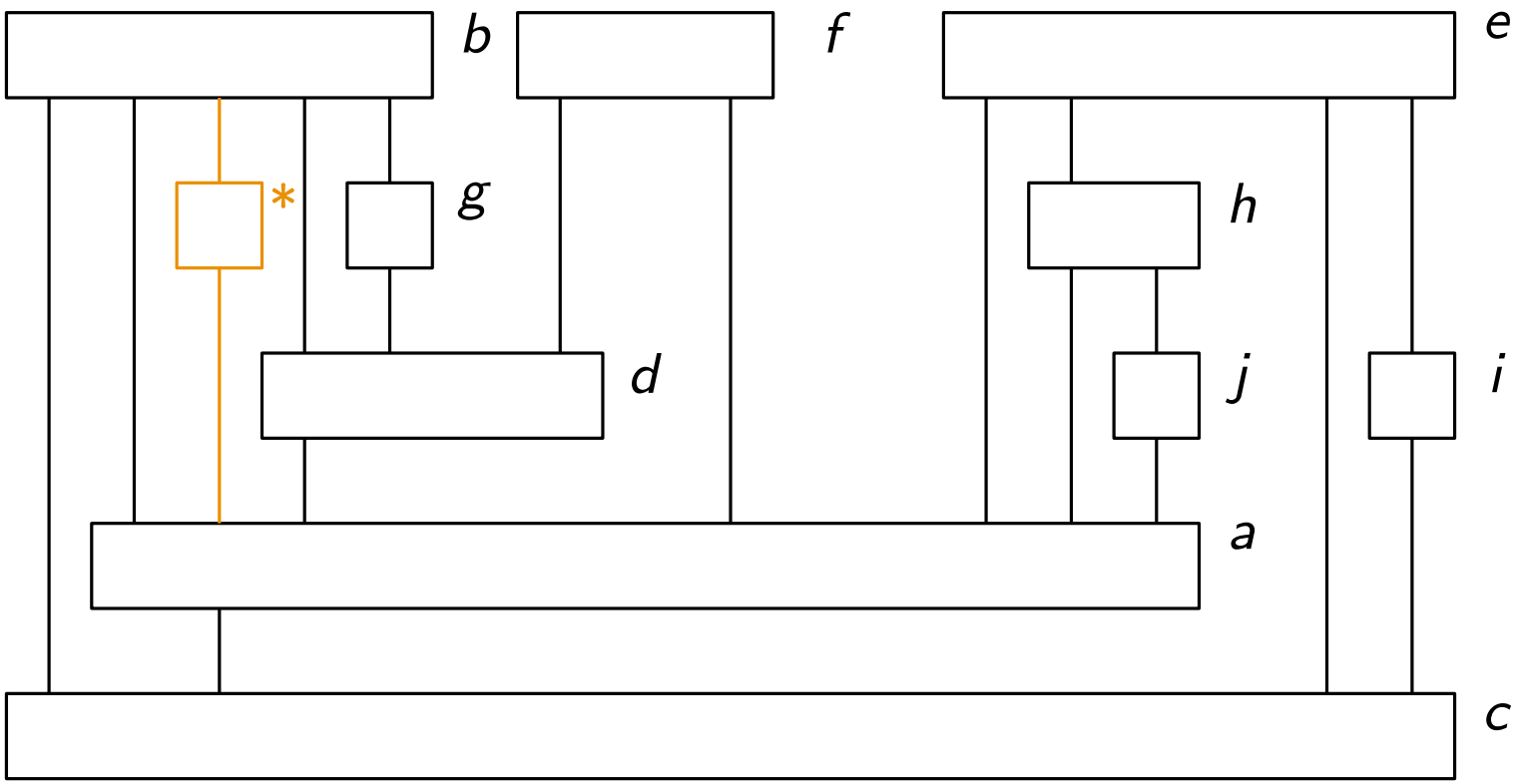

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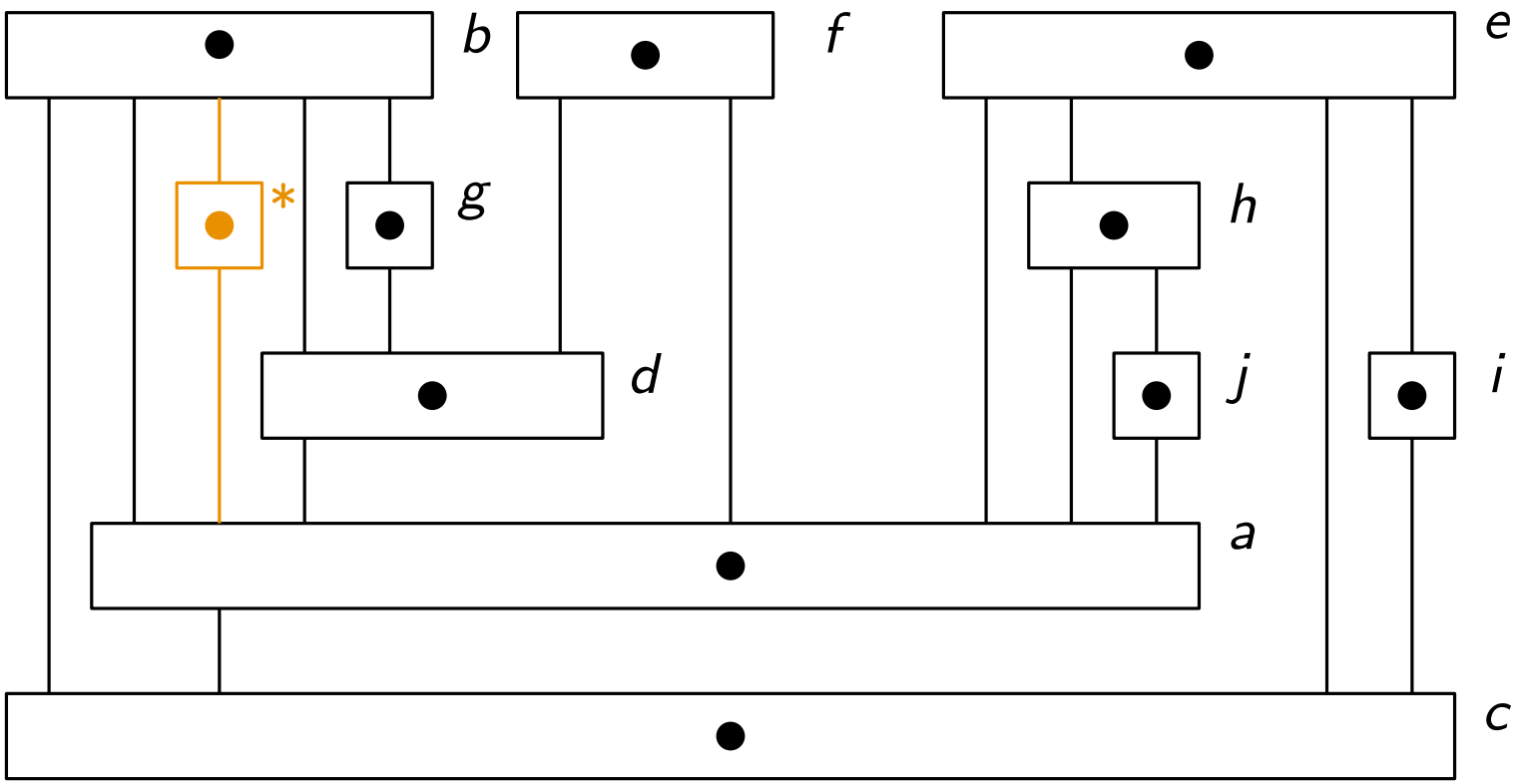

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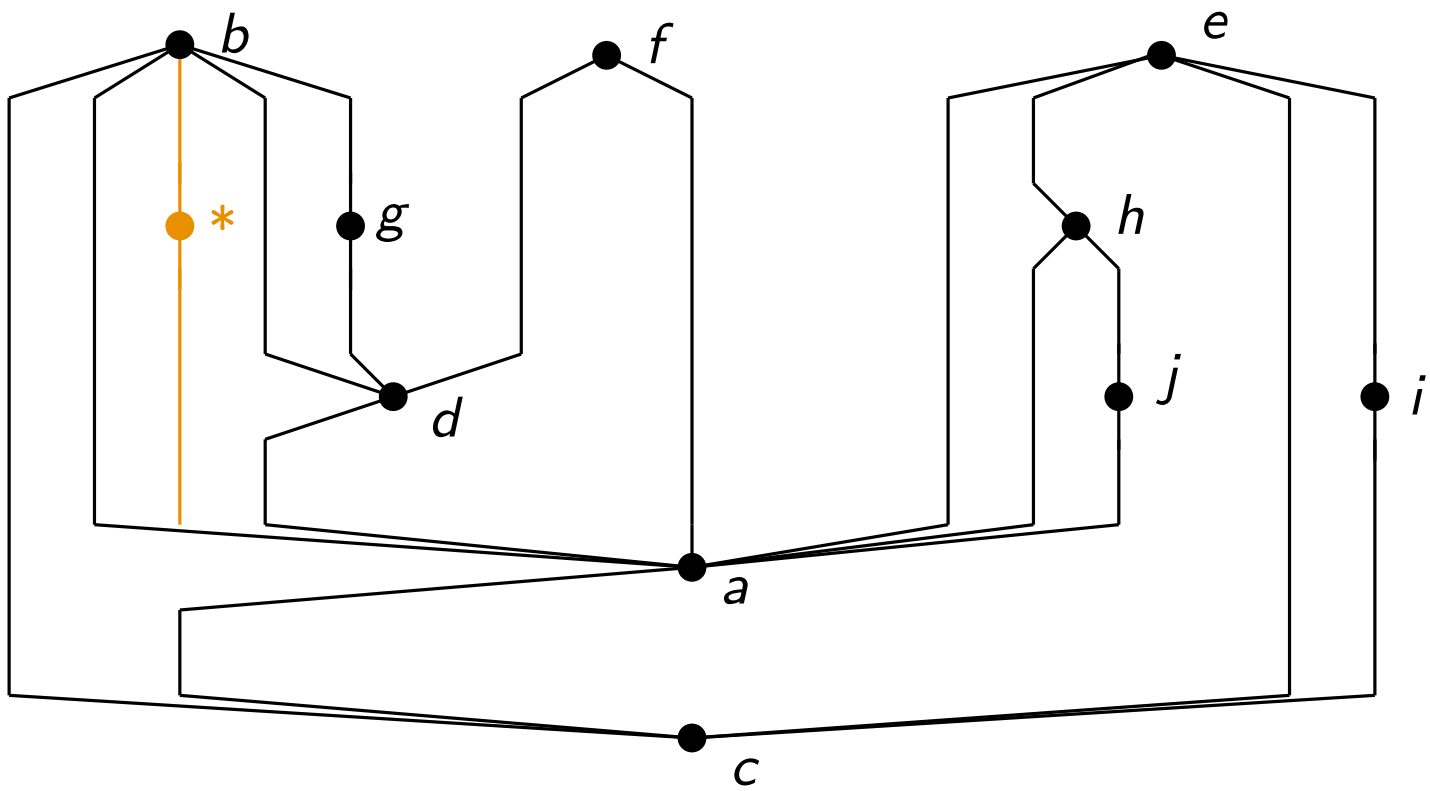

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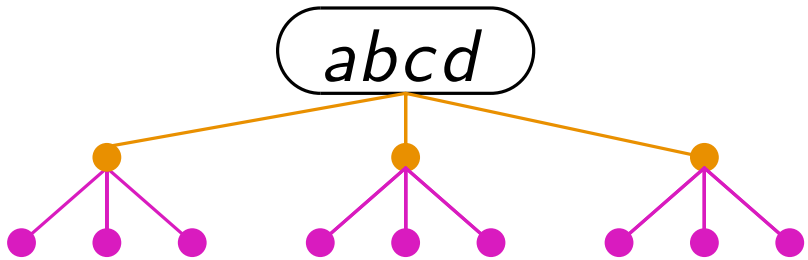

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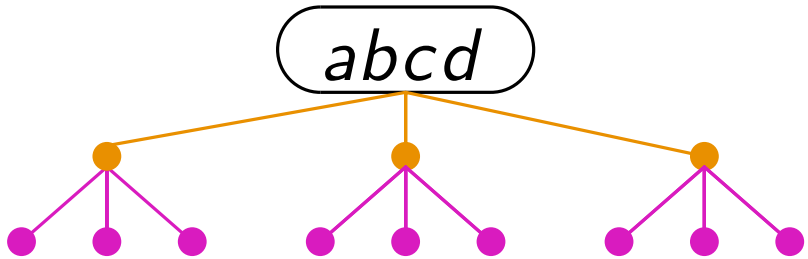

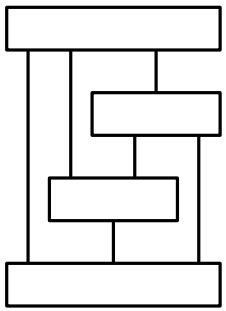

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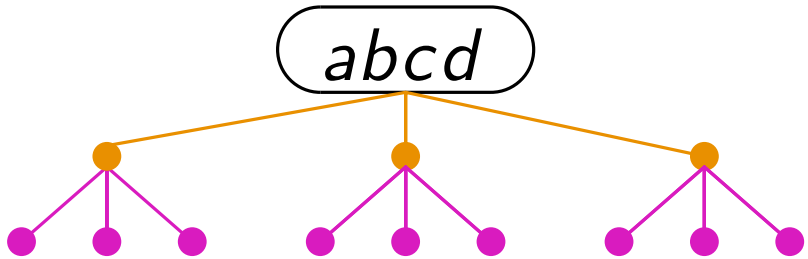

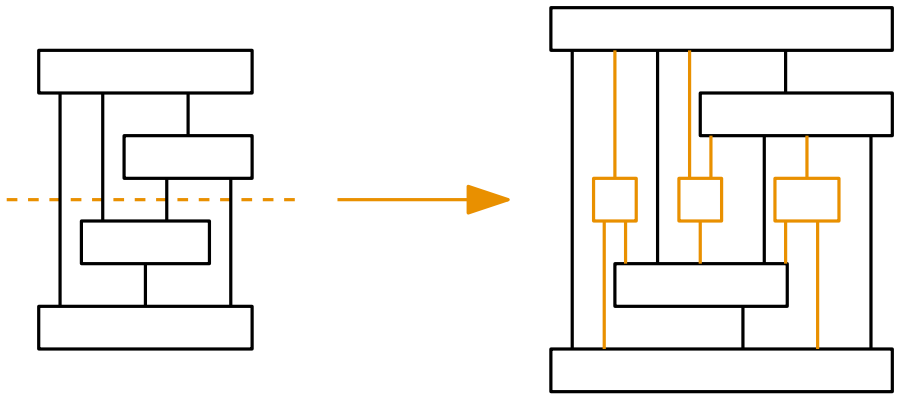

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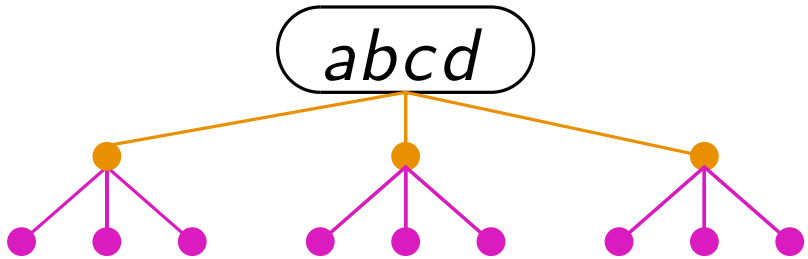

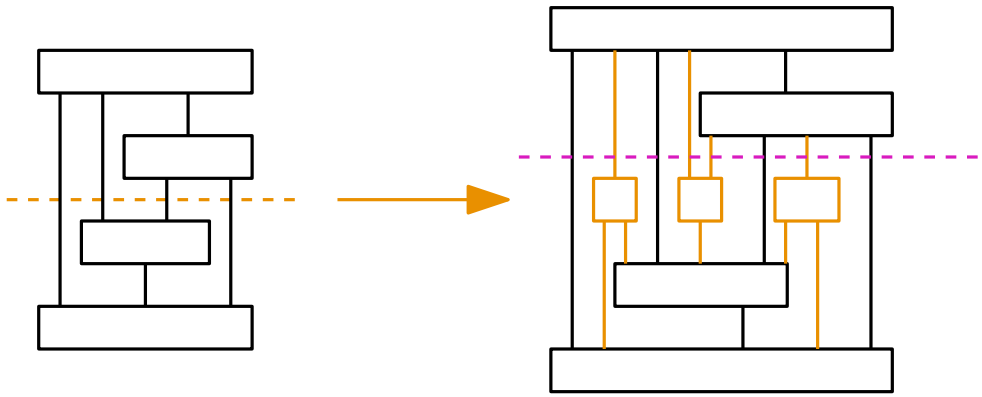

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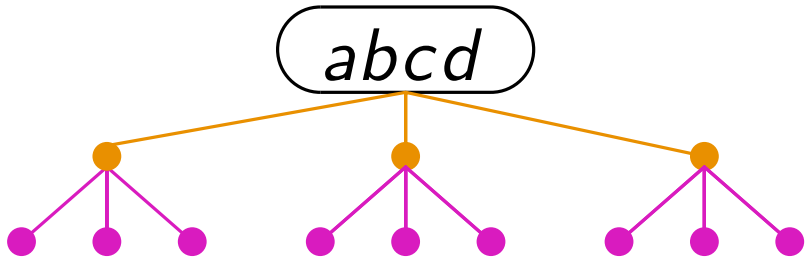
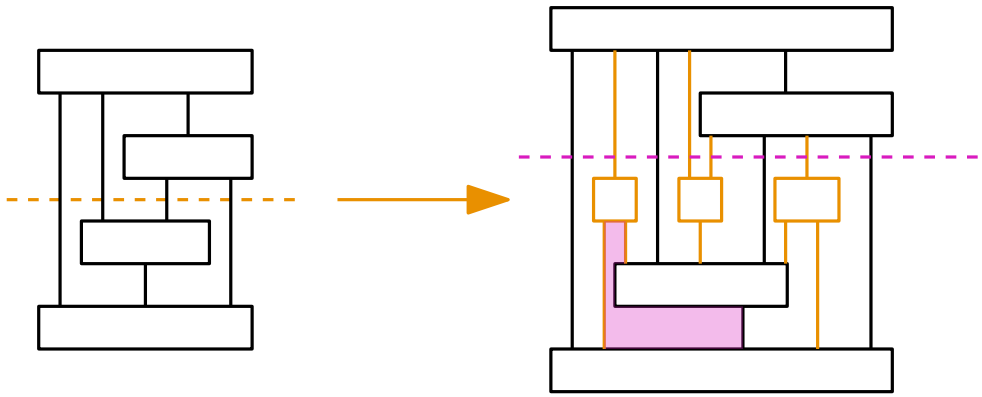
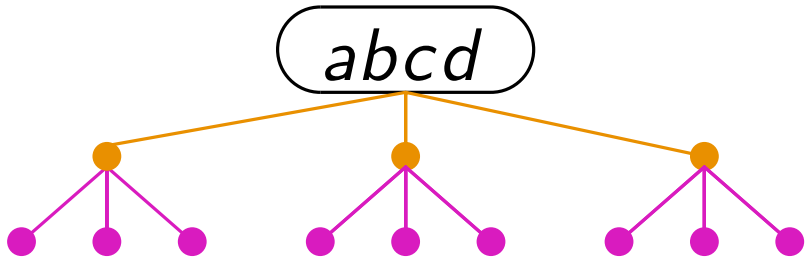
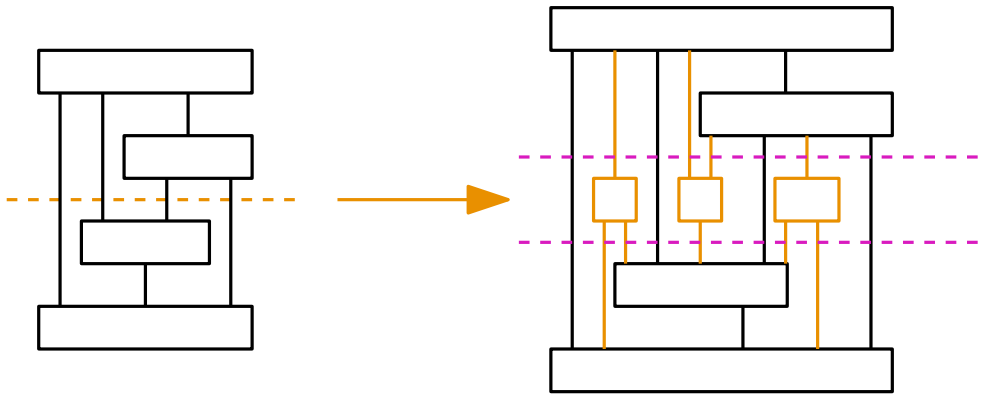




Planar 3-trees

- Recursively defined
- Start with K_4
- Add vertex, connect to 3 adjacent vertices
- Treewidth of three





- Amount of new layers at height h : $\mathcal{O}(2^h)$

Planar 3-trees

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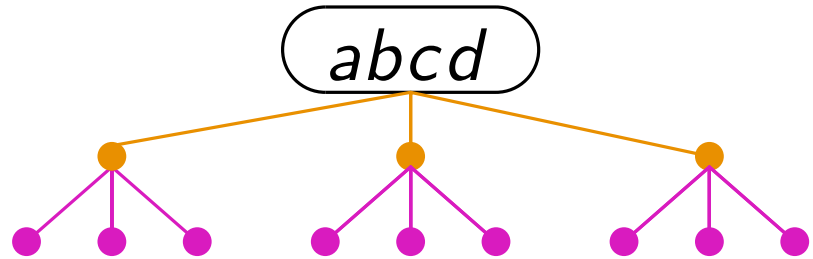
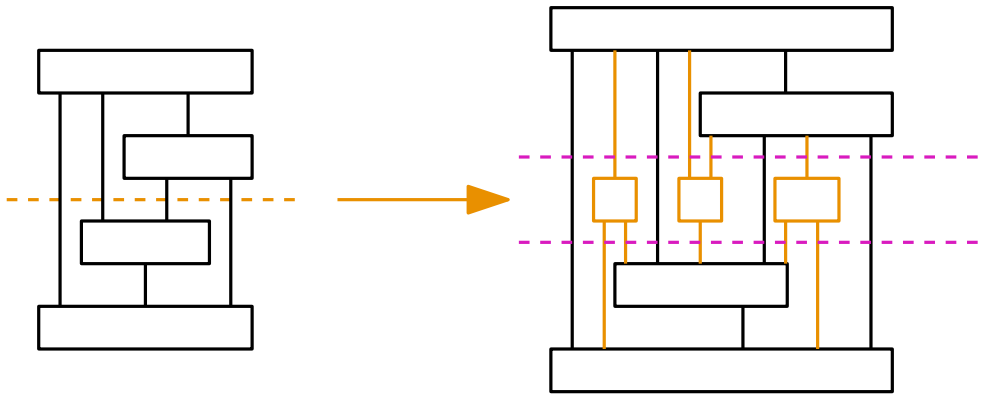
- Add vertex, connect to 3 adjacent vertices
- Treewidth of three





- Amount of new layers at height h : $\mathcal{O}(2^h)$

- $h \in \mathcal{O}(\log n) \Rightarrow \mathcal{O}(n)$ layers

Planar 3-trees

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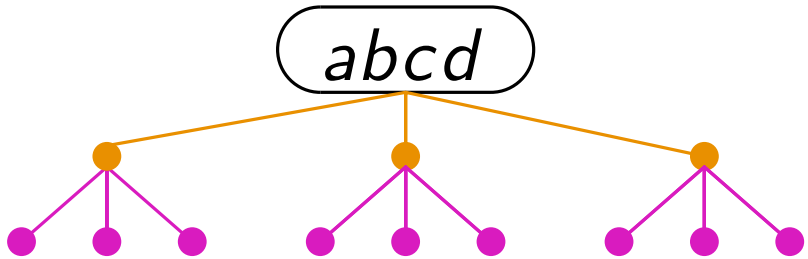

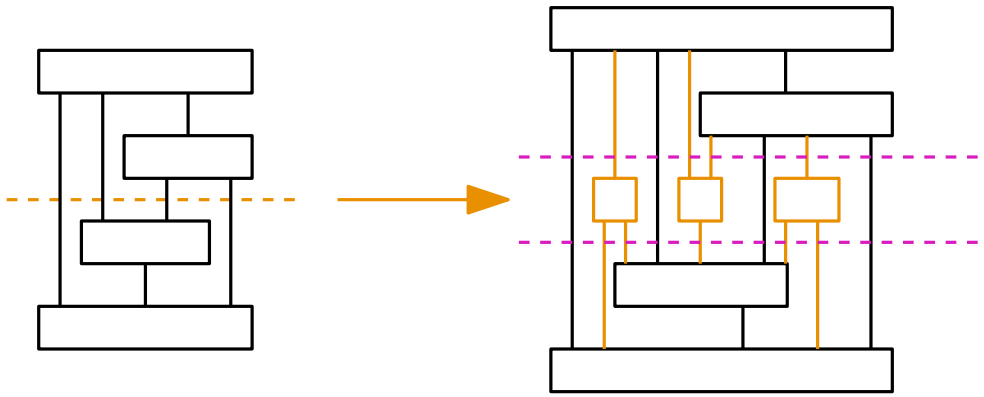

- Amount of new layers at height h : $\mathcal{O}(2^h)$

- $h \in \mathcal{O}(\log n) \Rightarrow \mathcal{O}(n)$ layers :(

Complete k-ary Tree Drawer Implementation

- Implemented in `python` $\geq$ `3.8`
- `networkx` library used for a graph structure representation
- `matplotlib` library used for plotting
- Coordinate computation can be found in thesis

Thank you for participating!

:)