

On Maximizing the Euclidian Distance between Adjacent Vertices in Planar Drawings of Small Area

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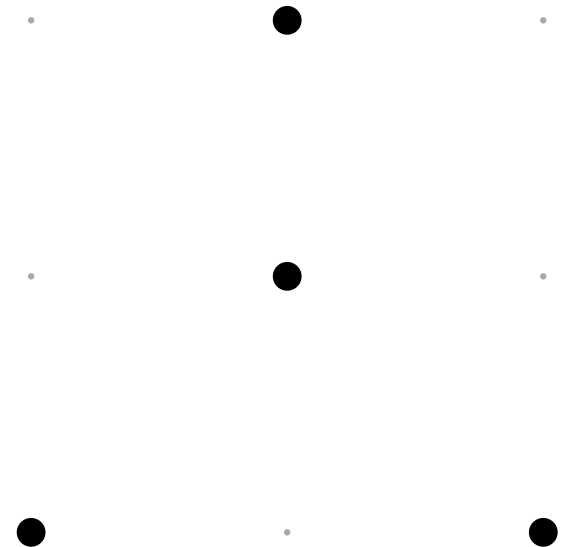
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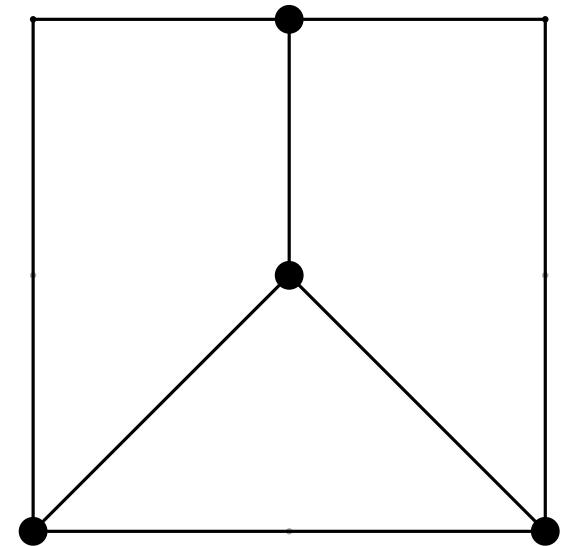
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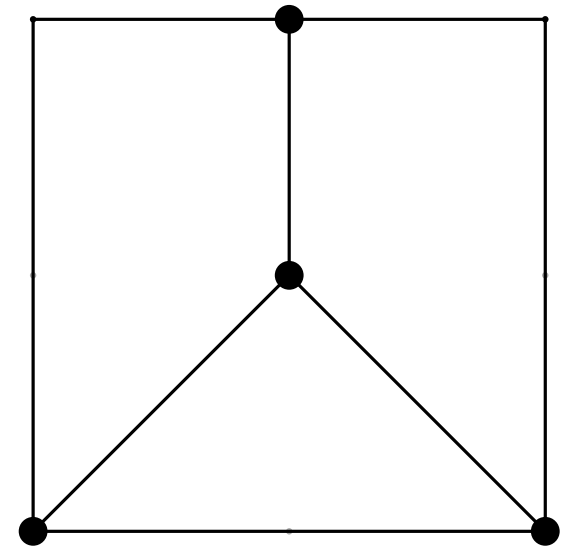
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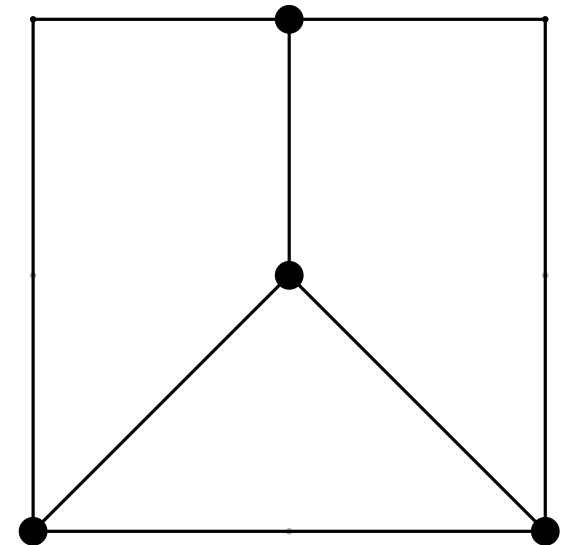
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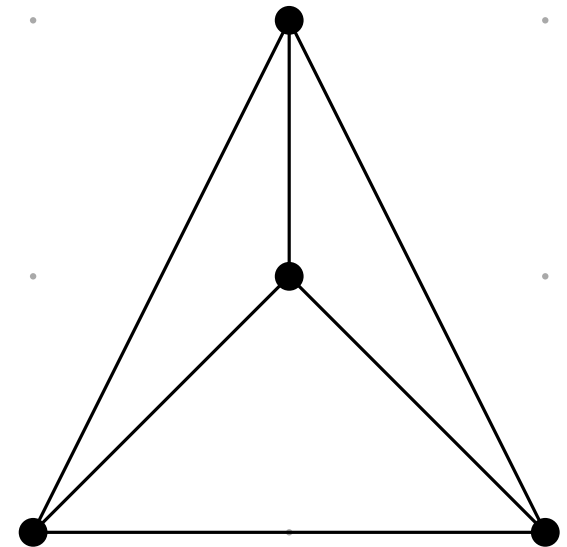
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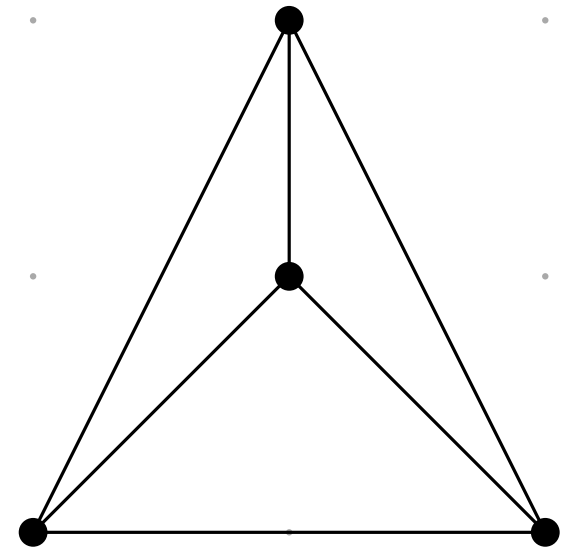
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- ▶ Total grid size determines geometrical properties



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 - ▶ *Live Challenge* as part of the contest includes heuristic approaches for geometrical problems of graph drawings on a fixed grid size
- ▶ This work addresses producing drawing algorithms for specific graph classes which will approach the geometrical problems with results described asymptotically

Live Challenge 2021

- ▶ Minimize the *edge length* ratio of the *longest* and *shortest* polyline

Live Challenge 2021

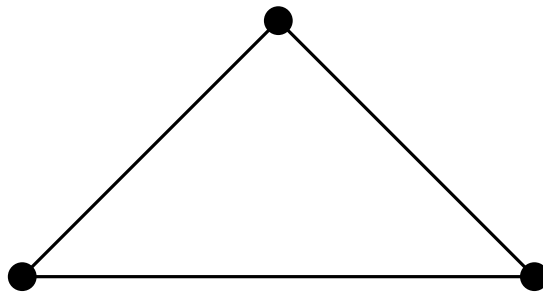
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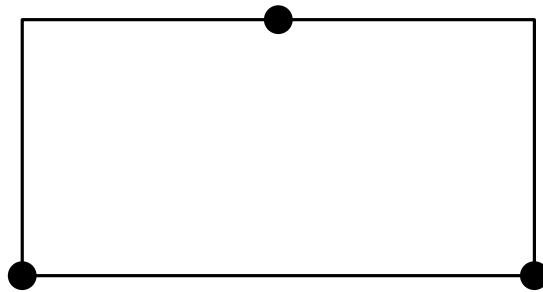
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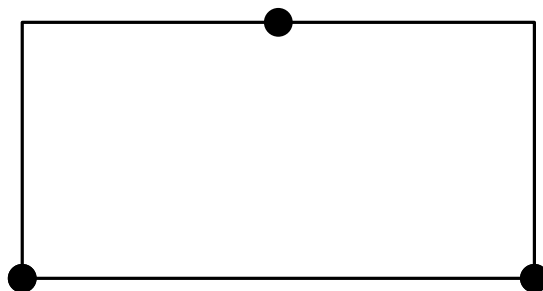
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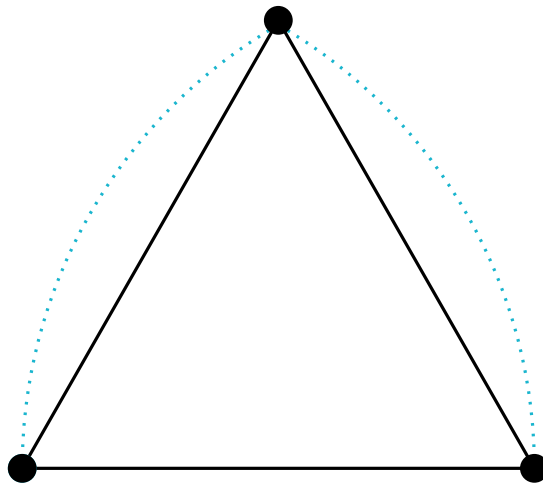
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$$r = \sqrt{2}$$

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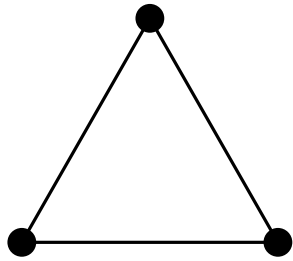
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$r = 1 + \varepsilon$, ε depends on grid size

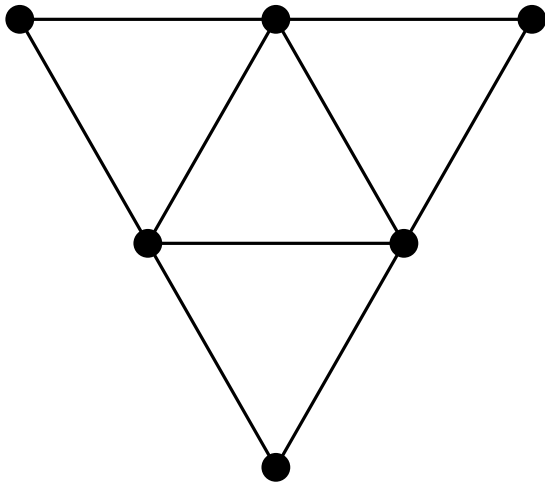
Adjacent and Nested Triangles

► $r = 1 + \varepsilon$



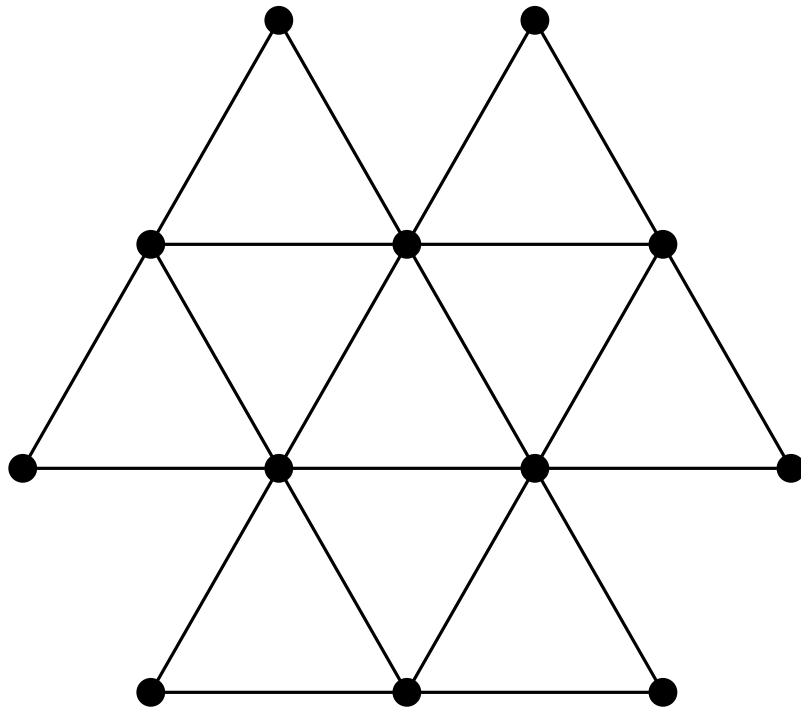
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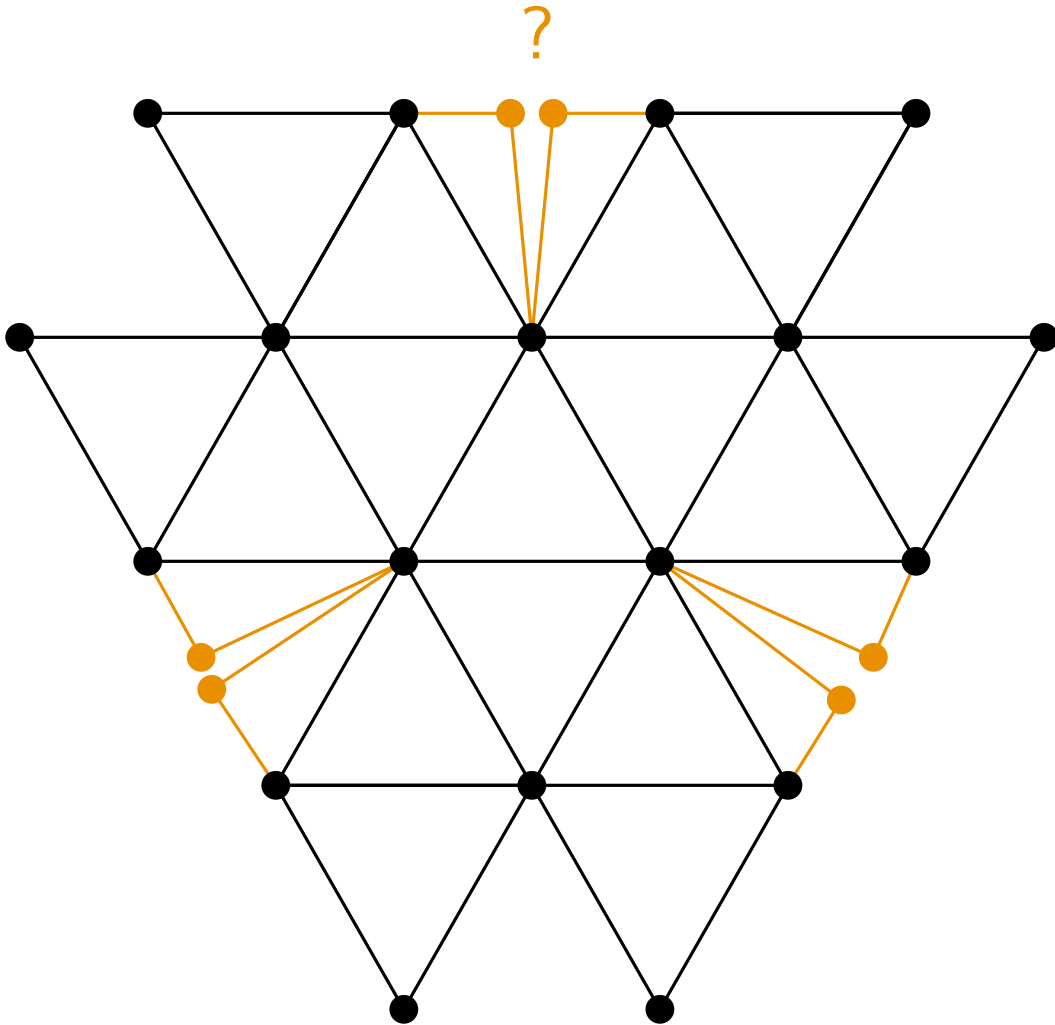
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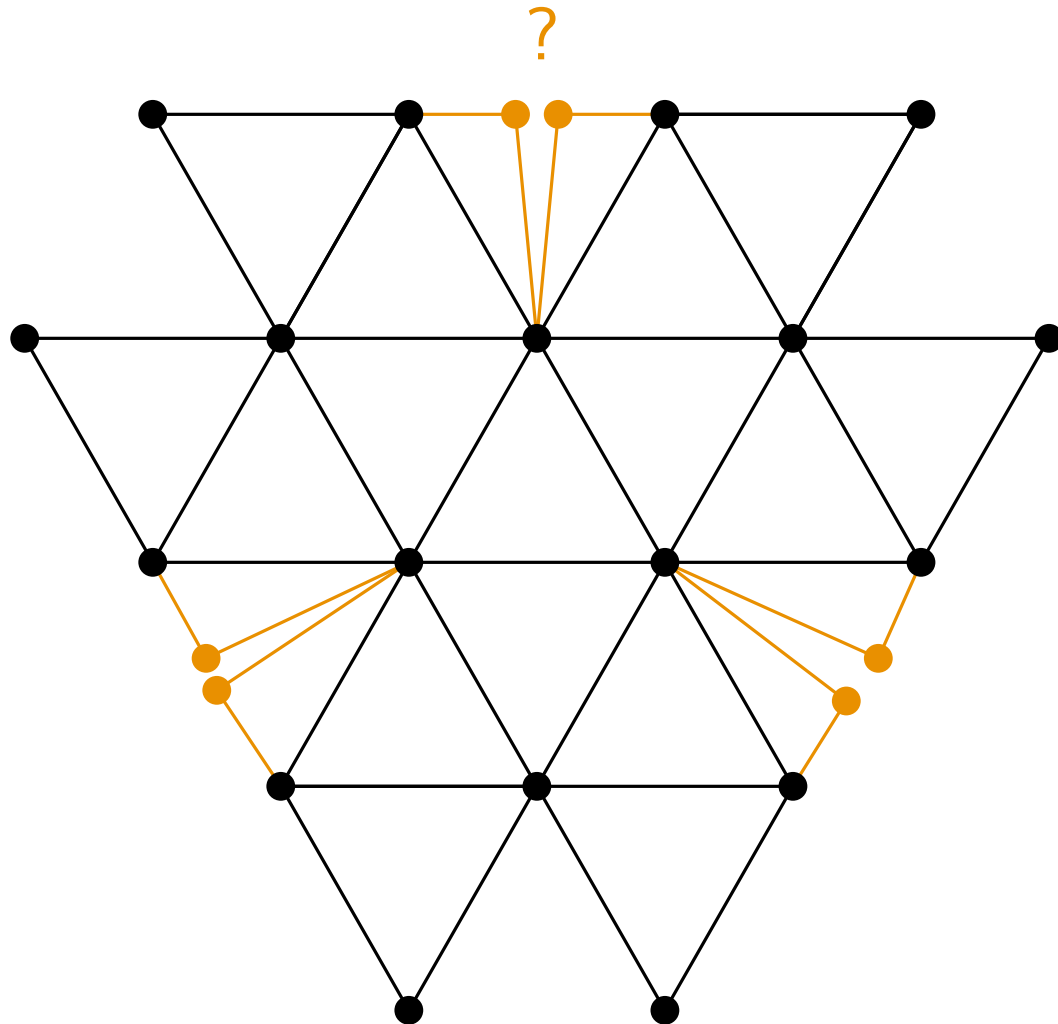
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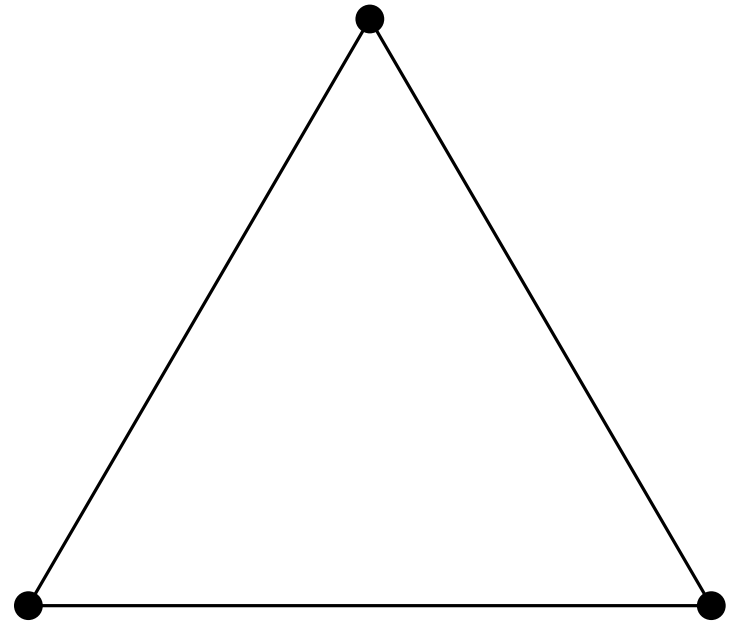


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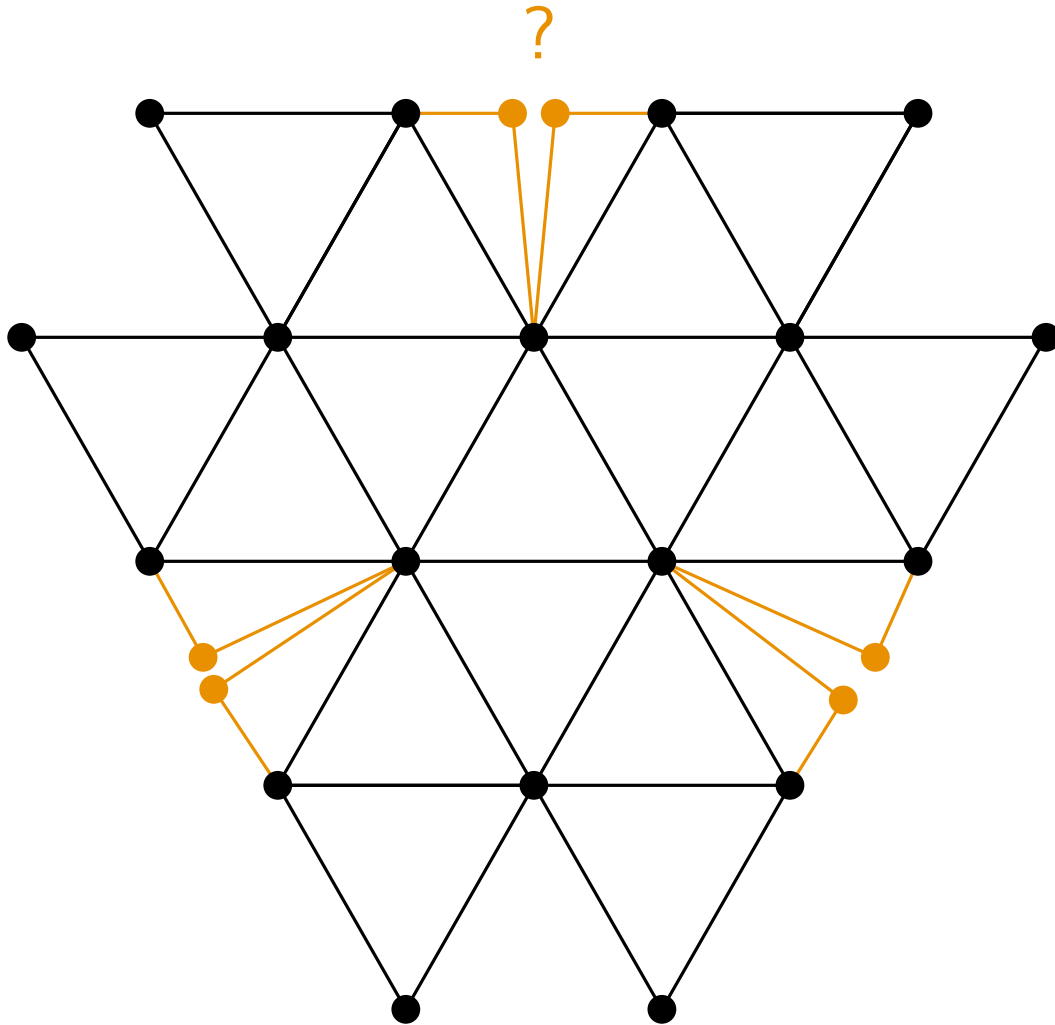


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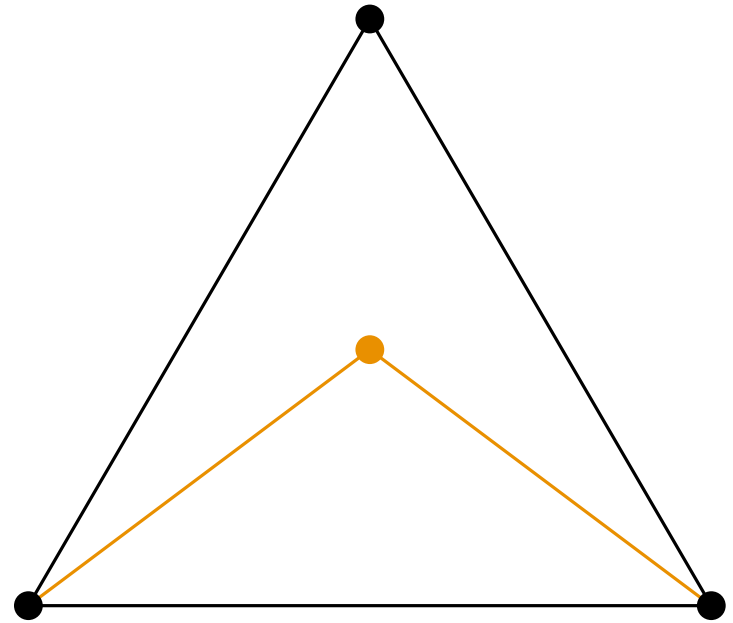


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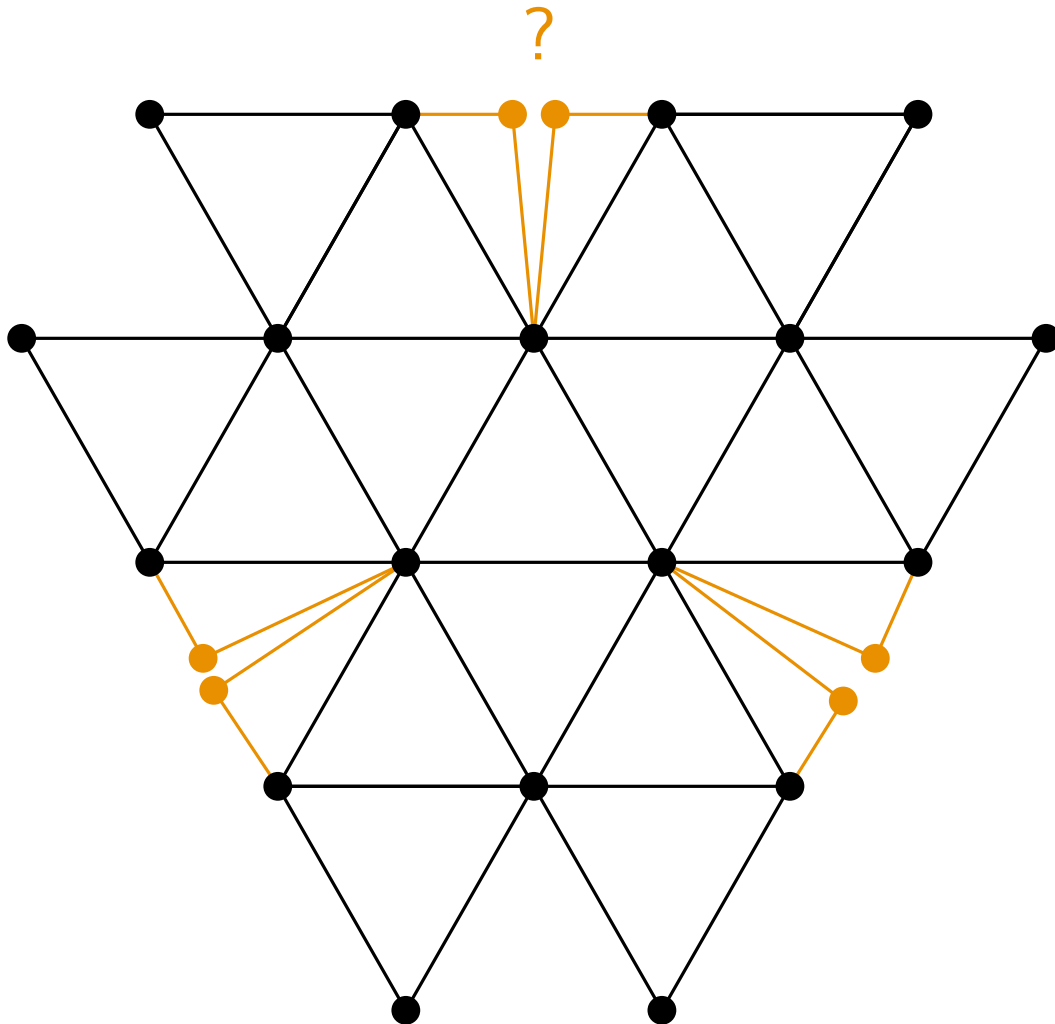


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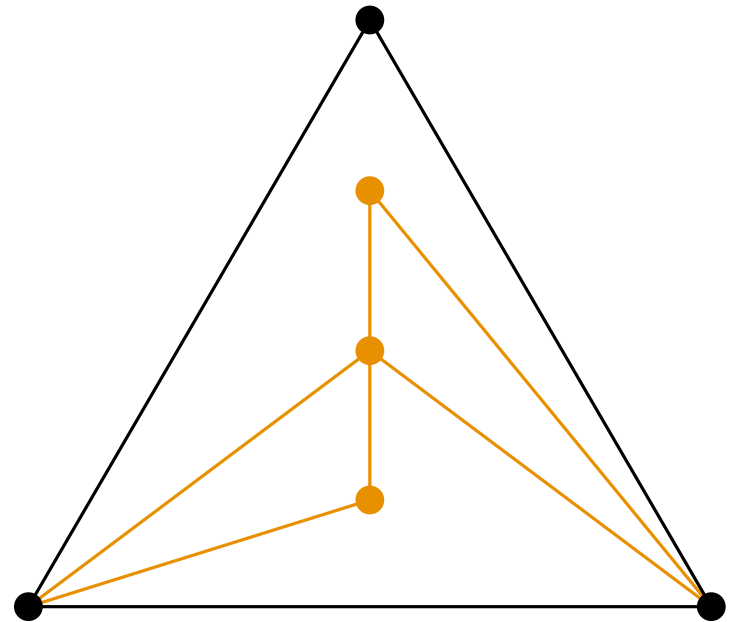


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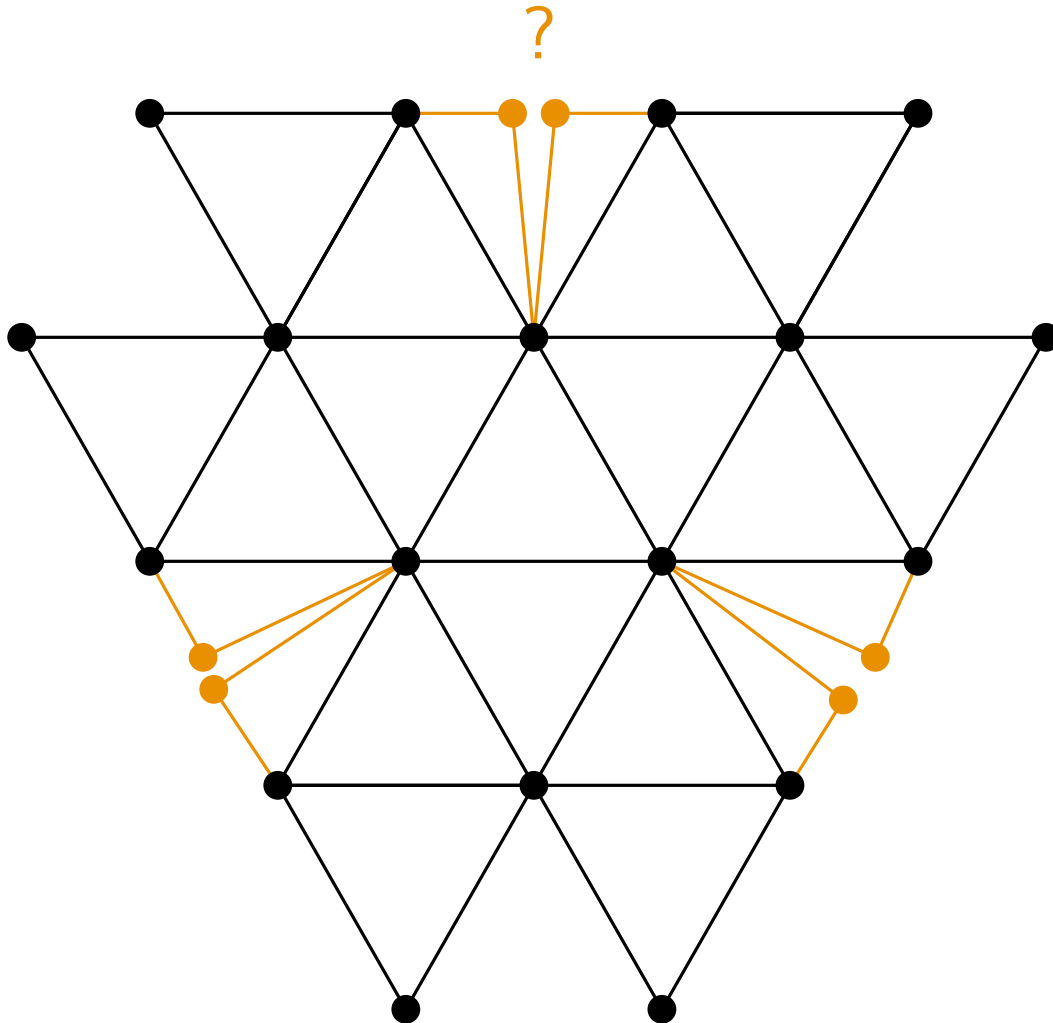


► $r = 4 + \varepsilon$

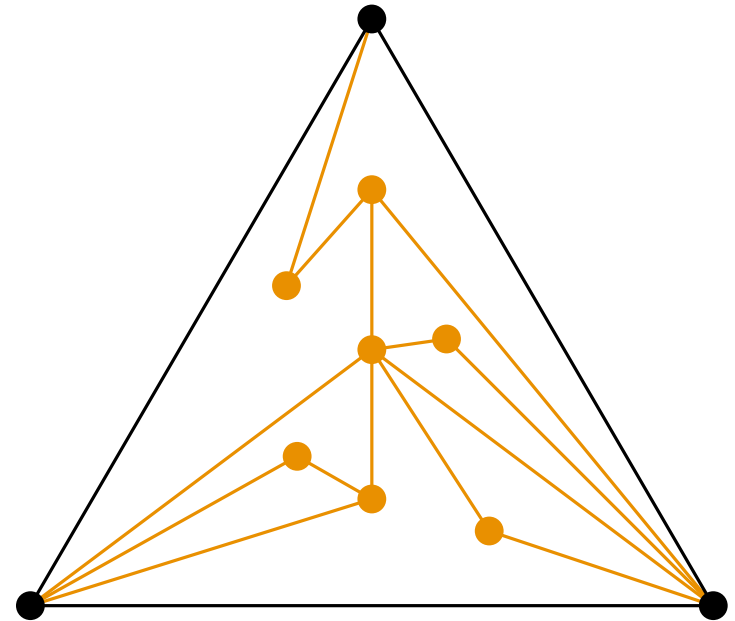


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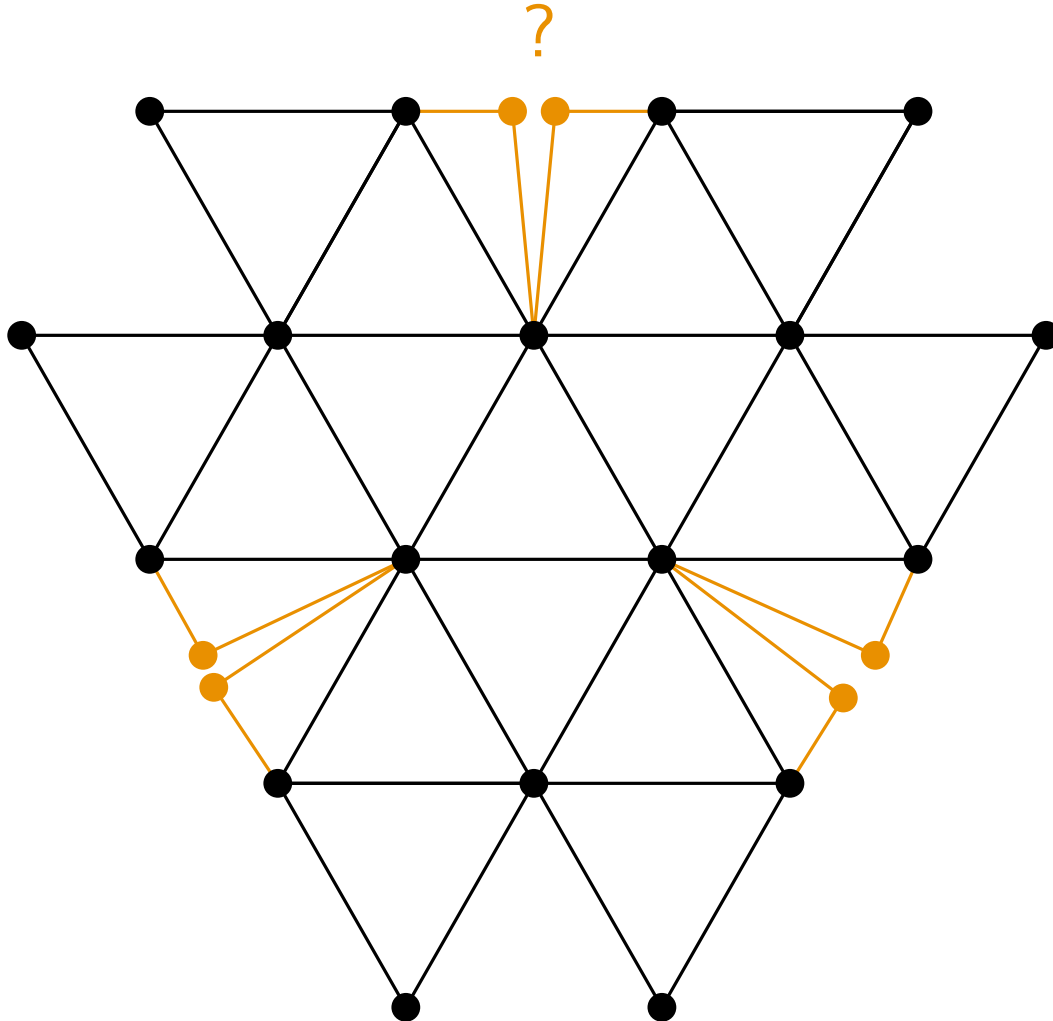


► $r = 8 + \varepsilon$

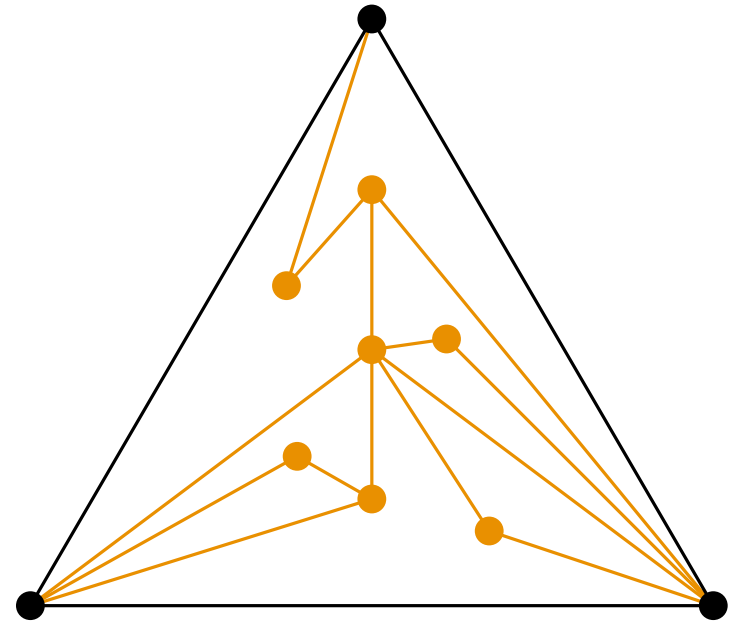


Adjacent and Nested Triangles

- ▶ $r = 2 + \varepsilon$
- ▶ *outerplanar graphs*

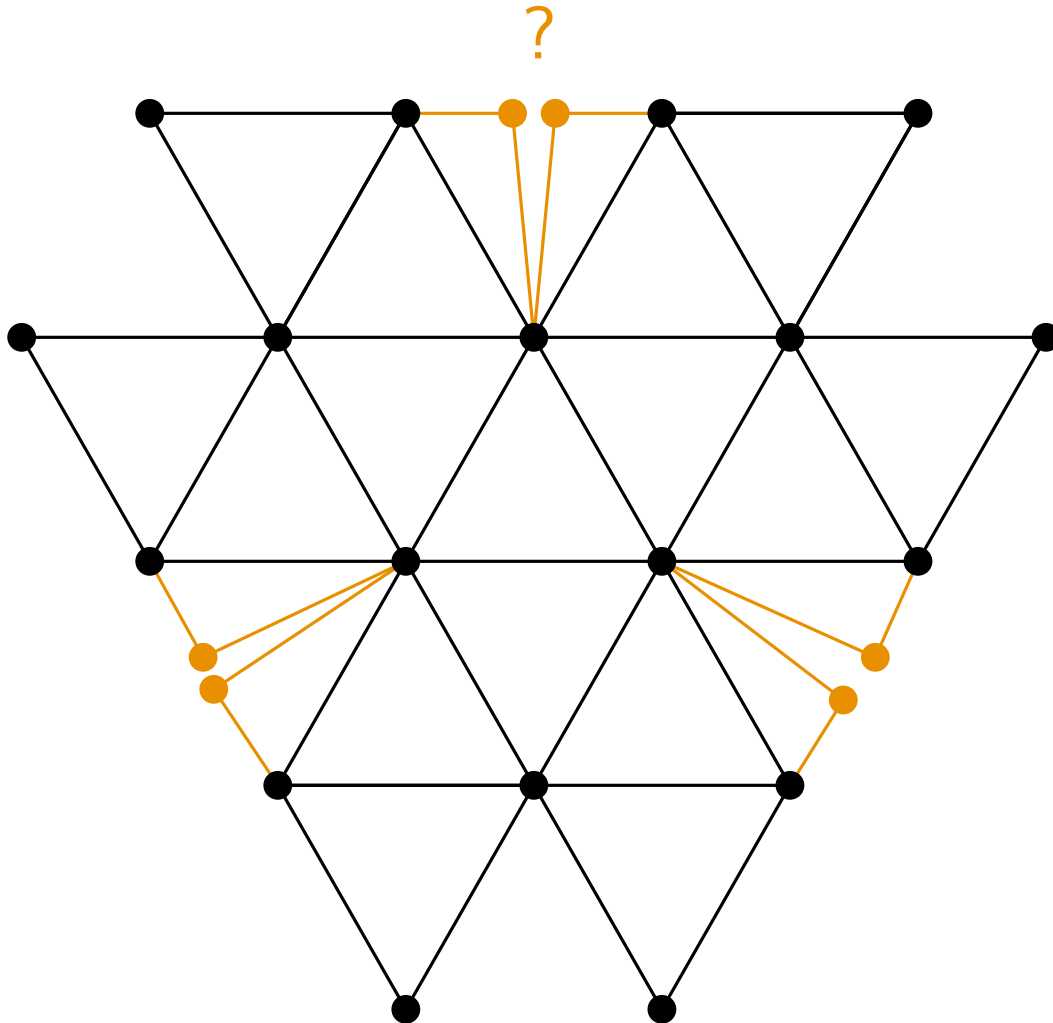


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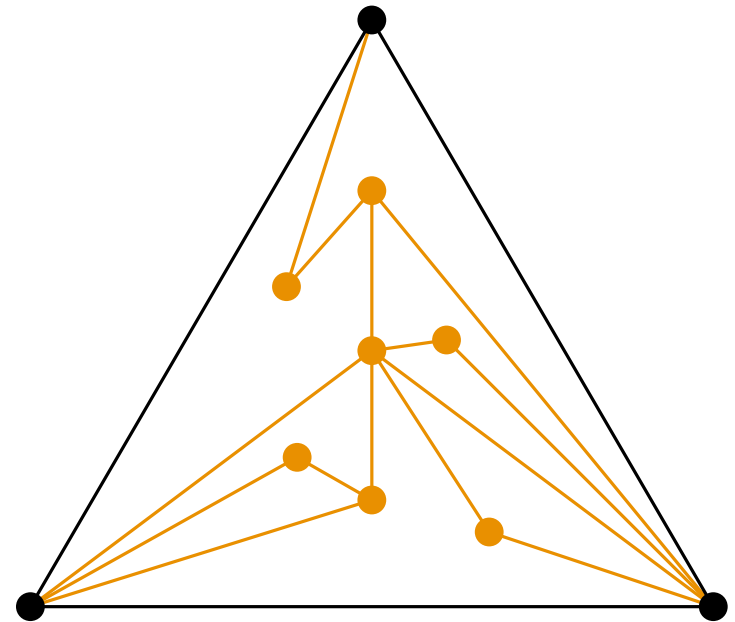


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...logarithmic?

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- ▶ Appendix
 - ▶ Planar 3-trees
 - ▶ Implementation of Complete k -ary Tree Drawer

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 - ▶ The height h of a rooted tree is defined by vertex with greatest height
- ▶ k -ary tree is a rooted tree with at most k children per inner node
- ▶ k -ary tree is *complete* if every inner node has k children and all leaves are on the same height

Drawing A Complete k -ary Tree T

- ▶ Unidirectional from top to bottom

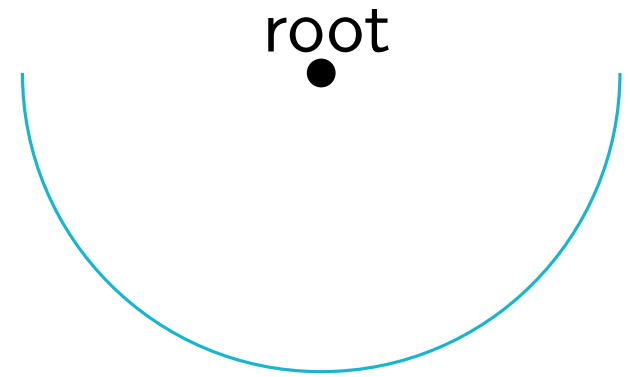
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- ▶ Unidirectional from top to bottom
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root
●

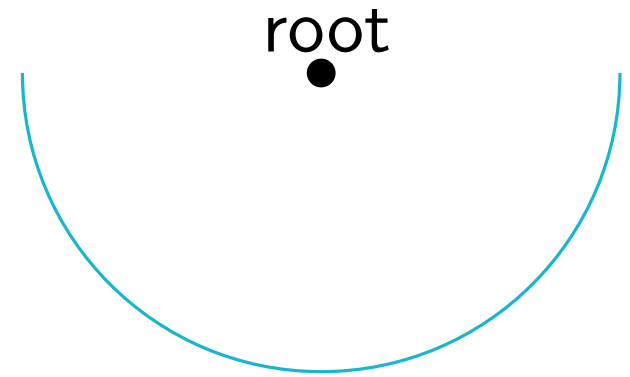
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- ▶ Draw circular arc C downwards



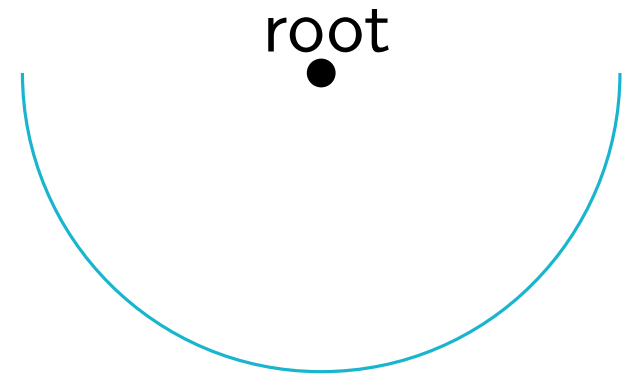
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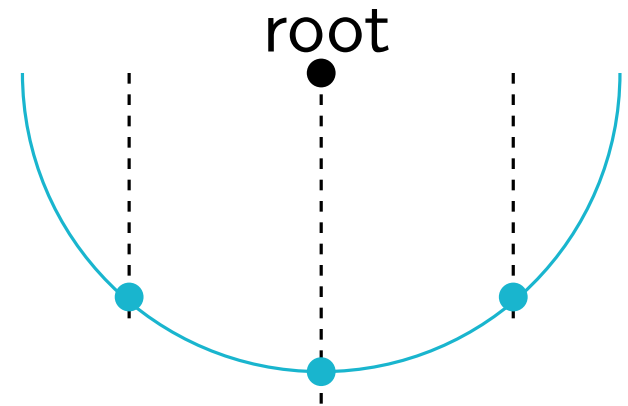
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- ▶ Place k children on C equidistantly



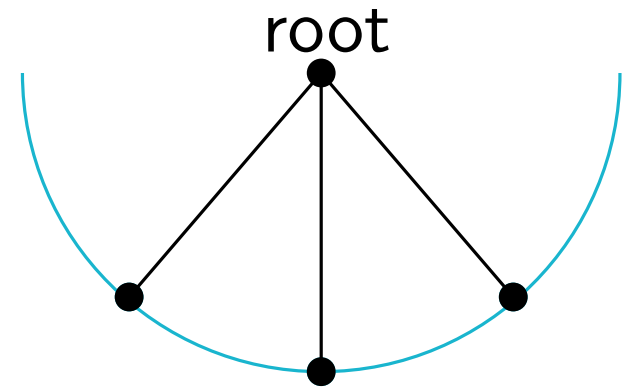
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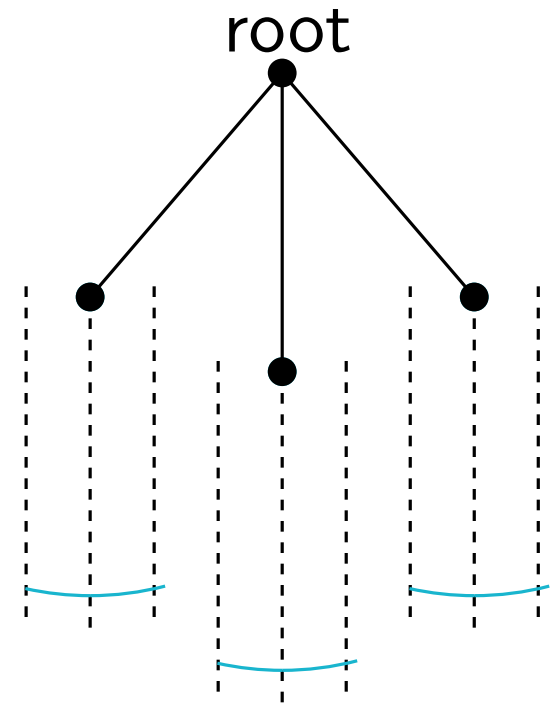
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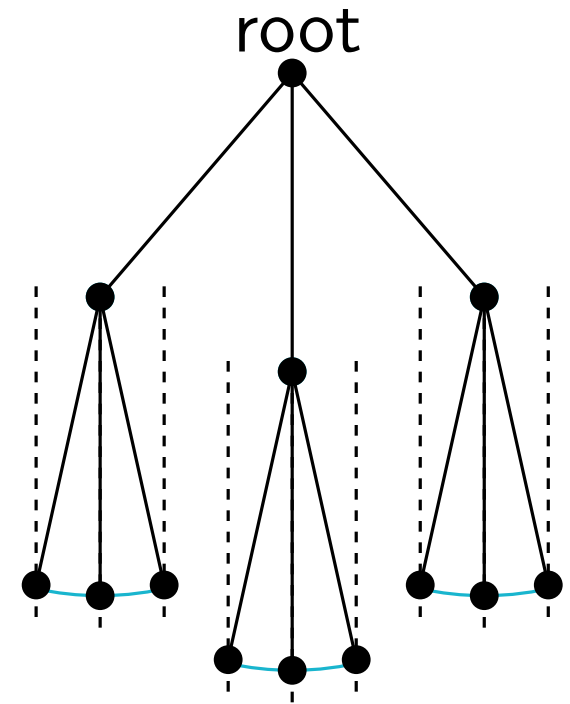
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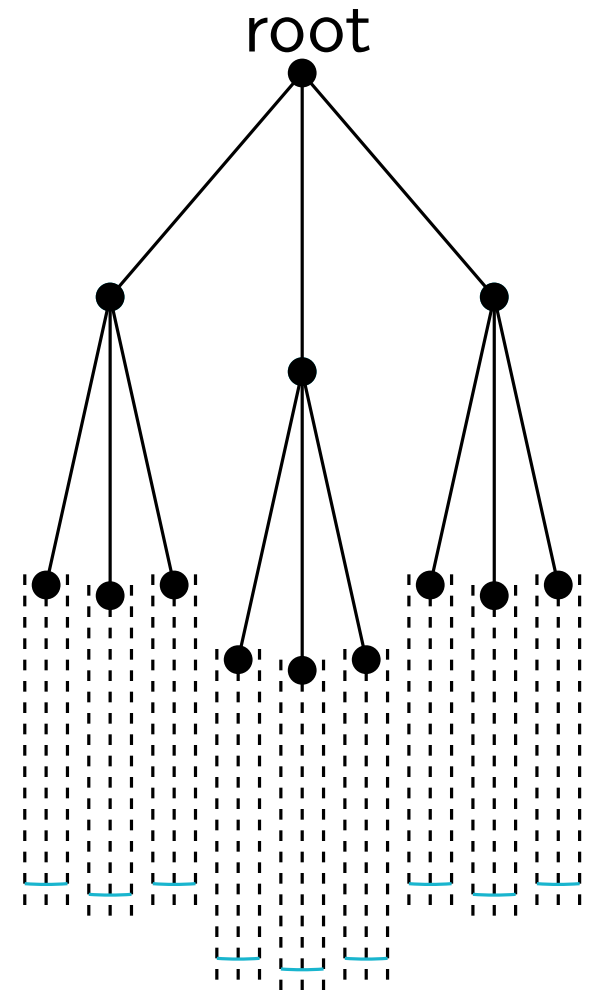
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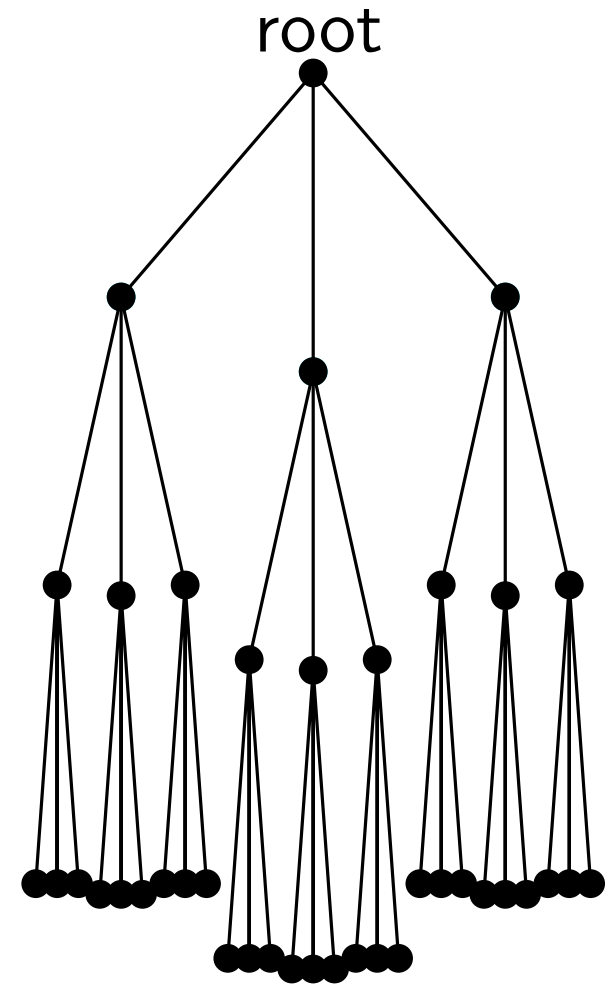
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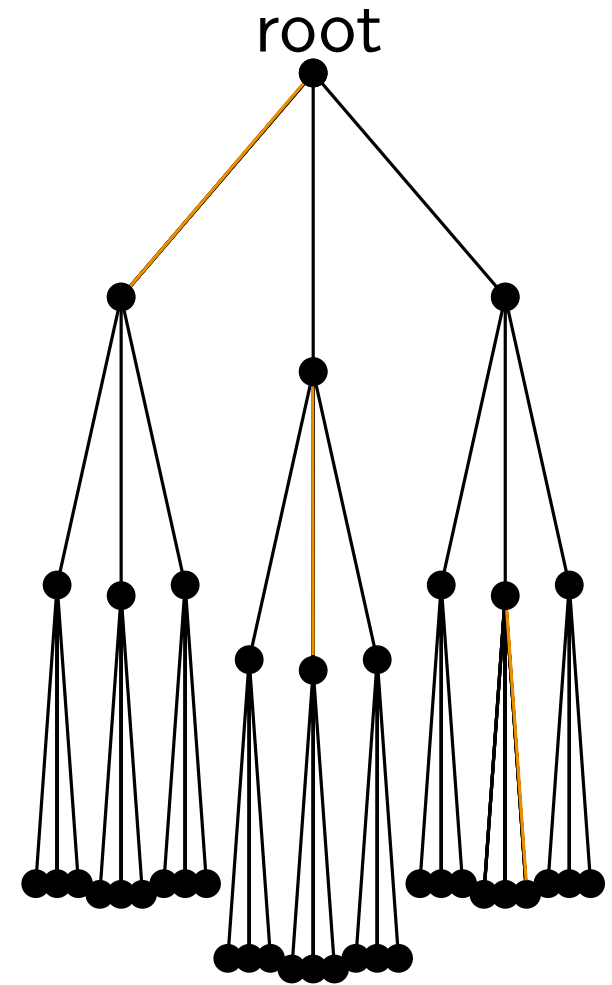
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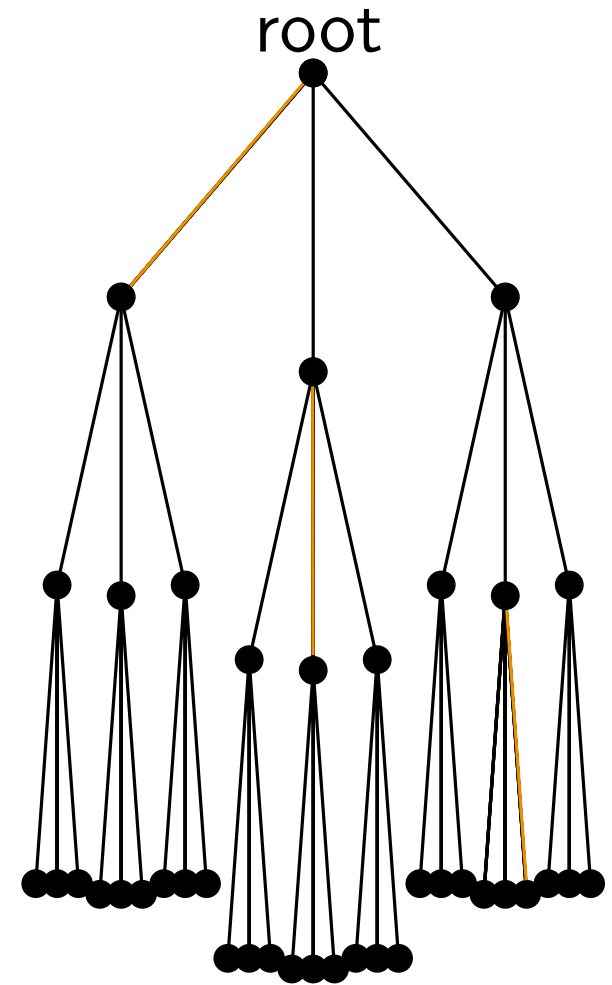
Results: Complete k -ary Trees

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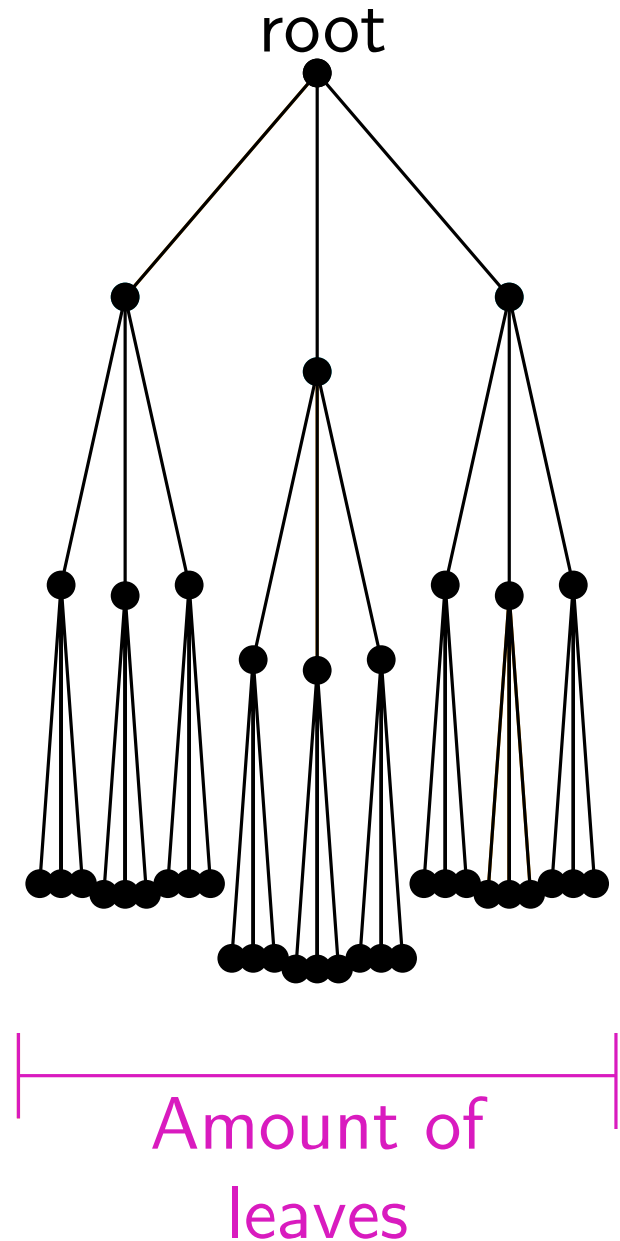
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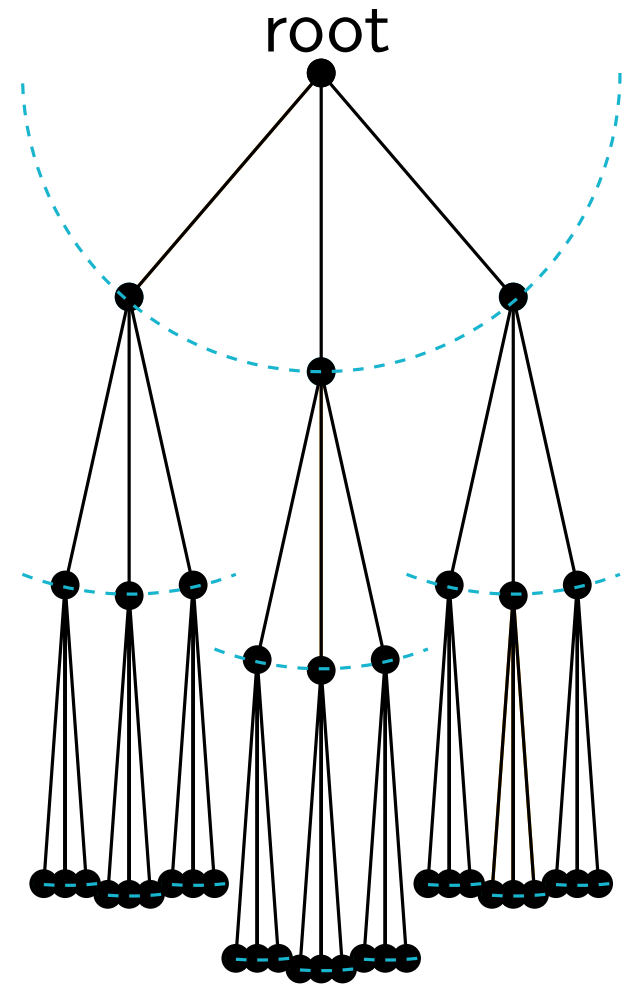
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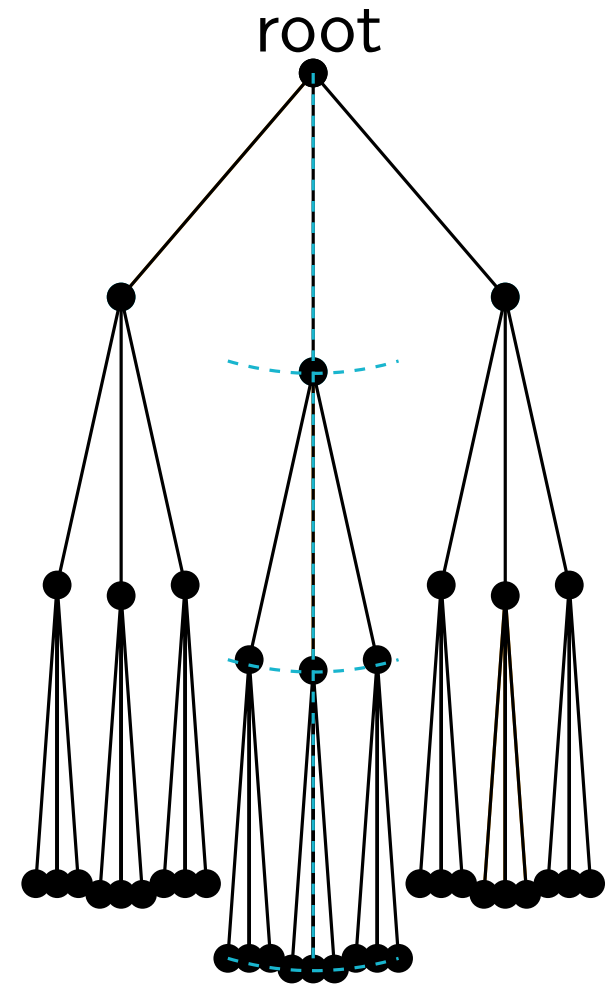
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- ▶ Radius of circular arcs: n



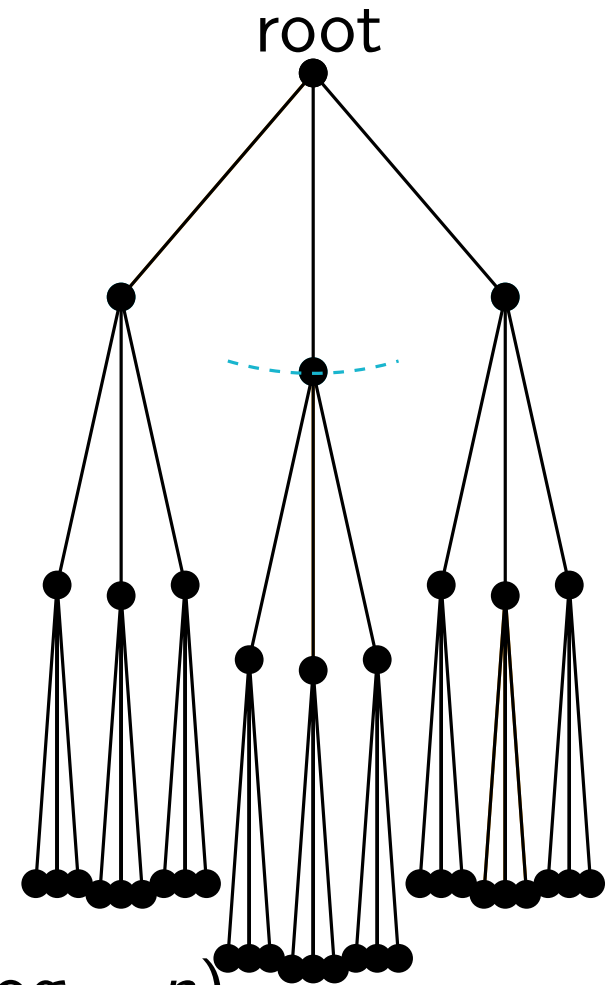
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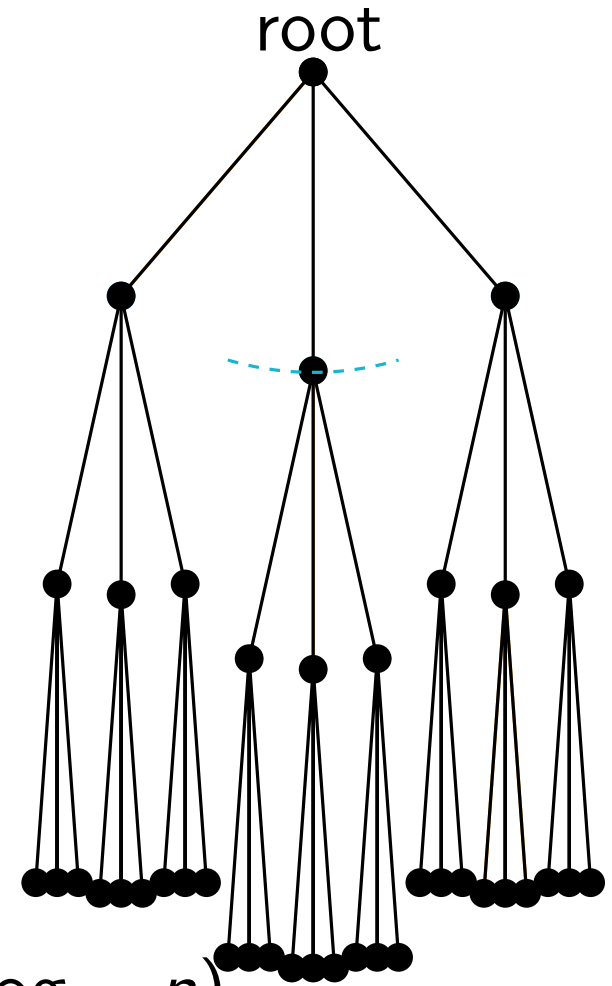
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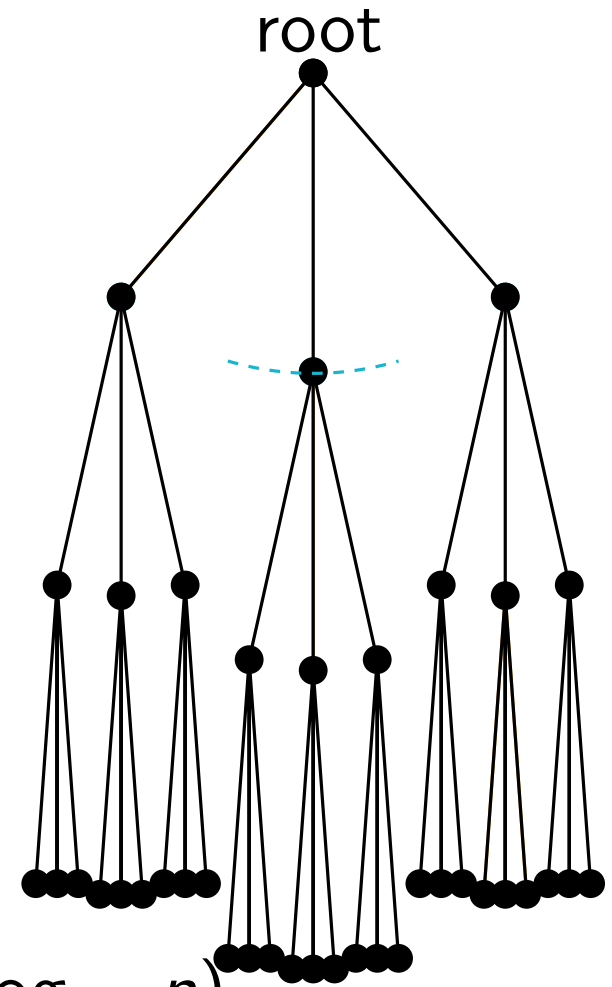
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admit a straight-line drawing with nearly-optimal ratio
on area $\mathcal{O}(n^2 \log n)$
-



Results: Complete k -ary Trees

- ▶ Every straight-line is $\sim n$ long
- ▶ Ratio of drawings value $1 + \varepsilon$,
 $0 \leq \varepsilon < 1$
- ▶ Area width in $\mathcal{O}(n)$
- ▶ Height of rooted tree T determines
- ▶ Radius of circular arcs: n
- ▶ Height of the drawing: $h \cdot n$
- ▶ Complete k -ary tree with height $\mathcal{O}(\log_{(k)} n)$
admit a straight-line drawing with nearly-optimal ratio
on area $\mathcal{O}(n^2 \log n)$



Results: General Trees

- ▶ Consider height h between $\Omega(\log n)$ and up to $\mathcal{O}(n)$

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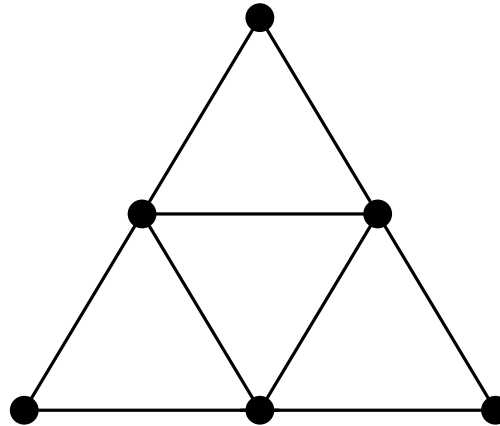
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Outerplanar Graphs and Series-Parallel Graphs

Outerplanar Graphs

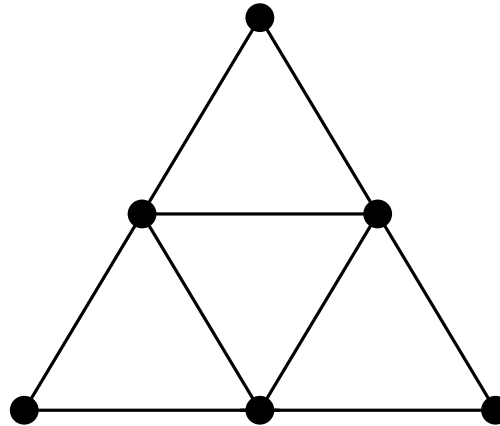
Outerplanar Graphs

- ▶ G admitting a drawing where every vertex lies on the outer face is *outerplanar*



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2-Terminal Series-Parallel Graphs

- ▶ Recursively defined class of planar graphs

2-Terminal Series-Parallel Graphs

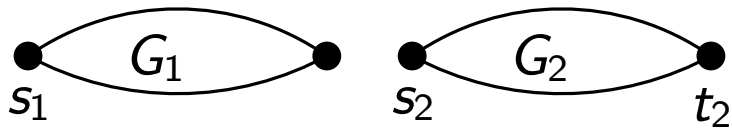
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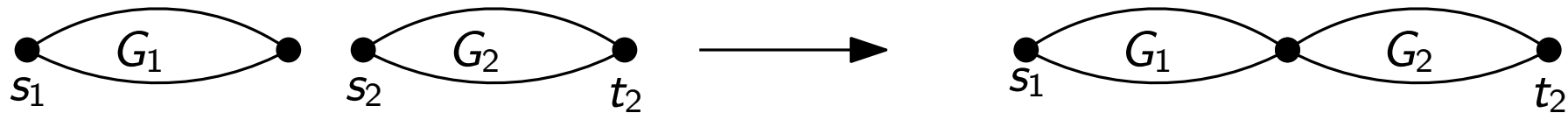
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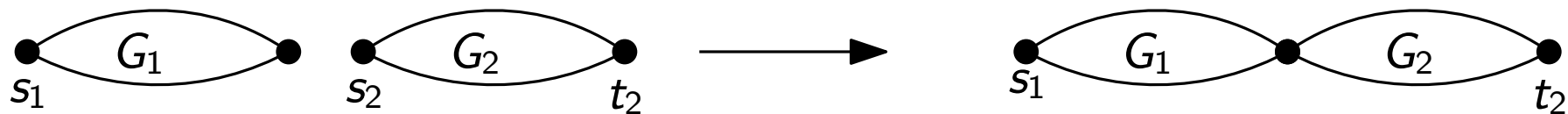
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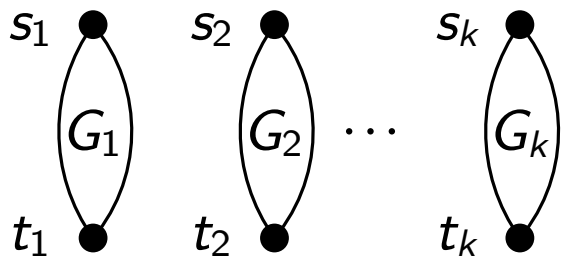


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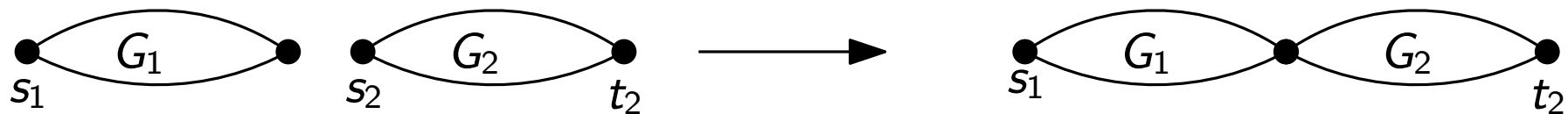


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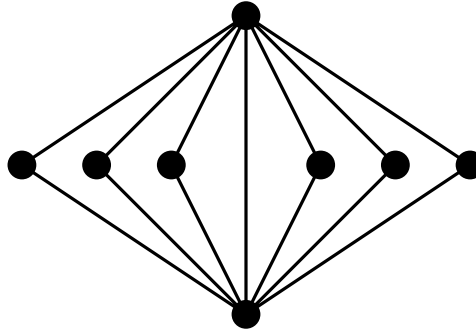


Series-Parallel Graph

- ▶ a *series-parallel* graph has 2-terminal SP-graphs as biconnected components

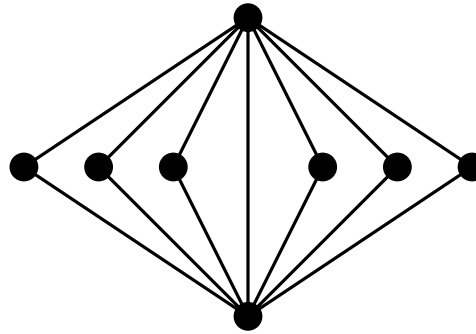
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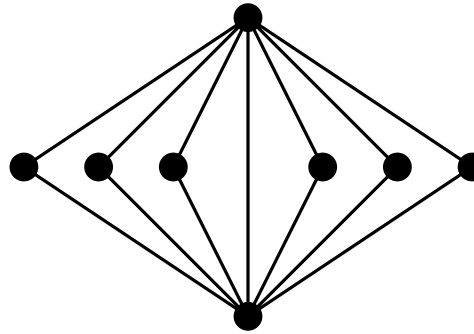
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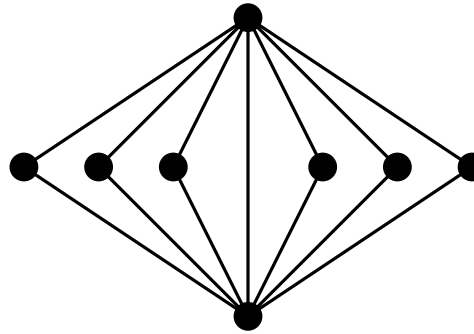
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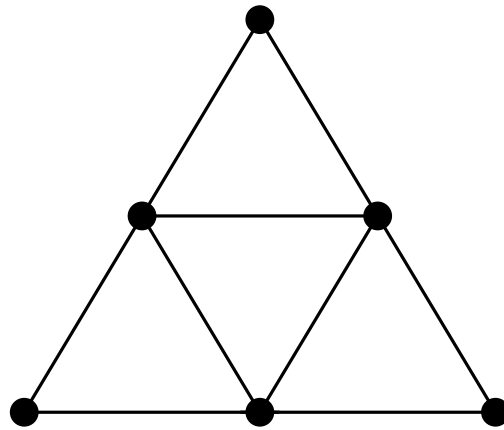
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- ▶ maximal outerplanar graphs are maximal SP-graphs
 - ▶ Drawing approaches deal with outerplanar graphs first
 - ▶ Extended to SP-graphs, if possible

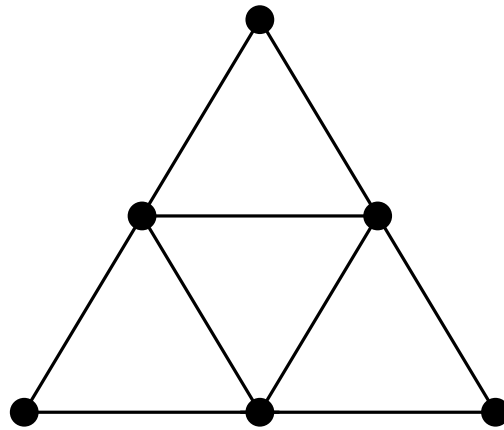
Approach I: Using Weak Dual Graph

- ▶ outerplanar graph *maximal* if no edges can be inserted without destroying outerplanarity



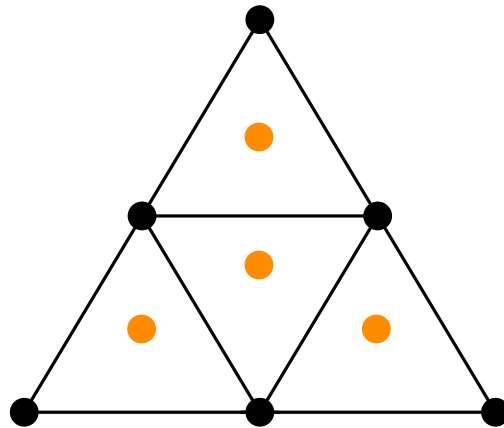
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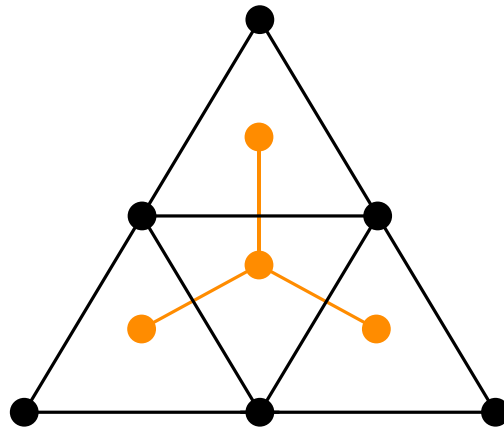
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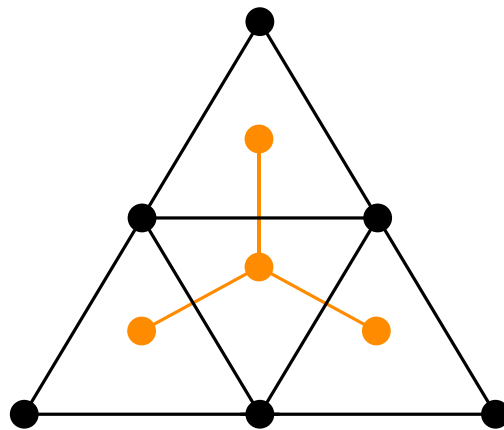
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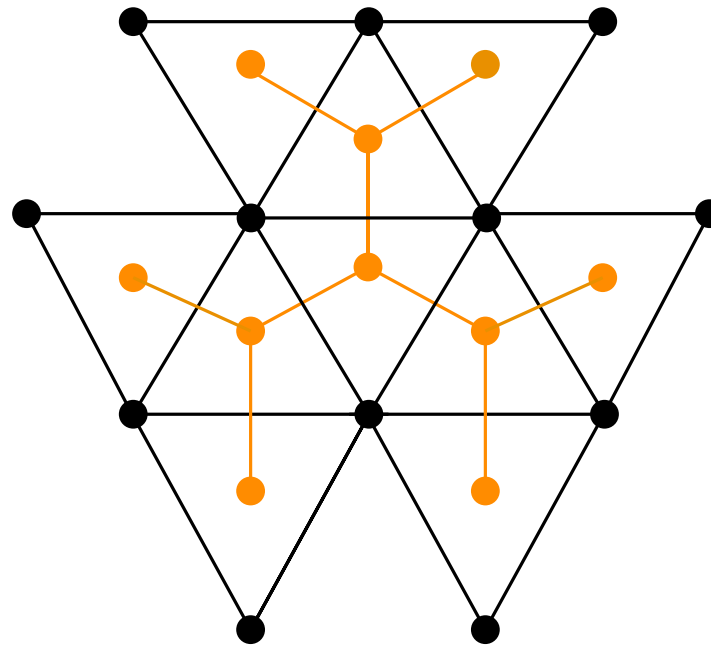
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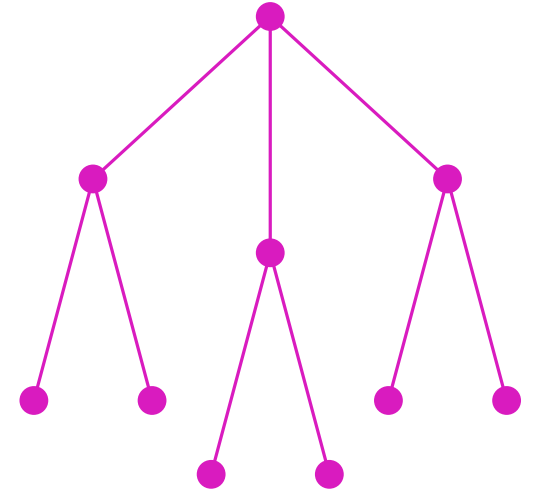
- ▶ G^* for maximal outerplanar graph is *simple tree*
- ▶ Degree of G^* is at most 3

Approach I: Using Weak Dual Graph

- ▶ Make G maximal

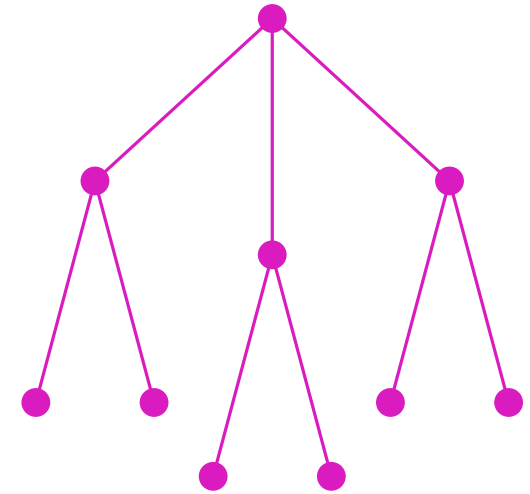
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- ▶ Make G maximal
- ▶ Draw G^*



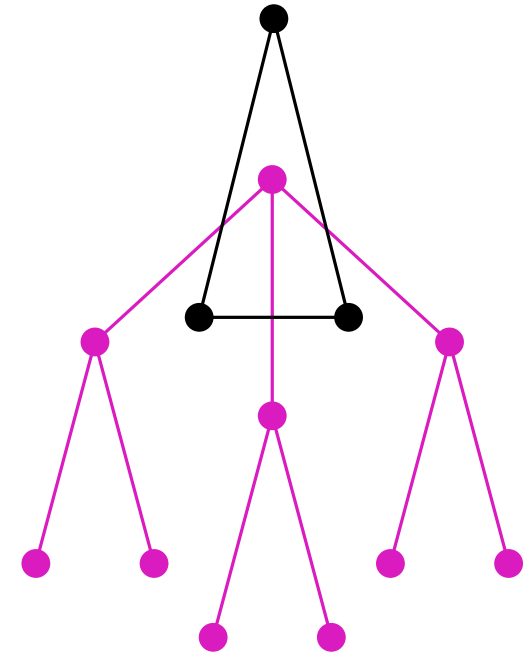
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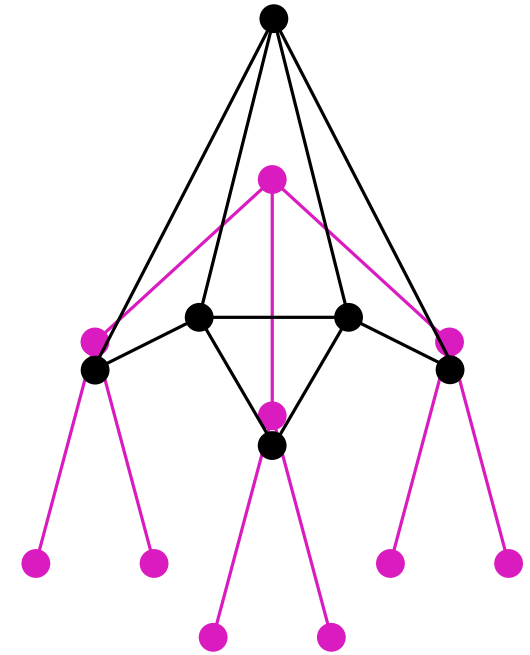
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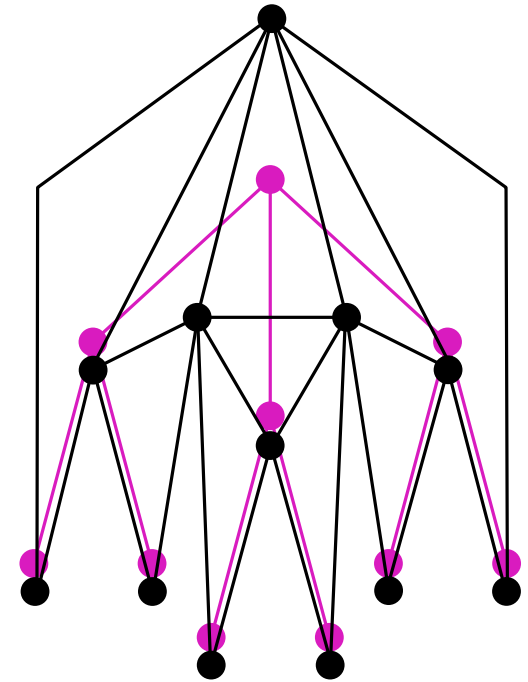
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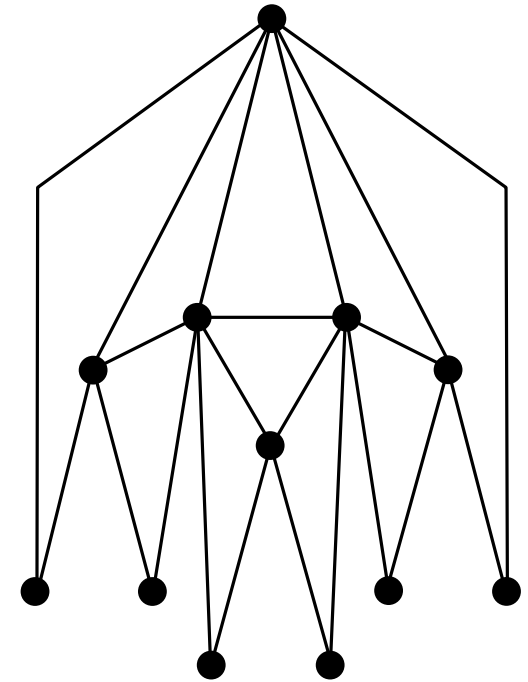
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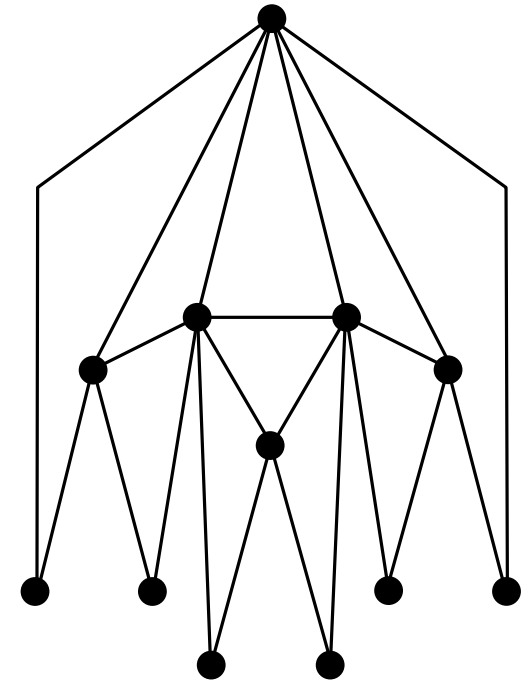
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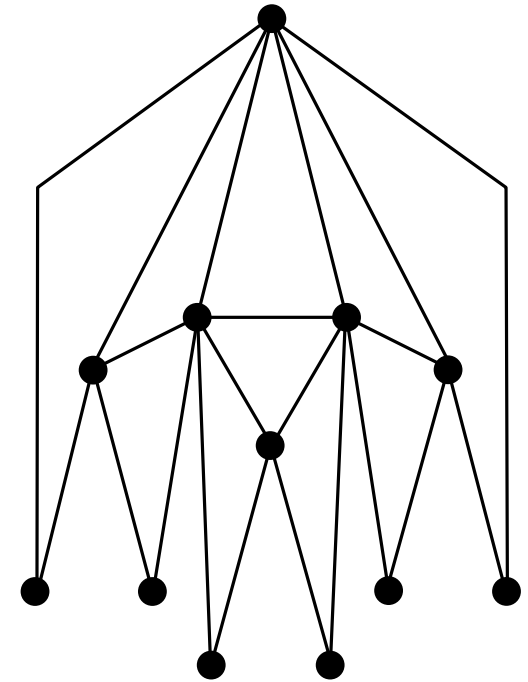
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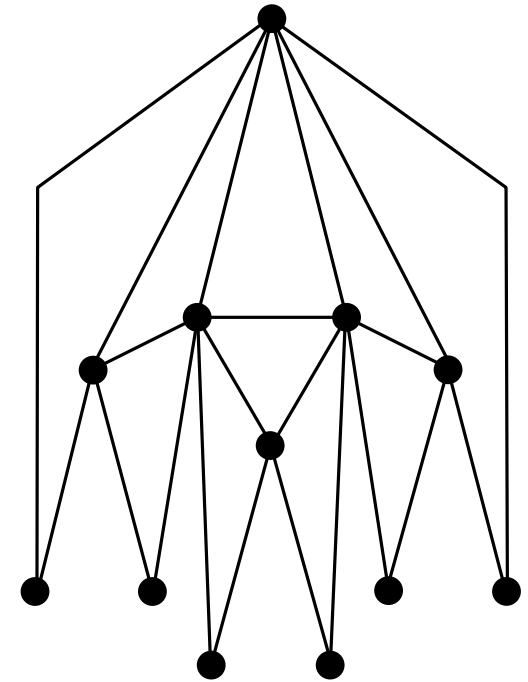
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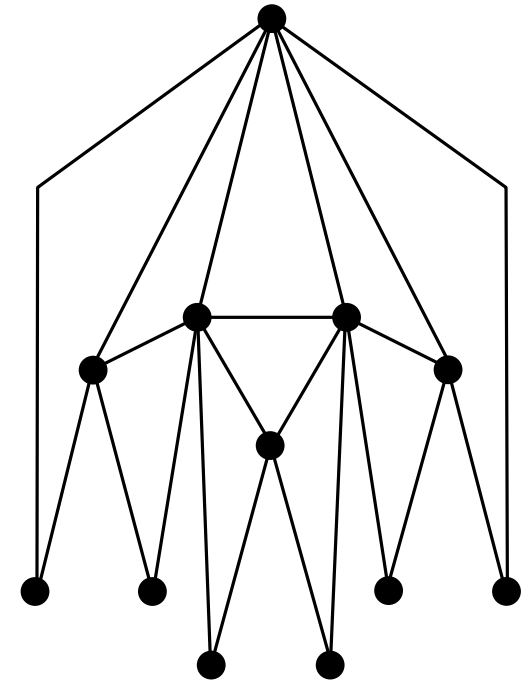
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- ▶ If height of G^* in $\mathcal{O}(\log n)$
 - ▶ Radius in $\Theta(n)$, drawing height in $\mathcal{O}(n \log n)$
 - ▶ ratio in $\mathcal{O}(\log n)$ on area $\mathcal{O}(n^2 \log n)$, 1 bend per edge



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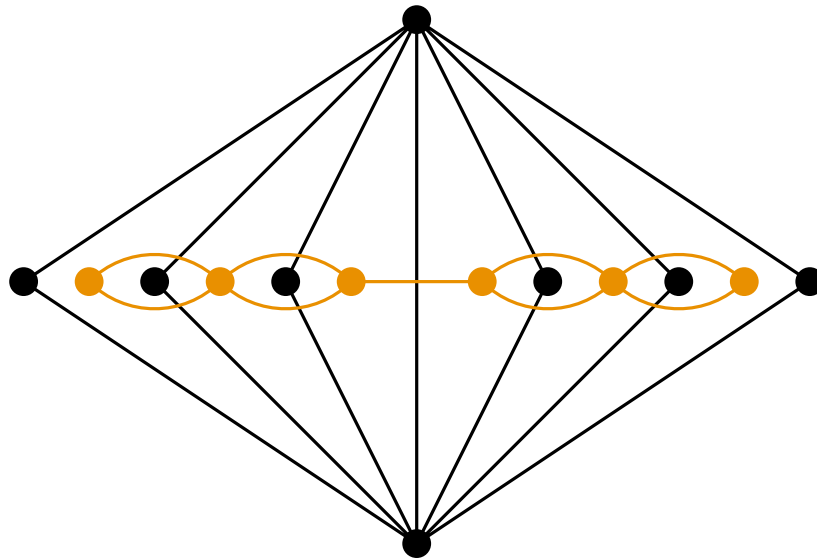
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Approach II: Using tree decomposition

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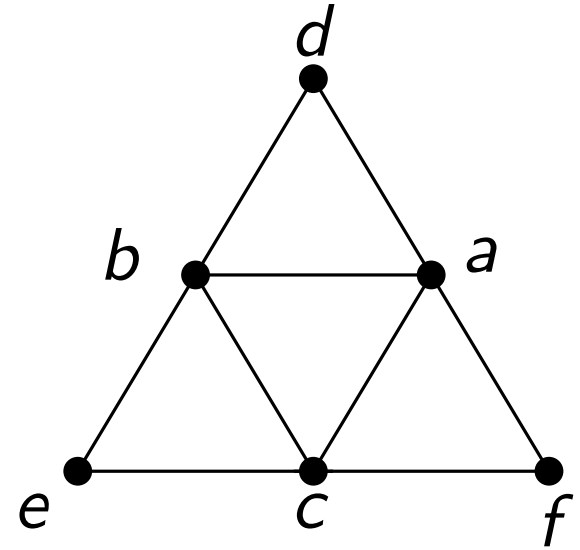
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- ▶ *treewidth* is least width of any tree decomposition of G

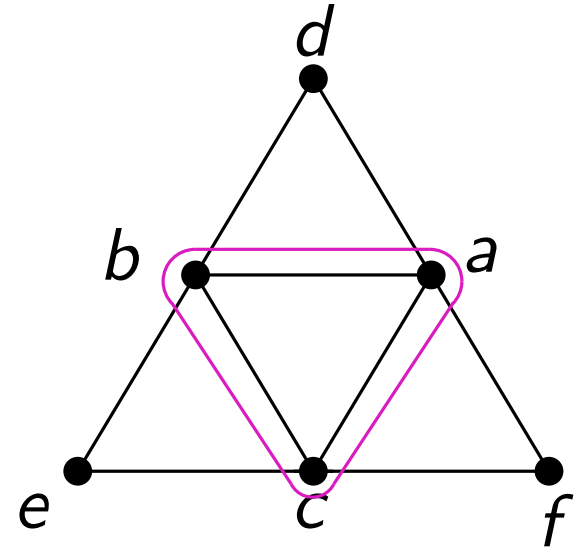
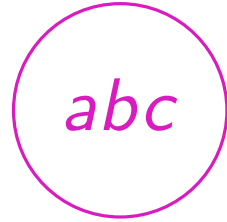
Tree Decomposition Of SP-Graphs

► (T, W)



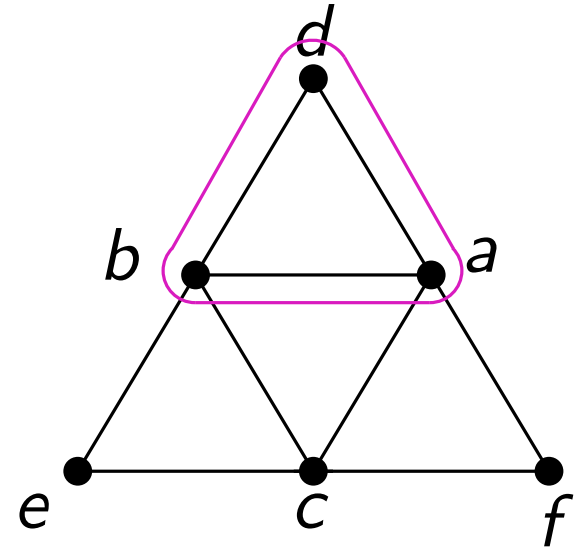
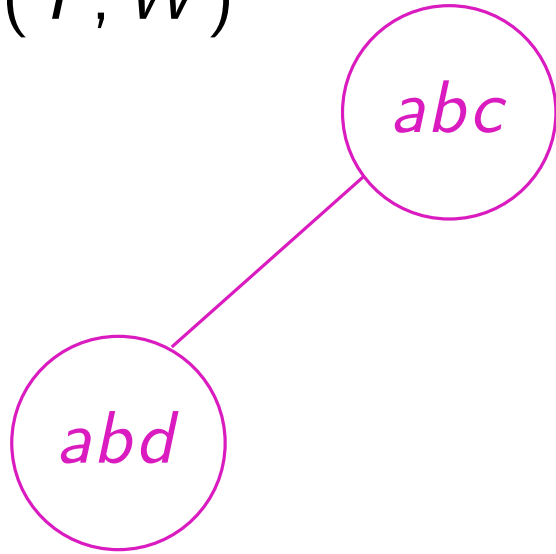
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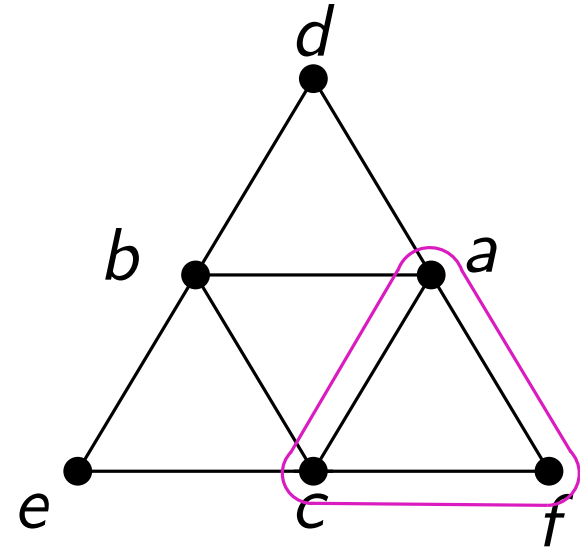
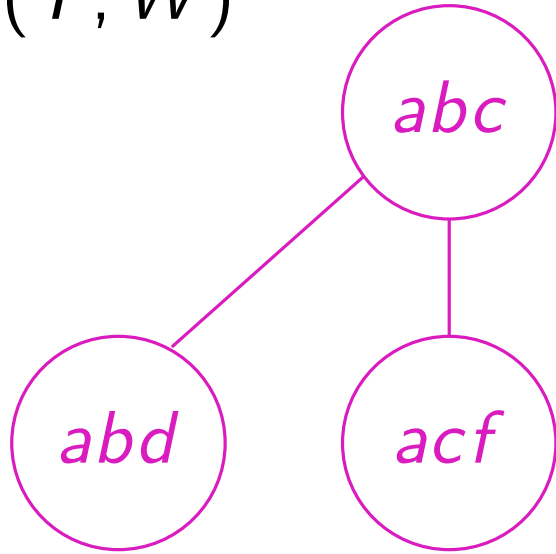
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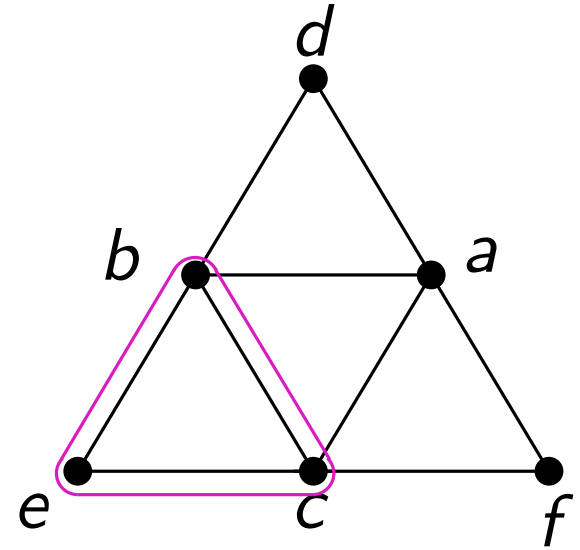
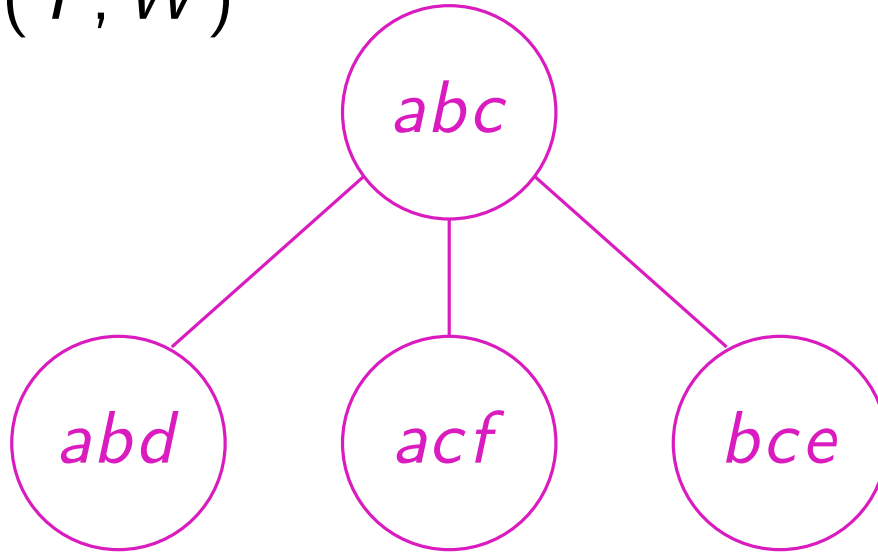
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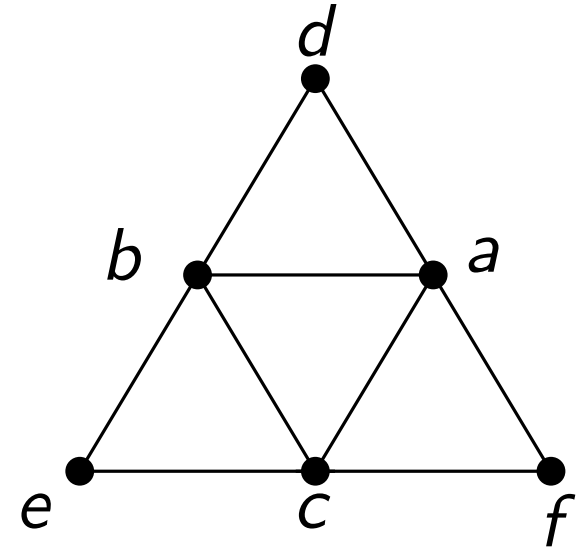
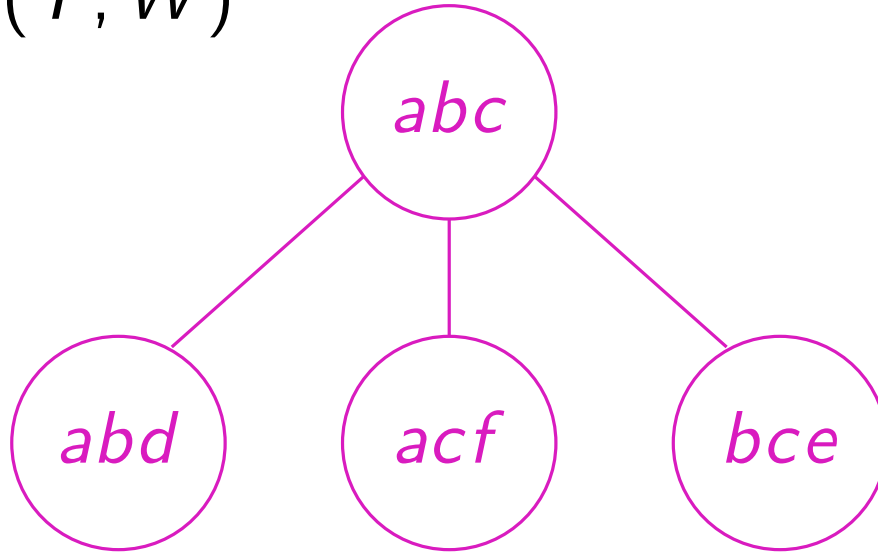
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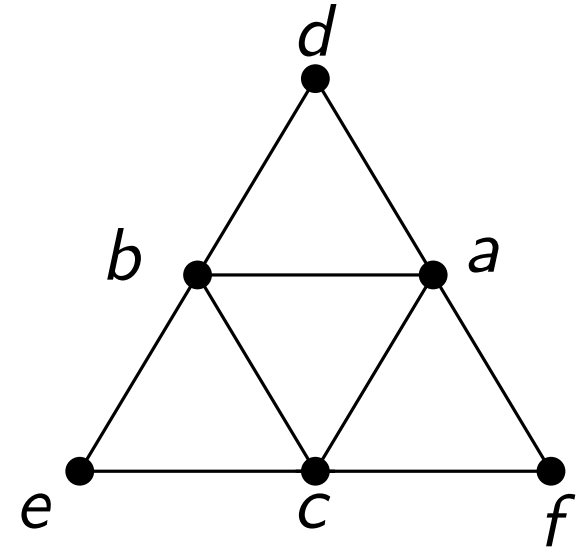
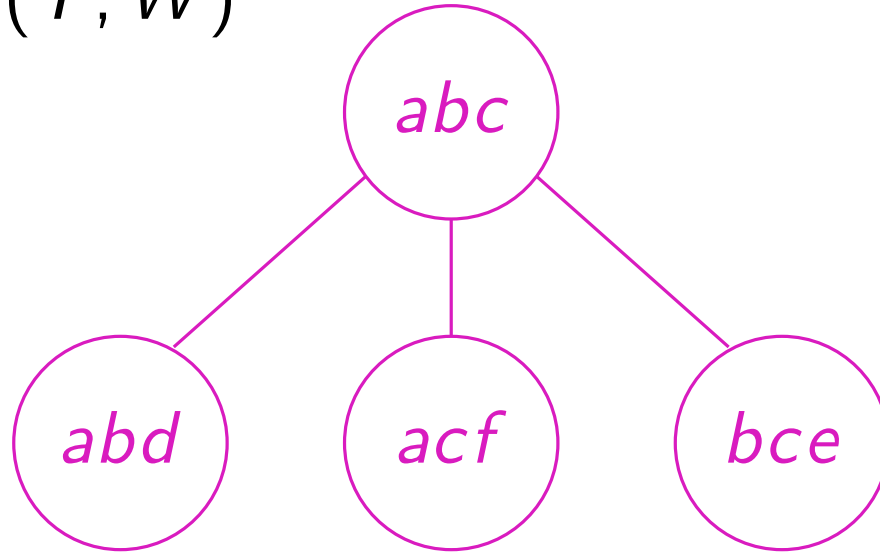
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► treewidth of 2 for any maximal series-parallel graph

Tree Decomposition Of SP-Graphs

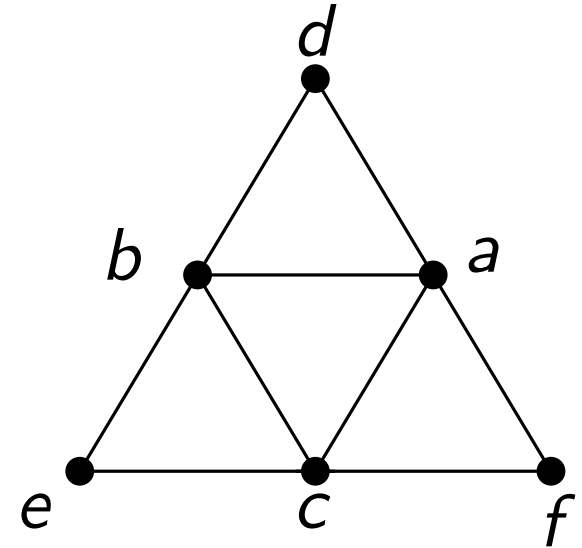
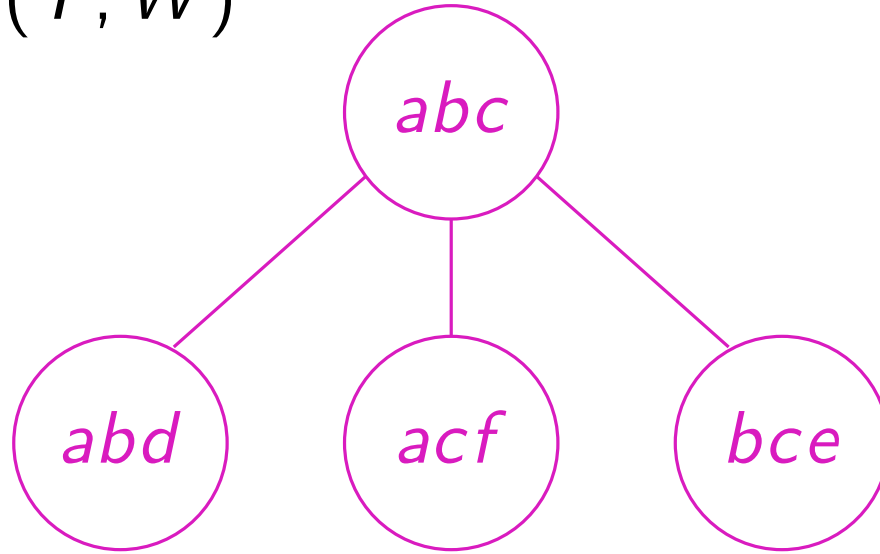
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- treewidth of 2 for any maximal series-parallel graph
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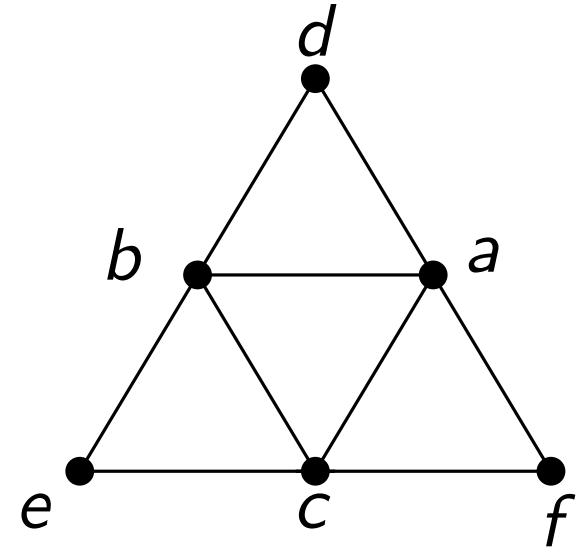
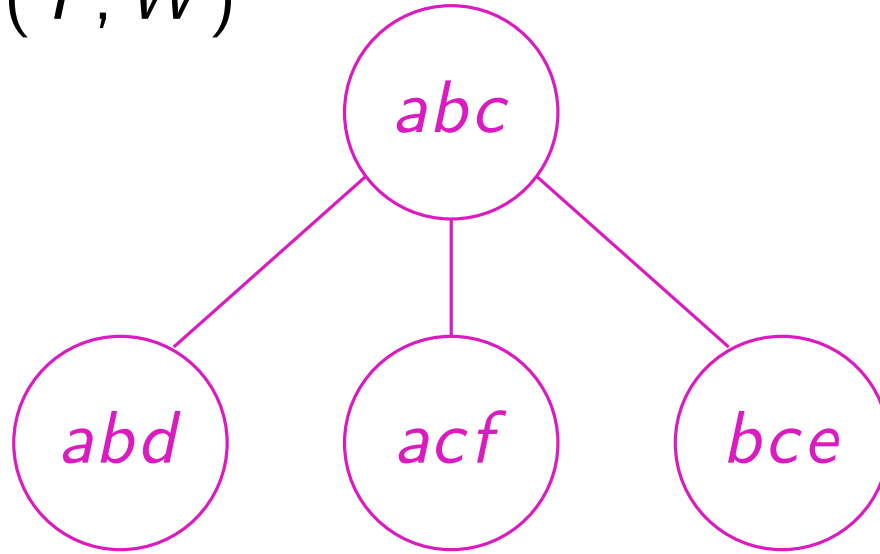
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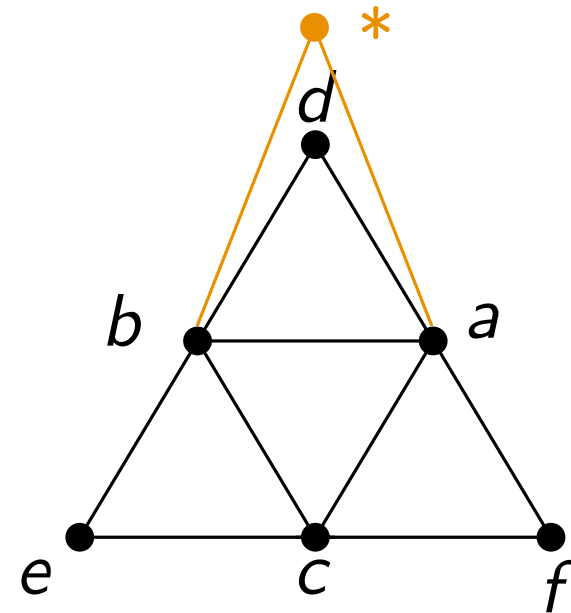
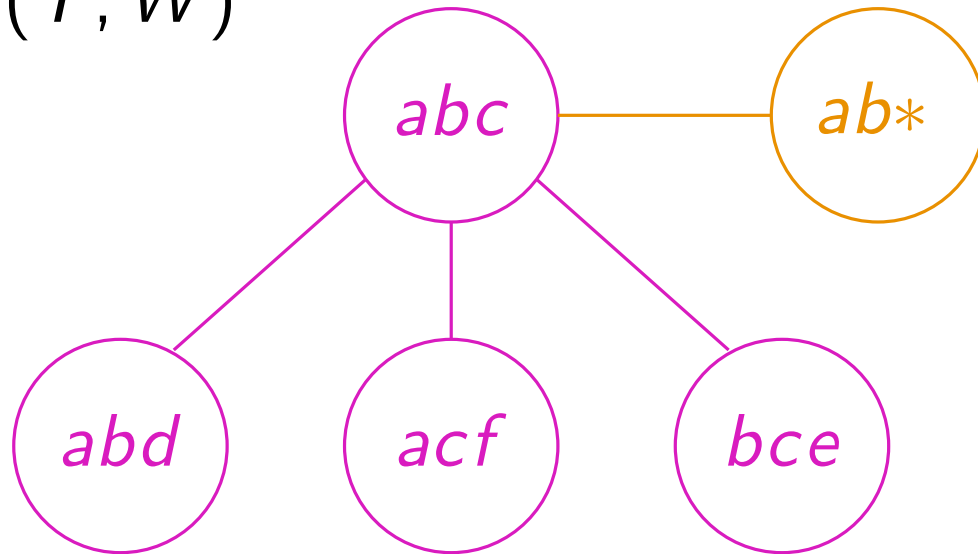
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Tree Decomposition Of SP-Graphs

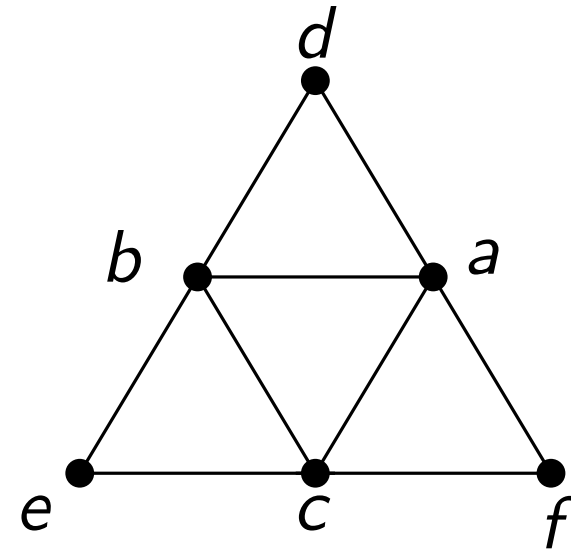
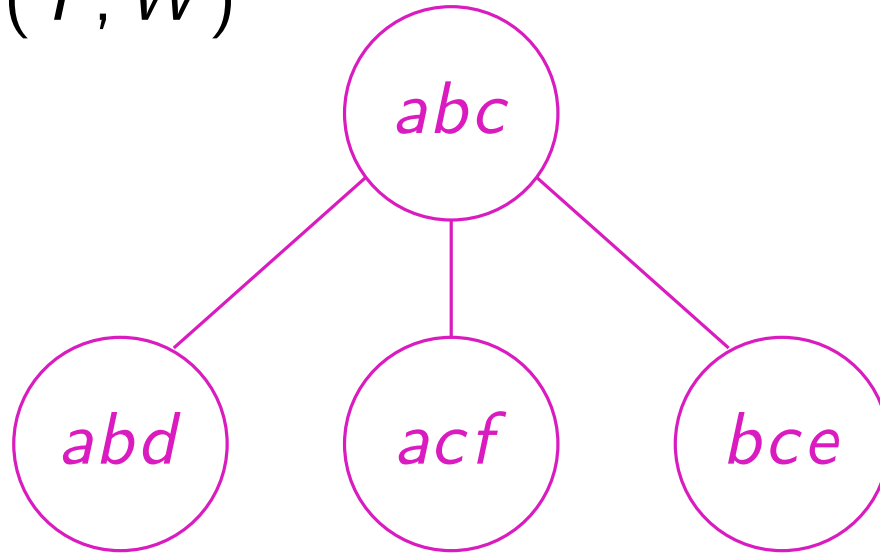
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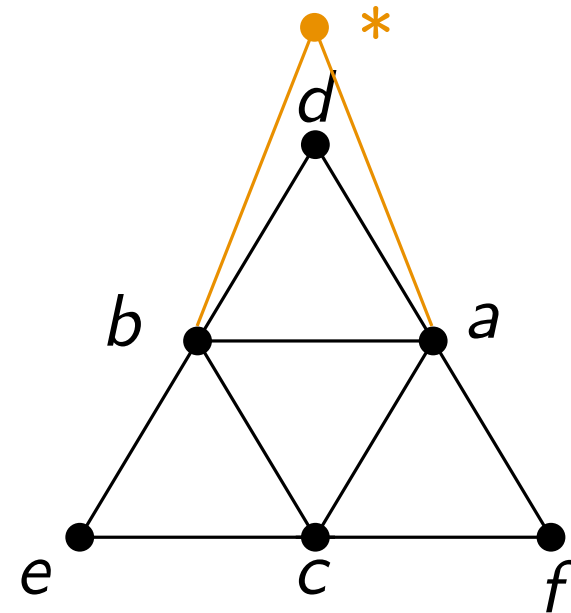
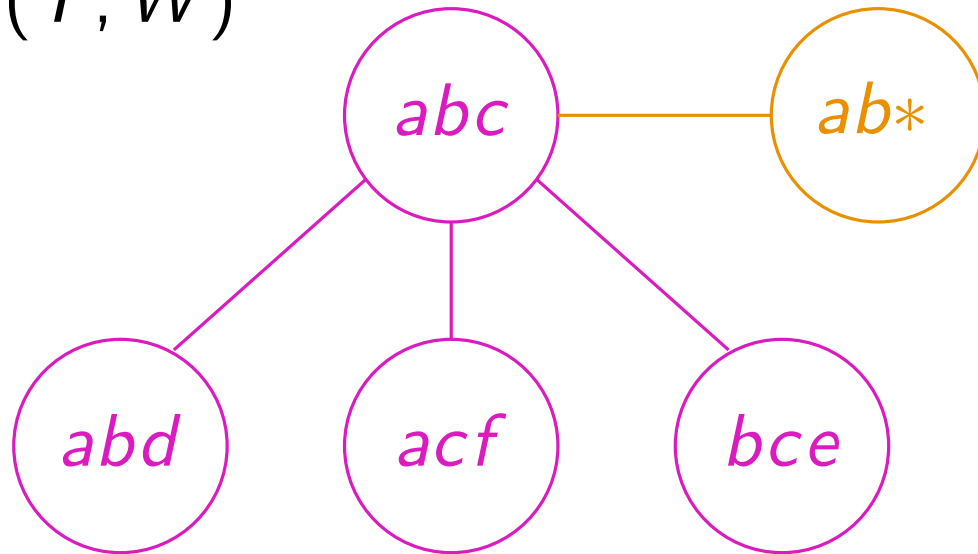
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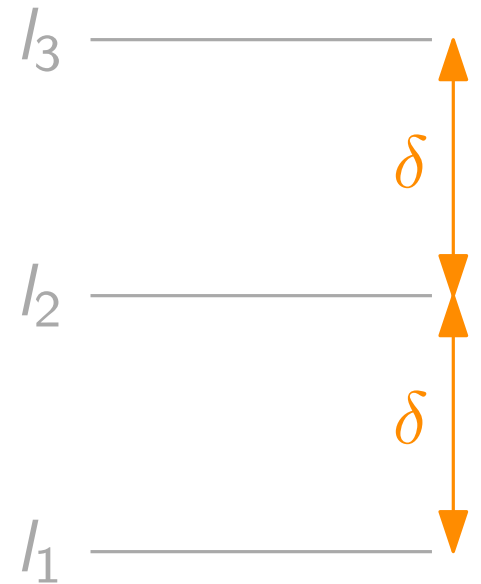
l_3 —————

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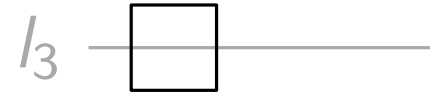
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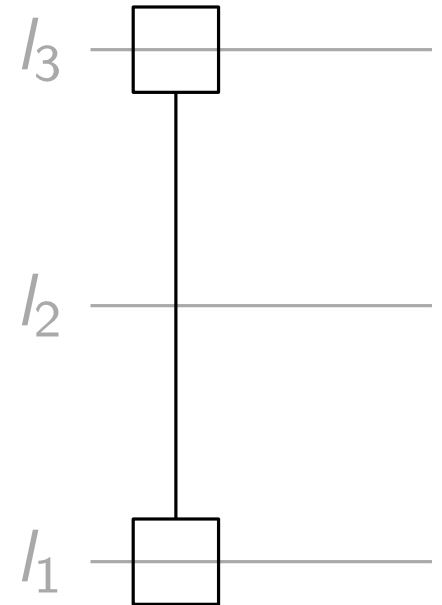
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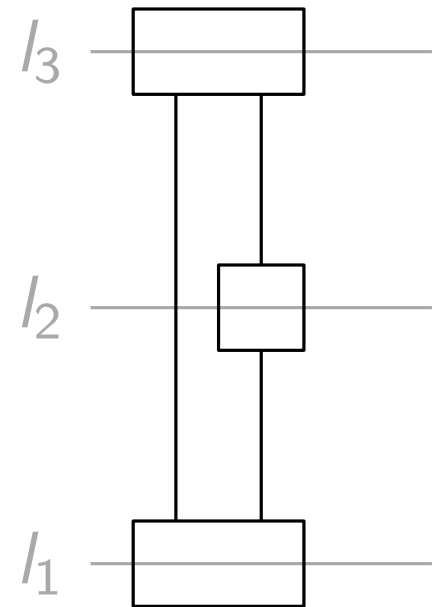
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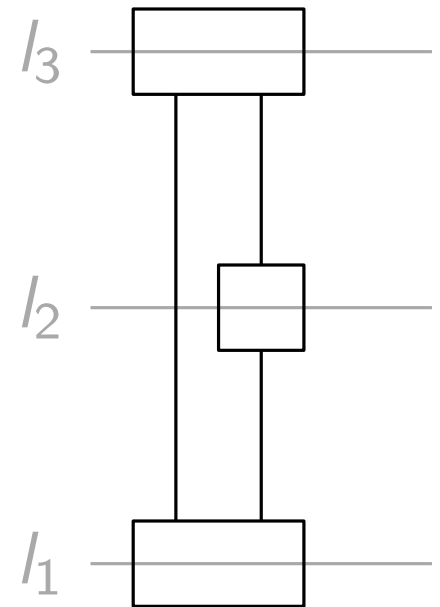
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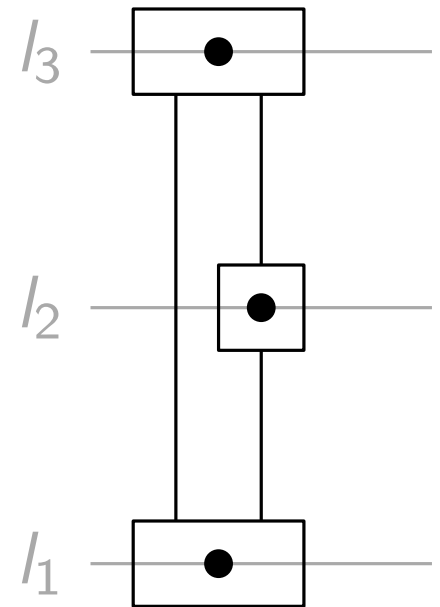
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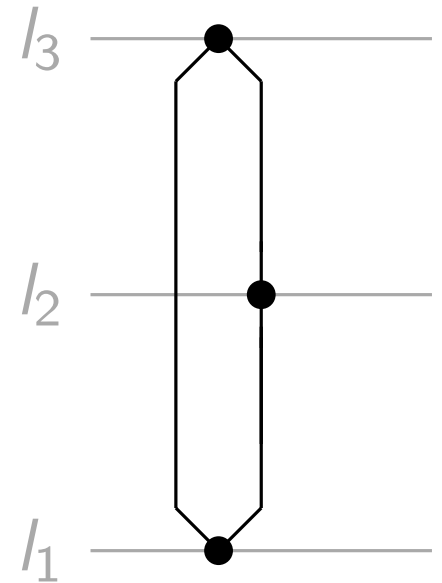
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- ▶ Transfer B into a polyline drawing, remove edges



Vertex Insertions In Box Drawing

- ▶ Vertices inserted on layers as extendable boxes

Vertex Insertions In Box Drawing

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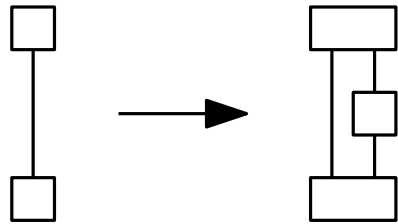
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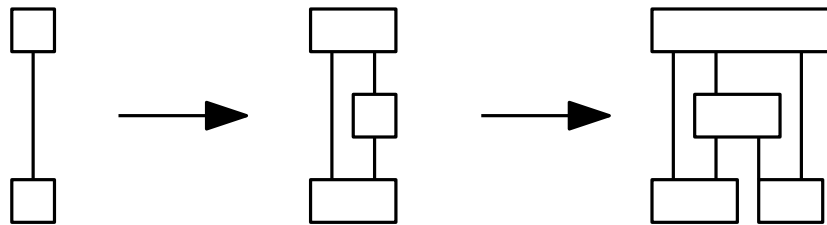
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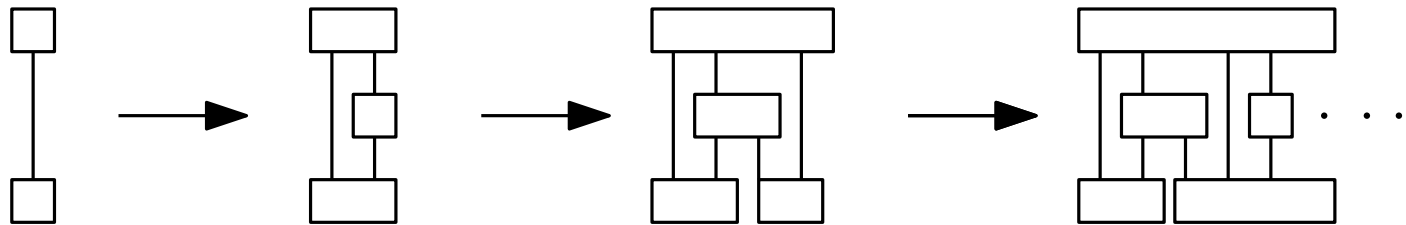
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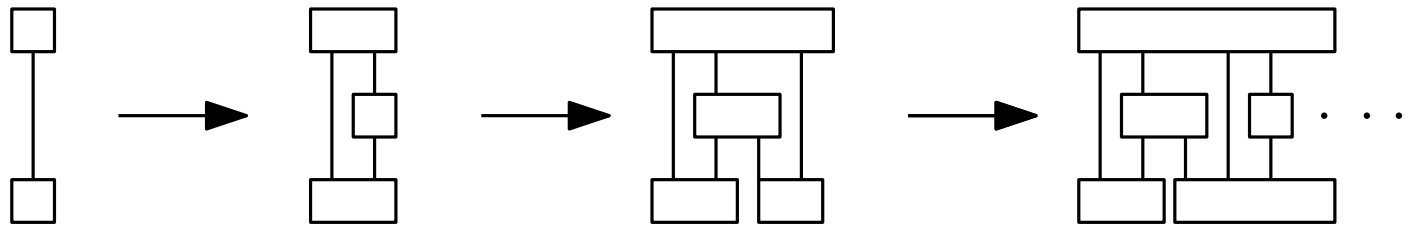
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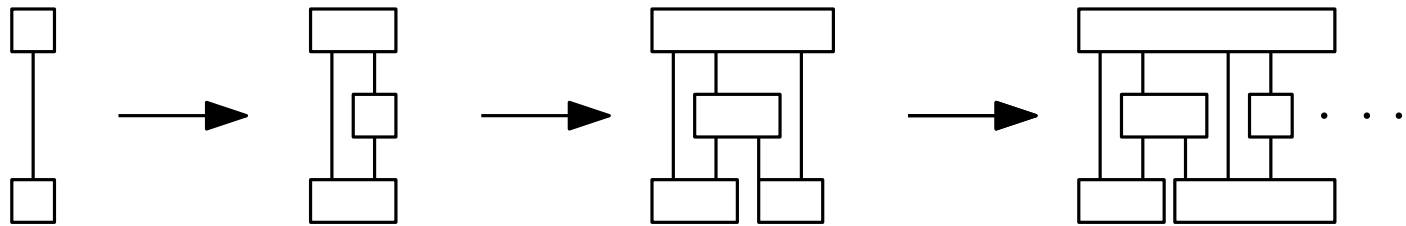
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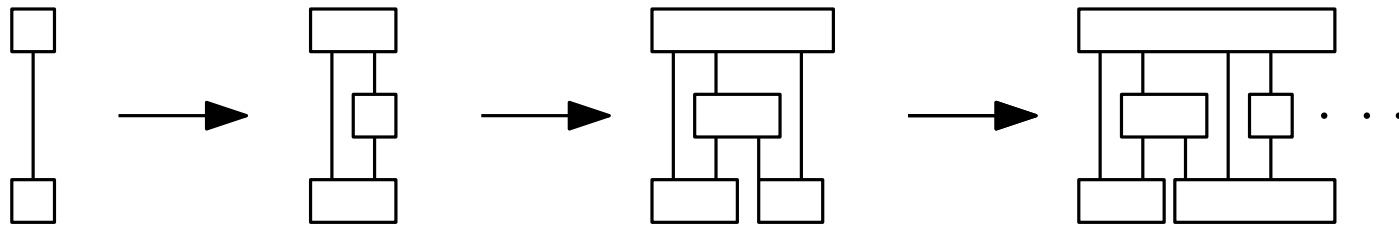
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- ▶ A layer is *reachable* if it is free, v can be placed on it and edges to a, b can be drawn without destroying planarity

Reachability of Layers

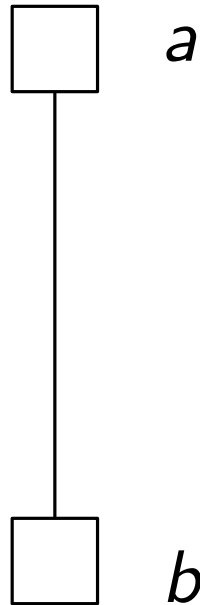
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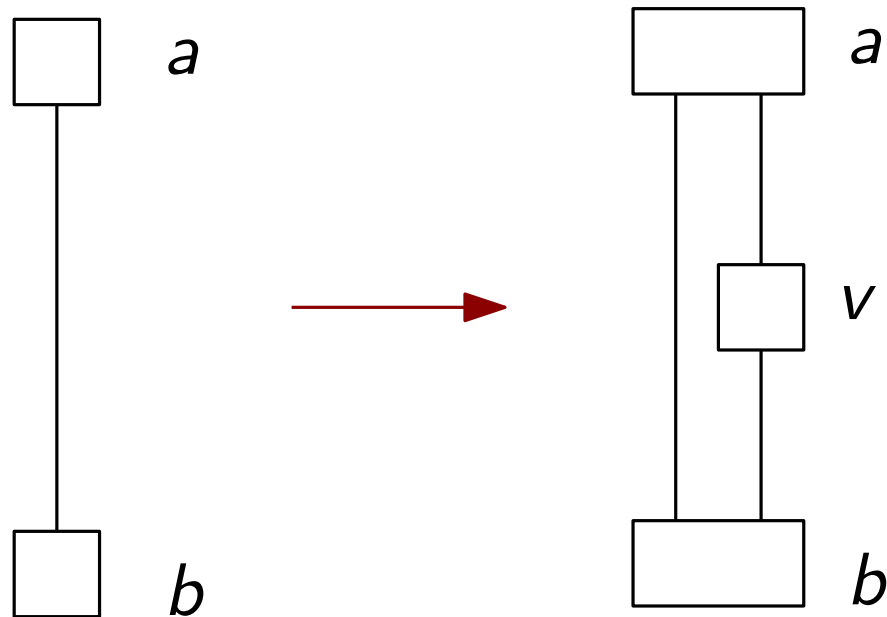
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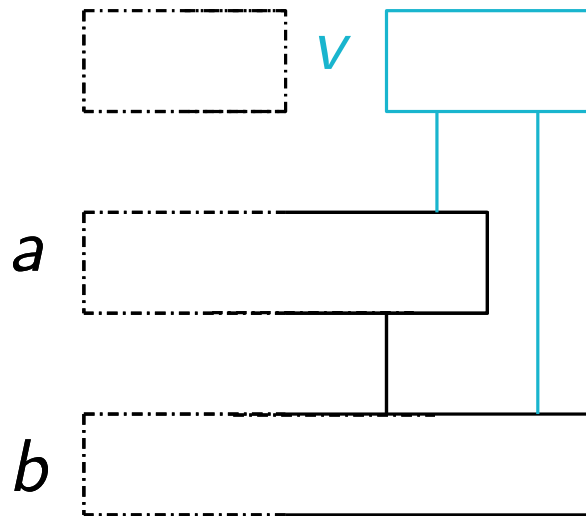


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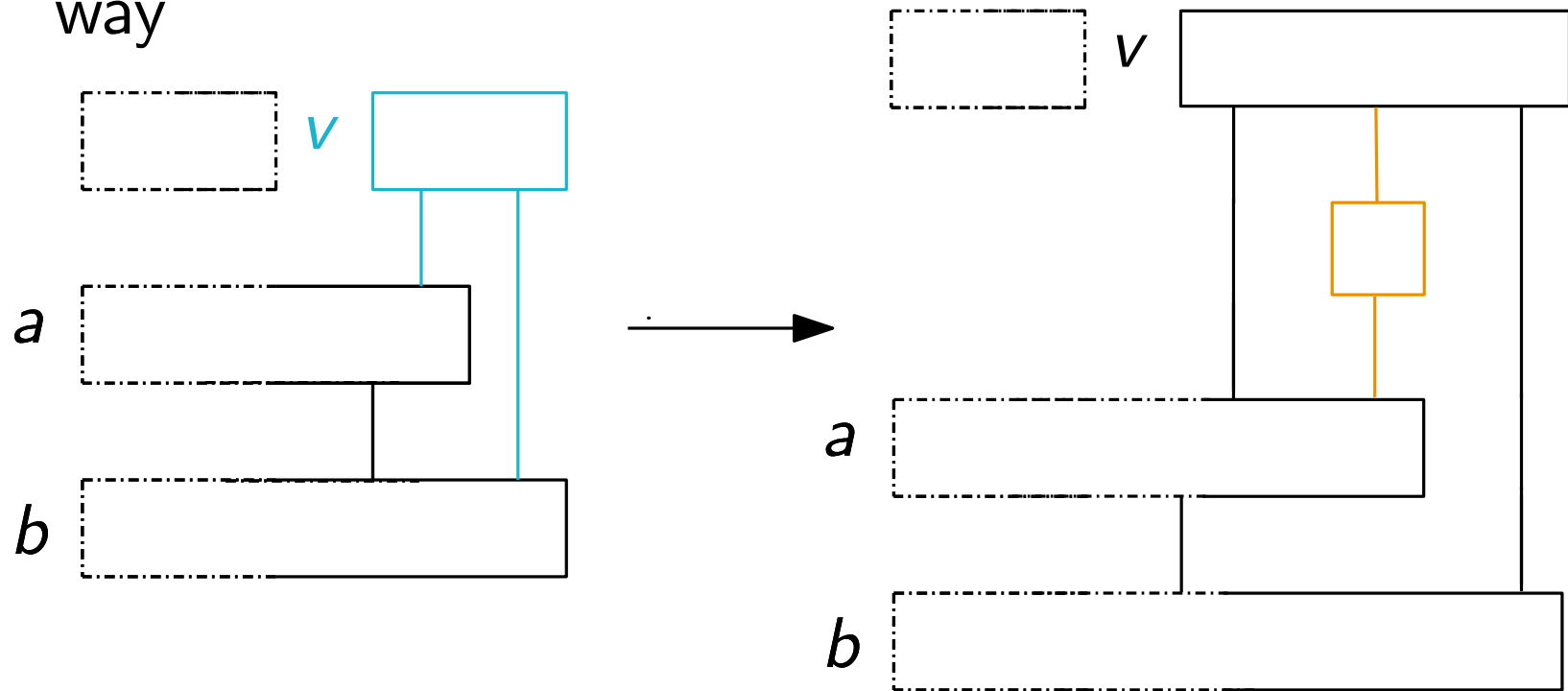
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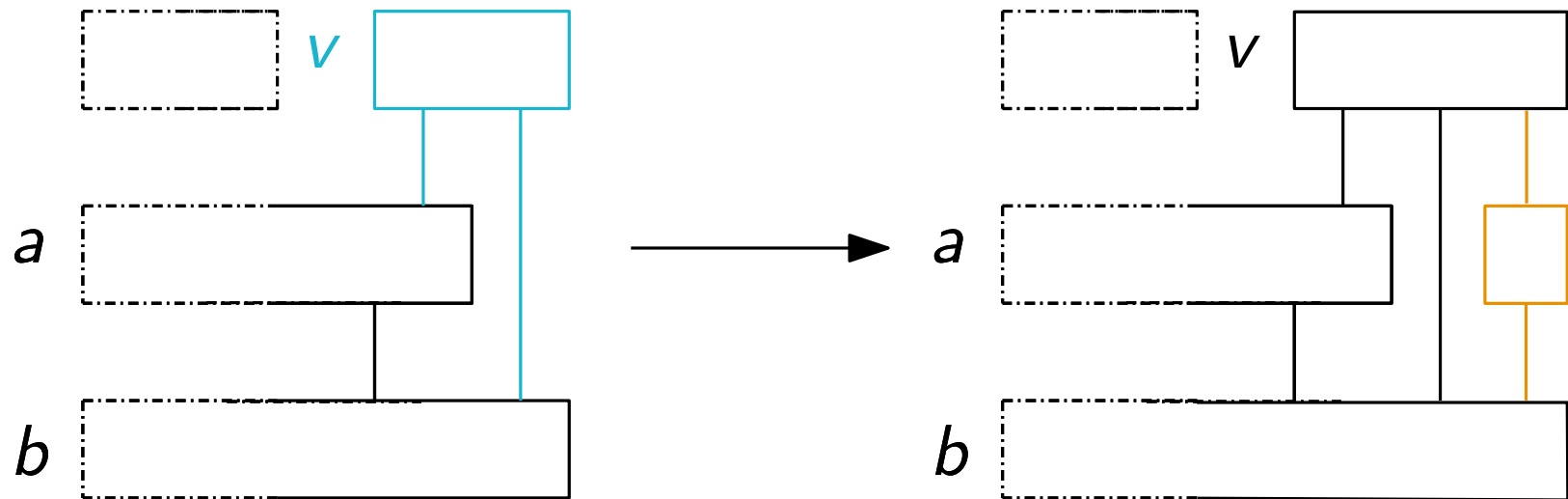
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- ▶ Subsequent vertex insertions between v and a
- ▶ Destroy outerplanarity property

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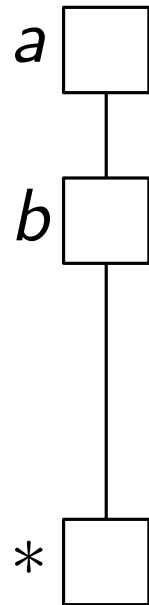
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Reachability of Layers

- ▶ Case 3: Layer becomes reachable by layer reassignments of already drawn vertices

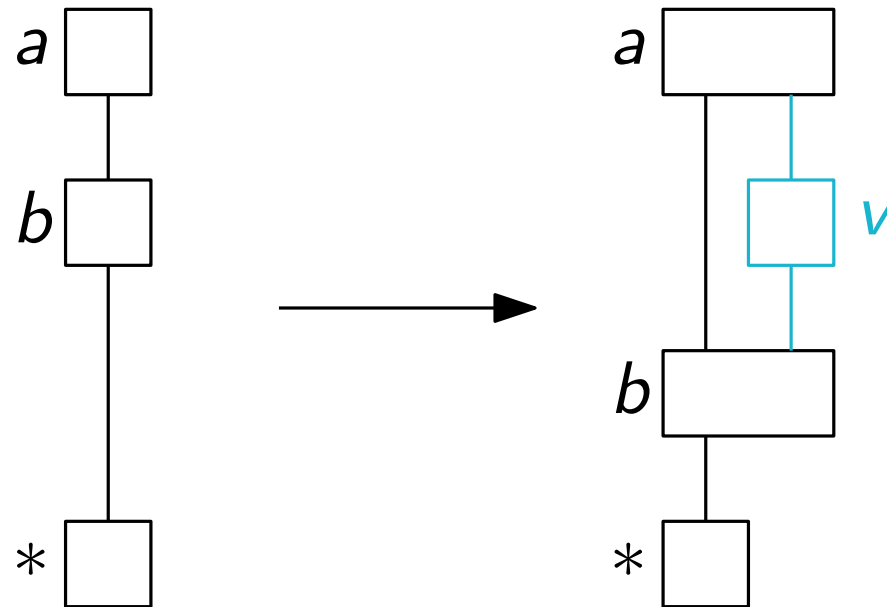
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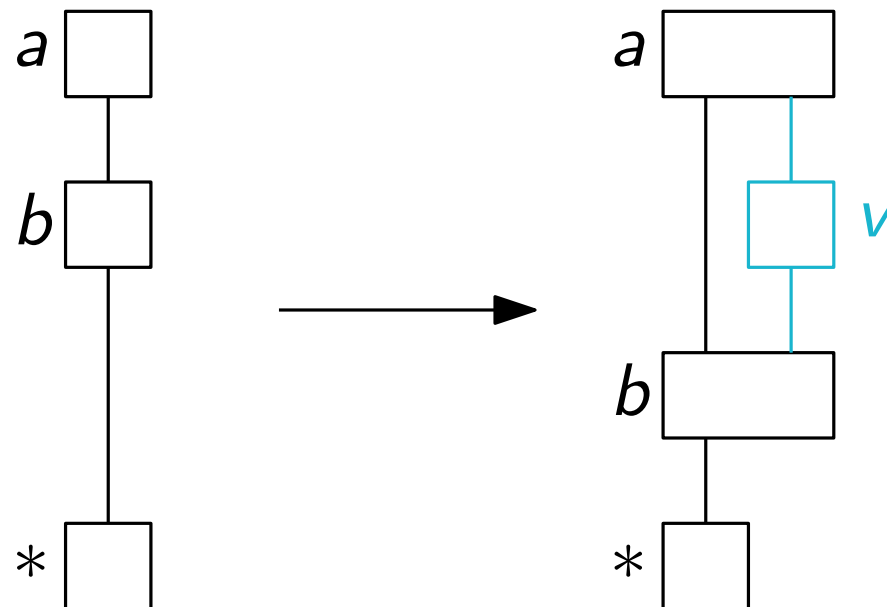
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- ▶ Then, either case 1 or case 2 will apply
- ▶ If neither of those cases possible, insert a new layer between a and b

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Traversal Of Tree Decomposition

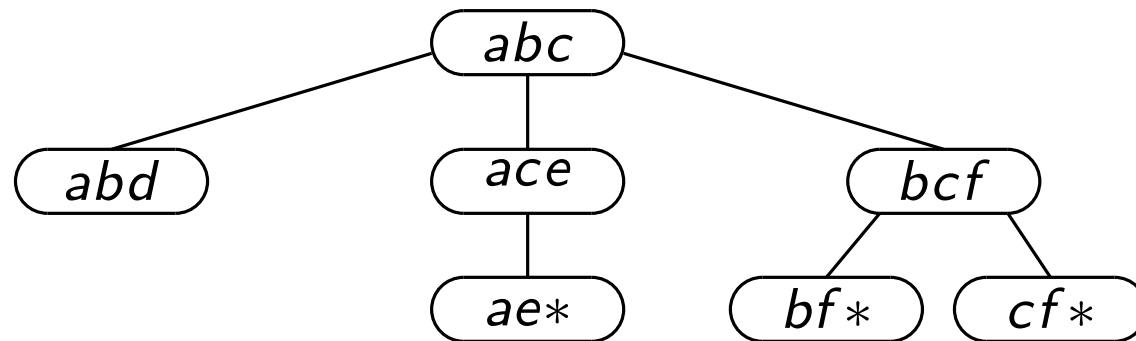
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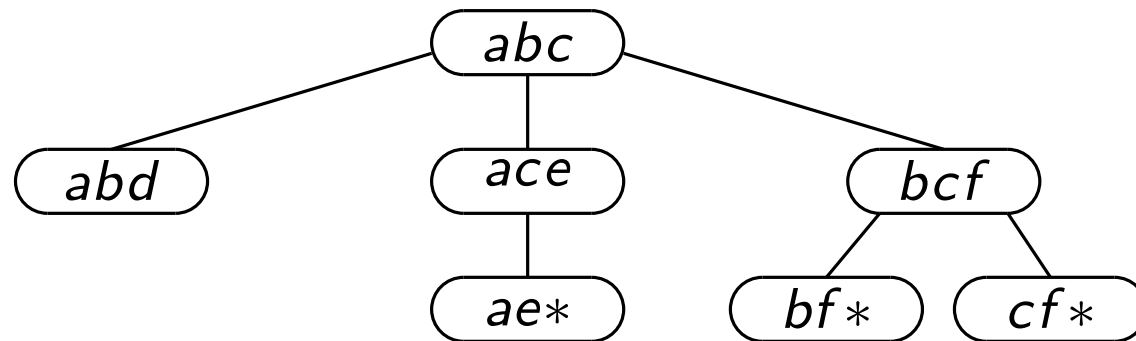
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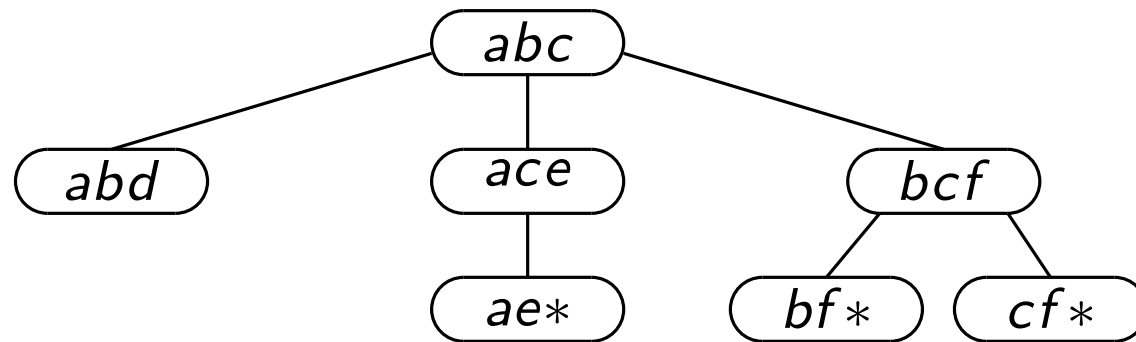
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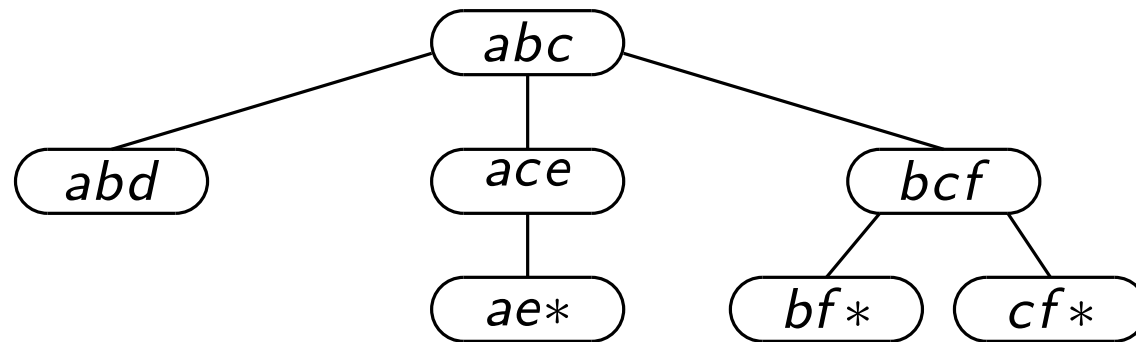
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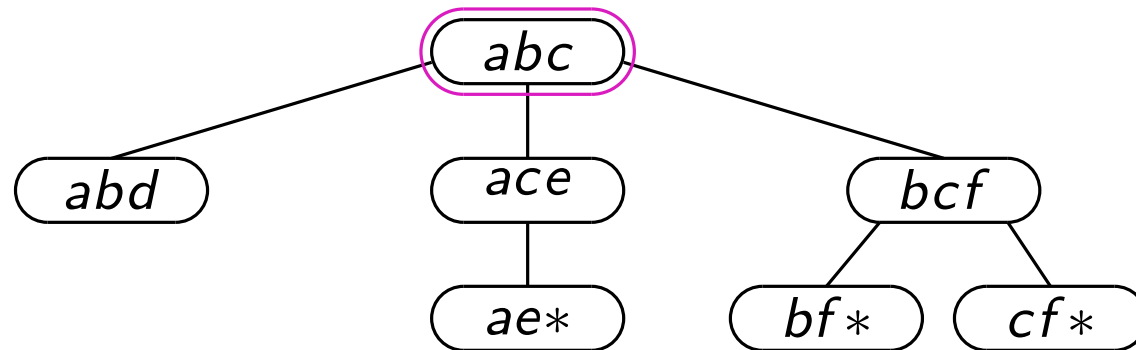
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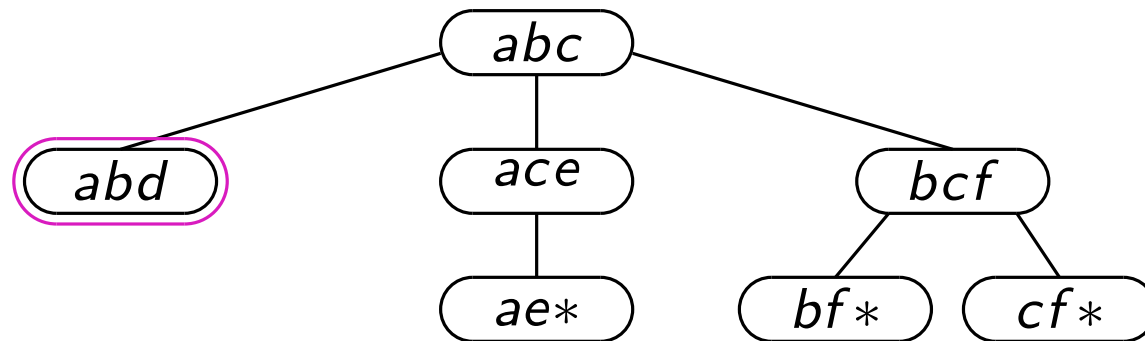
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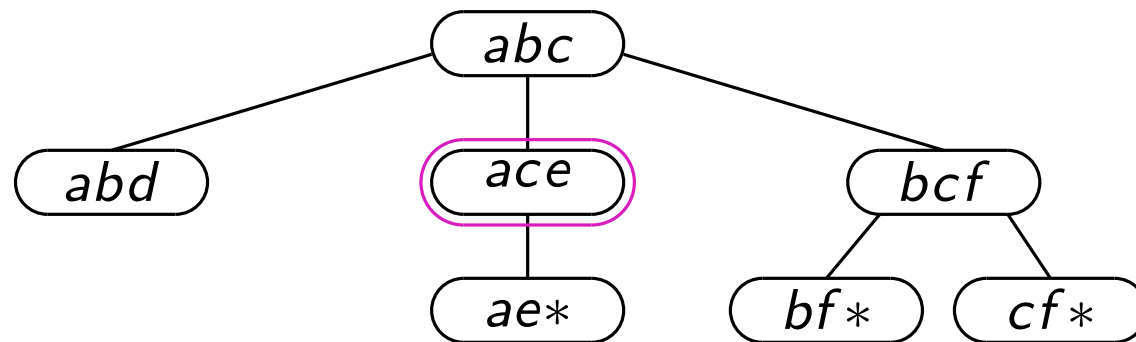
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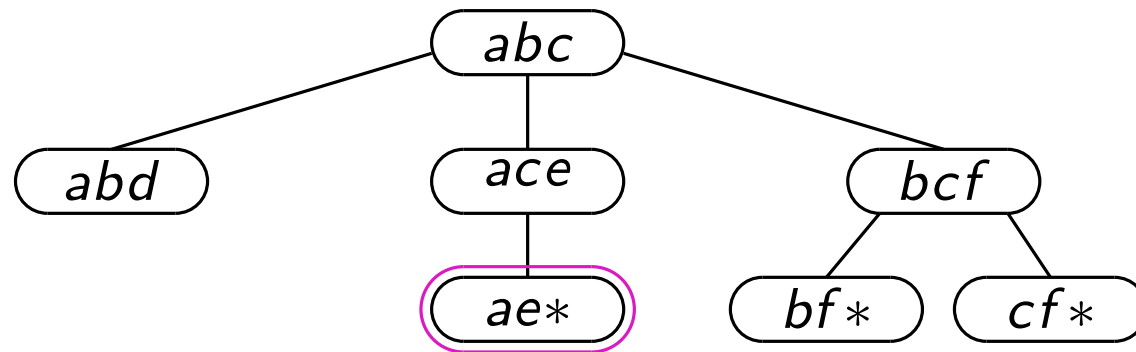
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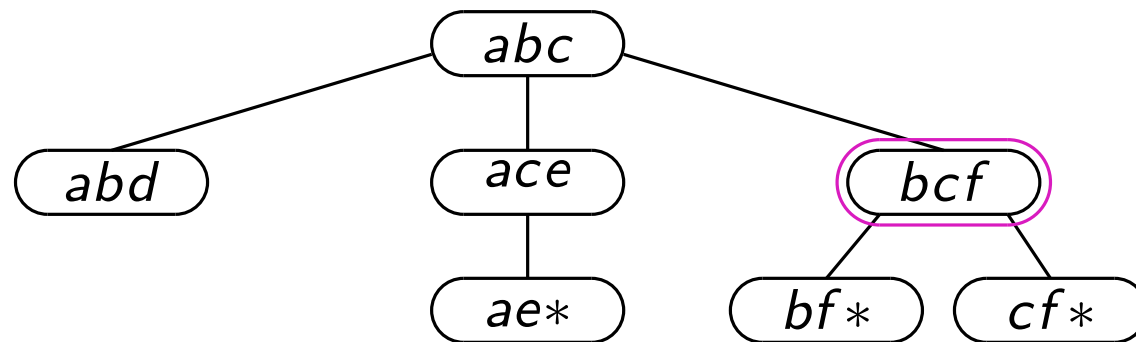
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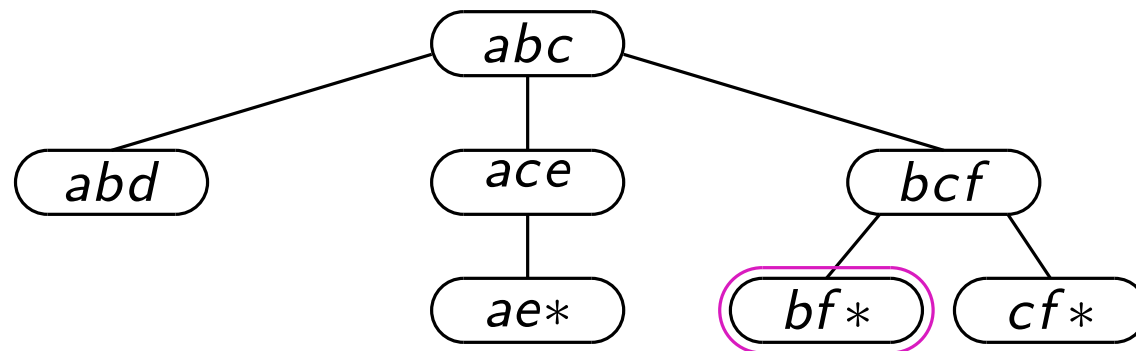
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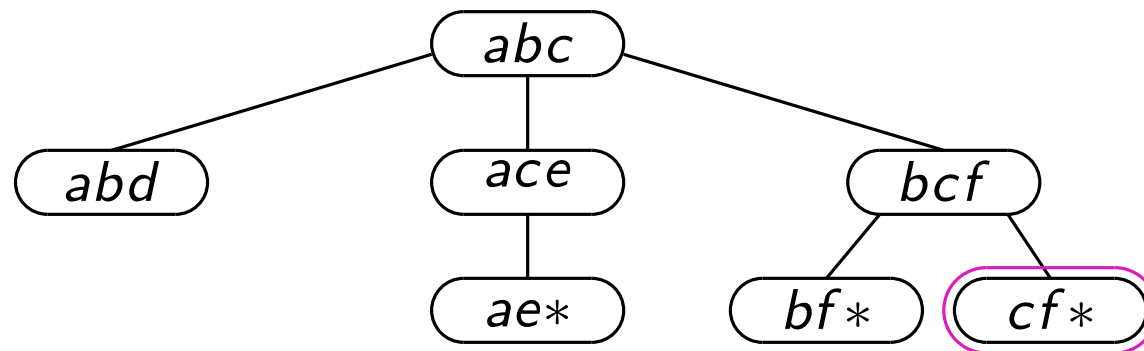
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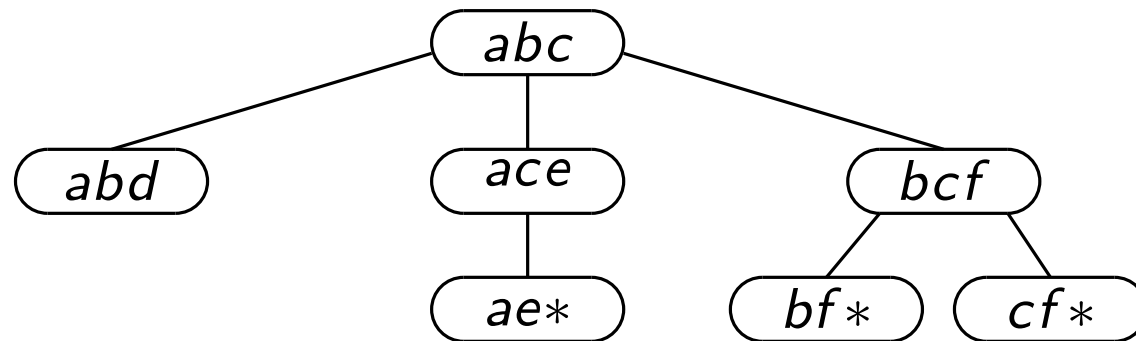
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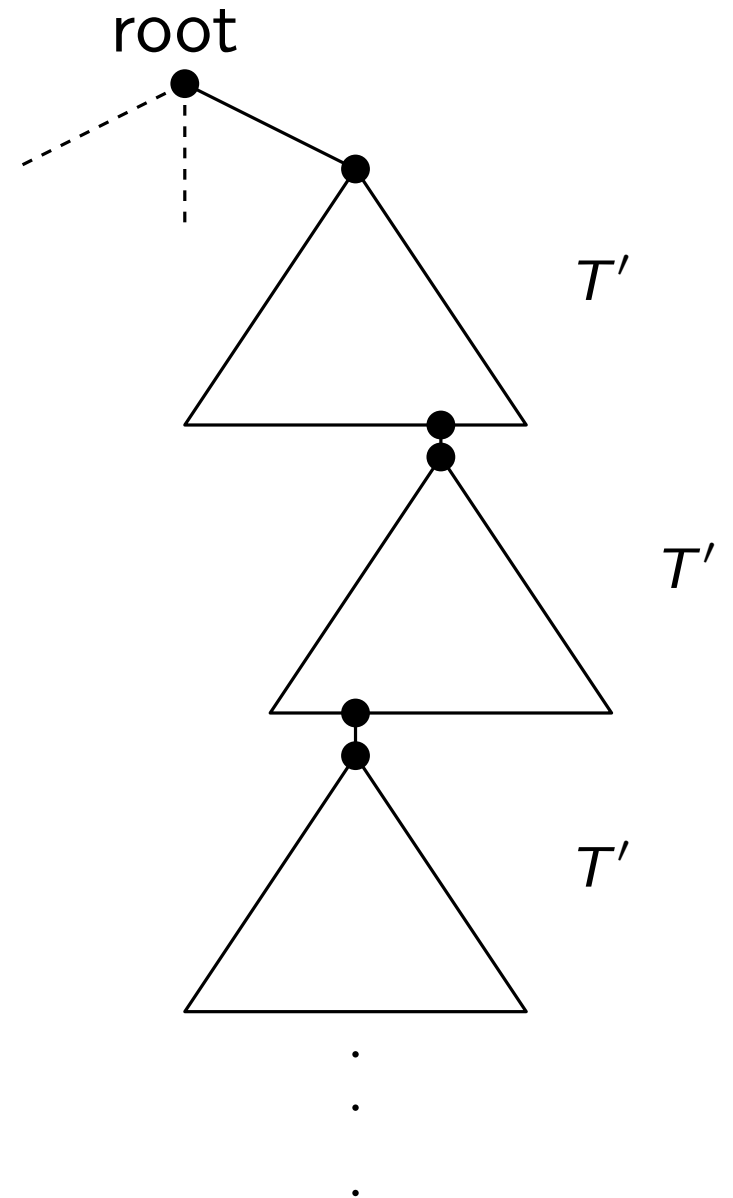
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- ▶ Realizable by DFS traversal

Advantage of Prioritized DFS Traversal

- ▶ Consider following subtree of T

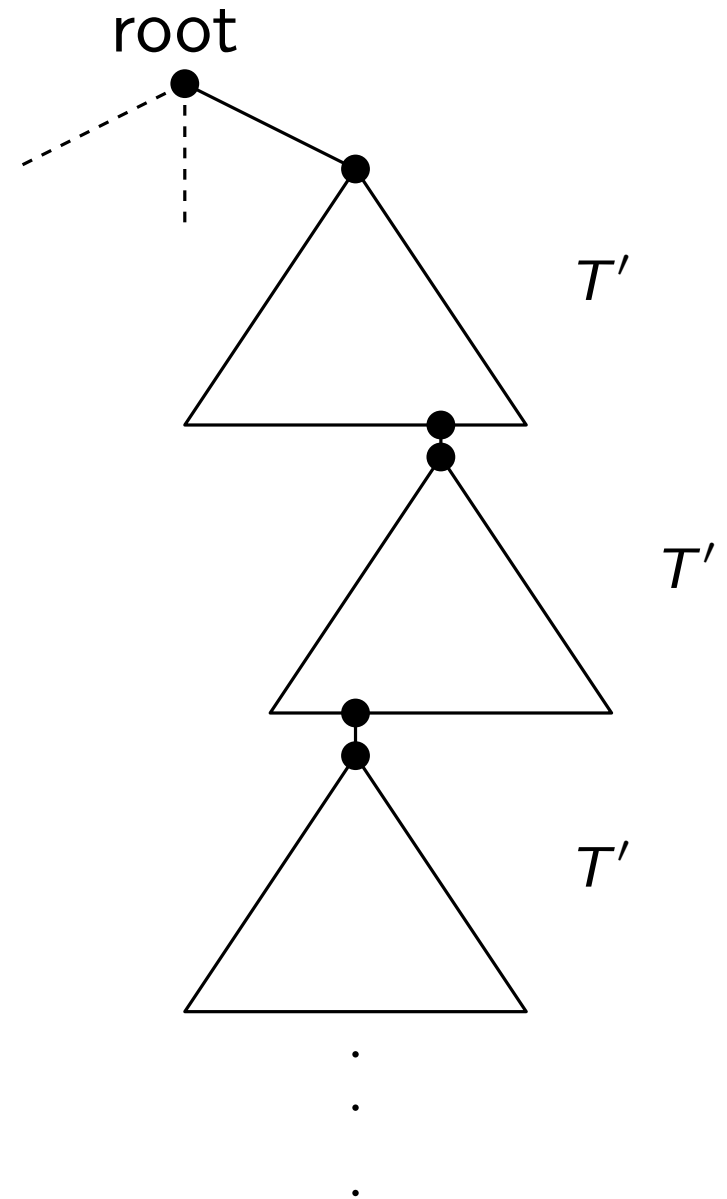
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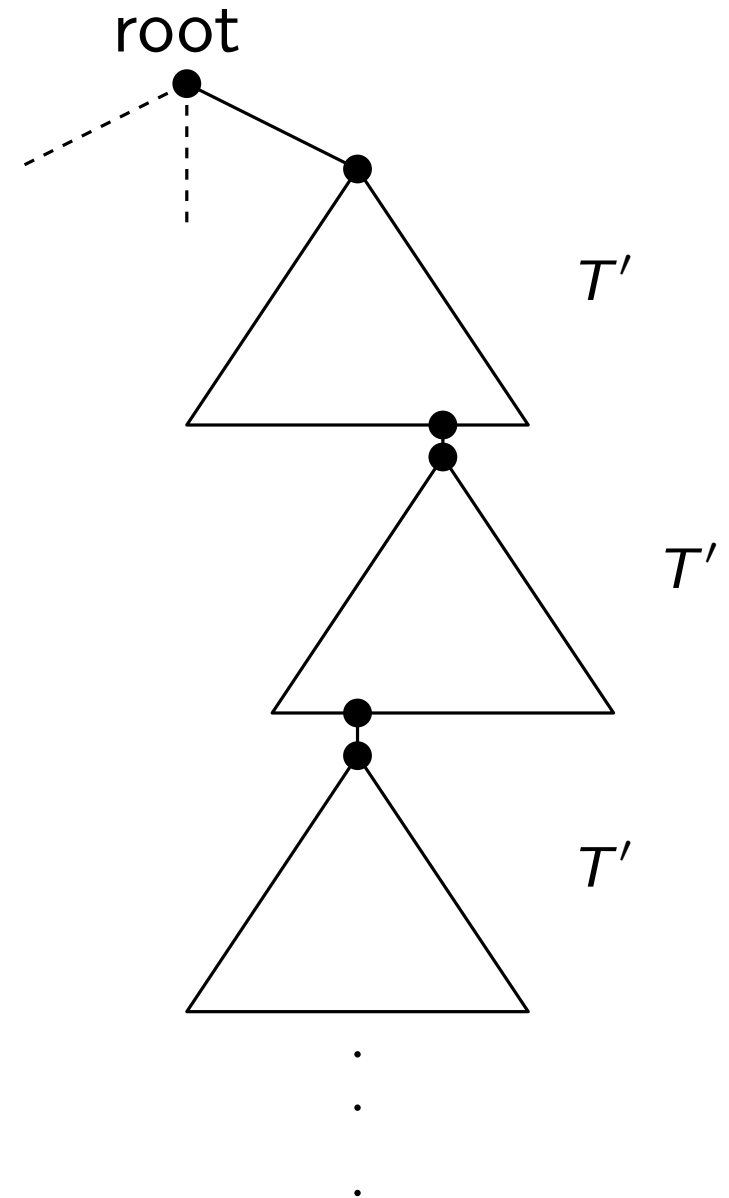
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- ▶ Consider following subtree of T
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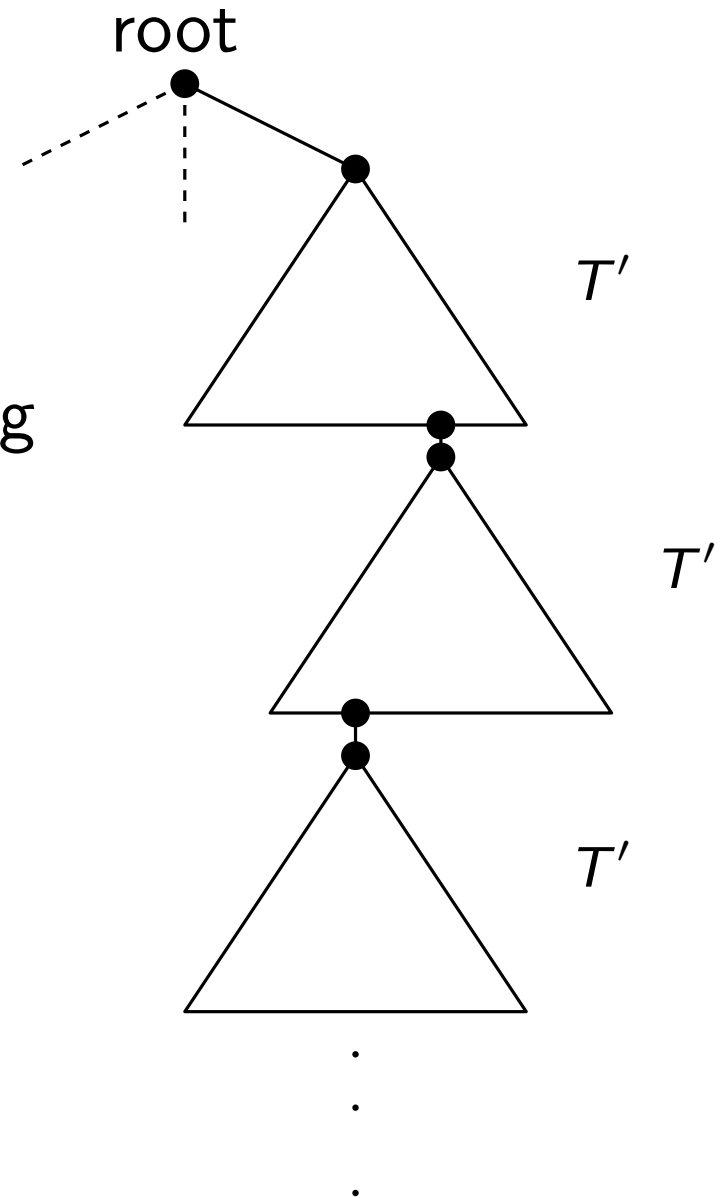
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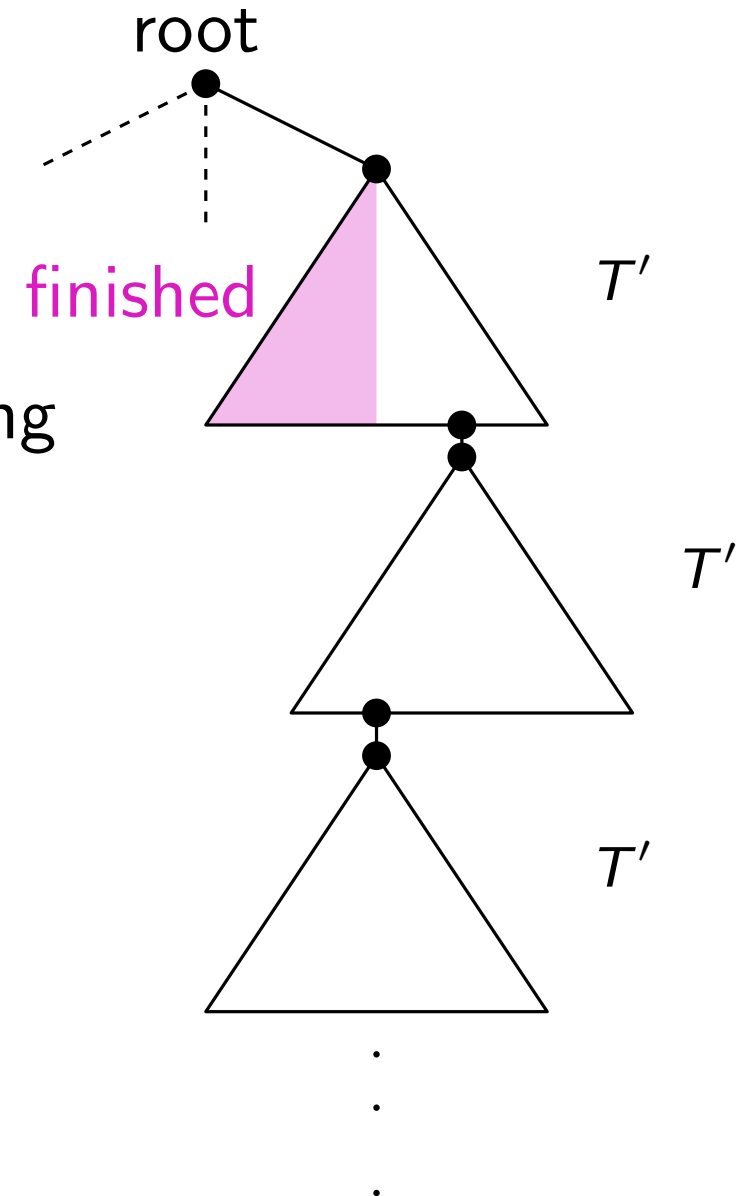
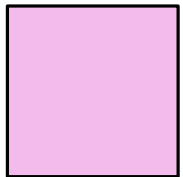
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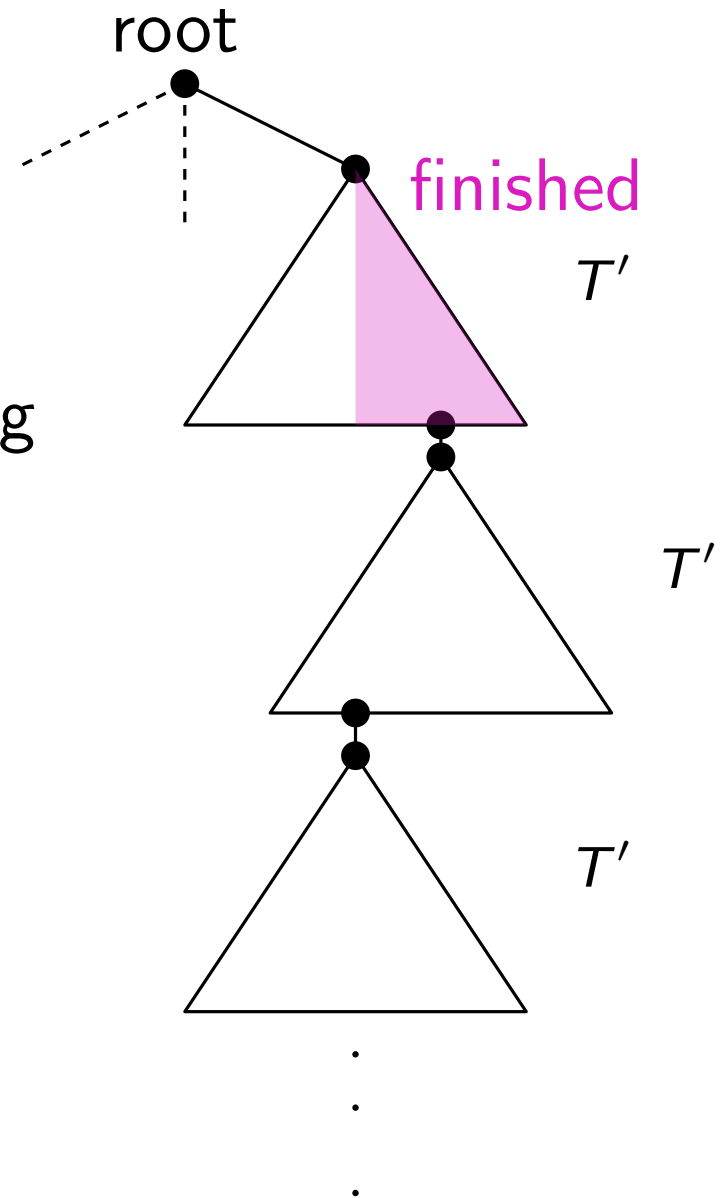
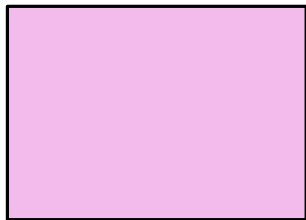
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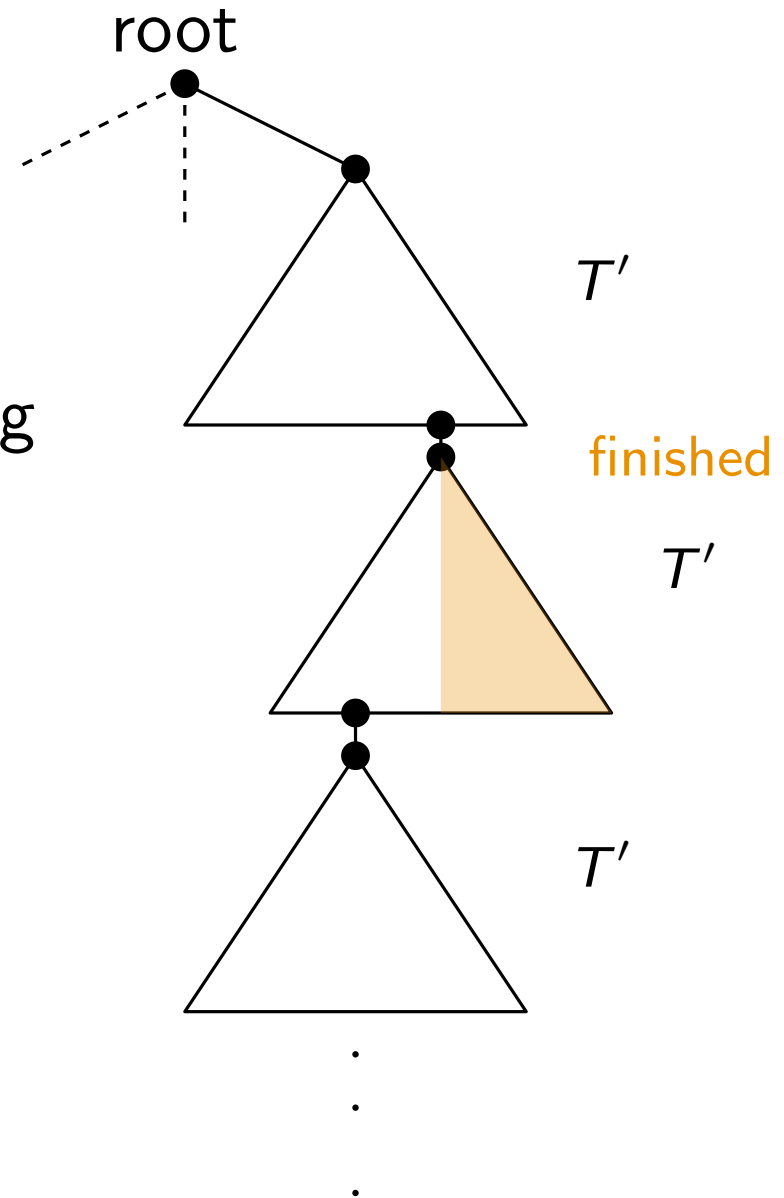
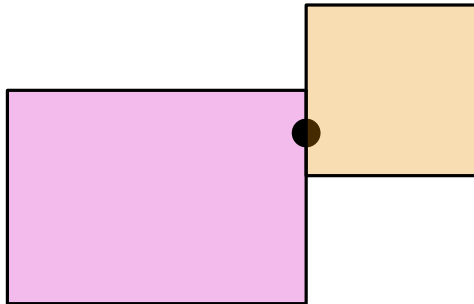
Advantage of Prioritized DFS Traversal

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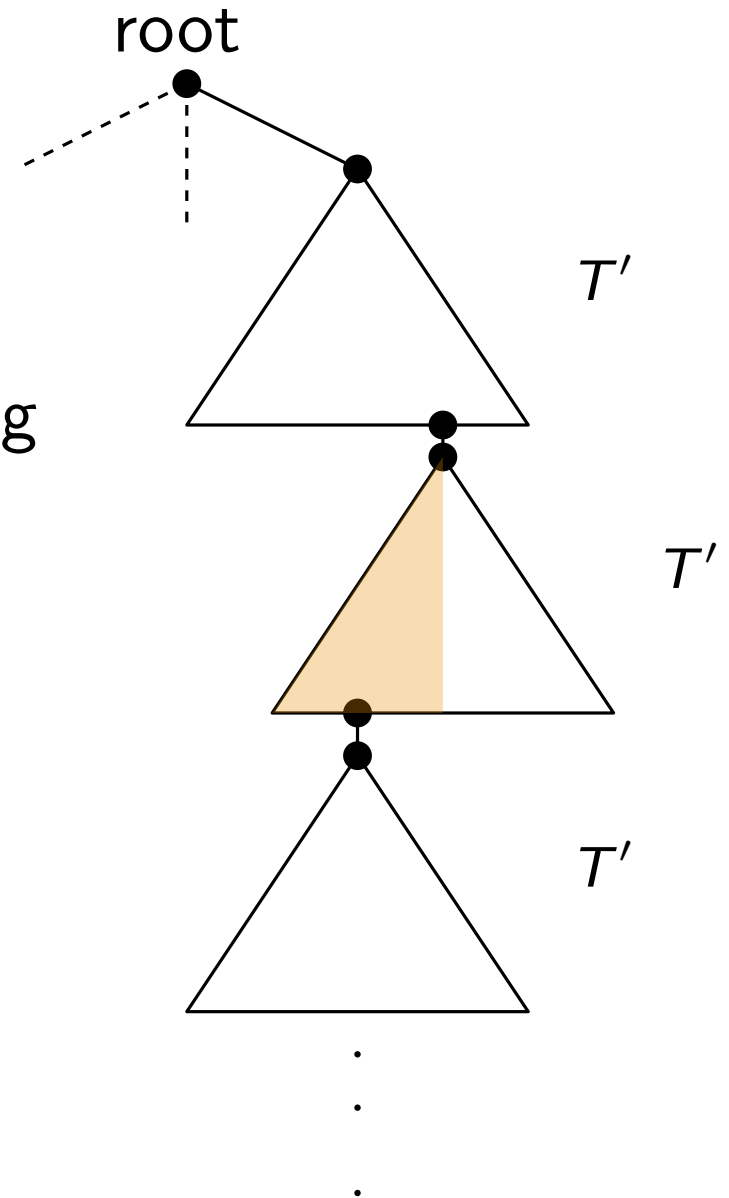
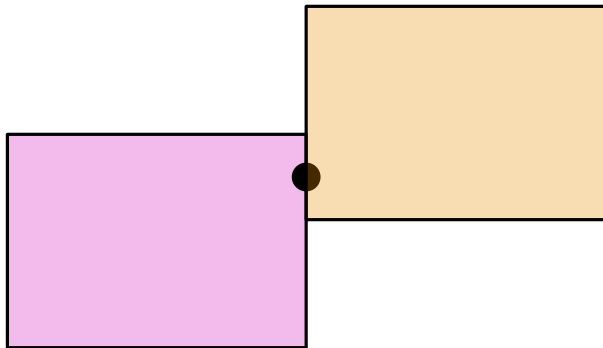
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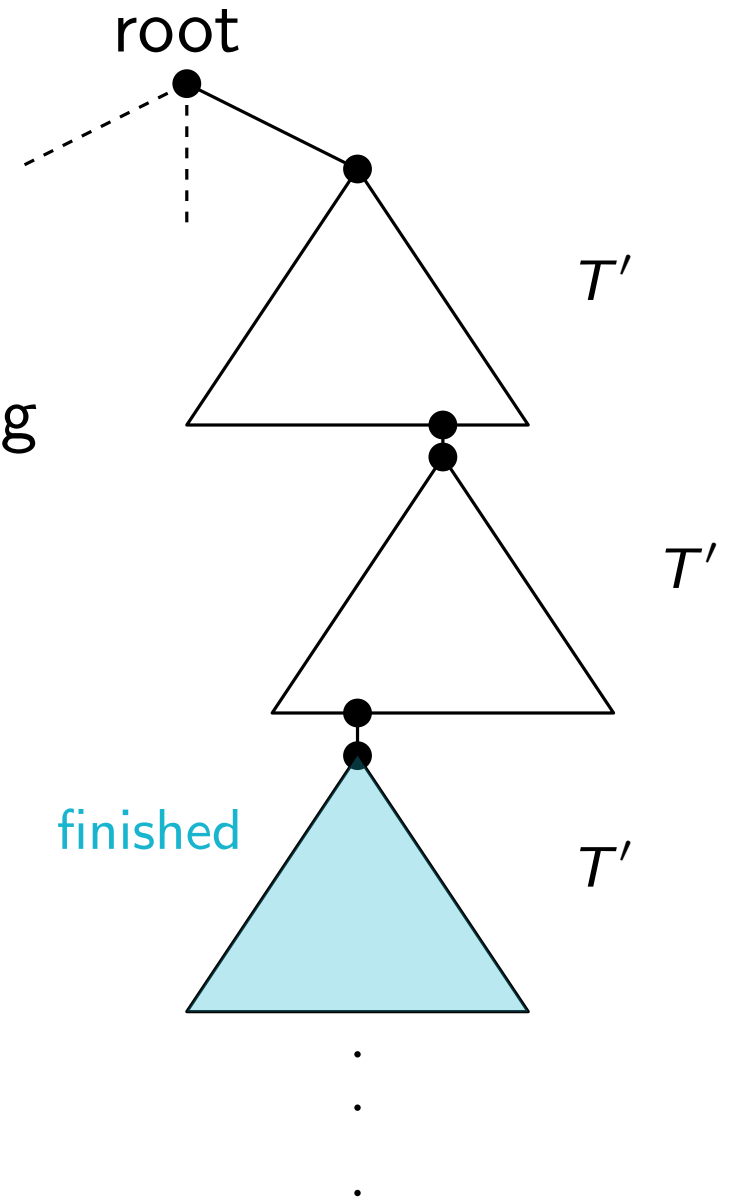
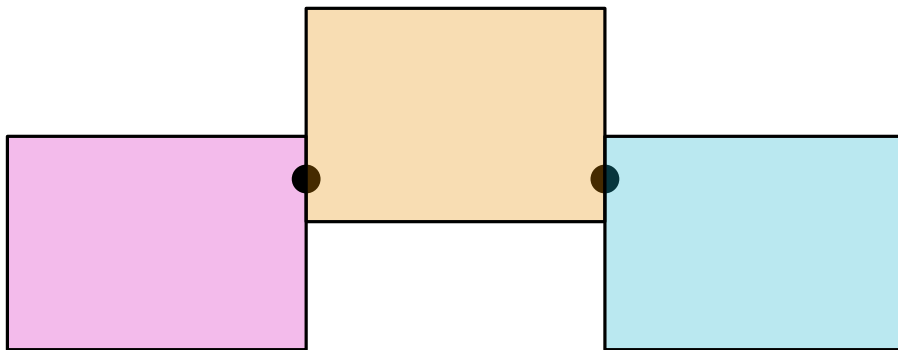
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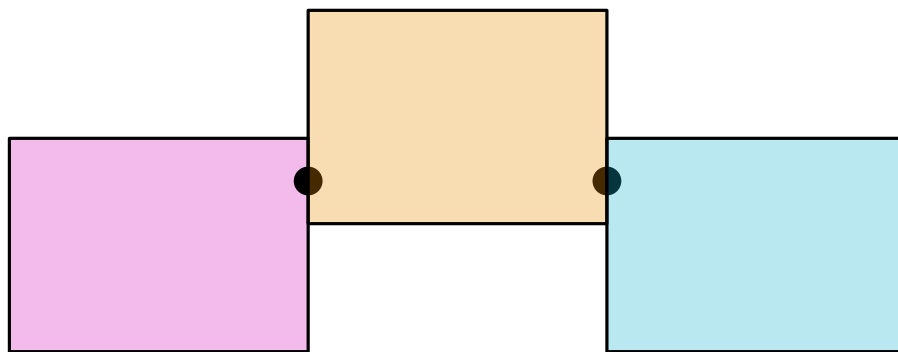
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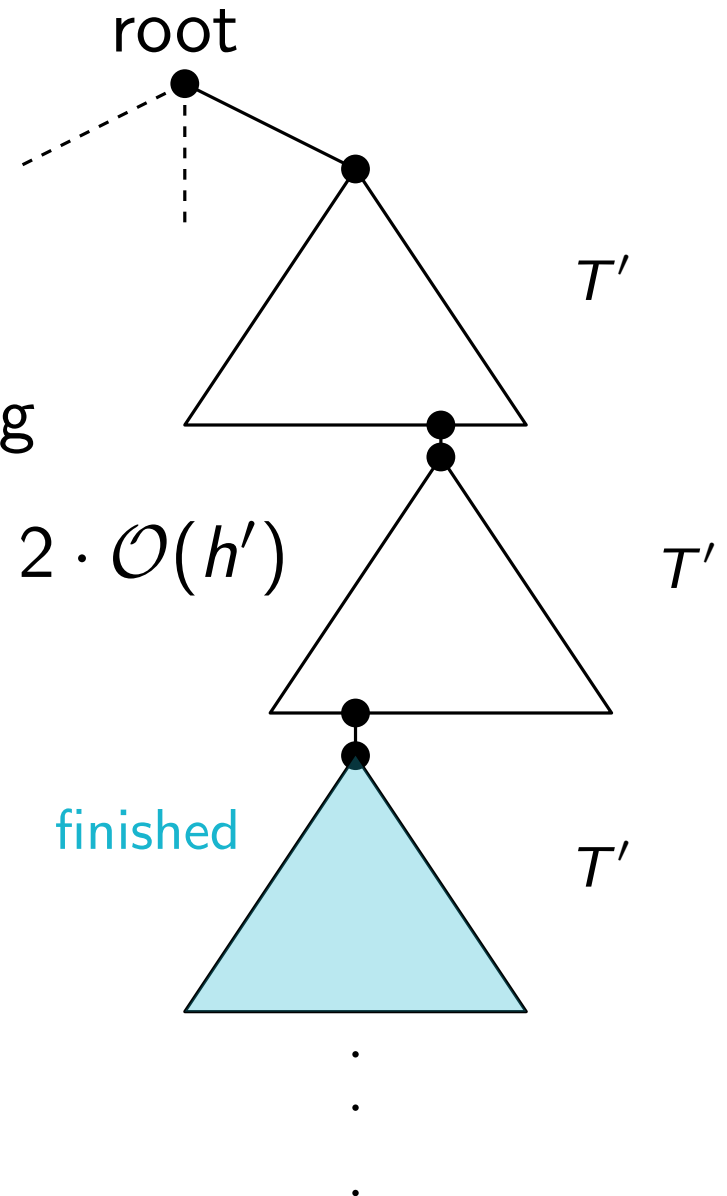


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$$\leq 2 \cdot \mathcal{O}(h')$$



- ▶ Total amount of layers equals constant multiple of amount of layers induced by drawing T'

Total Height of Box Drawing

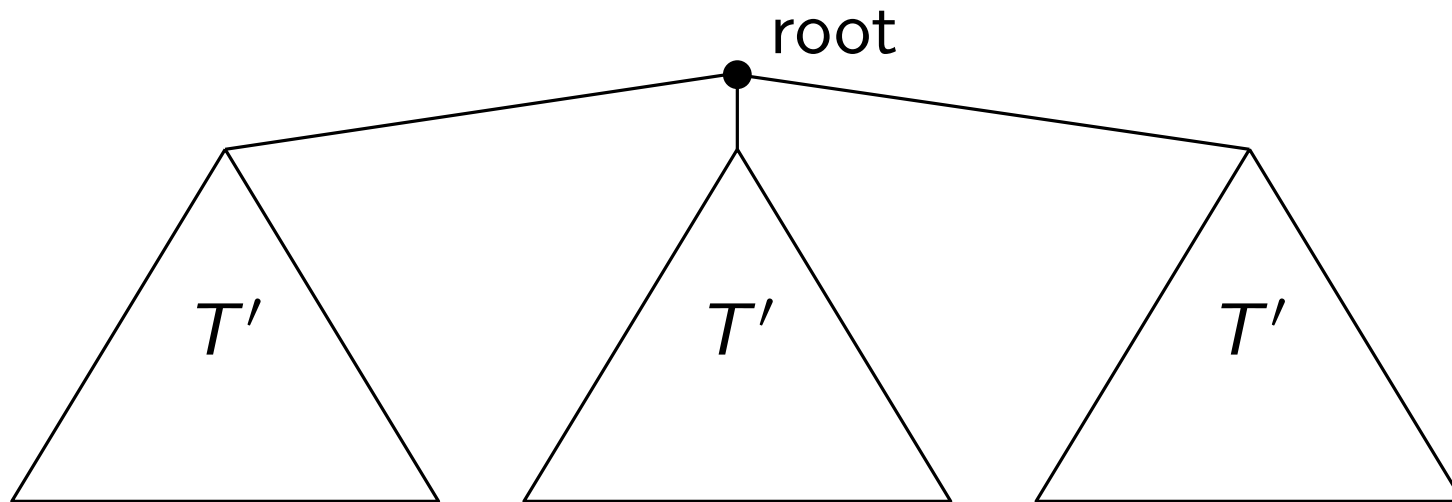
- ▶ Height of Box Drawing: $\delta \cdot \text{\#layers}$

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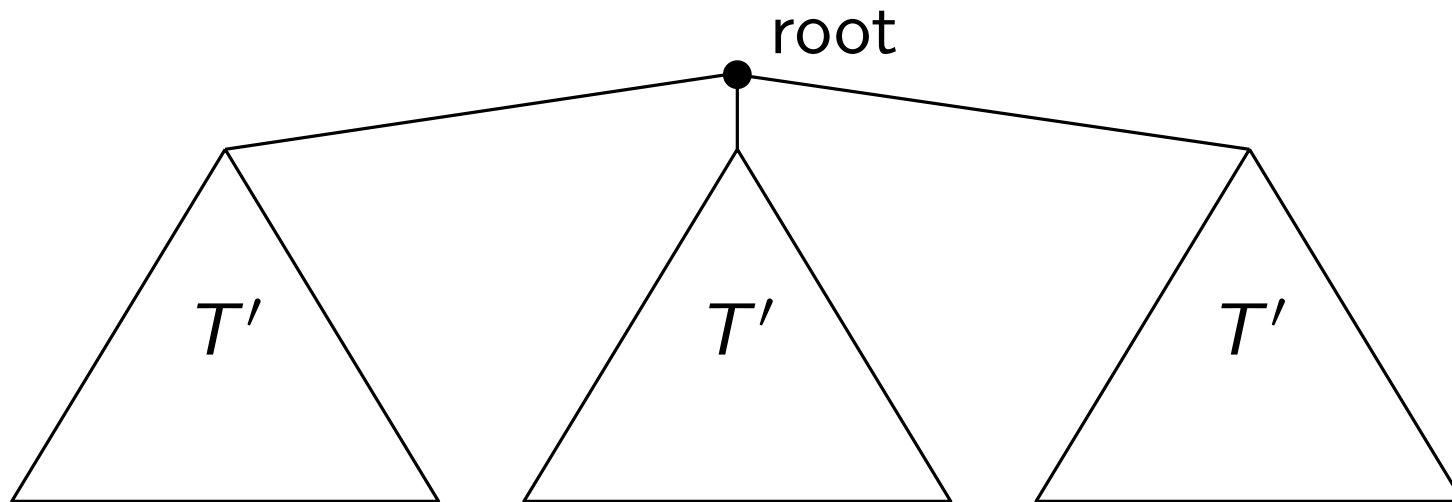
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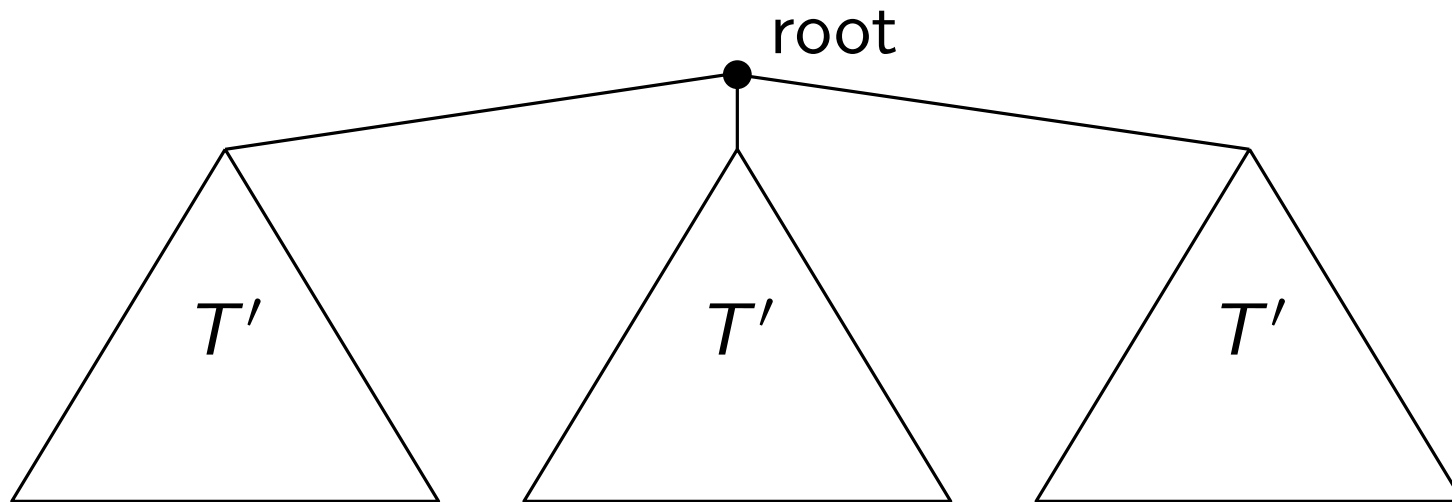
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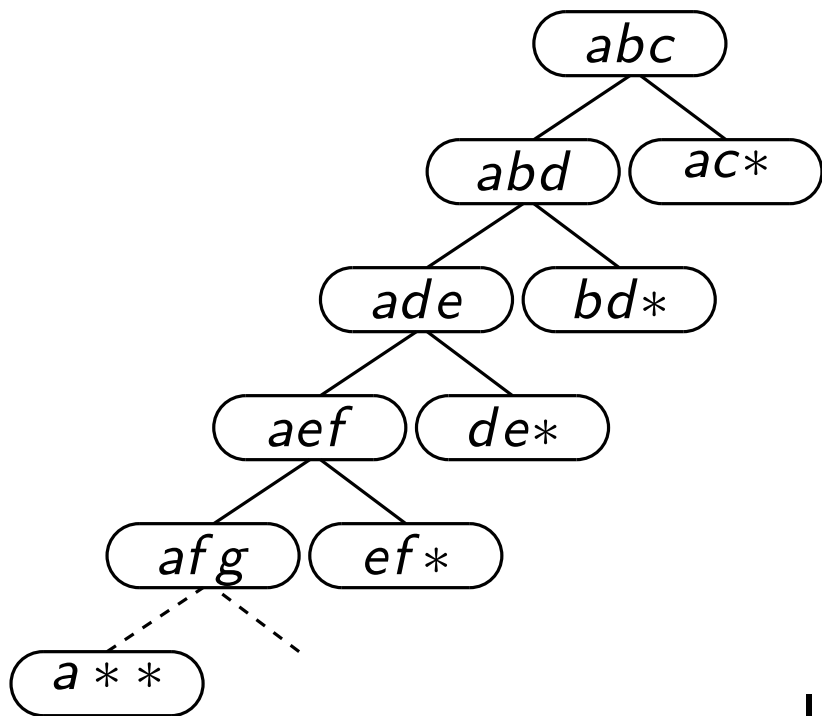
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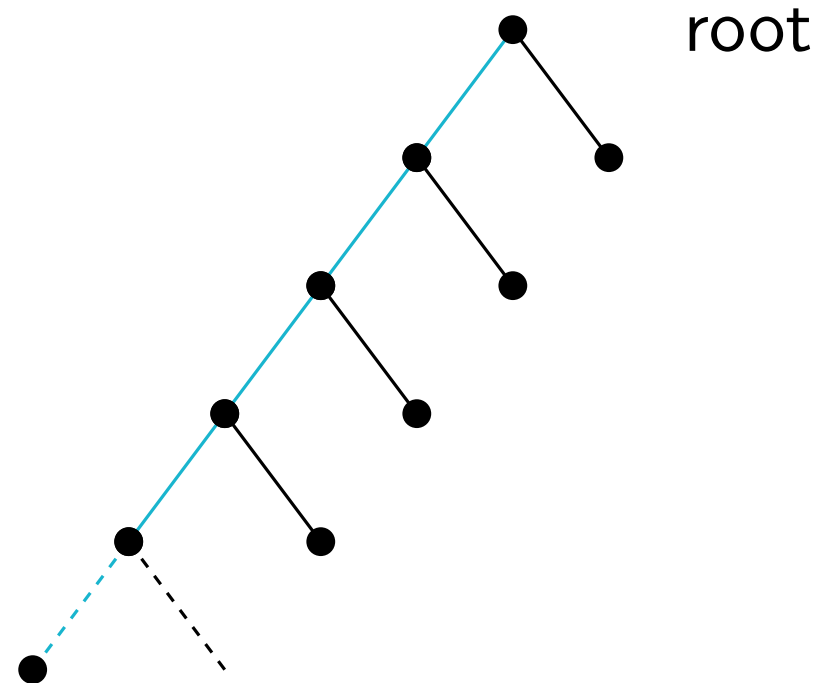
- ▶ Subtree priority does not take effect at any stage of DFS
 - ▶ Enforcing new layers

Layer Amount in T'

- ▶ In T' , consider path from root to a leaf

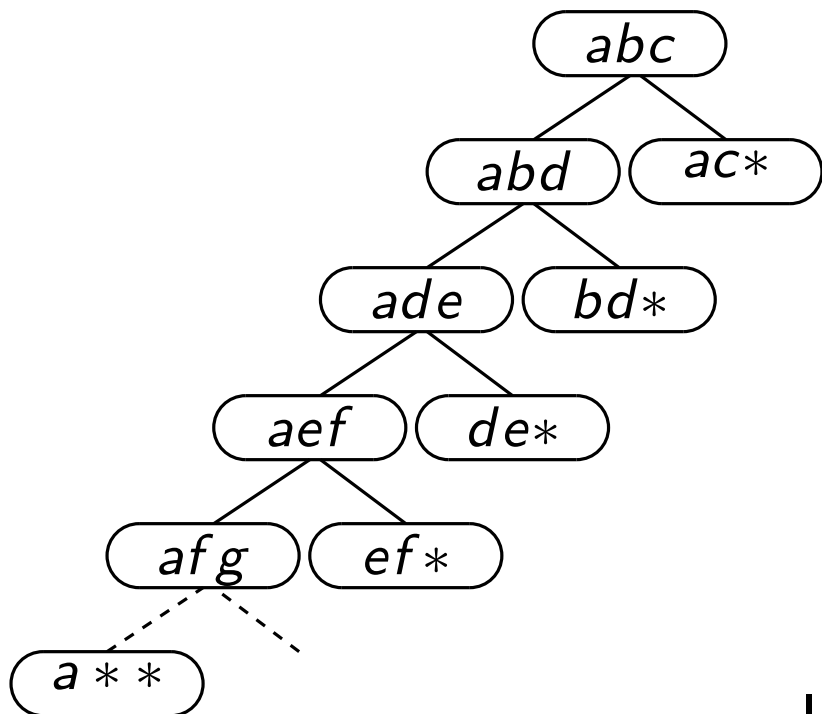


leaf

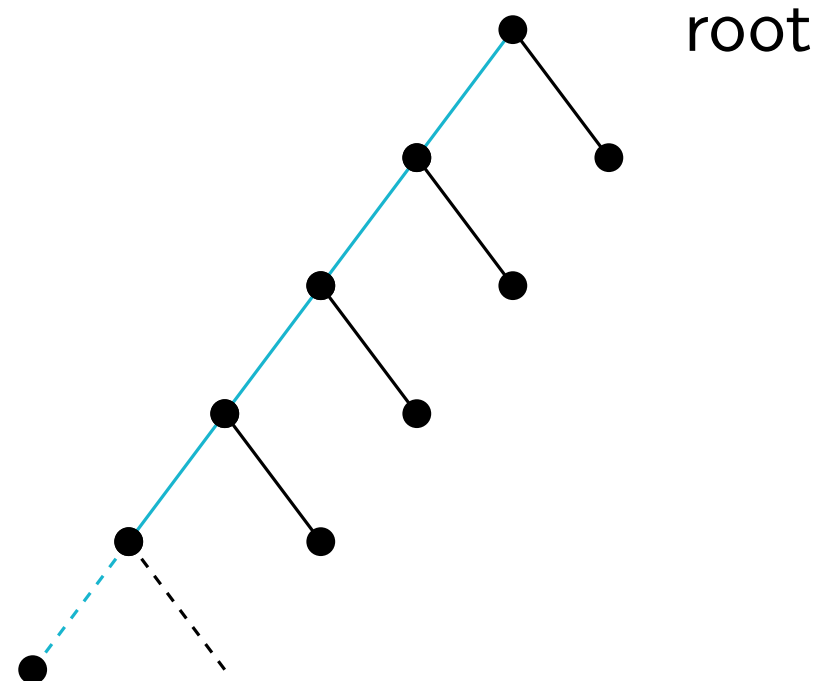


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- ▶ In T' , consider path from root to a leaf
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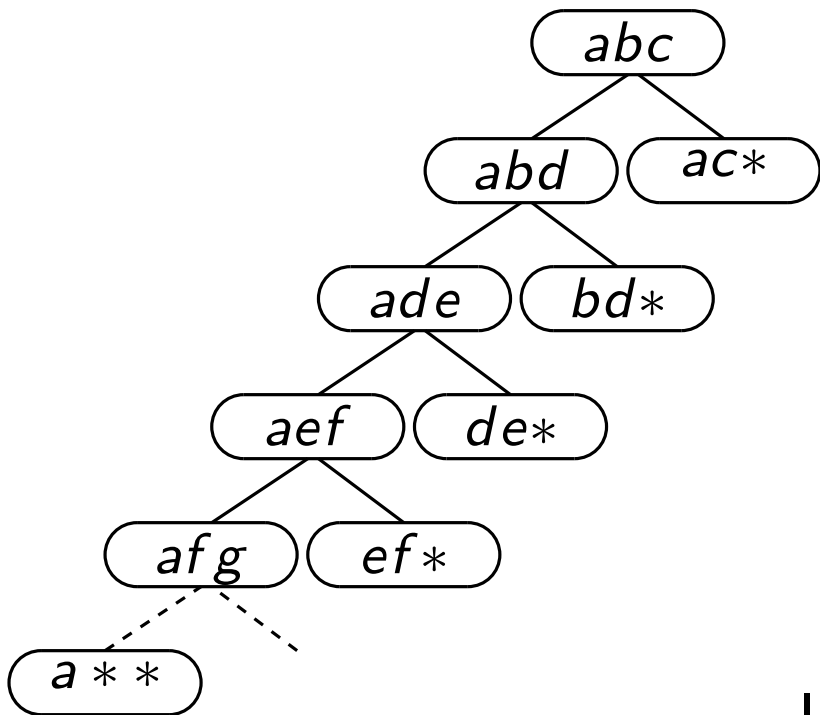


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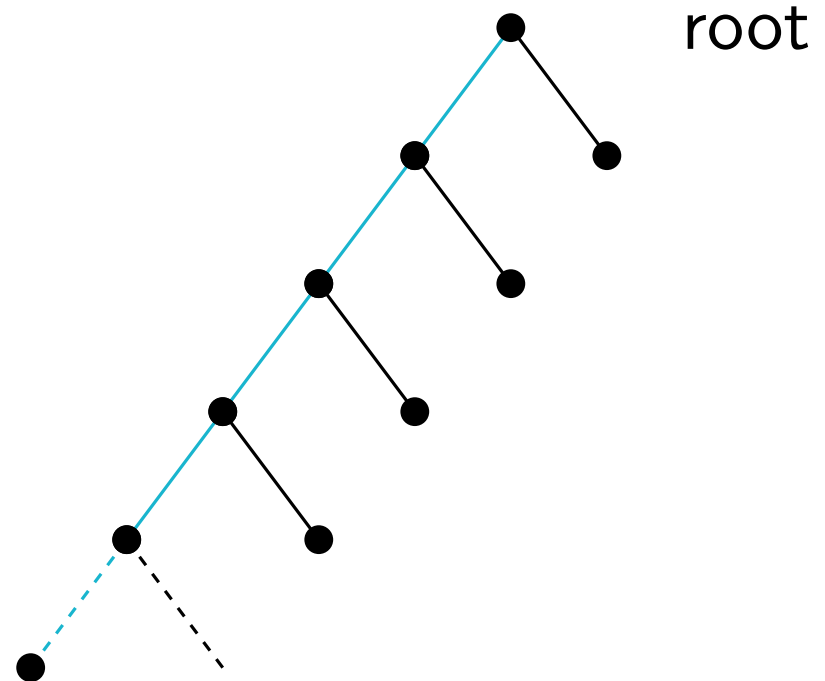


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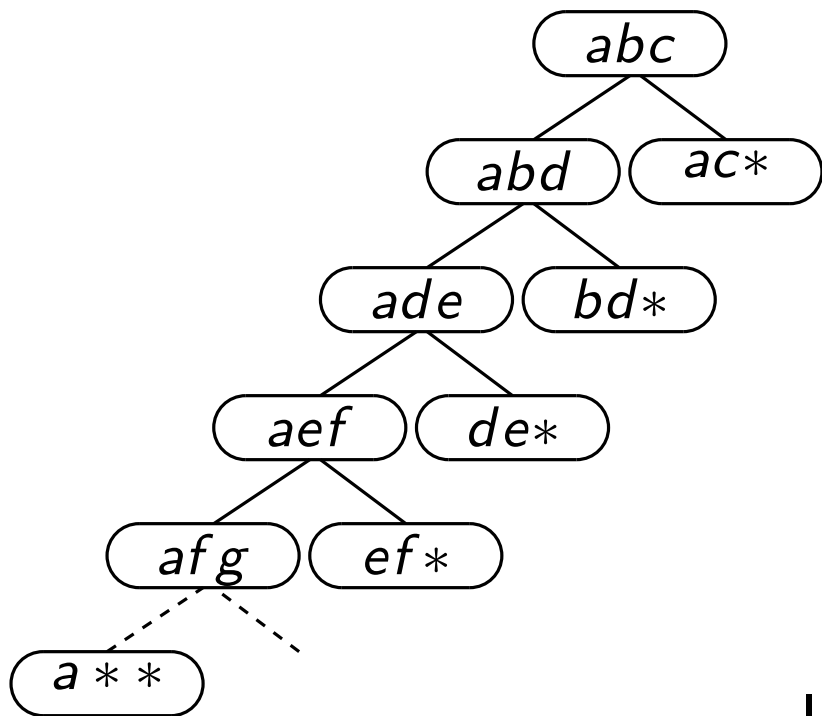
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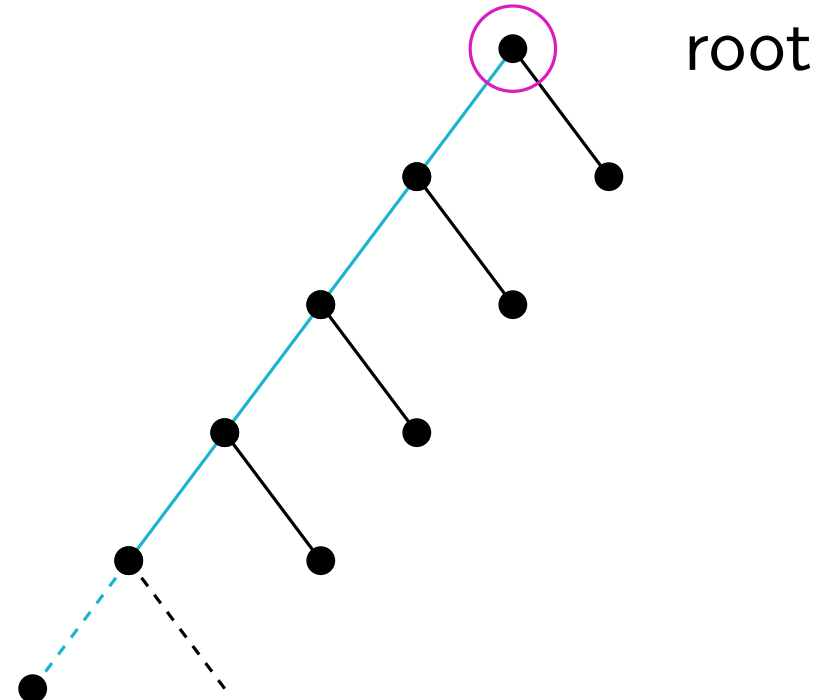
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Amount of layers occupied: ≥ 3



leaf

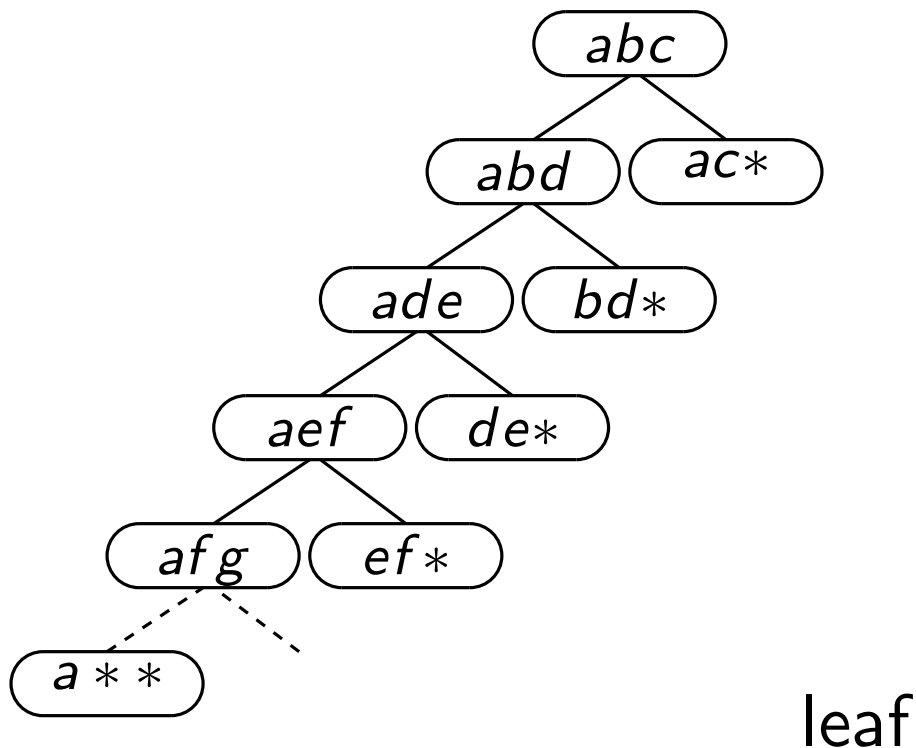
DFS starts at root



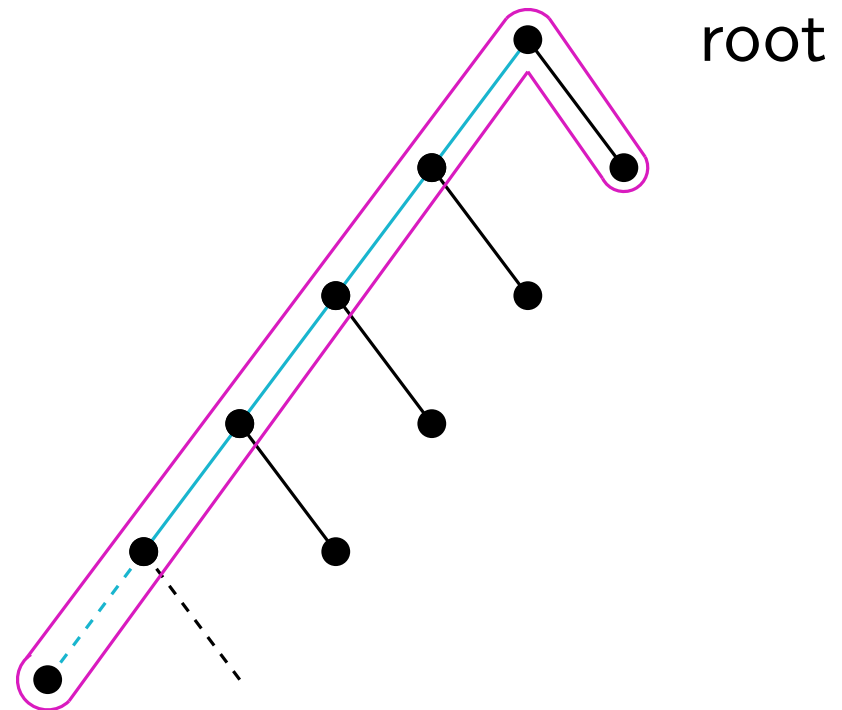
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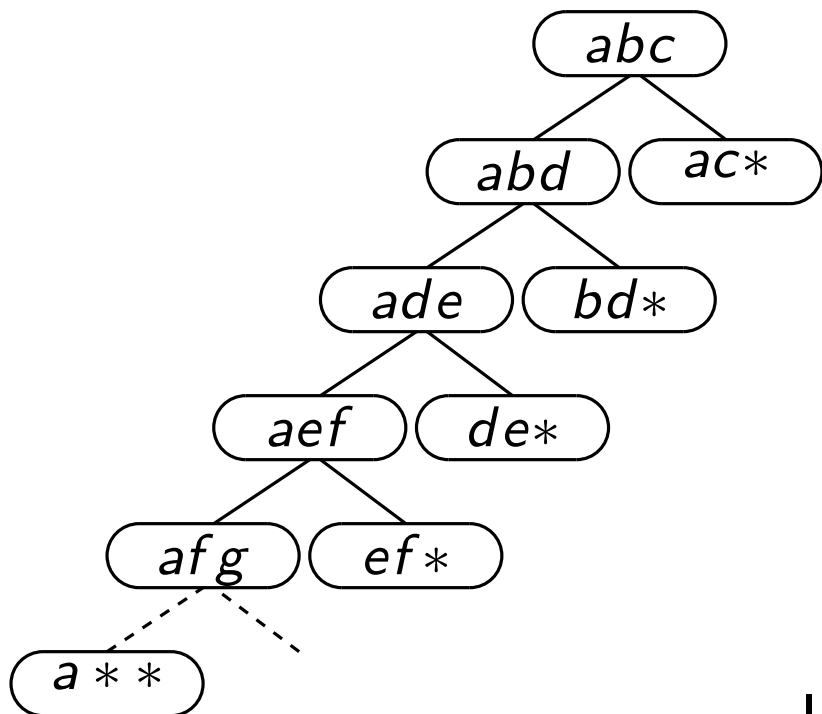
Occupation of a



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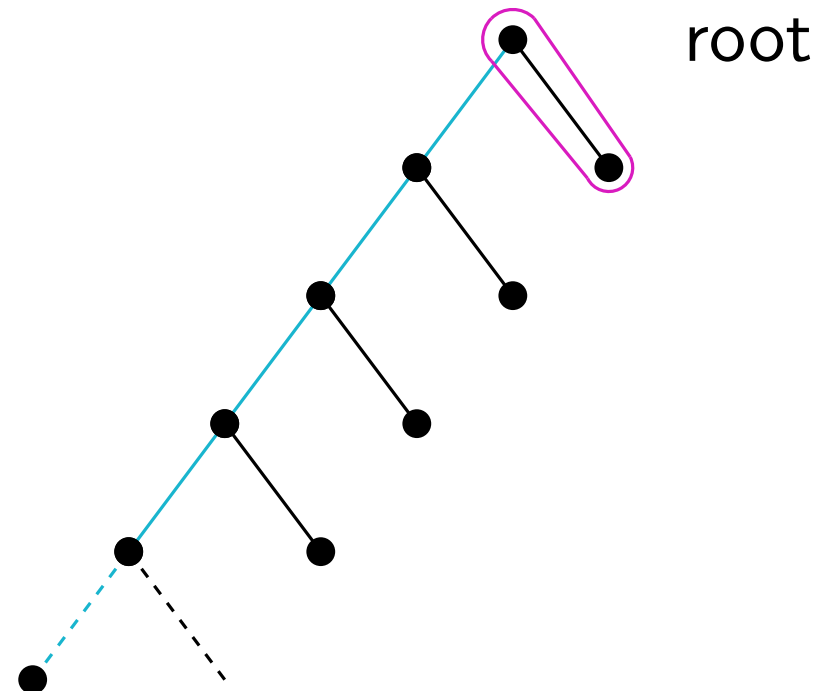
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leaf

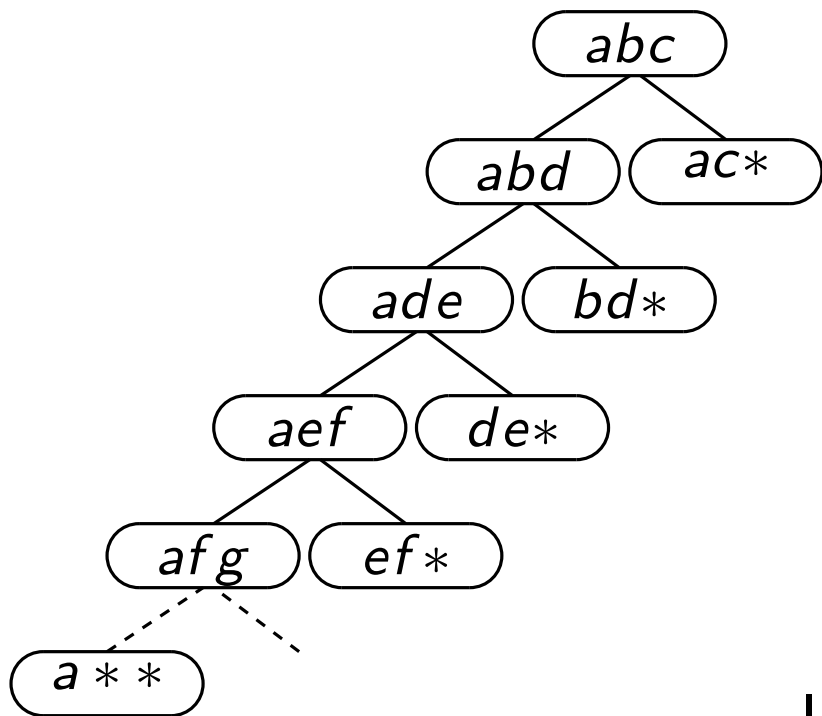
Occupation of c



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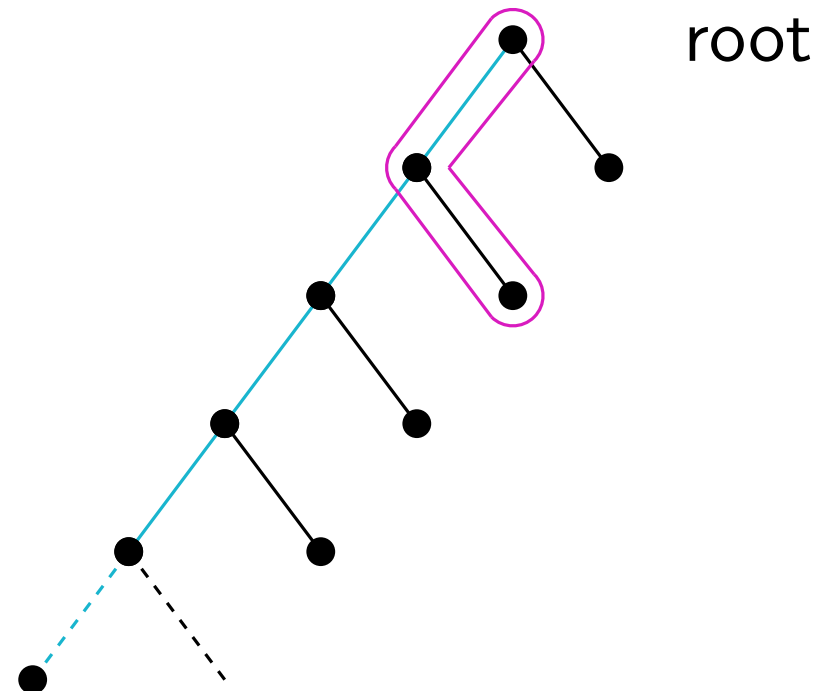
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Amount of layers occupied: ≥ 3



leaf

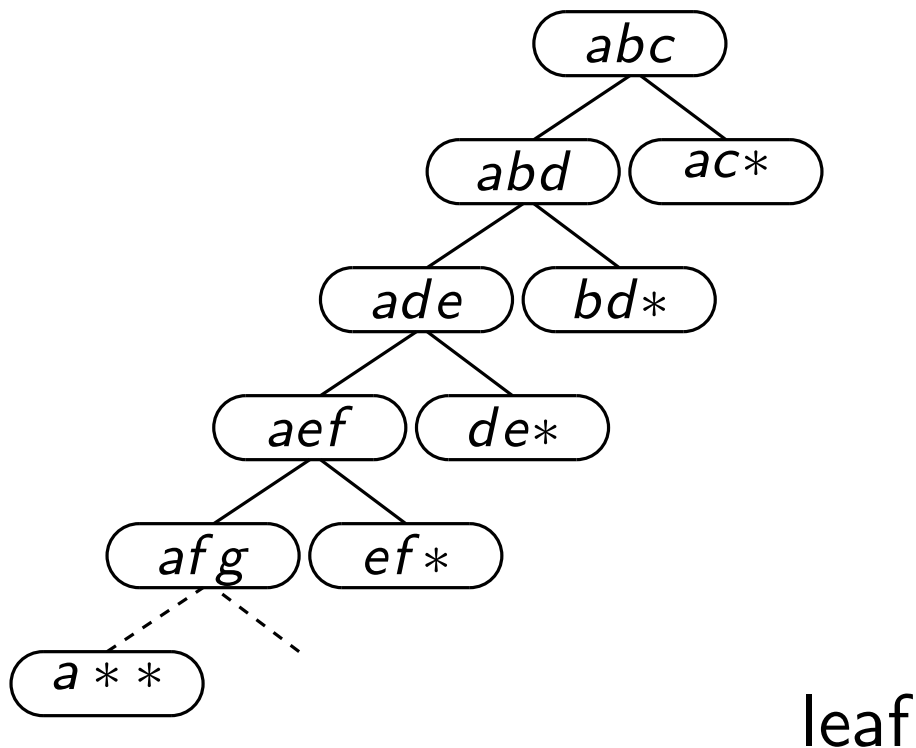
Occupation of b



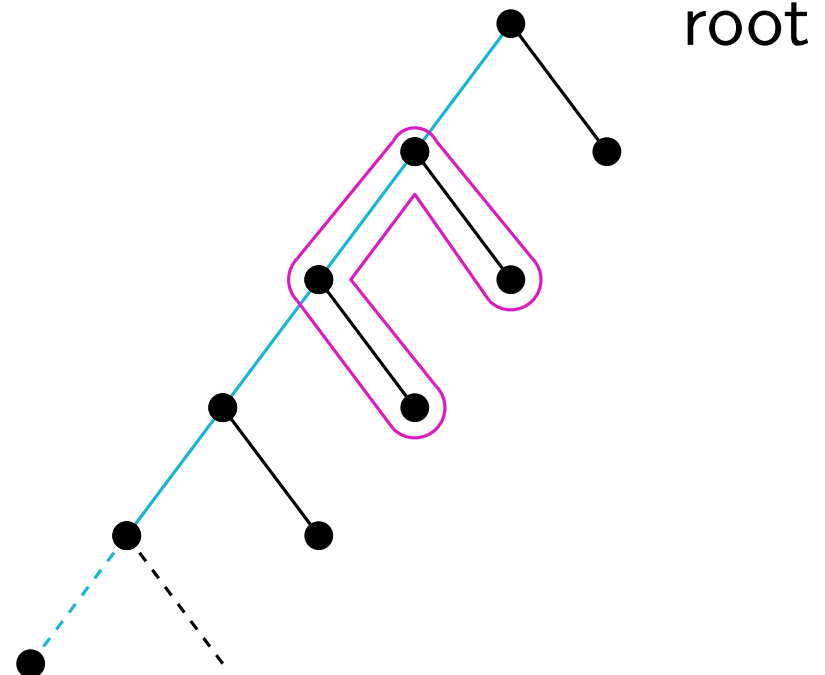
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Amount of layers occupied: ≥ 4



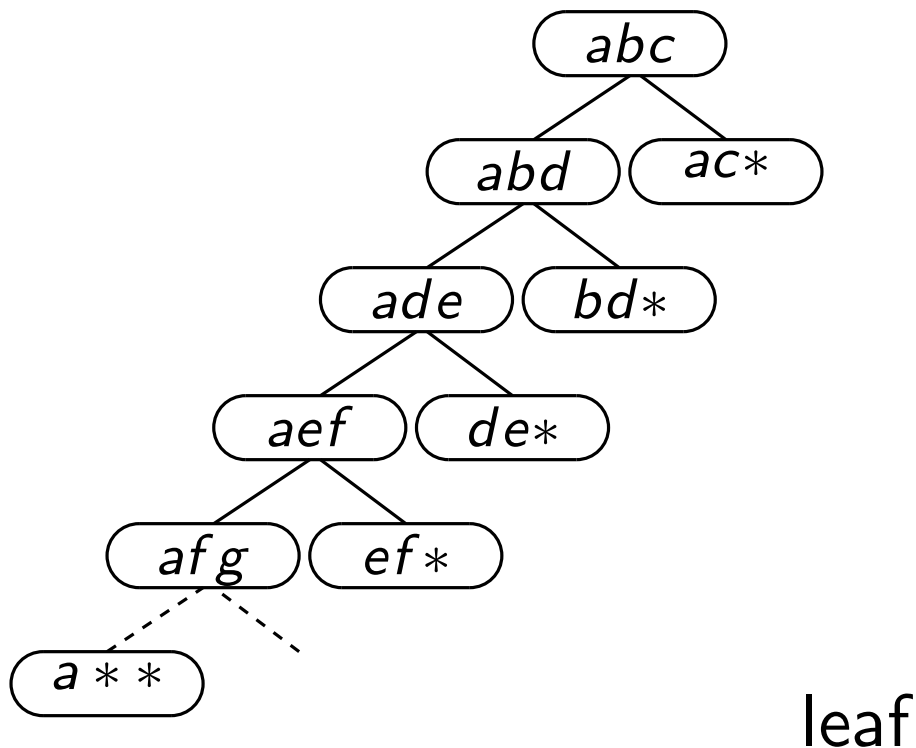
Occupation of d



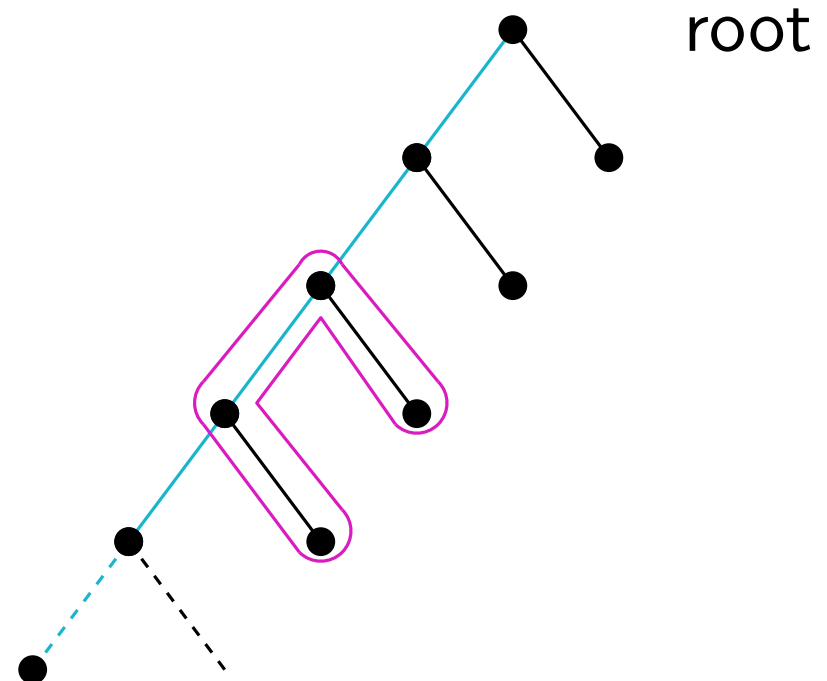
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Amount of layers occupied: ≥ 5



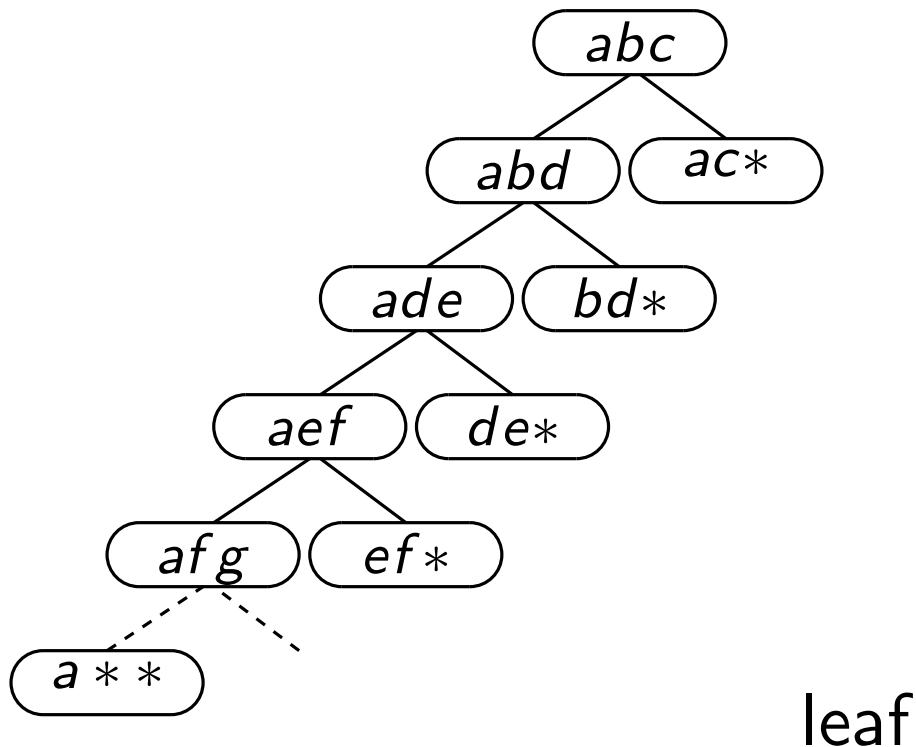
Occupation of e



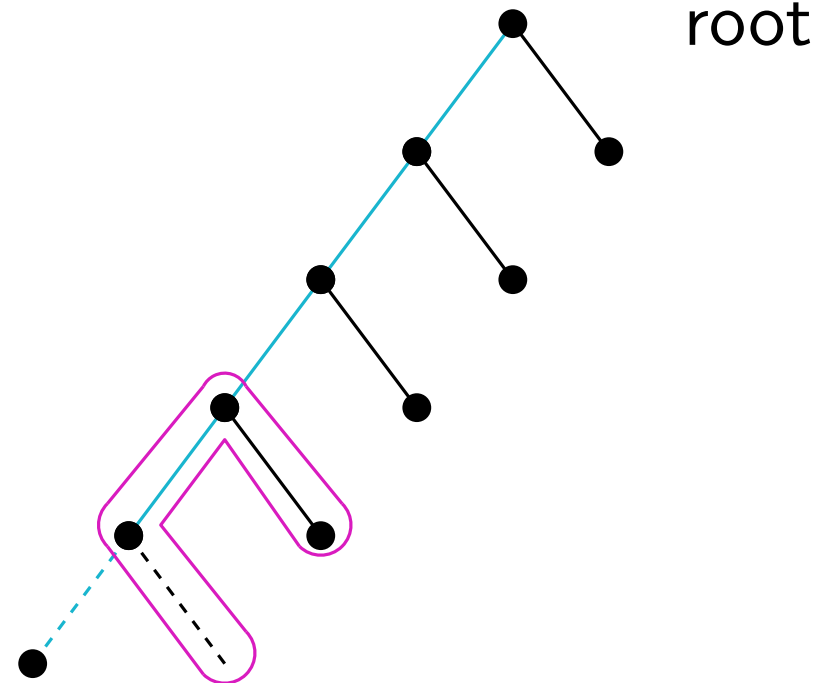
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Amount of layers occupied: ≥ 6



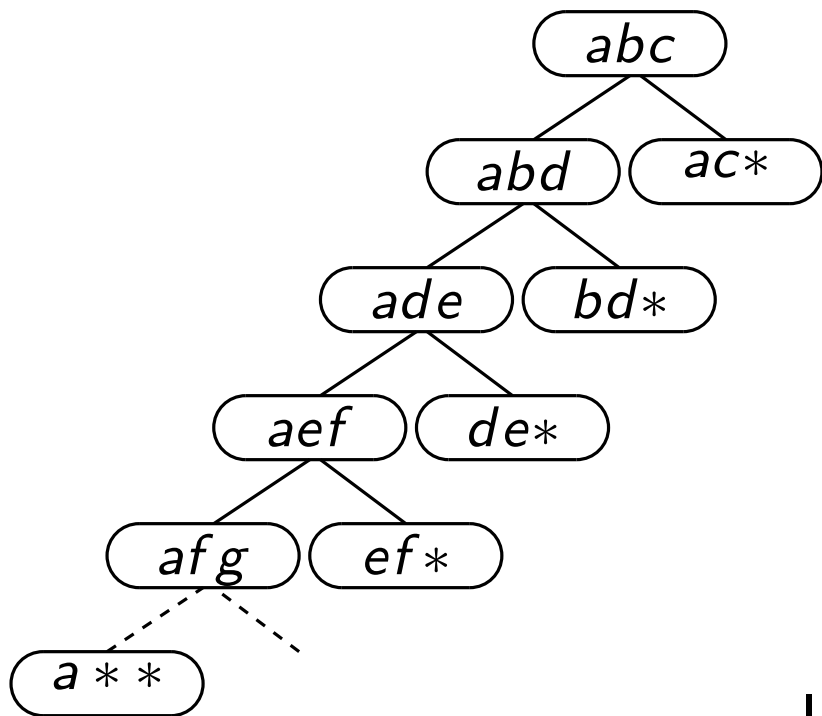
Occupation of f



Layer Amount in T'

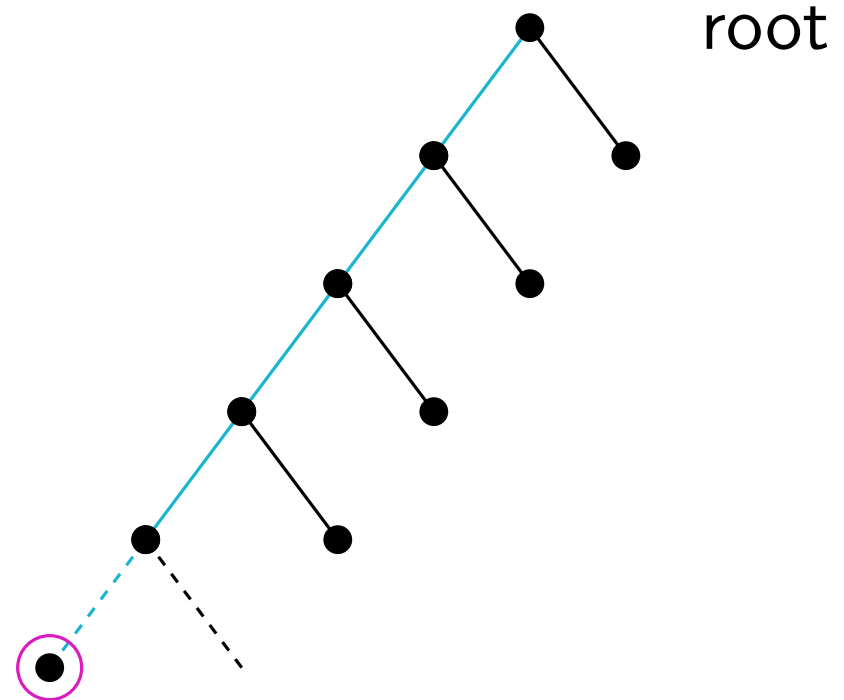
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Amount of layers occupied: $\mathcal{O}(\log n')$



leaf

DFS ends at leaf



Total Amount of Layers

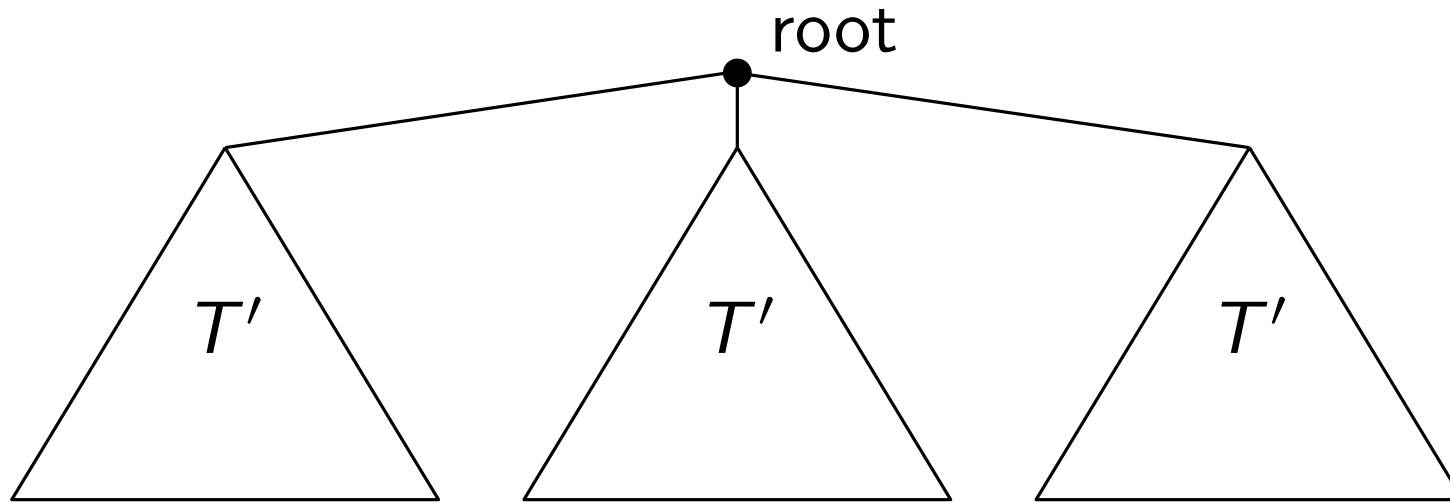
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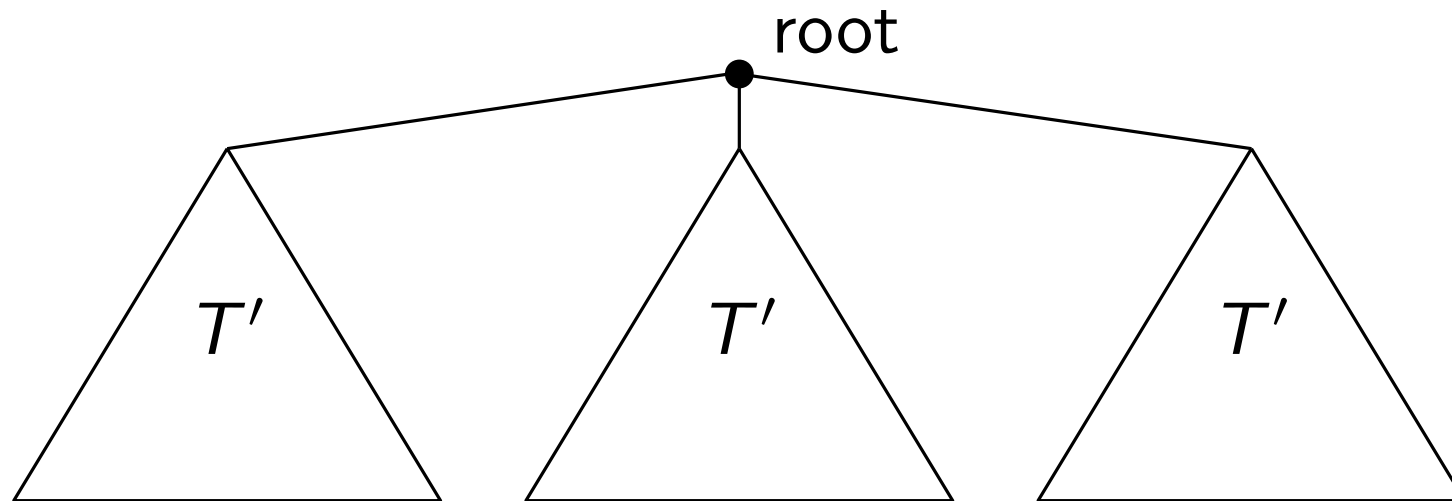
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Worst Case Subtree Structure

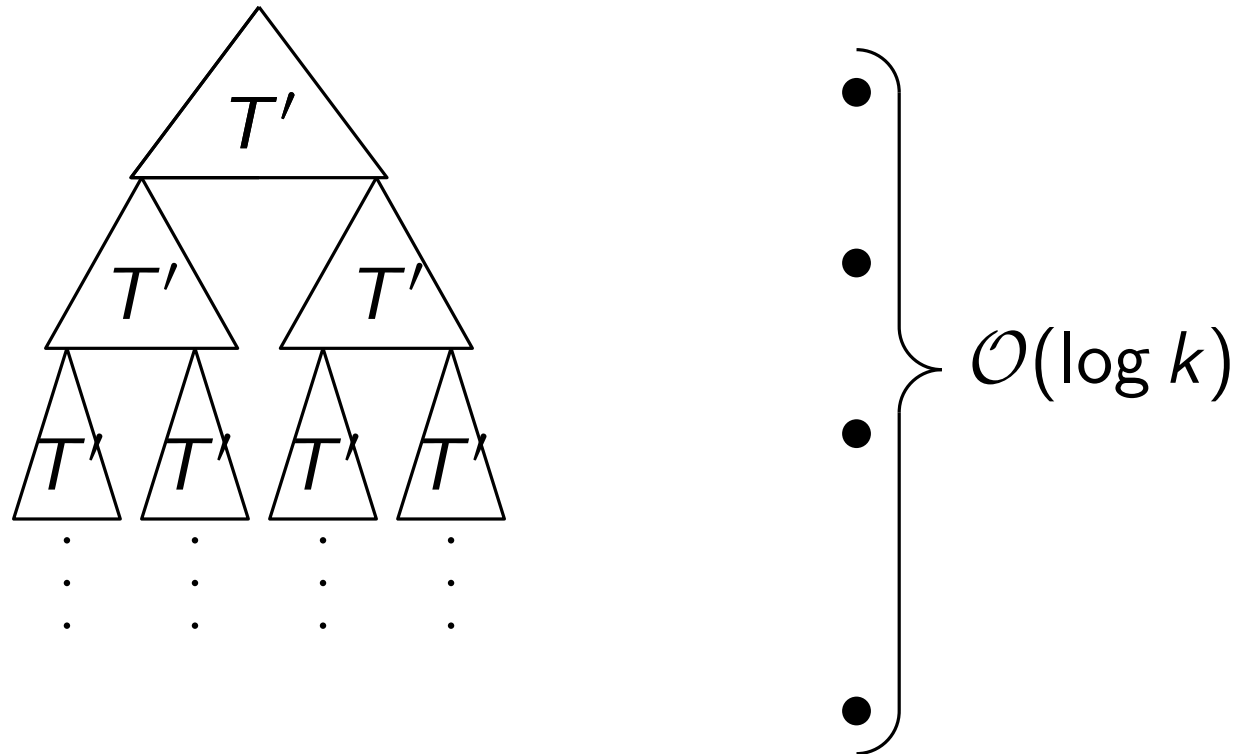
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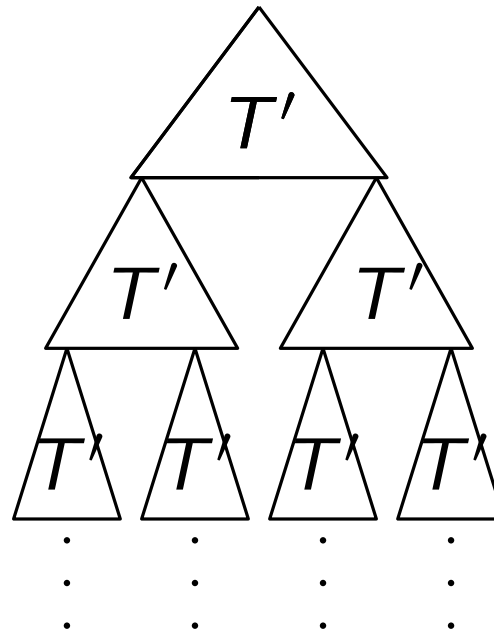
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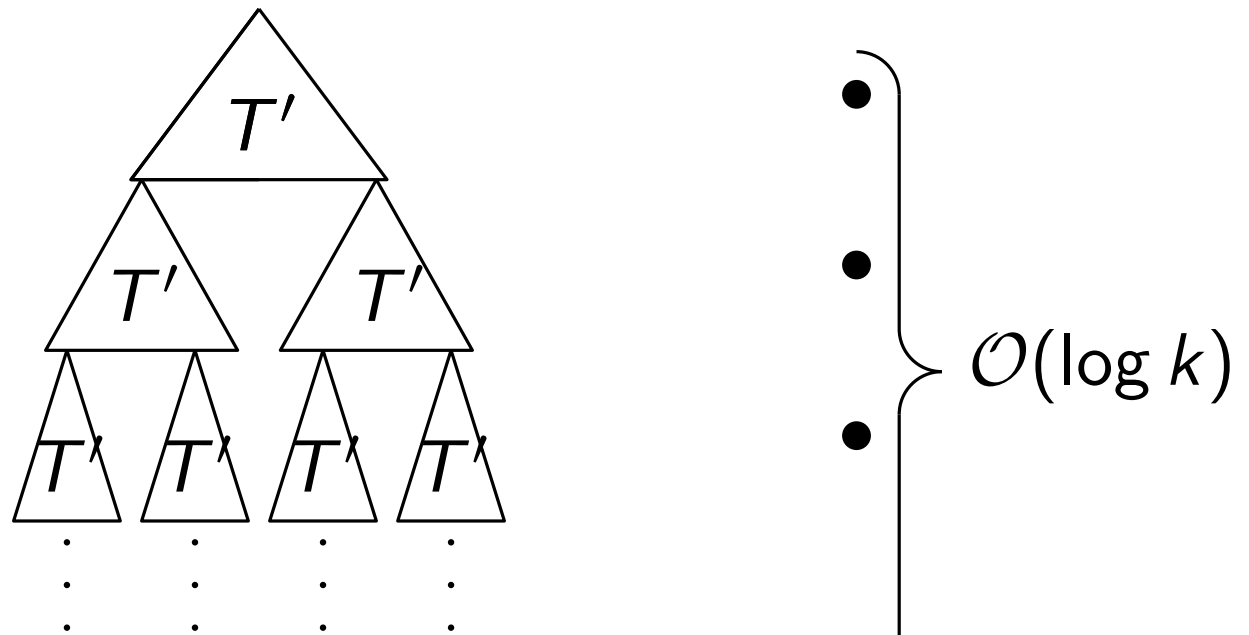
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- ▶ Any outerplanar graph G admits a polyline drawing in $\mathcal{O}(n^2 \log^2 n)$ with $r \in \mathcal{O}(\log^2 n)$ and two bends per edge

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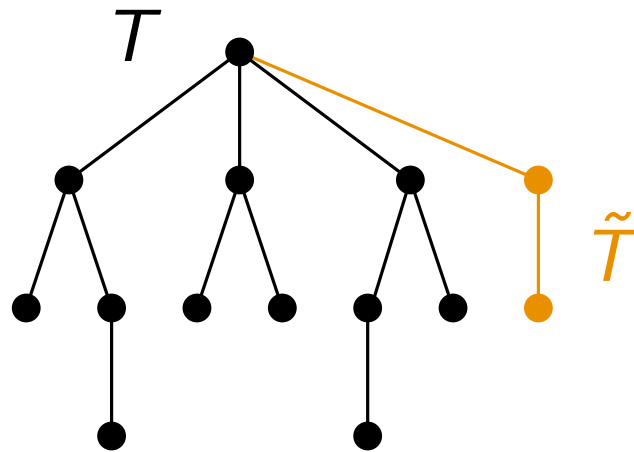
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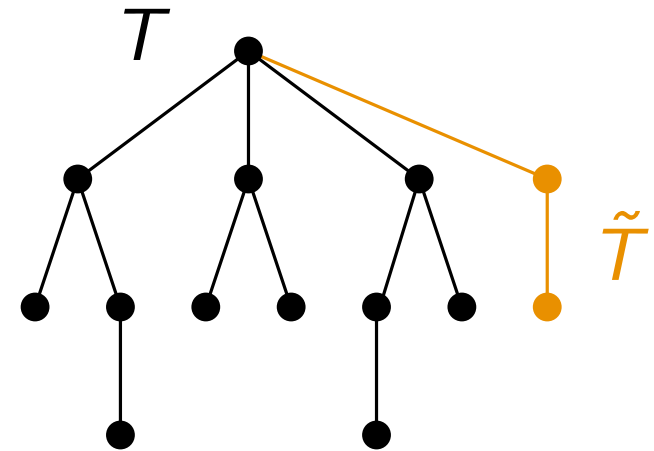
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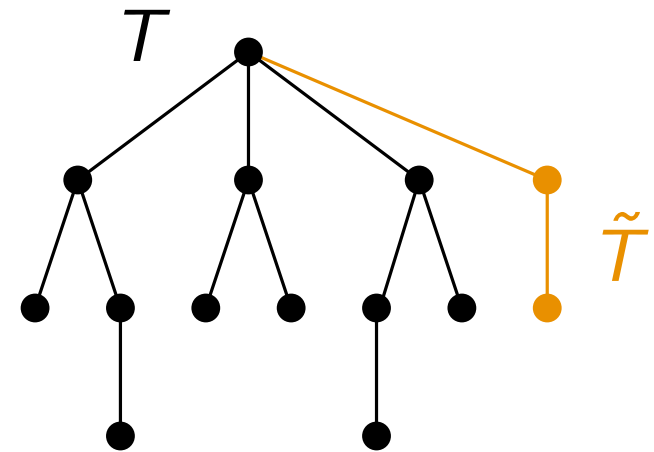
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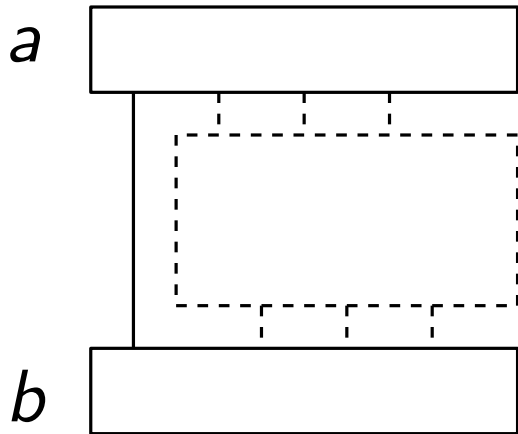
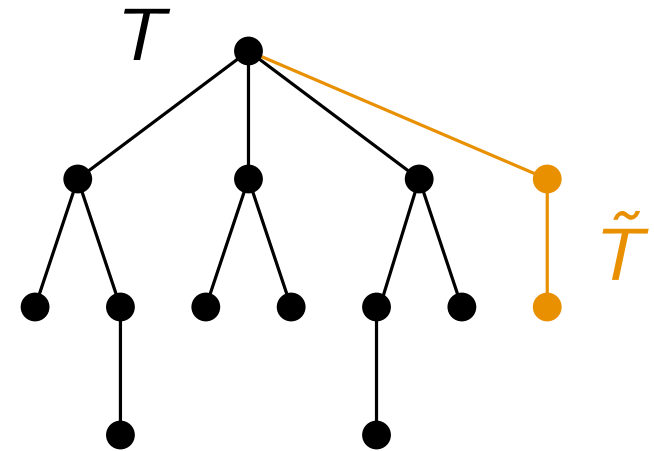
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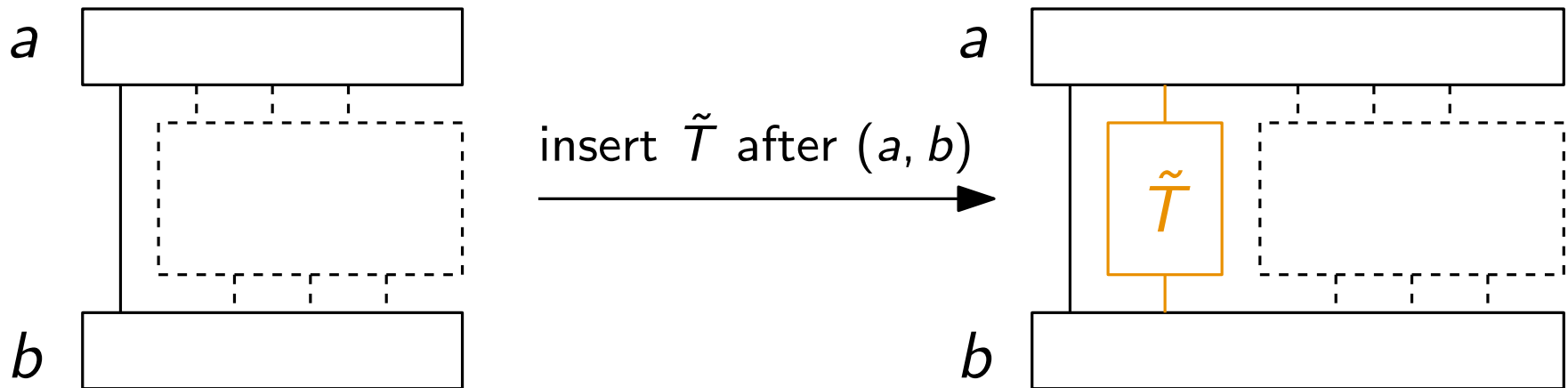
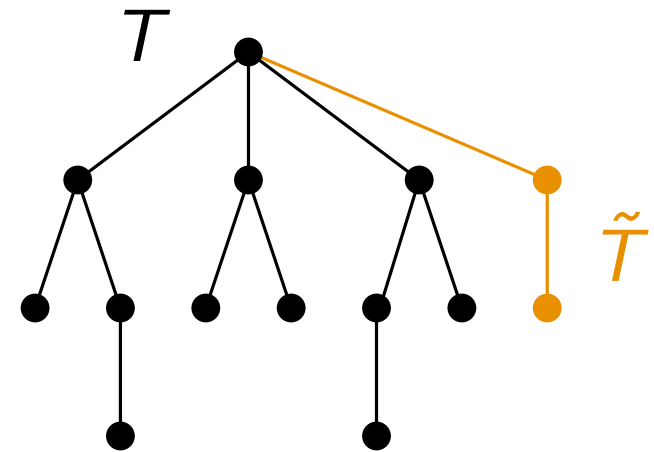
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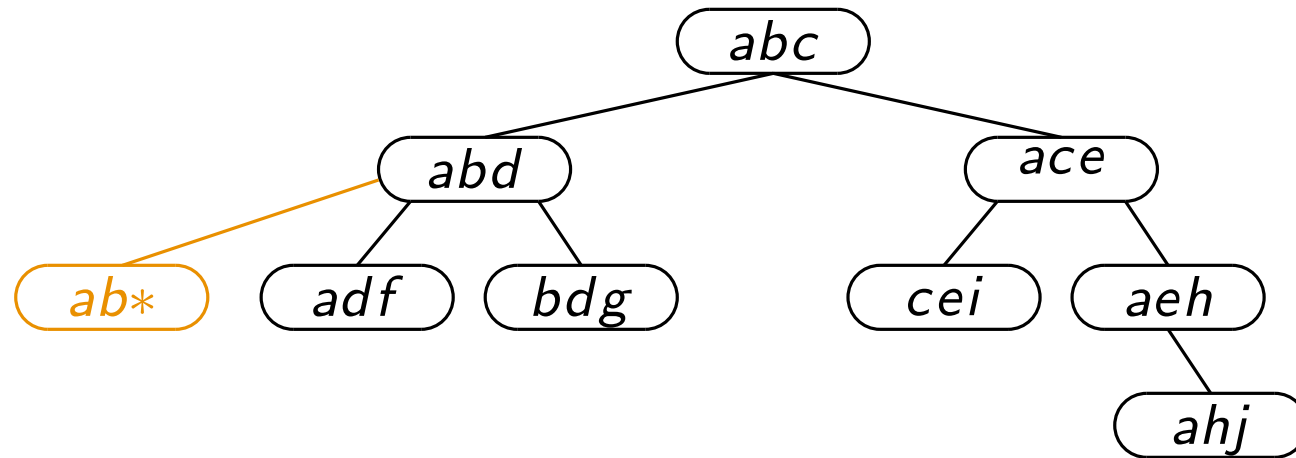


- ▶ Does not alter the area bounds of $B_{G'}$ asymptotically!

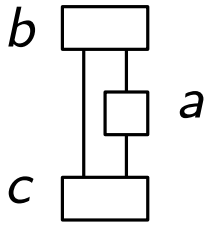
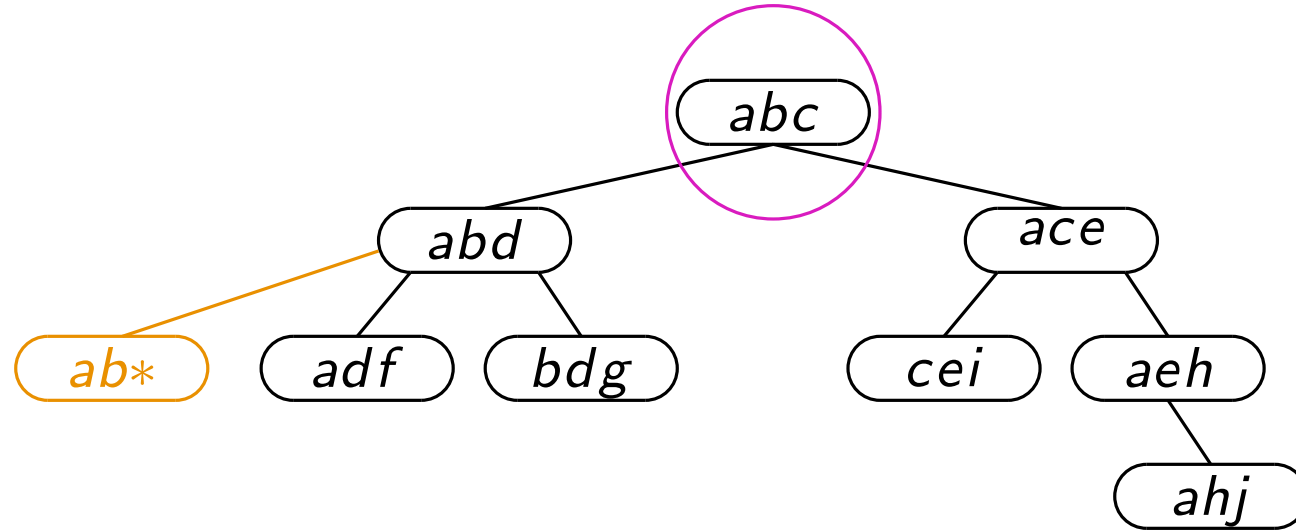
Results for Series-Parallel Graphs

- ▶ Since the area bounds stay the same, the results of outerplanar graphs also hold for series-parallel graphs
- ▶ Any series-parallel graph G admits a polyline drawing in $\mathcal{O}(n^2 \log^2 n)$ with $r \in \mathcal{O}(\log^2 n)$ and two bends per edge

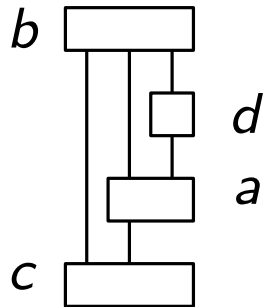
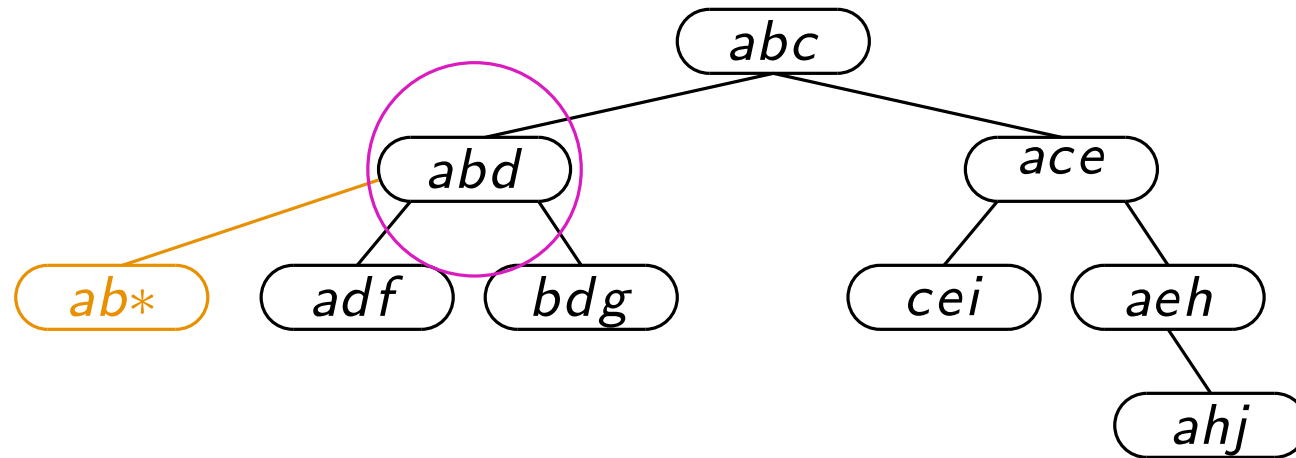
Example drawing



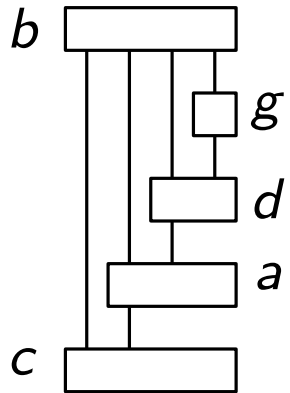
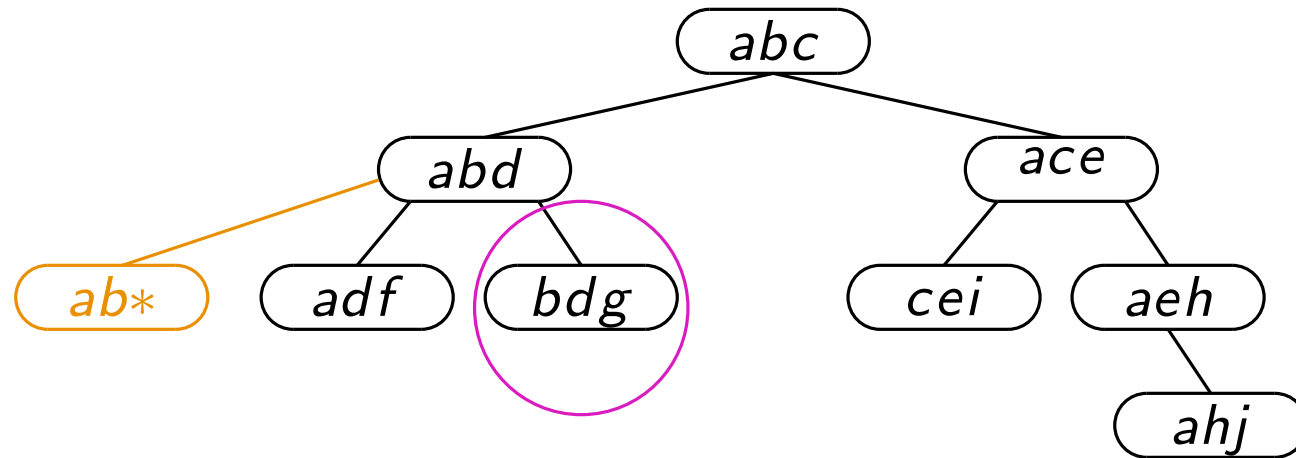
Example drawing



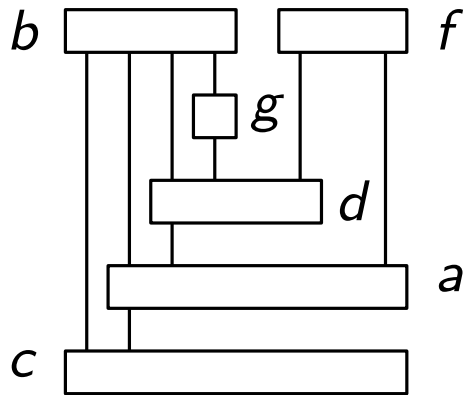
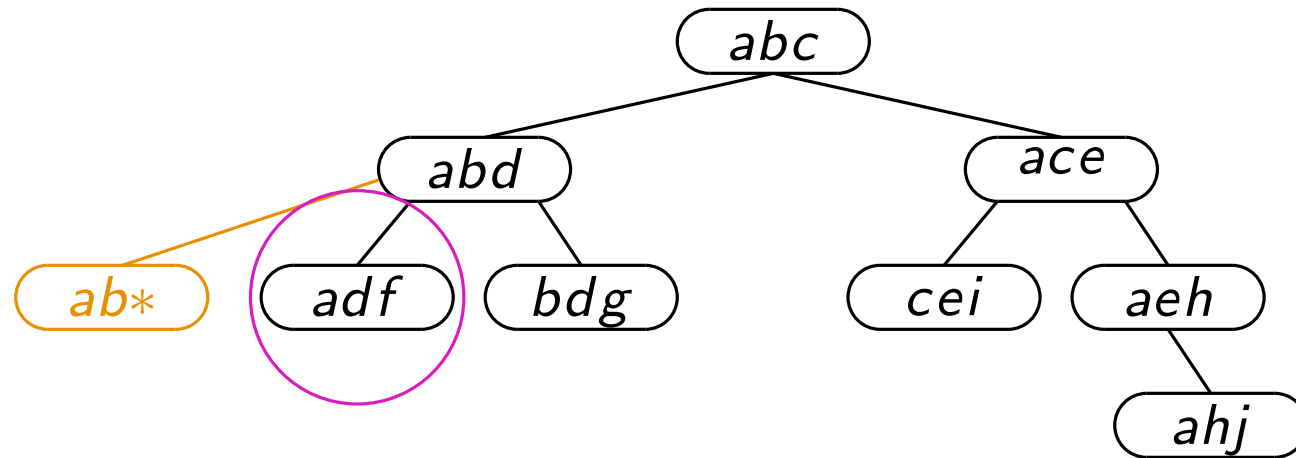
Example drawing



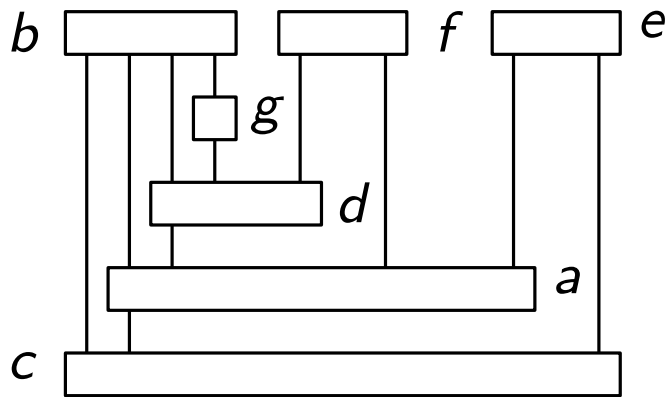
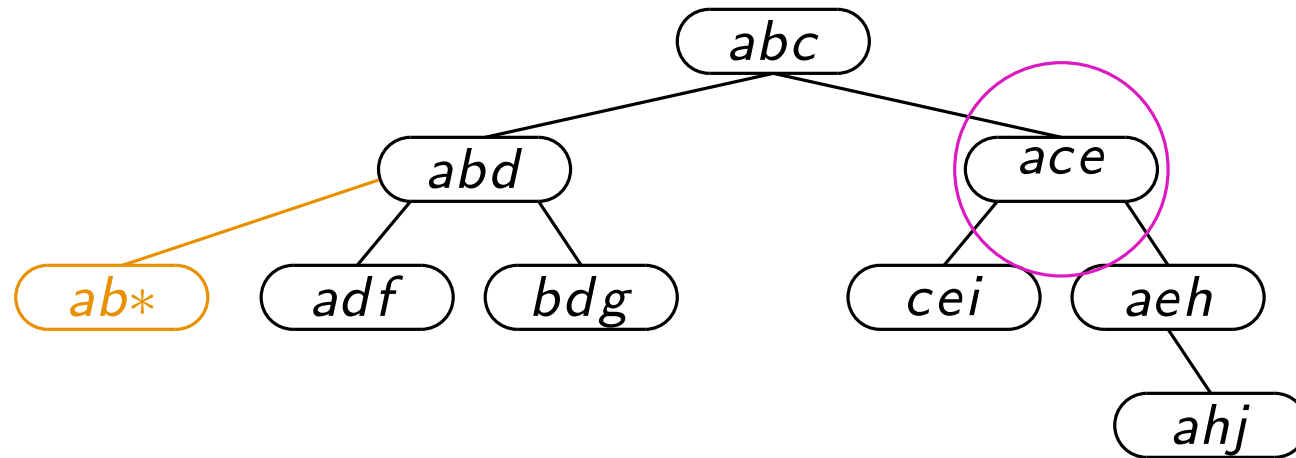
Example drawing



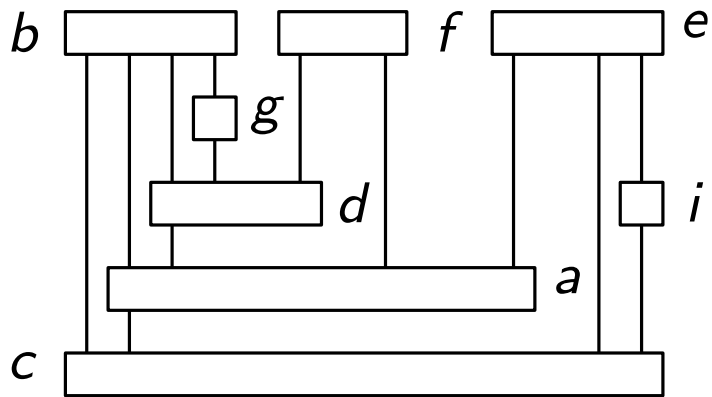
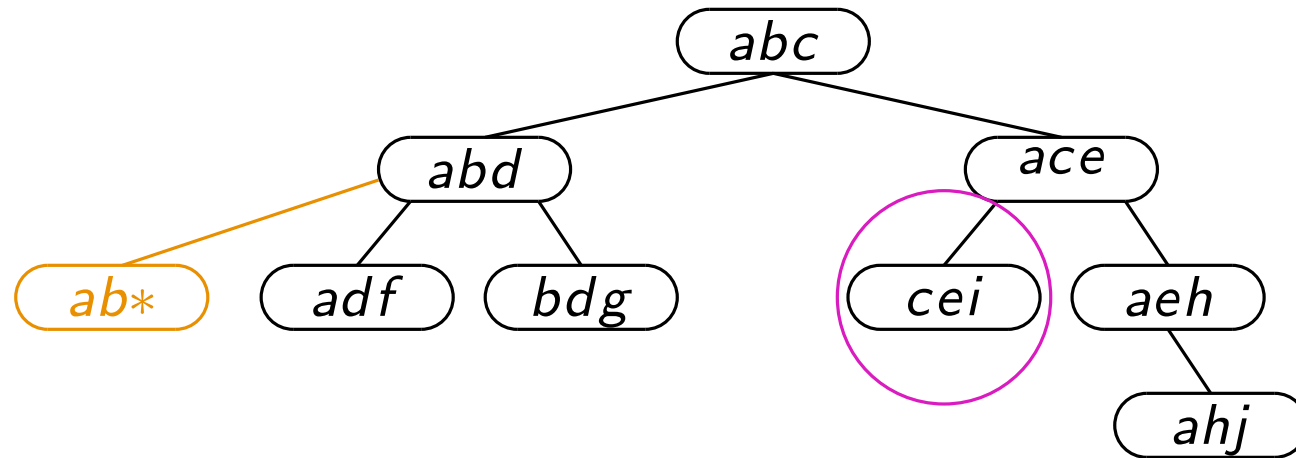
Example drawing



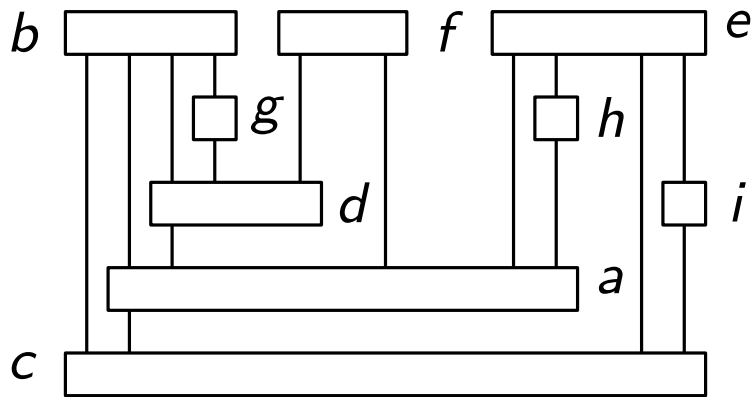
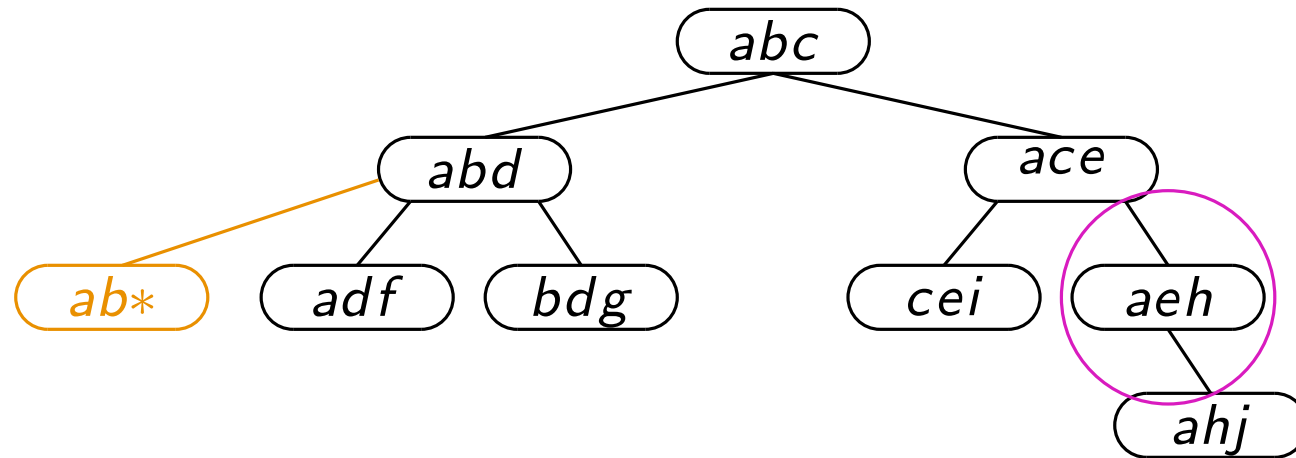
Example drawing



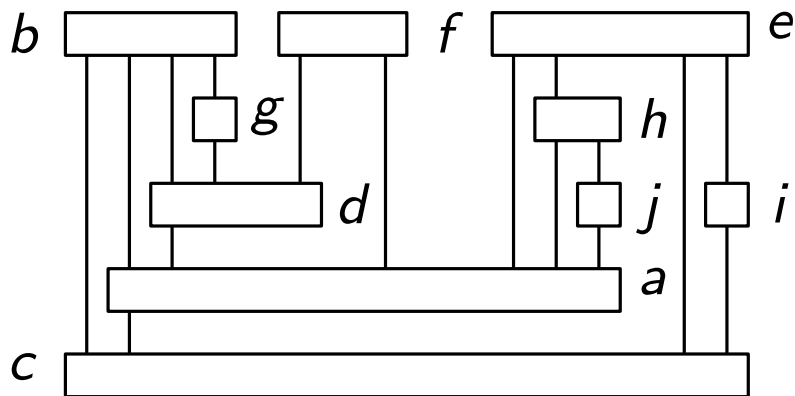
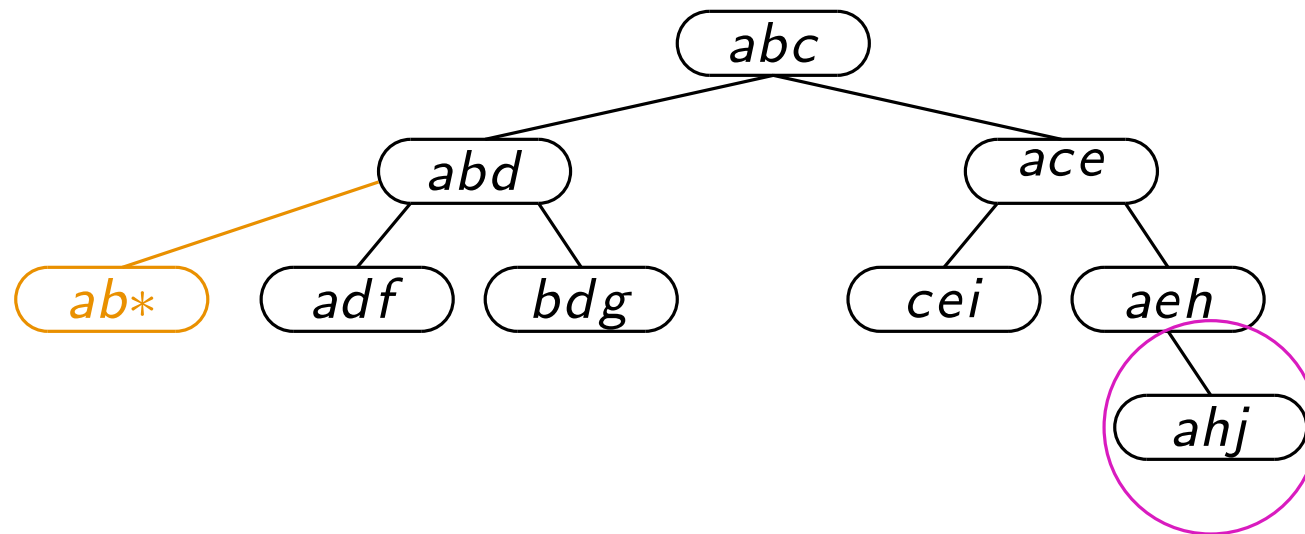
Example drawing



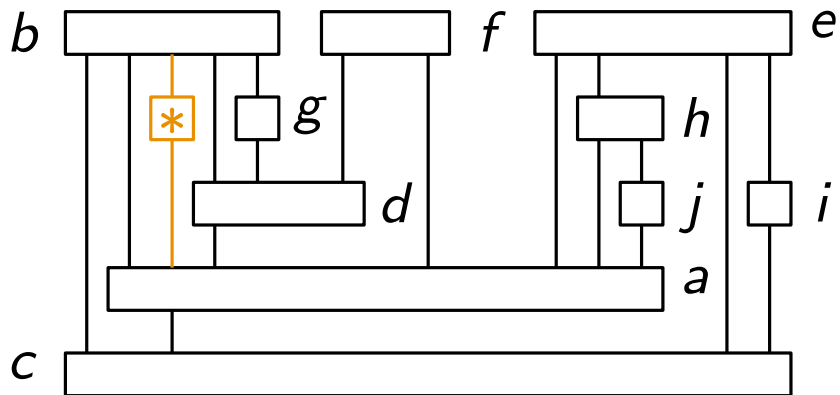
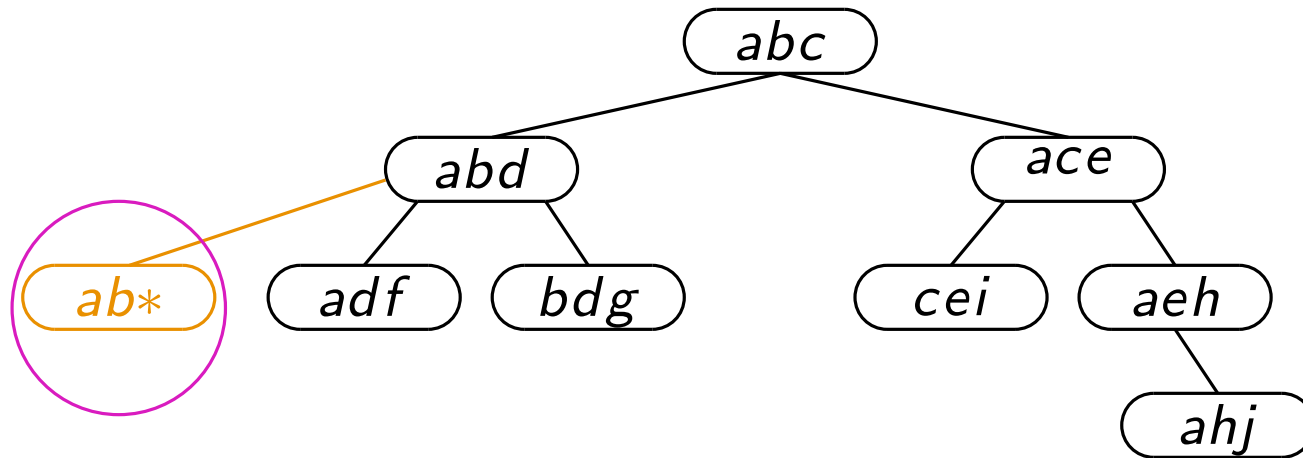
Example drawing



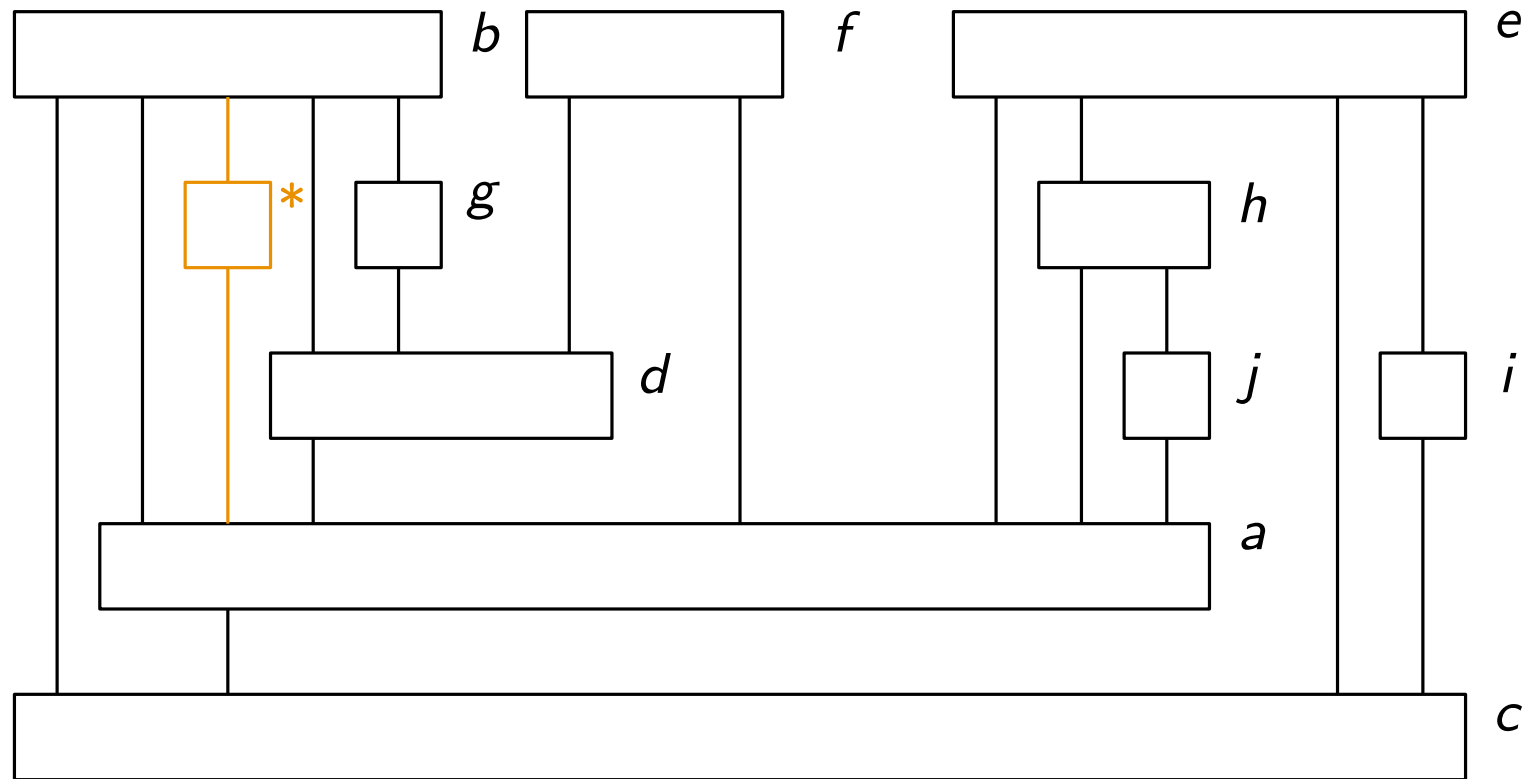
Example drawing



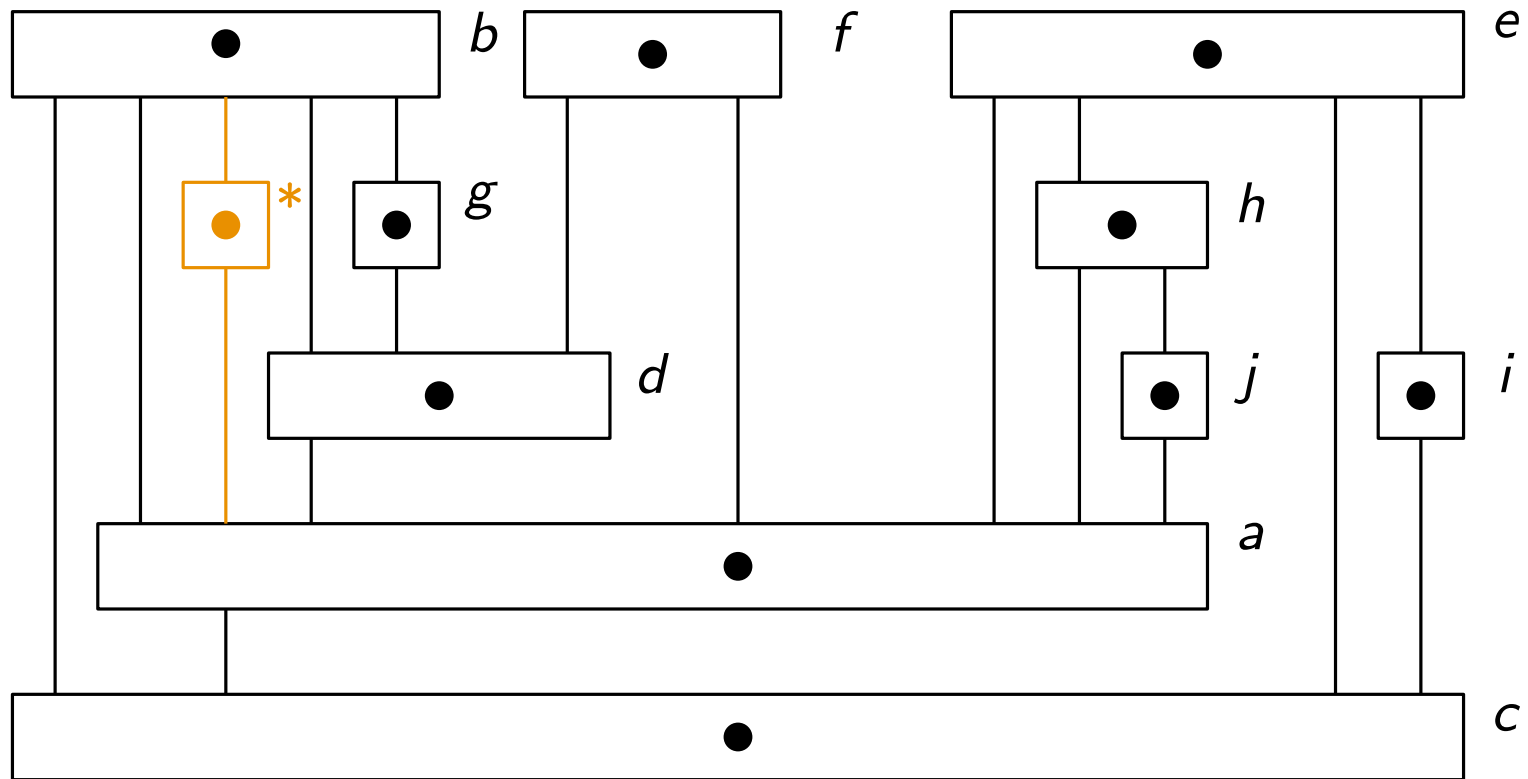
Example drawing



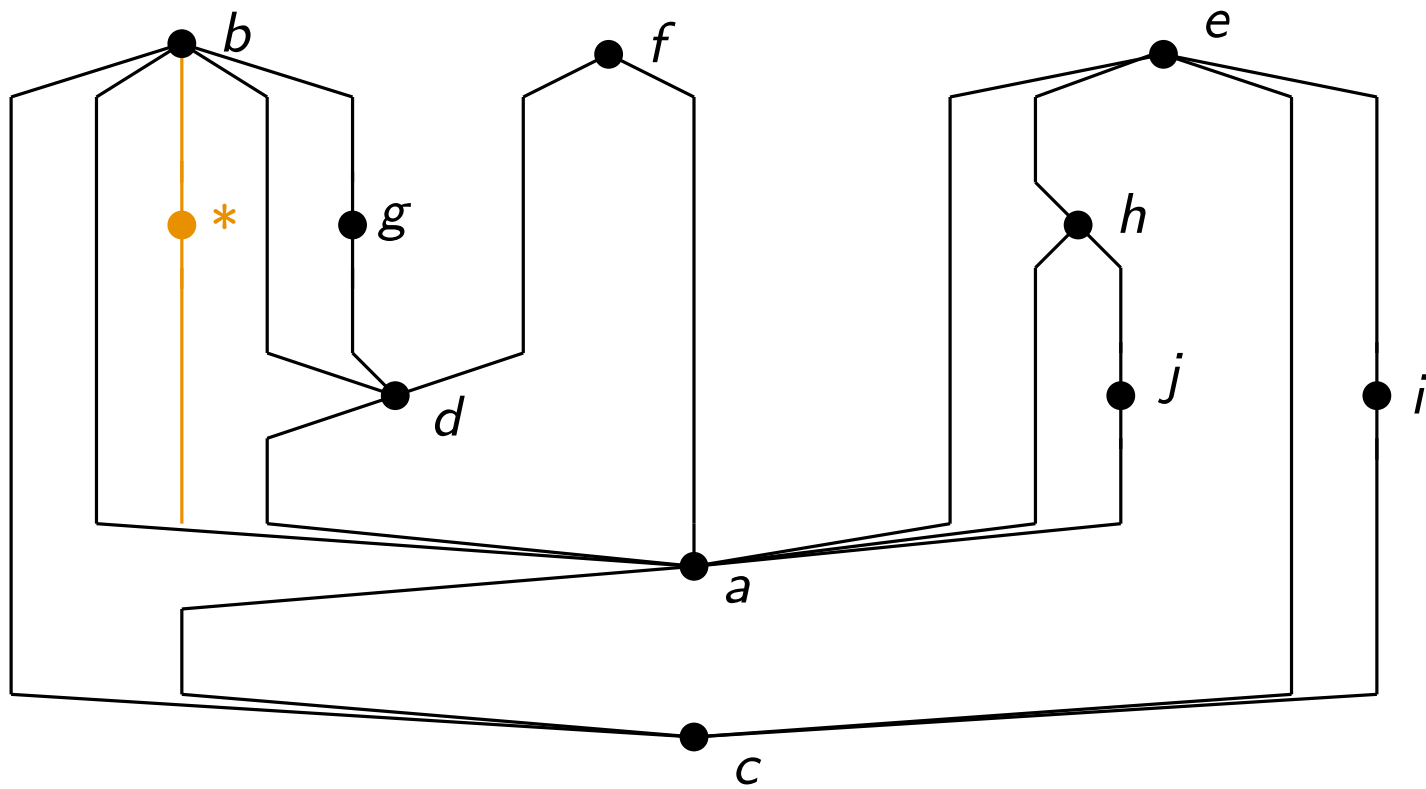
Example drawing



Example drawing



Example drawing



Overall Results

Graph Class	Ratio r	Area	Bends per Edge

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Complete k -ary Trees	$1 + \varepsilon$	$\mathcal{O}(n^2 \log n)$	-

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Outerplanar Graphs	$\mathcal{O}(\log^2 n)$	$\mathcal{O}(n^2 \log^2 n)$	2
Series-parallel Graphs	$\mathcal{O}(\log^2 n)$	$\mathcal{O}(n^2 \log^2 n)$	2

Questions?



Thank you for participating!

:)

Planar 3-trees

Complete k -ary Tree Implementation

- ▶ Implemented in python ≥ 3.8
- ▶ `networkx` library used for a graph structure representation
- ▶ `matplotlib` library used for plotting
- ▶ Coordinate computation can be found in thesis