On Maximizing the Euclidian Distance between Adjacent Vertices in Planar Drawings of Small Area

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Introduction

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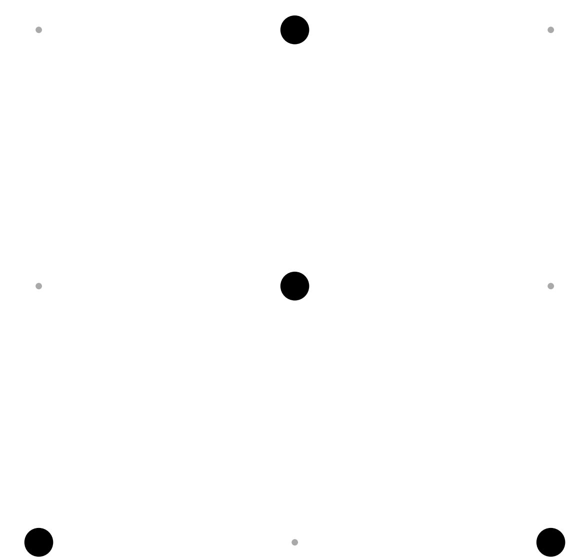

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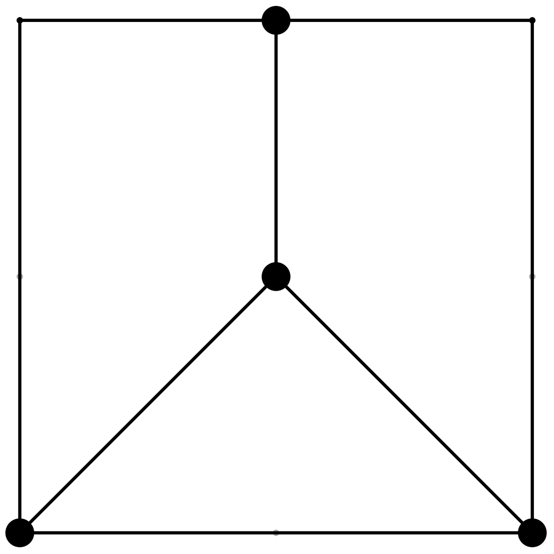

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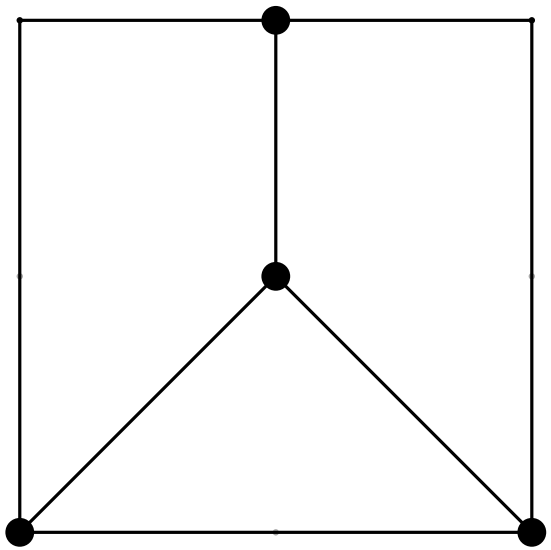

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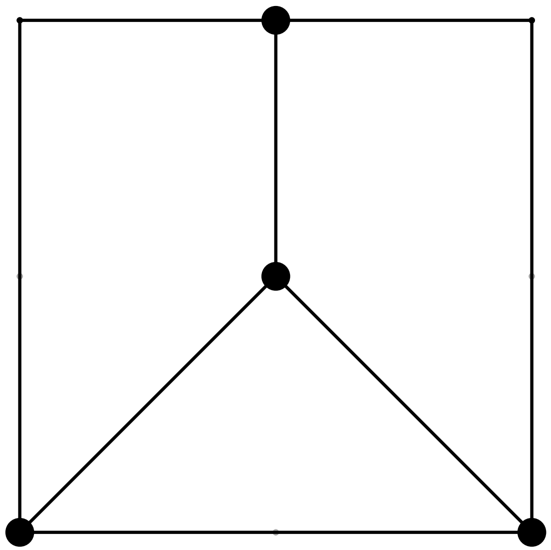

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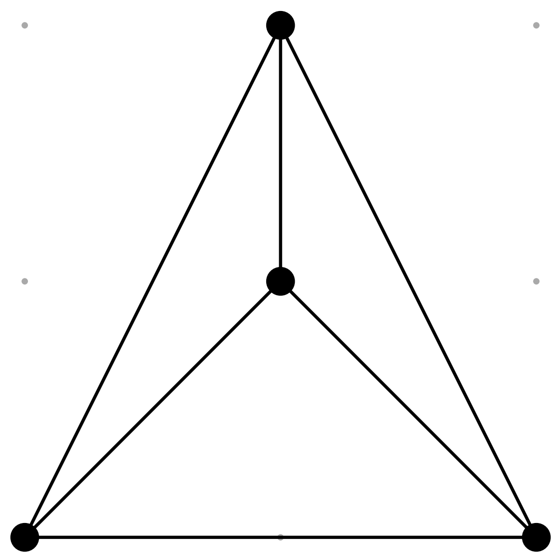

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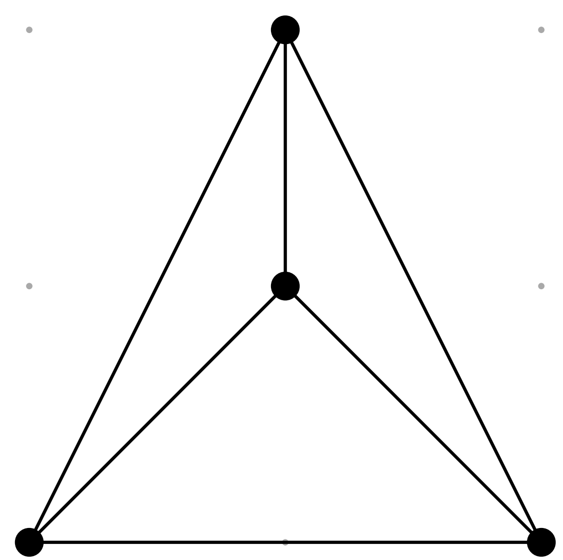

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- Straight-line drawing is a polyline drawing without bends
- Total grid size determines geometrical properties

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  - *Live Challenge* as part of the contest includes heuristic approaches for geometrical problems of graph drawings on a fixed grid size
- This work addresses producing drawing algorithms for specific graph classes which will approach the geometrical problems with results described asymptotically

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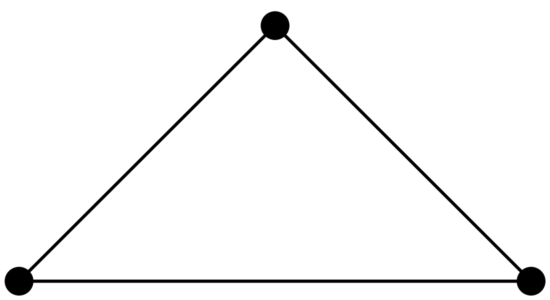

$r_{2021} \neq 1$

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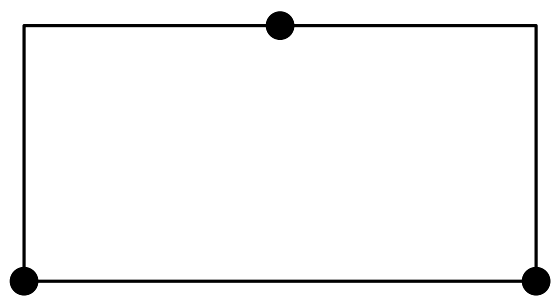

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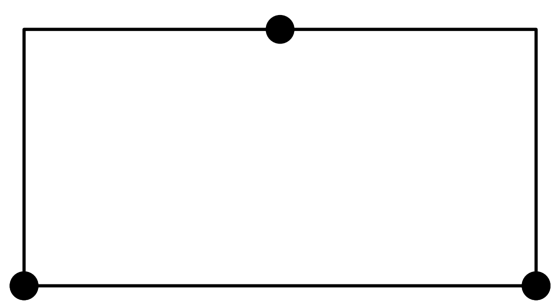

$$r = \sqrt{2}$$

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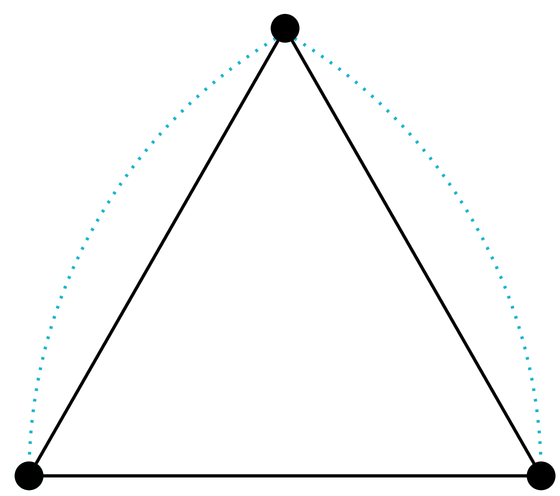

$r = 1 + \varepsilon$, ε depends on grid size

Adjacent and Nested Triangles

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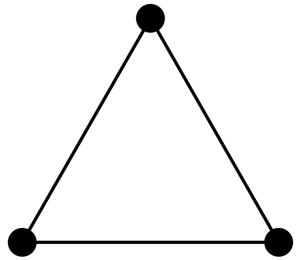

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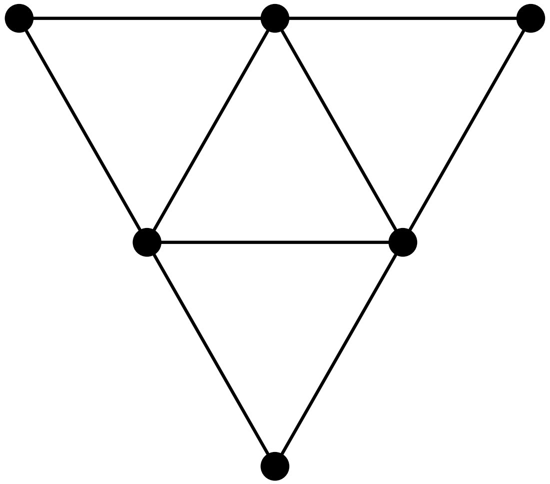

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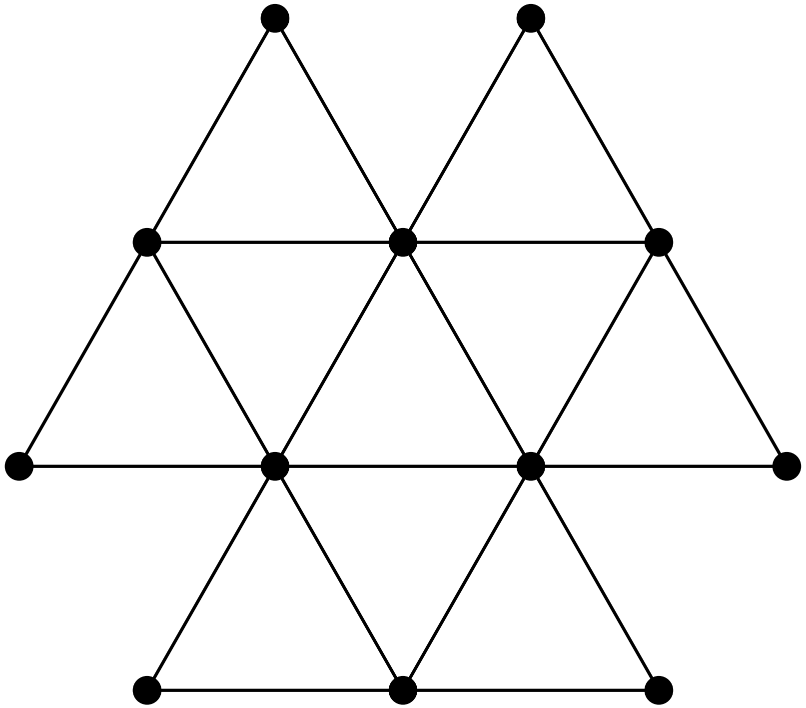

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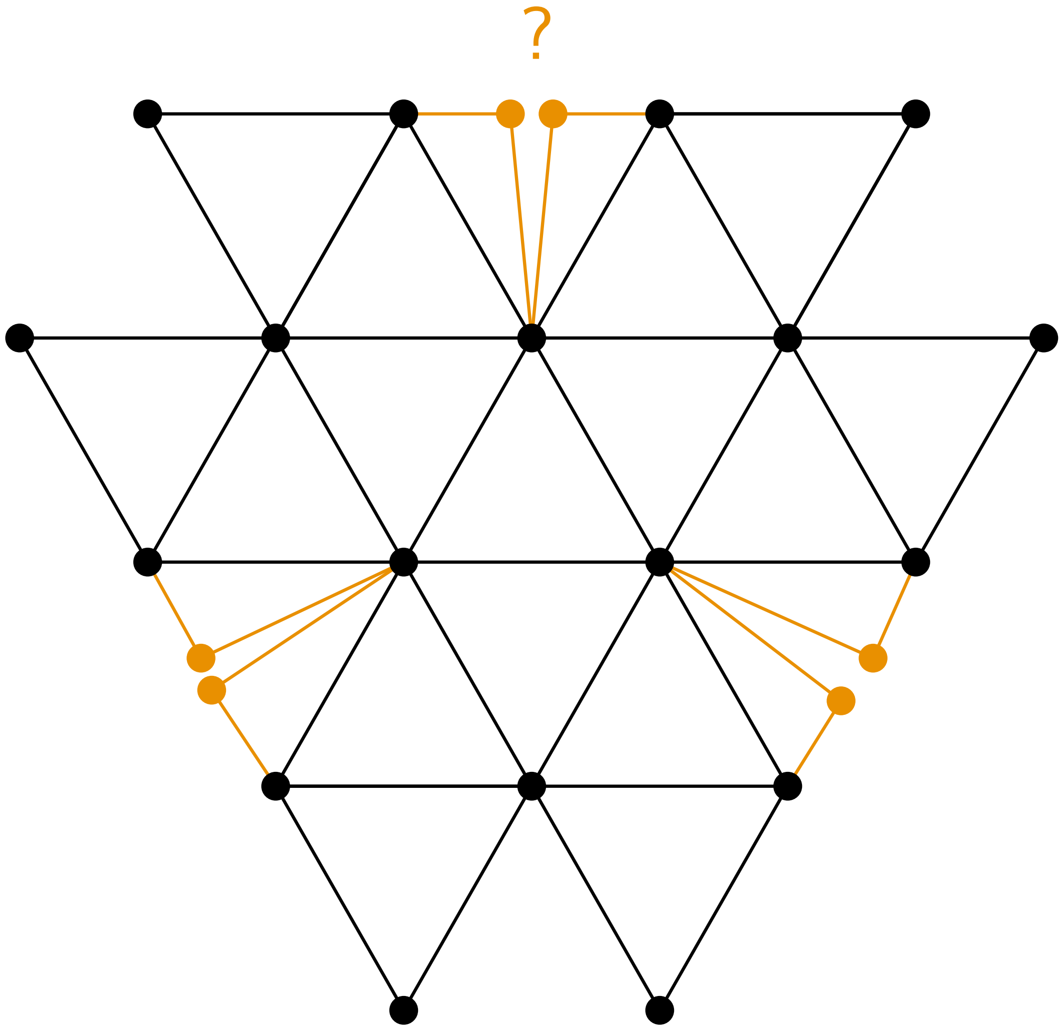

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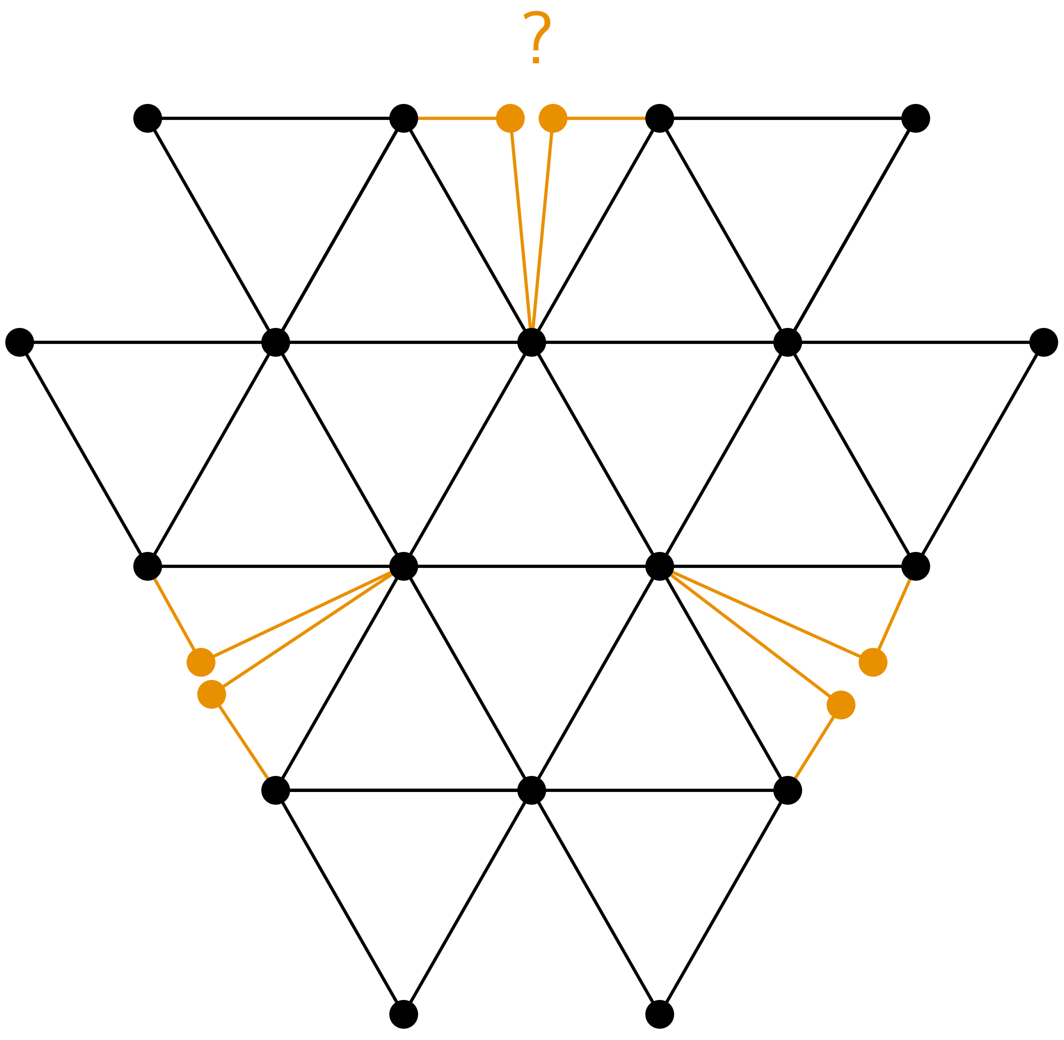

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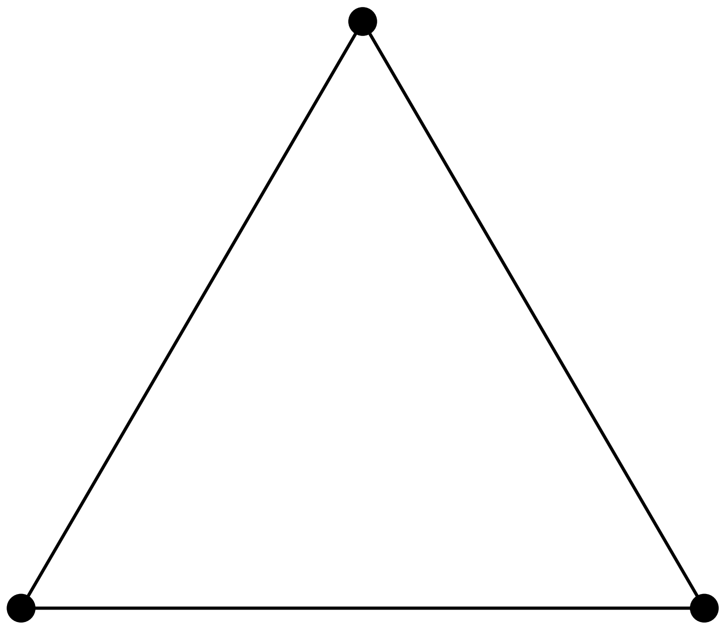

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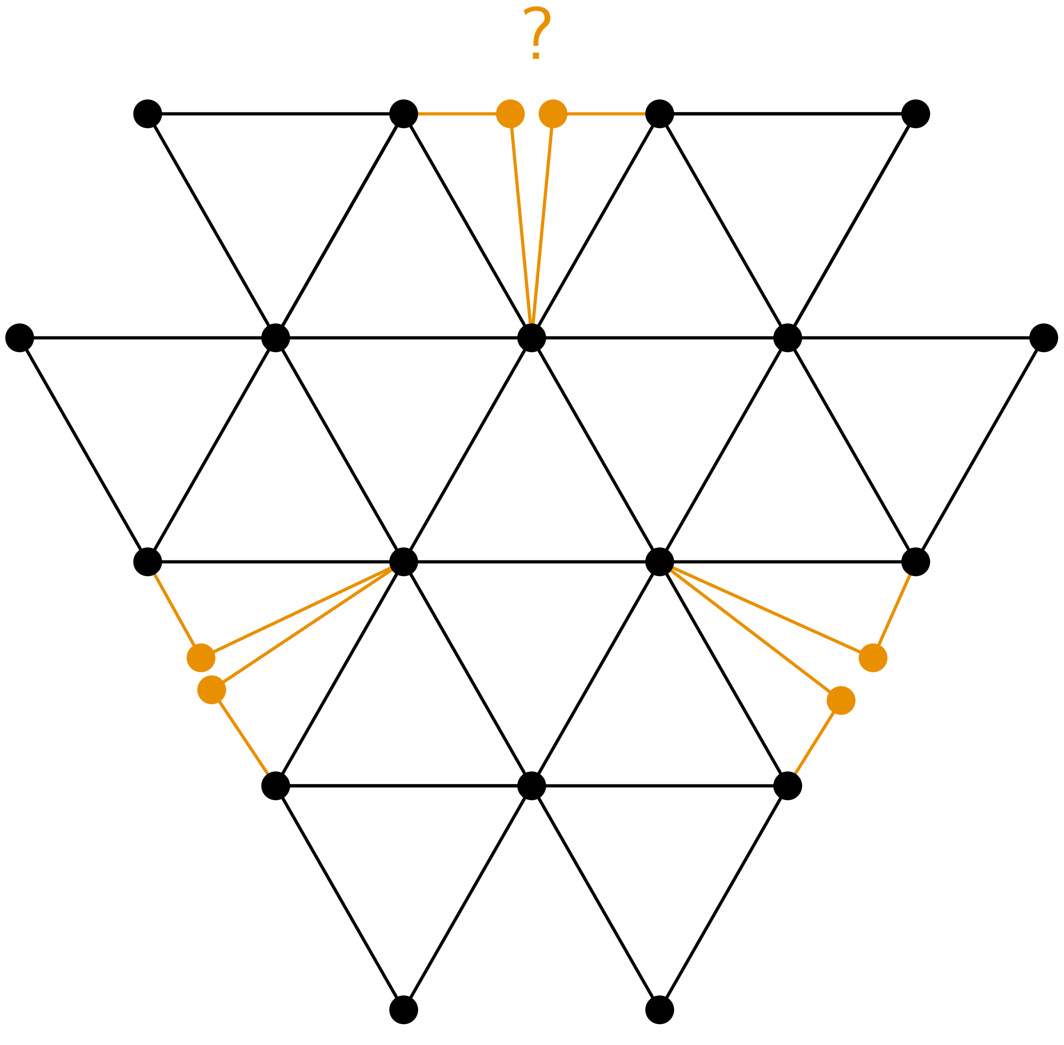

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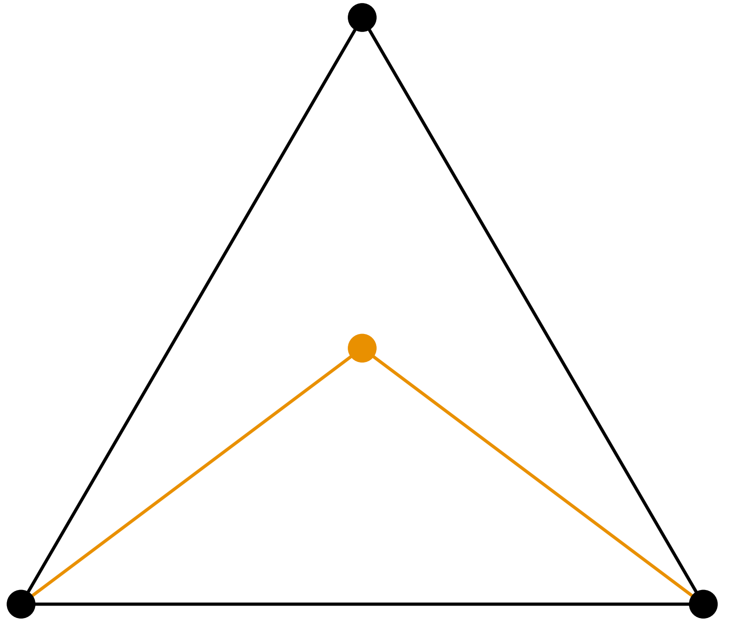
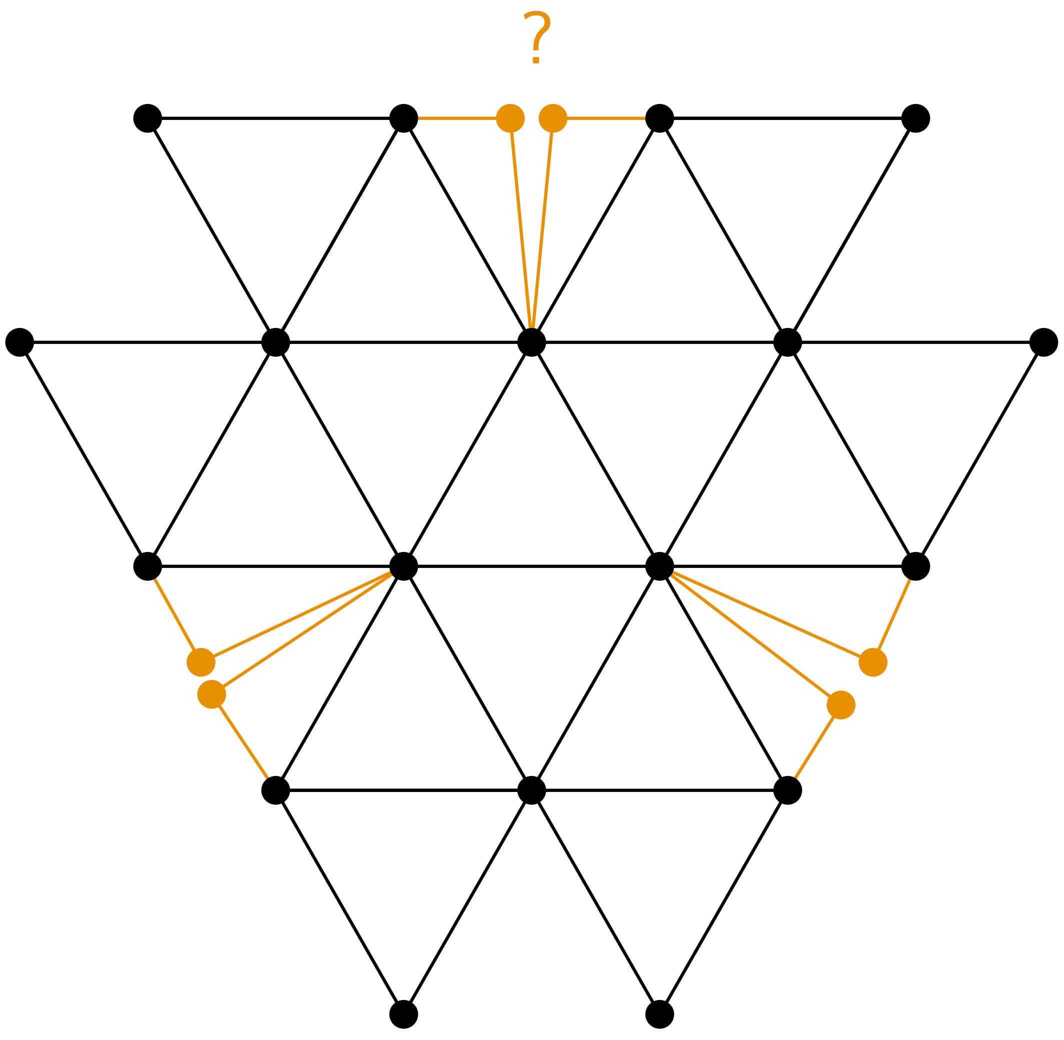
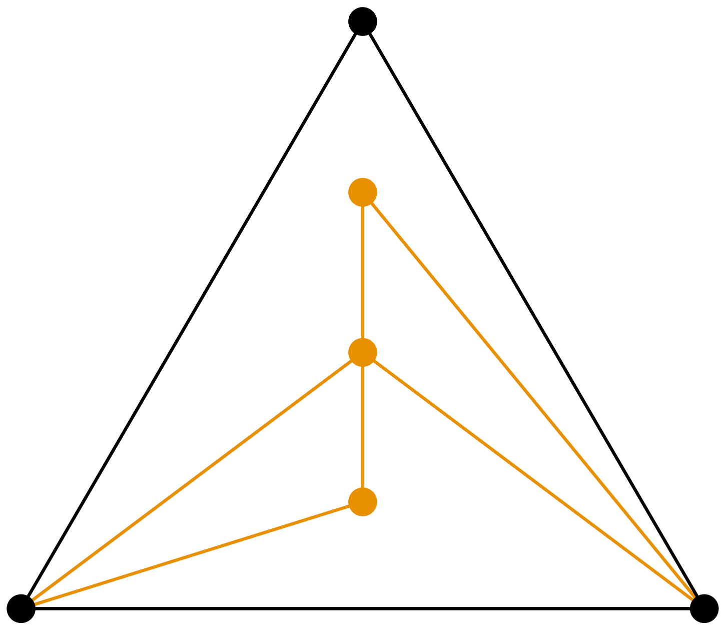


Adjacent and Nested Triangles

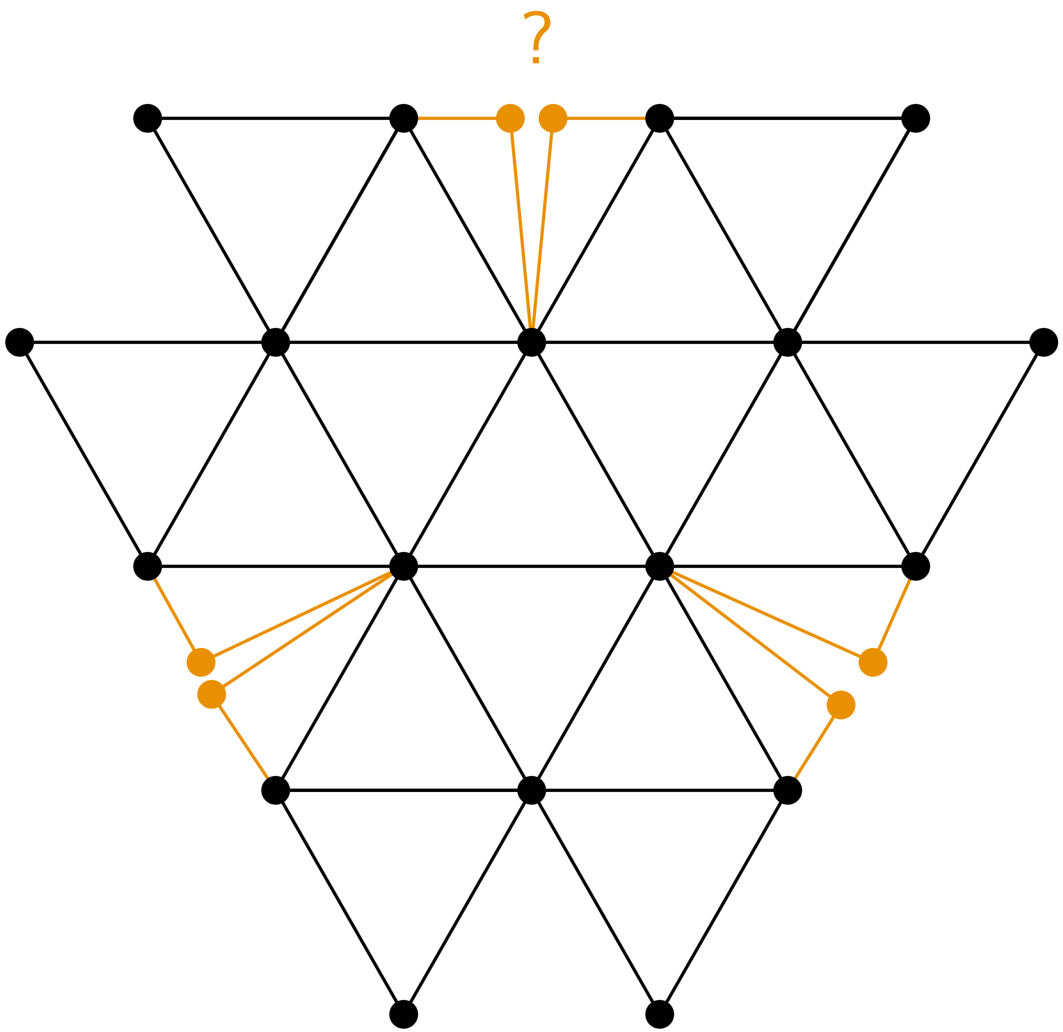
- $r = 2 + \varepsilon$
- $r = 4 + \varepsilon$

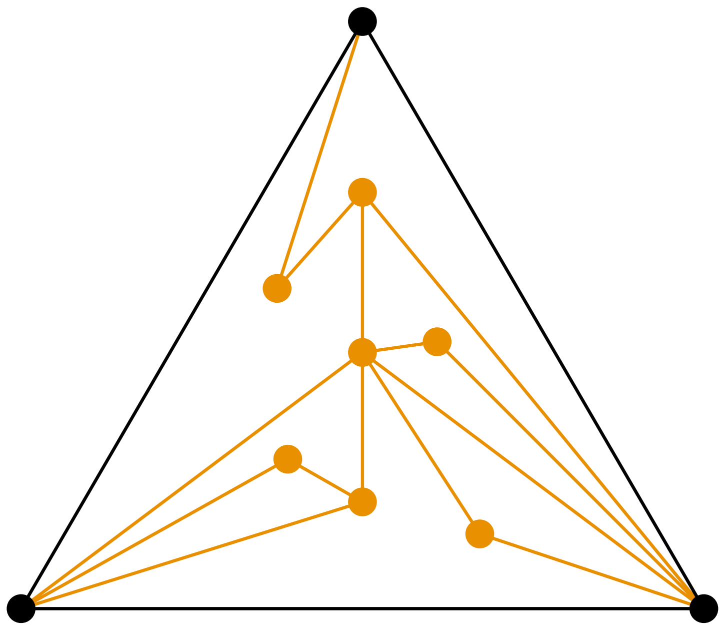




Adjacent and Nested Triangles

- $r = 2 + \varepsilon$
- $r = 8 + \varepsilon$





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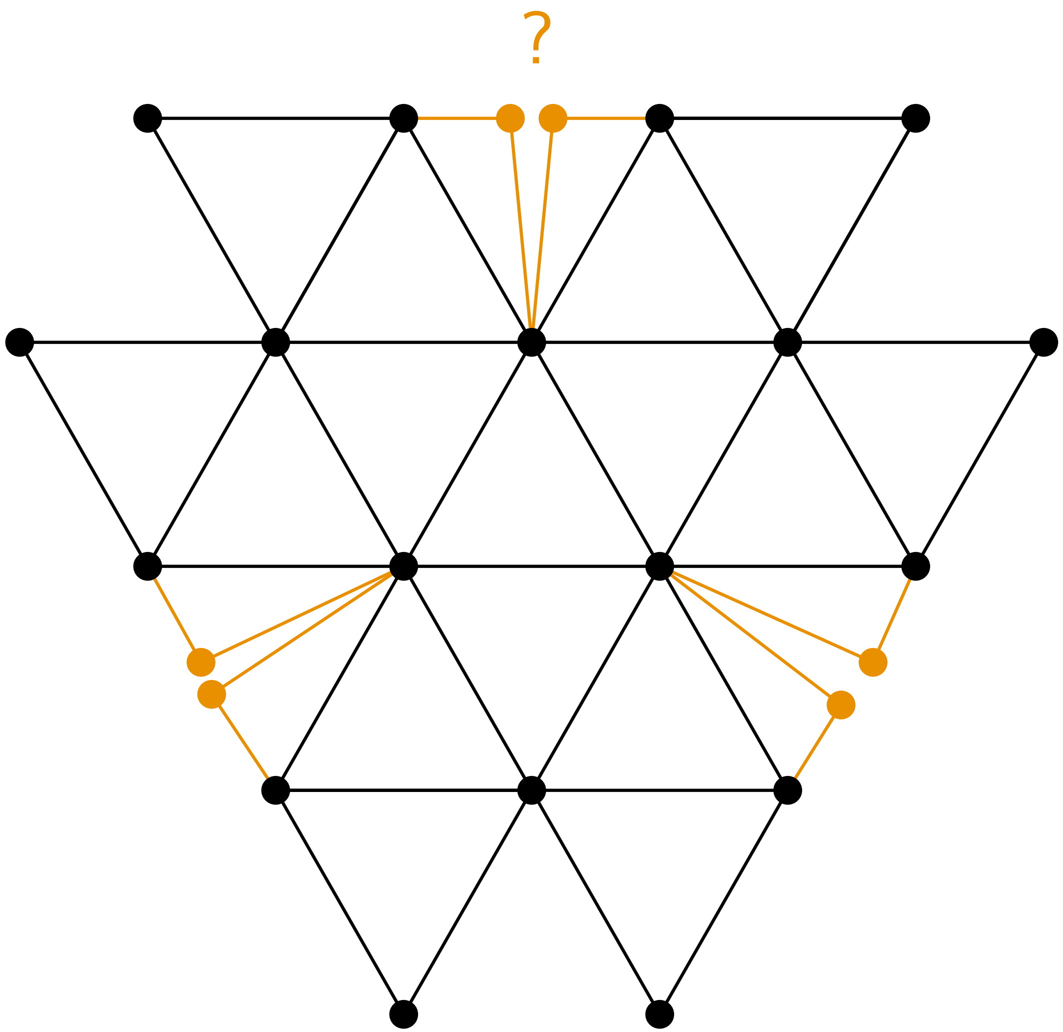
- $r = 2 + \varepsilon$
- *outerplanar graphs*



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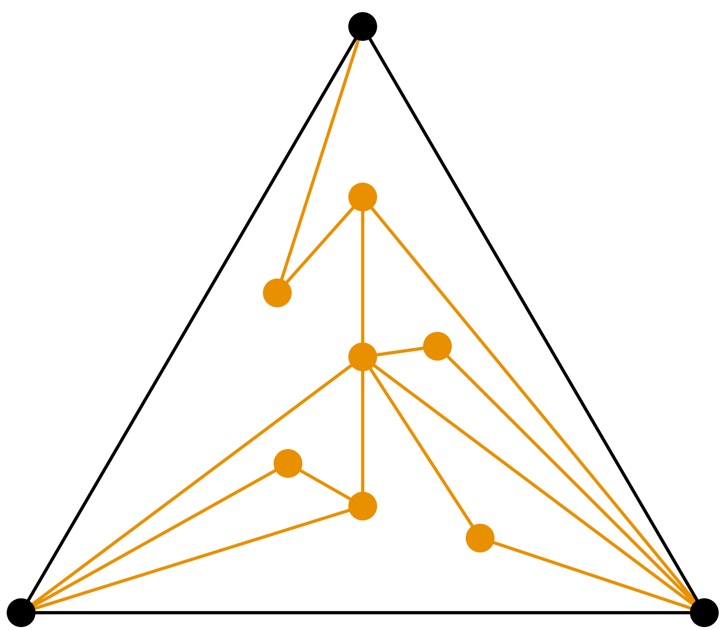

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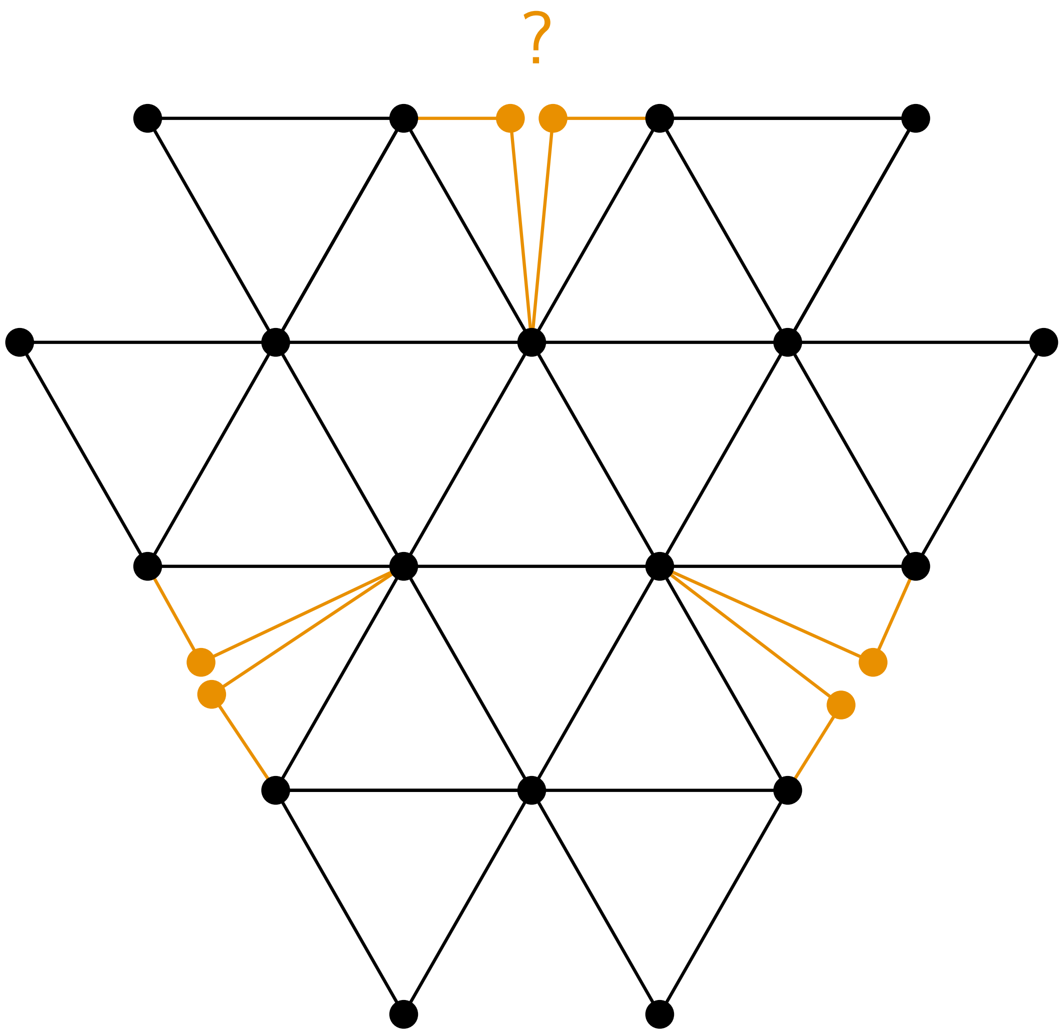

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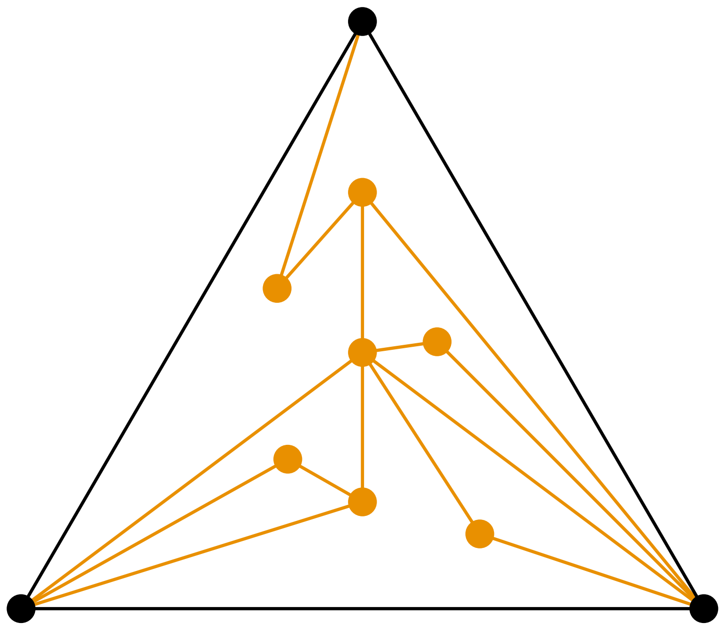

...logarithmic?

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- Appendix
  - Planar 3-trees
  - Implementation of Complete k-ary Tree Drawer

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- k-ary tree is *complete* if every inner node has k children and all leaves are on the same height

Drawing A Complete k-ary Tree T

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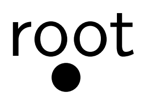
- Place the root of T on the grid



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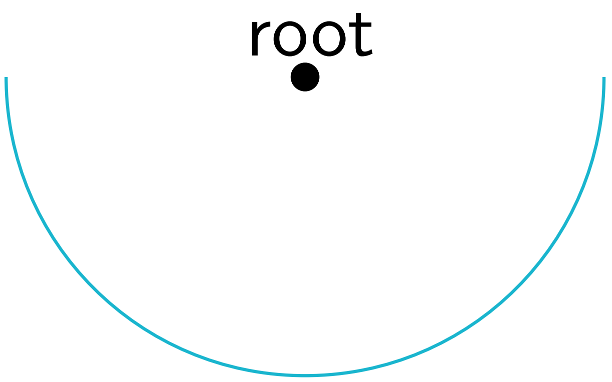

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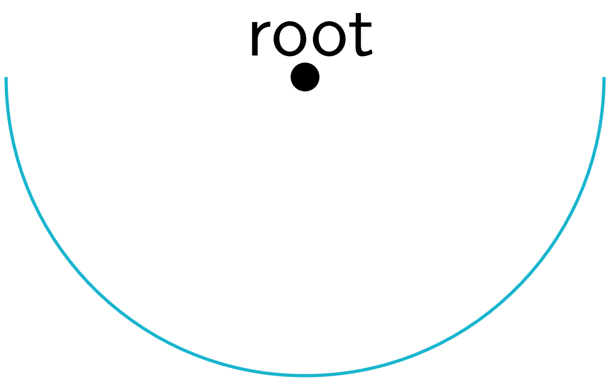

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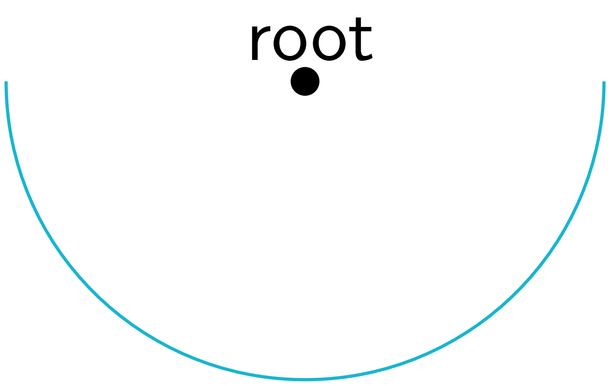

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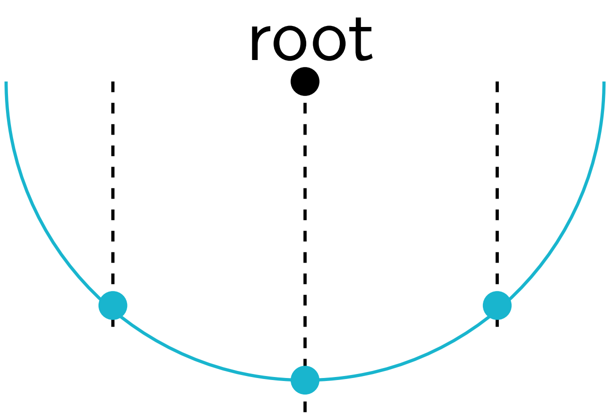

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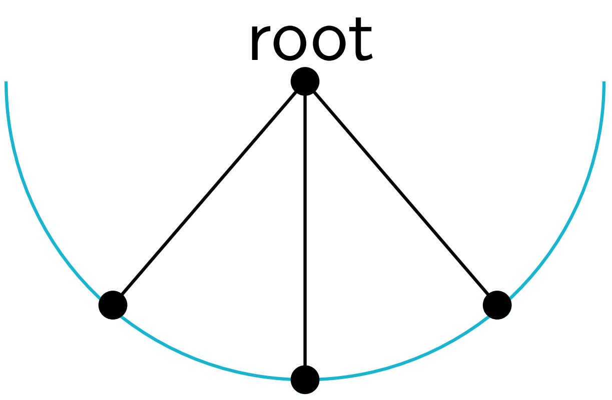

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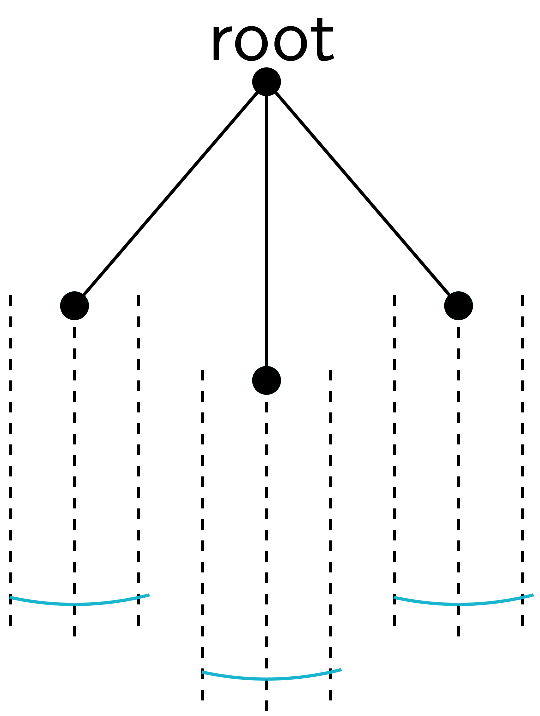

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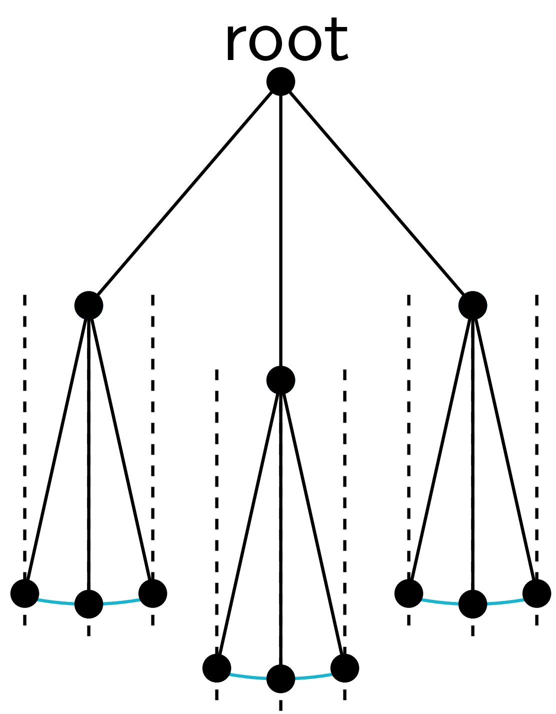

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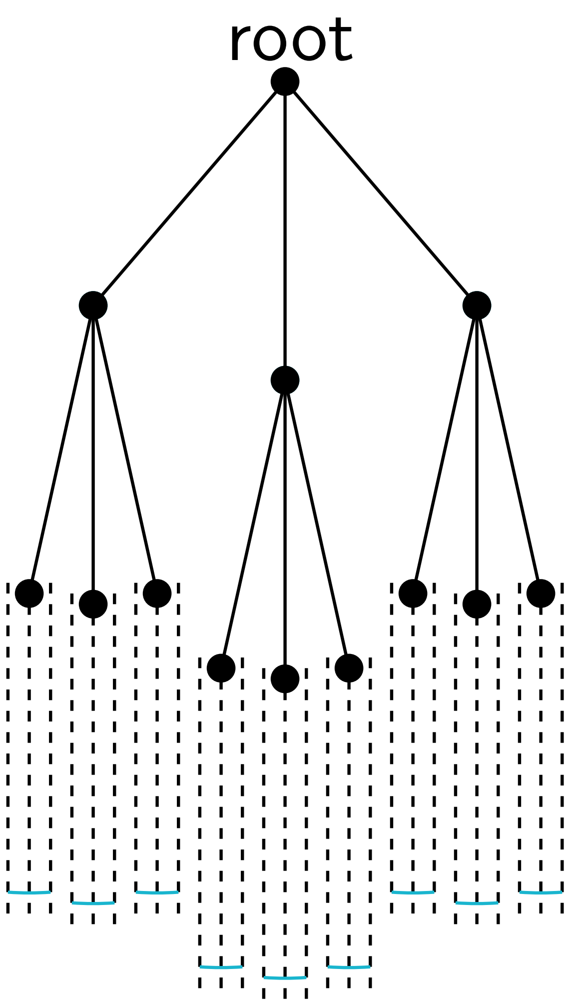

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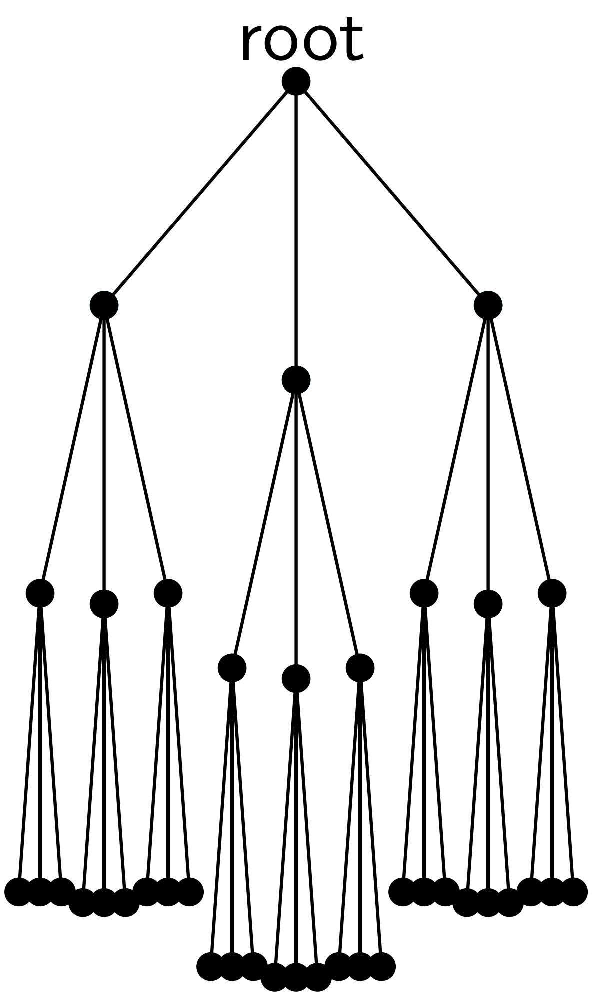

Results: Complete k-ary Trees

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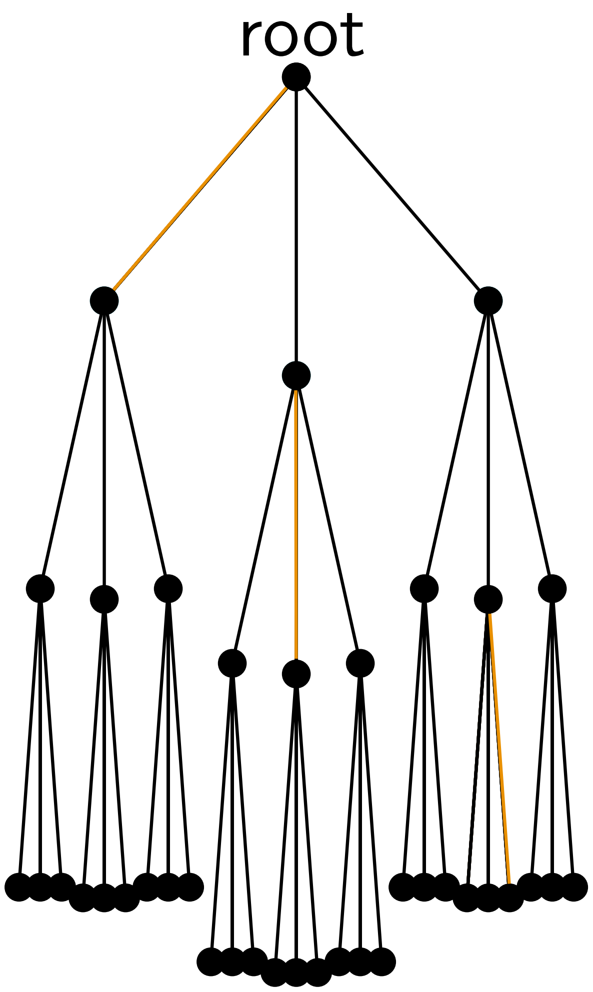

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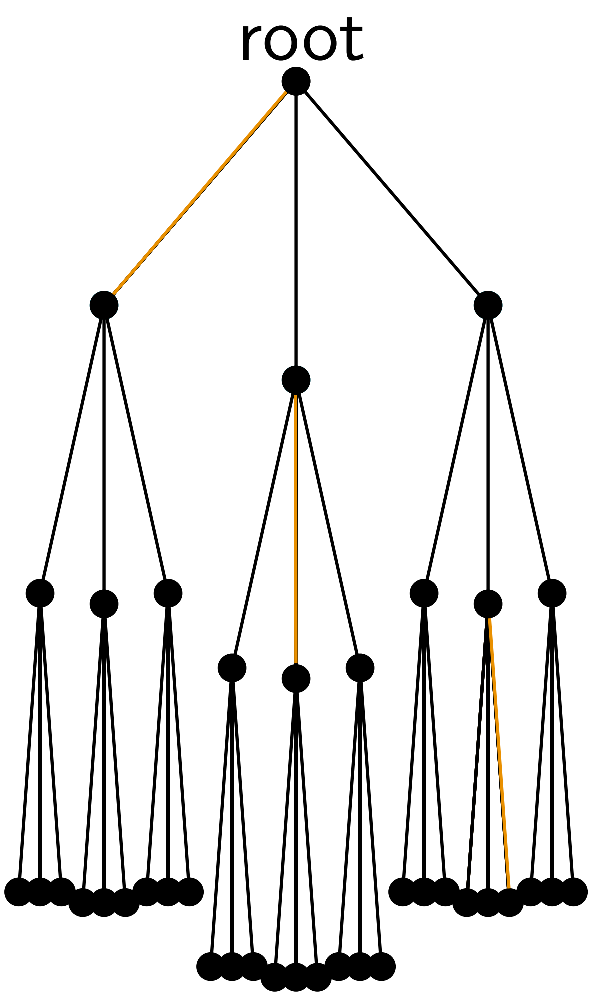
  - Ratio of drawings value $1 + \varepsilon$, $0 \leq \varepsilon < 1$



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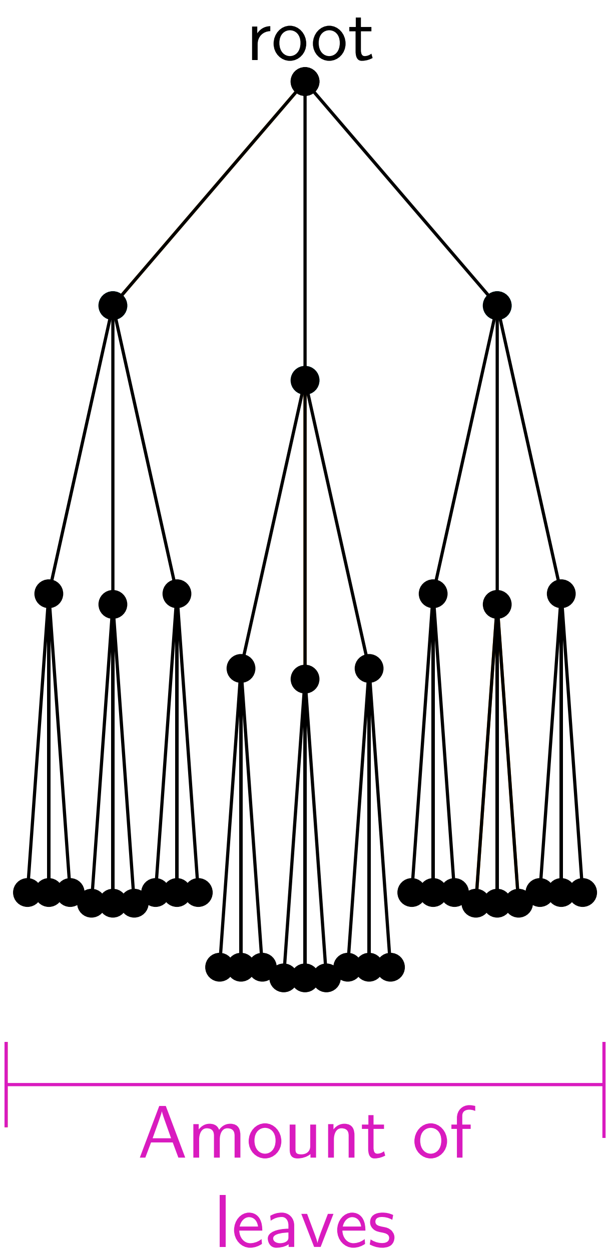

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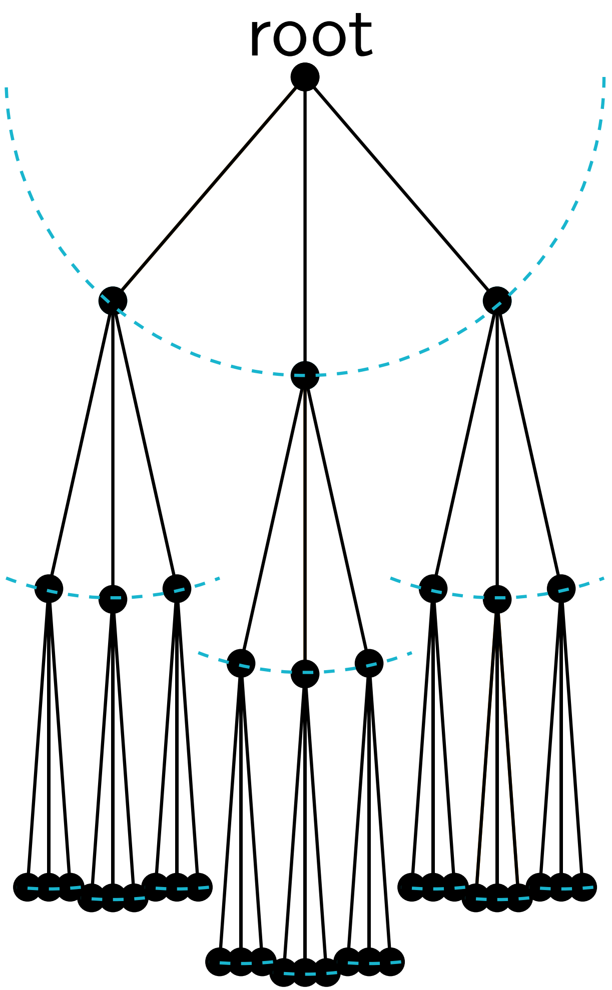

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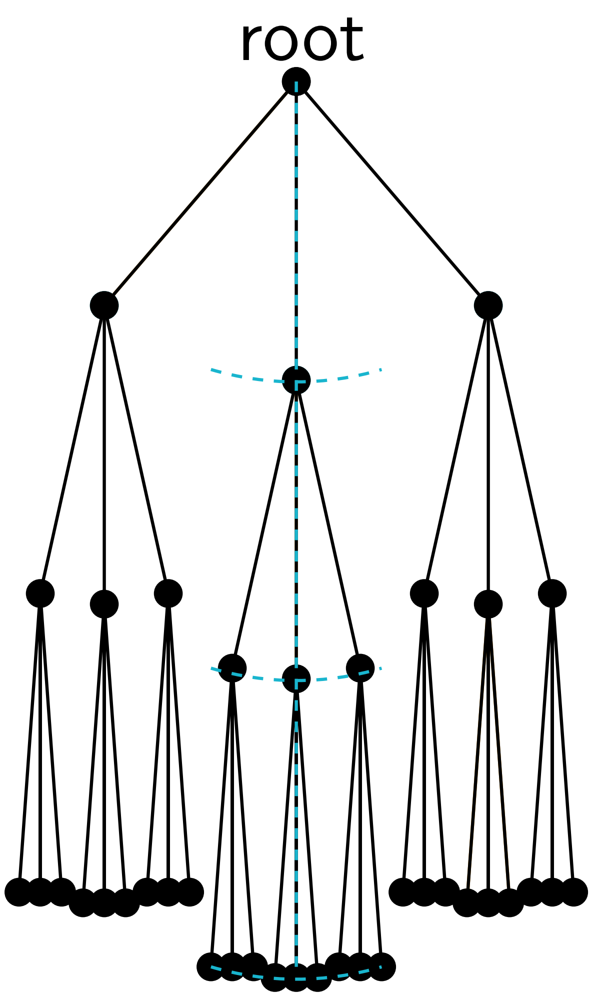

Results: Complete k-ary Trees

- Every straight-line is $\sim n$ long
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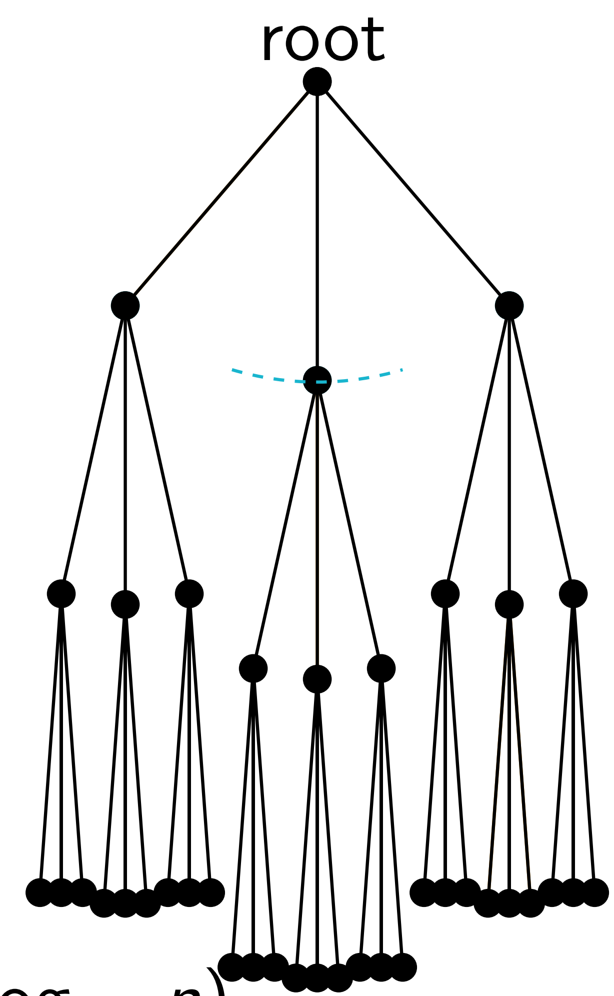

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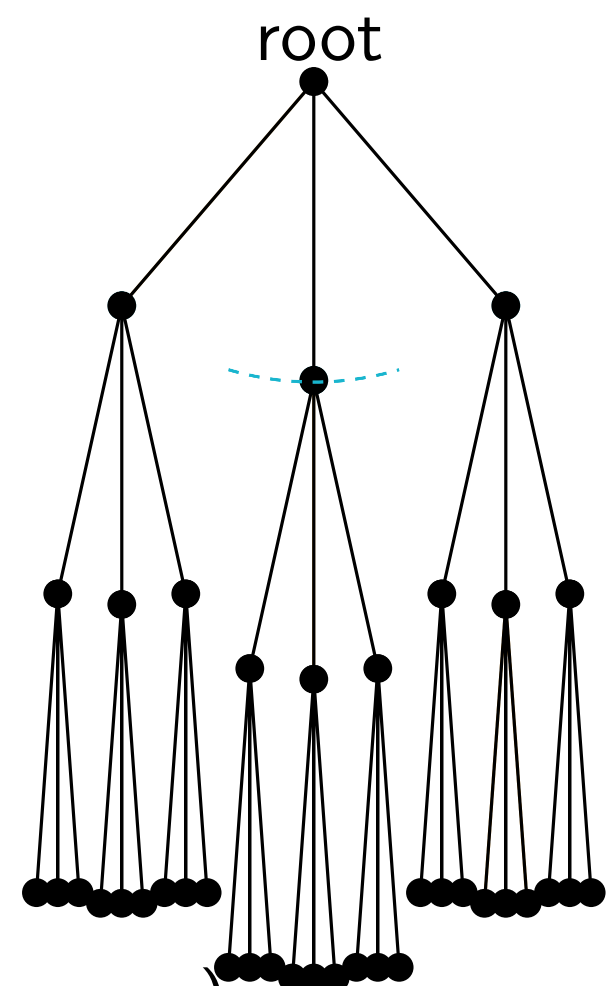

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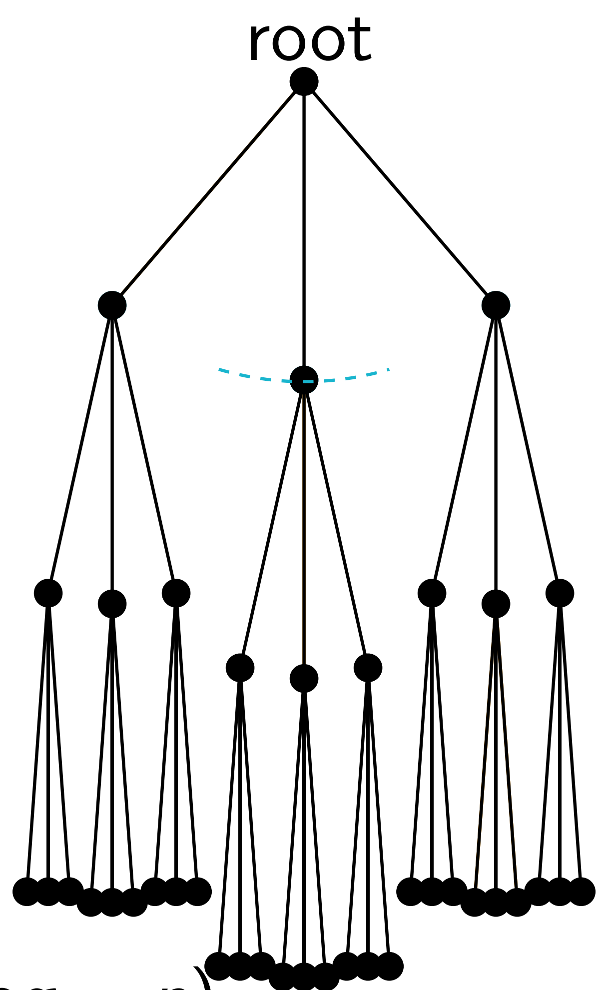

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Outerplanar Graphs and Series-Parallel Graphs

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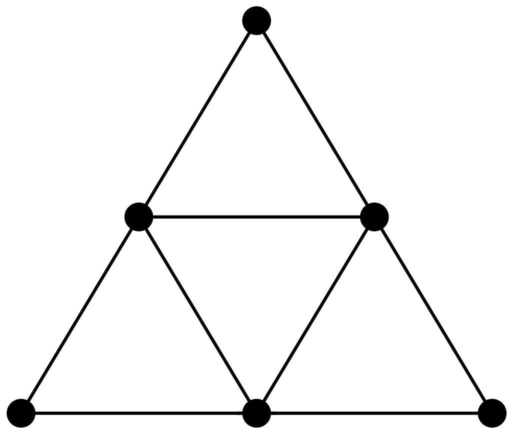

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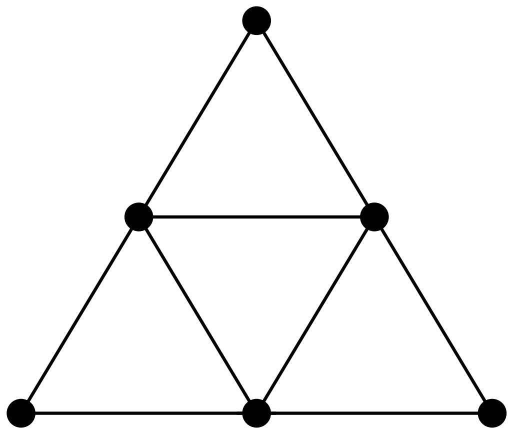

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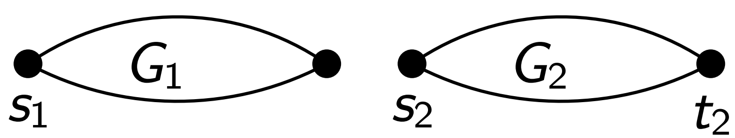

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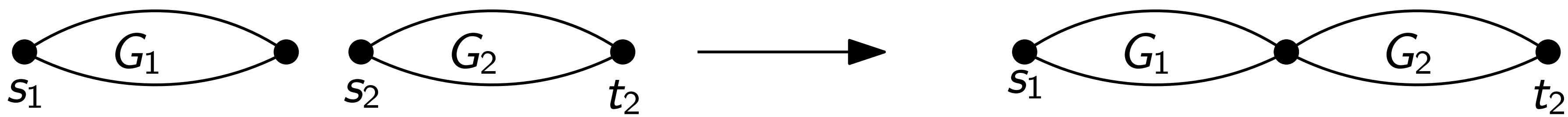

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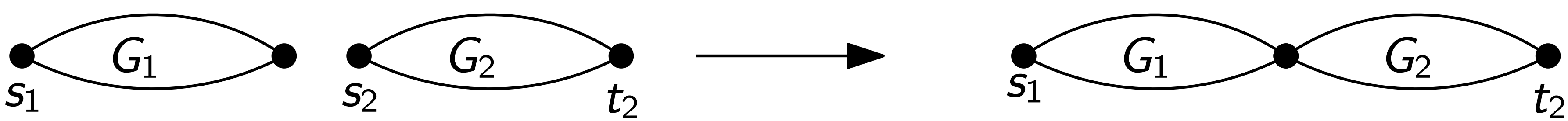

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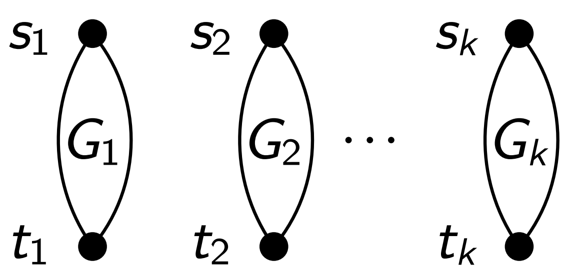

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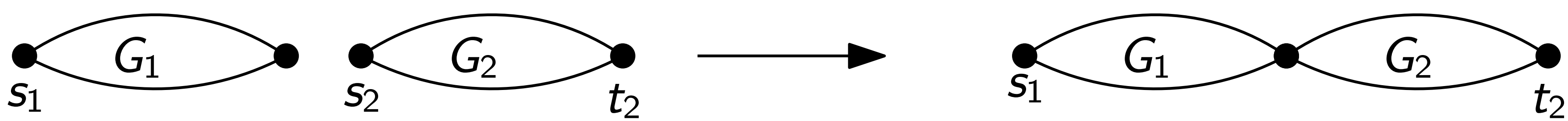

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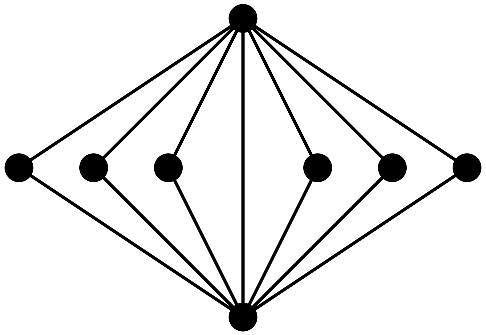

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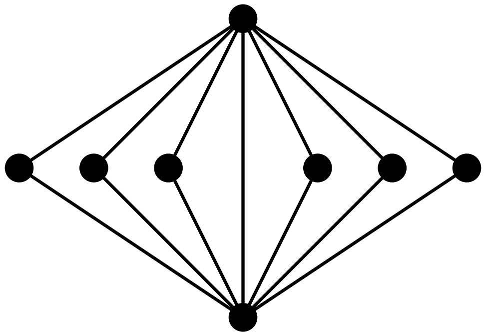

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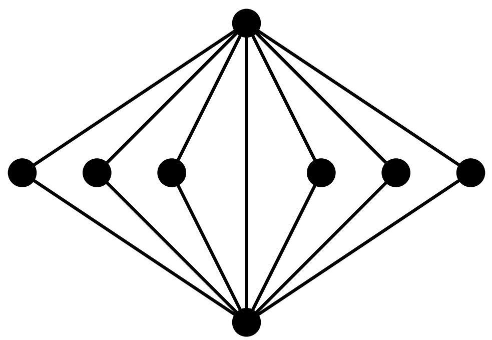

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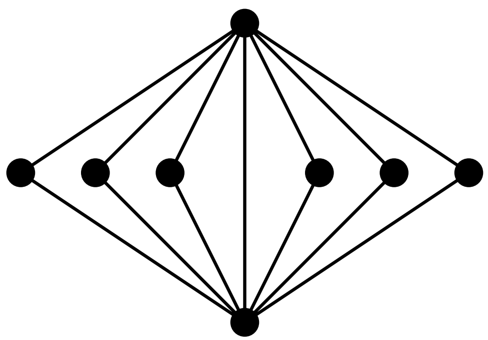
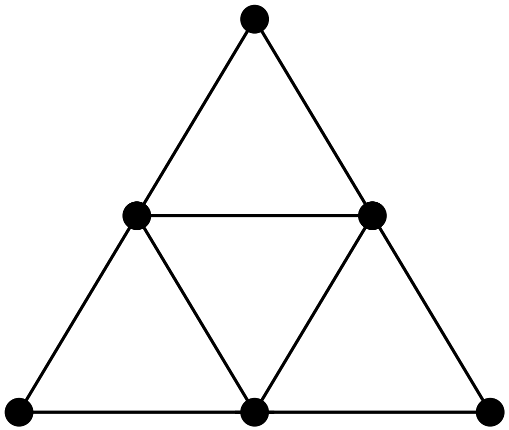


- a series-parallel graph is *maximal* if no edges can be inserted while maintaining a series-parallel graph
- maximal outerplanar graphs are maximal SP-graphs
  - Drawing approaches deal with outerplanar graphs first
  - Extended to SP-graphs, if possible

Approach I: Using Weak Dual Graph

- outerplanar graph *maximal* if no edges can be inserted without destroying outerplanarity



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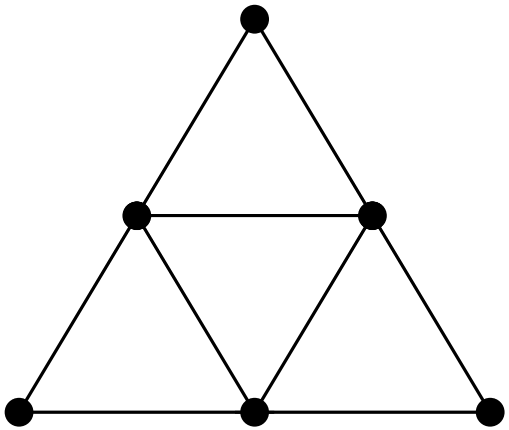

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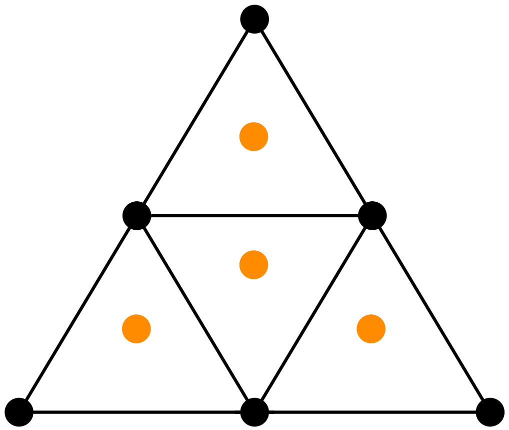

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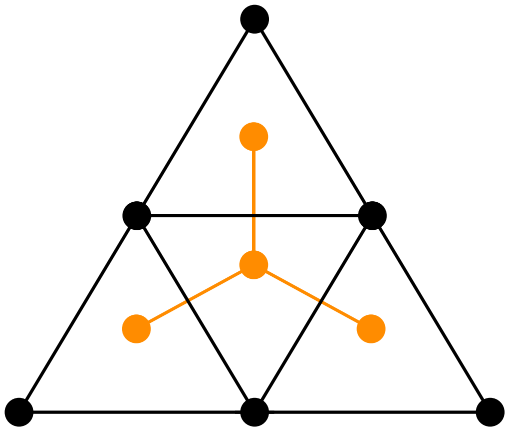

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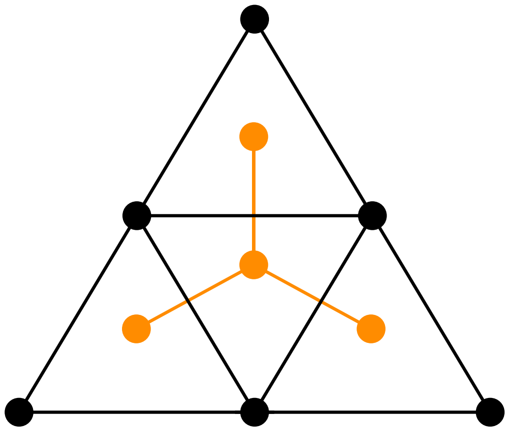

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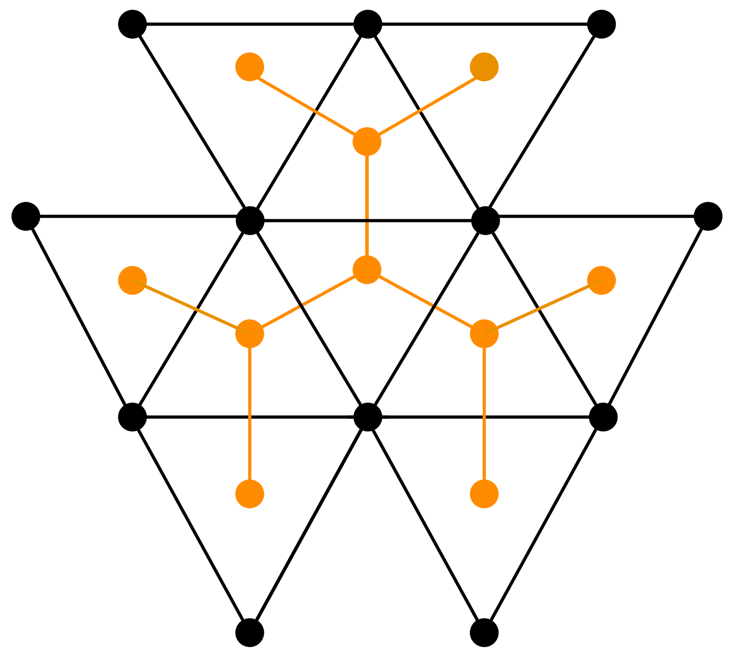

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- Degree of G^* is at most 3

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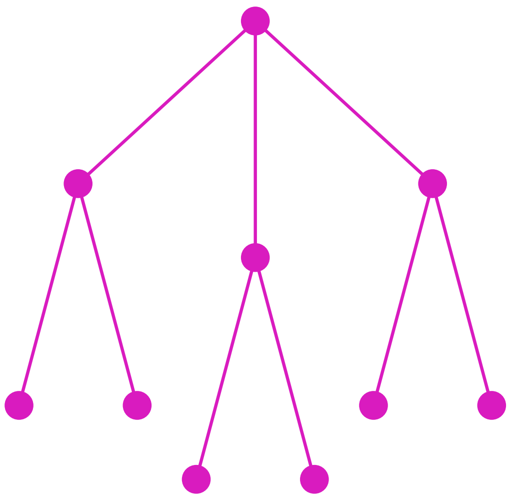

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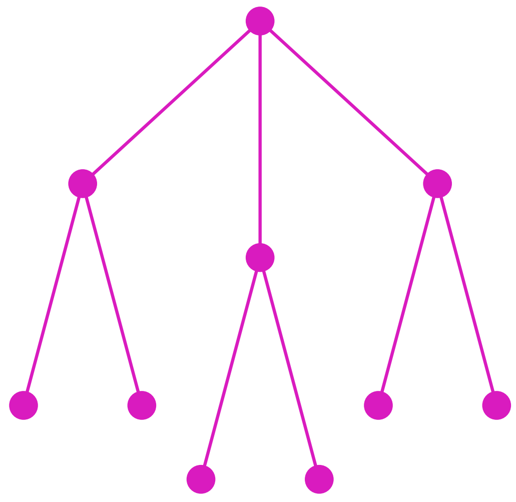

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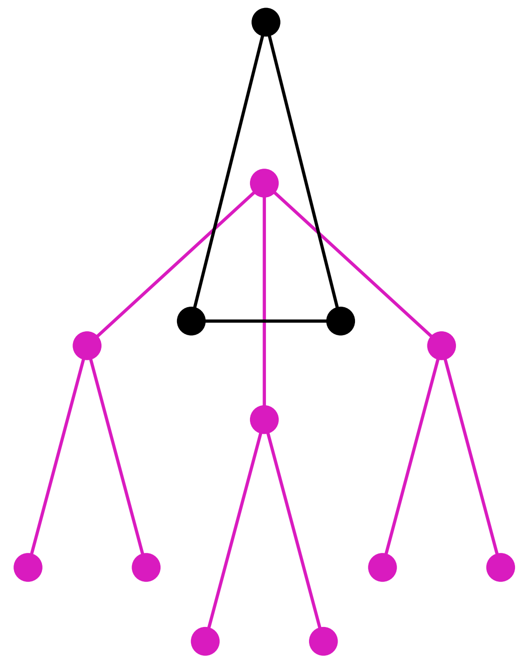

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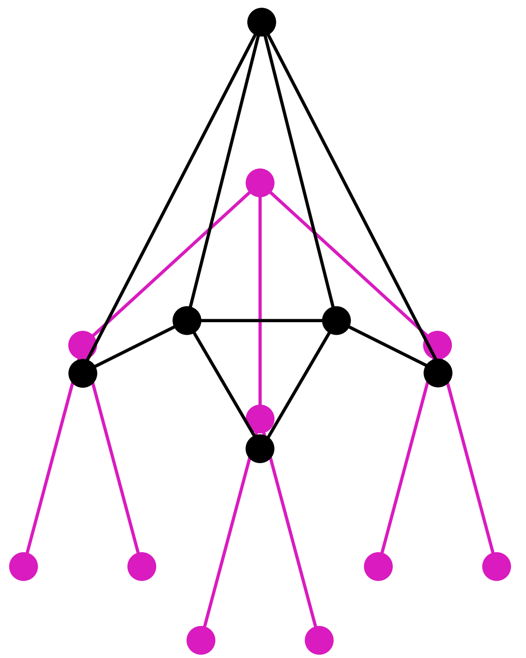

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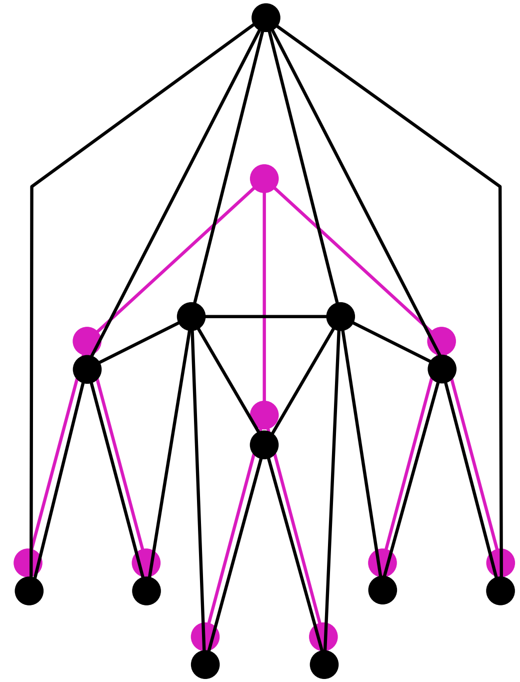

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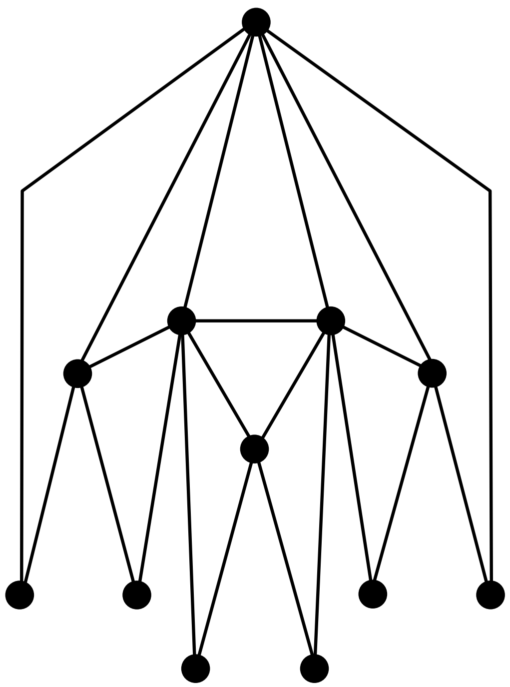

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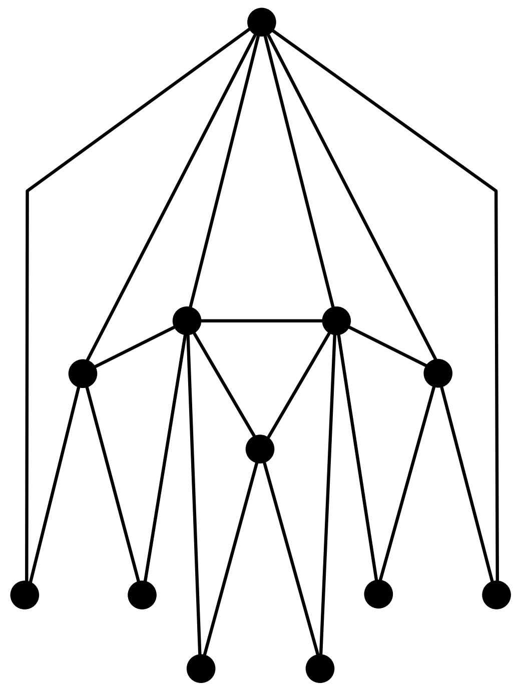

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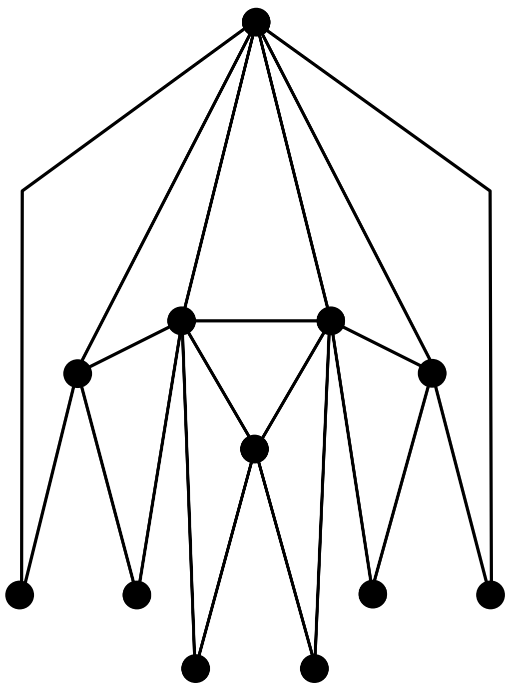

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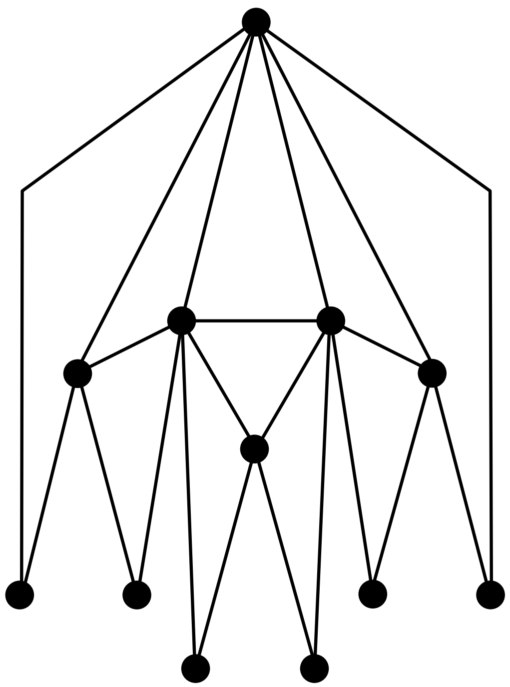

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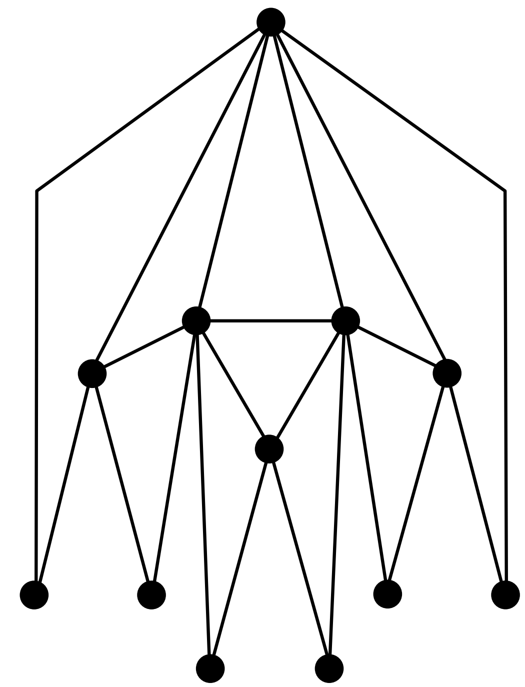

- If height of G^* in $\mathcal{O}(\log n)$
  - Radius in $\Theta(n)$, drawing height in $\mathcal{O}(n \log n)$
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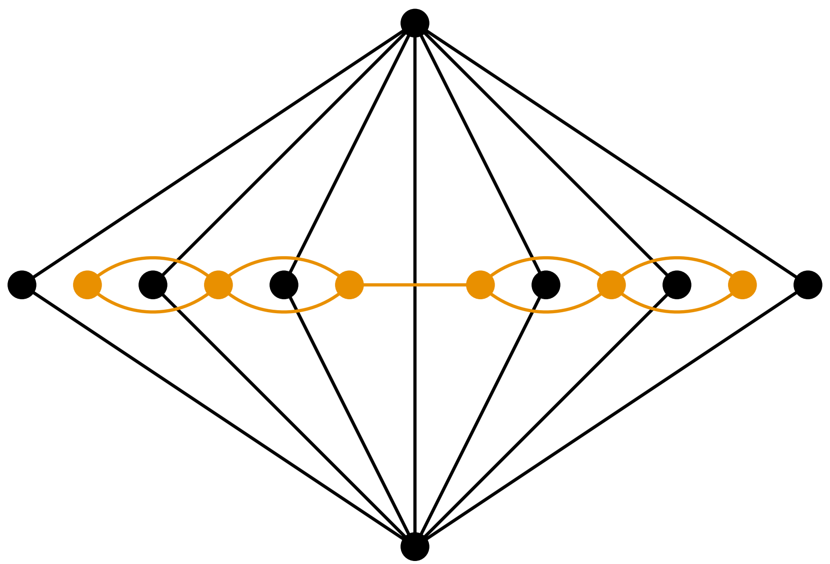

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- *treewidth* is least width of any tree decomposition of G

Tree Decomposition Of SP-Graphs

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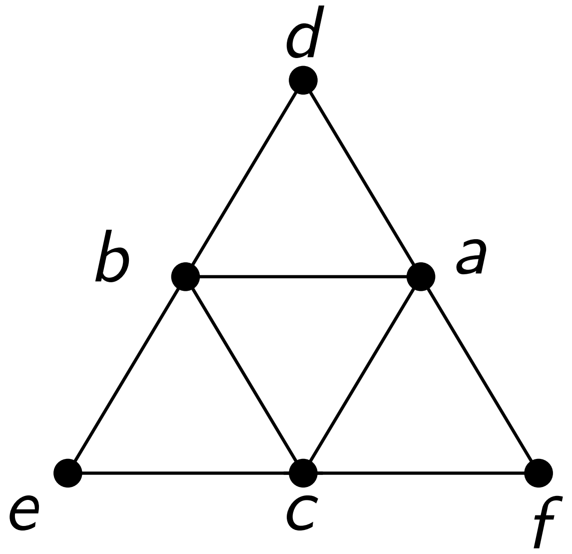

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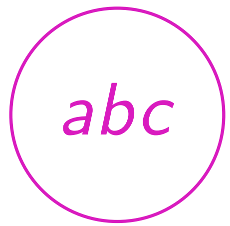

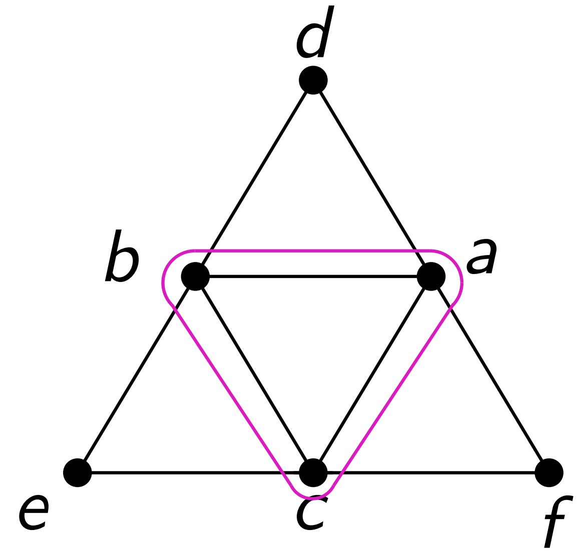

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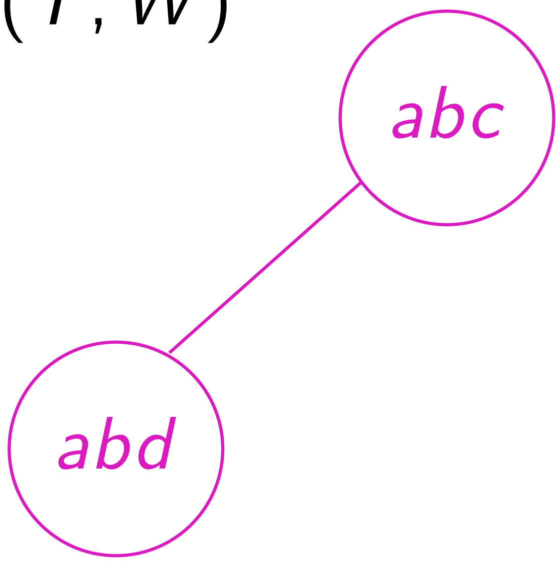

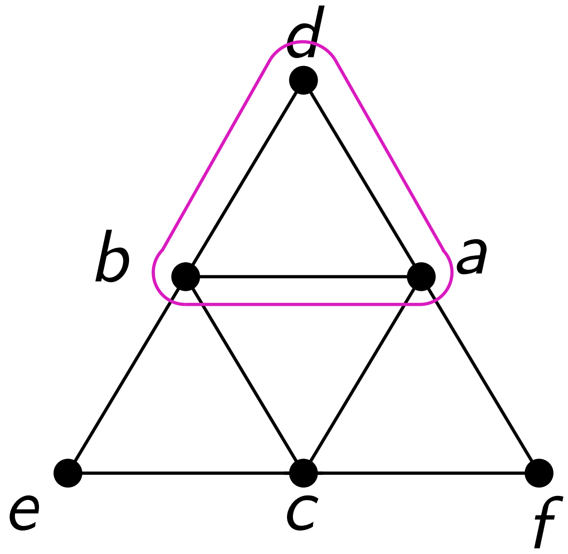

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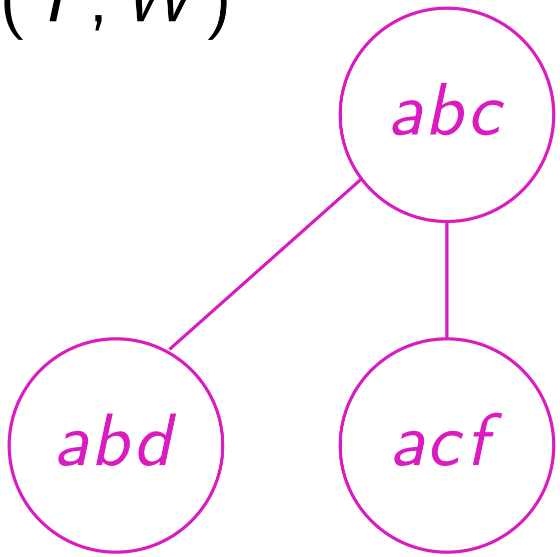

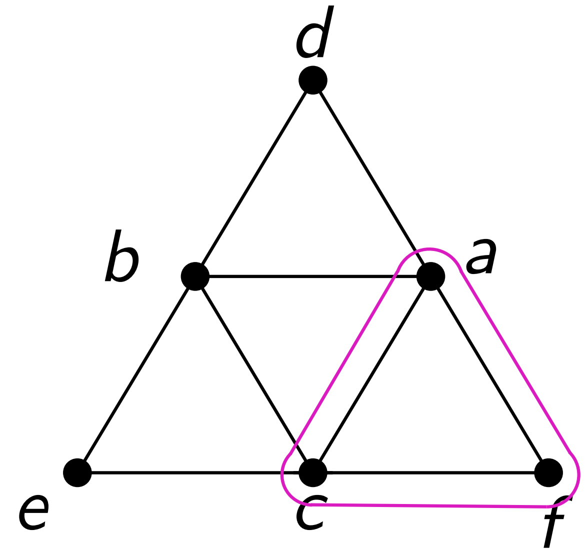

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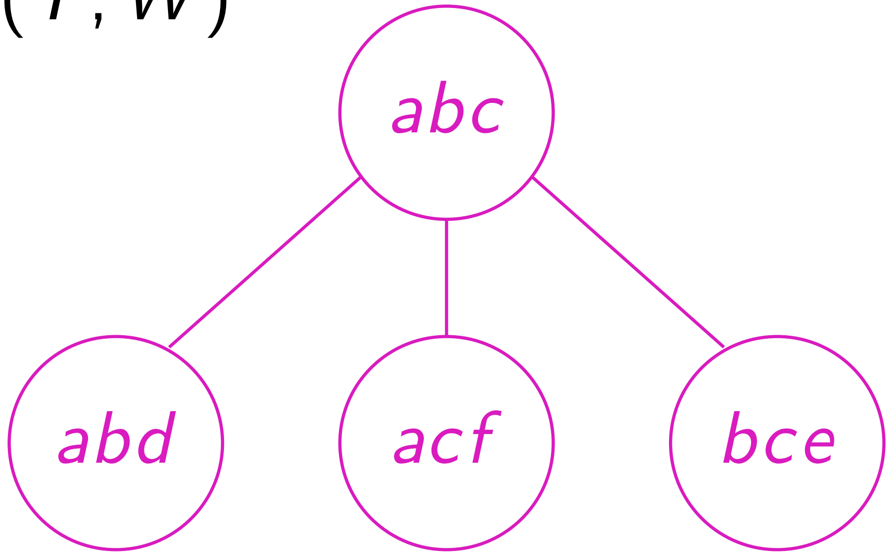

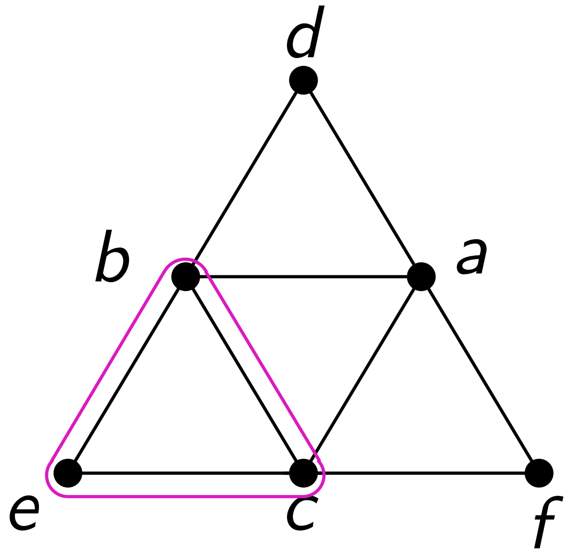

Tree Decomposition Of SP-Graphs

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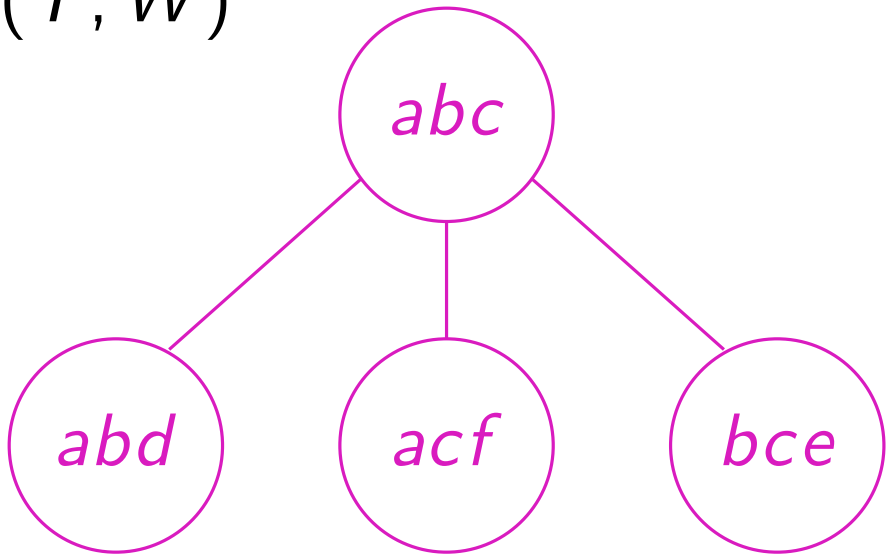

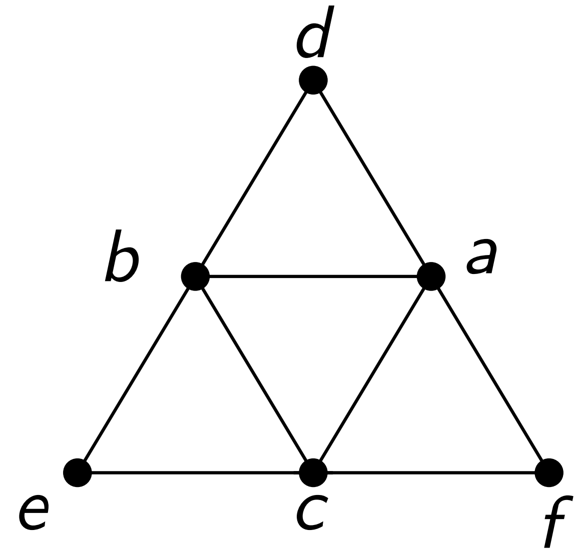

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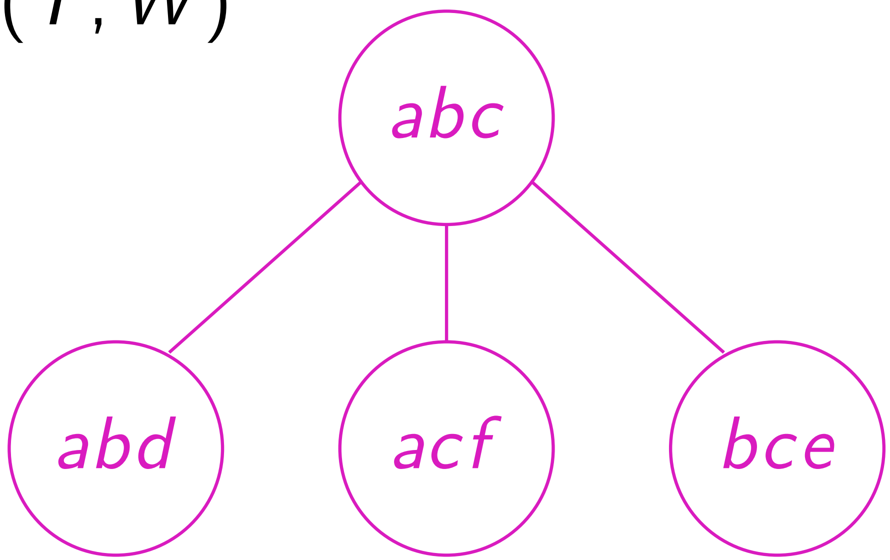

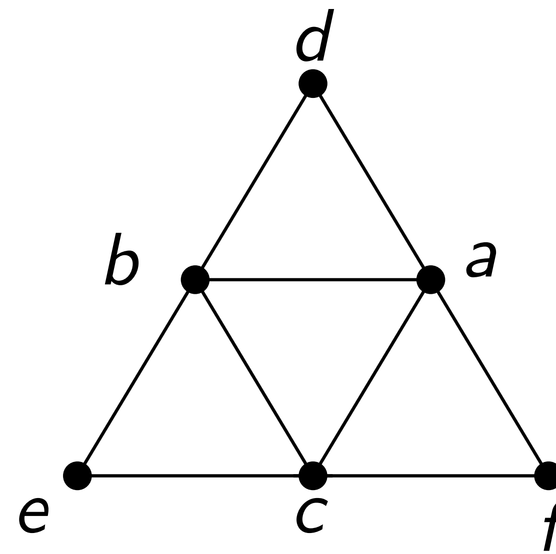

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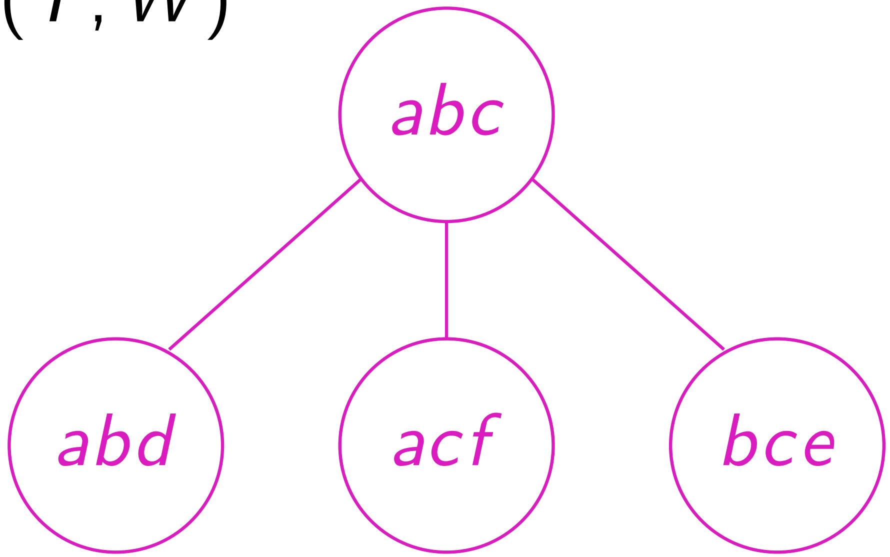

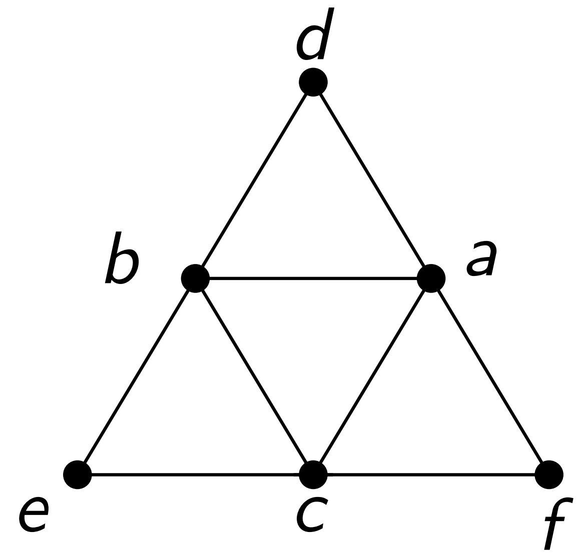

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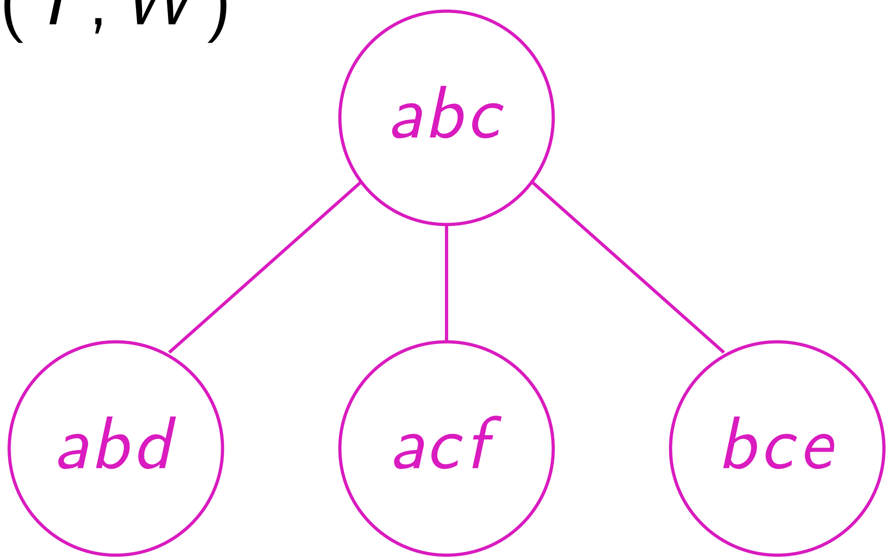

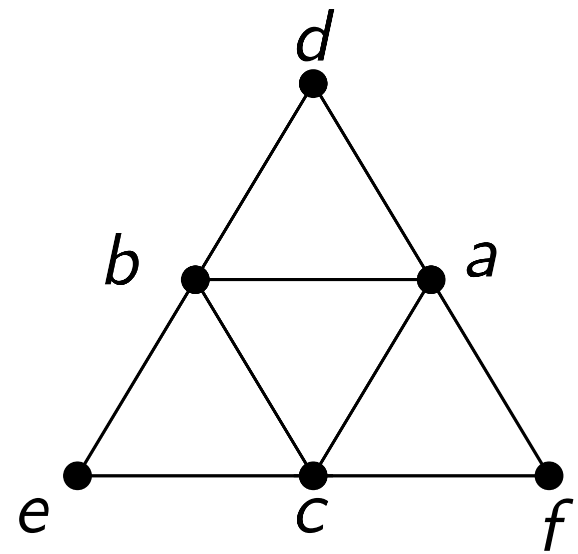

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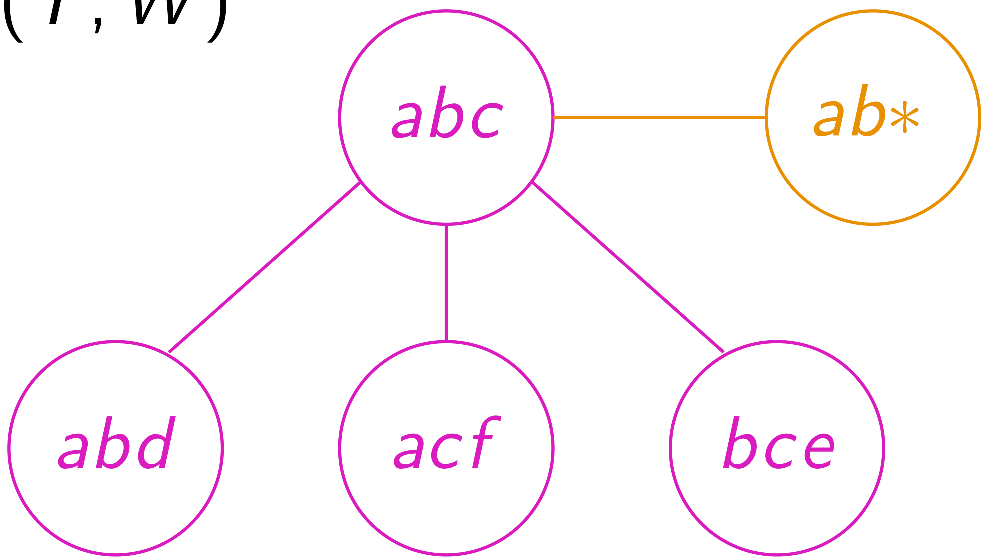

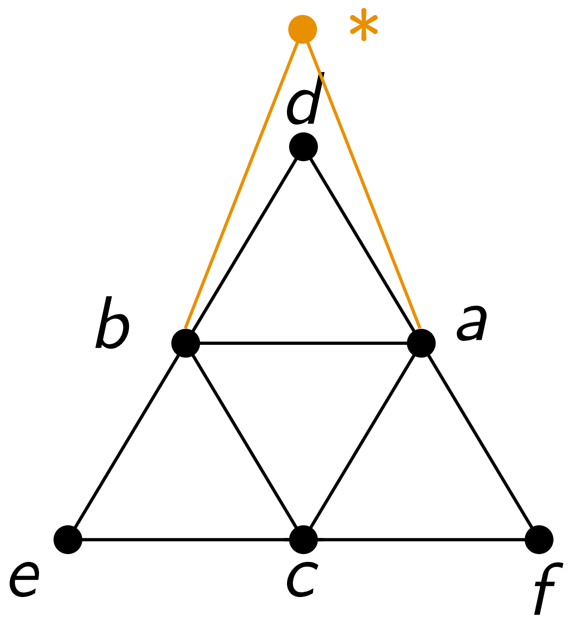

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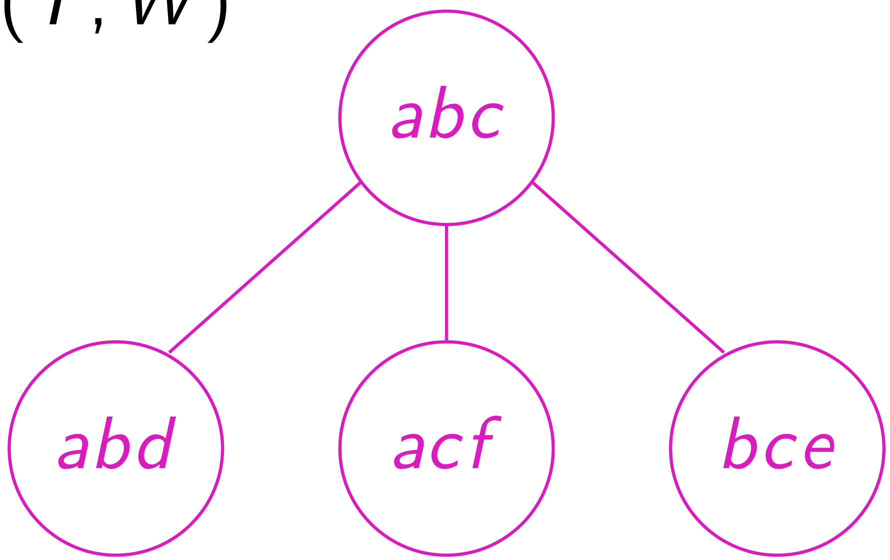

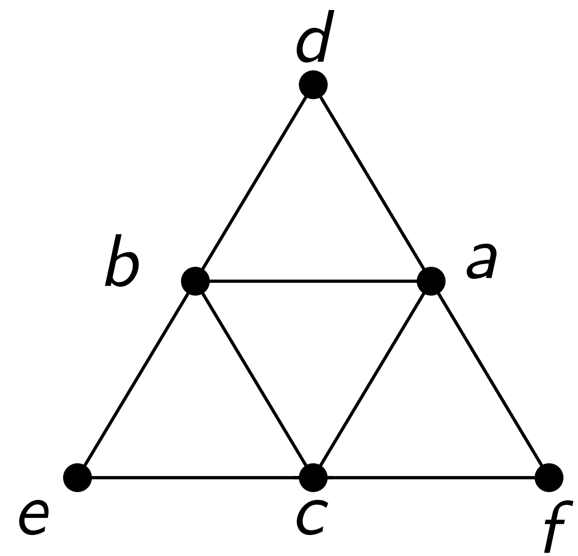

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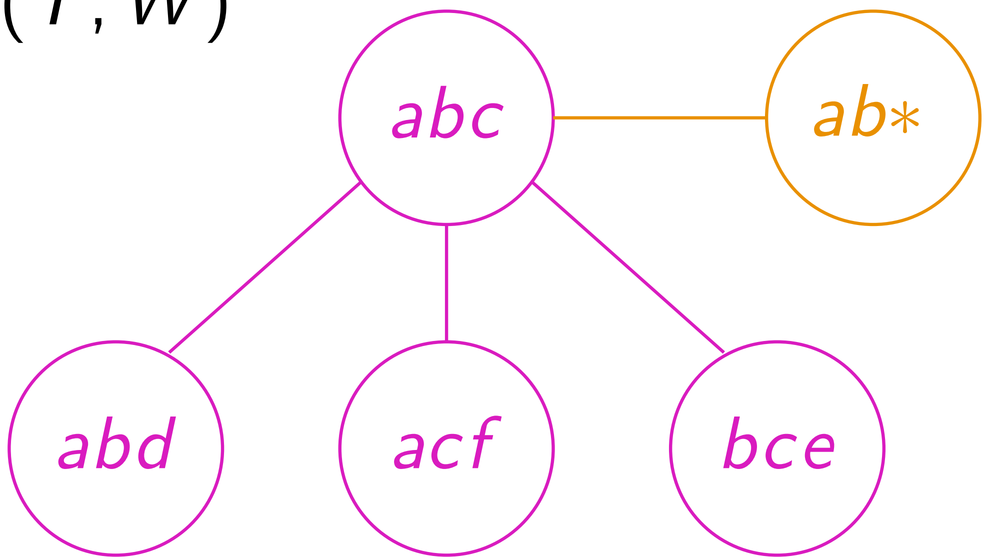

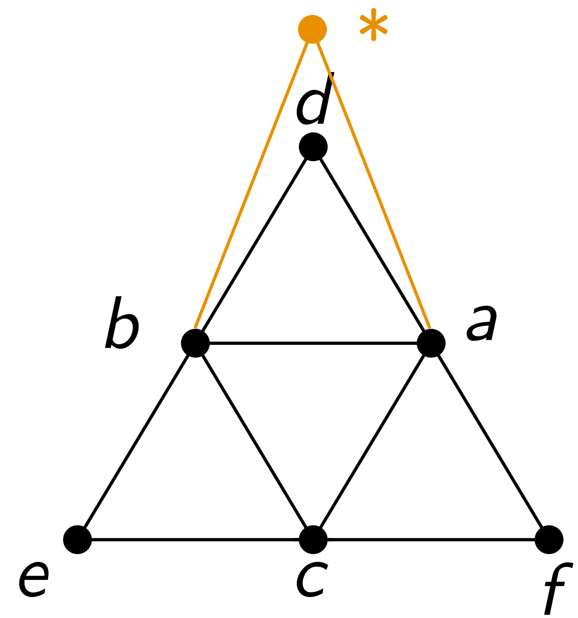

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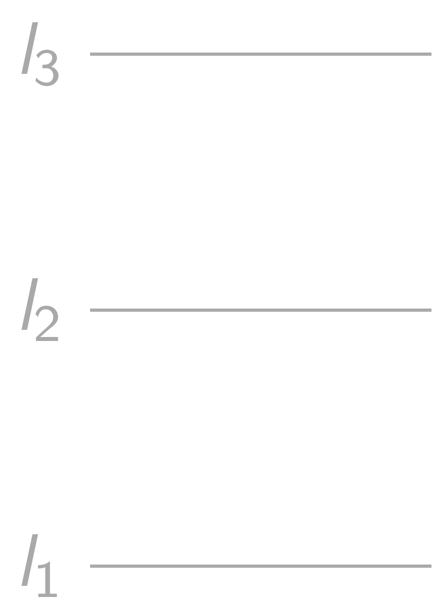

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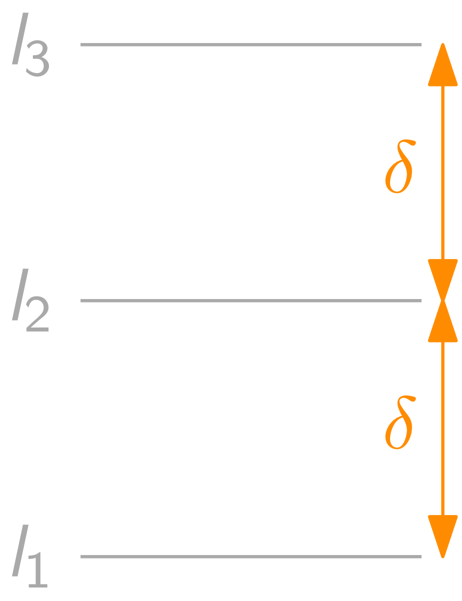

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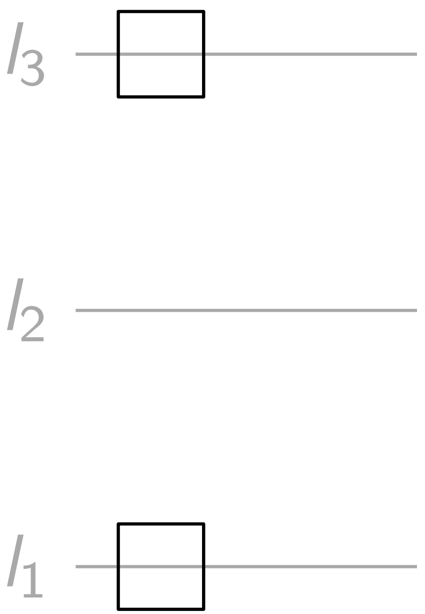

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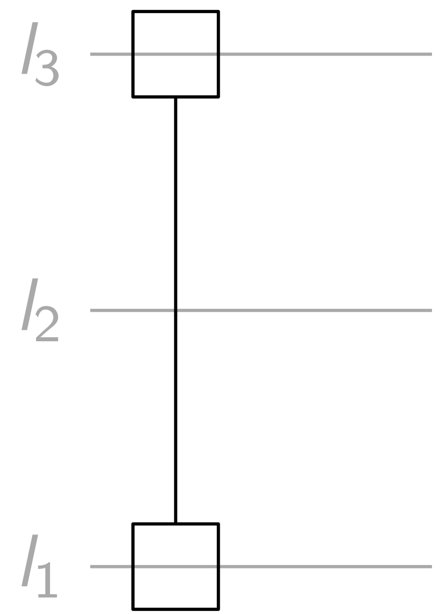

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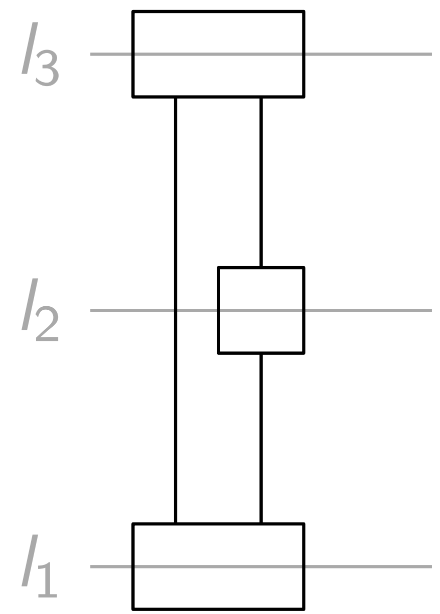

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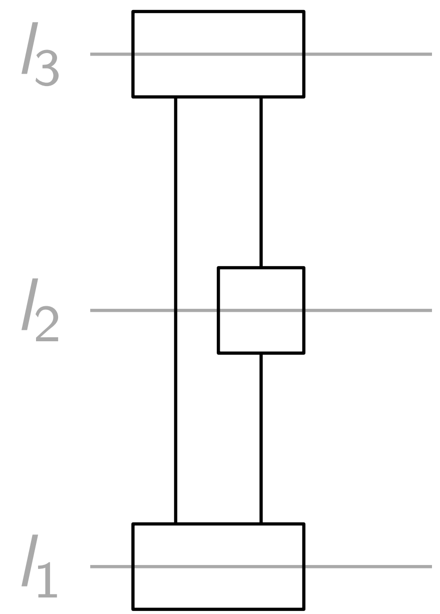

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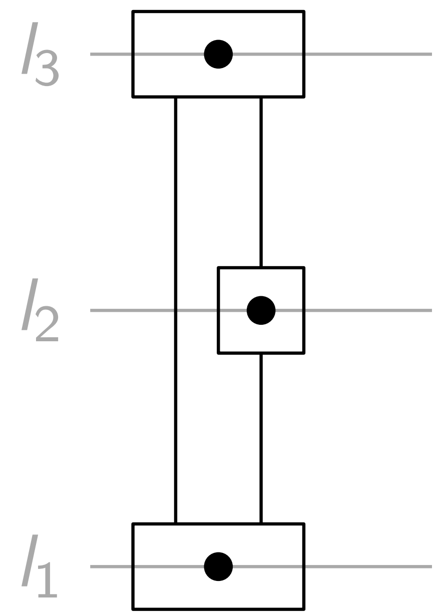

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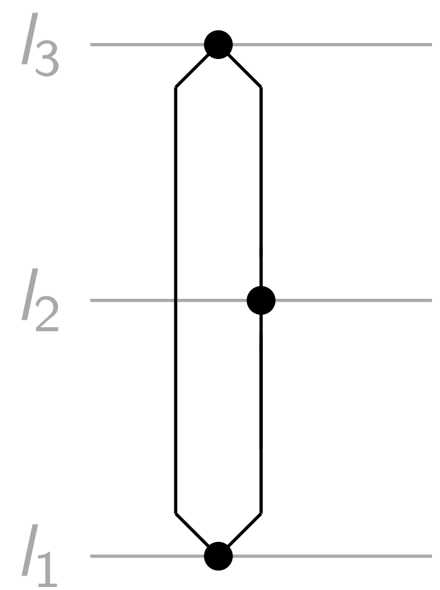

- Transfer B into a polyline drawing, remove edges

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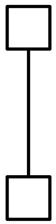

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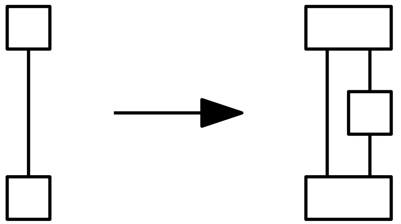

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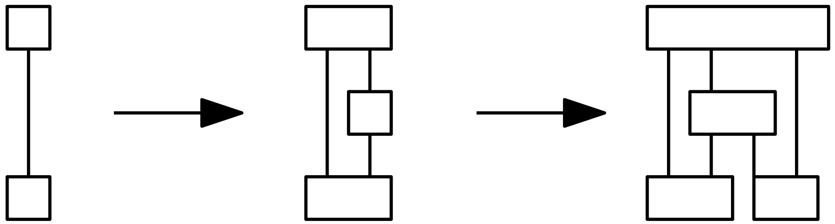

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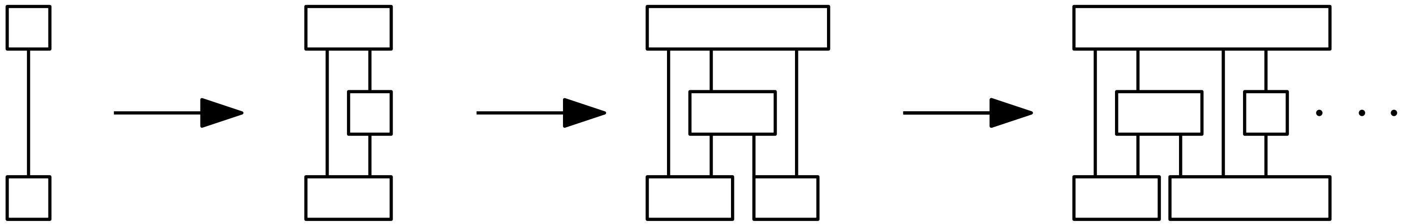

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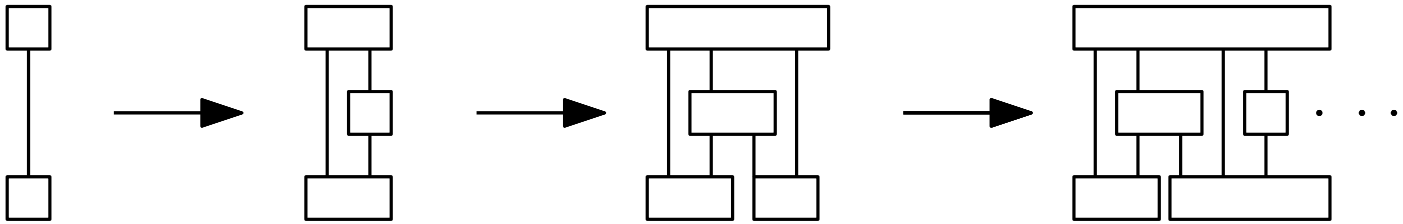

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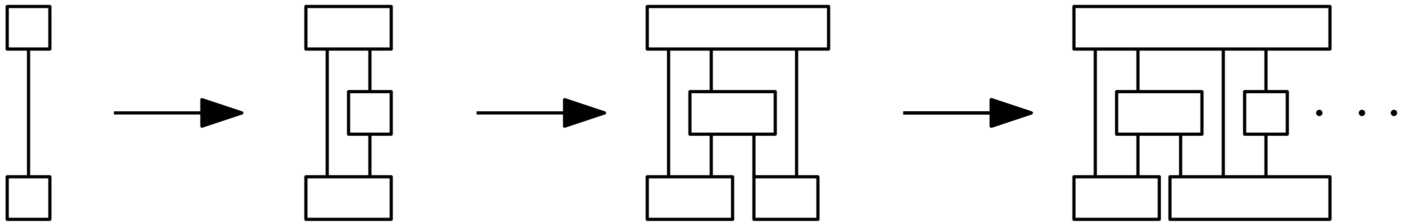

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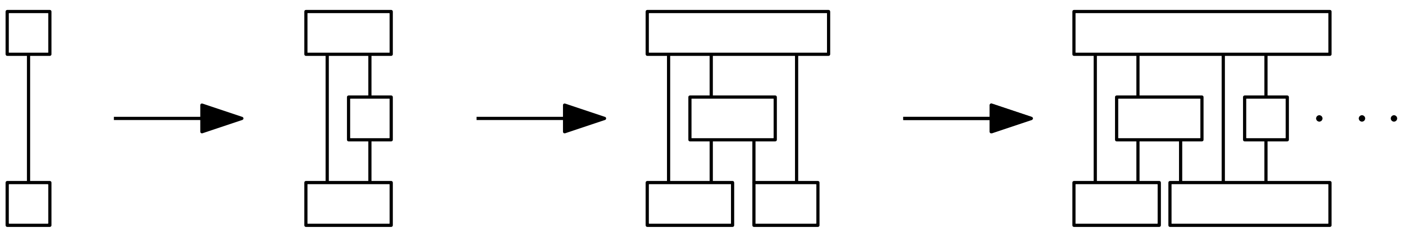

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- A layer is *reachable* if it is free, v can be placed on it and edges to a, b can be drawn without destroying planarity

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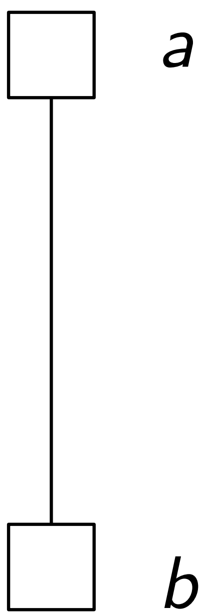

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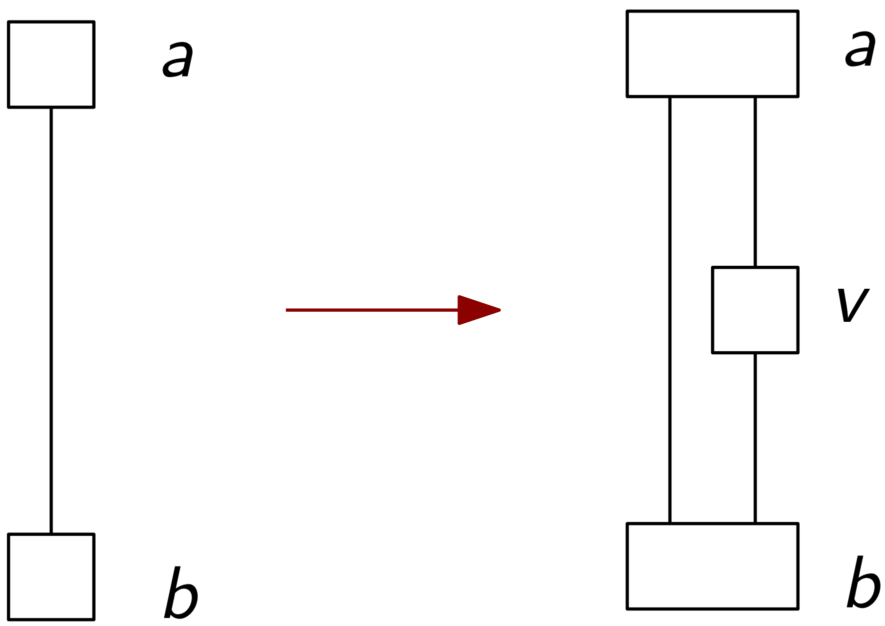

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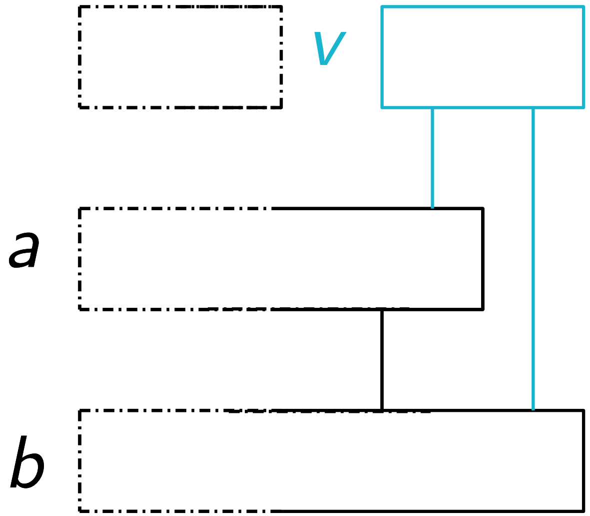

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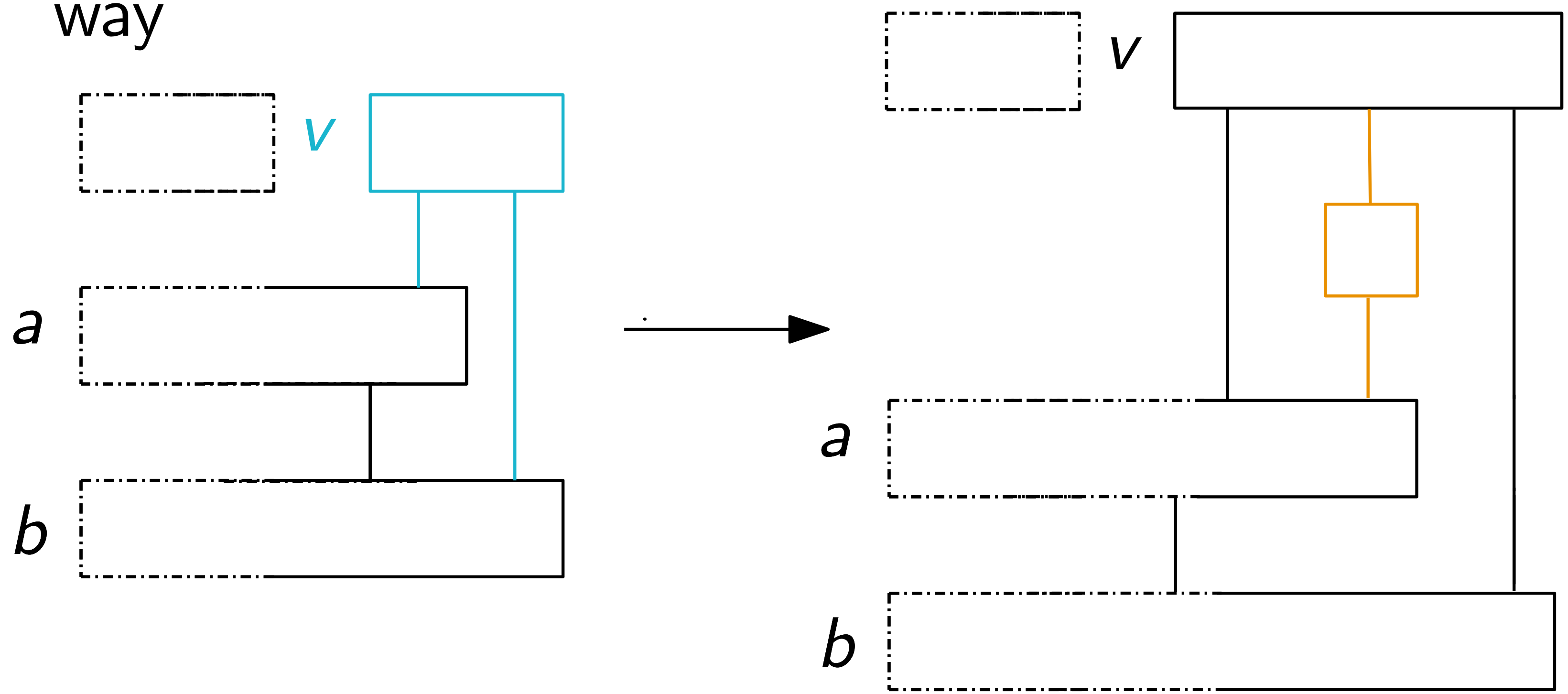

  - Subsequent vertex insertions between v and a
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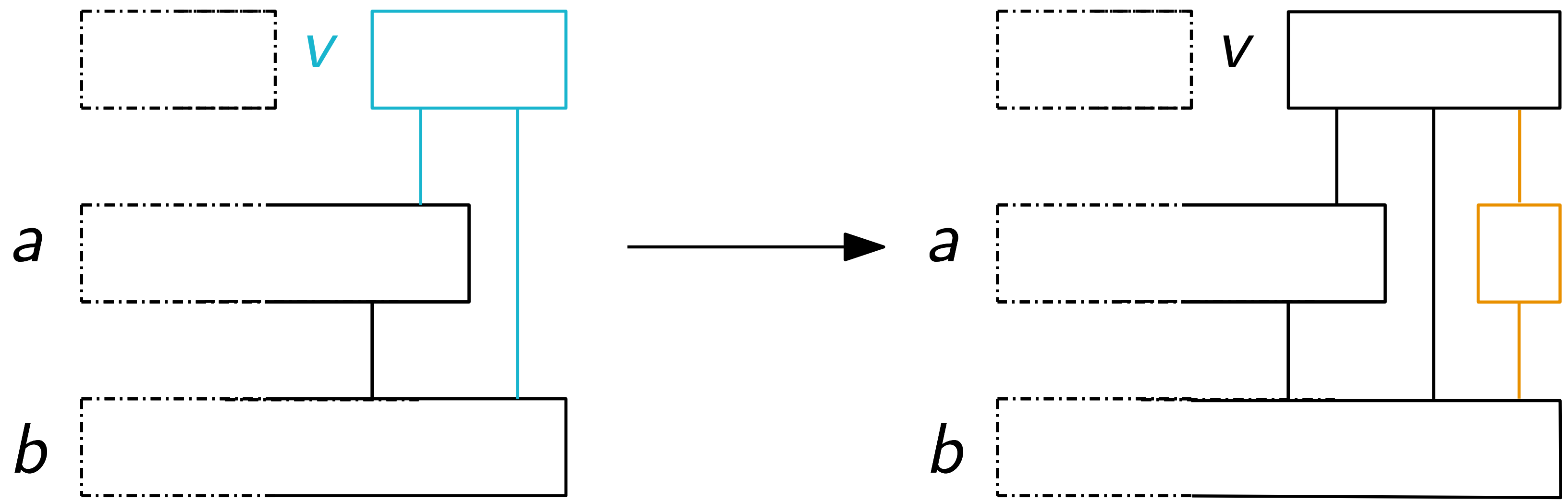

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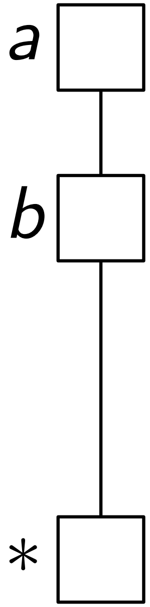

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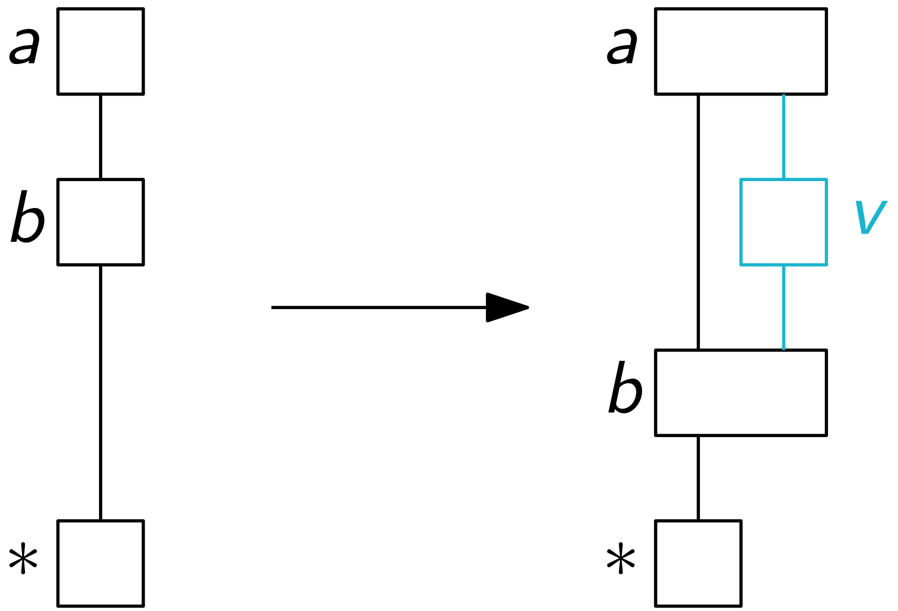

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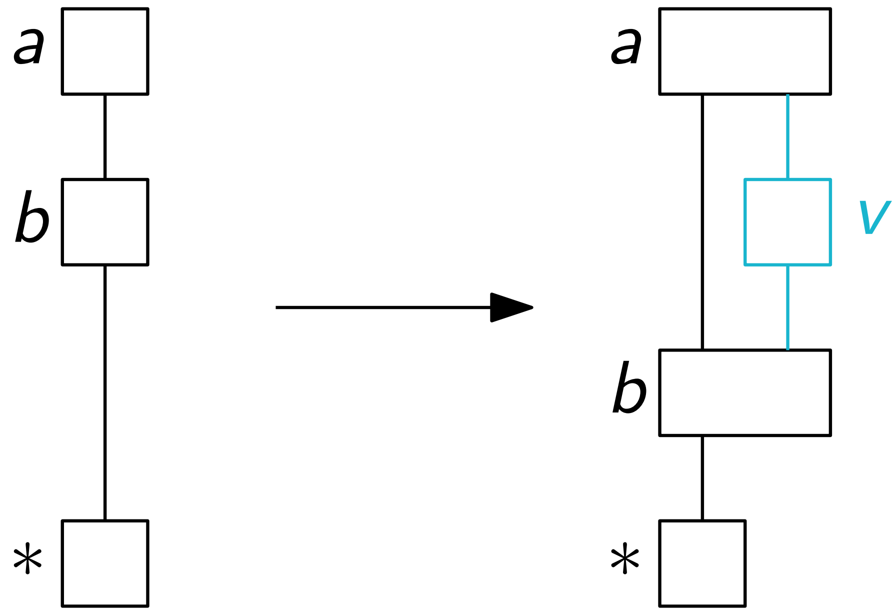

- Then, either case 1 or case 2 will apply
- If neither of those cases possible, insert a new layer between a and b

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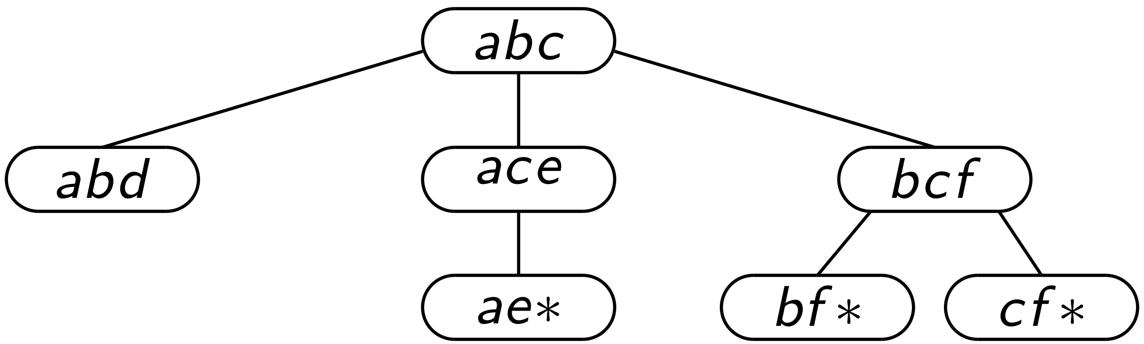

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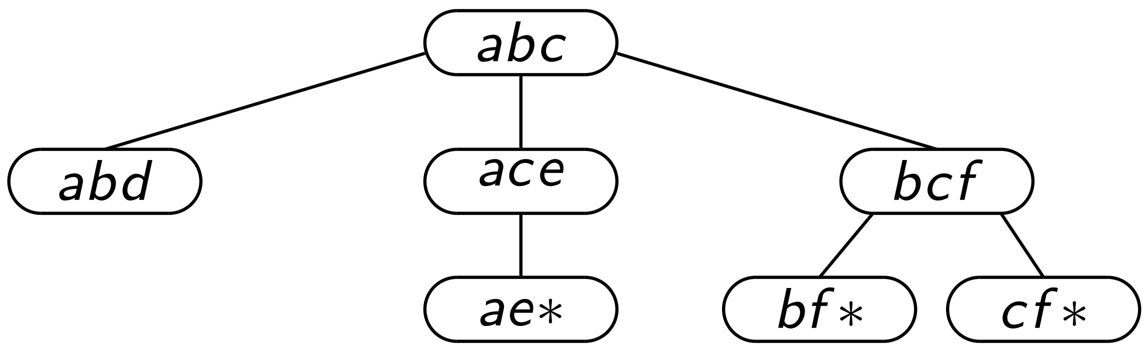

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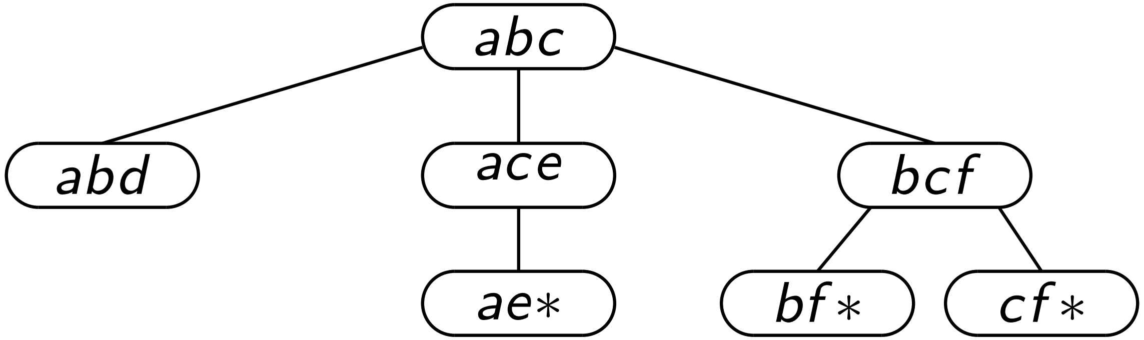

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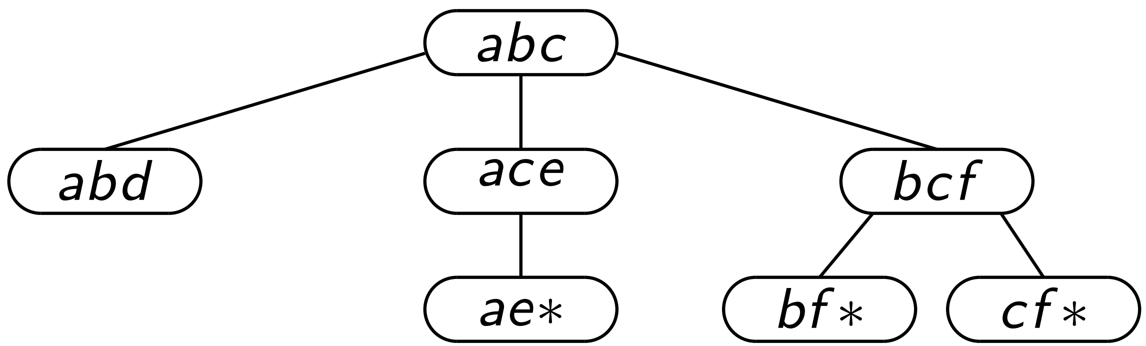

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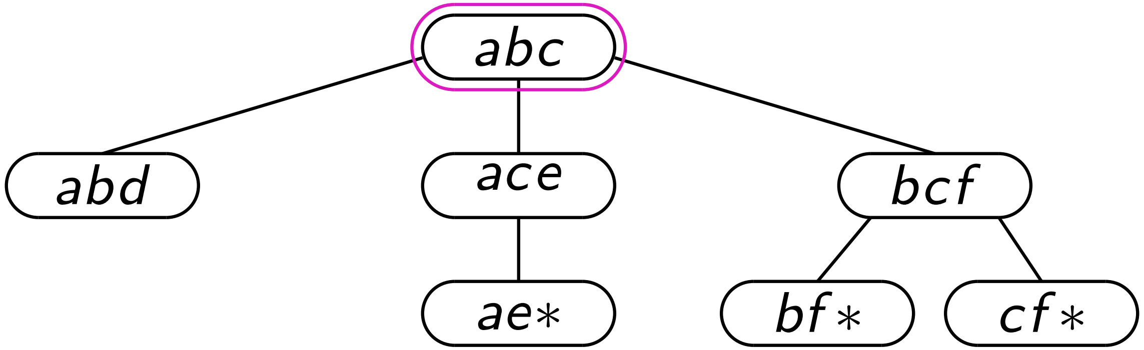

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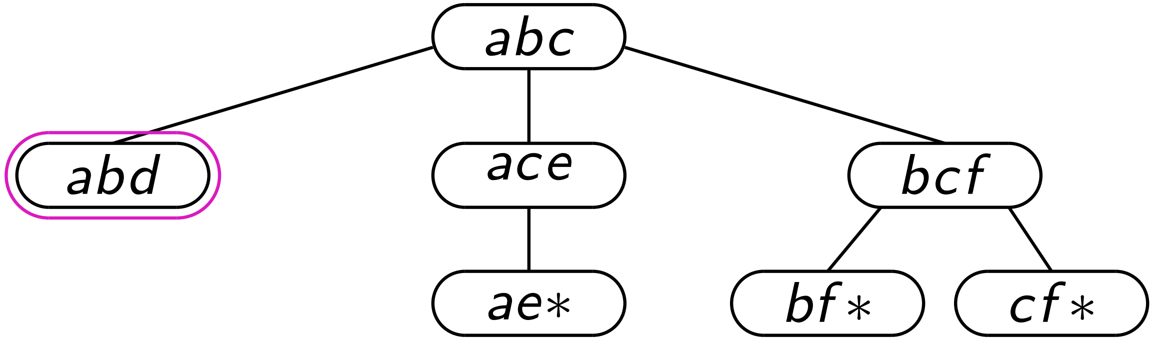

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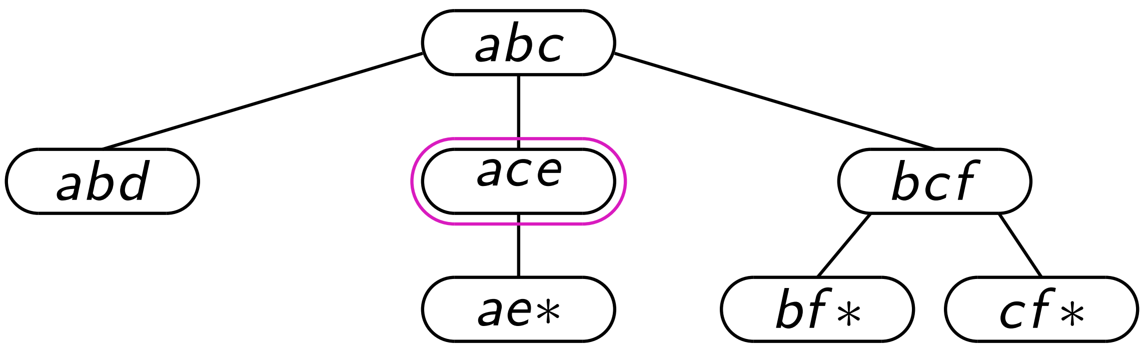

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Traversal Of Tree Decomposition

- Traversal of T determines the order of vertices inserted
- subtree priority + DFS
  - Siblings of T share exactly one vertex in their bags
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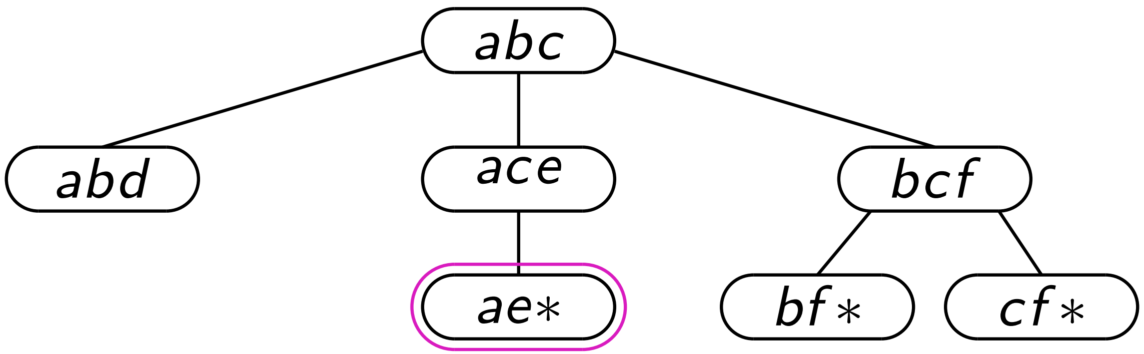

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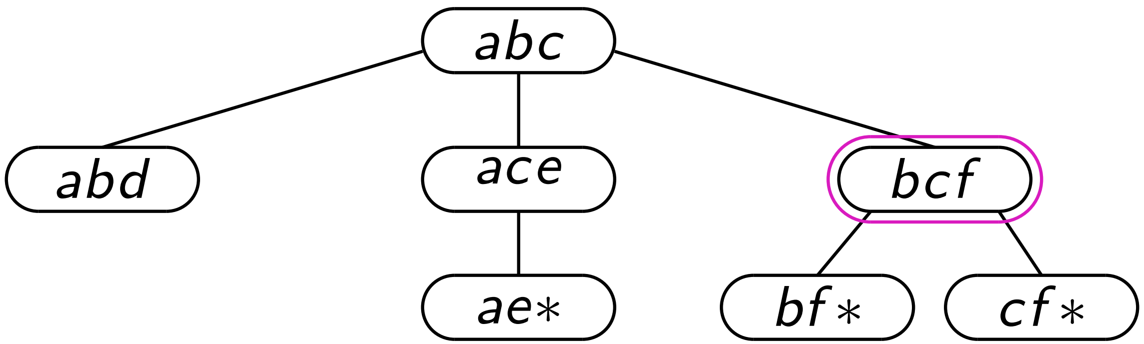

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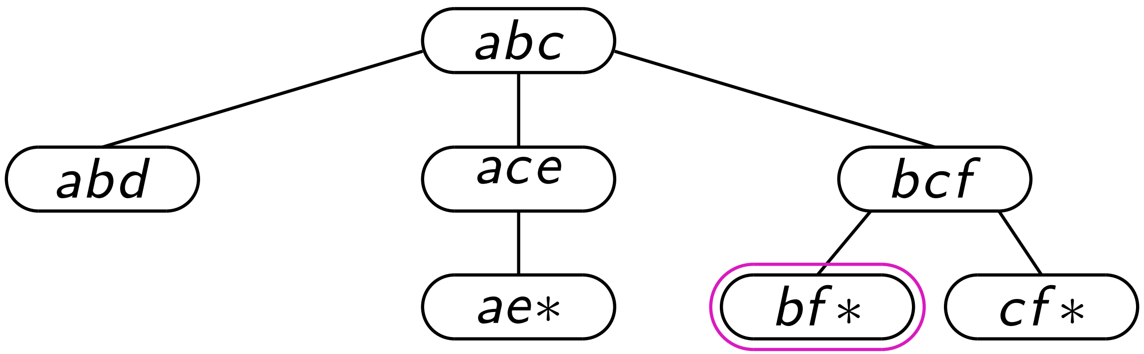

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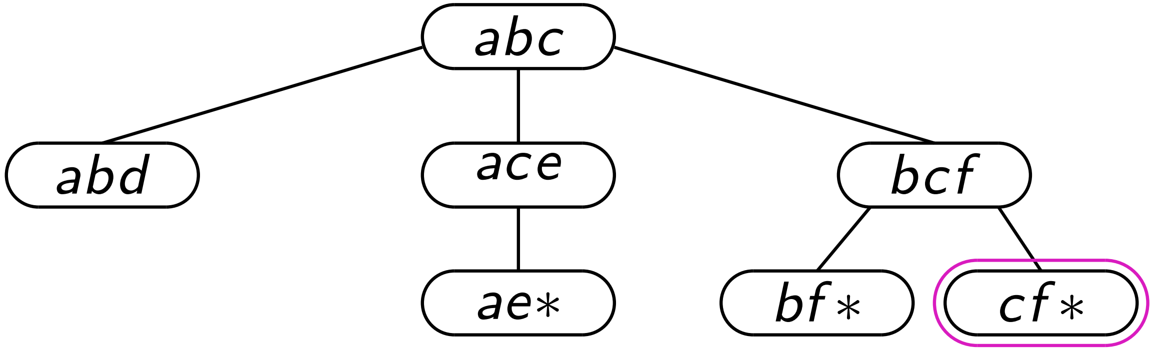

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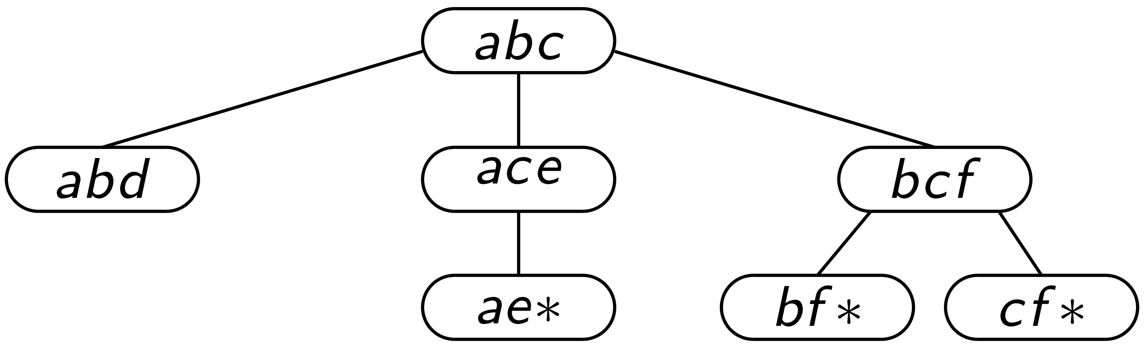

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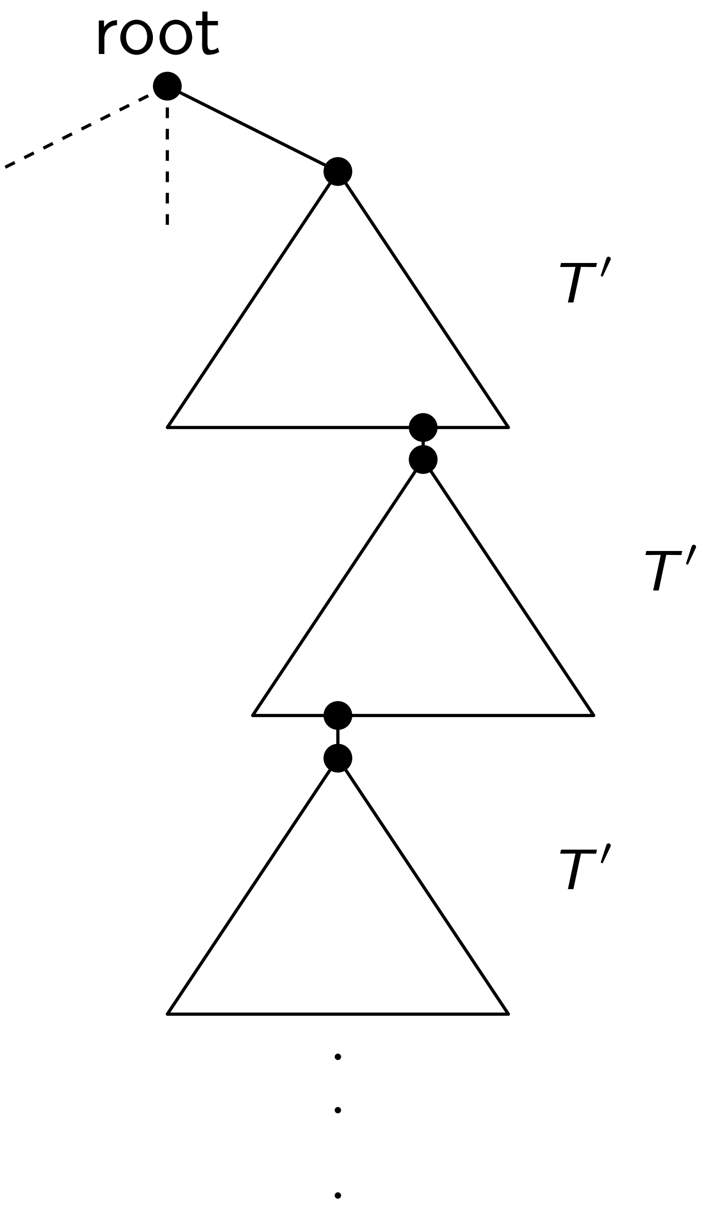

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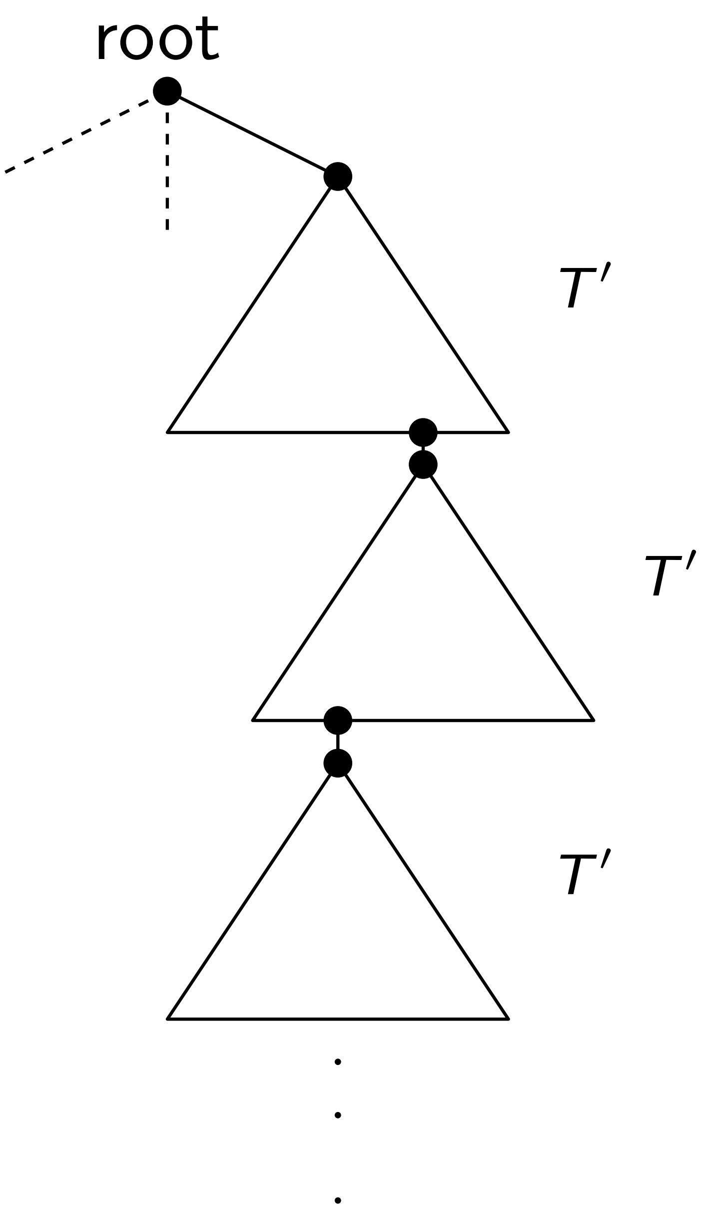

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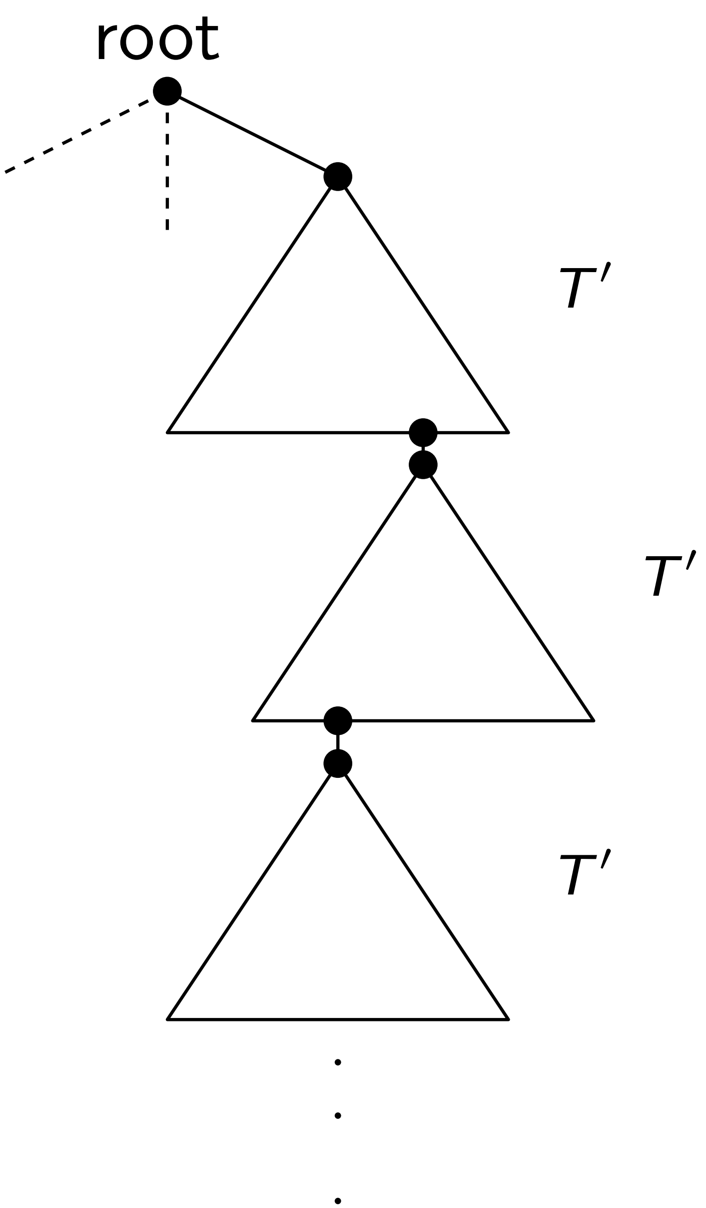

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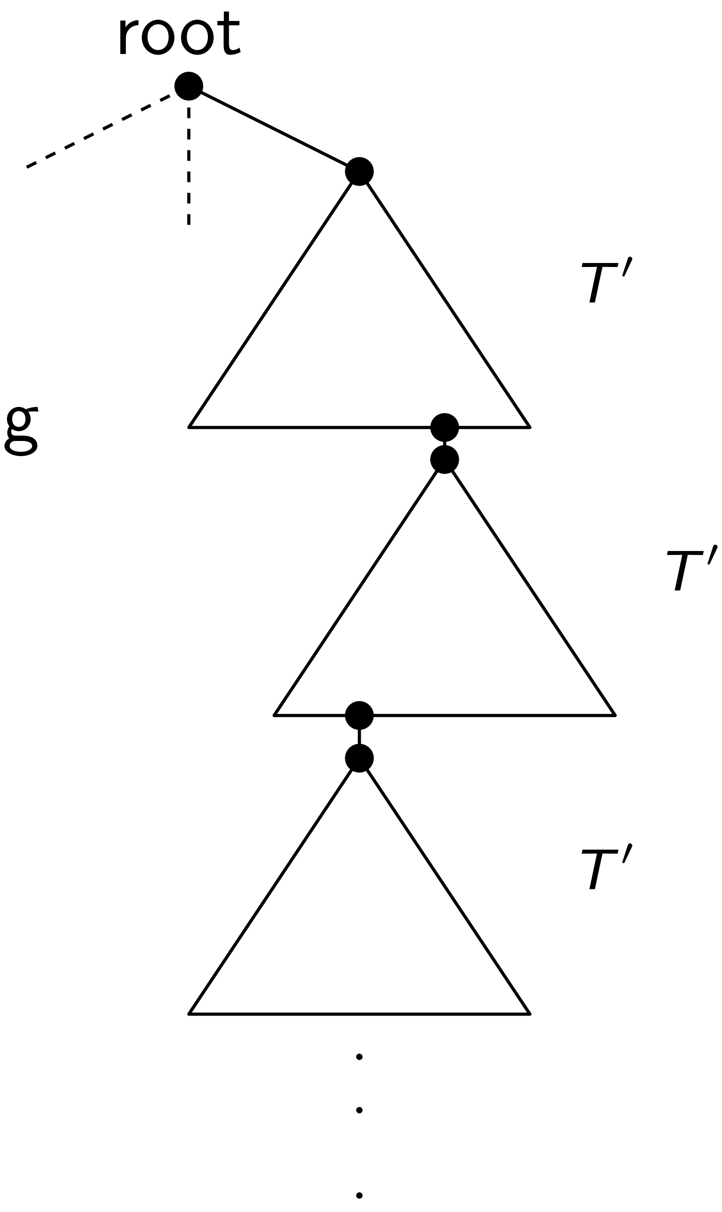

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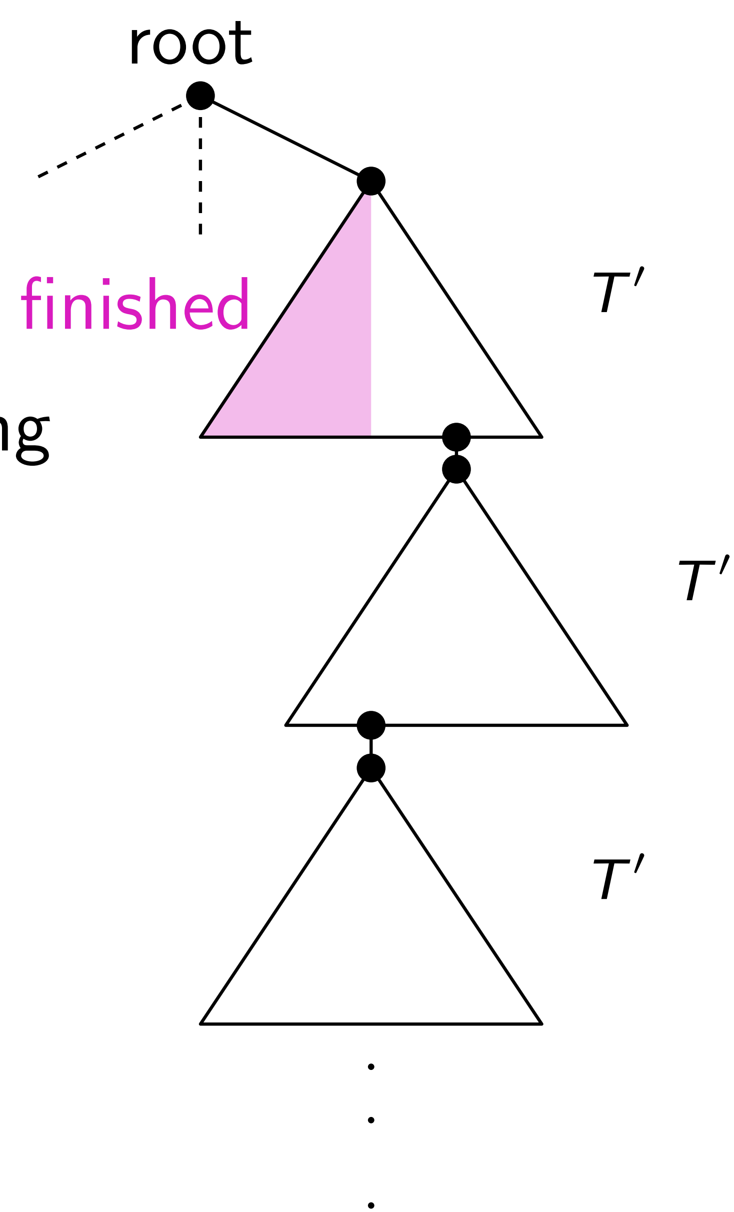

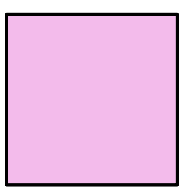

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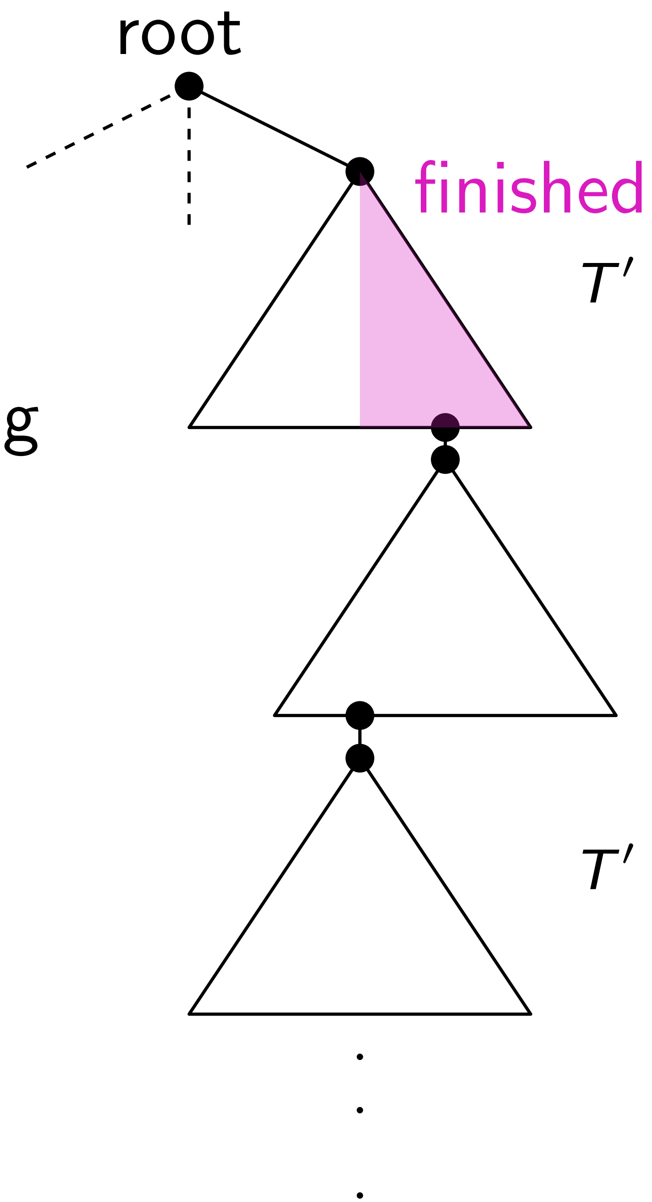

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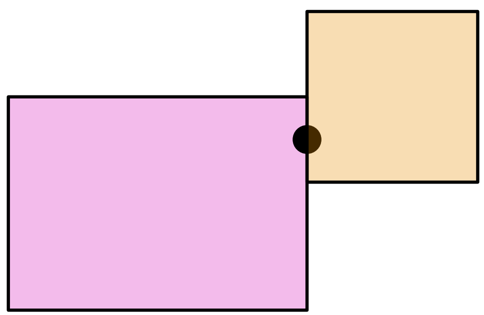

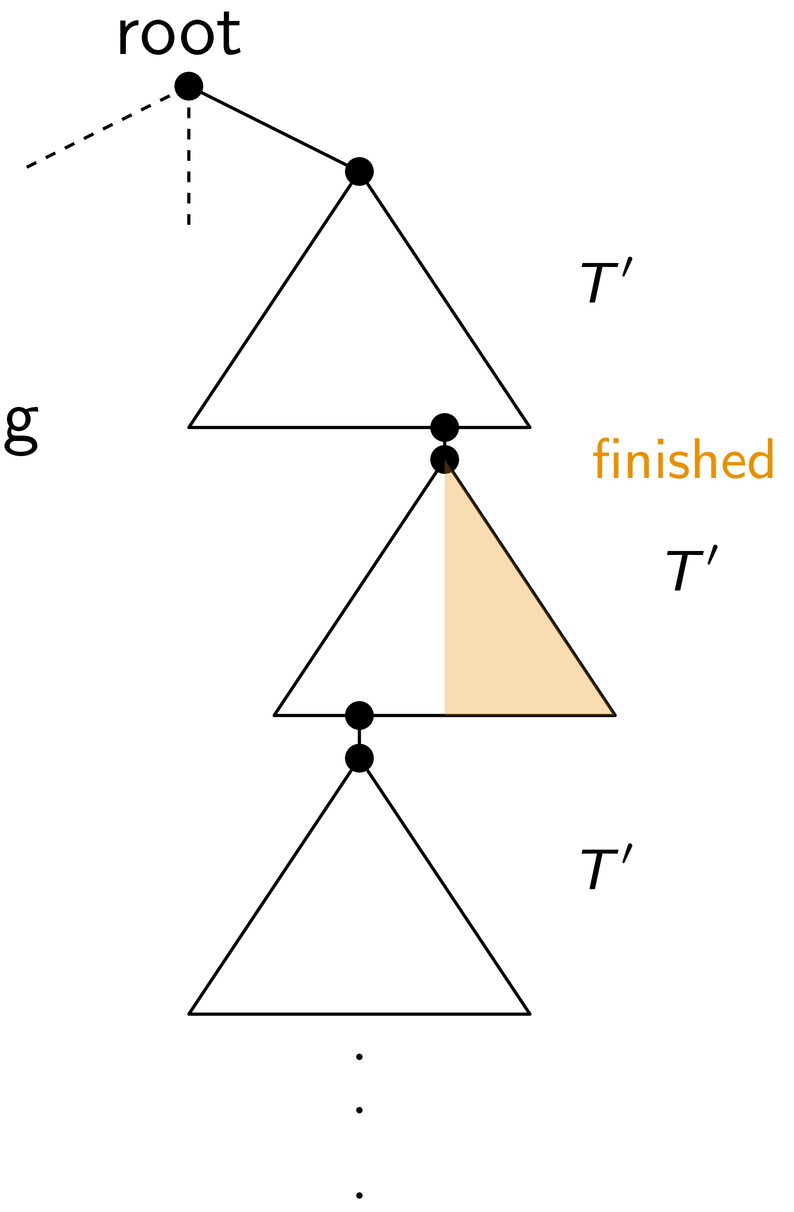

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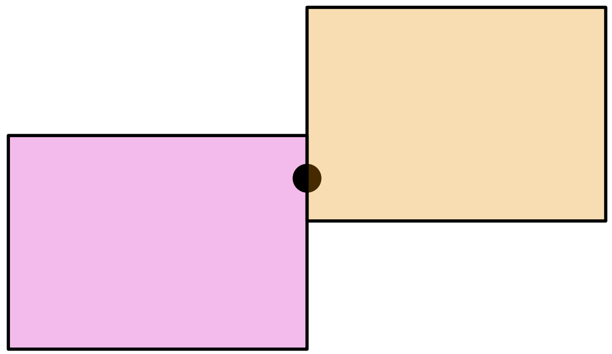

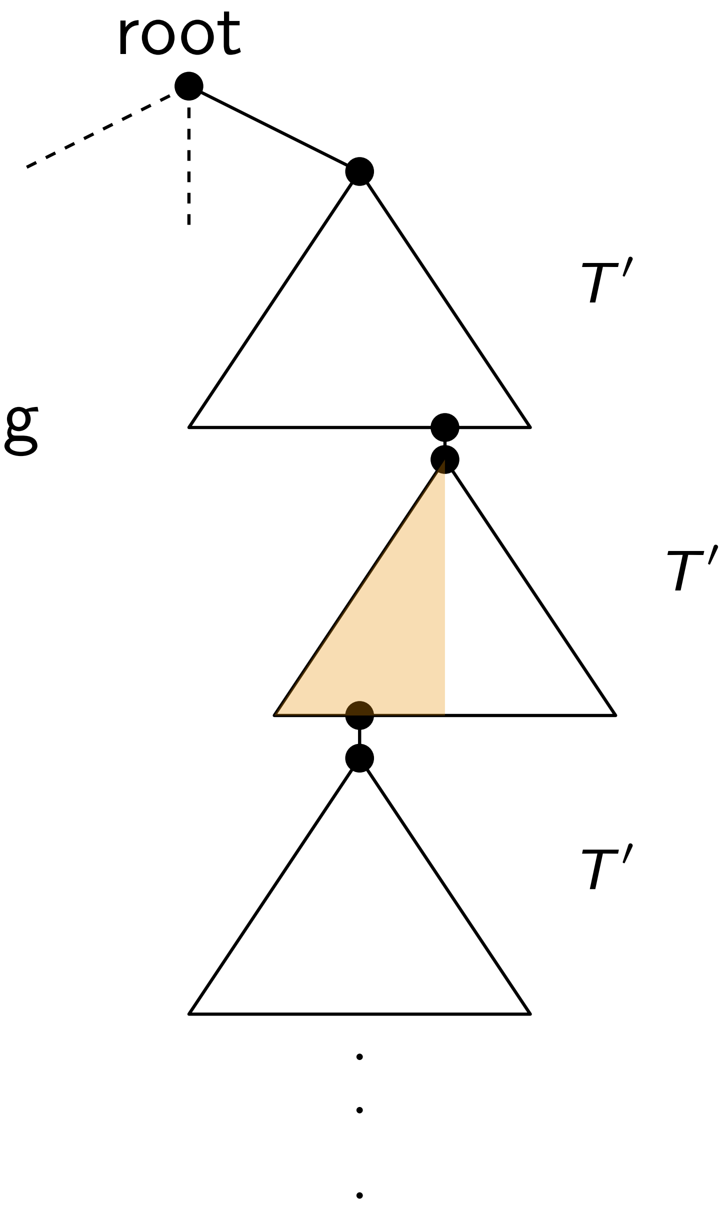

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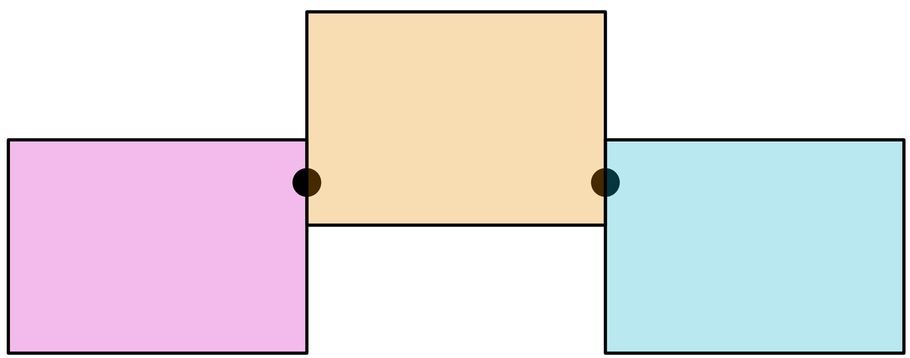

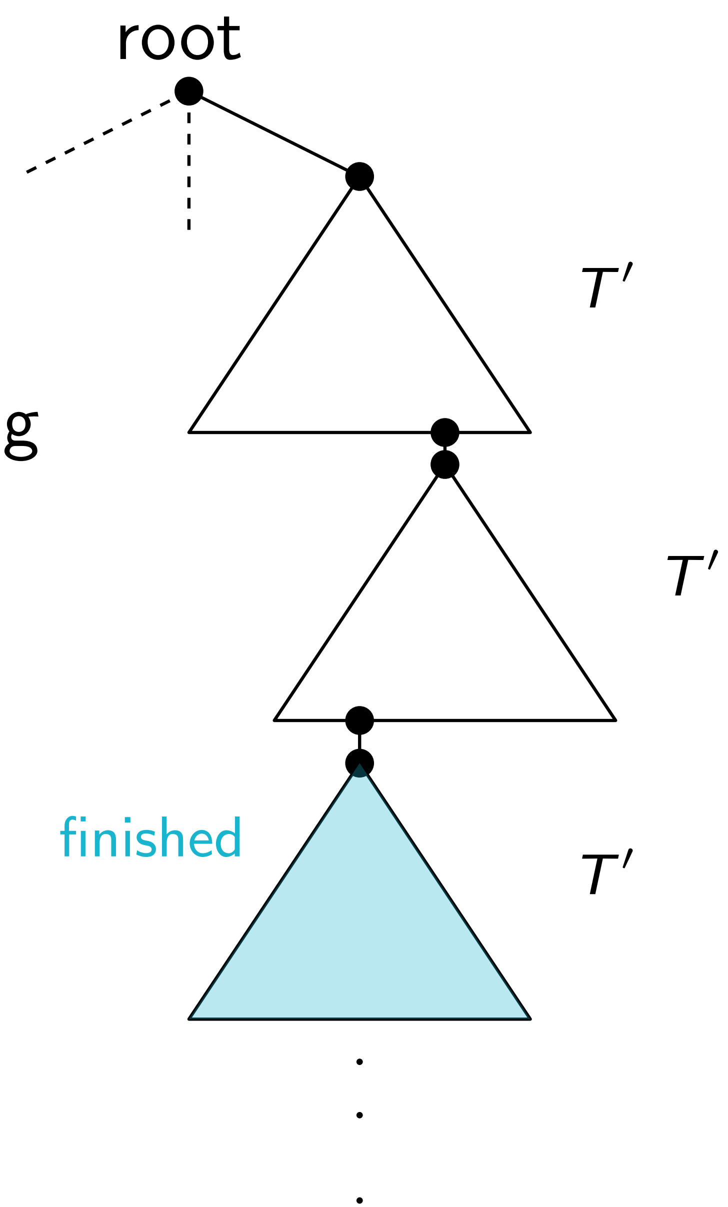

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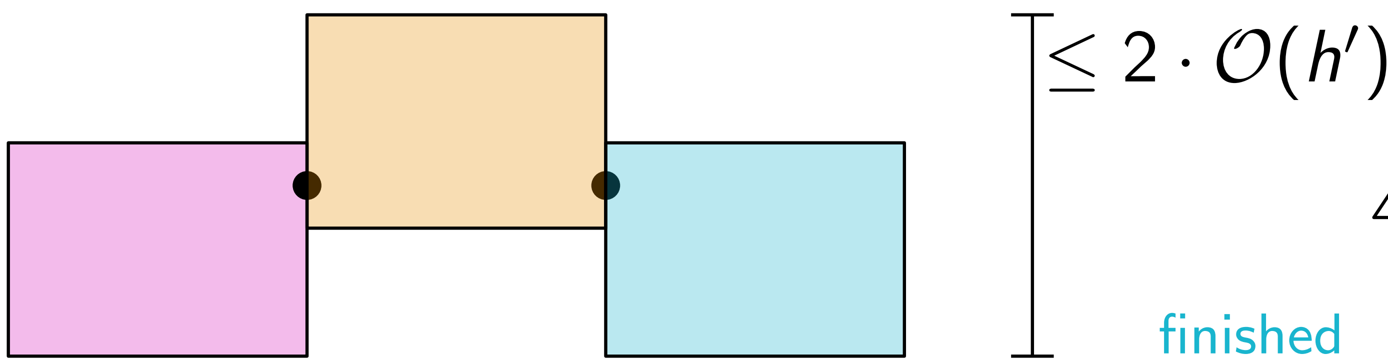

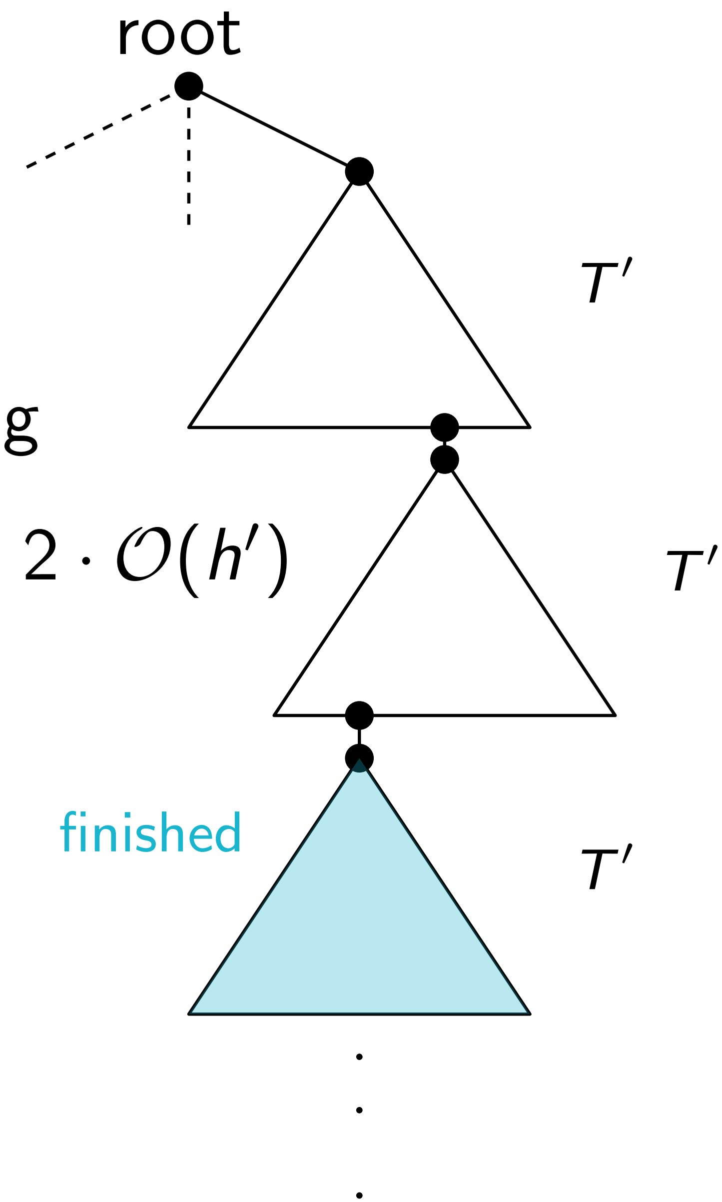

- Total amount of layers equals constant multiple of amount of layers induced by drawing T'

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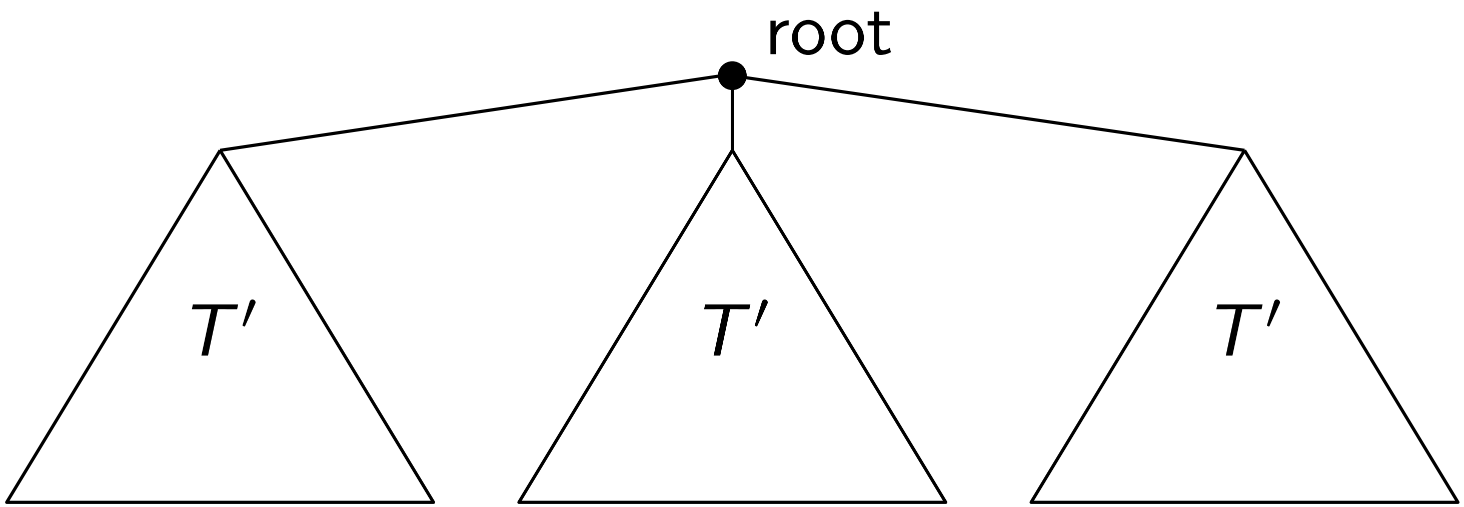

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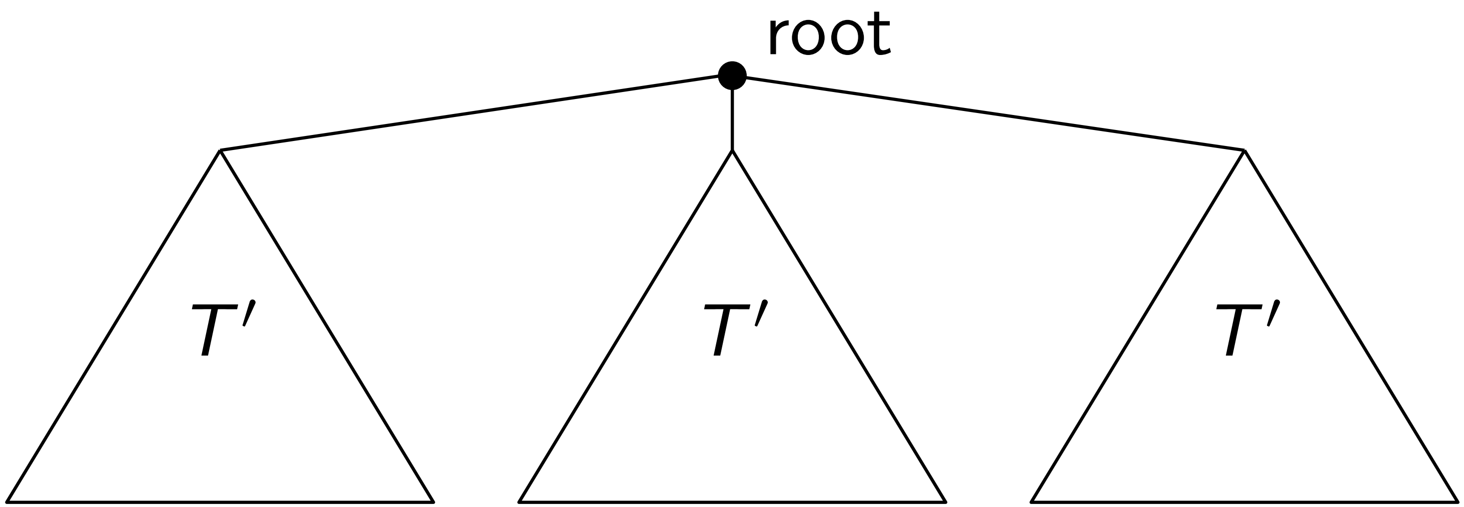

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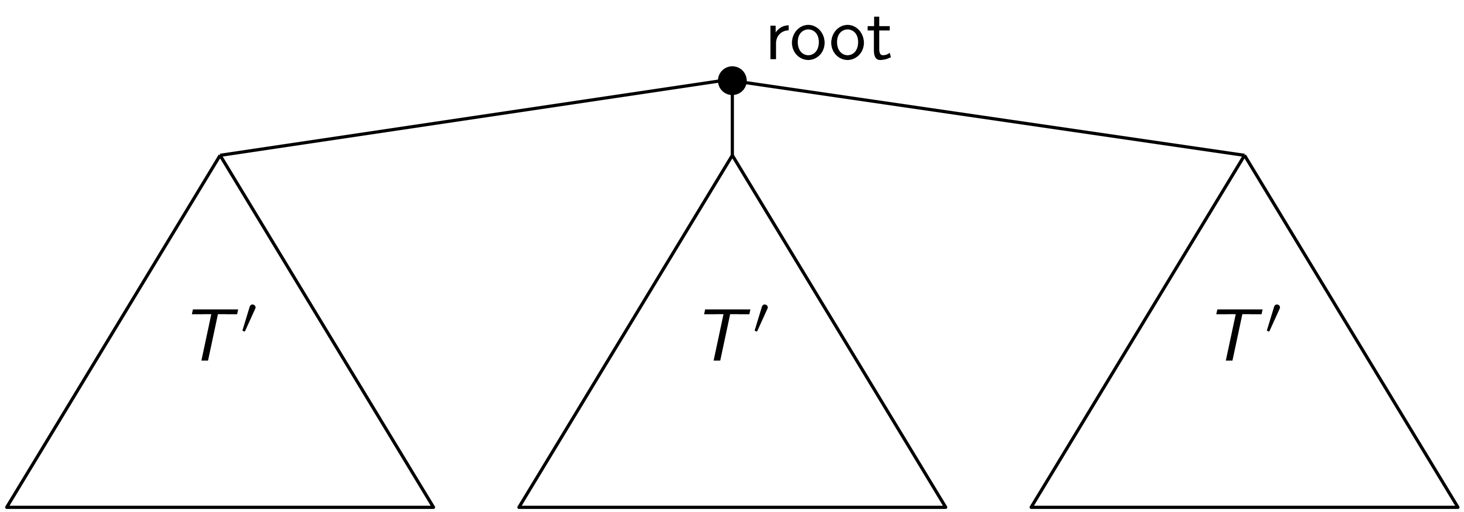
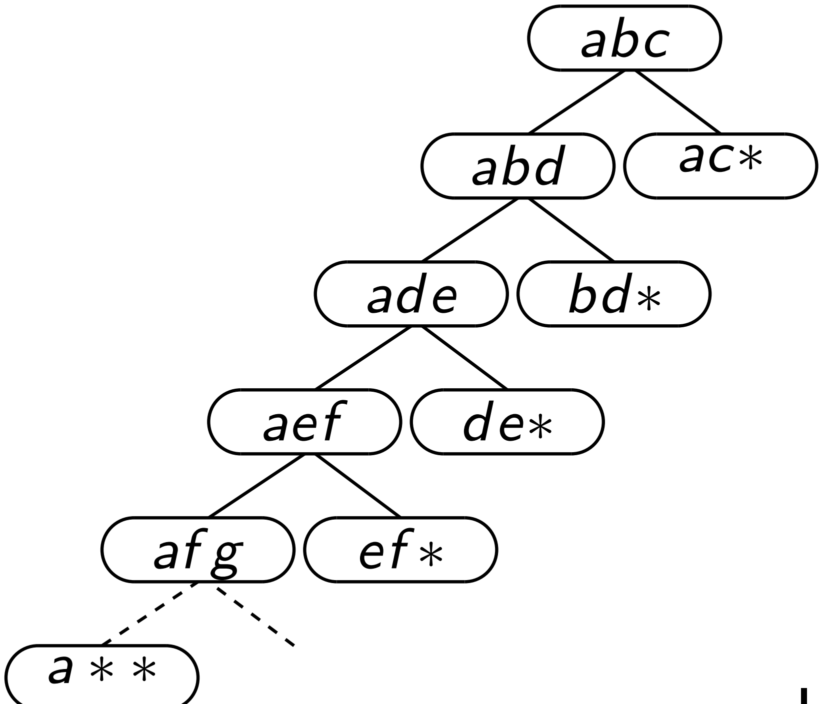
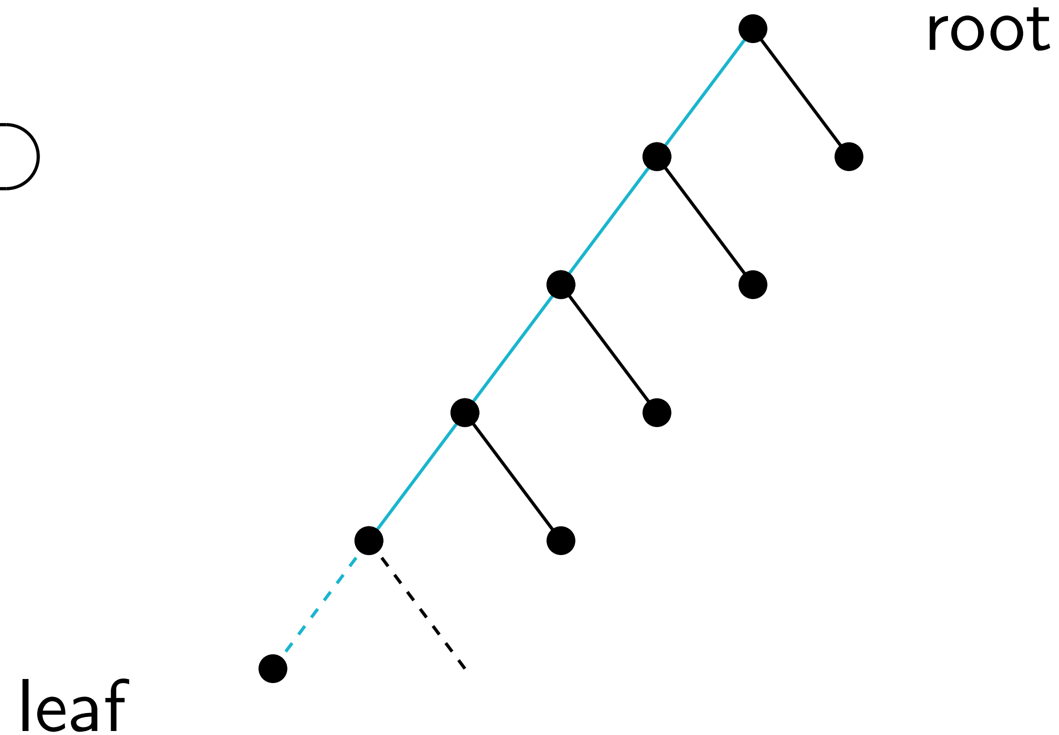


- Subtree priority does not take effect at any stage of DFS
  - Enforcing new layers

Layer Amount in T'

- In T', consider path from root to a leaf





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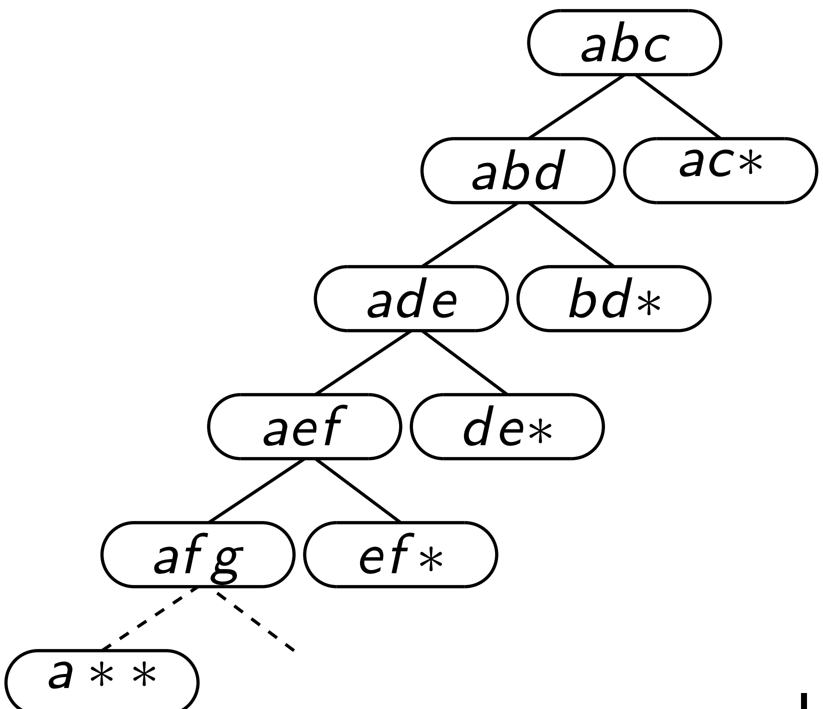

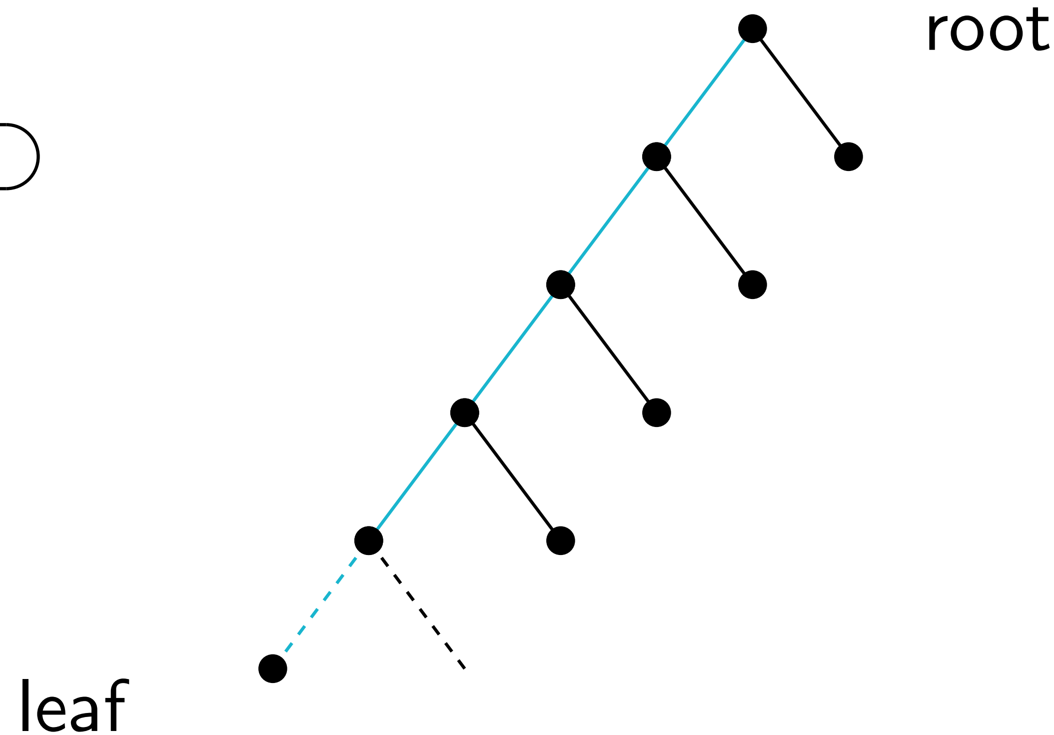

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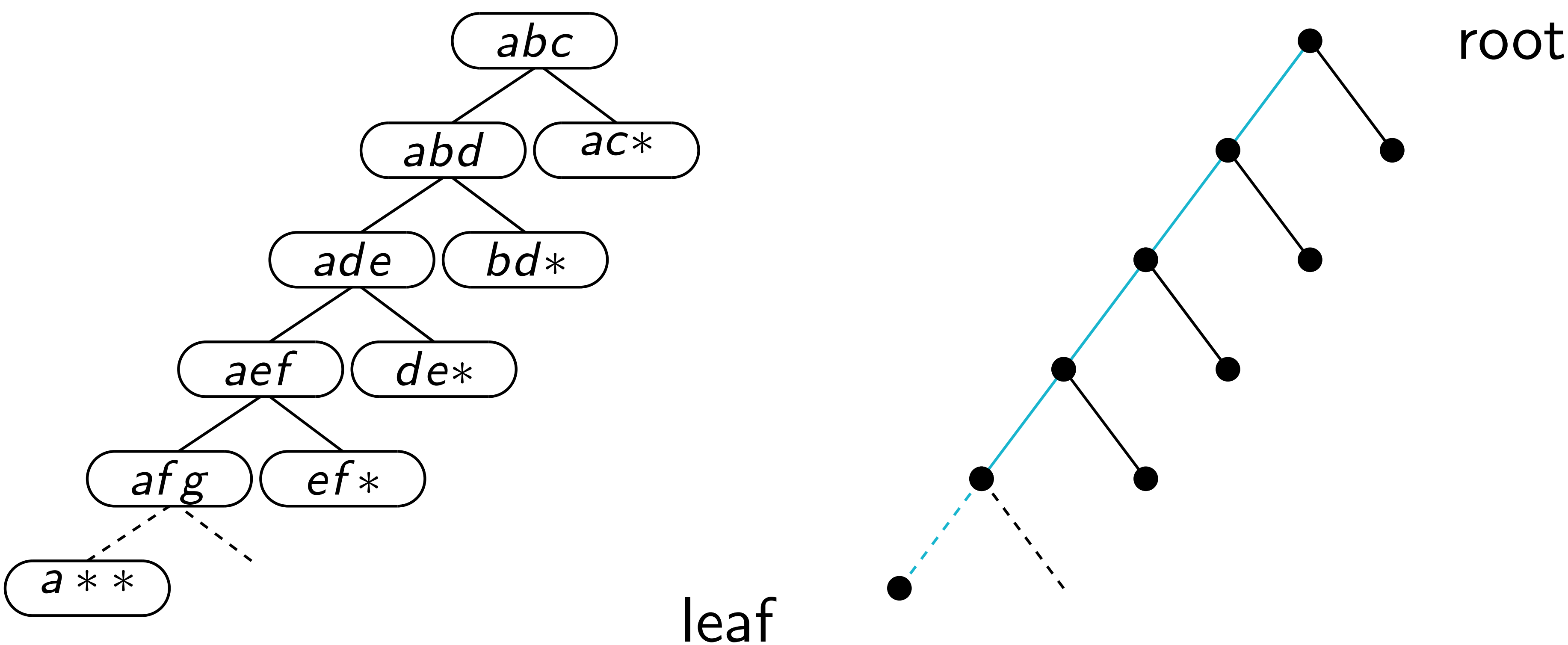

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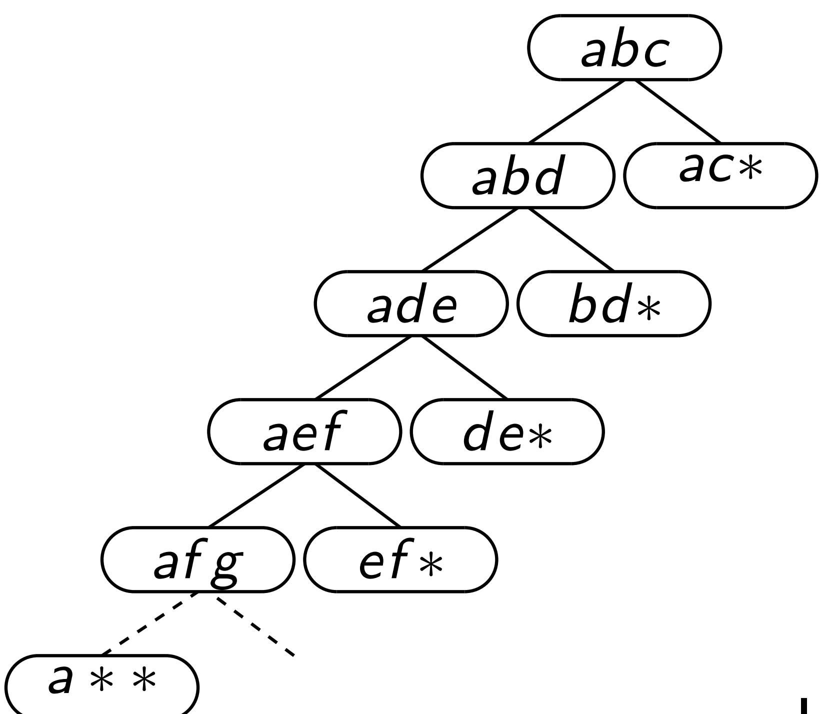
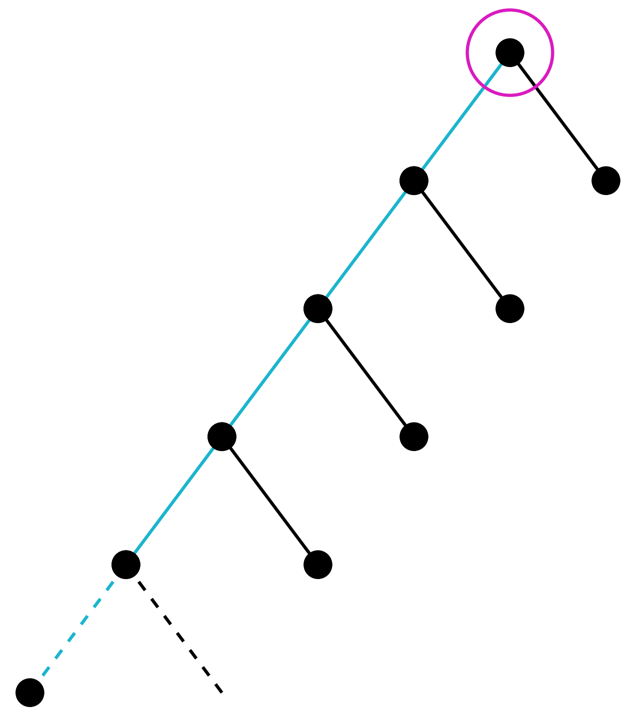


DFS starts at root

root



leaf

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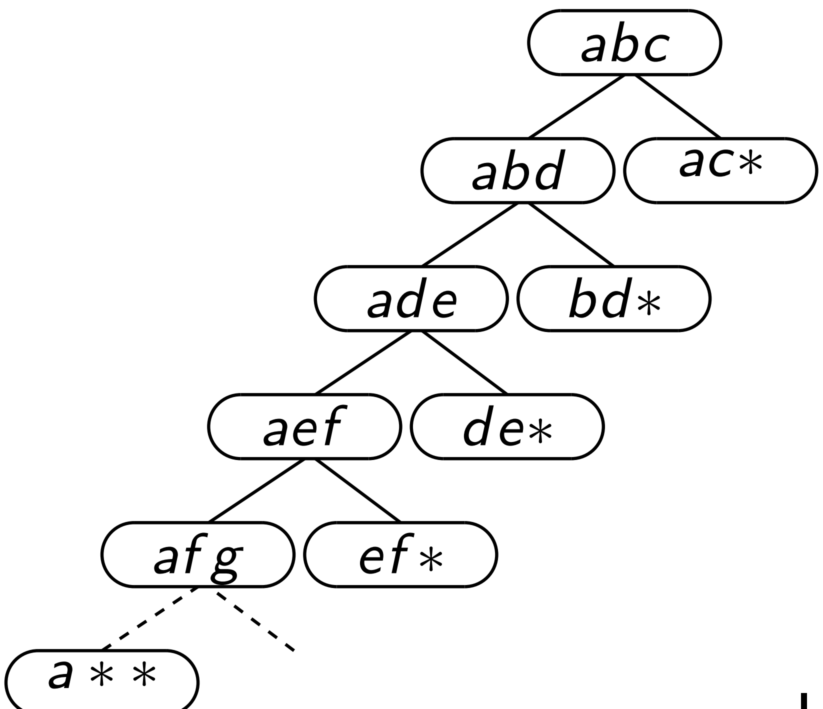
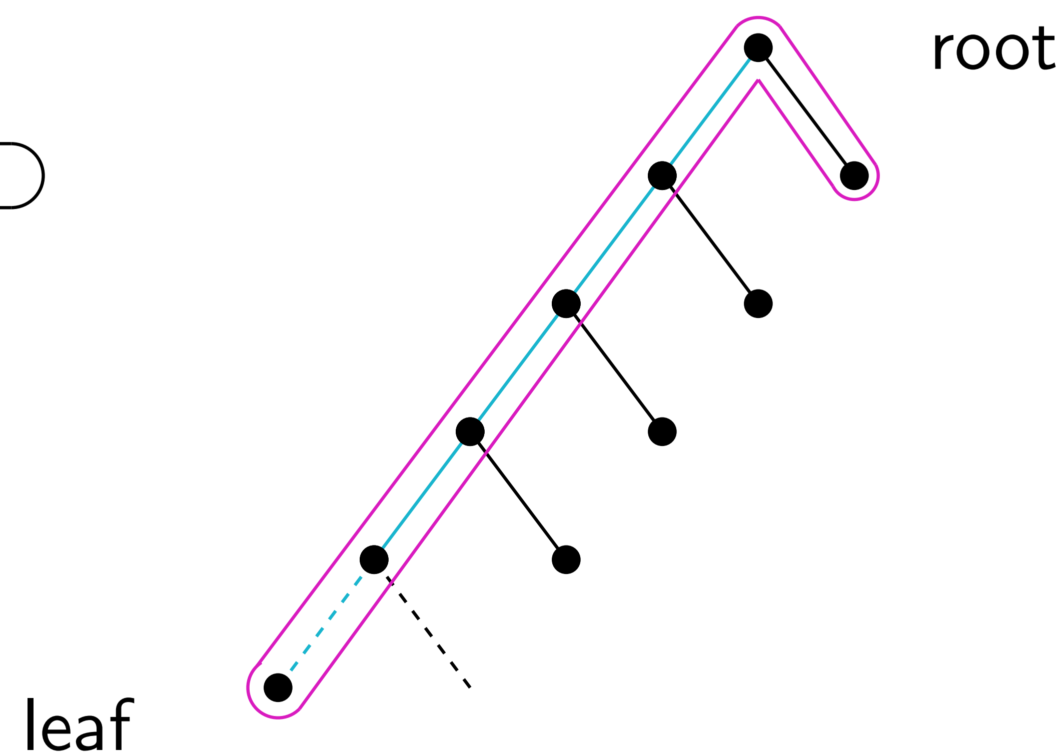
Occupation of a





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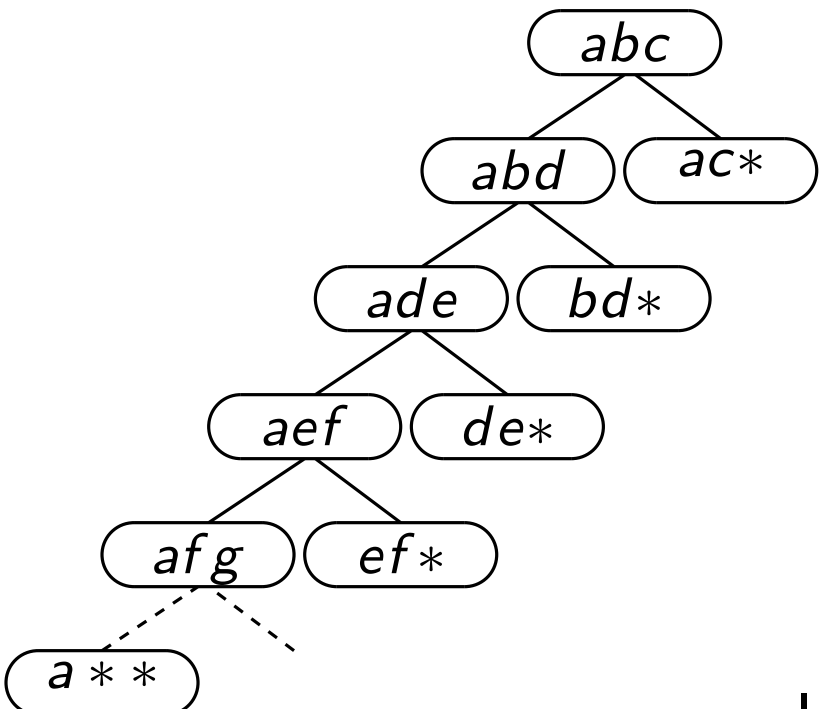

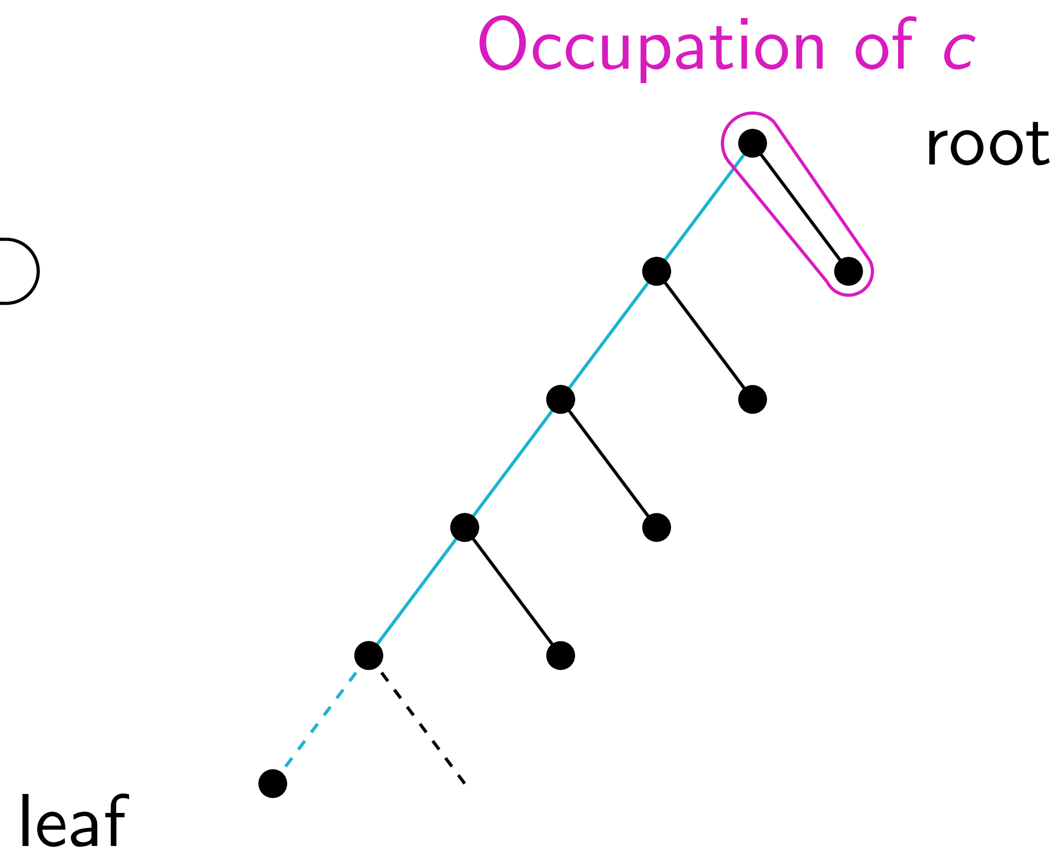

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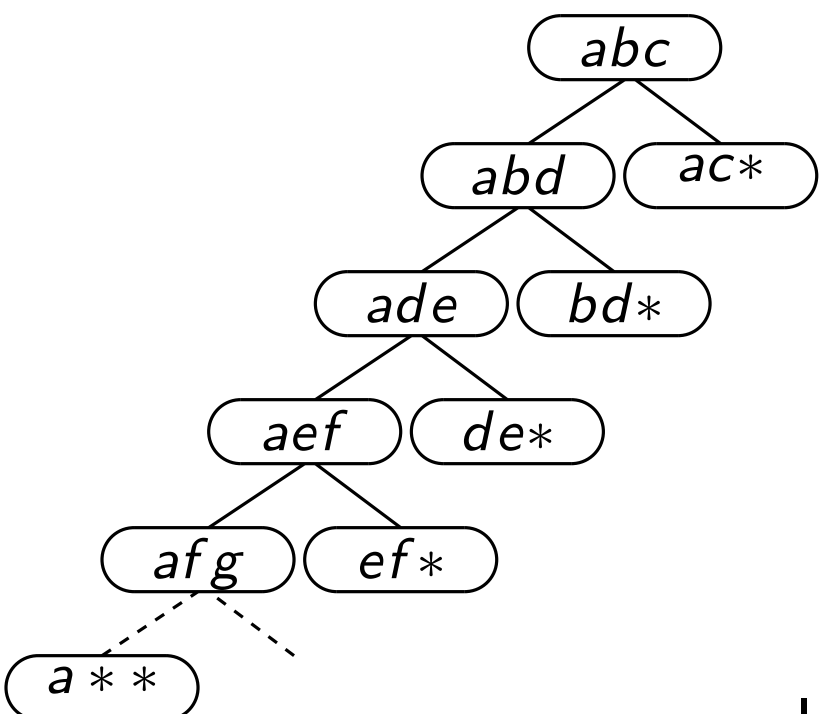

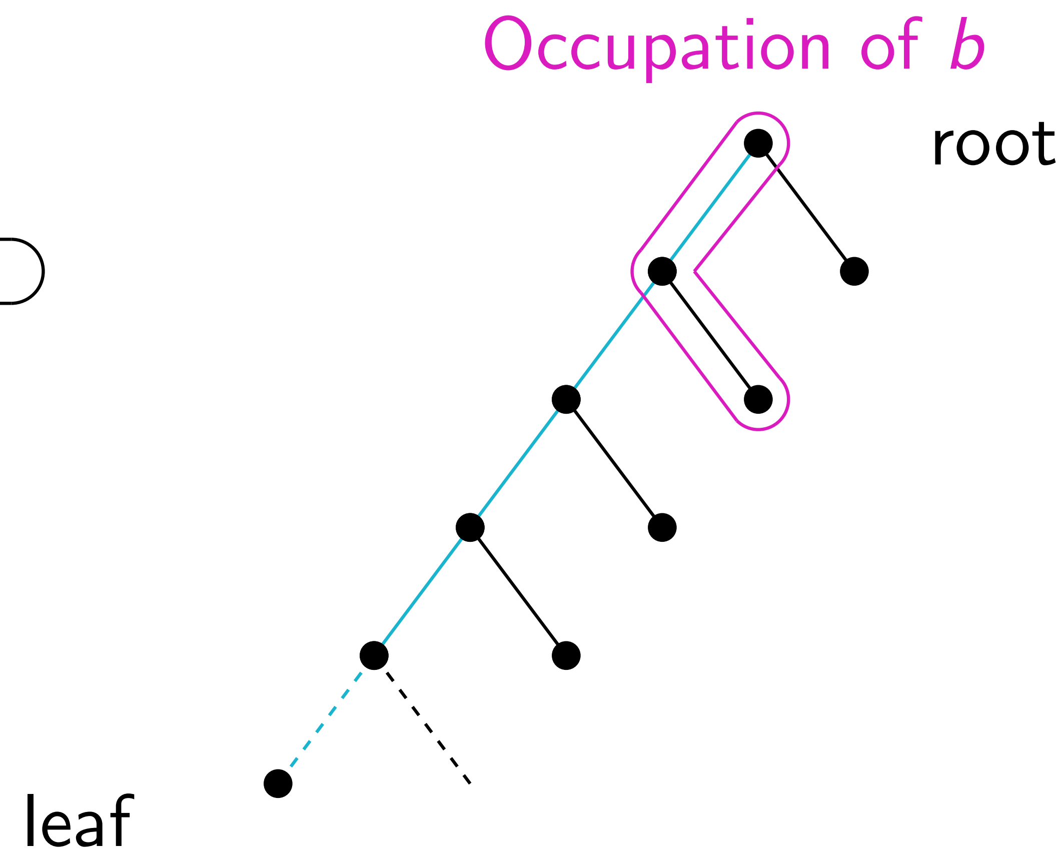

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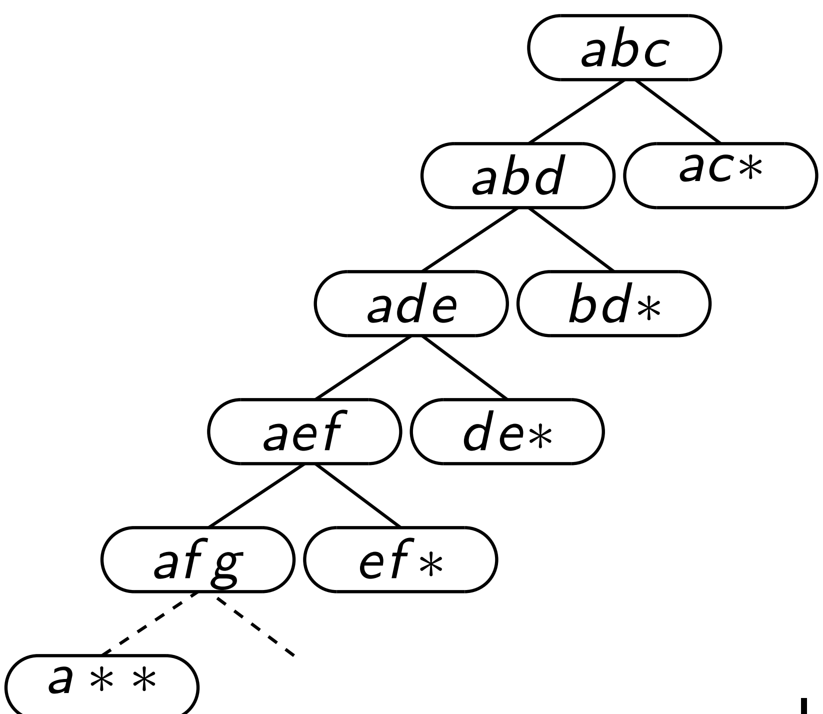
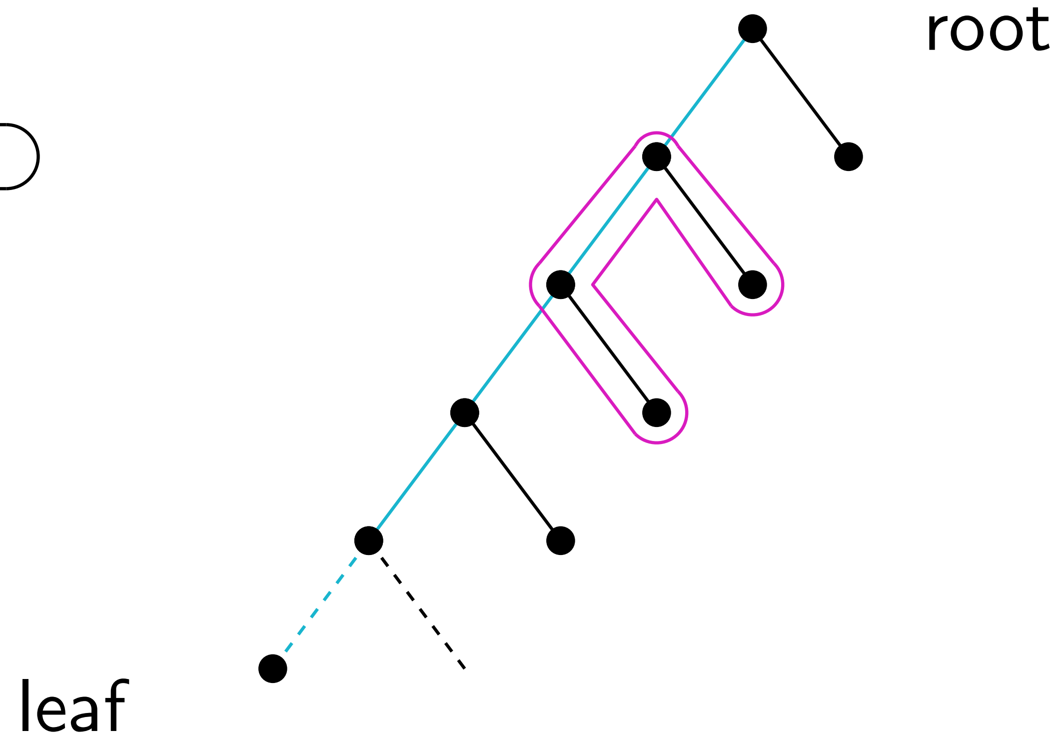


Occupation of d



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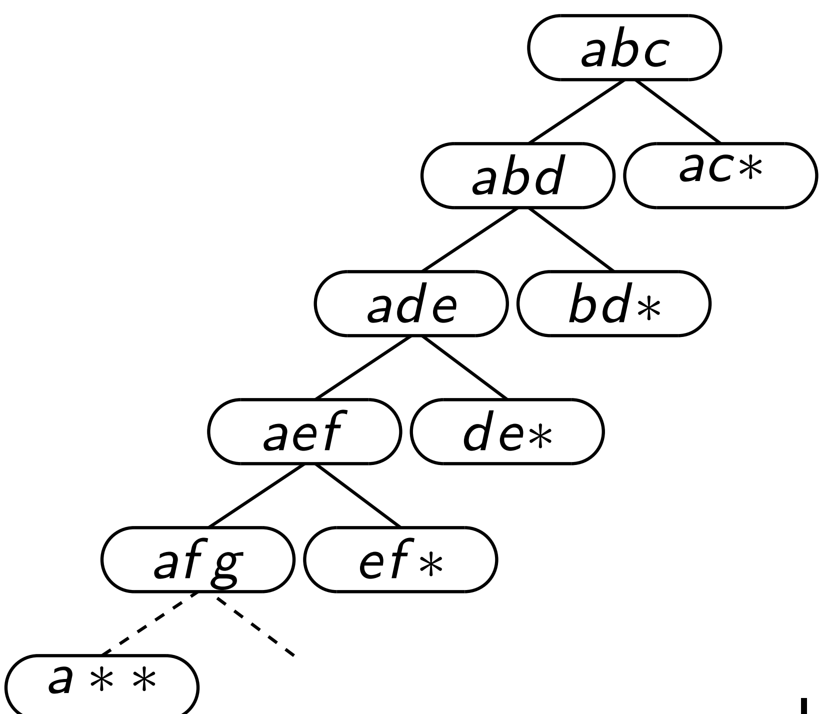

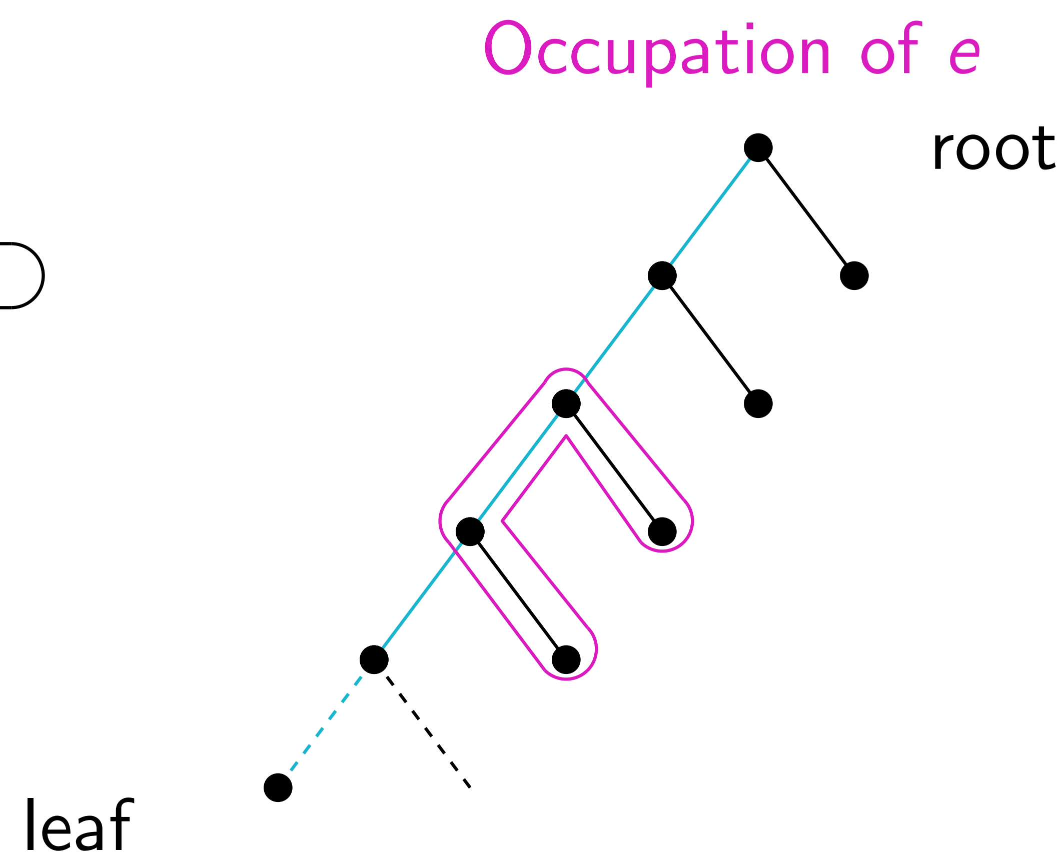

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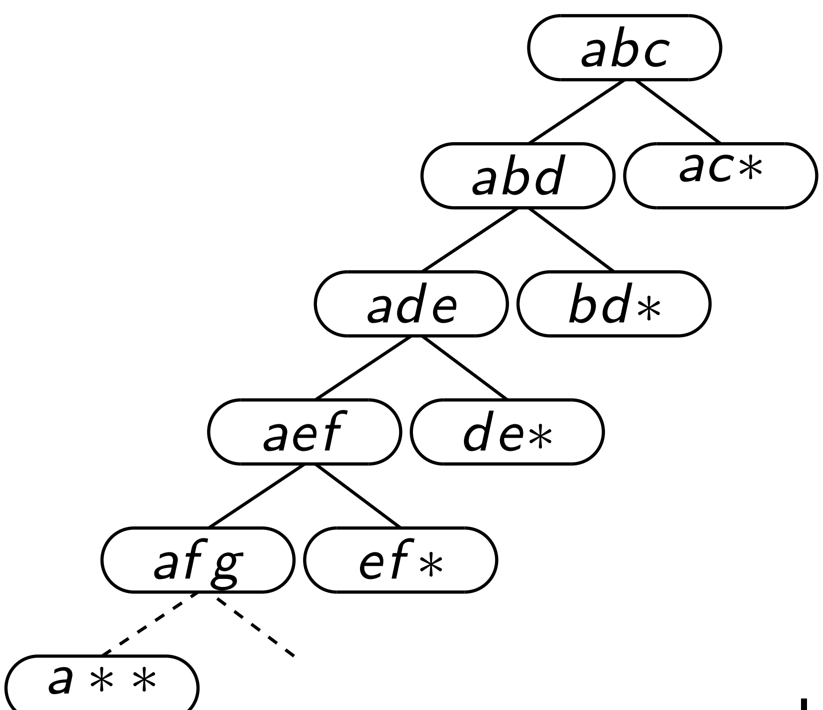

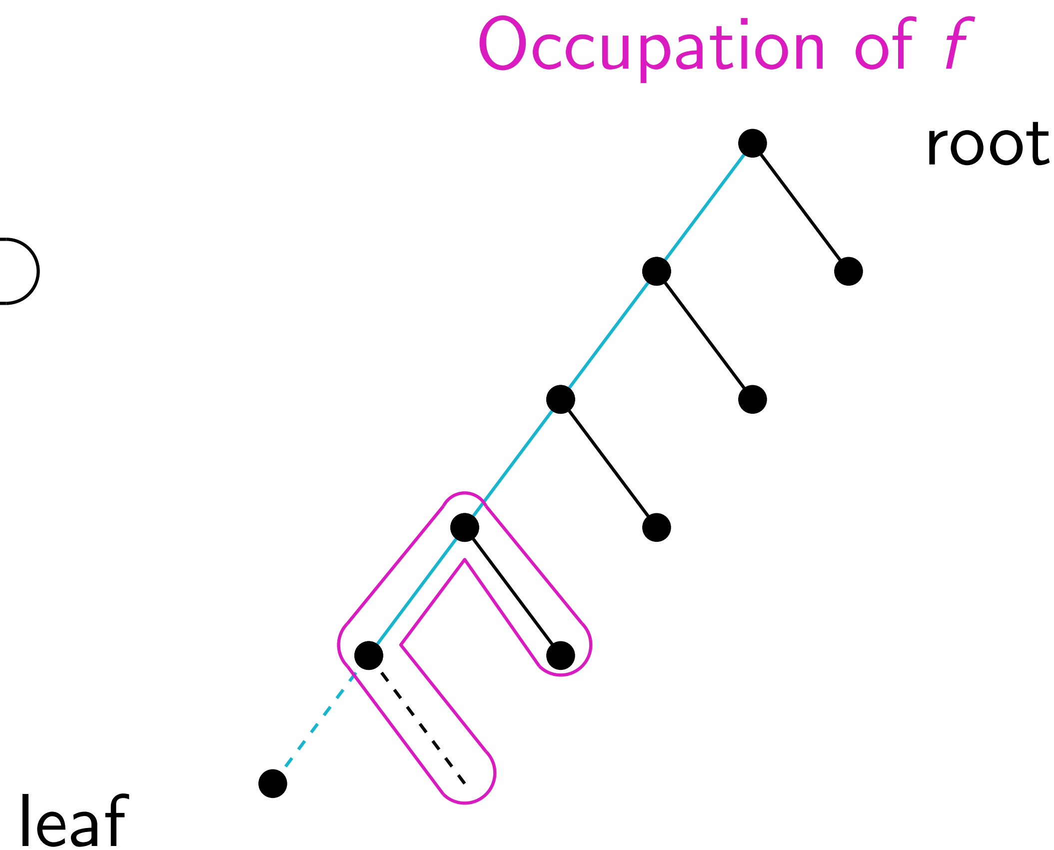

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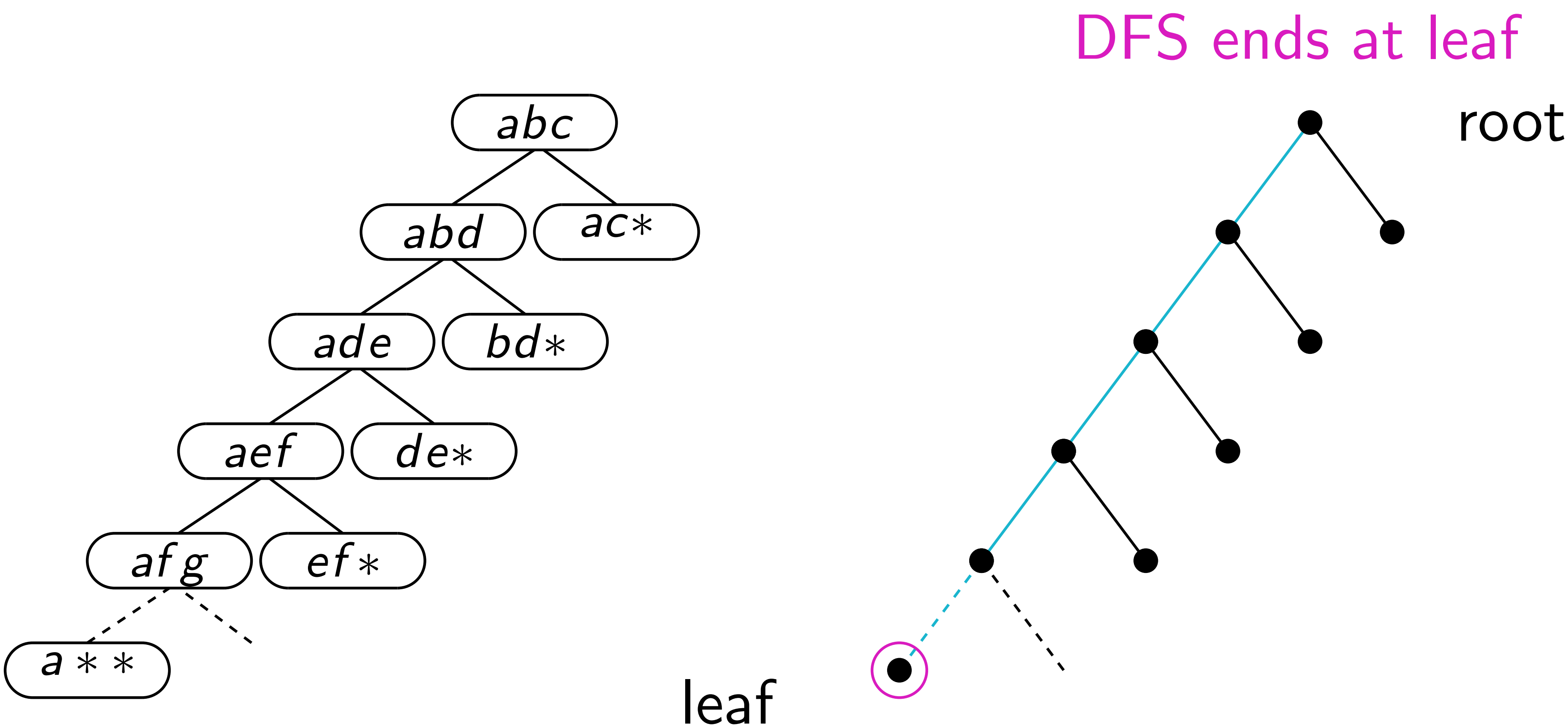
`Amount of layers occupied:` $\mathcal{O}(\log n')$



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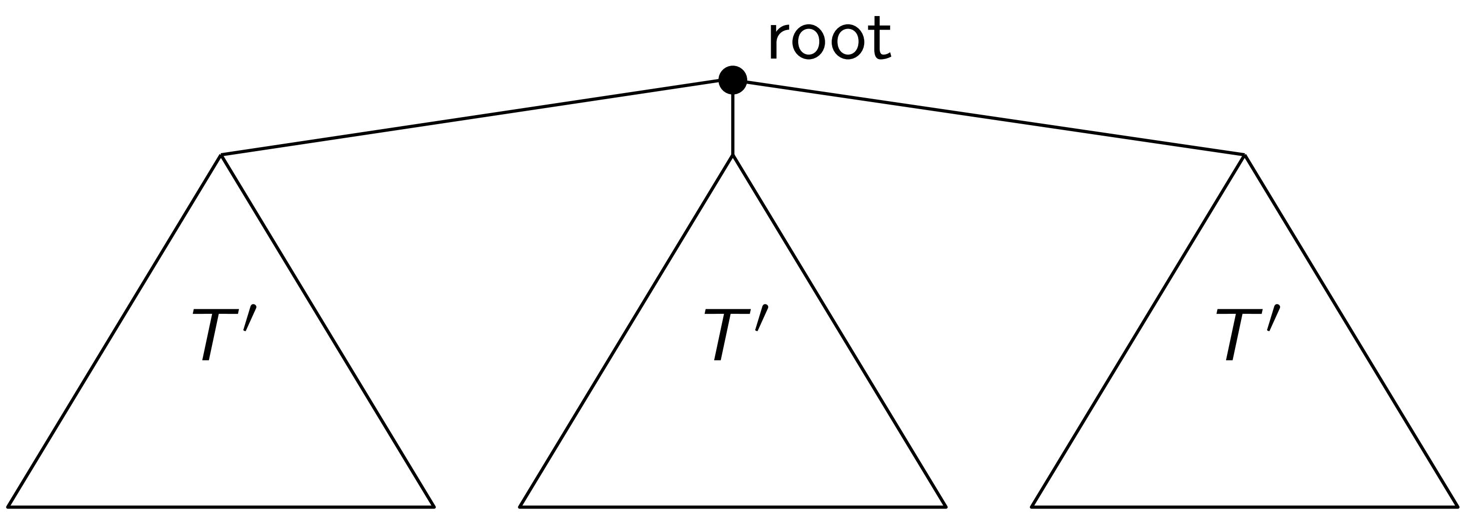

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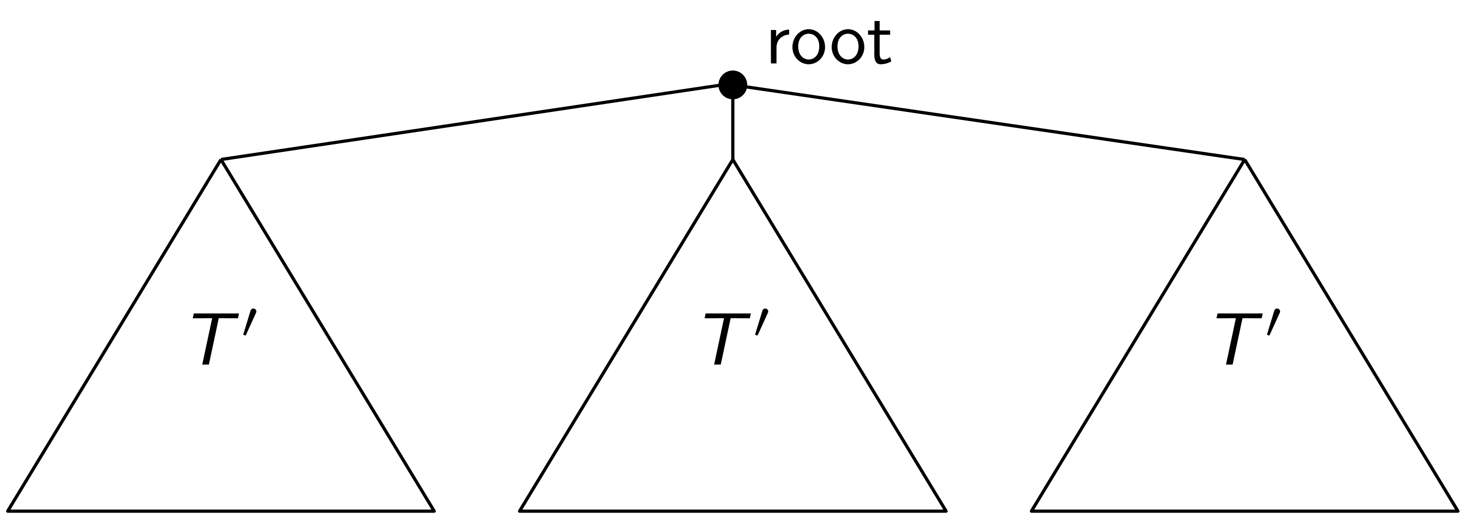

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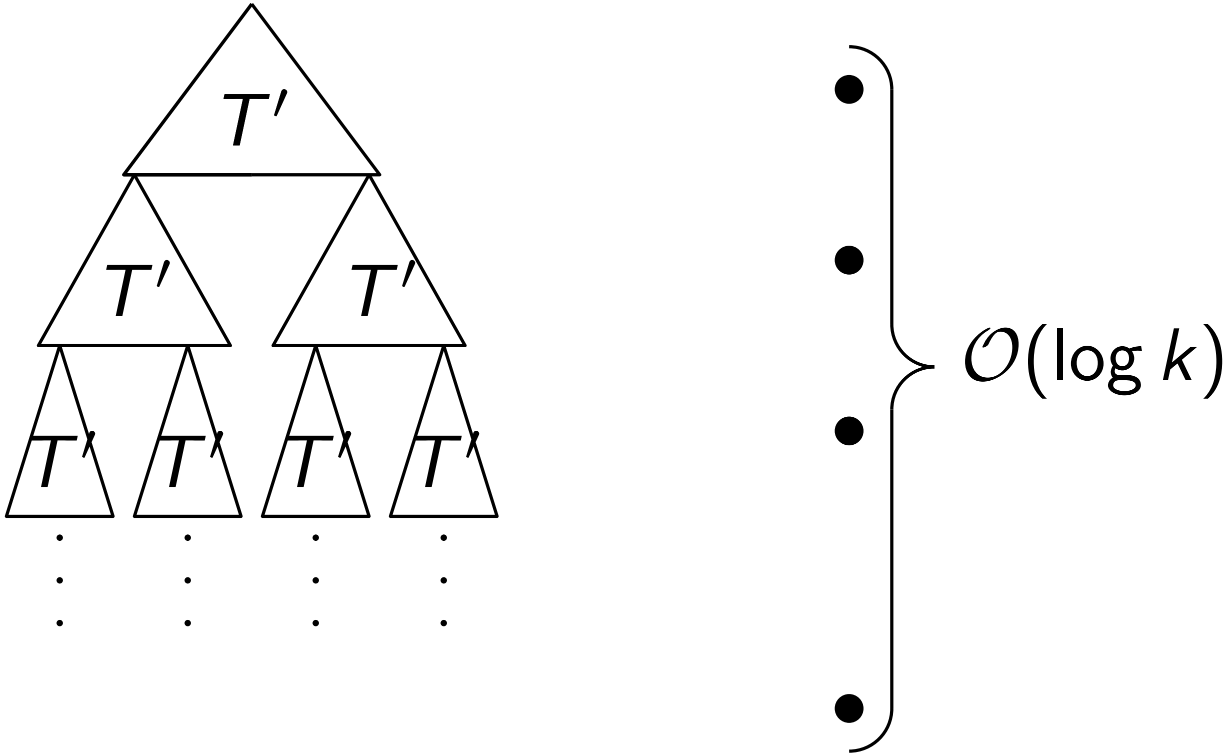

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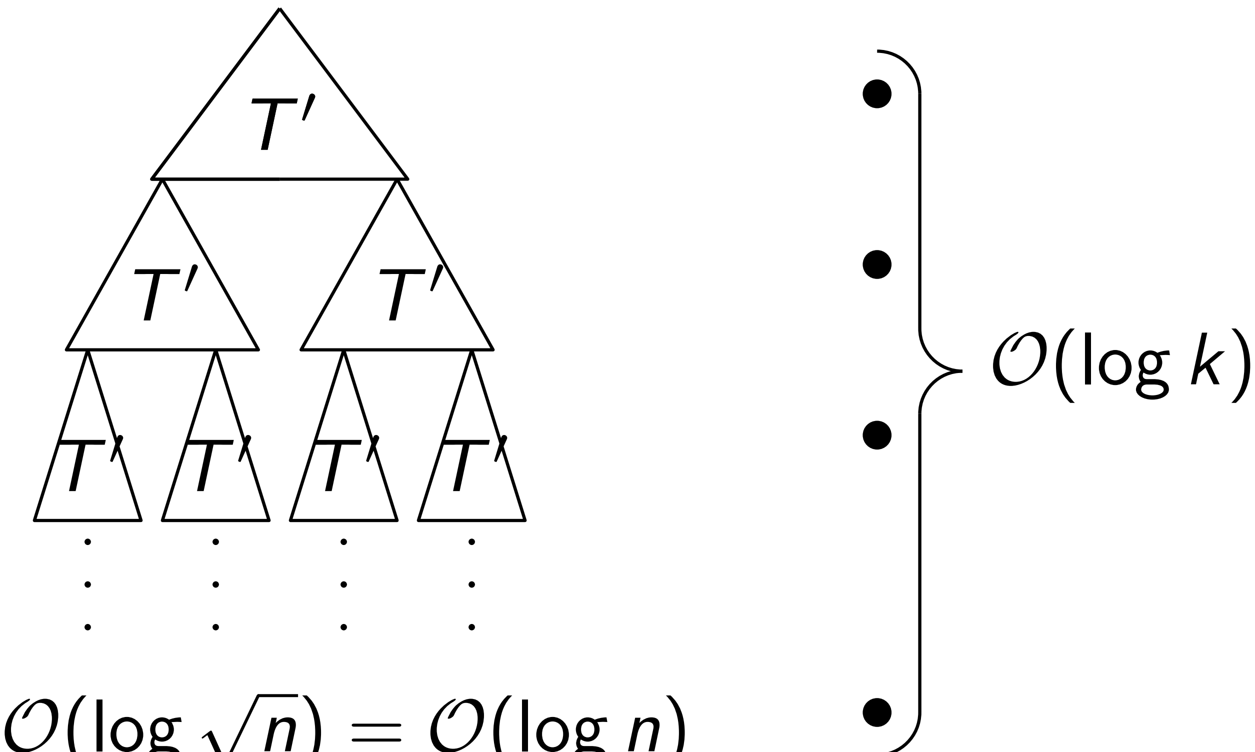

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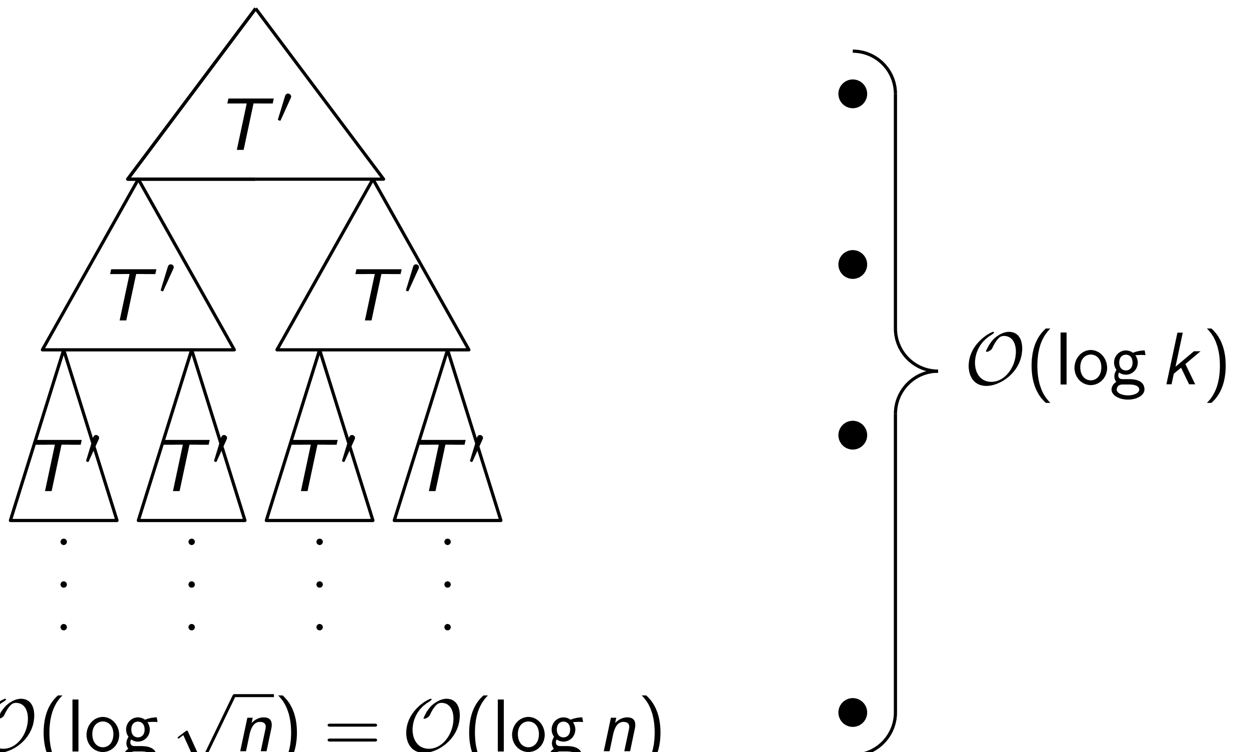

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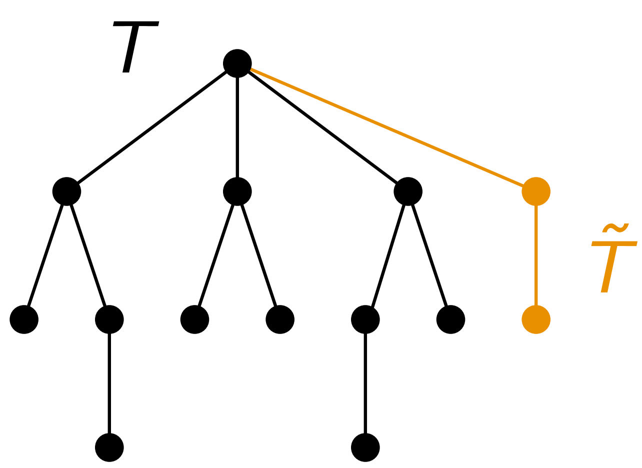

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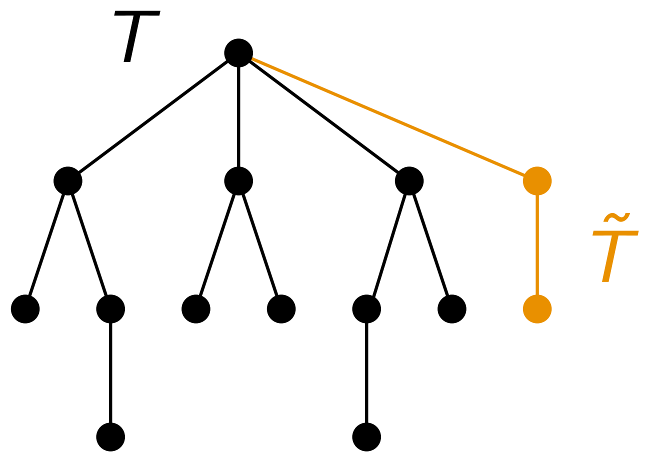

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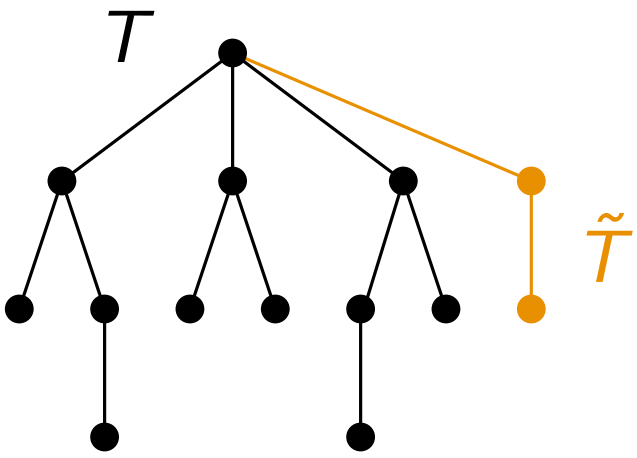

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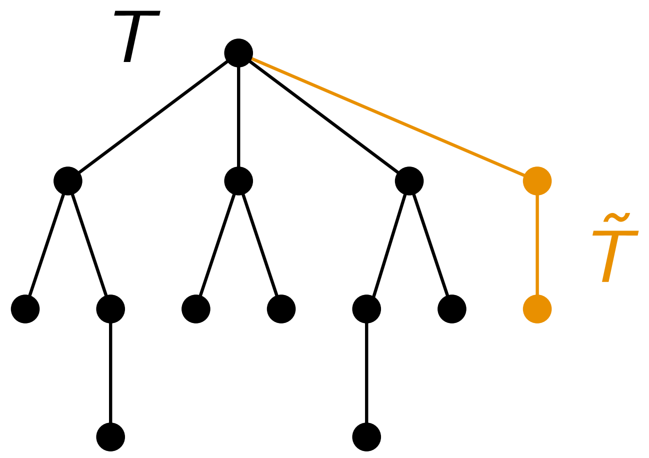

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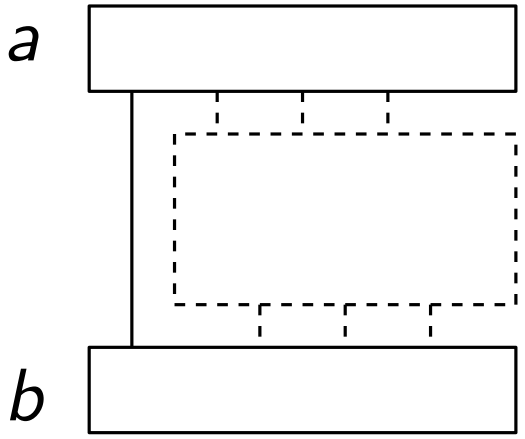

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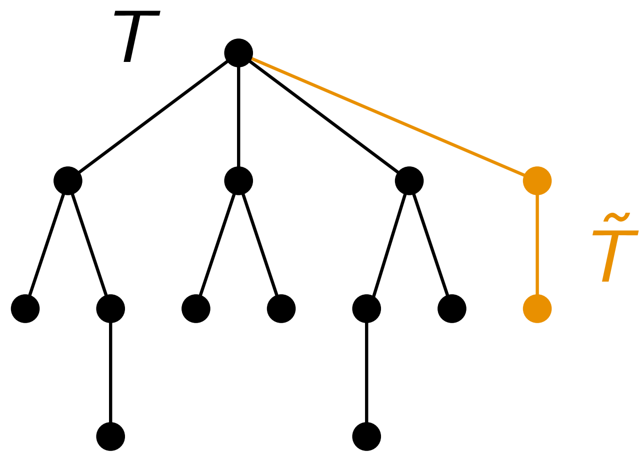

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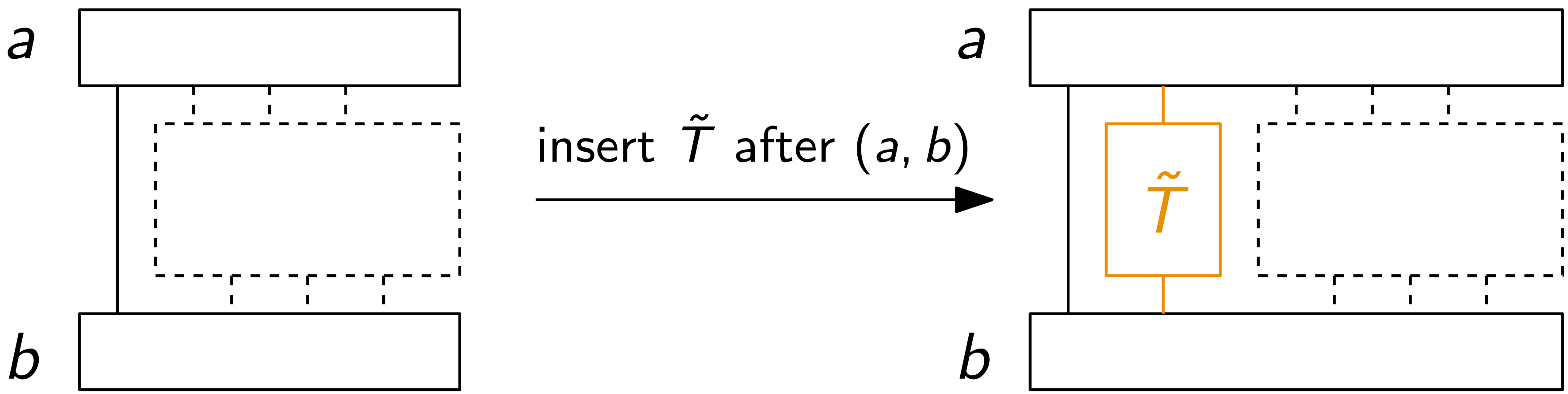
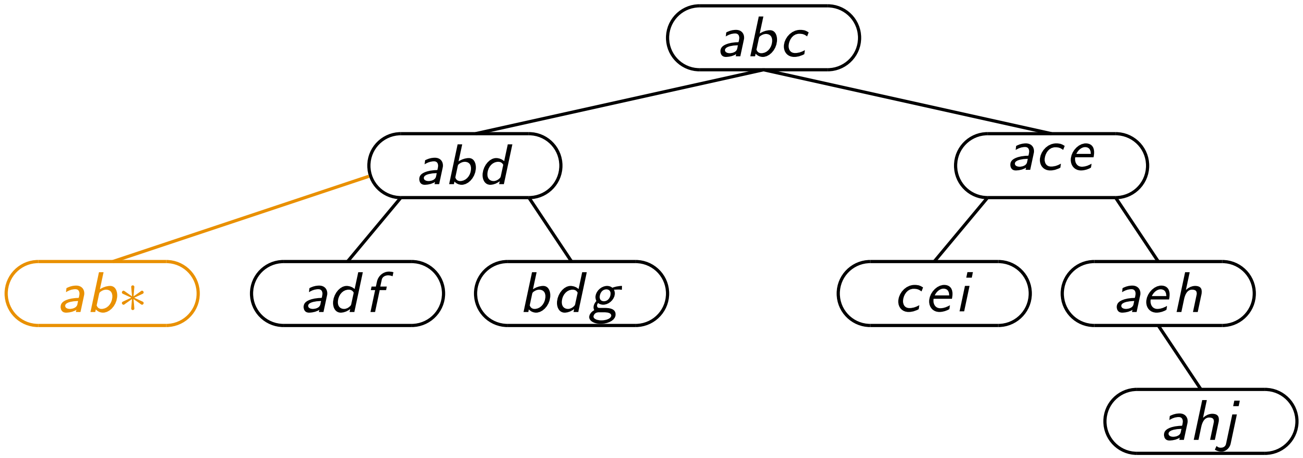


- Does not alter the area bounds of $B_{G'}$ asymptotically!

Results for Series-Parallel Graphs

- Since the area bounds stay the same, the results of outerplanar graphs also hold for series-parallel graphs
- Any series-parallel graph G admits a polyline drawing in $\mathcal{O}(n^2 \log^2 n)$ with $r \in \mathcal{O}(\log^2 n)$ and two bends per edge

Example drawing



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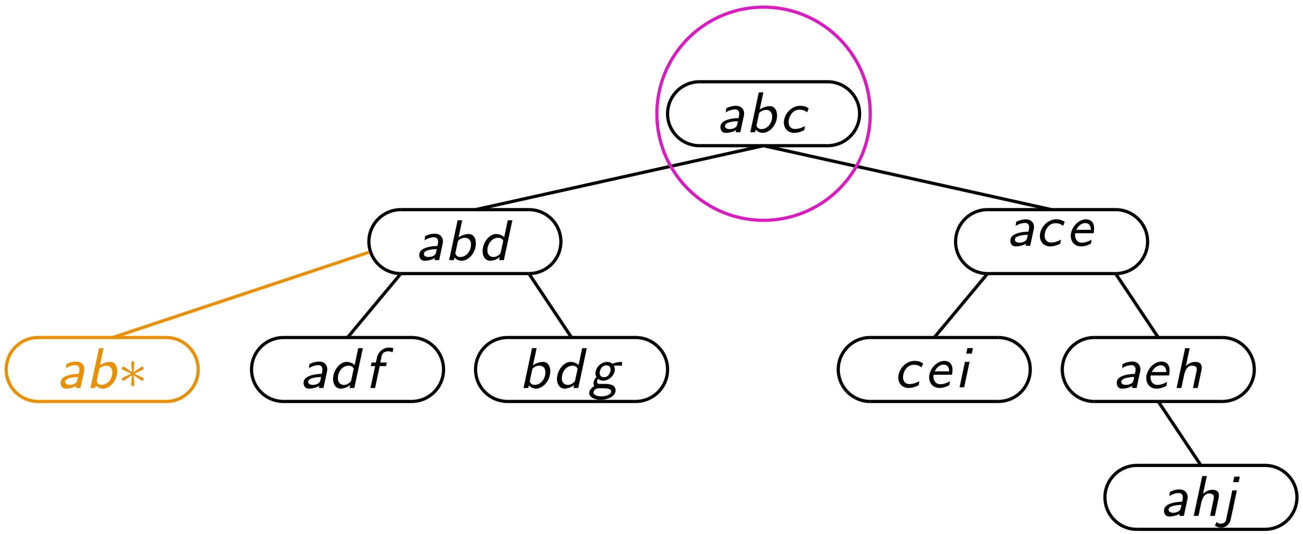

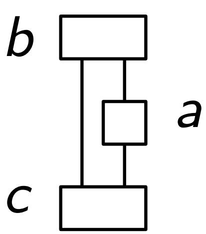

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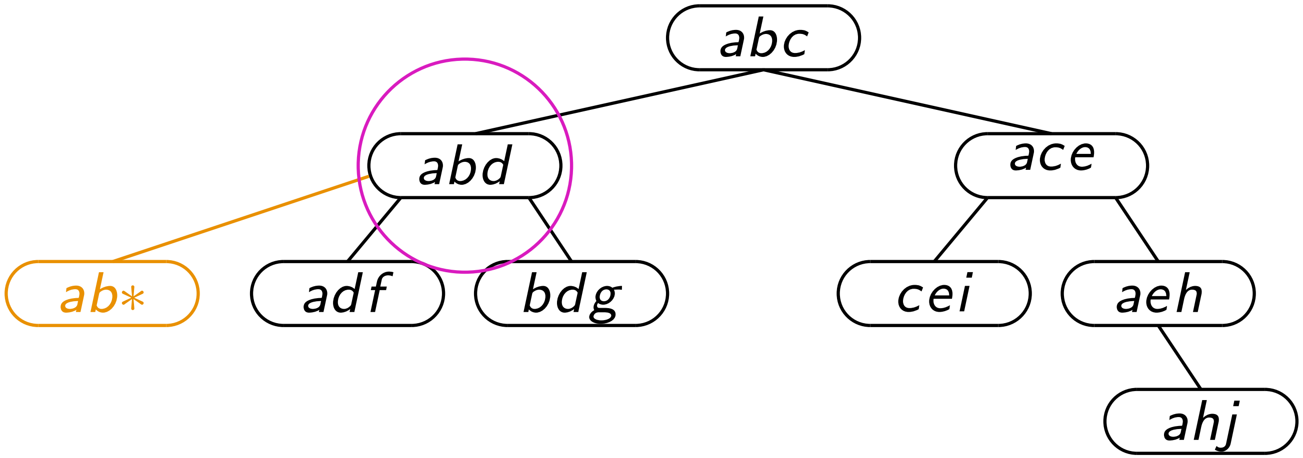

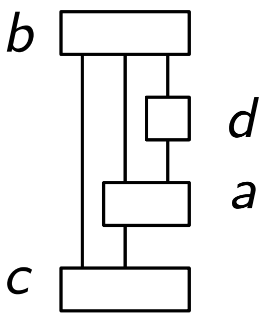

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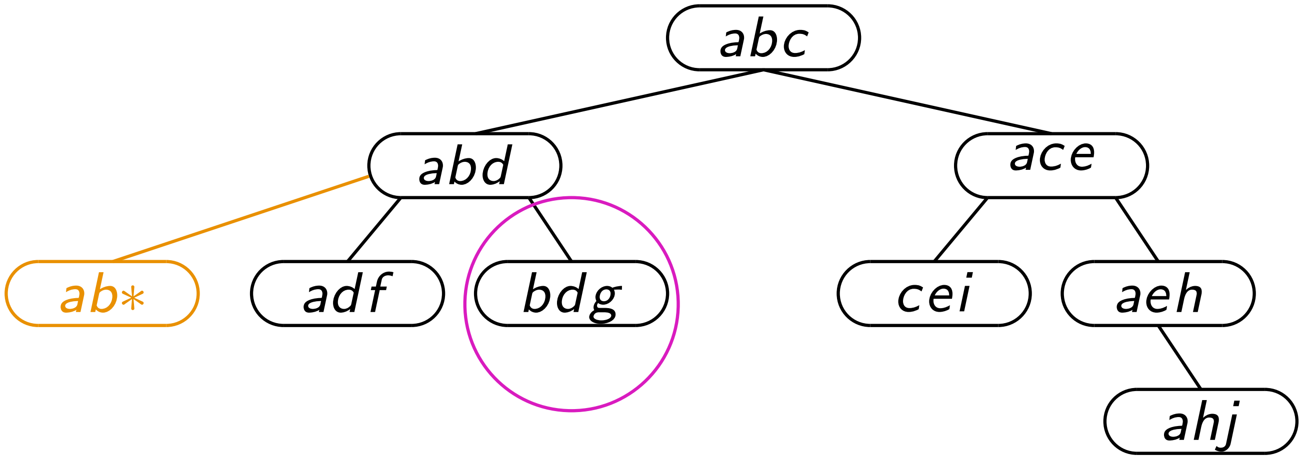

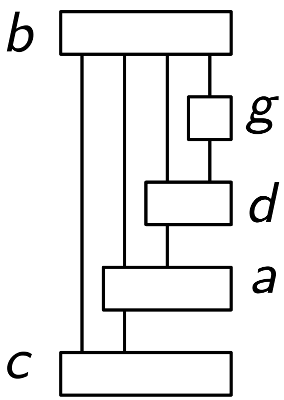

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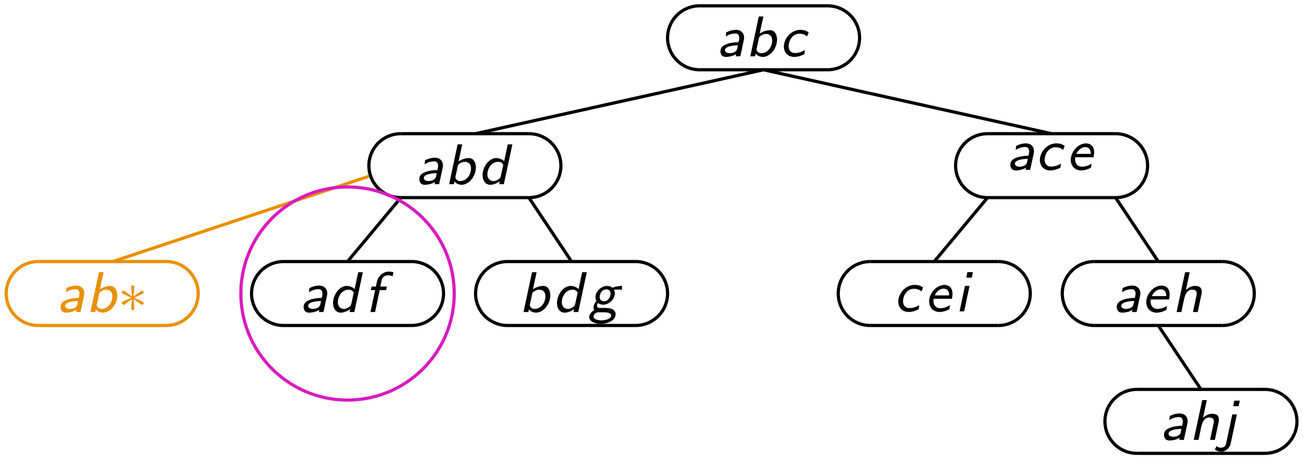

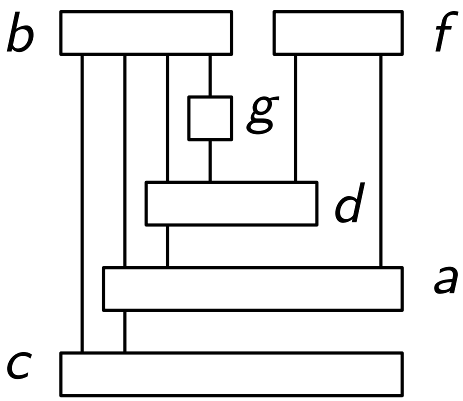

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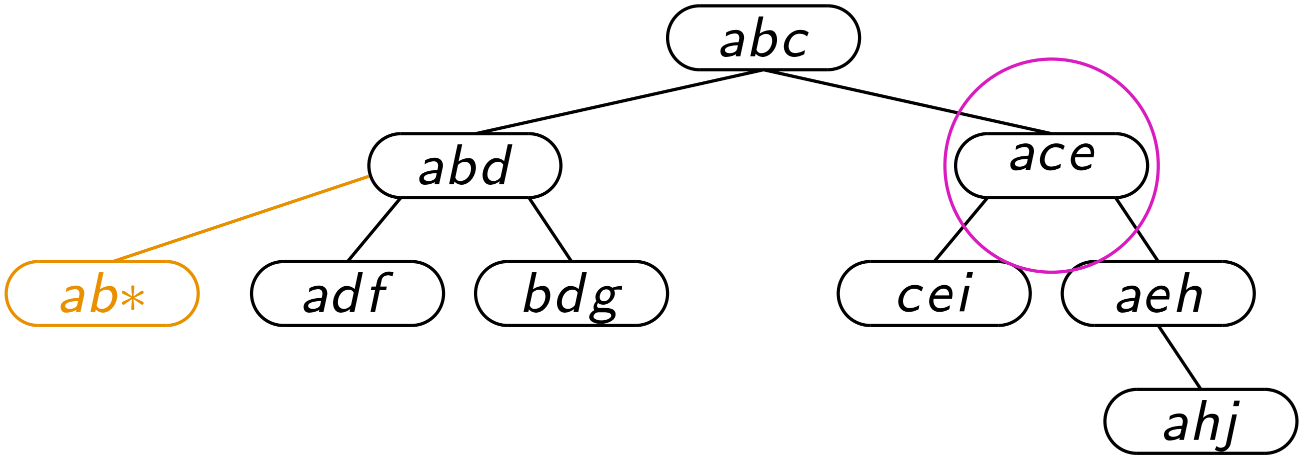

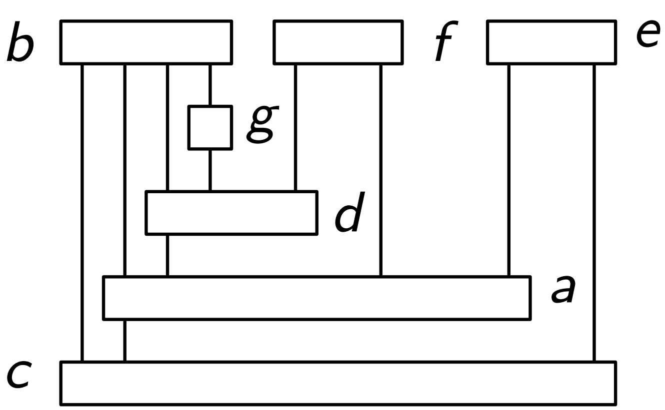

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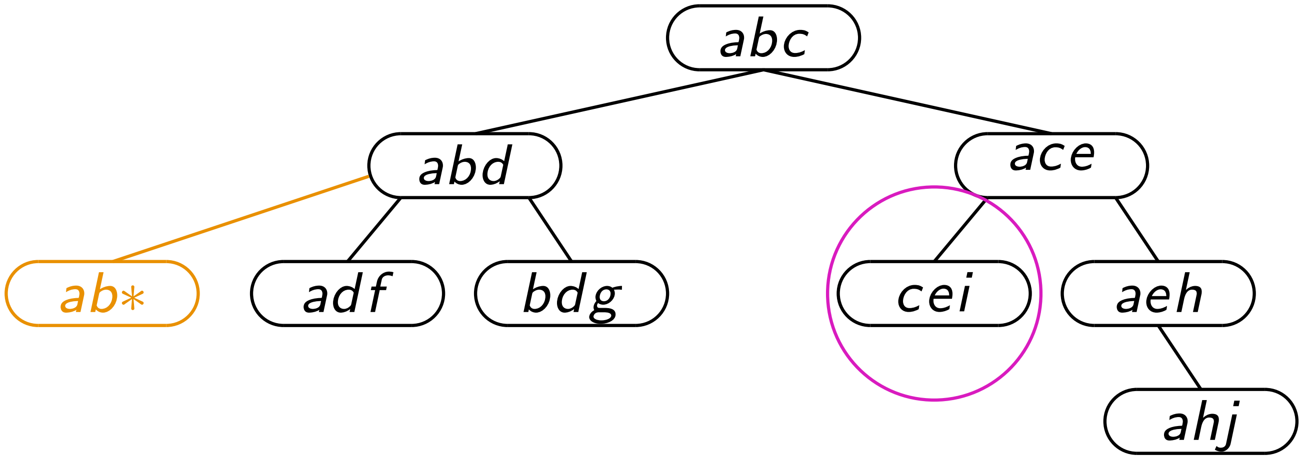

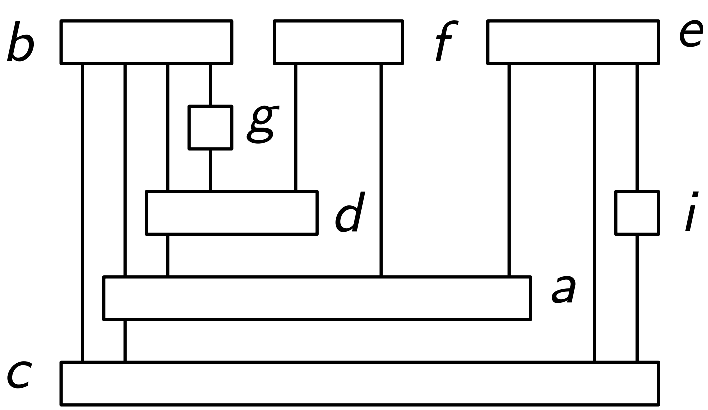

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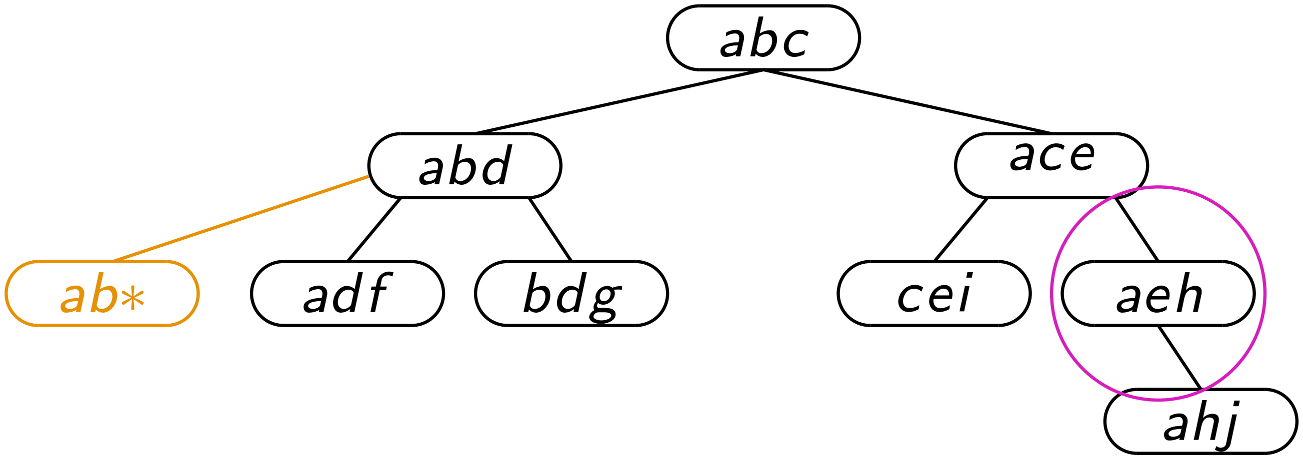

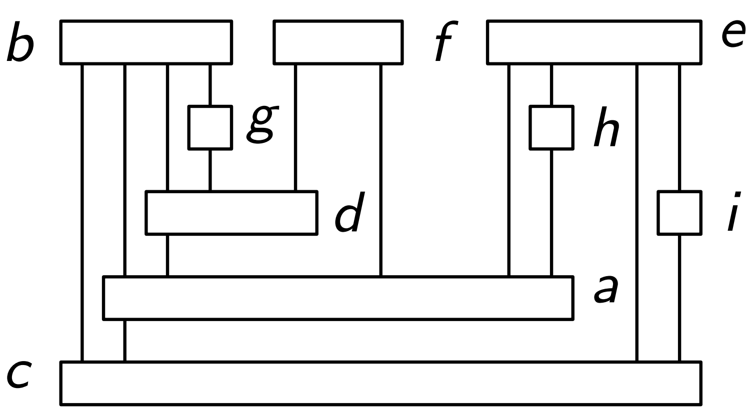

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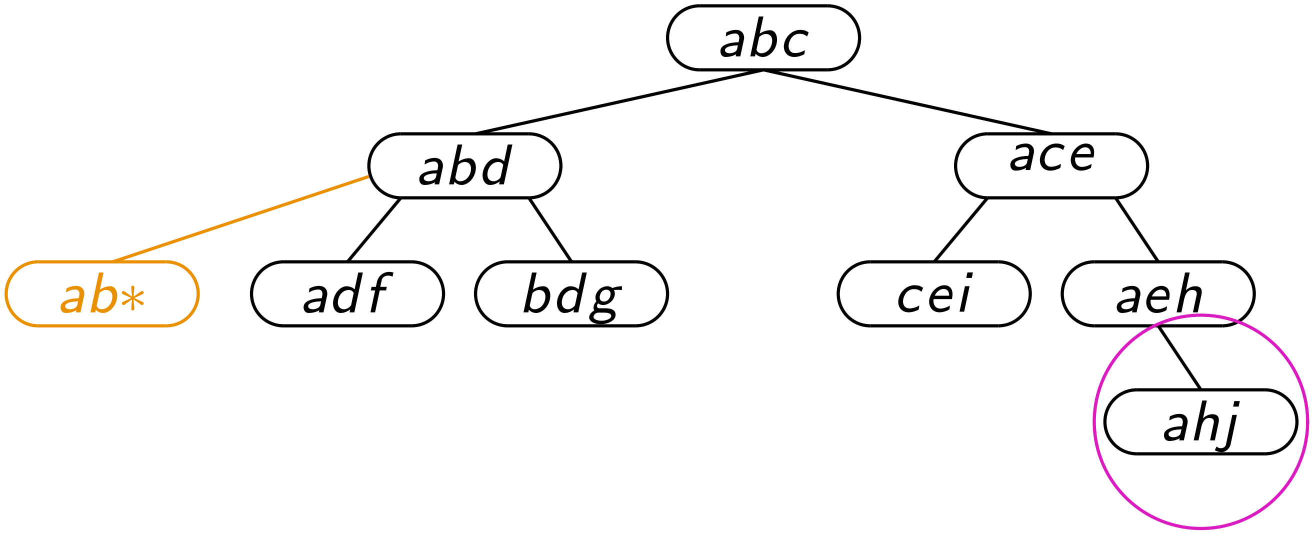

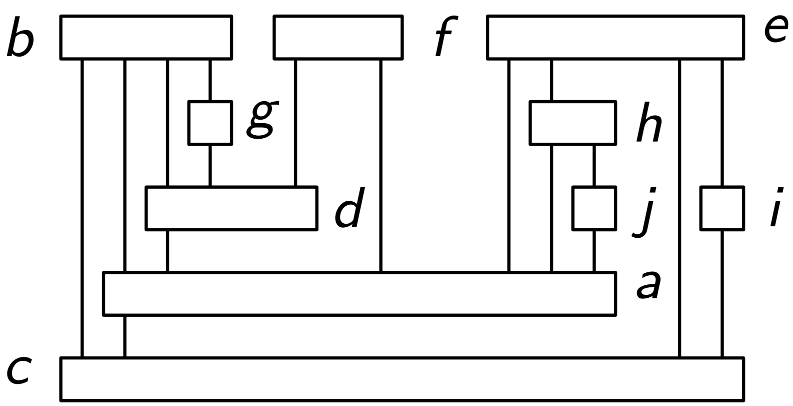

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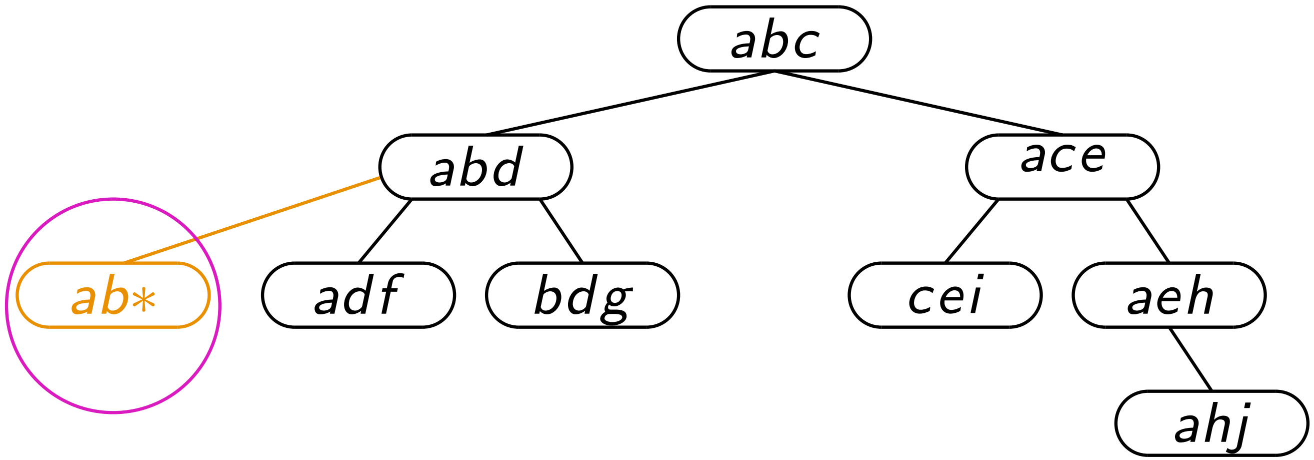

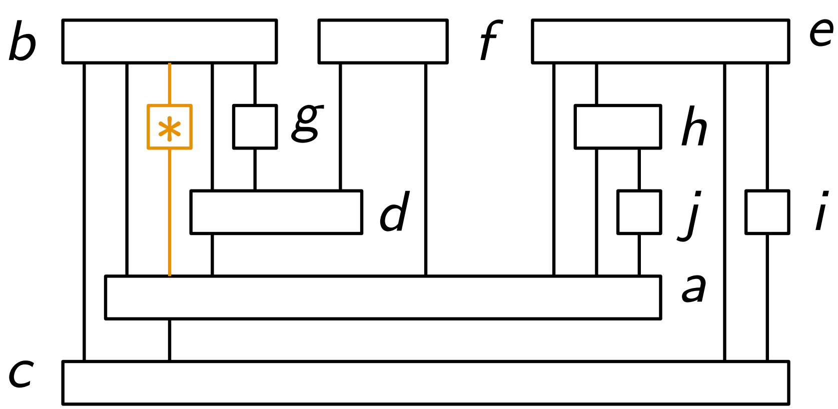

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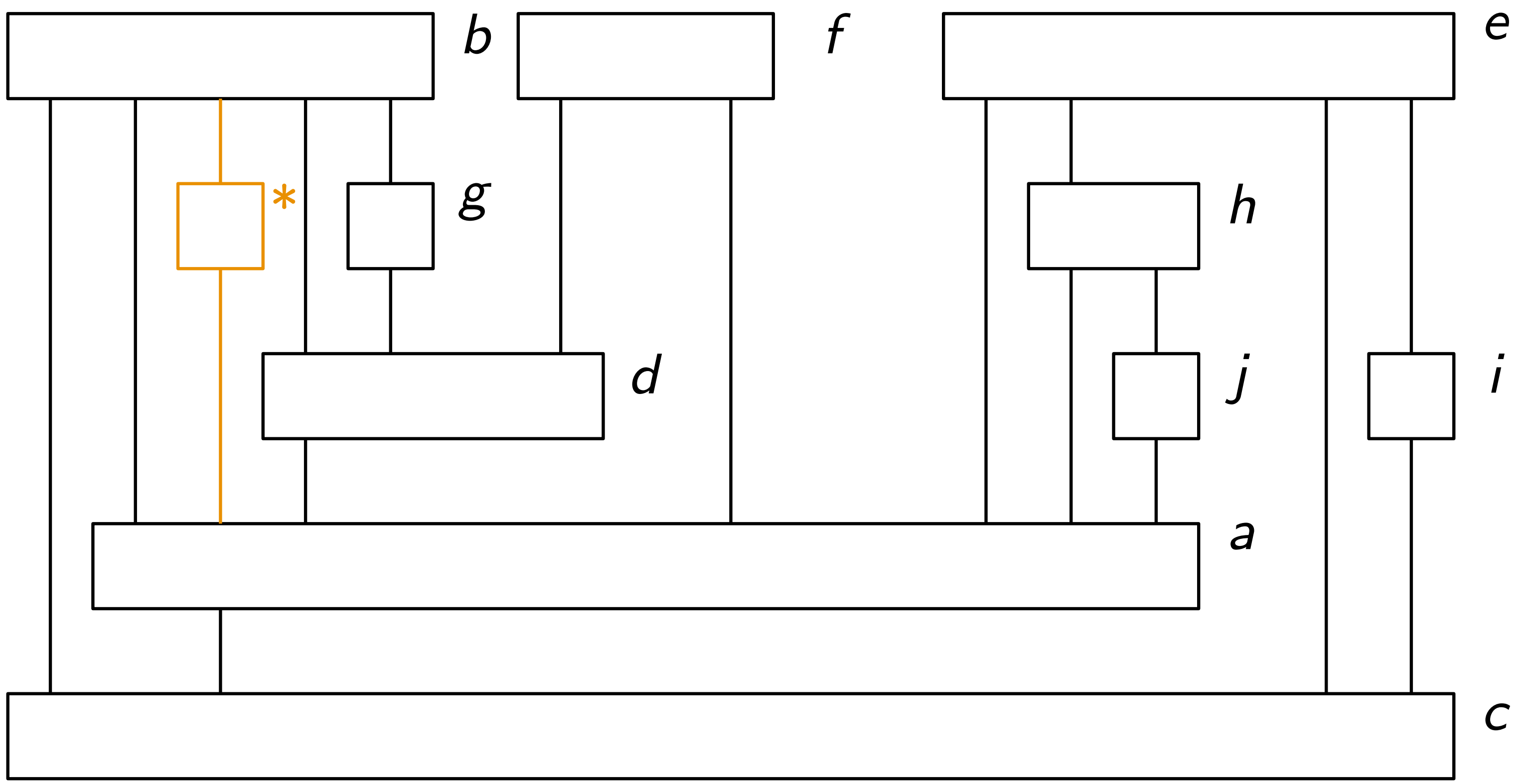

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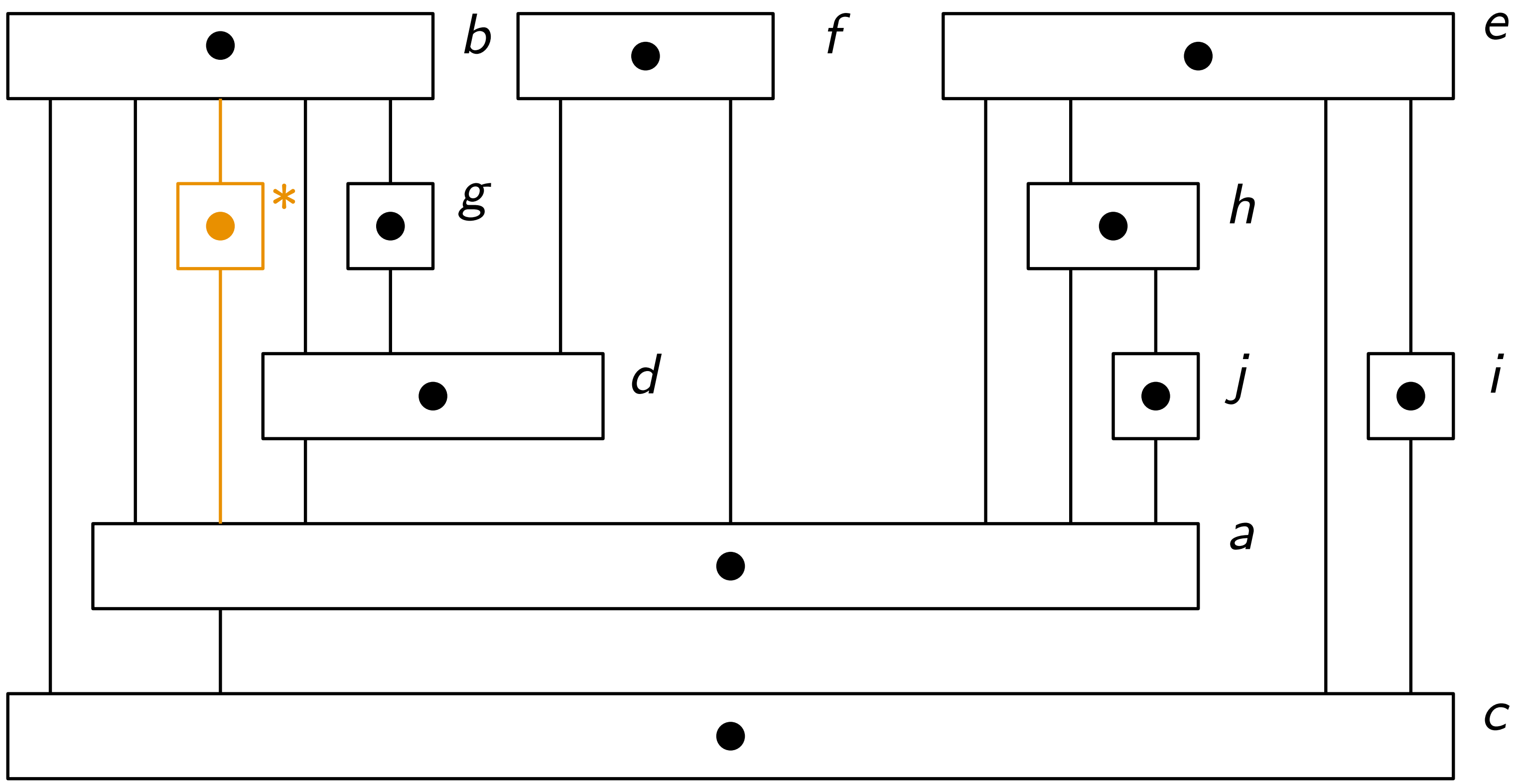

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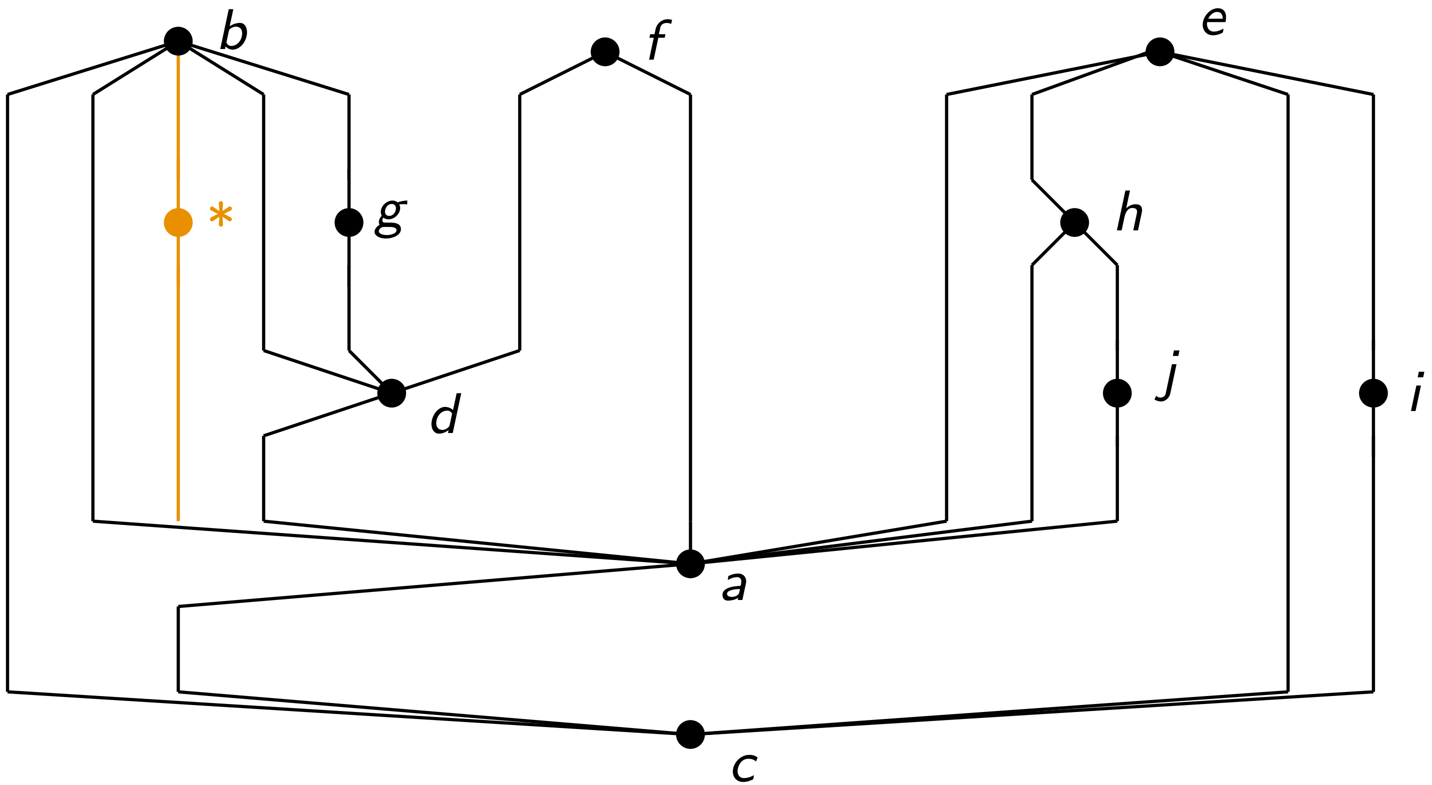

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Questions?

Thank you for participating!

:)

Planar 3-trees

Complete *k*-ary Tree Implementation

- Implemented in `python` $\geq$ `3.8`
- `networkx` library used for a graph structure representation
- `matplotlib` library used for plotting
- Coordinate computation can be found in thesis