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Two models of two-dimensional bandwidth problems ☆

Lan Lin [a,b], Yixun Lin [c,*]

[a] The Key Laboratory of Embedded System and Service Computing, Ministry of Education, Tongji University, Shanghai 200092, China
[b] School of Electronics and Information Engineering, Tongji University, Shanghai 200092, China
[c] Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China

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ABSTRACT

The two-dimensional bandwidth problem is to embed a graph G into an $n \times n$ grid in the plane such that the maximum distance between adjacent vertices is as small as possible. Here, the "distance" has two different meanings: the L_1-norm distance and L_∞-norm distance. So we have two models of two-dimensional bandwidth problem. This paper investigates the basic properties and relations of these two models. Some lower bounds, upper bounds, and exact results are presented.

1. Introduction

The bandwidth minimization problem of graphs and its variants have a wide range of applications. In particular, the following generalized bandwidth problem has significant background in circuit layout of VLSI designs and network communication [1,4,5]. Given a host graph H, a labelling (or embedding) of a guest graph G is an injection $f: V(G) \to V(H)$. The bandwidth of labelling f of G to H is

$$B_H(G, f) = \max_{uv \in E(G)} d_H\big(f(u), f(v)\big),$$

where $d_H(x, y)$ denotes the distance between x and y in graph H. The bandwidth of G (relative to H) is

$$B_H(G) = \min_f B_H(G, f),$$

where the minimum is over all labellings f. When H is a path P_n, this is the bandwidth $B(G)$ in the ordinary sense, which has been extensively studied in the literature (see surveys [3–6]). When H is a grid graph (mesh) $P_n \times P_n$, $B_H(G)$ is the *two-dimensional bandwidth* [4,5,7].

This paper considers the latter case. To make sure, let us formulate the problem more precisely as follows. The grid graph H has the $n \times n$ grid $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ as its vertex set, in which each vertex is a grid (lattice) point. In circuit layout models, the wires are usually placed along the vertical and horizontal directions. So the distance usually means the rectilinear distance, namely the L_1-norm distance. That is, the distance between two points $(i, j), (i', j') \in V(H)$ is

$$d_{L_1}\big((i, j), (i', j')\big) = |i - i'| + |j - j'|.$$

Then the two-dimensional bandwidth of a labelling f for G is

$$B_2(G, f) = \max_{uv \in E(G)} d_{L_1}\big(f(u), f(v)\big)$$

and the two-dimensional bandwidth of G is

$$B_2(G) = \min_f B_2(G, f).$$

A labelling f attaining the above minimum is called an *optimal labelling*.

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* Corresponding author.
E-mail address: linyixun@zzu.edu.cn (Y. Lin).

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It should be emphasized that the bandwidth embedding here is related to the VLSI layout, but to say rigorously, they are not the same thing. In fact, the edge routings in VLSI layout are not allowed to overlap from each other (with congestion 1) and people are mainly interested in some other objectives (such as layout area, expansion ratio) [1,10,12]. However, in the bandwidth problem mentioned above, we need not consider the edge routings and only the distances are taken into account.

As a graph labelling problem, Chung [4] first presented an introduction on $B_2(G)$ and especially proved the following *density lower bound*:

Theorem 1.1. *For any graph G on n vertices,*

$$B_2(G) \geqslant \left\lceil \frac{\sqrt{n}-1}{D(G)} \right\rceil,$$

where $D(G)$ stands for the diameter of G.

This is a generalization of the one-dimensional density lower bound [3,4]:

$$B(G) \geqslant \left\lceil \frac{n-1}{D(G)} \right\rceil.$$

This lower bound plays an important role in the study of $B(G)$, as many classes of graphs attain this bound, for example, paths P_n, cycles C_n, stars $K_{1,n-1}$, complete graphs K_n, and a number of trees. However, for two-dimensional bandwidth $B_2(G)$, the bound of Theorem 1.1 is attainable only by a few graphs, such as paths P_n and cycles C_n (with n being even). So it remains to be improved.

Motivated by other application backgrounds, we may consider the problem under the distance of L_∞-norm. For this, the distance between two points $(i, j), (i', j') \in V(H)$ is

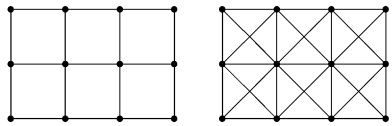
$$d_{L_\infty}\big((i,j),(i',j')\big) = \max\big\{|i-i'|, |j-j'|\big\}.$$

This distance does not mean the length of wire connecting points (i, j) and (i', j'), but may represent the "dilation" along the vertical and horizontal directions, which is meaningful in some interconnection network architectures. In fact, in a parallel computation system, the host graph H represents the parallel computer architecture on which the computation is to be performed. In general, the host graph H takes the form of grid graphs (or hypercubes). So we have the above definition of embedding with respect to the L_1-distance. However, in some systems (say, MasPar computer), one may consider the host graph of "extended grid graph", in which each interior node is allowed to communicate directly with eight nodes in its neighborhood (see, e.g., [10]). This gives rise to the distance of L_∞-norm. Based on this type of distance in the plane, we have another model of two-dimensional bandwidth problem: The bandwidth of labelling f for G is

$$B_2'(G, f) = \max_{uv \in E(G)} d_{L_\infty}\big(f(u), f(v)\big)$$

and the bandwidth of G is

$$B_2'(G) = \min_f B_2'(G, f).$$



(a) $P_3 \times P_4$ (b) $P_3 \otimes P_4$

Fig. 1. Grid graph and extended grid graph.

So we obtain two minimization problems of $B_2(G)$ and $B_2'(G)$. They have close relations. For instance, by $d_{L_\infty}(x, y) \leqslant d_{L_1}(x, y) \leqslant 2d_{L_\infty}(x, y)$, the following is obvious:

Proposition 1.2. $B_2'(G) \leqslant B_2(G) \leqslant 2B_2'(G)$.

It turns out that $B_2'(G)$ has better combinatorial properties and is easier to determined. So we can get benefit by studying them together.

The goal of this paper is to study two graph-theoretic invariants $B_2(G)$ and $B_2'(G)$ in a way to generalize results of one-dimensional bandwidth $B(G)$. In particular, we describe some basic properties by comparing these two models. The paper is organized as follows. In Section 2 we sketch preliminaries from a combinatorial geometry approach. In Section 3 some lower and upper bounds are presented. Section 4 is concerned with results of special graphs, including the one on caterpillars.

2. Geometry of grid points

From a geometrical point of view, we consider how to embed a graph G into the $n \times n$ grid $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. For convenience, the set $\{(i, j)\colon 1 \leqslant i \leqslant n\}$ is called the *j-th vertical line*; the set $\{(i, j)\colon 1 \leqslant j \leqslant n\}$ is called the *i-th horizontal line*.

We need the notion of graph product. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent if and only if either $[u = u'$ and $vv' \in E_2]$ or $[v = v'$ and $uu' \in E_1]$. Moreover, the *strong product* $G_1 \otimes G_2$ is the graph with vertex set $V_1 \times V_2$ and two vertices (u, v) and (u', v') are adjacent if and only if $[u = u'$ and $vv' \in E_2]$ or $[v = v'$ and $uu' \in E_1]$ or $[uu' \in E_1$ and $vv' \in E_2]$.

For example, $P_n \times P_n$ is the ordinary grid graph (see Fig. 1(a)). And we call $P_n \otimes P_n$ the *extended grid graph* in the plane (following the version of [10]), which can be obtained from $P_n \times P_n$ by adding two diagonal edges in each cell (see Fig. 1(b)). By definition, $B_2(G)$ is the bandwidth with host graph $H = P_n \times P_n$ and $B_2'(G)$ is the bandwidth with host graph $H = P_n \otimes P_n$. From this, we have the following observation.

Proposition 2.1. $B_2(G) = 1$ *if and only if G is a subgraph of* $P_n \times P_n$; $B_2'(G) = 1$ *if and only if G is a subgraph of* $P_n \otimes P_n$.

The problem of deciding whether $B_2(G) = 1$ is NP-complete even for a binary tree (see [2] and [4], Theorem 5.8). The problem of deciding $B_2'(G) = 1$ may be similar, but the exact complexity result is unknown yet.

Now, we consider the diameter of a set of grid points. We first observe the case of rectilinear distance (L_1-distance). For a finite set S in the plane, the L_1-*diameter* of S is

$d_{L_1}(S) = \max_{x,y\in S} d_{L_1}(x, y)$.

The following geometrical fact was presented in [7]. Here is a short proof.

Proposition 2.2. *The minimum L_1-diameter of a set S with n grid points in the plane is*

$$\delta(n) = \min\left\{2\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil, 2\left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1\right\}.$$

Proof. Suppose that $|S| = n$ and $d_{L_1}(S) = d$. We distinguish two cases.

Case 1: $d = 2r$. By making suitable translation and rotation on the plane, S can be put into a set

$$D = \left\{(x, y) \in \mathbf{Z}^2\colon\ |x| + |y| \leqslant r\right\}.$$

A simple calculation gives $|D| = r^2 + (r+1)^2$. By $|D| = r^2 + (r+1)^2 \geqslant n$, we have $r \geqslant (\sqrt{2n-1} - 1)/2$. So

$$d \geqslant 2\left\lceil \frac{\sqrt{2n-1}-1}{2} \right\rceil.$$

Case 2: $d = 2r + 1$. By making suitable translation and rotation on the plane, S can be put into a set

$$D = \left\{(x, y) \in \mathbf{Z}^2\colon\ |x| + |y| \leqslant r + 1 \text{ for } x > 0,\ |x| + |y| \leqslant r \text{ for } x \leqslant 0\right\}.$$

It can be easily computed that $|D| = 2(r + 1)^2$. By $|D| = 2(r + 1)^2 \geqslant n$, we have $r \geqslant \sqrt{n/2} - 1$. So

$$d \geqslant 2\left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1.$$

Combining above two cases gives $d \geqslant \delta(n)$. □

We next consider the L_∞-distance. For a finite set S in the plane, the L_∞-*diameter* of S is

$d_{L_\infty}(S) = \max_{x,y\in S} d_{L_\infty}(x, y)$.

The following is the counterpart of Proposition 2.2.

Proposition 2.3. *The minimum L_∞-diameter of a set S with n grid points in the plane is*

$\delta'(n) = \lceil \sqrt{n} - 1 \rceil$.

Proof. Suppose that $|S| = n$ and $d_{L_\infty}(S) = d$. By making suitable translation on the plane, S can be put into a square

$$D = \left\{(x, y) \in \mathbf{Z}^2\colon\ 0 \leqslant x \leqslant d,\ 0 \leqslant y \leqslant d\right\}.$$

It is obvious that $|D| = (d + 1)^2$. By $|D| = (d + 1)^2 \geqslant n$, we have $d \geqslant \sqrt{n} - 1$, as required. □

3. Lower and upper bounds

We first state an improvement of the density lower bound of Theorem 1.1.

Theorem 3.1. *For any graph G with n vertices and diameter $D(G)$,*

$$B_2(G) \geqslant \max\left\{B'_2(G), \left\lceil \frac{\delta(n)}{D(G)} \right\rceil\right\} \geqslant B'_2(G) \geqslant \left\lceil \frac{\sqrt{n}-1}{D(G)} \right\rceil.$$

Proof. We show the first inequality. $B_2(G) \geqslant B'_2(G)$ has been known in Proposition 1.2. Moreover, for any labelling f of G which embeds G into n grid points, there must be two vertices $u, v \in V(G)$ such that $d_{L_1}(f(u), f(v)) \geqslant \delta(n)$ (by Proposition 2.2). Then $B_2(G, f)D(G) \geqslant B_2(G, f)d_G(u, v) \geqslant d_{L_1}(f(u), f(v)) \geqslant \delta(n)$. So $B_2(G, f) \geqslant \lceil \delta(n)/D(G) \rceil$ for any labelling f, as required. The second inequality is trivial. For the third inequality, we need only replace the L_1-distance by the L_∞-distance and replace $\delta(n)$ by $\delta'(n) = \lceil \sqrt{n} - 1 \rceil$ in the previous proof. □

In this theorem, we combine the results of [8] and [9] and give a short proof. These two bounds are tight for a number of special graphs (see the next section). The following are some upper bounds.

Proposition 3.2. *For any graph G with n vertices, $B_2(G) \leqslant \delta(n)$, $B'_2(G) \leqslant \lceil \sqrt{n} - 1 \rceil$.*

Proof. We need only embed $V(G)$ in a set $S \subseteq \mathbf{Z}^2$ of minimum diameter. □

Proposition 3.3. *For any graph G with n vertices,*

$$\left\lceil \frac{n-1}{D(G)} \right\rceil \leqslant B'_2(G \otimes G) \leqslant B(G).$$

Proof. Since $G \otimes G$ has n^2 vertices and has diameter $D(G \otimes G) = D(G)$, the first inequality is due to Theorem 3.1. We next show the second inequality. Let f be an optimal labelling of $B(G)$. We construct a labelling f^* of $G \otimes G$ as follows. We embed $V(G \otimes G)$ on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and in each vertical line or horizontal line, we make a copy of G in the way of embedding f. It is easy to check that all vertical edges, horizontal edges, and diagonal edges of this embedding have maximum length $B(G)$. So $B'_2(G \otimes G) \leqslant B'_2(G \otimes G, f^*) \leqslant B(G)$, proving the result. □

As said before, the one-dimensional density lower bound is sharp for many special graphs. So both inequalities in Proposition 3.3 hold with equalities for these graphs, say we have $B'_2(C_n \otimes C_n) = 2$, $B'_2(K_{1,n} \otimes K_{1,n}) = \lceil n/2 \rceil$, $B'_2(K_n \otimes K_n) = n - 1$.

4. Special graphs

By the above lower and upper bounds, the following results of special graphs are straightforward.

Proposition 4.1.

(1) *For complete graph* K_n, $B_2(K_n) = \delta(n)$, $B'_2(K_n) = \lceil \sqrt{n} - 1 \rceil$.
(2) *For star* $K_{1,n}$, $B_2(K_{1,n}) = \lceil \delta(n+1)/2 \rceil$, $B'_2(K_{1,n}) = \lceil (\sqrt{n+1} - 1)/2 \rceil$.
(3) *For Cartesian product* $P_m \times C_n$, $B_2(P_m \times C_n) = 2$, $B'_2(P_m \times C_n) = 1$.
(4) *For Cartesian product* $C_m \times C_n$, $B_2(C_m \times C_n) = B'_2(C_m \times C_n) = 2$.

Now we proceed to study an important class of trees. A caterpillar T is a tree which yields a path $p(T)$ (called *spine*) when all its pendant vertices are removed. The one-dimensional bandwidth problem for caterpillars was solved by Syslo and Zak [11]. Their result is as follows. Let $p(T) = (w_1, w_2, \ldots, w_m)$ and let T_{ij} denote the subtree of T consisting of $w_i, w_{i+1}, \ldots, w_{j-1}, w_j$ and all their neighbors. Then the bandwidth is $B(T) = \max_{i \leqslant j} \lceil \frac{|V(T_{ij})|-1}{j-i+2} \rceil$. Now we consider the two-dimensional bandwidth $B'_2(T)$. We first observe the case of $m = 2$, that is, the so-called double stars (or trees of diameter 3). Let n_i be the number of neighbors of w_i $(i = 1, 2)$. Then T is formed by two stars $T_1 = K_{1,n_1}$ and $T_2 = K_{1,n_2}$ with edge $w_1 w_2$ in common, where $n = |V(T)| = n_1 + n_2$. In the sequel, we need the notion of neighborhood with respect to the L_∞-distance. We call the set of grid points

$$S(v,r) = \{x \in \mathbf{Z}^2 \colon d_{L_\infty}(v,x) \leqslant r\}$$
$$= \{(x_1, x_2) \in \mathbf{Z}^2 \colon |x_1 - v_1| \leqslant r,\ |x_2 - v_2| \leqslant r\}$$

the *square neighborhood* (or just *square* in short) of radius r centered at $v = (v_1, v_2)$.

Proposition 4.2. *For a double star* $T = K_{1,n_1} \cup K_{1,n_2}$ *with* $n = |V(T)| = n_1 + n_2$ *and* $n_1 \geqslant n_2$,

$$B'_2(T) = \max\left\{ \left\lceil \frac{\sqrt{n_1+1} - 1}{2} \right\rceil, \left\lceil \frac{\sqrt{7n+2} - 3}{7} \right\rceil \right\}.$$

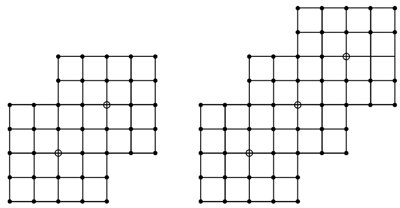
Proof. We first show the lower bound. Since K_{1,n_1} is a subgraph of T, we have $B'_2(T) \geqslant B'_2(K_{1,n_1}) = \lceil (\sqrt{n_1+1} - 1)/2 \rceil$. On the other hand, let f be an optimal embedding of T with $r = B'_2(T)$. Then T_i is embedded in a square $S_i = S(f(w_i), r)$ with $i = 1, 2$, and T is embedded in $S_1 \cup S_2$ in which the L_∞-distance between $f(w_1)$ and $f(w_2)$ is at most r. Observe that $|S_1 \cup S_2|$ is maximum (i.e., $|S_1 \cap S_2|$ is minimum) if and only if the L_1-distance of two centers $f(w_1)$ and $f(w_2)$ is as large as possible (see the 2-square in Fig. 2(a)). So the maximum value of $|S_1 \cup S_2|$ is $2(2r+1)^2 - (r+1)^2$. Therefore

$$n \leqslant |S_1 \cup S_2| \leqslant 2(2r+1)^2 - (r+1)^2$$
$$= 7r^2 + 6r + 1.$$

Hence

$$r \geqslant \left\lceil \frac{\sqrt{7n+2} - 3}{7} \right\rceil,$$

proving the lower bound.



(a) 2-square (b) 3-square

Fig. 2. String of squares $Q(r,k)$.

Conversely, let r^* denote the right-hand side of the equality in the proposition. We construct two squares $S_1 = S(w'_1, r^*)$ and $S_2 = S(w'_2, r^*)$ such that w'_1 and w'_2 have distance r^* in both coordinate directions (as the 2-square in Fig. 2(a)). Then we make an embedding f^* of T into these two squares with w_i at w'_i $(i = 1, 2)$ and each star T_i in each square S_i. By the definition of r^*, each square S_i has enough room to hold star T_i and $S_1 \cup S_2$ to hold the double star T. Therefore $B'_2(T, f^*) \leqslant r^*$. This completes the proof. □

In order to generalize this result to m-section caterpillars, we define a string of squares as follows. For a given integer $r > 0$, take k grid points in the plane $v_0 = (0,0)$, $v_1 = (r,r)$, $\ldots$, $v_{k-1} = ((k-1)r, (k-1)r)$. The union of k squares

$$Q(r,k) = S(v_0, r) \cup S(v_1, r) \cup \cdots \cup S(v_{k-1}, r)$$

is called a *k-square*. For example, a 2-square and a 3-square with $r = 2$ are shown in Fig. 2. By simple calculation, we have

$$|Q(r,k)| = k(2r+1)^2 - (k-1)(r+1)^2$$
$$= (3k+1)r^2 + (2k+2)r + 1.$$

Proposition 4.3. *Suppose that* T *is a caterpillar with spine path* $p(T) = (w_1, w_2, \ldots, w_m)$. *Let* T_{ij} *denote the subtree of* T *consisting of* $w_i, w_{i+1}, \ldots, w_{j-1}, w_j$ *and all their neighbors* $(i \leqslant j)$ *and* $n_{ij} = |V(T_{ij})|$. *Then*

$$B'_2(T)$$
$$= \max_{i \leqslant j} \left\lceil \frac{\sqrt{(3(j-i+1)+1)n_{ij} + (j-i)(j-i+1)}}{3(j-i+1)+1} - \frac{(j-i+2)}{3(j-i+1)+1} \right\rceil.$$

Proof. We first show the lower bound. Let f be an optimal embedding of T with $r = B'_2(T)$. Then for any $i \leqslant j$, T_{ij} is embedded in a union of $j-i+1$ squares $S_i \cup S_{i+1} \cup \cdots \cup S_j$, where the star T_α with center w_α is embedded in a square $S_\alpha = S(f(w_\alpha), r)$ $(i \leqslant \alpha \leqslant j)$. Note that $|S_i \cup S_{i+1} \cup \cdots \cup S_j|$ is no more than the cardinality of a $(j-i+1)$-square $Q(r, j-i+1)$. So

$$n_{ij} \leqslant |Q(r, j-i+1)|$$
$$= \big(3(j-i+1)+1\big)r^2 + \big(2(j-i+1)+2\big)r + 1.$$

Hence

$$r \geqslant \left\lceil \frac{\sqrt{(3(j-i+1)+1)n_{ij}+(j-i)(j-i+1)}}{3(j-i+1)+1} - \frac{(j-i+2)}{3(j-i+1)+1} \right\rceil.$$

By the arbitrariness of i and j, we have the required lower bound.

Conversely, let r^* denote the right-hand side of the equality in the proposition. We construct an m-square $Q(r^*, m)$ and then define an embedding f^* of T into these m squares by putting each star T_i in each square S_i for $i = 1, 2, \ldots, m$. We claim that this procedure works correctly. If not, then at some step j, S_j had no enough room to hold T_j. So we could backtrack to the first full square S_i and get a subtree T_{ij}. For this T_{ij} we have $n_{ij} = |V(T_{ij})| > |Q(r^*, j-i+1)|$, contradicting the definition of r^*. Hence this procedure indeed provides an embedding with $B'_2(T, f^*) \leqslant r^*$. This completes the proof. □

It is hard to obtain explicit formula of $B_2(T)$ for caterpillars. However, relative to $B'_2(T)$, it may increase by at most a factor of 2.

Finally, we mention some asymptotic results.

- If G is a planar graph, then $B_2(G) = O(\sqrt{n}\log n/\log\log n)$ (Theorem 5.7 of [4]). This evaluation is based on the VLSI layout. However, we can do better by means of bandwidth embedding, namely $B_2(G) = O(\sqrt{n})$ in Proposition 3.2.
- If $T_{2,k}$ is a complete binary tree with $n = 2^{k+1} - 1$ vertices, then $B_2(T_{2,k}) = \Theta(\sqrt{n}/\log n)$ (Theorem 5.5 of [4]). In fact, $D(T_{2,k}) = 2k = O(\log n)$. The lower bound can be obtained by our Theorem 3.1. And the upper bound can be obtained easier by using bandwidth embedding instead of VLSI layout.

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