Calculus I Notes

MATH 1190

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Contents

Introduction

What is this?

This is a project to keep all of my notes for my Calculus I class in a nice PDF file.

About this project

This PDF is created using a combination of LaTeX and Wolfram Mathematica. All code for this project can be found on its GitHub page. This project is intended to be built on a Windows computer because it relies on running a `.bat` file at compile time.

Also, if any errors are found in the PDF please create an issue on this project's GitHub page here. This includes any errors in the math or spelling or grammar errors as my editor that I use for this project does not have spell check.

Pull requests and controbutions are wlecome.

Chapter 2

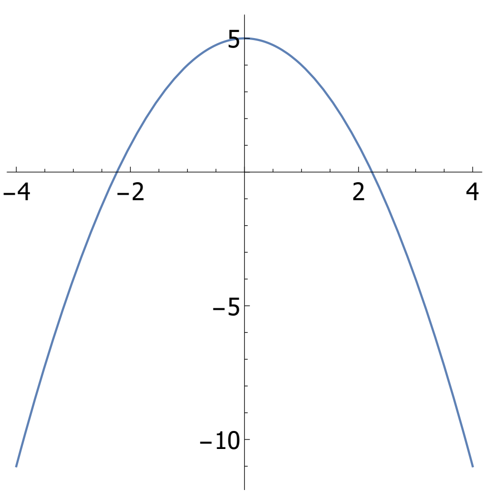
The Derivative

2.1 Rates of Change and The Derivative

A particle's rectilinear (1D) motion has its position defined by the function $s(t) = 5 - t^2$, where s is measured in meters and t in seconds.

1. Sketch the graph of the function on the interval from $t = 0$ to $t = 4$.
2. Find the *average* velocity over the time over the time interval from $t = 0$ to $t = 4$. On your graph, draw what this quantity represents.
3. Approximate the *instantaneous* velocity when $t = 2$ by finding the average velocity over the intervals $t = 2$ to $t = 3$, $t = 2$ to $t = 2.5$, and $t = 2$ to $t = 2.1$.
4. Write a general expression that represents the average velocity over the time interval from $t = 2$ to $t = 2 + h$.
5. Find the instantaneous velocity when $t = 2$ by finding the limit of the above expression as $h \to 0$.

Answers:

1. 

2.

$$\frac{\Delta s}{\Delta t}$$
$$\frac{s(4)-s(0)}{4-0}$$
$$\frac{(5-4^2)-(5-0^2)}{4}$$
$$\frac{-11-5}{4}$$
$$-4[m/s]$$

3.
- $t=2$ to $t=3$:

$$\frac{s(3)-s(2)}{3-2}$$
$$\frac{(5-3^2)-(5-2^2)}{1}$$
$$-5[m/s]$$

- $t=2$ to $t=2.5$:

$$\frac{s(2.5)-s(2)}{3-2}$$
$$\frac{(5-2.5^2)-(5-2^2)}{0.5}$$
$$-4.5[m/s]$$

- $t=2$ to $t=2.1$:

$$\frac{s(2.1)-s(2)}{3-2}$$
$$\frac{(5-2.1^2)-(5-2^2)}{0.1}$$
$$-4.1[m/s]$$

Guess: velocity <u>at</u> $t=2$ is approximately $4[m/s]$.

4.

$$\frac{s(2+h)-s(2)}{2+h-2}$$
$$\frac{\left(5-(2+h)^2\right)-(5-2^2)}{h}$$
$$\frac{5-(4+4h+h^2)-(1)}{h}$$
$$\frac{-4h-h^2}{h}$$
$$-4-h$$

5.

$$\lim_{h\to 0}\left(\frac{s(2+h)-s(2)}{h}\right)$$
$$\lim_{h\to 0}(-4-h)$$
$$-4-0$$
$$-4$$

2.1.1 Definitions

The slope of a curve can be found using the following equations:

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) \tag{2.1}$$

$$\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right) \tag{2.2}$$

These are also known as:

- The *instantaneous* velocity of an object at time c whose position is given by the function $f(x)$.
- The *slope of the tangent line* to the curve $y = f(x)$ at $x = c$.
- The *instantaneous rate of change* of the function $f(x)$ at $x = c$.
- The *derivative* of f at c.
- $f'(c)$

Example 2.1.0.1. Find the slope of the line tangent to $y = \dfrac{1}{x+5}$ when $x = 1$. Then find the equation for the tangent line at that point.

$$\lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{\dfrac{1}{1+h+5}-\dfrac{1}{1+5}}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{\left(\dfrac{6}{6}\cdot\dfrac{1}{1+h+5}\right)-\left(\dfrac{1}{1+5}\cdot\dfrac{6+h}{6+h}\right)}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{\dfrac{6}{6\,(6+h)}-\dfrac{6+h}{6\,(6+h)}}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{\dfrac{-h}{6\,(6+h)}}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{1}{\cancel{h}}\cdot\frac{-\cancel{h}}{6\,(6+h)}\right)$$
$$\lim_{h\to 0}\left(\frac{-1}{6\,(6+h)}\right)$$
$$\frac{-1}{36}$$

Equation of line: We have the slope, all we need is a point (substitute 1 in for x).

$$y = \frac{1}{1+6}$$
$$y = \frac{1}{6}$$

So the point is $(1, \frac{1}{6})$. Equation:

$$y - \frac{1}{6} = -\frac{1}{36}\,(x-1)$$
$$y = -\frac{1}{36}x + \frac{1}{36} + \frac{1}{6}$$
$$y = -\frac{1}{36}x + \frac{7}{36}$$

2.2 The Derivative of a Function

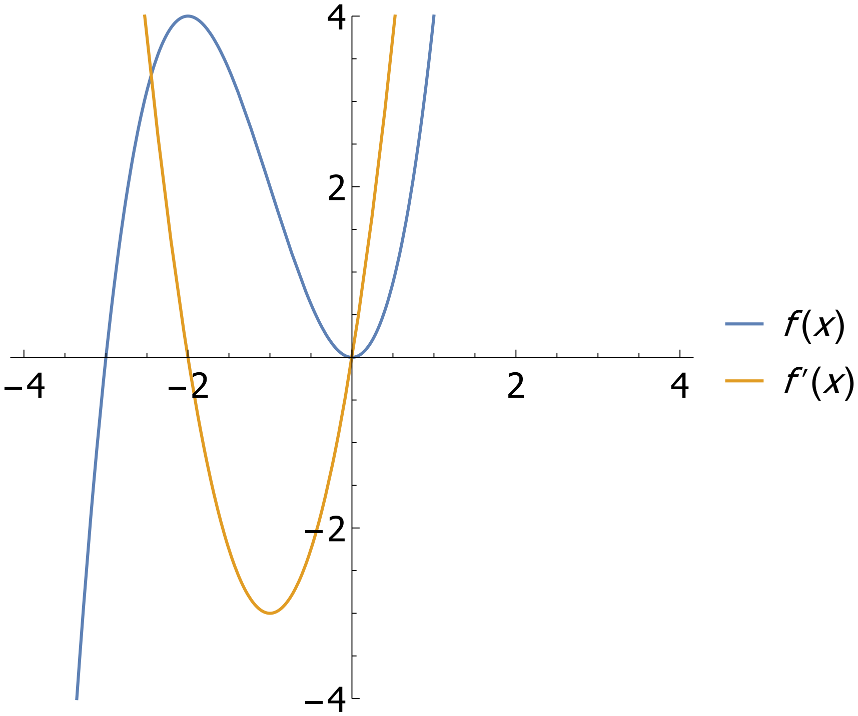
Theorem 2.2.1 (Derivative)**.** *The derivative of f is the function*

$$f'(x) = \lim_{h \to 0} \left(\frac{f\,(x+h) - f(x)}{h} \right) \tag{2.3}$$

Remark. This is only true if the limit exists.

Corollary 2.2.1.1. *If the limit does exist at $x = c$ then f is differentiable at c*

Corollary 2.2.1.2. *If the limit exists at every point in interval $[a, b]$ then f is differentiable on $[a, b]$.*



Example 2.2.1.1. If $f(x) = x^2 + 2x + 1$ find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \left(\frac{f\,(x+h) - f(x)}{h} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\left((x+h)^2 + 2\,(x+h) + 1\right) - (x^2 + 2x + 1)}{h} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h + \cancel{1} - \cancel{x^2} - \cancel{2x} - \cancel{1}}{h} \right)$$

$$f'(x) = \lim_{h \to 0} \left(\frac{2xh + h^2 + 2h}{h} \right)$$
$$f'(x) = \lim_{h \to 0} (2x + h + 2)$$
$$f'(x) = 2x + 2$$

Example 2.2.1.2. Let $f(x) = \sqrt{x}$. Find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \right)$$
$$f'(x) = \lim_{h \to 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}} \right)$$
$$f'(x) = \frac{1}{\sqrt{x+0} + \sqrt{x}}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$

Corollary 2.2.1.3. *Functions will fail to be differentiable at*

- *Cusps*
- *Corners*
- *Vertical Tangents*
- *Any point where it is discontinuous*

Lemma 2.2.2. *If f is differentiable at $x = c$ then f is continuous at c. However a function can be continuous but not differentiable (e.g. $y = |x|$ at $x = 0$).*

2.3 The Derivative of Polynomial Functions and $y = e^x$

Recall that if $f(x) = x^2 + 2x + 1$, then $f'(x) = 2x + 2$. We could write this in different ways.

- If $y = x^2 + 2x + 1$ then $y' = 2x + 2$.
- If $y = x^2 + 2x + 1$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2$.
- If $y = x^2 + 2x + 1$ then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 2x + 1\right) = 2x + 2$.

Remark. The last one, $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is an instruction to take a derivative of what comes after it.

Theorem 2.3.1 (Derivative of a Constant)**.** *If A is a constant and $f(x) = A$ then $f'(x) = 0$.*

Theorem 2.3.2 (Derivative of a line with a slope of 1)**.** *If $f(x) = x$ then $f'(x) = 1$.*

Review 2.3.1. *Use the definition of a derivative to find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)$.*

$$\lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{x^2 + h^2 + 2xh - x^2}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{h^2 + 2xh}{h}\right)$$
$$\lim_{h\to 0}(h + 2x)$$
$$0 + 2x$$
$$2x$$

2.3.1 Basic Rules

Power Rule

Theorem 2.3.3 (Power Rule)**.** *If $n \geq 1$ is an integer, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1} \tag{2.4}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Cx^n\right) = (C \cdot n)x^{n-1} \qquad \textit{If } C \textit{ is a constant.} \tag{2.5}$$

Review 2.3.2. *If $y = 5x^2$ find y' using the definition of a derivative.*

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\left(\frac{5\,(x+h)^2 - 5x^2}{h}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{5\left((x+h)^2 - x^2\right)}{h}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{5\,(x^2 + h^2 + 2xh - x^2)}{h}\right)\\
f'(x) &= 5\lim_{h\to 0}\left(\frac{x^2 + h^2 + 2xh - x^2}{h}\right)\\
f'(x) &= 5\lim_{h\to 0}\left(\frac{h^2 + 2xh}{h}\right)\\
f'(x) &= 5\lim_{h\to 0}(h + 2x)\\
f'(x) &= 5\cdot 2x\\
f'(x) &= 10x
\end{aligned}
$$

Constant Multiplication Rule

Theorem 2.3.4 (Constant Multiplication Rule)**.** *Suppose $F(x) = k\cdot f(x)$ for some real number k if $f(x)$ is differentiable then $F(x)$ is also differentiable, and*

$$F'(x) = k\cdot f'(x) \tag{2.6}$$

Example 2.3.4.1. If $f(x) = \pi x^7$ find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \pi\frac{\mathrm{d}}{\mathrm{d}x}\left(x^7\right)\\
f'(x) &= \pi\cdot\left(7x^6\right)\\
f'(x) &= 7\pi x^6
\end{aligned}
$$

Addition Rule

Theorem 2.3.5 (Addition Rule)**.** *If $F(x) = f(x) + g(x)$ and f and g are differentiable then $F(x)$ is also differentiable.*

$$F'(x) = f'(x) + g'(x) \tag{2.7}$$

Example 2.3.5.1. If $y = 3x^5 - 7x^2 - \frac{1}{2}x + 5$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned}
y' &= 3 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right) - 7 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(5) \\
y' &= 3 \cdot 5x^4 - 14x - \frac{1}{2} \cdot 1 + 0 \\
y' &= 15x^4 - 14x - \frac{1}{2}
\end{aligned}$$

2.3.2 Derivative of $f(x) = e^x$

Theorem 2.3.6 (Derivative of $f(x) = e^x$)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x \tag{2.8}$$

If $f(x) = e^x$, then $f'(x) = e^x$.

2.4 Product Rule, Quotient Rule, and Higher Order Derivatives

2.4.1 Basic Rules Continued

Product Rule

Theorem 2.4.1 (Product Rule)**.** *If f and g are differentiable then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) \cdot g(x) \tag{2.9}$$

Restated:

$$(f \cdot g)' = f \cdot g' + f' \cdot g \tag{2.10}$$

Example 2.4.1.1. If $f(x) = e^x x^4$ find $f'(x)$.

$$\begin{aligned}
f'(x) &= e^x \cdot 4x^3 + x^4 \cdot e^x \\
f'(x) &= 4e^x x^3 + e^x x^4 \\
f'(x) &= e^x x^3 (4 + x)
\end{aligned}$$

Example 2.4.1.2. If $y = 4(x^2 - 7)$, find y'.

$$\begin{aligned}
f'(x) &= 4x^2 - 28 \\
f'(x) &= 8x
\end{aligned}$$

Quotient Rule

Theorem 2.4.2 (Quotient Rule)**.** *If f and g are differentiable at x and $g(x) \neq 0$ then $\dfrac{f}{g}$ is differentiable at x and*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) - f(x)\dfrac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)}{g(x)^2} \tag{2.11}$$

Restated:

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2} \tag{2.12}$$

Remark. The order of the quotient rule can be remembered with the rhyme "lo d hi minus hi d lo all over the square of what's below".

Example 2.4.2.1. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{3x^3 - 5x}{5e^x + 2}\right)$

$$\begin{aligned} f' &= 9x^2 - 7 \\ g' &= 5e^x \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^3 - 5x}{5e^x + 2}\right) = \frac{(5e^x + 2)(9x - 7) - (3x^3 - 7x)(5e^x)}{(5e^x + 2)^2}$$

Remark. In some cases it is okay to not simplify the answer.

2.4.2 Revising the Power Rule

$$\begin{aligned} f(x) &= x^n \\ f'(x) &= nx^{x-1} \end{aligned} \qquad\qquad \begin{aligned} f(x) &= Ax^n \\ f'(x) &= Anx^{n-1} \end{aligned}$$

Where n is <u>any</u> integer.

Example 2.4.2.2. Find $f'(x)$ if $f(x) = \dfrac{1}{3x^4}$.

$$\begin{aligned} f(x) &= \frac{1}{3x^4} \\ f(x) &= \frac{1}{3}x^{-4} \\ f(x) &= -\frac{4}{3}x^{-5} \end{aligned}$$

2.4.3 Higher Order Derivatives

Definition 2.4.1. The derivative if f' is the second derivative of f.

Notation: $f''(x)$

Read: "f double prime of x"

Can also consider third derivative f''', fourth derivatives $f^{(4)}$, etc.

Example 2.4.2.3. If $f(x) = 5x^3$, find f', f'', and f'''

$$\begin{aligned} f'(x) &= 15x^2 \\ f''(x) &= 30x \\ f'''(x) &= 30 \end{aligned}$$

Why do we care?

We know $f'(x)$ tells us the rate of change of f. What does $f''(x)$ tell us?

- Rate of change of the rate of change...
- In the context of f = position:
  - f' is velocity (how fast the position is changing)
  - f'' is acceleration (how fast the velocity is changing)
- In the context of f = number of unemployed people in the U.S.:
  - f' is how quickly unemployment is growing or shrinking
  - Suppose we are in a recession where unemployment is increasing. As f'' decreases, it means that jobs are being more slowly.

Example 2.4.2.4. A rock thrown vertically from the surface of the moon at an initial velocity of 24 [m/s] reaches a height of $s = 24t - 0.8t^2$ meters in t seconds

1. What is the velocity at time t? What is the acceleration?
2. How long before the rock reaches its highest point?
3. How high does the rock go?
4. How long before the rock reaches half of it's maximum height?
5. How long is the rock aloft?
6. What is the rock's speed on impact?

Answers:

1.

$$\begin{aligned} v &= s' = 24 - 16t[m/s] \\ a &= s'' = -1.6[m/s^2] \end{aligned}$$

2.

$$\begin{aligned} 24 - 1.6t &= 0 \\ 24 &= 1.6t \\ t &= 15[s] \end{aligned}$$

3.

$$\begin{gathered} 24\,(15) - 0.8\,(15)^2 \\ 180[m] \end{gathered}$$

4.

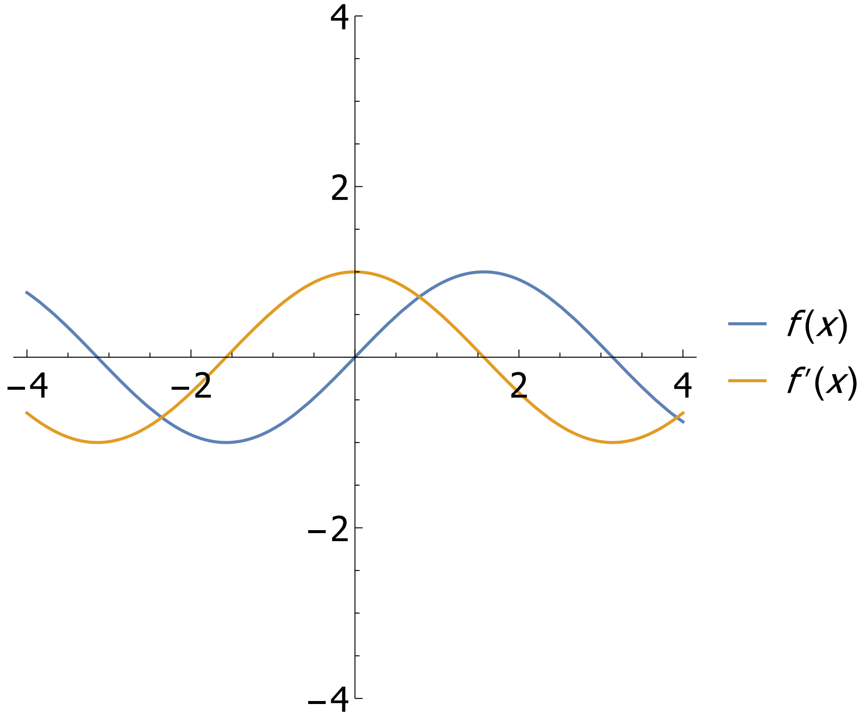
$$\begin{gathered} 90 = 24\,(t) - 0.8\,(t)^2 \\ 0 = -0.8t^2 + 24t - 90 \\ \frac{-24 \pm \sqrt{576 - 4 \cdot -0.8 \cdot -90}}{-1.6} \\ \frac{-24 \pm \sqrt{288}}{-1.6} \\ \frac{-24 \pm 12\sqrt{2}}{-1.6} \\ t = \underline{4.3934}, 25.6066 \end{gathered}$$

5. $30[s]$ (Two times the time to reach the peak (see #2))

6. $-24[m/s]$ (Same as initial velocity but negative)

2.5 The Derivative of Trigonometric Functions

Looking at the graph of $y = \sin(x)$, Can we get an idea of how the derivative looks?



The derivative of $y = \sin(x)$ is $y' = \cos(x)$

2.5.1 Derivatives of $f(x) = \sin(x)$ and $f(x) = \cos(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x) \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x) \tag{2.14}$$

Example 2.5.0.1. Find $\frac{\mathrm{d}}{\mathrm{d}x}(x\cos(x))$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$$

$$x \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \cos(x)$$

$$x(-\sin(x)) + 1 \cdot \cos(x)$$

$$-x\sin(x) + \cos(x)$$

2.5.2 Derivative of $f(x) = \tan(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x) \tag{2.15}$$

Proof. Proof that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$\frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot - \sin(x)}{\cos^2(x)}$$

$$\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$\frac{1}{\cos^2(x)} \quad \text{-OR-} \quad 1 + \tan^2(x)$$

$$\sec^2(x)$$

□

Example 2.5.0.2. Find the derivative of $y = \cot(x)$ in two ways: Using $\sin(x)$ and $\cos(x)$, and using $\tan(x)$.
Method 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos(x)}{\sin(x)}\right)$$

$$\frac{\sin(x) \cdot - \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)}$$

$$\frac{- \sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$\frac{- \left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)}$$

$$-\frac{1}{\sin^2(x)}$$

$$- \csc^2(x)$$

Method 2:

$$
\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\tan(x)}\right) \\
\frac{\tan(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(1) - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x))}{\tan^2(x)} \\
\frac{\cancel{\tan(x)\cdot 0} - \sec^2(x)}{\tan^2(x)} \\
-\frac{\sec^2(x)}{\tan^2(x)} \\
\frac{\frac{1}{\cancel{\cos^2(x)}}}{\frac{\sin^2(x)}{\cancel{\cos^2(x)}}} \\
-\frac{1}{\sin^2(x)} \\
-\csc^2(x)
\end{gathered}
$$

2.5.3 Derivatives of SIX basic trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x) \tag{2.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x) \tag{2.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x) \tag{2.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot(x)) = -\csc^2(x) \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec(x)) = \sec(x)\tan(x) \tag{2.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc(x)) = -\csc(x)\cot(x) \tag{2.21}$$

Chapter 3

More About Derivatives

3.1 The Chain Rule

Suppose we have

$$f(x) = \sin(x^2)$$

It is a composite function

$$g(x) = \sin(x)$$
$$h(x) = x^2$$
$$g(h(x)) = f(x)$$
$$(g \circ h)(x) = f(x)$$

3.1.1 The Chain Rule

Note. This is helpful in Calculus II

Theorem 3.1.1. *Suppose we have f and g are both differentiable then*

$$(f \circ g)' = f'(g(x)) \cdot g'(x) \tag{3.1}$$

Example 3.1.1.1.

$$f(x) = \sin(x^2)$$
$$f'(x) = \cos(x^2) \cdot 2x$$

Example 3.1.1.2.

$$y = \tan^2(\theta)$$
$$y = (\tan(\theta))^2$$

$$f(x) = x^2 \qquad\qquad f'(x) = 2x$$
$$g(\theta) = \tan(\theta) \qquad\qquad g'(\theta) = \sec^2(\theta)$$
$$y' = 2\left(\tan(\theta)\right) \cdot \sec^2(\theta)$$

Example 3.1.1.3.

$$s(x) = \csc(\cos(x))$$

$$\begin{aligned} f(x) &= \csc(x) & f'(x) &= -\csc(x)\cot(x) \\ g(x) &= \cos(x) & g'(x) &= -\sin(x) \end{aligned}$$

$$\begin{gathered} s'(x) = -\csc(\cos(x))\cot(\cos(x)) \cdot -\sin(x) \\ s'(x) = \sin(x)\csc(\cos(x))\cot(\cos(x)) \end{gathered}$$

Example 3.1.1.4.

a) find y'

$$y = \left(\frac{3x^2+1}{2x^2-x}\right)^4$$

$$\begin{aligned} f(x) &= x^4 & f'(x) &= -\csc(x)\cot(x) \\ g(x) &= \frac{3x^2+1}{2x^2-x} & g'(x) &= \frac{-3x^2-4x+1}{\left(2x^2-x\right)^2} \end{aligned}$$

$$\begin{aligned} y' &= 4\left(\frac{3x^2+1}{2x^2-x}\right)^3\left(-\frac{3x^2-4x+1}{\left(2x^2-x\right)^2}\right) \\ y' &= \frac{-4\left(3x^2+1\right)^3\left(3x^2+4x-1\right)}{\left(2x^2-x\right)^5} \end{aligned}$$

b) Find where the curve has horizontal tangents. ($y' = 0$)

$$\begin{gathered} \frac{-4\left(3x^2+1\right)^3\left(3x^2+4x-1\right)}{\left(2x^2-x\right)^5} = 0 \\ -4\left(3x^2+1\right)^3\left(3x^2+4x-1\right) = 0 \\ \cancel{-4 = 0} \qquad \text{no solutions} \\ 3x^2+1 = 0 \qquad \text{no solutions} \\ 3x^2+4x-1 = 0 \\ x = \frac{-4 \pm \sqrt{16-4(3)(-1)}}{2(3)} \\ x = \frac{-4 \pm \sqrt{38}}{6} \end{gathered}$$

Example 3.1.1.5.

$$\begin{aligned}
y &= (x)\left(\sec(e^x)\right)\\
y' &= x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(e^x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right)\\
y' &= x \cdot \sec(e^x)\tan(e^x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right)\\
y' &= x \cdot e^x \sec(e^x)\tan(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right)\\
y' &= x \cdot e^x \sec(e^x)\tan(e^x) + 1 \cdot \left(\sec(e^x)\right)\\
y' &= \sec(e^x)\left(xe^x \tan(e^x) + 1\right)
\end{aligned}$$

Note. Do you chain rule or product rule "first"? It depends!

3.1.2 Constant to the Power of x Rule

We want to derive $y = 3^x$.
Start by writing 3^x in terms of e^x:

$$\begin{aligned}
3^x &= e^{\ln(3^x)}\\
3^x &= e^{x\ln(3)}
\end{aligned}$$

Use the chain rule:

$$\begin{aligned}
f(x) &= e^x & f'(x) &= e^x\\
g(x) &= x\ln(3) & g'(x) &= \ln(3)
\end{aligned}$$

$$\begin{aligned}
y' &= e^{x\ln(3)} \cdot \ln(3)\\
y' &= 3^x \cdot \ln(3)
\end{aligned}$$

Theorem 3.1.2 (Constant to the Power of x Rule)**.** *If we assume that a is constant where $a > 0$ and $a \neq 1$ then:*

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \cdot \ln(a) \tag{3.2}$$

Example 3.1.2.1.

$$y = 2^{x^5}$$

$$\begin{aligned}
f(x) &= 2^x & f'(x) &= 2^x \cdot \ln(2)\\
g(x) &= x^5 & g'(x) &= 5x^4
\end{aligned}$$

$$\begin{aligned}
y' &= 2^{x^5}\ln(2) \cdot 5x^4\\
y' &= 5 \cdot \ln(2) \cdot x^4 \cdot 2^{x^5}
\end{aligned}$$

Example 3.1.2.2.

$$y = 3\sec(2^x)$$
$$y' = 3\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(2^x)\right)$$

$$f(x) = \sec(x) \qquad f'(x) = \sec(x)\tan(x)$$
$$g(x) = 2^x \qquad g'(x) = 2^x \cdot \ln(2)$$

$$f'(x) = 3\sec(2^x)\tan(2^x) \cdot 2^x \ln(2)$$
$$f'(x) = (3\ln(2))\, 2^x \sec(2^x)\tan(2^x)$$

3.2 Implicit Differentiation

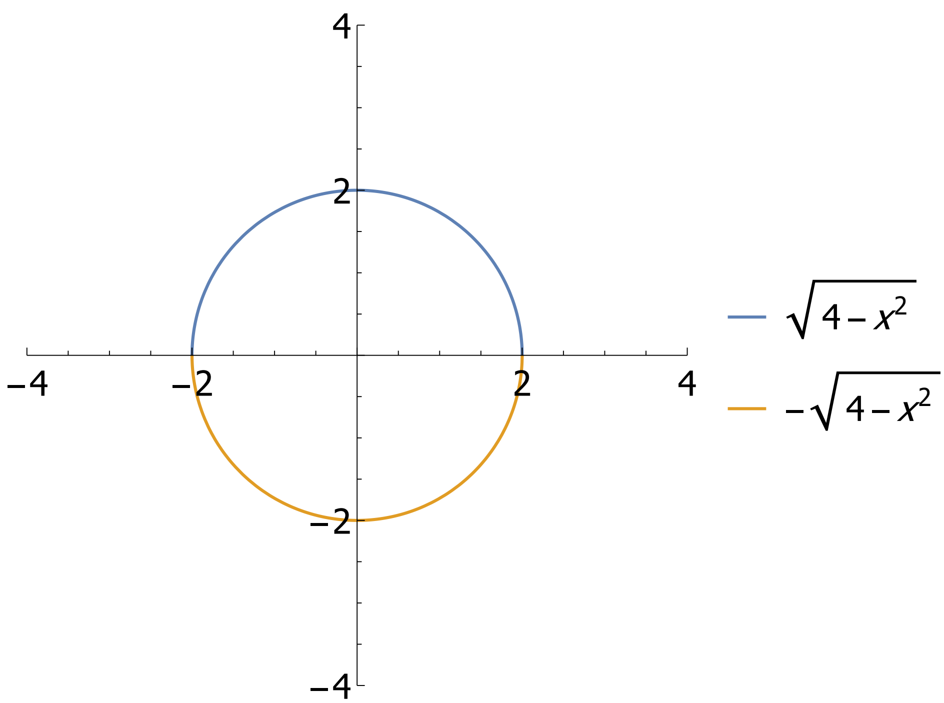
Some curves' equations can't be solved for y (or maybe not easily), but we should still be able find the tangent line and its slope.

Example 3.2.0.1.

$$x^2 + y^2 = 4$$

- Not a function!
- Can solve for y

$$y = \pm\sqrt{4 - x^2}$$



Given an expression with xs and ys to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

1. Treat y as a function of x and differentiate both sides of the equation with respect to x
2. Solve for $\frac{dy}{dx}$
3. Win

Note. When doing step one (1), you can think "Whenever I thake the derivative of y multiply that term by $\frac{dy}{dx}$."

Example 3.2.0.2. If $x^2 + y^2 = 4$ use implicit differentiation to find $\frac{dy}{dx}$

a)

$$\begin{aligned}
\frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(4) \\
2x + 2y\frac{dy}{dx} &= 0 \\
\frac{\cancel{2y}\frac{dy}{dx}}{\cancel{2y}} &= \frac{-2x}{2y} \\
\frac{dy}{dx} &= \frac{-2x}{2y} \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}$$

b) Find the slope of the tangent line at $\left(1, \sqrt{3}\right)$

$$\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = -\frac{1}{\sqrt{3}}$$

Note. The line after $\frac{dy}{dx}$ is read as "$\frac{dy}{dx}$ evaluated with $x = 1$ and $y = \sqrt{3}$"

c) What is the equation of the tangent at $\left(1, \sqrt{3}\right)$

$$\begin{aligned}
y - y_1 &= m\left(x - x_1\right) \\
y - \sqrt{3} &= -\frac{1}{\sqrt{3}}(x - 1)
\end{aligned}$$

3.3 Derivatives of Logarithmic Functions

3.3.1 Derivatives of Logarithmic Functions

Theorem 3.3.1 (Derivatives of Logarithmic Functions). *If we recall that* $\log_a(x) = y$*, then* $a^y = x$ *as well as* $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln(a)$ *then the following must be true:*

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a(x)) = \frac{1}{x\ln(a)} \tag{3.3}$$

Proof.

$$\begin{aligned} \log_a(x) &= y \\ a^y &= x \\ \frac{\mathrm{d}}{\mathrm{d}x}(a^y) &= \frac{\mathrm{d}}{\mathrm{d}x}(x) \\ a^y \ln(a) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a^y \ln(a)} \end{aligned}$$

This is fine but we don't want y in our answer.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\ln(a)}$$

This is valid because we know that $a^y = x$

□

Theorem 3.3.2 (Derivatives of Natrual Log). *Similar to the derivative of a* log *function the derivative of* ln *is:*

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x} \tag{3.4}$$

Proof.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) \\ &= \frac{1}{x\ln(e)} \\ &= \frac{1}{x} \end{aligned}$$

Note. The power rule will never give an exponent of -1, since $n = 0$ would mean that it's a constant.

$\square$

Example 3.3.2.1. Differentiate: $f(x) = \dfrac{x}{\ln(x)}$

$$\begin{aligned}
f'(x) &= \frac{(\ln(x))\,\frac{\mathrm{d}}{\mathrm{d}x}(x) - (x)\,\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x))}{(\ln(x))^2} \\
&= \frac{(\ln(x))\,(1) - (x)\left(\frac{1}{x}\right)}{(\ln(x))^2} \\
&= \frac{\ln(x) - 1}{(\ln(x))^2} \\
&= \frac{\ln(x)}{(\ln(x))^2} - \frac{1}{(\ln(x))^2} && \text{(Optional)} \\
&= \frac{1}{\ln(x)} - \frac{1}{(\ln(x))^2} && \text{(Optional)}
\end{aligned}$$

Example 3.3.2.2. Differentiate $y = |x|$

$$y = \begin{cases} \ln(x) & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

Note. Domain is all real number except 0

$$\begin{aligned}
&x > 0: \\
&y = \ln(x) \\
&y' = \frac{1}{x}
\end{aligned}$$

$$\begin{aligned}
&x < 0: \\
&y = \ln(-x) \\
&y' = \frac{1}{-x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(-x) \\
&y' = \frac{1}{-x}(-1) \\
&y' = \frac{1}{x}
\end{aligned}$$

Conclusion: because y' was the same in both cases then the following must be true:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(|x|)) = \frac{1}{x}$$

Example 3.3.2.3. Use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\ln\left(x^2 - y^2\right) = x - y$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2 - y^2\right)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(x - y) \\
\frac{1}{x^2 - y^2} \cdot \left(2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 1 - \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x}}{x^2 - y^2} &= 1 - \frac{\mathrm{d}y}{\mathrm{d}x} \\
2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= \left(1 - \frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(x^2 - y^2\right) \\
2x - 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 - y^2 - x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y^2\frac{\mathrm{d}y}{\mathrm{d}x} \\
-2y\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}y}{\mathrm{d}x} - y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 - y^2 - 2x \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(-2y + x^2 - y^2\right) &= x^2 - y^2 - 2x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2 - y^2 - 2x}{-2y + x^2 - y^2}
\end{aligned}$$

It is OK to leave a mix of xs and ys, because we can't write $y = (x \text{ stuff})$

3.3.2 Logarithmic Differentiation

Remember that question from the review?

$$y = \left(\frac{x^4 \sin^2(x)}{\sqrt{1 - x^2}}\right)$$

Yeah that one. To derive this one normally would be a lot of chain, quotient, and power rules. However an esier way of taking the derivative by taking the natural log of both sides, then implicit derive of both sides and solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$. This process is known as "logarithmic differentiation".

Example 3.3.2.4.

$$\begin{aligned}
y &= \left(\frac{x^4\sin^2(x)}{\sqrt{1-x^2}}\right)\\
\ln(y) &= \ln\left(\frac{x^4\sin^2(x)}{(1-x^2)^{\frac{1}{2}}}\right)\\
\ln(y) &= \ln\left(x^4\left(\sin^2(x)\right)\right) - \ln\left(1-x^2\right)^{\frac{1}{2}}\\
\ln(y) &= \ln\left(x^4\right) + \ln\left(\sin(x)\right)^2 - \frac{1}{2}\ln\left(1-x^2\right)\\
\ln(y) &= 4\ln(x) + 2\ln(\sin(x)) - \frac{1}{2}\ln\left(1-x^2\right)\\
\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= 4\cdot\frac{1}{x} + 2\frac{1}{\sin(x)}\cos(x) - \frac{1}{\cancel{2}}\cdot\frac{1}{1-x^2}\cdot\cancel{2}x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{4}{x} + 2\cot(x) - \frac{x}{1-x^2}\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^4\sin^2(x)}{\sqrt{1-x^2}}\left(\frac{4}{x} + 2\cot(x) - \frac{x}{1-x^2}\right)\\
&\qquad\text{(Substitue the original function back in for } y\text{)}
\end{aligned}$$

Another use of logarithmic differentiation

Suppose we to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right)$

- We can't use the fact that $\frac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right) = a^x\ln(a)$, since the base is not a constant.
- We can't use the fact that $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^r\right) = rx^{r-1}$, since the exponent is not a constant.

Here is how we solve it:

$$\begin{aligned}
y &= x^x\\
\ln(y) &= \ln\left(x^x\right)\\
\ln(y) &= x\ln(x)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(y)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\ln(x)\right)\\
\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= x\cdot\frac{1}{x} + 1\cdot\ln(x)\\
\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= 1+\ln(x)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(1+\ln(x)\right)
\end{aligned}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^x\left(1 + \ln(x)\right)$$

3.3.3 The Most Powerful Power Rule

Theorem 3.3.3 (The Most Powerful Power Rule)**.** *If r is any real number, then:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^r\right) = rx^{r-1} \tag{3.5}$$

Proof. If $y = x^a$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned}
\ln(y) &= \ln(x^a)\\
\ln(y) &= a\ln(x)\\
\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= a\cdot\frac{1}{x}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(a\cdot\frac{1}{x}\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= x^a\left(a\cdot\frac{1}{x}\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= x^a\cdot a\cdot x^{-1}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= ax^{a-1}
\end{aligned}$$

□

Example 3.3.3.1. Find the equation for the line tangent to $y = \left(x^3 - x + 1\right)^e$ when $x = 1$.

$$\begin{aligned}
y' &= e\left(x^3 - x + 1\right)^{e-1}\left(3x^2 - 1\right)\\
y'\big|_{x=1} &= e\left(1^3 - 1 + 1\right)^{e-1}\left(3\cdot 1^2 - 1\right)\\
&= e\,(1)^{e-1}\,(2)\\
&= 2e
\end{aligned}$$

$2e$ is our slope. To find the point we can substitute $x = 1$ into the original function:

$$\begin{aligned}
y\big|_{x=1} &= \left(1^3 - 1 + 1\right)^e\\
&= 1^e\\
&= 1
\end{aligned}$$

Point $= (1, 1)$
Tangent line: $y - 1 = 2e\,(x - 1)$

3.4 Newton's Method

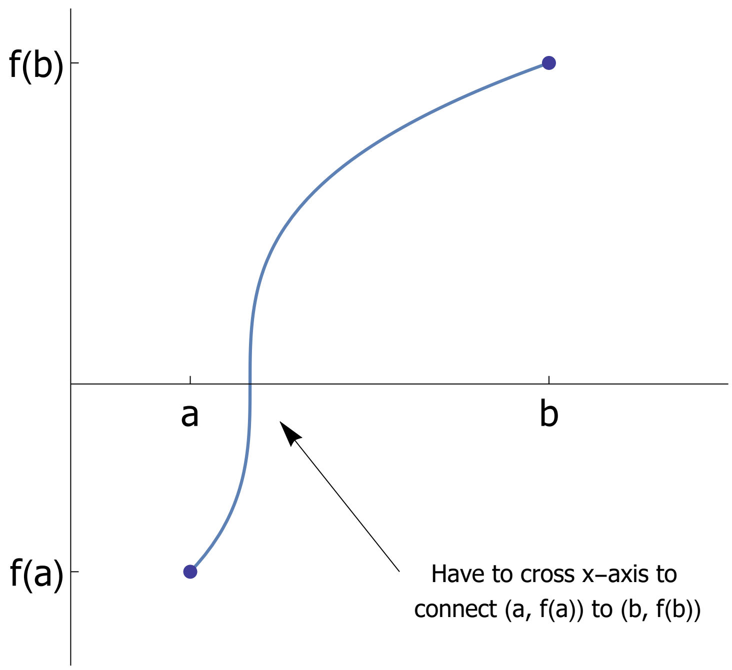
How do we solve equations like $x^3 - 4x + 2 = 0$?

- We could use a calculator... But what about before calculators? And HOW do calculators solve it?
- We could factor it. But factoring polynomials of high degrees really isn't fun. Maybe it doesn't factor (i.e. $\sin^2(x) + 2x - 5$)
- There are a variety of *numerical methods* that calculators and computers use to find roots of equations.
- One such method id Newton's Method.

3.4.1 Intermediate Value Theorem

Theorem 3.4.1 (Intermediate Value Theorem)**.** *If f is continouts on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there is some $c \in (a, b)$ such that $f(c) = N$.*

Note. $c \in (a, b)$ means that the value c is between a and b



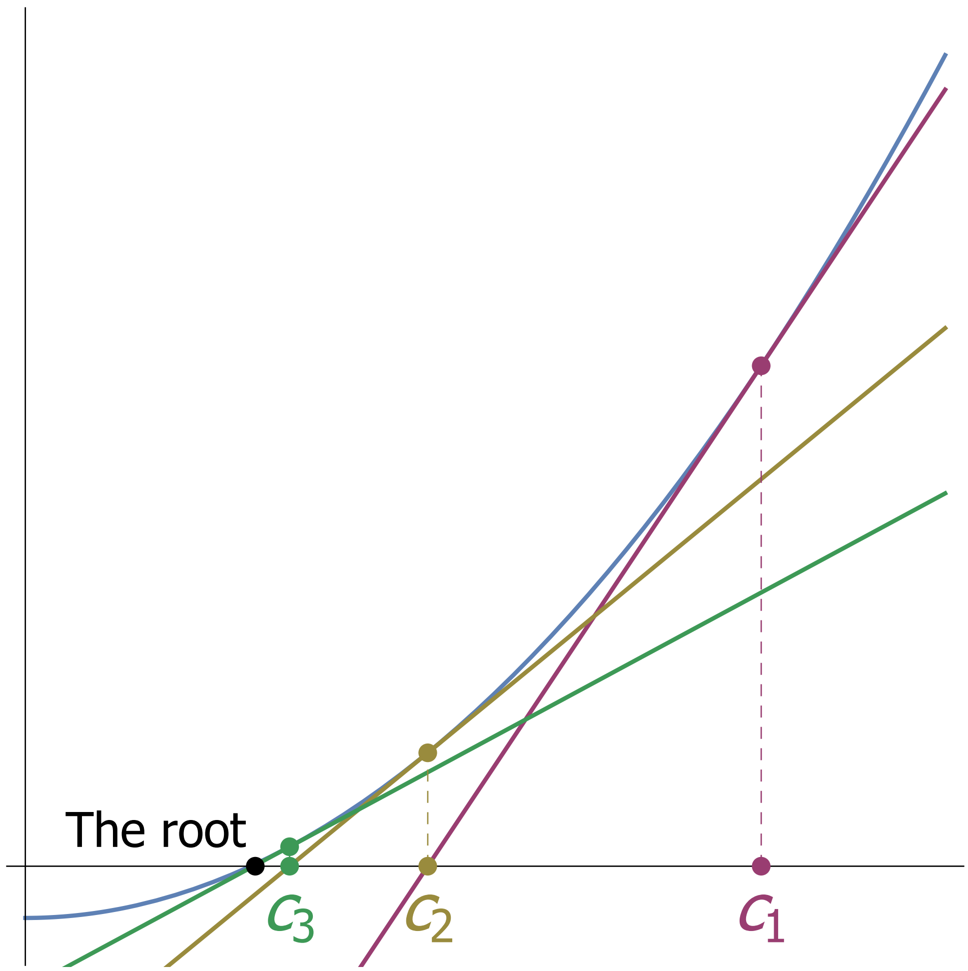
3.4.2 The idea behind Newtom's Method

Suppose $f(x)$ is a polynomial.

- Suppose we can apply the IVT to conclude that f has a root between a and b.

- We pick any value, c_1, between a nd b, and use that as our first guess of what the zero might be.
- We probably won't be right because there are an infinite amount of numbers between a and b.
- We can evaluate $f(c_1)$ using a calculator.
- We can find $f'(c_1)$ because we are awesome at derivatives. . . right?

Because c_1 is an approximation and is probably wrong we need to be able to ge a more accruate estimate. To do that we take the tangent line of $f(c_1)$ and where it intersects the x-axis is c_2. Repeat the process to get a progressively more and more accurate estimate for the root of the function.



3.4.3 A Recursive Formula

If c_1 is our first estimate the we can get a better approximation c_2 (called the "second approximation") by finding

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}$$

The third approximation is

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)}$$

We can keep going to get the nth approximation

$$c_n = c_{n-1} - \frac{f(c_{n-1})}{f'(c_{n-1})} \tag{3.6}$$

Note. In *most* cases when computers (calculators, laptops, etc.) are asked to find the square root of a function they actually use Newton's Method to find the square root. For example, if we asked a computer to find $\sqrt{S}$ it would have the function $f(x) = x^2 - S$ and then applies Newton's Method to find the square root of S. In fact, *most* processors have an instruction that is used to take the square root of a number and the most common implementation of this instruction uses Newton's Method and it is accurate up to 5 bits.

Example 3.4.1.1. Verify that $f(x) = x^3 - 4x + 2$ has a root on the interval $(1, 2)$. Then use a first approximation of $c_1 = 1.5$ to find the third approximation of that root.

$$f(1) = 1^3 - 4(1) + 2 = -1$$
$$f(2) = 2^3 - 4(2) + 2 = 2$$

The IVT says that this function *must* have a root in $(1, 2)$.

$$f'(x) = 3x^2 - 4$$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_2)}$$
$$c_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$$
$$c_2 = 1.5 - \frac{(1.5)^3 - 4(1.5) + 2}{3(1.5)^2 - 4} \approx 1.727$$

$$c_3 = 1.727 - \frac{(1.727)^3 - 4(1.727) + 2}{3(1.727)^2 - 4} \approx 1.678$$

The actual root is approximately 1.675, so we're correct up to two decimal places in just two iterations.

Example 3.4.1.2. Verify that $f(x) = x^4 - x^3 + x - 2$ has a root on the interval $(1, 2)$. Then use a first approximation of $c_1 = 1.5$ to find the fourth approximation of that root.

$$f(1) = 1^4 - 1^3 + 1 - 2 < 0$$
$$f(2) = 2^4 - 2^3 + 2 - 2 > 0$$

The IVT says that this function *must* have a root in $(1, 2)$.

$$f'(x) = 4x^3 - 3x^2 + 1$$

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)}$$
$$c_2 = 1.5 - \frac{(1.5)^4 - (1.5)^3 + 1.5 - 2}{4(1.5)^3 - 3(1.5)^2 + 1}$$
$$c_2 \approx 1.3468$$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)}$$
$$c_3 = 1.3468 - \frac{(1.3468)^4 - (1.3468)^3 + 1.3468 - 2}{4(1.3468)^3 - 3(1.3468)^2 + 1}$$
$$c_3 \approx 1.3104$$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)}$$
$$c_4 = 1.3104 - \frac{(1.3104)^4 - (1.3104)^3 + 1.3104 - 2}{4(1.3104)^3 - 3(1.3104)^2 + 1}$$
$$c_4 \approx 1.3086$$

This is actually correct up to 4 decimal places!

3.4.4 Good News and Bad News

Bad News

Sometimes Newton's Method fails. For example, your c_1 might be at a point where there is a horizontal tangent and it won't intersect the x-axis

Good News

It doesn't fail very often, and when it works, it gets close to the right answer "very quickly."

Chapter 4

Applications of the Derivative

4.1 Related Rates

Problem asks for a rate of change of some quantity; use the rate of change of related quantities.

Example 4.1.0.1. The volume of a right circular cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

How does the volume change over time with respect to the change in height if r is constant?

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\pi r^2 h\right)$$

$$1 \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \underbrace{\frac{1}{3}\pi r^2}_{\text{constant}} \cdot 1 \cdot \frac{\mathrm{d}h}{\mathrm{d}t}$$

How is $\frac{\mathrm{d}V}{\mathrm{d}t}$ related to $\frac{\mathrm{d}r}{\mathrm{d}t}$ if h is constant?

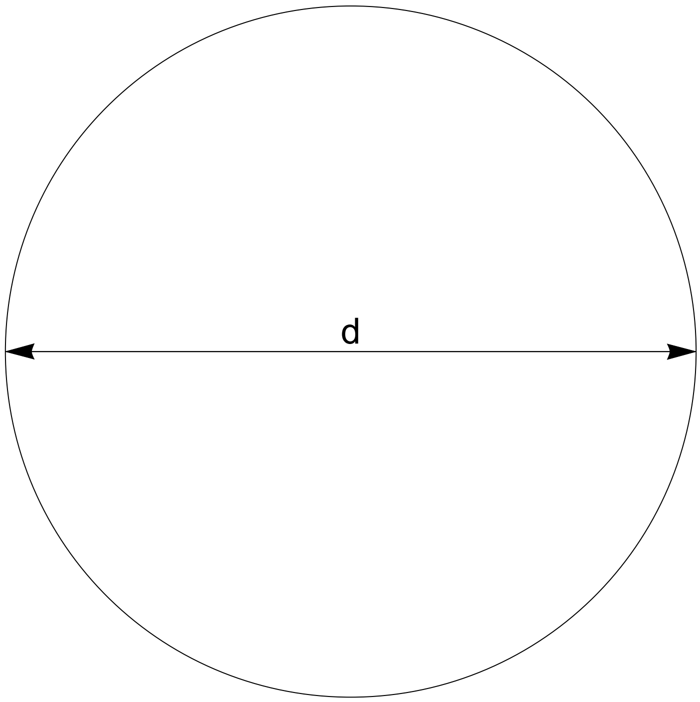
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \underbrace{\frac{1}{3}\pi h}_{\text{constant}} \cdot 2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t}$$

How is the volume changing over time if both the radius AND the height are changing?

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \underbrace{\frac{1}{3}\pi}_{\text{constant}} \left(r^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}t} + 2r \cdot \frac{\mathrm{d}r}{\mathrm{d}t}h\right)$$

4.1.1 Practice

If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm}^2/\,\mathrm{min}$, find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$. (Surface area of a sphere: $SA = 4\pi r^2$)



Solution

Variables we know:

$$\begin{aligned} d &= \text{ diameter} \\ SA &= \text{ surface area} \\ \frac{\mathrm{d}SA}{\mathrm{d}t} &= 1\,\mathrm{cm}^2/\,\mathrm{min} \end{aligned}$$

Variables we want:

$$\left.\frac{\mathrm{d}d}{\mathrm{d}t}\right|_{d=10\,\mathrm{cm}}$$

Equation:

$$\begin{aligned} SA &= 4\pi r^2 \\ SA &= 4\pi \left(\frac{d}{2}\right)^2 \\ SA &= \pi d^2 \end{aligned}$$

Derive:

$$\frac{\mathrm{d}SA}{\mathrm{d}t} = \pi \cdot 2d \cdot \frac{\mathrm{d}d}{\mathrm{d}t}$$

Substitute:

$$1\,\mathrm{cm}^2/\,\mathrm{min} = \pi \cdot 2\,(10\,\mathrm{cm}) \cdot \frac{\mathrm{d}d}{\mathrm{d}t}$$
$$\frac{\mathrm{d}d}{\mathrm{d}t} = \frac{-1\,\mathrm{cm}^2/\,\mathrm{min}}{20\pi\,\mathrm{cm}}$$
$$\frac{\mathrm{d}d}{\mathrm{d}t} = \frac{-1}{20\pi}\,\mathrm{cm}/\,\mathrm{min}$$

The diameter is decreasing at $\frac{-1}{20\pi}\,\mathrm{cm}/\,\mathrm{min}$.
Important: Be sure to analyze whether each peice of information given is a RATE or QUANTITY (distance, volume, etc).

- **Rates** are "d(blah)/dt"
- **Other quantities** are NOT
- Remember to define your vairables!
- Include units.
- Answer the question! Don't just do the math.

Example 4.1.0.2. A 5ft ladder, leaning against a wall, slips so that its base moves away from the wall at a rate of 2 ft/ sec. How fast will the top of the ladder be moving down the wall when the base is 4ft from the wall?

$$x = \text{base distance from the wall}$$
$$y = \text{height of contact with wall}$$
$$5\text{ft} = \text{Ladder length}$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\,\mathrm{ft}/\,\mathrm{sec}$$

We want $\frac{\mathrm{d}y}{\mathrm{d}t}$ when $x = 4\text{ft}$.
Note. We will be expecting a negative value.

Equation relating information:

$$x^2 + y^2 = (5\text{ft})^2$$

Note. 5ft is a constant x and y are changing.

Take the derivetive with relation to t:

$$2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

Note. It makes sense that the equation is equal to 0 because 5ft is constant.

Substitute:

$$2\,(4\text{ft})\,(2\,\text{ft/sec}) + 2\,(y)\,\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

We have two unknowns, y and $\frac{\mathrm{d}y}{\mathrm{d}t}$, but we can find y when $x = 4\text{ft}$:

$$\begin{aligned} x^2 + y^2 &= (5\text{ft})^2 \\ (4\text{ft})^2 + y^2 &= 25\,\text{ft}^2 \\ y^2 &= 9\,\text{ft}^2 \\ y &= \pm 3\text{ft} \\ y &= 3\text{ft} \end{aligned}$$

Now back to the substitution:

$$\begin{aligned} 2\,(4\text{ft})\,(2\,\text{ft/sec}) + 2\,(3\text{ft})\,\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ 16\,\text{ft}^2/\text{sec} + 6\text{ft} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} 6\text{ft} &= -16\,\text{ft}^2/\text{sec} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{-16\,\text{ft}^2/\text{sec}}{6\text{ft}} &= -\frac{8}{3}\text{ft/sec} \end{aligned}$$

The ladder slides down at $\frac{8}{3}$ft/sec.

Example 4.1.0.3. Suppose ther is a camera mounted 3000ft from the base of a rocket launching pad. Assume the rocket rises vertically and the camera is to take a series of photographs of the rocket. Because the rocket will be rising, the elevation angle of the camera will have to vary at just the right rate to keep the rocket in sight. Because the camera-to-rocket distance will be changing constantly, the camera focusing mechanism will also have to vary at just the right rate to keep the picture sharp. Suppose the rocket is rising vertically at a rate of 880 ft/sec when it is 4000ft up. How fast is the camera-to-rocket distance changing at that instant?

$$\begin{aligned} z &= \text{camera-to-rocket distance} \\ y &= \text{rocket height} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 880\,\text{ft/sec when } y = 4000\text{ft} \end{aligned}$$

We want $\frac{\mathrm{d}z}{\mathrm{d}t}$.
Equation:

$$(3000\text{ft})^2 + y^2 = z^2$$

Note. y and z are changing.

Derive with respect to t:

$$0 + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = 2z \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

Substitute: z is replaced by finding:

$$(3000\text{ft})^2 + (4000\text{ft})^2 = z^2$$
$$z = 5000\text{ft}$$

$$2\,(4000\text{ft}) \cdot 880\,\text{ft}/\sec = 2\,(5000\text{ft})\,\frac{\mathrm{d}z}{\mathrm{d}t}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2 \cdot 4000 \cdot 880\,\text{ft}^2/\sec}{2 \cdot 5000\text{ft}}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = 4.176\,\text{ft}/\sec = 704\,\text{ft}/\sec$$

The distance from the camera to the rocket is increasing at $704\,\text{ft}/\sec$.

4.2 Maximum and Minimum Values

—`Not yet implemented`—

4.3 The Mean Value Theorem

—`Not yet implemented`—

4.4 Local Extrema and Concavity

—`Not yet implemented`—

4.5 Indterminate Forms and L'Hopital's Rule

4.5.1 Indeterminate Forms

Let's look at the first two forms we're going care about here.
If

$$\lim_{x \to c} (f(x)) = \lim_{x \to c} (g(x)) = 0$$

then $\dfrac{f(x)}{g(x)}$ is an indeterminate form of the type $\dfrac{0}{0}$.
If

$$\lim_{x\to c}(f(x)) = \lim_{x\to c}(g(x)) = \infty$$

then $\dfrac{f(x)}{g(x)}$ is an indeterminate form of the type $\dfrac{\infty}{\infty}$.
Note. $c = \infty$ is allowed

View these cases as we get *no information* when we substitute to try and find the limit.

Example 4.5.0.1. Evaluate: $\lim_{x\to 0}\left(\dfrac{\tan(x)}{x}\right)$
Remember:

$$\lim_{x\to 0}\left(\frac{\sin(\theta)}{\theta}\right) = 1$$

$$\lim_{x\to 0}\left(\frac{\tan(x)}{x}\right) \stackrel{\text{sub}}{=} \frac{0}{0}$$

$$\lim_{x\to 0}\left(\frac{\frac{\sin(x)}{\cos(x)}}{x}\right)$$

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}\right)$$

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\cdot\lim_{x\to 0}\left(\frac{1}{\cos(x)}\right)$$

$$1\cdot\frac{1}{1}$$

$$1$$

4.5.2 L'Hopital's Rule

Theorem 4.5.1 (L'Hopital's Rule)**.** *Suppose f and g are differentiable near c, and $g'(x) \neq 0$ for all $x \neq c$ near c. Let L be a real number (or $\pm\infty$), and suppose $\dfrac{f(x)}{g(x)}$ in an indeterminate form at c of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. If*

$$\lim_{x\to c}\left(\frac{f'(x)}{g'(x)}\right) = L \tag{4.1}$$

then

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = L \tag{4.2}$$

The rule basically says **if you satart with a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indterminate form**, then taking the derivative of the top and the derivative of the bottom will not change the limit.
We are **NOT** taking the derivative of $\frac{f(x)}{g(x)}$!. Never take the derivative of a quotient without using the quotient rule.
You many need to apply the rule multiple times. You might get another indeterminate form when you take the derivative; so apply the rule again.
YOU HAVE TO START WITH AN INDETERMINATE FORM!

Example 4.5.1.1. Evaluate: $\lim_{x\to 0}\left(\frac{\tan(x)}{x}\right)$

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\tan(x)}{x}\right) &\overset{\text{sub}}{=} \frac{0}{0}\\ &\overset{\text{L'H}}{=} \lim_{x\to 0}\left(\frac{\sec^2(x)}{1}\right)\\ &\overset{\text{sub}}{=} \frac{\sec^2(0)}{1}\\ &= \frac{1}{1}\\ &= 1\end{aligned}$$

Note. The "L'H" in $\overset{\text{L'H}}{=}$ is required so we know that L'Hopital's Rule was used.

Note. It is **NOT** true that $\frac{\tan(x)}{x} = \frac{\sec^2(x)}{1}$, so you really need the limit opperator to keep the staments equivelent.

Appendix A

Buddy Chart

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right) = a^x \ln(a)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\right) = \cos(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right) = -\sin(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(x)\right) = \sec^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc(x)\right) = -\csc(x)\cot(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(x)\right) = \sec(x)\tan(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot(x)\right) = -\csc^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1}(x)\right) &= -\frac{1}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\log_a(x)\right) &= \frac{1}{x\ln(a)} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right) &= \frac{1}{x}
\end{aligned}$$