Calculus I Notes

MATH 1190

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Contents

Chapter 2

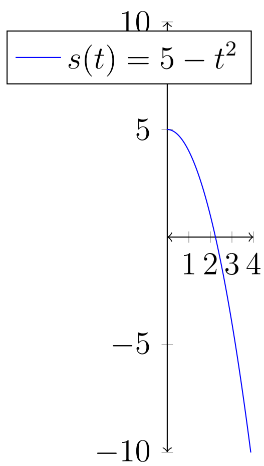
The Derivative

2.1 Rates of Change and The Derivative

A particle's rectilinear (1D) motion has its position defined by the function $s(t) = 5 - t^2$, where s is measured in meters and t in seconds.

1. Sketch the graph of the function on the interval from $t = 0$ to $t = 4$.
2. Find the *average* velocity over the time over the time interval from $t = 0$ to $t = 4$. On your graph, draw what this quantity represents.
3. Approximate the *instantaneous* velocity when $t = 2$ by finding the average velocity over the intervals $t = 2$ to $t = 3$, $t = 2$ to $t = 2.5$, and $t = 2$ to $t = 2.1$.
4. Write a general expression that represents the average velocity over the time interval from $t = 2$ to $t = 2 + h$.
5. Find the instantaneous velocity when $t = 2$ by finding the limit of the above expression as $h \to 0$.

Answers:

1. 

2.

$$\frac{\Delta s}{\Delta t}$$
$$\frac{s(4) - s(0)}{4 - 0}$$
$$\frac{(5 - 4^2) - (5 - 0^2)}{4}$$
$$\frac{-11 - 5}{4}$$
$$-4[m/s]$$

3.

- $t = 2$ to $t = 3$:

$$\frac{s(3) - s(2)}{3 - 2}$$
$$\frac{(5 - 3^2) - (5 - 2^2)}{1}$$
$$-5[m/s]$$

- $t = 2$ to $t = 2.5$:

$$\frac{s(2.5) - s(2)}{3 - 2}$$
$$\frac{(5 - 2.5^2) - (5 - 2^2)}{0.5}$$
$$-4.5[m/s]$$

- $t = 2$ to $t = 2.1$:

$$\frac{s(2.1) - s(2)}{3 - 2}$$
$$\frac{(5 - 2.1^2) - (5 - 2^2)}{0.1}$$
$$-4.1[m/s]$$

Guess: velocity at $t = 2$ is approximately $4[m/s]$.

4.

$$\frac{s(2+h)-s(2)}{2+h-2}$$
$$\frac{\left(5-(2+h)^2\right)-(5-2^2)}{h}$$
$$\frac{5-(4+4h+h^2)-(1)}{h}$$
$$\frac{-4h-h^2}{h}$$
$$-4-h$$

5.

$$\lim_{h\to 0}\left(\frac{s(2+h)-s(2)}{h}\right)$$
$$\lim_{h\to 0}(-4-h)$$
$$-4-0$$
$$-4$$

2.1.1 Definitions

The slope of a curve can be found using the following equations:

$$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) \tag{2.1}$$

$$\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right) \tag{2.2}$$

These are also known as:

- The *instantaneous* velocity of an object at time c whose position is given by the function $f(x)$.
- The *slope of the tangent line* to the curve $y = f(x)$ at $x = c$.
- The *instantaneous rate of change* of the function $f(x)$ at $x = c$.
- The *derivative* of f at c.
- $f'(c)$

Example 2.1.0.1. Find the slope of the line tangent to $y = \dfrac{1}{x+5}$ when $x = 1$. Then find the equation for the tangent line at that point.

$$\lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{\frac{1}{1+h+5} - \frac{1}{1+5}}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{\left(\frac{6}{6} \cdot \frac{1}{1+h+5} \right) - \left(\frac{1}{1+5} \cdot \frac{6+h}{6+h} \right)}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{\frac{6}{6\,(6+h)} - \frac{6+h}{6\,(6+h)}}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{\frac{-h}{6\,(6+h)}}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{1}{\cancel{h}} \cdot \frac{-\cancel{h}}{6\,(6+h)} \right)$$

$$\lim_{h \to 0} \left(\frac{-1}{6\,(6+h)} \right)$$

$$\frac{-1}{36}$$

Equation of line: We have the slope, all we need is a point (substitute 1 in for x).

$$y = \frac{1}{1+6}$$

$$y = \frac{1}{6}$$

So the point is $(1, \frac{1}{6})$. Equation:

$$y - \frac{1}{6} = -\frac{1}{36}\,(x-1)$$

$$y = -\frac{1}{36}x + \frac{1}{36} + \frac{1}{6}$$

$$y = -\frac{1}{36}x + \frac{7}{36}$$

2.2 The Derivative of a Function

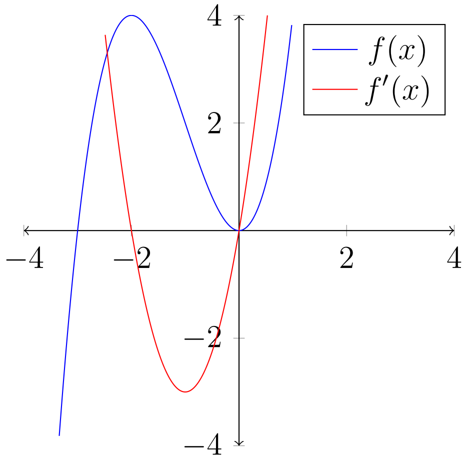
Theorem 2.2.1 (Derivative)**.** *The derivative of f is the function*

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \tag{2.3}$$

Remark. This is only true if the limit exists.

Corollary 2.2.1.1. *If the limit does exist at $x = c$ then f is differentiable at c*

Corollary 2.2.1.2. *If the limit exists at every point in interval $[a, b]$ then f is differentiable on $[a, b]$.*



Example 2.2.1.1. If $f(x) = x^2 + 2x + 1$ find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\
f'(x) &= \lim_{h \to 0} \left(\frac{\left((x+h)^2 + 2(x+h) + 1\right) - (x^2 + 2x + 1)}{h} \right) \\
f'(x) &= \lim_{h \to 0} \left(\frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h + \cancel{1} - \cancel{x^2} - \cancel{2x} - \cancel{1}}{h} \right) \\
f'(x) &= \lim_{h \to 0} \left(\frac{2xh + h^2 + 2h}{h} \right) \\
f'(x) &= \lim_{h \to 0} (2x + h + 2) \\
f'(x) &= 2x + 2
\end{aligned}
$$

Example 2.2.1.2. Let $f(x)=\sqrt{x}$. Find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\left(\frac{f\left(x+h\right)-f(x)}{h}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{\cancel{x}+h-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}\right)\\
f'(x) &= \lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right)\\
f'(x) &= \frac{1}{\sqrt{x+0}+\sqrt{x}}\\
f'(x) &= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Corollary 2.2.1.3. *Functions will fail to be differentiable at*

- *Cusps*
- *Corners*
- *Vertical Tangents*
- *Any point where it is discontinuous*

Lemma 2.2.2. *If f is differentiable at $x=c$ then f is continuous at c. However a function can be continuous but not differentiable (e.g. $y=|x|$ at $x=0$).*

2.3 The Derivative of Polynomial Functions and $y=e^x$

Recall that if $f(x)=x^2+2x+1$, then $f'(x)=2x+2$. We could write this in different ways.

- If $y=x^2+2x+1$ then $y'=2x+2$.
- If $y=x^2+2x+1$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+2$.

- If $y = x^2 + 2x + 1$ then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 2x + 1\right) = 2x + 2$.

Remark. The last one, $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is an instruction to take a derivative of what comes after it.

Theorem 2.3.1 (Derivative of a Constant)**.** *If A is a constant and $f(x) = A$ then $f'(x) = 0$.*

Theorem 2.3.2 (Derivative of a line with a slope of 1)**.** *If $f(x) = x$ then $f'(x) = 1$.*

Review 2.3.1. *Use the definition of a derivative to find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)$.*

$$\lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{x^2 + h^2 + 2xh - x^2}{h}\right)$$
$$\lim_{h\to 0}\left(\frac{h^2 + 2xh}{h}\right)$$
$$\lim_{h\to 0}(h + 2x)$$
$$0 + 2x$$
$$2x$$

2.3.1 Basic Rules

Power Rule

Theorem 2.3.3 (Power Rule)**.** *If $n \geq 1$ is an integer, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1} \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Cx^n\right) = (C \cdot n)x^{n-1} \qquad \text{If } C \text{ is a constant.} \tag{2.5}$$

Review 2.3.2. *If $y = 5x^2$ find y' using the definition of a derivative.*

$$f'(x) = \lim_{h\to 0}\left(\frac{5(x+h)^2 - 5x^2}{h}\right)$$
$$f'(x) = \lim_{h\to 0}\left(\frac{5\left((x+h)^2 - x^2\right)}{h}\right)$$
$$f'(x) = \lim_{h\to 0}\left(\frac{5\left(x^2 + h^2 + 2xh - x^2\right)}{h}\right)$$
$$f'(x) = 5\lim_{h\to 0}\left(\frac{x^2 + h^2 + 2xh - x^2}{h}\right)$$

$$
\begin{aligned}
f'(x) &= 5\lim_{h\to 0}\left(\frac{h^2+2xh}{h}\right)\\
f'(x) &= 5\lim_{h\to 0}(h+2x)\\
f'(x) &= 5\cdot 2x\\
f'(x) &= 10x
\end{aligned}
$$

Constant Multiplication Rule

Theorem 2.3.4 (Constant Multiplication Rule)**.** *Suppose $F(x) = k\cdot f(x)$ for some real number k if $f(x)$ is differentiable then $F(x)$ is also differentiable, and*

$$F'(x) = k\cdot f'(x) \tag{2.6}$$

Example 2.3.4.1. If $f(x) = \pi x^7$ find $f'(x)$.

$$
\begin{aligned}
f'(x) &= \pi\frac{\mathrm{d}}{\mathrm{d}x}\left(x^7\right)\\
f'(x) &= \pi\cdot\left(7x^6\right)\\
f'(x) &= 7\pi x^6
\end{aligned}
$$

Addition Rule

Theorem 2.3.5 (Addition Rule)**.** *If $F(x) = f(x)+g(x)$ and f and g are differentiable then $F(x)$ is also differentiable.*

$$F'(x) = f'(x)+g'(x) \tag{2.7}$$

Example 2.3.5.1. If $y = 3x^5-7x^2-\frac{1}{2}x+5$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$
\begin{aligned}
y' &= 3\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right)-7\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x)+\frac{\mathrm{d}}{\mathrm{d}x}(5)\\
y' &= 3\cdot 5x^4-14x-\frac{1}{2}\cdot 1+0\\
y' &= 15x^4-14x-\frac{1}{2}
\end{aligned}
$$

2.3.2 Derivative of $f(x) = e^x$

Theorem 2.3.6 (Derivative of $f(x) = e^x$)**.**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x \tag{2.8}$$

If $f(x) = e^x$, then $f'(x) = e^x$.

2.4 Product Rule, Quotient Rule, and Higher Order Derivatives

2.4.1 Basic Rules Continued

Product Rule

Theorem 2.4.1 (Product Rule)**.** *If f and g are differentiable then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot g(x)\right) = f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) \cdot g(x) \tag{2.9}$$

Restated:

$$(f \cdot g)' = f \cdot g' + f' \cdot g \tag{2.10}$$

Example 2.4.1.1. If $f(x) = e^x x^4$ find $f'(x)$.

$$\begin{aligned} f'(x) &= e^x \cdot 4x^3 + x^4 \cdot e^x \\ f'(x) &= 4e^x x^3 + e^x x^4 \\ f'(x) &= e^x x^3(4 + x) \end{aligned}$$

Example 2.4.1.2. If $y = 4(x^2 - 7)$, find y'.

$$\begin{aligned} f'(x) &= 4x^2 - 28 \\ f'(x) &= 8x \end{aligned}$$

Quotient Rule

Theorem 2.4.2 (Quotient Rule)**.** *If f and g are differentiable at x and $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable at x and*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)}{g(x)^2} \tag{2.11}$$

Restated:

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2} \tag{2.12}$$

Remark. The order of the quotient rule can be remembered with the rhyme "hi d lo lo d hi all over the square of what's below".

Example 2.4.2.1. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{3x^3-5x}{5e^x+2}\right)$

$$\begin{aligned} f' &= 9x^2-7 \\ g' &= 5e^x \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^3-5x}{5e^x+2}\right) = \frac{\left(5e^x+2\right)\left(9x-7\right)-\left(3x^3-7x\right)\left(5e^x\right)}{\left(5e^x+2\right)^2}$$

Remark. In some cases it is okay to not simplify the answer.

2.4.2 Revising the Power Rule

$$\begin{aligned} f(x) &= x^n & \qquad f(x) &= Ax^n \\ f'(x) &= nx^{x-1} & \qquad f'(x) &= Anx^{n-1} \end{aligned}$$

Where n is <u>any</u> integer.

Example 2.4.2.2. Find $f'(x)$ if $f(x) = \dfrac{1}{3x^4}$.

$$\begin{aligned} f(x) &= \frac{1}{3x^4} \\ f(x) &= \frac{1}{3}x^{-4} \\ f(x) &= -\frac{4}{3}x^{-5} \end{aligned}$$

2.4.3 Higher Order Derivatives

Definition 2.4.1. The derivative if f' is the second derivative of f.

Notation: $f''(x)$

Read: "f double prime of x"

Can also consider third derivative f''', fourth derivatives $f^{(4)}$, etc.

Example 2.4.2.3. If $f(x) = 5x^3$, find f', f'', and f'''

$$\begin{aligned} f'(x) &= 15x^2 \\ f''(x) &= 30x \\ f'''(x) &= 30 \end{aligned}$$

Why do we care?

We know $f'(x)$ tells us the rate of change of f. What does $f''(x)$ tell us?

- Rate of change of the rate of change...
- In the context of $f =$ position:
  - f' is velocity (how fast the position is changing)
  - f'' is acceleration (how fast the velocity is changing)
- In the context of $f =$ number of unemployed people in the U.S.:
  - f' is how quickly unemployment is growing or shrinking
  - Suppose we are in a recession where unemployment is increasing. As f'' decreases, it means that jobs are being more slowly.

Example 2.4.2.4. A rock thrown vertically from the surface of the moon at an initial velocity of 24 [m/s] reaches a height of $s = 24t - 0.8t^2$ meters in t seconds

1. What is the velocity at time t? What is the acceleration?
2. How long before the rock reaches its highest point?
3. How high does the rock go?
4. How long before the rock reaches half of it's maximum height?
5. How long is the rock aloft?
6. What is the rock's speed on impact?

Answers:

1.

$$\begin{aligned} v &= s' = 24 - 16t[m/s] \\ a &= s'' = -1.6[m/s^2] \end{aligned}$$

2.

$$\begin{aligned} 24 - 1.6t &= 0 \\ 24 &= 1.6t \\ t &= 15[s] \end{aligned}$$

3.

$$\begin{gathered} 24\,(15) - 0.8\,(15)^2 \\ 180[m] \end{gathered}$$

4.

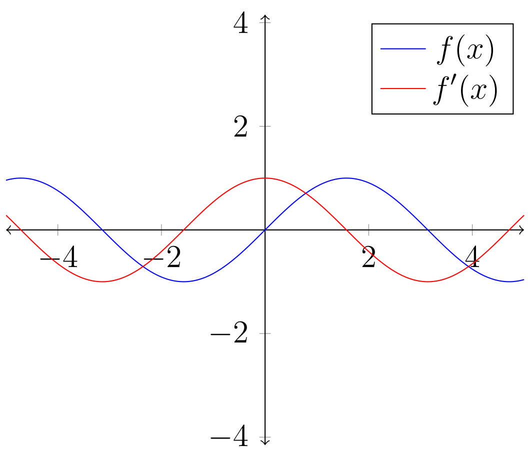
$$\begin{gathered}
90 = 24\,(t) - 0.8\,(t)^2 \\
0 = -0.8t^2 + 24t - 90 \\
\frac{-24 \pm \sqrt{576 - 4 \cdot -0.8 \cdot -90}}{-1.6} \\
\frac{-24 \pm \sqrt{288}}{-1.6} \\
\frac{-24 \pm 12\sqrt{2}}{-1.6} \\
t = \underline{4.3934}, 25.6066
\end{gathered}$$

5. $30[s]$ (Two times the time to reach the peak (see #2)

6. $-24[m/s]$ (Same as initial velocity but negative)

2.5 The Derivative of Trigonometric Functions

Looking at the graph of $y = \sin(x)$, Can we get an idea of how the derivative looks?



The derivative of $y = \sin(x)$ is $y' = \cos(x)$

2.5.1 Derivatives of $f(x) = \sin(x)$ and $f(x) = \cos(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x) \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x) \tag{2.14}$$

Example 2.5.0.1. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\cos(x)\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\right) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right) = -\sin(x)$$

$$x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x\right) \cdot \cos(x)$$
$$x\left(-\sin(x)\right) + 1 \cdot \cos(x)$$
$$-x\sin(x) + \cos(x)$$

2.5.2 Derivative of $f(x) = \tan(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(x)\right) = \sec^2(x) \tag{2.15}$$

Proof. Proof that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan(x)\right) = \sec^2(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(x)}{\cos(x)}\right)$$
$$\frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot -\sin(x)}{\cos^2(x)}$$
$$\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$
$$\frac{1}{\cos^2(x)} \quad \text{-OR-} \quad 1 + \tan^2(x)$$
$$\sec^2(x)$$

□

Example 2.5.0.2. Find the derivative of $y = \cot(x)$ in two ways: Using $sin(x)$ and $cos(x)$, and using $tan(x)$.

Method 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos(x)}{\sin(x)}\right)$$
$$\frac{\sin(x)\cdot -\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)}$$
$$\frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$
$$\frac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)}$$
$$-\frac{1}{sin^2(x)}$$
$$-\csc^2(x)$$

Method 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\tan(x)}\right)$$
$$\frac{\tan(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(1) - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x))}{\tan^2(x)}$$
$$\frac{\cancel{\tan(x)\cdot 0} - \sec^2(x)}{\tan^2(x)}$$
$$-\frac{\sec^2(x)}{\tan^2(x)}$$
$$\frac{\frac{1}{\cancel{\cos^2(x)}}}{\frac{\sin^2(x)}{\cancel{\cos^2(x)}}}$$
$$-\frac{1}{\sin^2(x)}$$
$$-\csc^2(x)$$

2.5.3 Derivatives of SIX basic trig functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x) \tag{2.16}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x) \tag{2.17}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \sec^2(x) \tag{2.18}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot(x)\right) = -\csc^2(x) \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(x)\right) = \sec(x)\tan(x) \tag{2.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc(x)\right) = -\csc(x)\cot(x) \tag{2.21}$$

Chapter 3

More About Derivatives

3.1 The Chain Rule

Suppose we have

$$f(x) = \sin(x^2)$$

It is a composite function

$$g(x) = \sin(x)$$
$$h(x) = x^2$$
$$g(h(x)) = f(x)$$
$$(g \circ h)(x) = f(x)$$

3.1.1 The Chain Rule

Note. This is helpful in Calculus II

Theorem 3.1.1. *Suppose we have f and g are both differentiable then*

$$(f \circ g)' = f'(g(x)) \cdot g'(x) \tag{3.1}$$

Example 3.1.1.1.

$$f(x) = \sin(x^2)$$
$$f'(x) = \cos(x^2) \cdot 2x$$

Example 3.1.1.2.

$$y = \tan^2(\theta)$$
$$y = (\tan(\theta))^2$$

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g(\theta) = \tan(\theta) \qquad g'(\theta) = \sec^2(\theta)$$
$$y' = 2\left(\tan(\theta)\right) \cdot \sec^2(\theta)$$

Example 3.1.1.3.

$$s(x) = \csc(\cos(x))$$

$$\begin{aligned} f(x) &= \csc(x) & \qquad f'(x) &= -\csc(x)\cot(x) \\ g(x) &= \cos(x) & \qquad g'(x) &= -\sin(x) \end{aligned}$$

$$s'(x) = -\csc(\cos(x))\cot(\cos(x)) \cdot -\sin(x)$$
$$s'(x) = \sin(x)\csc(\cos(x))\cot(\cos(x))$$

Example 3.1.1.4.

a): find y'

$$y = \left(\frac{3x^2+1}{2x^2-x}\right)^4$$

$$\begin{aligned} f(x) &= x^4 & \qquad f'(x) &= -\csc(x)\cot(x) \\ g(x) &= \frac{3x^2+1}{2x^2-x} & \qquad g'(x) &= \frac{-3x^2-4x+1}{\left(2x^2-x\right)^2} \end{aligned}$$

$$y' = 4\left(\frac{3x^2+1}{2x^2-x}\right)^3\left(-\frac{3x^2-4x+1}{\left(2x^2-x\right)^2}\right)$$
$$y' = \frac{-4\left(3x^2+1\right)^3\left(3x^2+4x-1\right)}{\left(2x^2-x\right)^5}$$

b) Find where the curve has horizontal tangents. (y' = 0)

$$\frac{-4\left(3x^2+1\right)^3\left(3x^2+4x-1\right)}{\left(2x^2-x\right)^5} = 0$$
$$-4\left(3x^2+1\right)^3\left(3x^2+4x-1\right) = 0$$
$$\cancel{-4 = 0} \qquad \text{no solutions}$$
$$3x^2+1 = 0 \qquad \text{no solutions}$$
$$3x^2+4x-1 = 0$$
$$x = \frac{-4 \pm \sqrt{16-4\,(3)\,(-1)}}{2\,(3)}$$
$$x = \frac{-4 \pm \sqrt{38}}{6}$$

Example 3.1.1.5.

$$
\begin{aligned}
y &= (x)\left(\sec(e^x)\right) \\
y' &= x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(e^x)\right) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right) \\
y' &= x \cdot \sec(e^x)\tan(e^x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right) \\
y' &= x \cdot e^x \sec(e^x)\tan(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x) \cdot \left(\sec(e^x)\right) \\
y' &= x \cdot e^x \sec(e^x)\tan(e^x) + 1 \cdot \left(\sec(e^x)\right) \\
y' &= \sec(e^x)\left(xe^x \tan(e^x) + 1\right)
\end{aligned}
$$

Note. Do you chain rule or product rule "first"? It depends!

3.1.2 Constant to the Power of x Rule

We want to derive $y = 3^x$.
Start by writing 3^x in terms of e^x:

$$
\begin{aligned}
3^x &= e^{\ln(3^x)} \\
3^x &= e^{x\ln(3)}
\end{aligned}
$$

Use the chain rule:

$$
\begin{aligned}
f(x) &= e^x & f'(x) &= e^x \\
g(x) &= x\ln(3) & g'(x) &= \ln(3)
\end{aligned}
$$

$$
\begin{aligned}
y' &= e^{x\ln(3)} \cdot \ln(3) \\
y' &= 3^x \cdot \ln(3)
\end{aligned}
$$

Theorem 3.1.2 (Constant to the Power of x Rule)**.** *If we assume that a is constant where $a > 0$ and $a \neq 1$ then:*

$$
\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \cdot \ln(a) \tag{3.2}
$$

Example 3.1.2.1.

$$
y = 2^{x^5}
$$

$$
\begin{aligned}
f(x) &= 2^x & f'(x) &= 2^x \cdot \ln(2) \\
g(x) &= x^5 & g'(x) &= 5x^4
\end{aligned}
$$

$$
\begin{aligned}
y' &= 2^{x^5}\ln(2) \cdot 5x^4 \\
y' &= 5 \cdot \ln(2) \cdot x^4 \cdot 2^{x^5}
\end{aligned}
$$

Example 3.1.2.2.

$$y = 3\sec(2^x)$$
$$y' = 3\frac{\mathrm{d}}{\mathrm{d}x}(\sec(2^x))$$

$$f(x) = \sec(x) \qquad f'(x) = \sec(x)\tan(x)$$
$$g(x) = 2^x \qquad g'(x) = 2^x \cdot \ln(2)$$

$$f'(x) = 3\sec(2^x)\tan(2^x) \cdot 2^x \ln(2)$$
$$f'(x) = (3\ln(2))\, 2^x \sec(2^x)\tan(2^x)$$

3.2 Implicit Differentiation

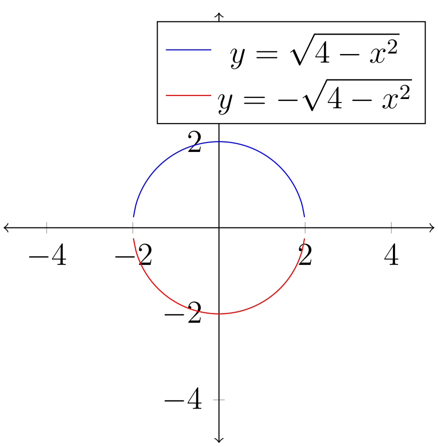
Some curves' equations can't be solved for y (or maybe not easily), but we should still be able find the tangent line and its slope.

Example 3.2.0.1.

$$x^2 + y^2 = 4$$

- Not a function!
- Can solve for y

$$y = \pm\sqrt{4 - x^2}$$



Given an expression with xs and ys to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

1. Treat y as a function of x and differentiate both sides of the equation with respect to x
2. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

3. Win

Note. When doing step one (1), you can thing "Whenever I thake the derivative of y multiply that term by $\frac{\mathrm{d}y}{\mathrm{d}x}$."

Example 3.2.0.2. If $x^2 + y^2 = 4$ use implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$

a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(4)\\ 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\cancel{2y}\frac{\mathrm{d}y}{\mathrm{d}x}}{\cancel{2y}} &= \frac{-2x}{2y}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2x}{2y}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}\end{aligned}$$

b) Find the slope of the tangent line at $\left(1, \sqrt{3}\right)$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,\sqrt{3})} = -\frac{1}{\sqrt{3}}$$

Note. The line after $\frac{\mathrm{d}y}{\mathrm{d}x}$ is read as "$\frac{\mathrm{d}y}{\mathrm{d}x}$ evaluated with $x = 1$ and $y = \sqrt{3}$"

c) What is the equation of the tangent at $\left(1, \sqrt{3}\right)$

$$\begin{aligned}y - y_1 &= m\left(x - x_1\right)\\ y - \sqrt{3} &= -\frac{1}{\sqrt{3}}(x-1)\end{aligned}$$

3.3 Derivatives of Logarithmic Functions

Theorem 3.3.1 (Derivatives of Logarithmic Functions)**.** *If we recall that* $\log_a(x) = y$*, then* $a^y = x$ *as well as* $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x\ln(a)$ *then the following must be true:*

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a(x)) = \frac{1}{x\ln(a)} \tag{3.3}$$

Proof.

$$\begin{aligned}
\log_a(x) &= y \\
a^y &= x \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(a^y\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\right) \\
a^y \ln(a) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{a^y \ln(a)}
\end{aligned}$$

This is fine but we don't want y in our answer.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln(a)}$$

This is valid because we know that $a^y = x$

□

What about $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right)$?

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\log_e(x)\right) \\
&= \frac{1}{x \ln(e)} \\
&= \frac{1}{x}
\end{aligned}$$

Note. The power rule will never give an exponent of -1, since $n = 0$ would mean that it's a constant.

Example 3.3.1.1. Differentiate: $f(x) = \dfrac{x}{\ln(x)}$

$$\begin{aligned}
f'(x) &= \frac{\left(\ln(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x\right) - \left(x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right)}{\left(\ln(x)\right)^2} \\
&= \frac{\left(\ln(x)\right)\left(1\right) - \left(x\right)\left(\dfrac{1}{x}\right)}{\left(\ln(x)\right)^2} \\
&= \frac{\ln(x) - 1}{\left(\ln(x)\right)^2}
\end{aligned}$$

$$= \frac{\ln(x)}{(\ln(x))^2} - \frac{1}{(\ln(x))^2} \qquad \text{(Optional)}$$

$$= \frac{1}{\ln(x)} - \frac{1}{(\ln(x))^2} \qquad \text{(Optional)}$$

Example 3.3.1.2. Differentiate $y = |x|$

$$y = \begin{cases} \ln(x) & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

Note. Domain is all real number except 0

$$\begin{aligned} x &> 0 : \\ y &= \ln(x) \\ y' &= \frac{1}{x} \end{aligned}$$

$$\begin{aligned} x &< 0 : \\ y &= \ln(-x) \\ y' &= \frac{1}{-x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(-x) \\ y' &= \frac{1}{-x}(-1) \\ &= \frac{1}{x} \end{aligned}$$

Conclusion: because y' was the same in both cases then the following must be true:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(|x|)) = \frac{1}{x}$$

Appendix A

Buddy Chart

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right) = a^x \ln(a)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\right) = \cos(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right) = -\sin(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(x)\right) = \sec^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc(x)\right) = -\csc(x)\tan(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec(x)\right) = \sec(x)\tan(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot(x)\right) = -\csc^2(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\csc^{-1}(x)\right) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{-1}(x)\right) = -\frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\log_a(x)\right) = \frac{1}{x\ln(a)}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right) = \frac{1}{x}$$