Group Rewriting

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1 Clifford Group

1.1 Pauli Group

1.1 Definition (P_n) *The Pauli Group on n qubits is the group generated by σ_x σ_y σ_z and i operating on each of n qubits. As a group it is isomorphic to $(D_4 \rtimes C_2)^n$*

1.2 Clifford Group

1.2 Definition (C_n) *The normalizer of P_n in $U(2^n)$.*

1.3 Lemma *It is generated by the generators of P_n along with the following ?.*

1.3 Cliff+T

1.4 Theorem (Universality)

1.5 Theorem (Solovay-Kitaev)

As a corolary of universality we know that we can take an arbitrary rotation of one qubit $SU(2)$ to an approximation using only Cliff+T gates. A desired error bound must be given. An algorithm for this is given by
`https://arxiv.org/pdf/1403.2975.pdf`
but it is faster than Solovay-Kitaev which is more general.

2 Coxeter Groups

2,3 qubit relations with weyl exceptional other fun groups
`https://hal.inria.fr/file/index/docid/420456/filename/E8Weyl.pdf`

https://arxiv.org/pdf/0807.3650.pdf

Rewriting system for coxeter group
https://ac.els-cdn.com/0022404994900191/1-s2.0-0022404994900191-main.pdf?_tid=1f07609c-04b2-11
acdnat=1517202550_f41a15336d6c1eb1d880008e0214ec41

Words with intervening property for a quick check of reducedness. Outputs a witness of nonreducedness. But no witness found does not imply reduced.
Given an expression and a coxeter graph, algorithm to find a segment of the word to show it is not reduced/null if this is inconclusive
Then feed that into the reducing algorithm. Repeat until intervening property says the expression might be reduced.
http://emis.ams.org/journals/EJC/Volume_17/PDF/v17i1n9.pdf

Algorithm for reducing words in Coxeter group
https://mathoverflow.net/questions/109071/algorithm-for-reducing-words-in-a-coxeter-group

2.1 X,Y,Z,H and Swaps

2.1 Lemma *The group generated by these gates has the following relations.*

$$
\begin{aligned}
(X_iY_i)^4 &= 1 \\
(X_iZ_i)^4 &= 1 \\
(Y_iZ_i)^4 &= 1 \\
(H_iX_i)^8 &= 1 \\
(H_iY_i)^4 &= 1 \\
(H_iZ_i)^8 &= 1 \\
(H_iS_{i,i+1})^4 &= 1 \\
(X_iS_{i,i+1})^4 &= 1 \\
(Y_iS_{i,i+1})^4 &= 1 \\
(Z_iS_{i,i+1})^4 &= 1 \\
(H_{i+1}S_{i,i+1})^4 &= 1 \\
(X_{i+1}S_{i,i+1})^4 &= 1 \\
(Y_{i+1}S_{i,i+1})^4 &= 1 \\
(Z_{i+1}S_{i,i+1})^4 &= 1 \\
(S_{i,i+1}S_{i+1,i+2})^3 &= 1
\end{aligned}
$$

There may be additional relations, but that would just be more quotienting, so reducing a word using only these relations would still be valid.

Proof See the Mathematica notebook □

Make a picture of the Coxeter graph with $i = 1, 2, 3$ for 3 qubits. 3 colors of edges for order 3, 4 and 8. As usual no edge for order 2. For more qubits just extend this graph by adding more basic units of the same structure.

3 Finite Complete Rewriting Systems

Surface groups but more importantly for here theorem about short exact sequences
https://arxiv.org/pdf/math/9611205.pdf

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https://ac.els-cdn.com/0022404994900191/1-s2.0-0022404994900191-main.pdf?_tid=1f07609c-04b2-11
acdnat=1517202550_f41a15336d6c1eb1d880008e0214ec41

Gap for all finitely presented groups
http://doc.sagemath.org/html/en/reference/groups/sage/groups/finitely_presented.html

4 Unsorted

https://arxiv.org/pdf/1509.02004.pdf
https://arxiv.org/pdf/1701.05200.pdf