Gate Identification

Ammar Husain

August 15, 2018

1 Josephson Oscillator

Recall a single qubit constructed from a Josephson junction. The qubits are the $\mid 0\rangle$ and $\mid 1\rangle$ states of the number operator. The gates are then constructed by adjusting E_J and E_C.

$$\begin{aligned}
H_T &= \frac{-1}{2} E_J \sum_m \mid m\rangle\langle m+1 \mid + \mid m+1\rangle\langle m \mid \\
\mid \phi\rangle &= \sum e^{im\phi} \mid m\rangle \\
H_T \mid \phi\rangle &= -E_J \cos\phi \mid \phi\rangle \\
v_{group}(\phi) &= \frac{1}{\hbar} \frac{\partial}{\partial\phi}(-E_J \cos\phi) \\
I(\phi) &= \frac{2e}{\hbar} E_J \sin\phi
\end{aligned}$$

This gives a maximal coherent (dissipationless) current $\frac{2e}{\hbar} E_J$. Any more current will cause the voltage to rise above the 2Δ gap and break the approximation that we only need the ground states determined by the number of pairs only.

Now with a Coulomb term as well for this little capacitor

$$H = 4E_C(n - n_g)^2 - E_J \cos\phi$$

1.1 Definition (Mathieu Equation)

$$\Big[\frac{d^2}{dx^2} + (a - 2q\cos(2x))\Big] y = 0$$

For fixed a and q, the solution can be expressed as $F(a,q,x) = e^{i\mu x} P(a,q,x)$ with P π-periodic.

1.2 Definition (Hill Operator) *More generally, let $q(x)$ be a periodic real function instead of just the $a - 2q_0 \cos 2x - \lambda$ as above.*

$$\begin{aligned}
Hy &= -y'' + q(x)y \\
Hy &= \lambda y
\end{aligned}$$

1.3 Definition (Mathieu cosine and sine) *The solution with value 1 at $x = 0$ and derivative 0 or vice versa.*

These are real valued solutions.

$$
\begin{aligned}
C(a, q, x) &= \frac{F(a, q, x) + F(a, q, -x)}{2F(a, q, 0)} \\
S(a, q, x) &= \frac{F(a, q, x) - F(a, q, -x)}{2F'(a, q, 0)}
\end{aligned}
$$

If $q = 0$ then these are $\cos \sqrt{a}x$ and $\frac{\sin \sqrt{a}x}{\sqrt{a}}$ respectively. In general they are aperiodic.

For a given value of q there are countably many a that give periodic solutions. For example, if $q = 0$, then $a = (n)^2$ with $n \in \mathbb{Z}$ in order to achieve 2π periodicity.

$$
\begin{aligned}
H &= -4E_C\frac{\partial^2}{\partial\phi^2} + 8E_C n_g i\frac{\partial}{\partial\phi} + 4E_C n_g^2 - E_J\cos\phi \\
H_2 &= \frac{-H}{4E_C} + n_g^2 = \frac{\partial^2}{\partial\phi^2} - 2n_g i\frac{\partial}{\partial\phi} + \frac{E_J}{4E_C}\cos\phi \\
H_2\psi = \lambda_{H_2}\psi &\implies -4E_C(\lambda_{H_2} - n_g^2) = \lambda_H
\end{aligned}
$$

Let $\phi = 2x$

$$
\begin{aligned}
\frac{1}{4}\frac{\partial^2\psi}{\partial x^2} - n_g i\frac{\partial\psi}{\partial x} + \frac{E_J}{4E_C}\cos(2x)\psi &= \lambda\psi \\
\frac{\partial^2\psi}{\partial x^2} - 4n_g i\frac{\partial\psi}{\partial x} + (-4\lambda + \frac{E_J}{E_C}\cos(2x))\psi &= 0
\end{aligned}
$$

If $n_g = 0$ then we get $\psi = a_S C(-4\lambda, \frac{-E_J}{2E_C}, \frac{\phi}{2}) + a_A S(-4\lambda, \frac{-E_J}{2E_C}, \frac{\phi}{2})$

Now keep n_g, but define w

$$
\begin{aligned}
w &= \psi e^{-2in_g x} \\
w &= C_1 C(-4\lambda + 4n_g^2, -\frac{E_J}{2E_C}, x) + C_2 S(-4\lambda + 4n_g^2, -\frac{E_J}{2E_C}, x) \\
\psi &= e^{2in_g x} w
\end{aligned}
$$

Proof

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= \frac{\partial w}{\partial x}e^{2in_gx} + (2in_g)we^{2in_gx} \\
\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial^2 w}{\partial x^2}e^{2in_gx} + 2in_g\frac{\partial w}{\partial x}e^{2in_gx} + 2in_g\frac{\partial w}{\partial x}e^{2in_gx} - 4n_g^2we^{2in_gx} \\
\frac{\partial^2\psi}{\partial x^2} - 4n_gi\frac{\partial\psi}{\partial x} + (-4\lambda + \frac{E_J}{E_C}\cos 2x)\psi &= \frac{\partial^2 w}{\partial x^2}e^{2in_gx} + 4n_g^2we^{2in_gx} - (4\lambda + \frac{E_J}{E_C}\cos 2x)we^{2in_gx} \\
\frac{\partial^2 w}{\partial x^2} + (-4\lambda + 4n_g^2 - 2\frac{E_J}{2E_C}\cos 2x)w &= 0 \\
w &= C_1C(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, x) + C_2S(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, x) \\
\psi &= C_1e^{in_g\phi}C(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, \frac{\phi}{2}) + C_2e^{in_g\phi}S(-4\lambda + 4n_g^2, \frac{E_J}{2E_C}, \frac{\phi}{2})
\end{aligned}$$

We want ψ to be periodic under $\phi \to \phi + 2\pi$ so $x \to x + \pi$. So w must go to $e^{-2in_g\pi}w$ upon the same shift. This gives countably many allowed values for $-4\lambda + 4n_g^2$ so countably many values for the energy λ_H. That is the spectrum as a function of $E_{C,J}$ and n_g.

Let's say $C_1 = 1$ and $C_2 = 0$. In Mathematica this is given by MathieuCharacteristicA[$-2n_g + k$,$-E_J/2E_C$]. Call it MCA for short. That is under $x \to x + 2\pi$ pick up a $e^{ir2\pi}$.

$$\lambda = n_g^2 - \frac{1}{4}MCA(k - 2n_g, -\frac{E_J}{2E_C})$$

Similarly for the odd functions we have

$$\lambda = n_g^2 - \frac{1}{4}MCB(k - 2n_g, -\frac{E_J}{2E_C})$$

Together the spectrum is the collection of all

$$\begin{aligned}
\lambda_H &= E_CMCA(k - 2n_g, -\frac{E_J}{2E_C}) \\
\lambda_H &= E_CMCB(k - 2n_g, -\frac{E_J}{2E_C})
\end{aligned}$$

2 More general 1 qubit

Notice that if we define $y = e^{i\phi}$, $\cos\phi$ has degrees ± 1 in y. In addition the Coulomb term is quadratic in $n = i\hbar\frac{d}{d\phi}$. Therefore, for unknown dynamics H let us guess a form

$$\begin{aligned}
H &= A_{0,0} + A_{0,1}n + A_{1,0}e^{i\phi} + A_{-1,0}e^{-i\phi} \\
&+ A_{0,2}n^2 + A_{1,1}e^{i\phi}n + A_{-1,1}e^{-i\phi}n \\
&+ A_{2,0}e^{2i\phi} + A_{-2,0}e^{-2i\phi} \\
&+ h.c.
\end{aligned}$$

which has all the $A_{m,n}$ such that $\mid m \mid +n \leq 2$. Generally let the maximal degree be D.

3 Multiple qubits

Let $n_1 \cdots n_k$ be the number operators and $\phi_i \cdots \phi_k$ corresponding phases. Again maximal degree D.

$$\begin{aligned} H &= \sum A_{a_1,b_1,a_2,b_2 \cdots a_k,b_k} \prod_i e^{ia_i\phi_i} n_i^{b_i} + h.c. \\ \sum_i \mid a \mid_i + b_i &\leq D \end{aligned}$$

The total number of parameters here is upper bounded by $2^k \sum_{n \leq D} p(n, 2k)$ where the 2^k takes care of choosing the signs on the a_i and the $p(n)$ are the number of compositions of D into $2k$ natural numbers. If D is too big, this is unmanageable, but for smaller D it is more reasonable.

See HamiltonianLearning repository for the corresponding classical problem with the same kind of constraint being used to make the inverse dynamic problem solvable.

$$\begin{aligned} \langle H \rangle &= \sum \langle A_{a_1,b_1,a_2,b_2 \cdots a_k,b_k} \prod_i e^{ia_i\phi_i} n_i^{b_i} + h.c. \rangle \\ &= \sum Re(A_{a_1,b_1,a_2,b_2 \cdots a_k,b_k}) \langle \prod_i e^{ia_i\phi_i} n_i^{b_i} + h.c. \rangle \\ &+ \sum Im(A_{a_1,b_1,a_2,b_2 \cdots a_k,b_k}) \langle i (\prod_i e^{ia_i\phi_i} n_i^{b_i} - h.c.) \rangle \end{aligned}$$

If we have the statistics on those expectation values, then we get a linear algebra problem over the reals for the unknown $Re(A)$ and $Im(A)$. WLOG we have demanded that $\langle H \rangle = 0$ by adjusting $A_{0 \cdots 0}$. In order to get these statistics we must do repeated observations where the state was initialized the same and then allowed to evolve for some time.

3.1 Bootstrapping

We first want a good guess for the Hamiltonian. This is done by identifying the case from the more general 1 qubit. Call that H_i for it only applied to qubit i. The initial guess for k qubits is then

$$H = \sum_{j=1}^{k} H_i$$

H_i have only $A_{0 \cdots 0, a_i, b_i, 0 \cdots 0}$ nonzero. Then do similar for H_{ij} to focus on the case of 2-qubits in isolation from the other $k-2$. 2-local Hamiltonians are typical so one might stop there.