Entanglement in the Grothendieck Ring

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Abstract

In this paper we calculate the elements in the Grothendieck ring over $\mathbb{C}$ given by N quantum particles with distinguishable,bosonic or fermionic statistics where each has a d dimensional Hilbert space with the constraint that there is nontrivial entanglement between all N. We then give the class for mixed states but no longer with a restriction on entanglement. The main motivation comes from the invariants of systems that come from understanding the topology of the state space via maps from parameter spaces M or BG. For this reason we define families of entangled states as potentially interesting mapping spaces to extract homotopic information from.

1 Notational Setup

In this section let us set up some notation for both the Grothendieck ring and the Segre embeddings.

1.1 Grothendieck Ring of Varieties

1.1 Definition ($K_0(Var_{\mathbb{C}})$) *The abelian group generated by symbols $[X]$ for a variety X over $\mathbb{C}$ subject to the relations, that $[X] = [Y]$ for isomorphic varieties and $[X] = [Y] + [X \setminus Y]$ for Y a closed subvariety of X. This is a ring with product $[X][Y] = [(X \times_{Spec\,\mathbb{C}} Y)_{red}]$. In particular, define L as $[\mathbb{A}^1]$.*

This can be done mutatis mutandis with other base fields. Many invariants factor through this ring. For example,

1.2 Definition (E-Polynomial) *For a complex algebraic variety X, let*

$$E(X) \quad = \quad \sum_{p,q=0}^{d} (\sum_{i=0}^{2d} (-1)^i h^{p,q}(H^i_{cpt}(X))) u^p v^q$$

If X is smooth and projective, then this simplifies to

$$\begin{aligned} E(X) &= \sum_{p,q} (-1)^{p+q} h^{p,q}(X) u^p v^q \\ h^{p,q}(X) &= \dim H^q(X, \Omega^p) \end{aligned}$$

From this definition on varieties and the Meyer-Vietoris sequence, this extends to a ring map $K_0(Var_{\mathbb{C}}) \to \mathbb{Z}[u, v]$.

1.3 Corollary *From the E-polynomial, one may plug in special values to get the Hirzebruch χ_y and further to the topological Euler characteristic χ, arithmetic genus or signature.*

$$\begin{aligned} E(X, u=-y, v=1) &= \chi_y(X) \\ E(X, u=1, v=1) &= \chi_{y=-1}(X) = \chi(X) \\ E(X, u=0, v=1) &= \chi_{y=0}(X) = \chi_{arithmetic}(X) \\ E(X, u=-1, v=1) &= \chi_{y=1}(X) = \mathrm{sign}(X) \end{aligned}$$

1.4 Remark We caution that when considering maps into X, we are not looking at in the Zarisiki topology as would seem natural for thinking of it as a variety. We are viewing it in the analytic topology instead. $\Diamond$

1.2 Projective Embeddings

1.2.1 Segre

1.5 Definition (Segre) *The embedding Seg $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \to \mathbb{P}^{nm-1}$ given by*

$$[\sum_{i=1}^{n} x_i \mid e_i\rangle] \times [\sum_{i=1}^{m} y_i \mid f_i\rangle] \quad \to \quad [\sum_{i,j} x_i y_j \mid e_i\rangle \otimes \mid f_j\rangle]$$

This gives the unentangled pure states on $\mathbb{C}^n \otimes \mathbb{C}^m$. We are interested in the converse situation of entangled states. By abuse of notation let *Seg* also denote the maps for $\prod_i \mathbb{P}^{x_i - 1} \to \mathbb{P}^{\prod_i x_i - 1}$ including the case $x_i = d^{\lambda_i}$.

In this case we have the embedding *Seg* $\mathbb{P}^{d^{\lambda_1}-1} \cdots \times \mathbb{P}^{d^{\lambda_{max}}-1} \to \mathbb{P}^{d^n-1}$. For example, this may be the case of a system in which each batch of λ_i have been put into separate cavities so we should expect the state to be a tensor product of many ground states for each cavity. Conversely, if they are connected it would be unexpected to find any subsystem unentangled.

1.6 Lemma ([1]) *This subvariety is cut out by the following master equation*

$$x_{12}x_{34} - x_{14}x_{32} \quad = \quad 0$$

where the set of $1 \cdots max$ is divided into 2 subsets A and B. 1 and 3 are replaced by indices in $\prod_{k \in A} D^{\lambda_k}$ and 2 and 4 are replaced by indices in $\prod_{k \in B} D^{\lambda_k}$ where D is the index set for the coordinate basis of $\mathbb{C}^d$ and D^{λ_k} for $(\mathbb{C}^d)^{\otimes \lambda_k}$. This means together each of the above x_{ij} are coefficents of one of the induced basis vectors of $(\mathbb{C}^d)^{\otimes n}$.

If a given point in $[\mid x\rangle] \in \mathbb{P}^{d^n-1}$ is in this subvariety, one may repeatedly refine the partition until this criterion fails. This informs you of the entanglement profile of the state.

Of course, because these are $2^{max} d^{2n}$ equations there are further syzigies to account for the redundancy.

1.2.2 Plücker

1.7 Definition (Plücker) *Consider the Grassmannian of k dimensional subspaces in N dimensional Hilbert space. Choose a basis $\mid \psi_1\rangle \cdots \mid \psi_k\rangle$ of the k dimensional subspace. Take the Slater determinant of these vectors to give a vector in the fermionic Fock space $\mathbb{C}^{\binom{N}{k}}$. Choosing a different basis gives a proportional vector, so there is a well defined map $Gr(k, N) \to \mathbb{P}^{\binom{N}{k}-1}$.*

2 Parameterized States

In this section, we quickly review the existing literature without any entanglement assumptions in order to give the motivation for studying the topology of spaces of states.

Consider a family of Hamiltonians $H(p)$ on $(\mathbb{C}^d)^{\otimes n}$ parameterized by $p \in M$ for some manifold M. Suppose for all p we provide a state. For example, this might be the ground state for each of the $H(p)$ if all the Hamiltonians are suitable and there is a unique ground state at all points throughout the family. The Berry connection serves to transport between different points p and q. So assume that the map $M \to \mathbb{P}^{d^n-1}$ is continuous.

2.1 Lemma ([2]) *Suppose there is a symmetry G such that on the states $\mid \psi(p)\rangle\langle\psi(p) \mid$ are all G invariant for all $p \in M$. This gives a map $M \to (\mathbb{P}^{d^n-1})^G$. We can weaken this homotopically into a map $BG \to \mathbb{P}^{d^n-1}$ defined up to homotopy. Let $d^n \to \infty$ which makes this a map $[BG, \mathbb{P}^\infty \simeq K(\mathbb{Z},2)]$. This leads to an invariant $H^2(G,\mathbb{Z})$ classifying $0+1$ bosonic SPTs without time reversal.*

If the $\mathbb{CP}^\infty$ was endowed with the structure of complex conjugation, that would be the case with time reversal.

For higher dimensions D, replace this by $[BG, S^D\mathbb{P}^N]$, and take the limit $N \to \infty$. The result is $[BG, K(\mathbb{Z}, 2+D)] \simeq H^{D+2}(G,\mathbb{Z})$ thanks to Ω-spectra.

2.2 Lemma ([3, 4]) *Let the parameter space be the Brillouin Zone T^D. For each such point, assign the ground state of k fermions with that quasiperiodic boundary condition. By a band insulator assumption, this is a continuous map*

$$T^D \longrightarrow Gr(k,N) \xrightarrow{Plücker} \mathbb{P}^{\binom{N}{k}-1} \quad \text{with } N \text{ taken to } \infty \text{ to exhaust all of } L^2(\mathbb{R}^D).$$

Homotopically this is a map $[T^D, BU(k)]$ classifying rank k vector bundles. Let k be indeterminant so it is allowed to be arbitrarily large. The result is $[T^D, BU]$ giving $K^0(T^D)$. Fixing the filling fraction k gives a strictly better invariant. Having the invariants for all k will show which filling fractions have new behaviors compared with those that come from a direct sum bundle $k_1 + k_2 = k$.
Equivariance with respect to point group symmetries $P \subset O(D)$ can also be imposed.
Time reversal with $T^2 = +1$ means all of the constituent $\mid \psi_i\rangle$ can be chosen to have real wavefunctions replacing $BU(k)$ by $BO(k)$ and a fortiori $Gr_{\mathbb{C}}(k,N)$ by $Gr_{\mathbb{R}}(k,N)$.

3 The class of entangled states

Now that we have reviewed the value of topology of spaces of states, let us consider the topology of spaces of states with some entanglement criterion fixed.

3.1 Definition ($P(\mathrm{dis}, n, d)$) *Let $P(dis,n,d)$ denote the class in the Grothendieck ring of the pure states on n distinguishable qudits which are in the complement of all the Segre embeddings of taking partitions of n. Let $X_{dis,n,d}$ be the underlying space. In particular $P(dis,1,d) = [\mathbb{P}^{d-1}]$ because it is vacuous to divide up one qudit into subsets.*

3.1 Small Examples

For 2 qubits:

$$\begin{aligned}
[\mathbb{P}^3] &= [\mathbb{P}^1][\mathbb{P}^1] + [\mathbb{P}^3 \setminus Seg(\mathbb{P}^1 \times \mathbb{P}^1)] \\
P(\mathrm{dis},2,2) &= [\mathbb{P}^3] - [\mathbb{P}^1]^2 = (1+L+L^2+L^3) - (1+L)^2 \\
&= L^3 - L
\end{aligned}$$

For 3 qubits:

$$\begin{aligned} P(\text{dis},3,2) &= (1+L+L^2\cdots L^7)-(1+L)^3-3*P(2,2)*(1+L) \\ &= L^7+L^6+L^5-2L^4-3L^3+L^2+L \end{aligned}$$

For 2 and 3 qutrits:

$$\begin{aligned} P(\text{dis},2,3) &= (1+L+L^2\cdots L^8)-(1+L+L^2)^2 \\ &= L^8+L^7+L^6+L^5-L^3-2L^2-L \\ P(\text{dis},3,3) &= (1+L+L^2\cdots L^{26})-(1+L+L^2)^3-3*P(2,3)*(1+L+L^2) \\ &= L^{26}+L^{25}+L^{24}+L^{23}+L^{22}+L^{21}+L^{20}+L^{19}+L^{18}+L^{17}+L^{16} \\ &+ L^{15}+L^{14}+L^{13}+L^{12}+L^{11}-2L^{10}-5L^9-8L^8-8L^7-6L^6 \\ &- 2L^5+4L^4+6L^3+4L^2+L \end{aligned}$$

3.2 General Case

3.2 Theorem

$$\begin{aligned} P(dis,n,d) &= \left(\sum_{i=0}^{d^n-1} L^i\right)-\sum_{\substack{\lambda\vdash n\\ \lambda\neq(n)}} F(\lambda)\prod P(dis,\lambda_i,d) \\ F(\lambda) &= \frac{1}{\alpha_1!\cdots\alpha_r!}\binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2}\cdots\binom{n-\sum_{i=1}^{max-1}\lambda_i}{\lambda_{max}} \end{aligned}$$

$F(\lambda)$ is the number of partitions of the set of n qudits whose block sizes correspond to λ.

Proof The first summation is the expansion of the ambient $[\mathbb{P}^{d^n-1}]$.

For set partitions $\nu_{1,2}$ of n the states whose entanglement profiles are exactly $\nu_{1,2}$ are disjoint from each other. Each one is an embedding of $\prod \mathbb{P}^{d^{\lambda_i}-1}$ where λ_i are the block sizes of ν. To avoid overcounting we only want the $P(\text{dis},\lambda_i,d)$ part of each.

The number of set partitions of n with the same block sizes is given by selecting λ_i from the n with the multinomial coefficient and then dividing by the symmetry that comes from exchanging blocks of the same sizes. If $\lambda=1^{\alpha_1}\cdots m^{\alpha_m}=\lambda_1\lambda_2\cdots\lambda_{max}$ in alternate notations for partitions with number of parts of lengths m and the standard part length parameterization, then this gives the formula above. □

3.3 Corollary *$P(dis,n,d)$ is polynomial of degree d^n-1 in L with integer coefficients and leading term 1. In particular, it's roots are algebraic integers.*

Proof The $[\mathbb{P}^{d^n-1}]$ is a polynomial of degree d^n-1. The question is whether any of the terms subtracted off have a large enough degree to cancel this. They are of degree

$$lt(term_\lambda) = \prod lt(P(\text{dis},\lambda_i,d))$$

Given the inductive hypothesis on all the $\lambda_i<n$:

$$
\begin{aligned}
lt(term_\lambda) &= \prod lt(P(\text{dis}, \lambda_i, d)) \\
&= \prod (d^{\lambda_i} - 1) \\
&< \prod (d^{\lambda_i}) - 1 = d^n - 1 \\
lt(term_\lambda) &< d^n - 1
\end{aligned}
$$

The base case is $P(\text{dis}, 1, d)$ which is of degree $d-1$ in L.

3.4 Corollary *Let χ denote the map $K_0(Var_{\mathbb{C}}) \to \mathbb{Z}$ for the topological Euler characteristic and similarly for $\chi_{arithmetic}$, χ_y and* sign. *Then:*

$$
\begin{aligned}
\chi P(dis, 1, d) &= d \\
\chi P(dis, n \neq 1, d) &= 0 \\
\chi_{arithmetic} P(dis, 1, d) &= 1 \\
\chi_{arithmetic} P(dis, n \neq 1, d) &= 0 \\
\chi_{y=\zeta_d} P(dis, n, d) &= 0
\end{aligned}
$$

where ζ_d is a d'th root of unity.

Proof

$$
\begin{aligned}
\chi P(\text{dis}, 1, d) &= \chi(1 + L \cdots L^{d-1}) = d \\
\chi_{arithmetic} P(\text{dis}, 1, d) &= \chi_{arithmetic}(1 + L \cdots L^{d-1}) = 1
\end{aligned}
$$

Now assume by induction that $\chi(1 < n < N, d)$ and $\chi_{arithmetic}(1 < n < N, d)$ are all 0.

$$
\begin{aligned}
\chi P(\text{dis}, N, d) &= \chi(\sum_{i=0}^{d^N - 1} L^i) - \sum_{\substack{\lambda \vdash N \\ \lambda \neq (N)}} F(\lambda) \prod \chi P(\text{dis}, \lambda_i, d) \\
&= d^N - \sum_{\substack{\lambda \vdash N \\ \lambda \neq (N), (1^N)}} F(\lambda) \prod \chi P(\text{dis}, \lambda_i, d) - \chi(P(\text{dis}, 1, d))^N \\
&= - \sum_{\substack{\lambda \vdash N \\ \lambda \neq (N), (1^N)}} F(\lambda) \prod \chi P(\text{dis}, \lambda_i, d)
\end{aligned}
$$

but $\lambda \neq (1^N)$ but being a partition of N means all the parts are $< N$ and at least one is greater than 1. That means every summand gives 0. For $N = 2$, it is the empty summation which is 0. Similarly:

$$
\begin{aligned}
\chi_{arithmetic}P(\text{dis},N,d) &= \chi_{arithmetic}(\sum_{i=0}^{d^N-1} L^i) - \sum_{\substack{\lambda\vdash N\\ \lambda\neq(N)}} F(\lambda)\prod \chi_{arithmetic}P(\text{dis},\lambda_i,d)\\
&= 1 - \sum_{\substack{\lambda\vdash N\\ \lambda\neq(N),(1^N)}} F(\lambda)\prod \chi_{arithmetic}P(\text{dis},\lambda_i,d) - \chi_{arithmetic}(P(\text{dis},1,d))^N\\
&= -\sum_{\substack{\lambda\vdash N\\ \lambda\neq(N),(1^N)}} F(\lambda)\prod \chi_{arithmetic}P(\text{dis},\lambda_i,d)
\end{aligned}
$$

and again each summand is 0 for the same reason.

Let $\chi_{y=\zeta_d}P(\text{dis},n<N,d)=0$ be the inductive hypothesis:

$$
\begin{aligned}
\chi_{y=\zeta_d}P(\text{dis},N,d) &= \chi_{y=\zeta_d}(\sum_{i=0}^{d^N-1} L^i) - \sum_{\substack{\lambda\vdash N\\ \lambda\neq(N)}} F(\lambda)\prod \chi_{y=\zeta_d}P(\text{dis},\lambda_i,d)\\
&= -\sum_{\substack{\lambda\vdash N\\ \lambda\neq(N)}} F(\lambda)\prod \chi_{y=\zeta_d}P(\text{dis},\lambda_i,d)
\end{aligned}
$$

$\chi_{y=\zeta_d}P(\text{dis},1,d)=0$ because it is evaluation of a cyclotomic polynomial. In particular for $d=2k$ even, $y=-1$ is a d'th root of unity so sign $P(\text{dis},n,d=2k)=0$

3.3 Parameterized Entangled States

3.5 Definition (Family of Entangled States) *The map $M \to X_{dis,d,n}$ given by taking the ground state of $H(p)$ as before, but with the additional assumption on nontrivial entanglement. The homotopy class of this map is defined to be the invariant for this family.*

We have not given the full homotopy type of $X_{\text{dis},d,n}$ so this computation is not possible. However, we have given some cohomological information.

3.6 Definition (Cohomological Invariants) *For a given map $\psi \in [M, X_{dis,d,n}]$ and a cohomology theory $h^\bullet$, then the induced map $\psi^* \in Hom(h^i(X_{dis,d,n}) \to h^i(M))$. This is weaker than the full $[M, X_{d,n}]$ as a set but more manageable. Without the entanglement criterion, this is the passage to Chern classes instead of the full K-theory.*

Note that this cohomology does not have to be ordinary cohomology. It is allowed to be extraordinary such as TMF.

3.7 Remark If the parameter space has the structure of a complex scheme and the map from parameters to states has algebraic structure, then more subtle structures like coherent sheaves can be pulled back as well. A different topology on X would be at play. This may be an unreasonable request on $H(p)$. $\Diamond$

3.4 Zeta Function

Consider the space of configurations of r points chosen from $X_{\text{dis},n,d}$. Extracting the coefficient of t^r will reveal the topology of configurations of r states. One may imagine them as r replicas like in a disorder average. The following definitions are available in many places, but we refer to [5] because that includes the information theoretic perspective that inspired this work (though with a different axiomatization).

3.8 Definition (Kapranov Zeta function) *For every quasi-projective variety X, define*

$$Z_{mot}(X,t) \quad = \quad \sum_{r \geq 0} [Sym^r X] t^r \in 1 + tK_0(Var/\mathbb{C})[[t]]$$

This defines a map of abelian groups which takes addition in $K_0(Var_{\mathbb{C}})$ to multiplcation of power series.

3.9 Definition (Exponentiable Measure) *A ring R and a ring map μ $K_0(Var_{\mathbb{C}}) \to R$ such that μZ_{mot} becomes a ring map from $K_0(Var_{\mathbb{C}})$ to the Witt ring $W(R)$ which has multiplication of power series for it's addition and a more complicated multiplication determined uniquely by:*

$$\frac{1}{1-at} \star_{W(R)} \frac{1}{1-bt} \quad = \quad \frac{1}{1-(ab)t}$$

3.10 Corollary *Let μ be an exponentiable measure and letting $x = \mu(L)$.*

$$\begin{aligned} \mu Z_{mot}(P(dis,n,d),t) &= A \prod_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} B_\lambda^{-F(\lambda)} \\ A &\equiv \prod_{i=0}^{d^n-1} \frac{1}{1-x^i t} \\ B_\lambda &\equiv P(dis,\lambda_1,d) \star_{W(R)} \cdots \star_{W(R)} P(dis,\lambda_{max},d) \end{aligned}$$

One can do without the complicated $\star_{W(R)}$ if we expand out the polynomial first.

$$\begin{aligned} P(dis,n,d) &= \sum_{i=0}^{d^n-1} b_i L^i \\ \mu Z_{mot}(P(dis,n,d),t) &= \prod_{i=0}^{d^n-1} (\frac{1}{1-x^i t})^{b_i} \end{aligned}$$

3.11 Example (2 and 3 qubits) *Since we already expanded out the polynomials for these examples, we may immediately write*

$$\begin{aligned} \mu Z_{mot}(P(dis,2,2),t) &= \frac{1-xt}{1-x^3 t} \\ \mu Z_{mot}(P(dis,3,2),t) &= \frac{(1-x^4t)^2(1-x^3t)^3}{(1-x^7t)(1-x^6t)(1-x^5t)(1-x^2t)(1-xt)} \end{aligned}$$

3.5 Maximally Entangled States

3.12 Definition (Perfect Tensor [6]) *A vector $\psi \in (\mathbb{C}^d)^{\otimes 2n}$ is called perfect if for all equal bipartitions of the $2n$ qudits the associated matrix from adjunction is in $U((\mathbb{C}^d)^{\otimes n})$. This ensures that the state ψ is absolutely maximally entangled. For the corresponding projective space call these perfect states.*

Let us weaken that to $GL((\mathbb{C}^d)^{\otimes n})$. This is because algebraically we don't have complex conjugation. Call these algebraically perfect. For each bipartitioning this means the state is in the complement of the determinantal hypersurface. This is no longer absolute maximally entangled, but still a much stronger level of entanglement then we did before with the Segre embeddings.

3.13 Definition (Determinantal Hypersurface) *A determinantal hypersurface D_{d^n} is the hypersurface in $A^{d^{2n}}$ defined by the equation $\det x_{ij} = 0$ where i, j run from 1 through d^n. We use the same notation for the hypersurface in $P^{d^{2n}-1}$ defined by this homogenous polynomial of degree d^n. This is allowed to be singular.*

3.14 Proposition ([7]) *For $n = 1$, the class of algebraically perfect tensors is*

$$
\begin{aligned}
[A^{d^2} \setminus D_{d^1}] &= L^{\binom{d}{2}} \prod_{i=1}^{d} (L^i - 1) \\
\left[P^{d^2-1} \setminus D_{d^1}\right] &= L^{\binom{d}{2}} \prod_{i=2}^{d} (L^i - 1)
\end{aligned}
$$

For higher n, we must look at the complement of many hypersurfaces. In fact there are $\binom{2n-1}{n-1}$ hypersurfaces indexed by choice of equal bipartitionings into A and A^c, but there are nontrivial intersections that we must take care of. Let us call these each of these equations $\det_A = 0$. The algebraically perfect states then form the complement of this hypersurface arrangement. These equations may be combined into $(\prod_A \det_A) = 0$ at the cost of irreducibility. Cohomology groups of similar constructions are well studied [8], but now all the hypersurfaces are singluar.

3.15 Example ($n = 2$) *Let ψ_{ijkl} be the coefficents of $\mid e_i\rangle \otimes \mid e_j\rangle \otimes \mid e_k\rangle \otimes \mid e_l\rangle$ where $ijkl$ index 1 through d. For $d = 2$ and the subdivision $A = 12$ and $A^c = 34$.*

$$
det_A = \det \begin{pmatrix} \psi_{0,0,0,0} & \psi_{0,0,0,1} & \psi_{0,0,1,0} & \psi_{0,0,1,1} \\ \psi_{0,1,0,0} & \psi_{0,1,0,1} & \psi_{0,1,1,0} & \psi_{0,1,1,1} \\ \psi_{1,0,0,0} & \psi_{1,0,0,1} & \psi_{1,0,1,0} & \psi_{1,0,1,1} \\ \psi_{1,1,0,0} & \psi_{1,1,0,1} & \psi_{1,1,1,0} & \psi_{1,1,1,1} \end{pmatrix}
$$

3.16 Proposition ([9]) *For the complement to a hypersurface defined by f of degree D, we may also state homotopy groups*

$$
\pi_i(\mathbb{P}^{d^{2n}-1} \setminus \{f = 0\}) = \begin{cases} \mathbb{Z}_D & i = 1 \\ 0 & i = 2 \cdots (d^{2n} - 2 - k - 1) \end{cases}
$$

where k is the dimension of the singular locus of $f = 0$. For $\det_A$ this is $d^{2n} - 5$ corresponding to corank 2. In particular let $f = (\det_A)$ with $D = d^n$ or $(\prod_A \det_A)_{red}$ with $D \mid d^{\binom{2n-1}{n-1} n}$. Because of reduction, this means the degrees merely divide the naive degrees and is computed by removing multiplicity from $\prod_A \det_A$.

3.17 Corollary *For a loop of holographic states in the sense of [6] based on $2n$ valent tensors and qudits on edges, we have a $\mathbb{Z}_D$ invariant with D as above.*

Proof Take the loop of perfect tensors used to decorate all the vertices and forget to a loop of algebraically perfect tensors. Then take it's homotopy class. □

4 Classical Probability

The space of classical probability distributions on a finite set (cardinality d^N) is embedded as $\sum_{i=1}^{d^N} \sqrt{p_i} \mid e_i\rangle$ into pure quantum states. In fact this lands in $\mathbb{P}(\mathbb{R}^{d^N})$. $A = \mathbb{Z}_2^{d^N-1}$ by sign flips so that we may find a totally nonnegative representative in each A orbit. There are fixed points when any of the $p_i = 0$.

4.1 Definition (Very Affine) *The complement of the locus $\sqrt{p_1} = 0$ in $\mathbb{P}^{d^N-1}$ is an affine space $\mathbb{R}^{d^N-1}$. Extending this to remove all the other $\sqrt{p_i} = 0$ hyperplanes, gives the definition of the very affine space $VA_{N,d}$.*
The action of A now has no fixed points and each point in an orbit lie on disjoint components.

4.2 Remark This no longer encodes topological information without the miracle of decomposition into even dimensional cells brought on by complex geometry. $\Diamond$

4.3 Theorem *Let $P(\textit{cl-nonzero}, N, d)$ be the class of $X_{cl-nonzero,N,d} \subset VA_{N,d} = (\mathbb{R}^*)^{d^N-1} \subset P_{\mathbb{R}}^{d^N-1}$ which is the complement of all the embeddings of $[\pm\sqrt{p_1} \cdots \pm \sqrt{p_{d^{\lambda_1}}}]$ and $[\pm\sqrt{q_1} \cdots \pm \sqrt{q_{d^{\lambda_2}}}]$. These are lifts against the A action of probability distributions on sets of cardinalities d^{λ_1} and d^{λ_2}.*

$$
\begin{aligned}
[VA_{N,d}] &= (L-1)^{d^N-1} \\
P(\textit{cl-nonzero}, N, d) &= [VA_{N,d}] - \sum_{\substack{\lambda \vdash N \\ \lambda \neq (N)}} F(\lambda) \prod P(\textit{cl-nonzero}, \lambda_i, d) \\
F(\lambda) &= \frac{1}{\alpha_1! \cdots \alpha_r!} \binom{N}{\lambda_1} \binom{N-\lambda_1}{\lambda_2} \cdots \binom{N - \sum_{i=1}^{max-1} \lambda_i}{\lambda_{max}} \\
P(\textit{cl-nonzero}, 1, d) &= [VA_{1,d}]
\end{aligned}
$$

The classical nonzero distributions are then a $\mathbb{Z}_2^{d^N-1}$ quotient. This no longer fits in $K_0(Var_{\mathbb{R}})$. Instead, they are elements $\tilde{P}(\textit{cl-nonzero}, N, d) \in K_0(Stck_{\mathbb{R}})$ instead [10]. This is true even without the nonzero assumption, but one has to do different stabilizers depending on how many zeroes there are.

Proof The first term accounts for the ambient $\mathbb{R}_{\neq 0}^{d^N-1}$. We can normalize this as $\vec{x} = [1, x_2 \cdots x_{d^N}]$ with all $x_i \neq 0$. If $1 + \sum_{i=2}^{d^N} x_i^2 = X^2$, then renormalizing gives $\frac{1}{X}\vec{x} = [\sqrt{p_1} \cdots \sqrt{p_{d^N}}]$. The signs chosen for the $\sqrt{p_i}$ are the same as the signs of the x_i. This gives the A orbit of the embedding of nonzero probability into pure quantum states.

The equations for the Segre embedding imply independence of the associated distributions in d^{N_1} and d^{N_2}. Nonzero coordinates carry over mutatis mutandis because there are no zero divisors. Therefore this class can be computed analogously as the quantum analog.

A is a finite group giving an affine constant group scheme. Therefore the quotient represents a class in $K_0(Stck_{\mathbb{R}})$. It is a relatively simple one because we can identify a fundamental domain for the action with the semi-algebraic set of nowhere zero probability distributions. The A action moves between disjoint pieces by sign flips. $\square$

5 Symmetries of Qudit Configurations

So far there has been no structure on the n qudits. However, more often there will be symmetries acting. The two types we will discuss are an S_n action to make the particles identical and lattice translation actions when we imagine the n qudits are arranged as atoms in a crystal with periodic boundary conditions.

Let G be a symmetry group acting unitarily on the Hilbert space. In our original motivation, for this to be a symmetry of the system means that the projective class of $\mid \psi(p)\rangle$ should be G invariant for all p. The group restricts to act on the tautological line above $\mid \psi(p)\rangle$ as some one dimensional representation. Let χ be that one dimensional representation. In general as p varies within a connected component this tautological line can be any line in the isotypic component of χ.

5.1 Definition ($X_{dis,n,d,\chi}$) *Therefore define $X_{dis,n,d,\chi} \subset X_{dis,n,d}$ for the subset such that $\mid \psi(p)\rangle$ is in the isotypic component of χ. So ideally we would study homotopy classes of maps $M \to X_{dis,n,d,\chi}$ for all χ.*

5.2 Remark There is also the possibility of having $g \in G$ that are implemented antiunitarily. But the examples below, do not have this complication. $\Diamond$

5.1 Identical Particles

Let the symmetric group S_n act.

5.3 Lemma *Let χ_1 be the trivial representation and χ_2 be the sign representation.*

$$\begin{aligned} X_{dis,n,d,\chi_1} &= \mathbb{P}^{\binom{d+n-1}{n}-1} \\ X_{dis,n,d,\chi_2} &= \mathbb{P}^{\binom{d}{n}-1} \end{aligned}$$

Proof $\mid \psi\rangle$ being in the isotypic components for these characters means the same thing as being in the bosonic or fermionic Fock space of n identical particles with $\mathbb{C}^d$ being the single particle Hilbert space. The entanglement condition becomes automatic because any Segre embedding will break the symmetry or antisymmetry restriction. They are already in the complement of all the Segre embeddings. $\Box$

5.4 Definition ([11]) *A pure bosonic/fermionic state is nonminimally entangled if and only if it is outside the respective Segre maps*

$$\mathbb{P}(\mathbb{C}^d) \xrightarrow{Seg^b} \mathbb{P}(\mathbb{C}^{\binom{d+n-1}{n}})$$

$$(\mathbb{P}(\mathbb{C}^d))^n_o \xrightarrow{Seg^f} \mathbb{P}(\mathbb{C}^{\binom{d}{n}})$$

where the subscript o denotes the open dense complement of the locus where there is a linear dependence among the n vectors. That is the Slater determinant shouldn't vanish. The image of these embeddings is as independent as bosons/fermions can be within their constraint of being identical particles.

5.5 Corollary *For $n > 1$, the class of nonminimally entangled bosonic states (underlying set $X_{bos,n,d}$) $P(bos, n, d) \in K_0(Var_{\mathbb{C}})$ is*

$$\begin{aligned} P(bos,n,d) &= (1 + L + \cdots L^{\binom{d+n-1}{n}-1}) - (1 + L + \cdots L^{d-1}) \\ &= L^d + \cdots L^{\binom{d+n-1}{n}-1} \end{aligned}$$

If we require a sum of more than r such minimally entangled states then we must remove a secant variety of a Veronese embedding as studied in [12].

5.6 Corollary *The class of nonminimally entangled fermionic states (underlying set $X_{fer,n,d}$) $P(fer,n,d)$ is*

$$P(fer,n,d) \quad = \quad (1+L+\cdots L^{\binom{d}{n}-1}) - \frac{[d]_L!}{[d-n]_L![n]_L!}$$

where $[n]_L! = 1(1+L)\cdots(1+\cdots L^{n-1})$ is the q-factorial. Despite it's appearance as a rational function of L, the q-binomial gives a polynomial of degree $n(d-n)$.

Proof The map Seg^f factors as

$$(\mathbb{P}(\mathbb{C}^d))^n_o \longrightarrow Gr(n,d) \xrightarrow{\text{Plücker}} \mathbb{P}(\mathbb{C}^{\binom{d}{n}})$$

so the nonminimally entangled fermionic states are $[\mathbb{P}^{\binom{d}{n}-1}] - [Gr(n,d)]$.

5.7 Definition (Open Matroid Variety [13]) *The set of single Slater determinant states on the fermionic Hilbert space $\wedge^n(\mathbb{C}^d)$ such that the associated probability distribution on the finite set of cardinality $\binom{d}{n}$ has vanishing probabilities specified by the bases of of the given matroid.*

The matroid tells you which of the $\binom{d}{n}$ choices are bases and precisely those must have nonzero amplitudes. Those that are nonbases must have amplitude 0.

5.8 Definition (Big Open Positroid Variety) *In particular, if all your bases belong to the matroid then the fermionic state will have all $\binom{d}{n}$ amplitudes/probabilities be nonzero. This is the subset of $Gr(n,d)$ which lands in the very affine portion of $\mathbb{P}(\mathbb{C}^{\binom{d}{n}})$.*

5.9 Definition (Positive Grassmannian) *Suppose there is a time reversal symmetry such that all single particle states can be taken from $\mathbb{R}^d \subset \mathbb{C}^d$ in the construction of the Slater determinant. This gives the real Grassmannian for minimally entangled fermions. If we further restrict to states of the fermionic Fock space that come from probability distributions, that is the positivity restriction of the positive Grassmannian as studied in [13]. Information geometric constructions then pull back along this inclusion.*

5.10 Remark (Box-Ball systems) We can imagine the single particle Hilbert space $\mathbb{C}^d$ as indexing the sites of a one dimensional lattice. A box-ball system with n balls then gives a state inside $\mathbb{P}(\mathbb{C}^{\binom{d}{n}})$ using the $\sqrt{p}$ embedding. If there is a free fermion representation like in TASEP, then this fits with the minimal entanglement assumption. $\diamondsuit$

One can strengthen the measure of entanglement by considering sums of $\leq k$ Slater determinants. With $k=1$, this is minimal entanglement, but as k increases we remove more and more.

5.11 Definition (*Secant*$^{k-1}$ [14]) *For $Gr(n,d) \subset \mathbb{P}(\mathbb{C}^{\binom{d}{n}})$, it's $k-1$'st Secant variety is the Zarisiki closure of the linear space spanned by sums of k Slater determinants. It has dimension $\leq min(\binom{d}{n}-1, (k-1)n(d-n)+n(d-n)+k-1)$*

So one can construct the complement of this as we did for $X_{fer,n,d}$ by removing $Secant^0 = Gr(n,d)$. Of course the singularities will mean this computation will require blowups just like the determinantal hypersurfaces for perfect tensors did.

5.2 Lattice Structure

Another possibility is if the n qudits are arranged as the positions of a lattice subject to periodic boundary conditions. For example, if $n = n_x n_y n_z$, the qudits could represent spins on atoms of the cubic lattice with n_i atoms in each direction. This case has all n positions related by symmetries, but that is not generally the case.

5.12 Definition (Brillouin Torus) *Let the symmetry group A be an abelian group $\mathbb{Z}_a$, $\mathbb{Z}_a \times \mathbb{Z}_b$ or $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$. These represent arranging the spins in one, two or three dimensions. These are the physical locations for the qudits so the corresponding Hilbert space decomposes along the Pontryagin dual $A^\vee$. When $a, b, c \to \infty$, this becomes a topological torus of the appropriate dimension.*

Let $\vec{k}$ be the label for an element of $A^\vee$. In this case define $X_{\mathrm{dis},n,d,\vec{k}}$. If we take a thermodynamic limit $A = \mathbb{Z}^D$, $\vec{k}$ would fill out a torus $U(1)^D$.

However for homotopic purposes this is too strict. Instead of demanding that the projective class of $\mid \psi(p)\rangle$ be A invariant we may simply ask for a map $[BA, X_{\mathrm{dis},n,d}]$. This also has the advantage of being much more manageable to compute given that we have already calculated the E-polynomial of the target. When $A = \mathbb{Z}^D$, BA has a realization as a torus T^D.

6 Mixed States

An alternative to looking the Segre embeddings for pure states to measure entanglement is to take pick a subset A of the n particles and then provide their reduced density matrix. If the full state was in the image of a Segre embedding $\mathbb{P}^{|A|} \times \mathbb{P}^{n-|A|}$, then the reduced density matrix will be pure as well and vice versa. Because A is arbitrary, $X_{d,n}$ is equivalent to asking for all the ρ_A to be mixed.

6.1 Lemma *For p_i a fixed set of r probabilities with block sizes λ_j. That is to say, there are $p_1 = p_2 \cdots p_{\lambda_1}$ and then $p_{\lambda_1+1} = \cdots p_{\lambda_1+\lambda+2}$ and so on. Adjoin a λ_0 to be $D-r$ to represent the rest of the Hilbert space presumably of such high energy that they don't participate in the state ρ. The space of mixed states on $\mathbb{C}^D$ with this spectrum has class*

$$\begin{aligned} [M_{\vec{p},D}] &= \frac{[D]_L!}{[\lambda_0]_L![D-\lambda_0]_L!} \frac{[D-\lambda_0]_L!}{[\lambda_1]_L![D-\lambda_0-\lambda_1]_L!} \cdots \frac{[D-\lambda_0 \cdots \lambda_{max-1}]_L!}{[\lambda_{max}]_L![D-\lambda_0-\cdots\lambda_{max}]_L!} \\ &= \frac{[D]_L!}{[\lambda_0]_L! \cdots [\lambda_{max}]_L!} \end{aligned}$$

This is a q-multinomial coefficient.

Proof The structure of a partial flag variety is standard. For example, it can be seen in [15]. The complex structure is transferred along $G_{\mathbb{C}}/P_\lambda \simeq U(D)/(U(\lambda_0) \times \cdots U(\lambda_{max}))$. This then gives the cohomology groups through the E polynomial. The Hodge decomposition is then forgotten to obtain the cohomology without the auxiliary complex structure. □

Fix r to be the maximum number of nonzero eigenvalues we will consider. At the extreme this can be D, but does not need to be. In fact D will be sent to ∞, but r will remain finite. Then we have an $r-1$ simplex Δ_{r-1} with polynomials for each point that depend on the stabilizer under the S_r action. They are given by the above L-multinomial coefficients.

6.1 Varying the probabilities

6.2 Definition (Reduced Simplex) *$p_1 \geq p_2 \cdots p_r \geq 0$ such that $p_1 + \cdots p_r = 1$. This is a single piece of the barycentric subdivision of Δ_{r-1} by the S_r action.*

The maximum entropy state is $(\lambda_0 = D - r, \lambda_1 = r)$. This has all probabilities $\frac{1}{r}$ giving $S = \log r$ which is the maximum allowed because r is fixed. This is a codimension $r-1$ corner of the reduced simplex.

$$\begin{aligned} [M_{\vec{p},D}] &= [Gr(r,D)] \\ &= \frac{[D]_L!}{[r]_L![D-r]_L!} \end{aligned}$$

6.3 Remark This provides an alternative interpretation of band insulator classification. Instead of taking the pure state in a fermionic Fock space, one can take the partial trace down to one of the fermions. The resulting mixed state has eigenvalues $\frac{1}{r}$ for the span of the states that went into the Slater determinant and 0 for the rest of the Hilbert space. $\Diamond$

The generic case is $(\lambda_0 = D - r, \lambda_1 = 1 \cdots \lambda_r = 1)$ for r distinct nonzero probabilities. This is a hook partition. The signature with probabilities ordered is $0 \subset \mathbb{C} \subset \mathbb{C}^2 \cdots \mathbb{C}^r \subset \mathbb{C}^D$. That is $\mathbb{C}^k$ corresponds to the subspace spanned by eigenvectors with the k largest eigenvaluesp_i.

$$\begin{aligned} [M_{\vec{p},D}] &= [Fl_\lambda(\mathbb{C}^D)] \\ &= \frac{[D]_L!}{[D-r]_L!} \end{aligned}$$

6.4 Lemma *If $D \to \infty$, $Fl_\lambda(\mathbb{C}^D)$ is a realization of $B(B_r)$ where B_r is upper triangular matrices in $GL(r, \mathbb{C})$.*

Proof Full rank $\infty \times r$ complex matrices is contractible and provides a model for $E(B_r)$. Reading off rows gives the corresponding flag. $\square$

There are 2 possibilities for the fiber as we go to the codimension 1 boundary of U. Either $p_r \to 0$, in which case we get another hook partition of $(\lambda_0 = D - r + 1, \lambda_1 = 1 \cdots \lambda_{r-1} = 1)$ or $p_i = p_{i+1}$ for some $i < r$. These are the two results of taking the partition $(D-r)(1)^r$ and combining two rows. In that case the associated partition has shape $(D-r)(2)(1)^{r-1}$.

6.5 Corollary *When 2 nonzero probabilities become equal resulting in distribution $\vec{q}$, then the cohomology changes by*

$$rkH^{2i}(M_{\vec{q},D}) = \sum_{j=0}^{i} (-1)^j rkH^{2i-2j}(M_{\vec{p},D})$$

For the other type of boundary where $p_r \to 0$, the rank of H^{2i} also has a formula in terms of the ranks $rkH^{2i-2j}(M_{\vec{p},D})$. They are not manifestly positive, but they turn out to be.

Proof

$$\begin{aligned} [M_{\vec{q},D}] &= \frac{[D]_L!}{[D-r]_L![2]_L!} \\ &= [M_{\vec{p},D}](1 - L + L^2 + \cdots) \\ Coeff(L^i, [M_{\vec{q},D}]) &= Coeff(L^i, [M_{\vec{p},D}]) - Coeff(L^{i-1}, [M_{\vec{p},D}]) + \cdots (-1)^i Coeff(L^{i-i}, [M_{\vec{p},D}]) \end{aligned}$$

$$
\begin{aligned}
[M_{\vec{q},D}] &= \frac{[D]_L!}{[D-r+1][D-r]_L!} \\
&= [M_{\vec{p},D}](1 - A + A^2 + \cdots) \\
A &\equiv L + L^2 \cdots L^{D-r}
\end{aligned}
$$

which when expanded out for the coefficent of L^i depends on the coefficents of L^{i-j} of $[M_{\vec{p},D}]$.

All the other partitions μ of D with $\lambda_0 \geq D - r$ show up in deeper codimension. The depth is determined by how many times two rows must be combined. $\lambda_{hook} = (D-r)(1)^r \to \mu_1 \to \mu_2 \cdots \mu$ where each arrow is the result of merging two rows of the preceding diagram. There are $\sum_{i=1}^{r} p(i)$ different isomorphism classes of partial flag varieties being considered in total.

$$
\begin{tikzcd}
\text{(r)-step} & & GL(D)/P_{\lambda_{hook}} \arrow[d] & \\
\text{k-step} & & GL(D)/P_{\mu} \arrow[d] & \\
\text{2-step} & & \cdots \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{1-step} & Gr(r,D) \arrow[dr] & Gr(r-1,D) \arrow[d] & \cdots Gr(1,D) \arrow[dl] \\
 & & pt &
\end{tikzcd}
$$

Figure 1: Quotient maps between the fibers of different facets.

6.6 Definition ($\boldsymbol{Mix_{V,r,D}}$) *Let V be any collection of some of the facets of the reduced $r-1$ simplex and U specifically only the generic cell of distinct probabilities. Then let $Mix_{V,r,D}$ and $Mix_{U,r,D}$ be the total space above V and U. This indicates all mixed states whose spectrum is a point in V or U respectively. It combines all the partial flag varieties over all the points with the various parabolic subgroups.*

6.7 Remark If U was complex, this would resemble the subject of variation of Hodge structures. $\diamondsuit$

6.8 Definition (Springer Fiber) *Let the mixed state of concern be a thermal state. Reorder the blocks to be of increasing energy. Keep λ_0 are the states with $E = +\infty$ for $p = 0$. Then the Springer fiber for L is the mixed states with the specified probabilities with the additional condition that L can not raise energy. In other words $L(V_i) \subset V_i$ for the flag V_i given by the energy inequalities. L is chosen instead of the traditional X to draw the relation with the stochastic wavefunction formalism of [16].*

6.2 Families of Mixed States

Let the system be either open or at a finite temperature so that for each value of the parameter we have a mixed state rather than a pure state. A reasonable expectation is for the mixed state to be in Mix_{U,D_1,D_1} for some r and D_1 because finite temperature with a generic Hamiltonian will have all $\frac{1}{Z}e^{-\beta E_i}$ distinct.

6.9 Definition (Family of Mixed States) *For the parameter space M or the homotopic space BG, the family of generic mixed states is the resulting map to Mix_{U,r,D_1}. U is contractible so the topological information is contained in $Fl(1, 2 \cdots r, D_1)$ whose E-polynomial we have given. When there is an operator L that does not raise energy, the target is an analog of a Springer fiber over U instead.*

6.10 Proposition *We may compose with the quotient maps shown in fig. 1. These result in maps of sets $[M, Fl_{\lambda_{hook,r}} \mathbb{C}^{D_1}] \to [M, Gr(r-k, D_1)]$ for all $k < r$. If $D_1 \to \infty$, then these become maps $[M, B(B_r)] \to [M, BU(r-k)]$. This is the spectral flattening where there are $D_1 + k$ high energy levels and $r - k$ low energy levels that in the limit turn into $p = 0$ and $p = \frac{1}{r-k}$ respectively.*

Set $k = 0$ so no states run away to $E = \infty$. Assume r is known from Hilbert space of the low energy topological theory, then $B(B_r)$ represents the finite temperature case using a generic Hamiltonian to lift the degeneracy at $E = 0$. For a parameter space M of dimension D, the codimension D corners of fig. 1 can be expected.

In the case of BG for discrete groups, $Hom_{Grp}(G, B_r) \to [BG, B(B_r^{\delta})]$. The change of topology from the discrete topology δ to the usual topology then gives $[BG, B(B_r)]$

6.11 Corollary *Letting $r \to \infty$ makes B_r into the group B_∞ and $BU(r-k)$ into BU. The latter leads to complex K theory $K^0(M)$ or $K^0(BG)$.*

Consider N interacting fermions per unit cell in a crystal in dimension D. Simultaneous translation by lattice vectors still breaks up the Hilbert space into sectors over the Brillouin torus. This gives a map $T^D \to \mathbb{P}^{\binom{d}{N}-1}$. When the fermions don't interact this factors through $Gr(N, d)$ as before. Conversely we may ask for the map to land in the nonminimally entangled $X_{fer,N,d}$ instead. However, we can define a more manageable invariant by taking the partial trace of all but one fermion.

6.12 Proposition (k-Slater Invariants) *Taking a combination of k Slater determinants (So k should be > 1 but not too big). This is a point of the k'th associated secant variety of the Grassmannian. This mixed state lands in $Mix_{W,kN,d}$ and W is the stratum defined by having partition with a part $d - kN + o$ for $p = 0$ and a remaining partition of $kN - o$ for some $o \geq 0$. Taking $d \to \infty$ turns the fibers of each cell in W into BP for $P \subset GL(kN - o)$ the appropriate parabolic subgroup. Generically this will be $B(B_{kN})$, but there will be degenerations at some walls.*

All these fibers combine to give the target space $Mix_{W,kN}$ for a homotopic invariant $[T^D, Mix_{W,kN}]$. If desired, equivariance for point groups can also be demanded on these maps.

Proof Let $c_1 SL(\psi_{1,1} \cdots \psi_{1,N}) + \cdots c_k SL(\psi_{k,1} \cdots \psi_{k,N})$ be a state given as a sum of k Slater determinants. There are obviously $\leq min(kN, d)$ eigenvectors for ρ because they will be in the span of $\psi_{i,j}$. The dimension of the span of all the $| \psi_{i,j} \rangle$ relative to kN gives a bound on o.

Some assumptions on the $\psi_{i,j}$ give more control over ρ. If $\forall i \neq j$, $\exists m_1, m_2$ such that $\langle \psi_{i,m} || \psi_{j,l} \rangle = 0$ for all l then there will be no cross terms when taking the reduced density matrix.

$$
\begin{aligned}
| k - SL \rangle &= \sum_{i=1}^{k} c_i \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_j^{\otimes} | \psi_{i,\sigma j} \rangle \\
\rho &= Tr_{2 \cdots N} | k - SL \rangle \langle k - SL | \\
&= \sum_i | c_i |^2 \frac{1}{N!} \sum_j | \psi_{i,j} \rangle \langle \psi_{i,j} | (N-1)! \\
&= \sum_i | c_i |^2 \frac{1}{N} \sum_j | \psi_{i,j} \rangle \langle \psi_{i,j} |
\end{aligned}
$$

If no such m exist, then both i and j give the same N dimensional subspace of $\mathbb{C}^d$ so they can be combined into a single Slater determinant so this possibility can be eliminated. If only a single m exists, then the terms are orthogonal, but there are cross terms with c_i and c_j in ρ. $\square$

6.13 Example *Let M be a Riemann surface of genus ≥ 1 or a closed hyperbolic manifold of higher dimension with a basepoint. M homotopically presents a $K(\pi_1(M), 1)$. This means that a family of r maximally nondegenerate mixed states (with $d \to \infty$) will be classified by a map $[K(\pi_1(M), 1), B(B_r)]$ which receives a map from $Hom_{Group}(\pi_1(M), B_r)$. However, for a family of states we should expect some eigenvalues to collide or go to 0 along a codimension 1 stratum.*

Without control over these strata all that is left is the map to $B(GL_r)$ from the spectral flattening argument.

7 Conclusion

For every n and d we can write a polynomial which gives the class for the pure states on $(\mathbb{C}^d)^n$ which do not have any decomposition into unentangled subsets. This is by successively subtracting the contributions from all the relevant Segre embeddings. Perfect tensors are addressed by determinantal hypersurfaces. We have further done the case of N bosons where the state is not of the form $\mid v\rangle^n$ and N fermions where the state requires linear combination of more than one Slater determinant. Requiring more than k Slater determinants means removing a higher secant variety.

We then gave the case of mixed states when the spectral decomposition gave specified probabilities. The cohomology groups of these are given through q multinomial coefficients. These fibers can be combined along with the base Δ_{r-1} to answer questions about the topology of mixed states with various constraints on up to r nonzero eigenvalues.

To connect with other literature, we have described how as the dimension of the Hilbert space goes to ∞ and entanglement conditions are dropped, this reproduces bosonic SPT classification by the magic of $K(\mathbb{Z}, 2)$. The classification of free fermion systems in lattices is accomplished through mapping from the parameter space of possible quasi-periodic boundary conditions to the allowed states of fermions. This philosophy of mapping parameter spaces into spaces associated with projective and flag spaces describes systems at the level of their states in Hilbert spaces. This generalizes when asking for families of extended theories. Instead of asking how a state changes as parameters are changed, categories of boundary conditions are tracked as parameters change. For example, in the three dimensional fully extended context, this is the problem of $[BG, B\underline{\underline{BrPic}}]$.

Along the way we have pointed interesting connections to independence in probability, the positive Grassmannian and unipotent representations. These are all lines for further research. Using the information from the $P(n, d)$ to the actual homotopy groups of $X_{d,n}$ is necessary in order to actual compute the invariants for any particular M or BG. The example of k-Slater determinants involves a great deal of computation. With $D = 3$, we have 230 space groups, choices of how many electrons per unit cell N and how many Slater determinants will be allowed k. We also hinted at a connection between the replica trick and Kapranov Zeta function. Studying this in more detail is another avenue of future research.

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