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Introduction

In this document we explicitly list the conjugation rules of Clifford operators on Pauli operators. Section 1 contains the basics for a rigorous description and Sec. 2 the lists the conjugation rules, making use of the relation between Clifford operators and the symplectic group, which directly represents how the library is implemented.

1. Preliminaries

In this section we document some well known properties of Pauli operator and Clifford operators. It is maybe unnecessary technical, but I think this way it ensures clarity.

Definition 1.0.1 *For an index set I, with $|I| = n \in \mathbb{N}$, let $(|0\rangle_i, |1\rangle_i)$ be an orthonormal basis of $\mathcal{Q}_{,i} \cong \mathbb{C}^2$ and $\sigma : \{0, \ldots, n-1\} \to I$ bijective. The states*

$$\bigotimes_{i\in I} |x_i\rangle_i =: \left|x_{\sigma(n-1)} \ldots x_{\sigma(0)}\right\rangle =: |x\rangle \quad \textit{with} \;\; x = \sum_{i=0}^{n-1} x_{\sigma(i)} 2^i \tag{1}$$

for $x_i \in \{0,1\}$, $i \in I$ $(x \in \{0, \ldots, 2^n - 1\})$, define the standard orthonormal computational basis of $\mathcal{Q}_n = \bigotimes_{i\in I} \mathcal{Q}_{,i} \cong \mathbb{C}^{2^n}$. For $n = 0$, we just have the state $|0\rangle$.
For $m, n \in \mathbb{N}_0$ we define the canonical matrix representation

$$\mathrm{mat}_{m,n} : \mathscr{L}(\mathcal{Q}_m, \mathcal{Q}_n) \to \mathbb{C}^{2^m \times 2^n},\, A \mapsto \begin{pmatrix} \langle 0\,|\,A\,|\,0\rangle & \cdots & \langle 0\,|\,A\,|\,2^m - 1\rangle \\ \vdots & & \vdots \\ \langle 2^n - 1\,|\,A\,|\,0\rangle & \cdots & \langle 2^n - 1\,|\,A\,|\,2^m - 1\rangle \end{pmatrix} . \tag{2}$$

Remark 1.0.2 *The freedom of σ just means that we can relabel and resort the qubits, in a closed context, which will be handy later on. However, usually, the index set is just $\{0, \ldots n-1\}$, $n \in \mathbb{N}_0$, (maybe shifted by 1, depending on the context) and when we define operators we will always choose the identity for σ, i.e., the qubits are sorted from highest to lowest (left to right), if not otherwise said.*

Definition and Proposition 1.0.3 *Let $\mathrm{M}_n(R)$ be the multiplicative monoid of all $n \times n$ matrices over a ring R, $n \in \mathbb{N}$, and $\mathrm{GL}_n(R) \subseteq \mathrm{M}_n(R)$ the group of invertible matrices. We define the conjugation action of $\mathrm{GL}_n(R)$ on $\mathrm{M}_n(R)$ via*

$$* : \mathrm{GL}_n(R) \times \mathrm{M}_n(R) \to \mathrm{M}_n(R),\, (A, B) \mapsto A * B = ABA^{-1} \,. \tag{3}$$

For all $A \in \mathrm{GL}_n(R)$*, the mapping*

$$\mathrm{inn}_A : M_n(R) \to \mathrm{M}_n(R),\, B \mapsto A * B \tag{4}$$

is an automorphism of the $\mathrm{M}_n(R)$*.*

Definition 1.0.4 *We define* $\mathrm{U}(\mathbb{C}^n)$*,* $n \in \mathbb{N}$*, to be the multiplicative group of unitary operators on* $\mathbb{C}^n$*, and write* $\mathrm{U}(n) = \mathrm{U}(\mathbb{C}^n)$*.*

Definition 1.0.5 *The Pauli operators are defined by*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5}$$

Alternatively, we write $\sigma_x = \sigma_1 = X, \sigma_y = \sigma_2 = Y, \sigma_z = \sigma_3 = Z$ *(sometimes with upper indices), respectively, and set* $\boldsymbol{\sigma} = (\sigma_x \;\, \sigma_y \;\, \sigma_z)^\intercal$*. We also consider the identity as Pauli operator and set* $\sigma_0 = \mathbb{1}$*. Usually, when indexing* σ *with Latin indices, we count from* 1 *to* 3*, and when using Greek indices we count from* 0 *to* 3*.*

Definition 1.0.6 *Let* $n \in \mathbb{N}$*. The unitary infinite Pauli group* $\hat{\mathscr{P}}_n \leq \mathrm{GL}(\mathscr{Q}_n)$ *is defined by*

$$\hat{\mathscr{P}}_n = \left\{ u \bigotimes_{j=1}^{n} \sigma_{\mu_j} \;\middle|\; \mu_j \in \{0, \ldots, 3\};\, u \in \mathrm{U}(1) \right\} . \tag{6}$$

Definition 1.0.7 *Let* $n \in \mathbb{N}$*. The unitary infinite Clifford group* $\hat{\mathscr{C}}_n \leq \mathrm{U}(\mathscr{Q}_n)$ *is the normalizer of the Pauli group, i.e.,* $\hat{\mathscr{C}}_n * \hat{\mathscr{P}}_n = \hat{\mathscr{P}}_n$*.*

Definition 1.0.8 *Let* $n \in \mathbb{N}$*. We define the quotient groups*

$$\overline{\hat{\mathscr{P}}}_n = \hat{\mathscr{P}}/\mathrm{U}(1) \; , \quad \overline{\hat{\mathscr{C}}}_n = \hat{\mathscr{C}}/\mathrm{U}(1) \; . \tag{7}$$

and set P_n *and* C_n *to be sets of representatives, respectively.*

Proposition 1.0.9 *A unitary* $U \in \mathrm{U}\big(\bigotimes_{i=1}^n (\mathbb{C}^2)_i\big)$*,* $n \in \mathbb{N}$*, is uniquely defined, up to a phase, by its conjugation of the Pauli operators* $X_1, Z_1, \ldots X_n, Z_n$*.*

Proposition 1.0.10 *Let* $n \in \mathbb{N}$*.* $\overline{\hat{\mathscr{P}}}_n$ *is isomorph to the abelian group* $\overline{H(\mathbb{Z}_2^n)} = (\mathbb{Z}_2^n \times \mathbb{Z}_2^n, +)$ *with the standard addition via*

$$\overline{\tau} : (\mathbb{Z}_2^n \times \mathbb{Z}_2^n) \to \overline{\hat{\mathscr{P}}}_n, \tag{8}$$

$$(z, x) \mapsto \bigotimes_{j=1}^{n} \overline{Z}_j^{z_j} \overline{X}_j^{x_j} \; . \tag{9}$$

Theorem 1.0.11 ([1, 2]) *Let* $n \in \mathbb{N}$*. The Clifford group, up to Pauli operators, is isomorph to the symplectic group, i.e.,*

$$\overline{\hat{\mathscr{C}}}_n / \overline{\hat{\mathscr{P}}}_n \cong \mathrm{Sp}_{2n}(\mathbb{Z}_2) \; , \tag{10}$$

where $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ *is the symplectic group of the* $\mathbb{Z}_2^{2n}$ *vector space with respect to the* $\left(\begin{smallmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{smallmatrix}\right)$ *symplectic form. More specifically, the isomorphism is defined by*

$$\kappa : \overline{\hat{\mathscr{C}}}_n / \overline{\hat{\mathscr{P}}}_n \to \mathrm{Sp}_{2n}(\mathbb{Z}_2),\; \overline{g}\overline{\mathscr{P}}_n \mapsto S_g = \overline{\tau}^{-1} \circ \mathrm{inn}_g \circ \overline{\tau} \; , \tag{11}$$

Corollary 1.0.12 *The number of Clifford operators modulo Pauli operators is given by*

$$\left|\overline{\hat{\mathcal{C}}}_n / \overline{\hat{\mathcal{P}}}_n\right| = 2^{n^2} \prod_{i=1}^{n} \left(2^{2i} - 1\right) \tag{12}$$

Theorem 1.0.13 *Let $n \in \mathbb{N}$. Then it is*

$$\hat{\mathcal{C}}_n = \langle u, H_i, S_i, \mathrm{CZ}_{ij} \mid u \in \mathrm{U}(1);\ i, j \in \{1, \dots n\};\ i \neq j \rangle\ . \tag{13}$$

2. Conjugating Paulis

Remark 2.0.1 *Note that is sufficient to calculate the conjugation rules of representatives of $\overline{\hat{\mathcal{C}}}_n / \overline{\hat{\mathcal{P}}}_n$, $n \in \mathbb{N}$, since Pauli operators only change the phase factor of the result (which is trivial to calculate).*

2.1. Single qubit Cliffords

Proposition 2.1.1 *The following mappings list all elements of $\overline{\hat{\mathcal{C}}}_1 / \overline{\hat{\mathcal{P}}}_1$ (representatives) and explicitly describe the isomorphism $\overline{\hat{\mathcal{C}}}_1 / \overline{\hat{\mathcal{P}}}_1 \cong \mathrm{Sp}_2(\mathbb{F}_2)$.*

Table 1: *Explicit description of the $\overline{\hat{\mathcal{C}}}_1 / \overline{\hat{\mathcal{P}}}_1 \cong \mathrm{Sp}_2(\mathbb{F}_2)$ isomophism for every element: The second column specifies the element in $\overline{\hat{\mathcal{C}}}_1 / \overline{\hat{\mathcal{P}}}_1$ with respect to the generators S and H. The first column shows the matrix description of the canonical representatives. The third column shows the symplectic matrix according to the isomorphism. The fourth column contains the additional phases one has when conjugating with representatives (left for conjugating Z, right for conjugating X).*

repr.	$\overline{\hat{\mathcal{C}}}_1 / \overline{\hat{\mathcal{P}}}_1$	$\mathrm{Sp}_2(\mathbb{F}_2)$	phase
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathbb{1}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	S	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -i \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	H	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \frac{1}{\sqrt{2}}$	SH	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -i & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \frac{1}{\sqrt{2}}$	HS	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & i \end{pmatrix}$
$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}}$	SHS	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} -i & 1 \end{pmatrix}$

Proof. We calculate the according conjugations with the representatives:

$$S * Z = Z \tag{14}$$
$$S * X = Y = -iZX \tag{15}$$
$$H * Z = X \tag{16}$$
$$H * X = Z \tag{17}$$
$$SH * Z = S * X = -iZX \tag{18}$$
$$SH * X = S * Z = Z \tag{19}$$
$$HS * Z = H * Z = X \tag{20}$$
$$HS * X = -iH * ZX = iZX \tag{21}$$
$$SHS * Z = S * X = -iZX \tag{22}$$
$$SHS * X = iS * ZX = X \tag{23}$$

□

Remark 2.1.2 *In the following we list some typical Clifford gates and how they relate to the representative elements of* $\widehat{\mathcal{C}}_1/\widehat{\mathcal{P}}_1$ *(*H^{ab} *denotes the hermitian change from the eigenbasis of* a *to the eigenbasis of* b*,* $a, b \in \{X, Y, Z\}$*):*

$$S^\dagger = SZ = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \tag{24}$$
$$HSH = \sqrt{i}SHS = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{i} & \sqrt{-i} \\ \sqrt{-i} & \sqrt{i} \end{pmatrix} \tag{25}$$
$$\sqrt{X} = HSH = \sqrt{i}SHS = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \tag{26}$$
$$\sqrt{X}^\dagger = HSZH = HSHX = \sqrt{i}SHSX = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix} \tag{27}$$
$$\sqrt{Y} = \sqrt{i}HZ = \frac{1}{2}\begin{pmatrix} 1+i & -1-i \\ 1+i & 1+i \end{pmatrix} \tag{28}$$
$$\sqrt{Y}^\dagger = \sqrt{-i}ZH = \sqrt{-i}HX = \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1+i & 1-i \end{pmatrix} \tag{29}$$
$$H^{xy} = \mathrm{e}^{-i\pi/4}SX = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix} \tag{30}$$
$$H^{yz} = SHSZ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \tag{31}$$
$$\tag{32}$$

Sorted according to Prop. 2.1.1 we have the following 24 single qubit Clifford operators:

$$\begin{array}{llll}
\mathbb{1} = \mathbb{1} & \mathbb{1}X = X & \mathbb{1}Y = Y & \mathbb{1}Z = Z \\
S = S & SX = \sqrt{i}H^{xy} & SY = . & SZ = S^\dagger \\
H = H & HX = \sqrt{i}\sqrt{Y}^\dagger & HY = . & HZ = \sqrt{-i}\sqrt{Y} \\
SH = . & SHX = . & SHY = . & SHZ = . \\
HS = . & HSX = . & HSY = . & HSZ = . \\
SHS = \sqrt{-i}\sqrt{X} & SHSX = \sqrt{-i}\sqrt{X}^\dagger & SHSY = . & SHSZ = H^{yz}
\end{array} \tag{33}$$

2.2. Two qubit Cliffords

Remark 2.2.1 *For the single qubit Cliffords, it made sense to define a set of canonical representatives through the generators S and H (since there are only 6 up to Paulis). Doing the same thing for two qubit Cliffords gets out of hands; the generator strings would be just too long. Instead we try to choose the most common ones.*

Proposition 2.2.2 *The following mappings list some elements of $\left(\overline{\hat{\mathcal{C}}}_2/\overline{\hat{\mathcal{P}}}_2\right)\Big\backslash\left(\overline{\hat{\mathcal{C}}}_1/\overline{\hat{\mathcal{P}}}_1\right)$ (representatives) and explicitly describe the isomorphism $\overline{\hat{\mathcal{C}}}_2/\overline{\hat{\mathcal{P}}}_2 \cong \mathrm{Sp}_4(\mathbb{F}_2)$.*

Table 2: *Explicit description of the $\overline{\hat{\mathcal{C}}}_2/\overline{\hat{\mathcal{P}}}_2 \cong \mathrm{Sp}_4(\mathbb{F}_2)$ isomophism for some element: cf. Prop. 2.1.1. We defined in Def. 1.0.1, Rem. 1.0.2, and Prop. 1.0.10 uniquely how we represent the operators in matrix form: For the repr. matrix the sorted basis is $(|0\rangle_2|0\rangle_1, |0\rangle_2|1\rangle_1, |1\rangle_2|0\rangle_1, |1\rangle_2|1\rangle_1)$ and for the symplectic matrix it is (Z_1, Z_2, X_1, X_2), under the according isomorphism, respectively.*

repr.	$\overline{\hat{\mathcal{C}}}_2/\overline{\hat{\mathcal{P}}}_2$	$\mathrm{Sp}_4(\mathbb{F}_2)$	phase
$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$	CZ	$\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1&1\end{pmatrix}$
$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	CX_{21}	$\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1&1\end{pmatrix}$
$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$	CY_{21}	$\begin{pmatrix}1&0&0&1\\1&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&1&-i\end{pmatrix}$
$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$	SWAP	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&1&1\end{pmatrix}$
$\begin{pmatrix}1&0&0&0\\0&0&i&0\\0&i&0&0\\0&0&0&1\end{pmatrix}$	iSWAP	$\begin{pmatrix}0&1&1&1\\1&0&1&1\\0&0&0&1\\0&0&1&0\end{pmatrix}$	$\begin{pmatrix}1&1&-i&-i\end{pmatrix}$

Proof. In the following we set define $c, t \in \mathbb{N}$, s.t., $c > t$, i.e., $c = 2$ and $t = 1$. When we use $a, b \in \mathbb{N}$ as indices, the order does not matter. We calculate the according conjugations with the representatives (we also use Prop. A.0.1):

$$\mathrm{CZ}_{ab} * X_b = \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} = Z_a X_b \tag{34}$$

$$\mathrm{CX}_{ct} * Z_t = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix} = Z_t Z_c \tag{35}$$

$$\mathrm{CX}_{ct} * Z_c = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z_c \tag{36}$$

$$\mathrm{CX}_{ct} * X_t = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} = X_t \tag{37}$$

$$\mathrm{CX}_{ct} * X_c = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = X_t X_c \tag{38}$$

$$\mathrm{CY}_{ct} * Z_t = \ldots = Z_t Z_c \tag{39}$$

$$\mathrm{CY}_{ct} * Z_c = \ldots = Z_c \tag{40}$$

$$\mathrm{CY}_{ct} * X_t = i\mathrm{CY}_{ct} * Z_t Y_t = \ldots = X_t Z_c \tag{41}$$

$$\mathrm{CY}_{ct} * X_c = i\mathrm{CY}_{ct} * Z_c Y_c = \ldots = -i Z_t X_t X_c \tag{42}$$

$$\mathrm{SWAP}_{ab} Z_a = (\mathrm{CX}_{ab} * \mathrm{CX}_{ba}) * Z_a = \mathrm{CX}_{ab}\mathrm{CX}_{ba} * (\mathrm{CX}_{ab} * Z_a) \tag{43}$$

$$= \mathrm{CX}_{ab} * (\mathrm{CX}_{ba} * Z_a) = \mathrm{CX}_{ab} * Z_a Z_b = Z_b \tag{44}$$

$$\mathrm{SWAP}_{ab} X_b = \ldots = X_a \tag{45}$$

$$\mathrm{iSWAP}_{ab} * Z_a = H_b \mathrm{CX}_{ba}\mathrm{CX}_{ab} H_a S_a S_b * Z_a \tag{46}$$

$$= H_b \mathrm{CX}_{ba}\mathrm{CX}_{ab} * X_a = H_b \mathrm{CX}_{ba} * X_a X_b = H_b * X_b = Z_b \tag{47}$$

$$\mathrm{iSWAP}_{ab} * X_a = H_b \mathrm{CX}_{ba}\mathrm{CX}_{ab} H_a S_a S_b * X_a = -i H_b \mathrm{CX}_{ba}\mathrm{CX}_{ab} H_a * Z_a X_a \tag{48}$$

$$= i H_b \mathrm{CX}_{ba} * Z_a X_a X_b = i H_b * Z_a Z_b X_b = -i Z_a Z_b X_b \tag{49}$$

□

Remark 2.2.3 *In the following we list some other typical Clifford gates which only differ by Pauli operators:*

$$(\mathrm{iSWAP})^\dagger = \mathrm{iSWAP}\, Z_1 Z_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{50}$$

2.3. Other operations provided by the library

The library also provides other operations, which are not conjugations, like the moving Paulis from one qubit to another. We define the operation $\mathrm{move\_x\_to\_x}_{sd}$ to move the X Pauli on the qubit s to the qubit d. Analog we define "x to z", "z to x" and "z to z". They are all homomorphisms, which is why it makes sense to use just like Clifford operations when tracking Paulis. This operations are often useful in an MBQC related context where they can be used to move dependencies (cf. how the T telportation is optimized in the “Streamed tracking” section here)

Definition 2.3.1 *In the following we define the action of the “move” operations.*

Table 3: *Move operations of Pauli operators. The operations are linear defined on the basis $\{Z_s, X_s, Z_d, X_d\}$ (under the $\overline{\tau}$ isomorph) for some $s, d \in \mathbb{N}$.*

name	operation
move_z_to_z$_{sd}$	$Z_s \mapsto Z_d$ $X_s \mapsto X_s$ $Z_d \mapsto Z_d$ $X_d \mapsto X_d$
move_z_to_x$_{sd}$	$Z_s \mapsto X_d$ $X_s \mapsto X_s$ $Z_d \mapsto Z_d$ $X_d \mapsto X_d$
move_x_to_z$_{sd}$	$Z_s \mapsto Z_s$ $X_s \mapsto Z_d$ $Z_d \mapsto Z_d$ $X_d \mapsto X_d$
move_x_to_x$_{sd}$	$Z_s \mapsto Z_s$ $X_s \mapsto X_d$ $Z_d \mapsto Z_d$ $X_d \mapsto X_d$

A. Other useful stuff

Proposition A.0.1 *Here are some more operator identities:*

$$\mathrm{CX}_{ct} = H_t * \mathrm{CZ}_{ct} = H_t * \mathrm{CZ}_{tc} \tag{51}$$

$$\mathrm{CY}_{ct} = H_t^{yz} * \mathrm{CZ}_{ct} = H_t^{yz} * \mathrm{CZ}_{tc} \tag{52}$$

$$\mathrm{SWAP}_{ab} = \mathrm{CX}_{ab} * \mathrm{CX}_{ba} \tag{53}$$

$$\mathrm{iSWAP}_{ab} = H_b \mathrm{CX}_{ba} \mathrm{CX}_{ab} H_a S_a S_b \tag{54}$$

Proof.

$$H_t * \mathrm{CZ}_{ct} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \mathrm{CX}_{ct} \tag{55}$$

$$\mathrm{CX}_{ab} * \mathrm{CX}_{ba} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathrm{SWAP}_{ab} \tag{56}$$

$$2H_b\mathrm{CX}_{ba}\mathrm{CX}_{ab}H_aS_aS_b = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ X & -X \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & iS \end{pmatrix} \tag{57}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ 0 & i & 0 & 1 \\ 1 & 0 & -i & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2\mathrm{iSWAP} \tag{58}$$

□

References

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