

Dynamic response of an RC beam to a drop weight impact (steel ball)

Ball mass - $M_s = 2.1 \text{ t}$

Ball material - steel

Modulus of elasticity - $E_s = 206 \text{ GPa}$

Poisson's ratio - $\nu_s = 0.3$

Mass density - $\rho_s = 7.85 \frac{\text{t}}{\text{m}^3}$

Ball volume - $V_s = \frac{M_s}{\rho_s} = \frac{2.1 \text{ t}}{7.85 \text{ t/m}^3} = 0.268 \text{ m}^3 = 4/3\pi R^3$

Ball radius - $R_s = \sqrt[3]{\frac{3}{4} \cdot \frac{V_s}{\pi}} = 400 \text{ mm}$

Height of bottom above the beam surface - $H = 2 \text{ m}$

Structure type - simply supported beam

Beam length - $L = 12 \text{ m}$

Material - reinforced concrete C20/25

Modulus of elasticity - $E = 20 \text{ GPa}$

Poisson's ratio - $\nu = 0.2$

Shear modulus - $G = \frac{E}{2 \cdot (1 + \nu)} = \frac{20 \text{ GPa}}{2 \cdot (1 + 0.2)} = 8.33 \text{ GPa}$

Unit weight - $\gamma_b = 25 \frac{\text{kN}}{\text{m}^3}$

Cross section - rectangular with dimensions:

Width - $b = 350 \text{ mm}$

Height - $h = 650 \text{ mm}$

Area - $A = b \cdot h = 350 \text{ mm} \cdot 650 \text{ mm} = 2275 \text{ cm}^2$

Second moment of area - $I = \frac{b \cdot h^3}{12} = \frac{350 \text{ mm} \cdot (650 \text{ mm})^3}{12} = 8009895833 \text{ mm}^4$

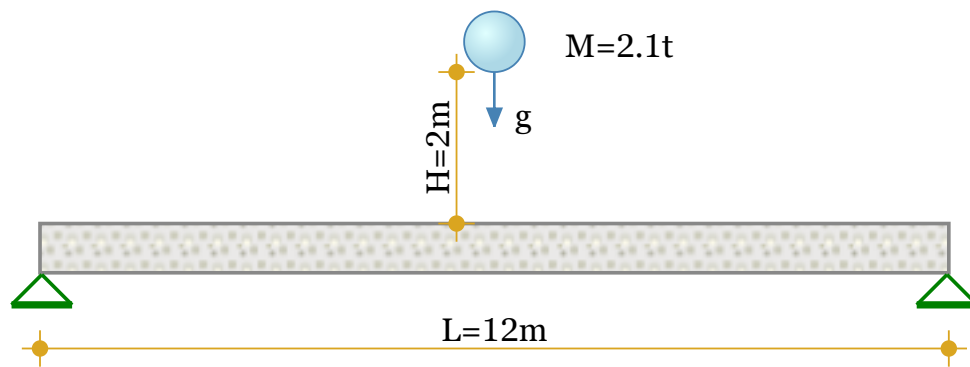
Shear area - $A_Q = \frac{5}{6} \cdot A = \frac{5}{6} \cdot 2275 \text{ cm}^2 = 1895.83 \text{ cm}^2$

Self-weight - $g_b = A \cdot \gamma_b = 2275 \text{ cm}^2 \cdot 25 \text{ kN/m}^3 = 5.69 \text{ kN/m}$

Uniform load - $q = 10 \frac{\text{kN}}{\text{m}}$

Gravity acceleration - $g = 9.81 \frac{\text{m}}{\text{s}^2}$

Uniform mass - $m = \frac{g_b + q}{g} = \frac{5.69 \text{ kN/m} + 10 \text{ kN/m}}{9.81 \text{ m/s}^2} = 1.6 \text{ t/m}$



Simple analytical solution

The structure is reduced to a SDOF system for simplicity

$$\text{Dynamically equivalent mass} - M_d = \frac{2 \cdot L}{\pi} \cdot m = \frac{2 \cdot 12 \text{ m}}{3.14} \cdot 1.6 \text{ t/m} = 12.22 \text{ t}$$

Potential energy of the ball before dropping

$$E_p = M_s \cdot g \cdot H = 2.1 \text{ t} \cdot 9.81 \text{ m/s}^2 \cdot 2 \text{ m} = 41.28 \text{ kJ}$$

$$\text{Kinetic energy immediately before the impact} - E_k = \frac{M_s \cdot v_0^2}{2}$$

The velocity at the moment before the impact is determined by the energy conservation law $E_k = E_p$:

$$v_0 = \sqrt{\frac{2 \cdot E_p}{M_s}} = \sqrt{\frac{2 \cdot 41.28 \text{ kJ}}{2.1 \text{ t}}} = 6.26 \text{ m/s}$$

Perfectly inelastic collision model is assumed.

$$\text{Total mass after contact} - M_{\text{tot}} = M_s + M_d = 2.1 \text{ t} + 12.22 \text{ t} = 14.33 \text{ t}$$

The velocity immediately after the contact is determined by the law of conservation of momentum:

$$v_1 = \frac{v_0 \cdot M_s}{M_{\text{tot}}} = \frac{6.26 \text{ m/s} \cdot 2.1 \text{ t}}{14.33 \text{ t}} = 0.92 \text{ m/s}$$

Structural stiffness for a vertical force applied at the middle point of the span

$$K = \frac{48 \cdot E \cdot I}{L^3} = \frac{48 \cdot 20 \text{ GPa} \cdot 8009895833 \text{ mm}^4}{(12 \text{ m})^3} = 4449.94 \text{ kN/m}$$

Deflection due to uniform load

$$z_0 = \frac{5 \cdot (g_b + q) \cdot L^4}{384 \cdot E \cdot I} = \frac{5 \cdot (5.69 \text{ kN/m} + 10 \text{ kN/m}) \cdot (12 \text{ m})^4}{384 \cdot 20 \text{ GPa} \cdot 8009895833 \text{ mm}^4} = 26.44 \text{ mm}$$

$$\text{Static displacement} - z_{\text{st}} = \frac{M_{\text{tot}} \cdot g}{K} = \frac{14.33 \text{ t} \cdot 9.81 \text{ m/s}^2}{4449.94 \text{ kN/m}} = 31.57 \text{ mm}$$

$$\text{Natural circular frequency} - \omega_1 = \sqrt{\frac{K}{M_{\text{tot}}}} = \sqrt{\frac{4449.94 \text{ kN/m}}{14.33 \text{ t}}} = 17.62 \text{ s}^{-1}$$

$$\text{Vibration period} - T_1 = \frac{2 \cdot \pi}{\omega_1} = \frac{2 \cdot 3.14}{17.62 \text{ s}^{-1}} = 0.356 \text{ s}$$

Dynamic factor

$$\mu = 1 + \sqrt{1 + \left(\frac{v_1 \cdot \omega_1}{g} \right)^2} = 1 + \sqrt{1 + \left(\frac{0.92 \text{ m/s} \cdot 17.62 \text{ s}^{-1}}{9.81 \text{ m/s}^2} \right)^2} = 2.93$$

Dynamic displacement - $z_d = \mu \cdot z_{st} = 2.93 \cdot 31.57 \text{ mm} = 92.58 \text{ mm}$

Dynamic force - $F_d = \mu \cdot M_s \cdot g = 2.93 \cdot 2.1 \text{ t} \cdot 9.81 \text{ m/s}^2 = 60.52 \text{ kN}$

(without self-weight and uniform load)

Simplified equation for the dynamic factor

$$\mu_1 = 1 + \frac{v_1 \cdot \omega_1}{g} = 1 + \frac{0.92 \text{ m/s} \cdot 17.62 \text{ s}^{-1}}{9.81 \text{ m/s}^2} = 2.65$$

The difference will be smaller for greater heights.

Elastic time history response of the structure as an SDOF system

Damped vibration is assumed with factor - $\xi = 0.05$

Vibration amplitude - $A = z_d - z_{st} = 92.58 \text{ mm} - 31.57 \text{ mm} = 61.01 \text{ mm}$ or

$$A = \frac{v_1}{\omega_1} = \frac{0.92 \text{ m/s}}{17.62 \text{ s}^{-1}} = 52.21 \text{ mm}$$

Theoretical equation of motion

$$y(t) = A \cdot e^{-\xi \cdot \omega_1 \cdot t} \cdot \sin(\omega_1 \cdot t)$$

Solution by direct integration of the impulse load

Duration of impulse transmission for a beam with infinite mass [1]

$$\tau_L = 2.94 \cdot \sqrt{\frac{\left(\frac{15}{16} \cdot M_s \cdot \left(\frac{1 - v^2}{E} + \frac{1 - v_s^2}{E_s} \right) \right)^2}{R_s \cdot v_0}} = 2.94 \cdot \sqrt{\frac{\left(\frac{15}{16} \cdot 2.1 \text{ t} \cdot \left(\frac{1 - 0.2^2}{20 \text{ GPa}} + \frac{1 - 0.3^2}{206 \text{ GPa}} \right) \right)^2}{400 \text{ mm} \cdot 6.26 \text{ m/s}}} = 3.93 \text{ ms}$$

Duration of impulse transmission for a beam with finite mass [2]

$$\tau_L = 2.94 \cdot \sqrt{\frac{\left(\frac{15}{16} \cdot \frac{M_s \cdot M_d}{M_s + M_d} \cdot \left(\frac{1 - v^2}{E} + \frac{1 - v_s^2}{E_s} \right) \right)^2}{R_s \cdot v_0}} = 2.94 \cdot \sqrt{\frac{\left(\frac{15}{16} \cdot \frac{2.1 \text{ t} \cdot 12.22 \text{ t}}{2.1 \text{ t} + 12.22 \text{ t}} \cdot \left(\frac{1 - 0.2^2}{20 \text{ GPa}} + \frac{1 - 0.3^2}{206 \text{ GPa}} \right) \right)^2}{400 \text{ mm} \cdot 6.26 \text{ m/s}}} = 3.69 \text{ ms}$$

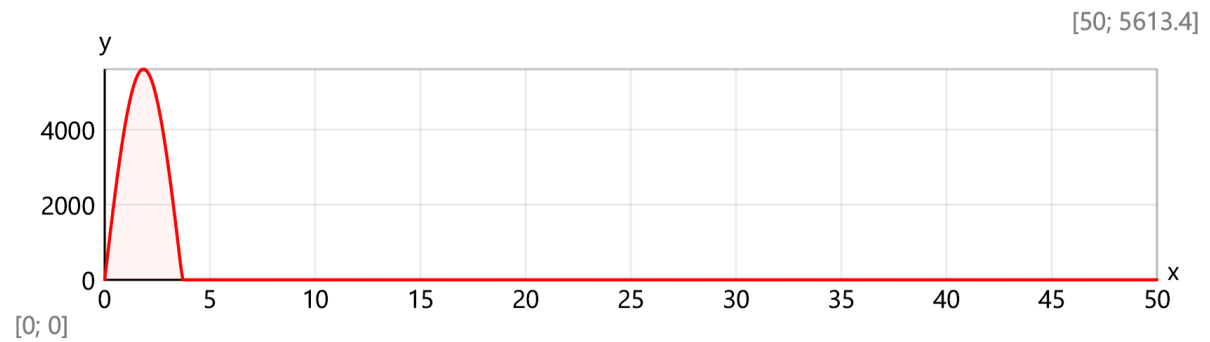
The above values correspond well to the experimental data in [3], where the recorded durations are of a similar magnitude.

The impulse force function will be determined by using the recommended expressions (9.20) - (9.22) in [1]

The coefficient of restitution for perfectly inelastic collision is - $e = 0$

$$F(t) = M_s \cdot v_0 \cdot (1 + e) \cdot \frac{\pi}{2 \cdot \tau_L} \cdot \sin\left(\frac{\pi}{\tau_L} \cdot t\right) \cdot (|t| \leq \tau_L)$$

Impulse load diagram



Maximal impulse load value - $F_{\max} = F\left(\frac{\tau_L}{2}\right) = F\left(\frac{3.69 \text{ ms}}{2}\right) = 5613.66 \text{ kN}$

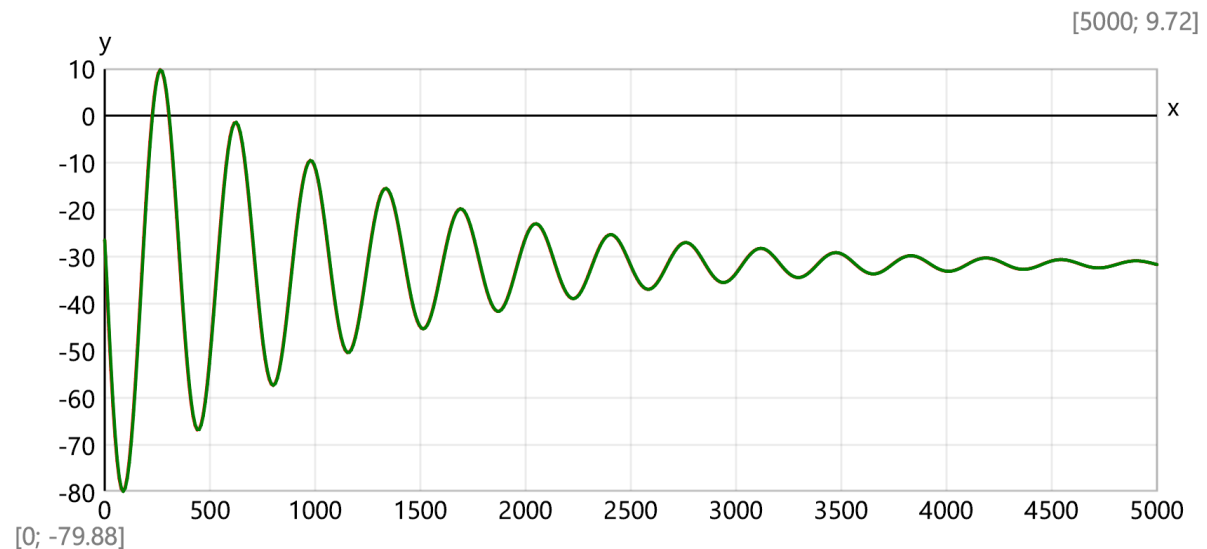
The equation of motion is expressed by the Duhamel's integral

$$y_D(t) = \frac{1}{M_{\text{tot}} \cdot \omega_1} \cdot \int_{0 \text{ ms}}^{\min(t; \tau_L)} F(\tau) \cdot e^{-\xi \cdot \omega_1 \cdot (t - \tau)} \cdot \sin(\omega_1 \cdot (t - \tau)) d\tau$$

Static displacement for the midpoint of the beam

$$y_0(t) = z_0 + (z_{\text{st}} - z_0) \cdot \begin{cases} \text{if } t < \frac{T_1}{4}: \sin\left(\frac{2 \cdot \pi \cdot t}{T_1}\right) \\ \text{else: } 1 \end{cases}$$

Dynamic displacement for the midpoint of the beam



Elastic time history response of the structure as an MDOF system

Number of intermediate joints - $n_j = 11$ (odd)

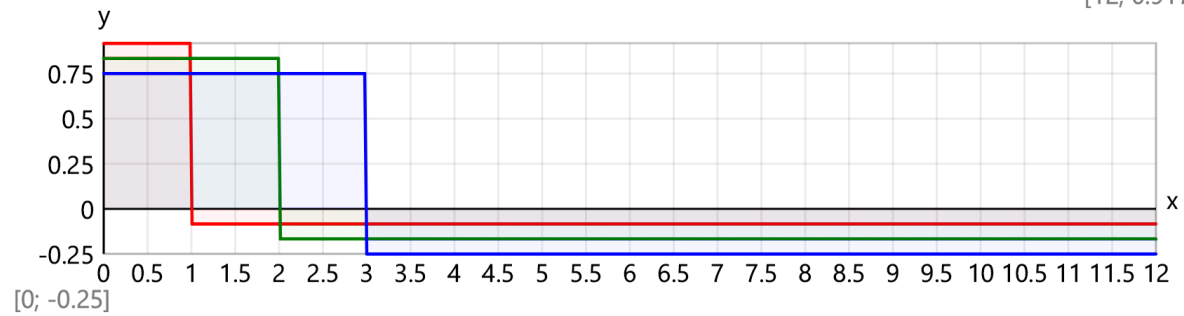
Length of one segment - $\Delta x = \frac{L}{n_j + 1} = \frac{12 \text{ m}}{11 + 1} = 1 \text{ m}$

Coordinate of joint j - $x_j(j) = \Delta x \cdot j$

Shear forces due to unit vertical load $F_j = 1$ at joint j

$$V_1(x; j) = \begin{cases} \text{if } x < x_J(j): 1 - \frac{x_J(j)}{L} \\ \text{else: } -\frac{x_J(j)}{L} \end{cases}$$

[12; 0.917]

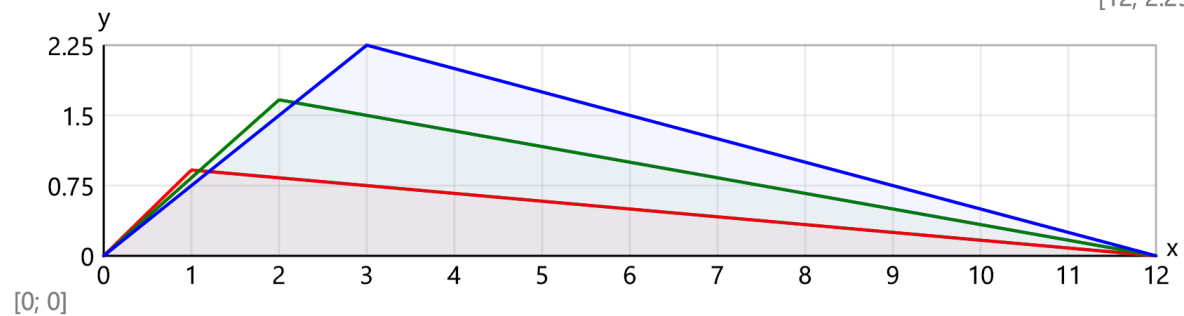


Bending moments due to unit vertical load $F_j = 1$ at joint j

$$M_{1,\max}(j) = \left(\frac{x_J(j)}{L} - 1 \right) \cdot x_J(j)$$

$$M_1(x; j) = M_{1,\max}(j) \cdot \begin{cases} \text{if } x < x_J(j): \frac{x}{x_J(j)} \\ \text{else: } \frac{L - x}{L - x_J(j)} \end{cases}$$

[12; 2.25]



Flexibility matrix

$$D(i; j) = \left(\int_{0m}^L M_1(x; i) \cdot M_1(x; j) dx \right) \cdot \frac{1}{E \cdot I} + \left(\int_{0m}^L V_1(x; i) \cdot V_1(x; j) dx \right) \cdot \frac{1}{G \cdot A_Q}$$

$$D = \begin{bmatrix} 0.0216 & 0.0378 & 0.0489 & 0.0552 & 0.0574 & 0.056 & 0.0514 & 0.0443 & 0.035 & 0.0242 & \cdots & 0.0124 \\ 0.0378 & 0.0704 & 0.093 & 0.106 & 0.111 & 0.109 & 0.1 & 0.0864 & 0.0685 & 0.0474 & \cdots & 0.0242 \\ 0.0489 & 0.093 & 0.128 & 0.149 & 0.158 & 0.155 & 0.144 & 0.124 & 0.0988 & 0.0685 & \cdots & 0.035 \\ 0.0552 & 0.106 & 0.149 & 0.179 & 0.193 & 0.193 & 0.18 & 0.156 & 0.124 & 0.0864 & \cdots & 0.0443 \\ 0.0574 & 0.111 & 0.158 & 0.193 & 0.214 & 0.217 & 0.205 & 0.18 & 0.144 & 0.1 & \cdots & 0.0514 \\ 0.056 & 0.109 & 0.155 & 0.193 & 0.217 & 0.227 & 0.217 & 0.193 & 0.155 & 0.109 & \cdots & 0.056 \\ 0.0514 & 0.1 & 0.144 & 0.18 & 0.205 & 0.217 & 0.214 & 0.193 & 0.158 & 0.111 & \cdots & 0.0574 \\ 0.0443 & 0.0864 & 0.124 & 0.156 & 0.18 & 0.193 & 0.193 & 0.179 & 0.149 & 0.106 & \cdots & 0.0552 \\ 0.035 & 0.0685 & 0.0988 & 0.124 & 0.144 & 0.155 & 0.158 & 0.149 & 0.128 & 0.093 & \cdots & 0.0489 \\ 0.0242 & 0.0474 & 0.0685 & 0.0864 & 0.1 & 0.109 & 0.111 & 0.106 & 0.093 & 0.0704 & \cdots & 0.0378 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.0124 & 0.0242 & 0.035 & 0.0443 & 0.0514 & 0.056 & 0.0574 & 0.0552 & 0.0489 & 0.0378 & \cdots & 0.0216 \end{bmatrix}$$

mm/kN

Mass matrix

$$d_{M,j} = \frac{m \cdot \Delta x}{t} = \frac{1.6 \text{ t/m} \cdot 1 \text{ m}}{t} = 1.6$$

$$M = \begin{bmatrix} 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.7 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.6 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1.6 \end{bmatrix}$$

Total mass of the structure - $\text{sum}(\vec{d}_M) = 19.7 \text{ t}$

Eigenvalues

$$M_{sq} = \sqrt{M} = \begin{bmatrix} 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.92 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.26 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1.26 \end{bmatrix}$$

$C = \text{copy}(M_{sq} \cdot D \cdot M_{sq}; \text{symmetric}(n_j); 1; 1) = \text{copy}(M_{sq} \cdot D \cdot M_{sq}; \text{symmetric}(11); 1;$

$$1) = \begin{bmatrix} 0.0345 & 0.0605 & 0.0781 & 0.0883 & 0.0918 & 0.136 & 0.0822 & 0.0708 & 0.056 & 0.0387 & \dots & 0.0198 \\ 0.0605 & 0.113 & 0.149 & 0.17 & 0.178 & 0.265 & 0.16 & 0.138 & 0.11 & 0.0758 & \dots & 0.0387 \\ 0.0781 & 0.149 & 0.204 & 0.238 & 0.252 & 0.378 & 0.23 & 0.199 & 0.158 & 0.11 & \dots & 0.056 \\ 0.0883 & 0.17 & 0.238 & 0.287 & 0.309 & 0.469 & 0.287 & 0.25 & 0.199 & 0.138 & \dots & 0.0708 \\ 0.0918 & 0.178 & 0.252 & 0.309 & 0.343 & 0.529 & 0.328 & 0.287 & 0.23 & 0.16 & \dots & 0.0822 \\ 0.136 & 0.265 & 0.378 & 0.469 & 0.529 & 0.839 & 0.529 & 0.469 & 0.378 & 0.265 & \dots & 0.136 \\ 0.0822 & 0.16 & 0.23 & 0.287 & 0.328 & 0.529 & 0.343 & 0.309 & 0.252 & 0.178 & \dots & 0.0918 \\ 0.0708 & 0.138 & 0.199 & 0.25 & 0.287 & 0.469 & 0.309 & 0.287 & 0.238 & 0.17 & \dots & 0.0883 \\ 0.056 & 0.11 & 0.158 & 0.199 & 0.23 & 0.378 & 0.252 & 0.238 & 0.204 & 0.149 & \dots & 0.0781 \\ 0.0387 & 0.0758 & 0.11 & 0.138 & 0.16 & 0.265 & 0.178 & 0.17 & 0.149 & 0.113 & \dots & 0.0605 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.0198 & 0.0387 & 0.056 & 0.0708 & 0.0822 & 0.136 & 0.0918 & 0.0883 & 0.0781 & 0.0605 & \dots & 0.0345 \end{bmatrix}$$

$$\vec{\lambda} = \text{reverse}(\text{last}(\text{eigenvals}(\mathbf{C} \cdot 10^{-3}); 7)) = [0.00261 \ 0.000137 \ 3.32 \times 10^{-5} \ 9.33 \times 10^{-6} \ 4.78 \times 10^{-6} \ 2.17 \times 10^{-6} \ 1.51 \times 10^{-6}]$$

$$\text{Natural circular frequencies} - \vec{\omega} = \sqrt{\frac{1}{\vec{\lambda}}} = [19.57 \ 85.55 \ 173.53 \ 327.32 \ 457.58 \ 678.76 \ 814.79] \text{ s}^{-1}$$

$$\text{Natural vibration frequencies} - \vec{f} = \frac{\vec{\omega}}{2 \cdot \pi} \cdot \text{Hz} = \frac{\vec{\omega}}{2 \cdot 3.14} \cdot \text{Hz} = [3.11 \text{ Hz} \ 13.61 \text{ Hz} \ 27.62 \text{ Hz} \ 52.09 \text{ Hz} \ 72.83 \text{ Hz} \ 108.03 \text{ Hz} \ 129.68 \text{ Hz}]$$

$$\text{Natural vibration periods} - \vec{T} = \frac{1}{\vec{f}} = [0.321 \text{ s} \ 0.0734 \text{ s} \ 0.0362 \text{ s} \ 0.0192 \text{ s} \ 0.0137 \text{ s} \ 0.00926 \text{ s} \ 0.00771 \text{ s}]$$

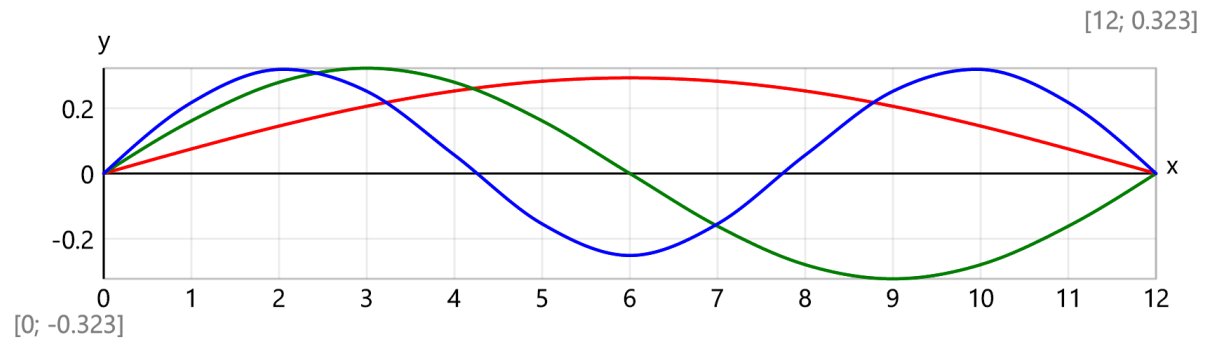
Eigenvectors

$$\Phi = \text{inverse}(\mathbf{M}_{\text{sq}}) \cdot \text{extract}_{\text{cols}}(\text{eigenvecs}(\mathbf{C} \cdot 10^{-3}); \text{range}(n_j; n_j - 7 + 1; -1)) = \text{inverse}(\mathbf{M}_{\text{sq}}) \cdot \text{extract}_{\text{cols}}(\text{eigenvecs}(\mathbf{C} \cdot 10^{-3}); \text{range}(11; 11 - 7 + 1; -1)) =$$

$$\begin{bmatrix} 0.0752 & 0.161 & 0.217 & 0.28 & 0.301 & 0.323 & 0.309 \\ 0.145 & 0.28 & 0.319 & 0.28 & 0.186 & -9.74 \times 10^{-13} & -0.113 \\ 0.206 & 0.323 & 0.252 & 2.84 \times 10^{-13} & -0.187 & -0.323 & -0.266 \\ 0.253 & 0.28 & 0.0565 & -0.28 & -0.309 & 1.15 \times 10^{-13} & 0.216 \\ 0.283 & 0.161 & -0.155 & -0.28 & -0.027 & 0.323 & 0.209 \\ 0.293 & 0 & -0.251 & -7.46 \times 10^{-15} & 0.22 & 5.01 \times 10^{-14} & -0.193 \\ 0.283 & -0.161 & -0.155 & 0.28 & -0.027 & -0.323 & 0.209 \\ 0.253 & -0.28 & 0.0565 & 0.28 & -0.309 & -2.16 \times 10^{-13} & 0.216 \\ 0.206 & -0.323 & 0.252 & -2.77 \times 10^{-13} & -0.187 & 0.323 & -0.266 \\ 0.145 & -0.28 & 0.319 & -0.28 & 0.186 & 1.09 \times 10^{-12} & -0.113 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0752 & -0.161 & 0.217 & -0.28 & 0.301 & -0.323 & 0.309 \end{bmatrix}$$

$$\mathbf{X} = \text{stack}(\text{matrix}(1; 3); \Phi; \text{matrix}(1; 3)) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0752 & 0.161 & 0.217 & 0.28 & 0.301 & 0.323 & 0.309 \\ 0.145 & 0.28 & 0.319 & 0.28 & 0.186 & -9.74 \times 10^{-13} & -0.113 \\ 0.206 & 0.323 & 0.252 & 2.84 \times 10^{-13} & -0.187 & -0.323 & -0.266 \\ 0.253 & 0.28 & 0.0565 & -0.28 & -0.309 & 1.15 \times 10^{-13} & 0.216 \\ 0.283 & 0.161 & -0.155 & -0.28 & -0.027 & 0.323 & 0.209 \\ 0.293 & 0 & -0.251 & -7.46 \times 10^{-15} & 0.22 & 5.01 \times 10^{-14} & -0.193 \\ 0.283 & -0.161 & -0.155 & 0.28 & -0.027 & -0.323 & 0.209 \\ 0.253 & -0.28 & 0.0565 & 0.28 & -0.309 & -2.16 \times 10^{-13} & 0.216 \\ 0.206 & -0.323 & 0.252 & -2.77 \times 10^{-13} & -0.187 & 0.323 & -0.266 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



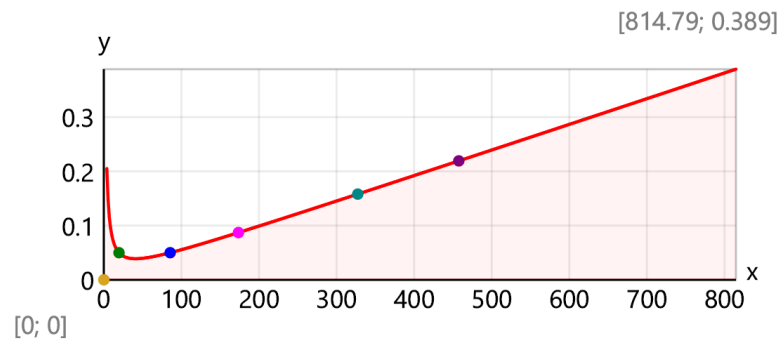
Modal masses - $\vec{m}_\phi = \text{diag2vec}(\text{transp}(\Phi) \cdot M \cdot \Phi) \cdot t = [1 \text{ t } 1 \text{ t } 1 \text{ t } 1 \text{ t } 1 \text{ t } 1 \text{ t }]$

Rayleigh damping model is assumed

$$\beta = \frac{2 \cdot \xi}{\vec{\omega}_1 + \vec{\omega}_2} = \frac{2 \cdot 0.05}{19.57 + 85.55} = 0.000951, \quad \alpha = \beta \cdot \vec{\omega}_1 \cdot \vec{\omega}_2 = 0.000951 \cdot 19.57 \cdot 85.55 = 1.59$$

$$\xi(\omega) = \frac{\alpha}{2 \cdot \omega} + \frac{\beta \cdot \omega}{2}$$

Modal damping factors - $\vec{\xi}_\phi = \xi(\vec{\omega}) = [0.05 \ 0.05 \ 0.0871 \ 0.158 \ 0.219 \ 0.324 \ 0.389]$



Damped natural frequencies

$$\vec{\omega}_D = \vec{\omega} \sqrt{1 - \vec{\xi}_\phi^2} \cdot s^{-1} = [19.54 \text{ s}^{-1} \ 85.44 \text{ s}^{-1} \ 172.87 \text{ s}^{-1} \ 323.2 \text{ s}^{-1} \ 446.43 \text{ s}^{-1} \ 642.14 \text{ s}^{-1} \ 750.77 \text{ s}^{-1}]$$

Dynamic load vector

$$F_\phi(i; t) = \Phi_{j_m} \cdot i \cdot F(t)$$

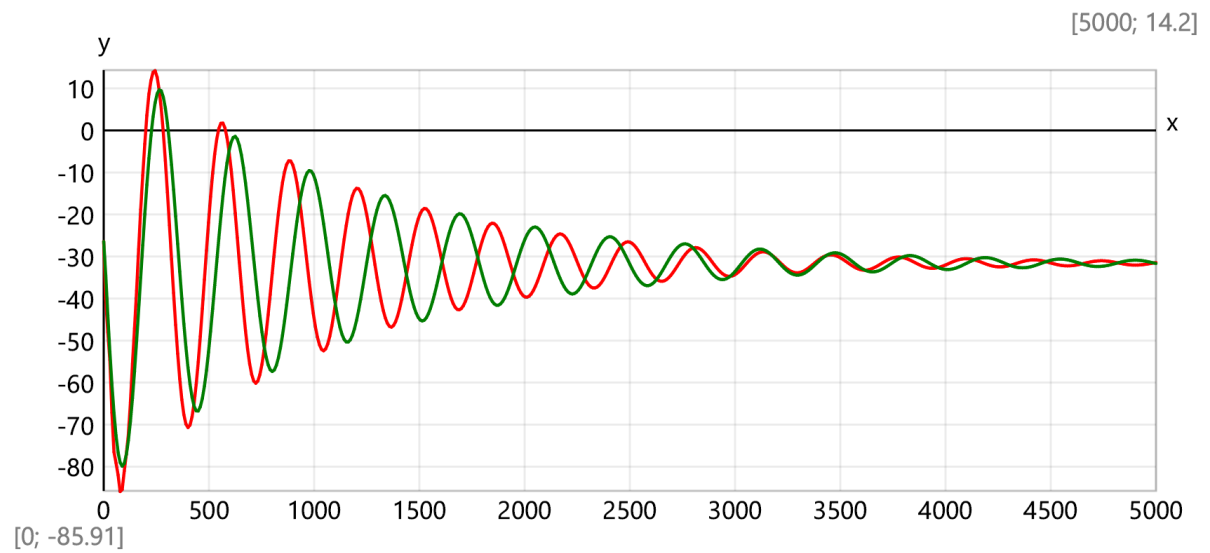
The equations of modal dynamic displacements are expressed by the Duhamel's integral

$$y_\phi(i; t) = \frac{1}{\vec{m}_{\phi, i} \cdot \omega_{D, i}} \cdot \int_{0 \text{ ms}}^{\min(t; \tau_L)} F_\phi(i; \tau) \cdot e^{-\vec{\xi}_{\phi, i} \cdot \vec{\omega}_{D, i} \cdot s^{-1} \cdot (t - \tau)} \cdot \sin(\omega_{D, i} \cdot (t - \tau)) \, d\tau$$

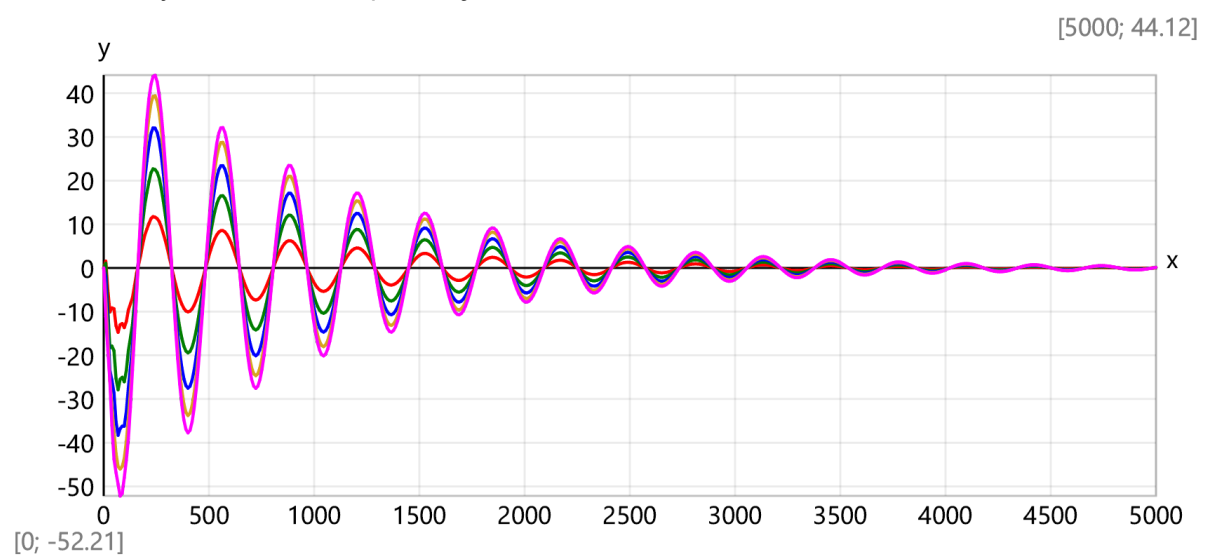
Joint displacements

$$y_J(j; t) = \sum_{i=1}^7 \Phi_{j, i} \cdot y_\phi(i; t)$$

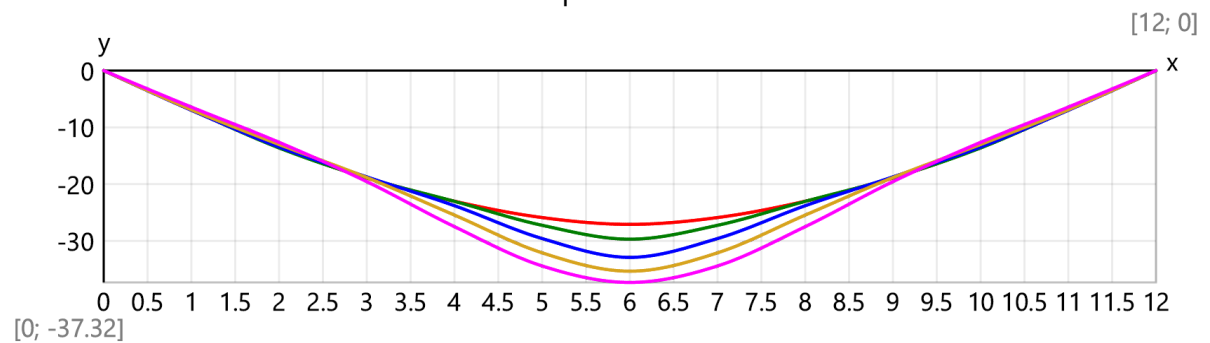
Comparison of time history records at midpoint for SDOF and MDOF systems



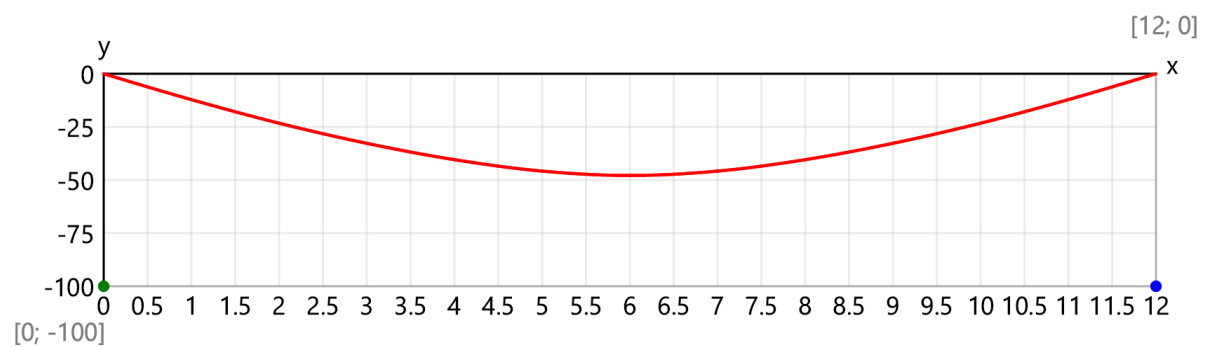
Time history records for separate joints



Beam deflections at the first five time steps of $\Delta t = 17.7$ ms



Animation of beam response (slowed down)



- [1] Harris C. M., Piersol A.G., HARRIS' SHOCK AND VIBRATION HANDBOOK, Fifth Edition, McGraw-Hill 2002, ISBN 0-07-137081-1
- [2] Qing Peng, Xiaoming Liu, Yueguang Wei, Elastic impact of sphere on large plate, Journal of the Mechanics and Physics of Solids, Volume 156, 2021, 104604, ISSN 0022 - 5096, <https://doi.org/10.1016/j.jmps.2021.104604>
- [3] Hong Hao and Thong M. Pham, Performance of RC Beams with or without FRP Strengthening Subjected to Impact Loading, Proceedings of the 2nd World Congress on Civil, Structural, and Environmental Engineering (CSEE'17) Barcelona, Spain – April 3– 4, 2017 ISSN:2371 - 5294 [DOI:10.11159/icsenm17.1](https://doi.org/10.11159/icsenm17.1)
- [4] Guban, D. "Inelastic collision and the Hertz theory of impact." American Journal of Physics 68 (2000): 920-924., <http://www.oxfordcroquet.com/tech/guban/index.asp>