

Ejercicio 2.1

Encuentra la expresión del espectro de Fourier (Forma exponencial y trigonométrica) para la señal:

$$x(t) = |6 \operatorname{Sen}(3t + \pi/4)|^2 \quad \text{con } t \in [-\pi, \pi]$$

Se tiene que:

$$x(t) = |6 \operatorname{Sen}(3t + \pi/4)|^2 = 6^2 \operatorname{Sen}^2(3t + \pi/4)$$

Por la propiedad:

$$\operatorname{Sen}^2(\theta) = \frac{1}{2} - \frac{\cos(2\theta)}{2}$$

Se obtiene:

$$x(t) = 36 \left(\frac{1}{2} - \frac{\cos(6t + \pi/2)}{2} \right) = \frac{36}{2} - 18 \cos(6t + \pi/2)$$

$$x(t) = 18 - 18 \cos(6t + \pi/2)$$

$$\text{Ahora: } \cos(\theta + \pi/2) = -\operatorname{Sen}(\theta)$$

Entonces:

$$x(t) = 18 + 18 \operatorname{Sen}(6t) //$$

Para Serie Trigonométrica:

$$x(t) = a_0 + \sum_{n=-N}^N a_n \cos(n\omega_0 t) + b_n \operatorname{Sen}(n\omega_0 t)$$

Como $x(t)$ tiene simetría IMPAR, por el seno, entonces:

$$x(t) = -x(-t)$$

$$\text{donde: } a_n = 0$$

Por lo tanto:

$$x(t) = 18 + 18 \operatorname{Sen}(6t) = a_0 + \sum_{n=-N}^N b_n \operatorname{Sen}(n\omega_0 t)$$

$$\text{Para } a_0: \quad a_0 = c_0 = \frac{1}{t_f - t_i} \int_{t_i}^{t_f} x(t) dt$$

$$a_0 = \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} (18 + 18 \operatorname{Sen}(6t)) dt$$

$$a_0 = \frac{1}{2\pi} \left[18t \right]_{-\pi}^{\pi} - \frac{18}{6\pi} \cos(6t) \Big|_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{2\pi} \left[18\pi - (-18\pi) - 3(\cos 6\pi - \cos(-6\pi)) \right]$$

$$a_0 = 18$$

Para b_n :

$$b_n = \frac{2}{t_f - t_i} \int_{t_i}^{t_f} x(t) \operatorname{Sen}(n\omega_0 t) dt$$

$$b_n = \frac{2}{\pi - (-\pi)} \int_{-\pi}^{\pi} (18 + 18 \operatorname{Sen}(6t)) \operatorname{Sen}(n\omega_0 t) dt$$

$$b_n = \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} 18 \operatorname{Sen}(n\omega_0 t) dt}_{(1)} + \underbrace{\int_{-\pi}^{\pi} 18 \operatorname{Sen}(6t) \operatorname{Sen}(n\omega_0 t) dt}_{(2)}$$

Para (1):

Se tiene que:

$$\omega_0 = \frac{2\pi}{T} \rightarrow T = 2\pi \rightarrow \omega_0 = \frac{2\pi}{2\pi} = 1$$

$$= \frac{18}{\pi} \int_{-\pi}^{\pi} \operatorname{Sen}(nt) dt = -\frac{18}{n\pi} \cos(nt) \Big|_{-\pi}^{\pi}$$

$$= -\frac{18}{n\pi} [\cos(n\pi) - \cos(-n\pi)]$$

$$(1) = 0$$

Para (2):

Con la identidad: $\operatorname{Sen}(\theta) \operatorname{Sen}(\alpha) = \frac{\cos(\theta - \alpha) - \cos(\theta + \alpha)}{2}$

Entonces: $\theta = 6t$, $\omega = n \text{ Wob}$
 $\alpha = nt$

Aplicando la identidad:

$$= \frac{\cos(6t - nt) - \cos(6t + nt)}{2}$$

$$= \frac{18}{\pi} \int_{-\pi}^{\pi} \frac{\cos(6t - nt) - \cos(6t + nt)}{2} dt$$

$$= \frac{9}{\pi} \left[\int_{-\pi}^{\pi} \cos(6t - nt) dt - \int_{-\pi}^{\pi} \cos(6t + nt) dt \right]$$

$$= \frac{9}{\pi} \left[\int_{-\pi}^{\pi} \cos(t(6-n)) dt - \int_{-\pi}^{\pi} \cos(t(6+n)) dt \right]$$

$$u = t(6-n)$$

$$du = (6-n) dt$$

$$\frac{du}{6-n} = dt$$

$$v = t(6+n)$$

$$dv = (6+n) dt$$

$$\frac{dv}{6+n} = dt$$

$$= \frac{9}{\pi} \left[\int_{-\pi}^{\pi} \frac{\cos(u)}{6-n} du - \int_{-\pi}^{\pi} \frac{\cos(v)}{6+n} dv \right]$$

$$= \frac{9}{\pi} \left[\frac{\text{Sen}(u)}{6-n} \Big|_{-\pi}^{\pi} - \frac{\text{Sen}(v)}{6+n} \Big|_{-\pi}^{\pi} \right]$$

$$= \frac{9}{\pi} \left[\frac{\text{Sen}(t(6-n))}{6-n} \Big|_{-\pi}^{\pi} - \frac{\text{Sen}(t(6+n))}{6+n} \Big|_{-\pi}^{\pi} \right]$$

$$= \frac{9}{\pi} \left[\left(\frac{\text{Sen}(\pi(6-n))}{6-n} - \frac{\text{Sen}(-\pi(6-n))}{6-n} \right) - \left(\frac{\text{Sen}(\pi(6+n))}{6+n} - \frac{\text{Sen}(-\pi(6+n))}{6+n} \right) \right]$$

$$= \left[\frac{9 (\text{Sen}(\pi(6-n)) - \text{Sen}(-\pi(6-n)))}{\pi(6-n)} - \frac{9 (\text{Sen}(\pi(6+n)) - \text{Sen}(-\pi(6+n)))}{\pi(6+n)} \right]$$

Para $n \neq 6$, $Q = n$; pero, para $n = 6$ se debe calcular el límite y aproximar la indeterminación $\frac{0}{0}$.

$$b_6 = 9 \lim_{n \rightarrow 6} \frac{\frac{d}{dn} ((\text{Sen}(\pi(6-n)) - \text{Sen}(-\pi(6-n))))}{\frac{d}{dn} (\pi(6-n))}$$

$$b_6 = 9 \lim_{n \rightarrow 6} \frac{\cos(6-n)\pi(-\pi) - \cos(-(6-n)\pi)(\pi)}{-\pi}$$

$$b_6 = 9 \lim_{n \rightarrow 6} \frac{\cos(0)(-\pi) - \cos(0)(\pi)}{-\pi}$$

$$b_6 = 9 \times \frac{(-2\pi)}{\pi} = 18$$

$$\boxed{b_6 = 18} //$$

$$a_n = \begin{cases} 18 & n=0 \\ 0 & \forall n \in \{0\} \end{cases}$$

$$b_n = \begin{cases} 18 & n=6 \\ 0 & \forall n \in \{6\} \end{cases}$$

Para Serie Exponencial Compleja:

$$C_0 = a_0 = 18$$

$$\text{Entonces: } C_n = \begin{cases} 18 & n=0 \\ -j9 & n = \{6, -6\} \\ 0 & \forall n \in \{0, 6, -6\} \end{cases}$$

$$C_n = \frac{a_n - j b_n}{2}$$

$$C_n = \frac{0 - j18}{2} = -\frac{j18}{2} = -j9$$

$$x(t) = \sum_{n=-N}^N C_n e^{jn\omega t}$$

$$x(t) = C_6 e^{-j6t} + C_0 e^0 + C_6 e^{j6t}$$

Para el error relativo:

$$E_r [\%] = \left[1 - \frac{1}{P_x} \sum_{n=-N}^N |C_n|^2 \right] 100\%$$

Para P_x

$$P_x = \frac{1}{T} \int_{t_1}^{t_2} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |18 + 18 \sin^2(6t)|^2 dt$$

$$P_x = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} 18^2 + 2(18)18 \sin(6t) + 18^2 \sin^2(6t) dt \right]$$

$$P_x = \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} 18^2 dt + \int_{-\pi}^{\pi} 648 \sin(6t) dt + \int_{-\pi}^{\pi} \left(\frac{1}{2} - \frac{\cos(12t)}{2} \right) dt \right]$$

$$P_x = \frac{1}{2\pi} \left[324 t \Big|_{-\pi}^{\pi} - \frac{648}{6} \cos(6t) \Big|_{-\pi}^{\pi} + 324 \left[\frac{1}{2} t \Big|_{-\pi}^{\pi} - \frac{\sin(12t)}{24} \Big|_{-\pi}^{\pi} \right] \right]$$

$$P_x = \frac{1}{2\pi} \left[[324(\pi) - 324(-\pi)] - 108 [\cos(6(\pi)) - \cos(6(-\pi))] + \right.$$

$$\left. 324 \left[\left[\frac{1}{2}(\pi) - \frac{1}{2}(-\pi) \right] - \left[\frac{\sin(12\pi)}{24} - \frac{\sin(12(-\pi))}{24} \right] \right] \right]$$

$$P_x = \frac{1}{2\pi} [648\pi - 0 + 324\pi] = \frac{972\pi}{2\pi} = 486$$

$$\boxed{P_x = 486} //$$

$$E_x = \left[1 - \frac{(-9)^2 + (18)^2 + (9)^2}{486} \right] \times 100\%$$

$$\boxed{E_x = 0\%} //$$

Ejercicio 2.2:

Sea la señal portadora $c(t) = A_c \cos(2\pi F_c t)$, con $A_c, F_c \in \mathbb{R}$, y la señal mensaje $m(t) \in \mathbb{R}$. Encuentre el espectro en frecuencia de la señal modulada en amplitud (AM), $y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t)$.

La transformada de Fourier de la señal modulada se puede encontrar como:

$$\begin{aligned} Y(\omega) &= F\{y(t)\} = F\left\{\left(1 + \frac{m(t)}{A_c}\right) c(t)\right\} \\ &= F\left\{\left(1 + \frac{m(t)}{A_c}\right)\right\} + F\{c(t)\} = F\{c(t)\} + \frac{1}{A_c} F\{m(t) c(t)\} \end{aligned}$$

Utilizando tablas de Fourier:

$$C(\omega) = F\{c(t)\} = F\{A_c \cos(2\pi f_c t)\}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$C(\omega) = A_c F\left\{\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right\}$$

$$A_c = \left[F\left\{\frac{e^{j2\pi f_c t}}{2}\right\} + F\left\{\frac{e^{-j2\pi f_c t}}{2}\right\} \right]$$

Entonces:

$$* F\{e^{\pm j\omega_0 t}\} = 2\pi \delta(\omega \mp \omega_0)$$

$$\frac{A_c}{2} [2\pi \delta(\omega - 2\pi F_c) + 2\pi \delta(\omega + 2\pi F_c)]$$

$$A_c \pi \delta(\omega - 2\pi F_c) + A_c \pi \delta(\omega + 2\pi F_c)$$

$$Y(\omega) = A_c \pi \delta(\omega - 2\pi F_c) + A_c \pi \delta(\omega + 2\pi F_c)$$

$$* F \left\{ \frac{m(t) (A_c \cos(2\pi F_c t))}{A_c} \right\} = F \{ \cos(2\pi F_c t) m(t) \}$$

$$F \left\{ \frac{m(t) e^{j2\pi F_c t}}{2} \right\} + F \left\{ \frac{m(t) e^{-j2\pi F_c t}}{2} \right\}$$

$$\frac{M(\omega - 2\pi F_c)}{2} + \frac{M(\omega + 2\pi F_c)}{2} = \frac{1}{2} M(\omega - 2\pi F_c) + \frac{1}{2} M(\omega + 2\pi F_c)$$

$$Y(\omega) = A_c \pi \delta(\omega - 2\pi F_c) + A_c \pi \delta(\omega + 2\pi F_c) + \frac{1}{2} M(\omega - 2\pi F_c) + \frac{1}{2} M(\omega + 2\pi F_c)$$