Ejercicio 2.1

Encuentre la expresión del espectro de Fourier (Forma exponencial y trigonométrica) para la señal: $x(t) = |6\,Sen(3t + \pi/4)|^2$ con $t \in [-\pi, \pi]$

Se tiene que:

$$x(t) = |6\,Sen(3t + \pi/4)|^2 = 6^2\,Sen^2(3t + \pi/4)$$

Por la propiedad:

$$Sen^2(\theta) = \frac{1}{2} - \frac{Cos(2\theta)}{2}$$

Se obtiene:

$$x(t) = 36\left(\frac{1}{2} - \frac{Cos(6t + \pi/2)}{2}\right) = \frac{\overset{18}{\cancel{36}}}{\cancel{2}_1} - 18\,Cos(6t + \pi/2)$$

$$x(t) = 18 - 18\,Cos(6t + \pi/2)$$

Ahora: $Cos(\theta + \pi/2) = -Sen(\theta)$

Entonces:

$$x(t) = 18 + 18\,Sen(6t) //$$

Para Serie Trigonométrica:

$$x(t) = a_0 + \sum_{n=-N}^{N} a_n Cos(nW_0t) + b_n Sen(nW_0t)$$

Como $x(t)$ tiene simetría IMPAR, por el seno, entonces:

$$x(t) = -x(-t)$$

donde: $a_n = 0$

Por lo tanto:

$$x(t) = 18 + 18\,Sen(6t) = a_0 + \sum_{n=-N}^{N} b_n Sen(nW_0t)$$

Para a_0: $a_0 = c_0 = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} x(t)\,dt$

$$a_0 = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi}\left(18+18\,\mathrm{Sen}(6t)\right)dt$$

$$a_0 = \frac{1}{2\pi}\left[18t\Big|_{-\pi}^{\pi} - \frac{18}{6}\cos(6t)\Big|_{-\pi}^{\pi}\right]$$

$$a_0 = \frac{1}{2\pi}\left[18\pi-(-18\pi) - 3\cancel{\left(\cos 6\pi - \cos(-6\pi)\right)}^{0}\right]$$

$$\boxed{a_0 = 18}$$

Para b_n:

$$\boxed{b_n = \frac{2}{t_f - t_i}\int_{t_i}^{t_f} x(t)\,\mathrm{Sen}(nW_0 t)\,dt}$$

$$b_n = \frac{2}{\pi-(-\pi)}\int_{-\pi}^{\pi}\left(18+18\,\mathrm{Sen}(6t)\right)\mathrm{Sen}(nW_0 t)\,dt$$

$$b_n = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi} 18\,\mathrm{Sen}(nW_0 t)\,dt}_{1} + \underbrace{\int_{-\pi}^{\pi} 18\,\mathrm{Sen}(6t)\,\mathrm{Sen}(nW_0 t)\,dt}_{2}\right]$$

Para 1:

Se tiene que:

$$W_0 = \frac{2\pi}{T} \longrightarrow T = 2\pi \longrightarrow W_0 = \frac{2\pi}{2\pi} = 1$$

$$= \frac{18}{\pi}\int_{-\pi}^{\pi}\mathrm{Sen}(nt)\,dt = -\frac{18}{n\pi}\cos(nt)\Big|_{-\pi}^{\pi}$$

$$= -\frac{18}{n\pi}\left[\cancel{\cos(n\pi) - \cos(-n\pi)}^{0}\right]$$

$$1 = 0$$

Para 2:

Con la identidad: $\mathrm{Sen}(\theta)\,\mathrm{Sen}(\alpha) = \dfrac{\cos(\theta-\alpha) - \cos(\theta+\alpha)}{2}$

Entonces: $\theta = 6t$, $\alpha = nW_0t$
$\alpha = nt$

Aplicando la identidad:

$$= \frac{\cos(6t-nt) - \cos(6t+nt)}{2}$$

$$= \frac{18}{\pi}\int_{-\pi}^{\pi} \frac{\cos(6t-nt) - \cos(6t+nt)}{2}\,dt$$

$$= \frac{9}{\pi}\left[\int_{-\pi}^{\pi} \cos(6t-nt)\,dt - \int_{-\pi}^{\pi} \cos(6t+nt)\,dt\right]$$

$$= \frac{9}{\pi}\left[\int_{-\pi}^{\pi} \cos(t(6-n))\,dt - \int_{-\pi}^{\pi} \cos(t(6+n))\,dt\right]$$

$u = t(6-n)$ $\qquad$ $v = t(6+n)$
$du = (6-n)\,dt$ $\qquad$ $dv = (6+n)\,dt$
$\frac{du}{6-n} = dt$ $\qquad$ $\frac{dv}{6+n} = dt$

$$= \frac{9}{\pi}\left[\int_{-\pi}^{\pi} \frac{\cos(u)}{6-n}\,du - \int_{\pi}^{\pi} \frac{\cos(v)}{6+n}\,dv\right]$$

$$= \frac{9}{\pi}\left[\frac{\operatorname{Sen}(u)}{6-n}\Big|_{-\pi}^{\pi} - \frac{\operatorname{Sen}(v)}{6-n}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{9}{\pi}\left[\frac{\operatorname{Sen}(t(6-n))}{6-n}\Big|_{-\pi}^{\pi} - \frac{\operatorname{Sen}(t(6+n))}{6+n}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{9}{\pi}\left[\left(\frac{\operatorname{Sen}(\pi(6-n))}{6-n} - \frac{\operatorname{Sen}(-\pi(6-n))}{6-n}\right) - \left(\frac{\operatorname{Sen}(\pi(6+n))}{6+n} - \frac{\operatorname{Sen}(-\pi(6+n))}{6+n}\right)\right]$$

$$= \boxed{\frac{9\left(\operatorname{Sen}(\pi(6-n)) - \operatorname{Sen}(-\pi(6-n))\right)}{\pi(6-n)} - \frac{9\left(\operatorname{Sen}(\pi(6+n)) - \operatorname{Sen}(-\pi(6+n))\right)}{\pi(6+n)}}$$ //

Para $n \neq 6$, $a = n$; pero, para $n = 6$ se debe calcular el límite y aproximar la indeterminación $\frac{0}{0}$.

$$b_6 = 9 \lim_{n\to 6} \frac{\frac{d}{dn}\left(\left(\operatorname{Sen}(\pi(6-n))\right) - \left(\operatorname{Sen}(-\pi(6-n))\right)\right)}{\frac{d}{dn}\left(\pi(6-n)\right)}$$

$$b_6 = 9 \lim_{n\to 6} \frac{\cos((6-n)\pi)(-\pi) - \cos(-(6-n)\pi)(\pi)}{-\pi}$$

$$b_6 = 9 \lim_{n\to 6} \frac{\cos(0)(-\pi) - \cos(0)(\pi)}{-\pi}$$

$$b_6 = \frac{9 \times (-2\pi)}{-\pi} = 18 \qquad \boxed{b_6 = 18}$$

$$a_n = \begin{cases} 18 & n=0 \\ 0 & \forall \setminus \{0\} \end{cases}$$

$$b_n = \begin{cases} 18 & n=6 \\ 0 & \forall \setminus \{6\} \end{cases}$$

Para Serie Exponencial Compleja:

$$C_0 = a_0 = 18$$

$$C_n = \frac{a_n - jb_n}{2}$$

$$C_n = \frac{0 - j18}{2} = \frac{-j18}{2} = -j9$$

Entonces: $C_n = \begin{cases} 18 & n=0 \\ -j9 & n=\{6,-6\} \\ 0 & \forall \setminus \{0,6,-6\} \end{cases}$

$$\boxed{x(t) = \sum_{n=-N}^{N} C_n e^{j0}}$$

$$x(t) = C_{-6} e^{-j6t} + C_0 e^{0} + C_6 e^{j6t}$$

Para el error relativo:

$$\boxed{E_r[\%] = \left[1 - \frac{1}{P_x} \sum_{n=-N}^{N} |C_n|^2\right] 100\%}$$

Para P_x

$$P_x = \frac{1}{T} \int_{t_i}^{t_f} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |18 + 18 \operatorname{Sen}^2(6t)|^2 dt$$

$$P_x = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 18^2 - 2(18)18\,Sen(6t) + 18^2 Sen^2(6t)\,dt\right]$$

$$P_x = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 18^2 dt - \int_{-\pi}^{\pi} 648\,Sen(6t)\,dt + \int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{Cos(12t)}{2}dt\right)\right]$$

$$P_x = \frac{1}{2\pi}\left[324t\Big|_{-\pi}^{\pi} + \frac{648}{6}Cos(6t)\Big|_{-\pi}^{\pi} + 324\left[\frac{1}{2}t\Big|_{-\pi}^{\pi} - \frac{Sen(12t)}{24}\Big|_{-\pi}^{\pi}\right]\right]$$

$$P_x = \frac{1}{2\pi}\Big[\left[324(\pi) - 324(-\pi)\right] + 108\left[Cos(6(\pi)) - Cos(6(-\pi))\right] +$$

$$324\left[\left[\frac{1}{2}(\pi) - \frac{1}{2}(-\pi)\right] - \left[\frac{Sen(12\pi)}{24} - \frac{Sen(12(-\pi))}{24}\right]\right]\Big]$$

$$P_x = \frac{1}{2\pi}\left[648\pi + 0 + 324\pi\right] = \frac{972\cancel{\pi}}{2\cancel{\pi}} = 486$$

$$\boxed{P_x = 486} \;//$$

$$E_x = \left[1 - \frac{(-9)^2 + (18)^2 + (9)^2}{486}\right] \times 100\%$$

$$\boxed{E_x = 0\%} \;//$$

Ejercicio 2.2:

Sea la señal portadora $c(t) = A_c \cos(2\pi F_c t)$, con $A_c, F_c \in \mathbb{R}$, y la señal mensaje $m(t) \in \mathbb{R}$. Encuentre el espectro en frecuencia de la señal modulada en amplitud (AM), $y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t)$.

la transformada de fourier de la señal modulada se puede encontrar como:

$$Y(\omega) = F\{Y(t)\} = F\left\{\left(1 + \frac{m(t)}{A_c}\right) c(t)\right\}$$

$$= F\left\{\left(1 + \frac{m(t)}{A_c}\right)\right\} + F\{c(t)\} = F\{C(t)\} + \frac{1}{A_c} F\{m(t)\, c(t)\}$$

Utilizando tablas de Fourier:

$$C(\omega) = F\{C(t)\} = F\{A_c \cos(2\pi F_c t)\}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$C(\omega) = A_c\, F\left\{\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right\}$$

$$A_c = \left[F\left\{\frac{e^{j2\pi F_c t}}{2}\right\} + F\left\{\frac{e^{-j2\pi F_c t}}{2}\right\}\right.$$

Entonces:

$$* \; F\{e^{\pm j\omega_0 t}\} = 2\pi\, \delta(\omega \mp \omega_0)$$

$$\frac{A_c}{\cancel{2}}\left[\cancel{2}\pi\,\delta\,(\omega - 2\pi F_c) + \cancel{2}\pi\,\delta\,(\omega + 2\pi F_c)\right]$$

$$A_c\pi\delta\,(\omega - 2\pi F_c) + A_c\pi\delta\,(\omega + 2\pi F_c)$$

$$C(\omega) = A_c\pi\delta\,((\omega - 2\pi F_c) + (\omega + 2\pi F_c))$$

$$* \; F\left\{\frac{m(t)\,(A_c \cos(2\pi F_c t))}{A_c}\right\} = F\{\cos(2\pi F_c t\}\; m(t)\}$$

$$F\left\{\frac{m(t)\,e^{j2\pi F_c t}}{2}\right\} + F\left\{\frac{m(t)\,e^{-j2\pi F_c t}}{2}\right\}$$

$$\frac{M(\omega - 2\pi F_c t)}{2} + \frac{M(\omega + 2\pi F_c t)}{2} = \frac{1}{2}M((\omega - 2\pi F_c t) + (\omega + 2\pi F_c t))$$

$$\boxed{Y(\omega) = A_c\pi\delta\left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)\right] + \frac{1}{2}M\left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)\right]}$$