

# Least Squares



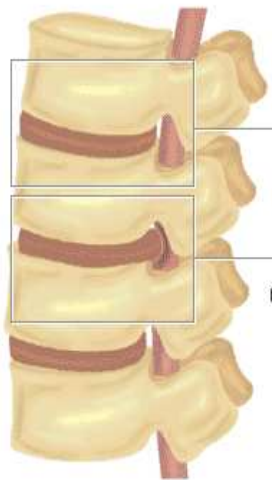
## Numerical Analysis

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2024-SISSA mathLab Trieste

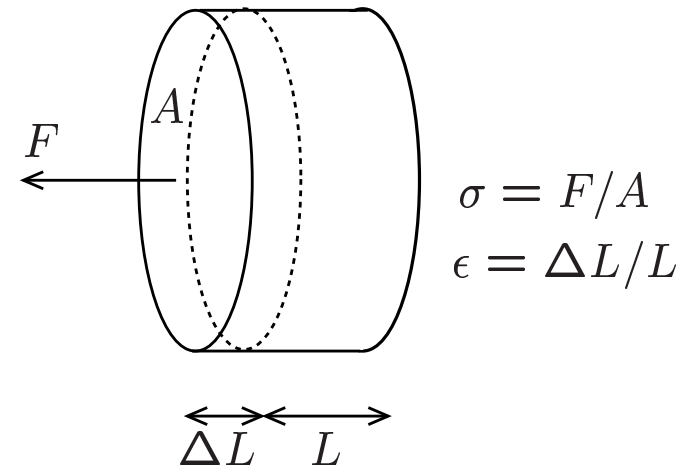
# Examples and motivations

**Example 1.** We consider a mechanical test to establish the link between stresses ( $MPa = 100N/cm^2$ ) and relative deformations (cm/cm) of a sample of biological tissue (an intervertebral disc, taken from P. Komarek, Chapt. 2 of *Biomechanics of Clinical Aspects of Biomedicine*, 1993, J. Valenta ed., Elsevier).



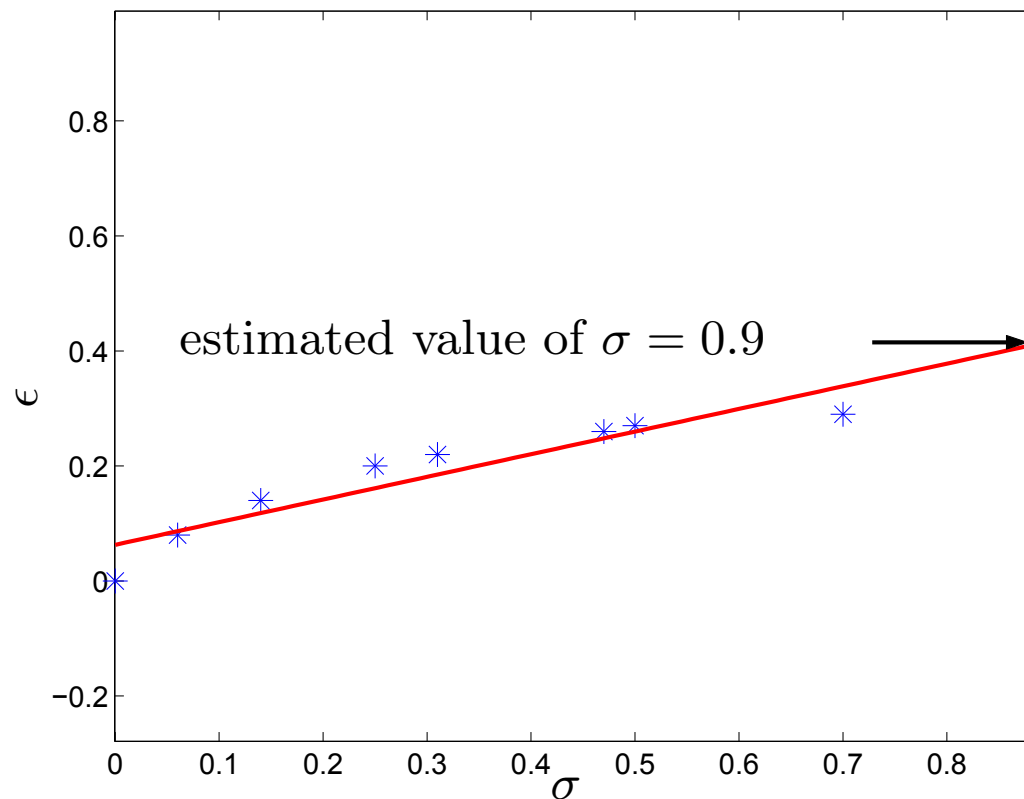
disques intervertébraux

test $i$	pressure $\sigma$	deformation $\epsilon$
1	0.00	0.00
2	0.06	0.08
3	0.14	0.14
4	0.25	0.20
5	0.31	0.23
6	0.47	0.25
7	0.60	0.28
8	0.70	0.29

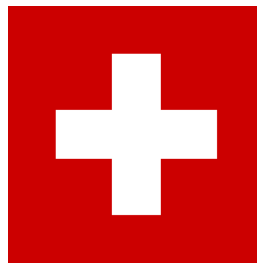


From these data, we want to estimate the deformation corresponding to a stress  $\sigma = 0.9$  MPa.

By the method of least squares, we get that best approximation for the data is  $p(x) = 0.3938x - 0.0629$ . The approximation can be used (called a *linear regression*) to estimate  $\epsilon$  when  $\sigma = 0.9$  MPa: We get  $p(0.9) \simeq 0.4$ .



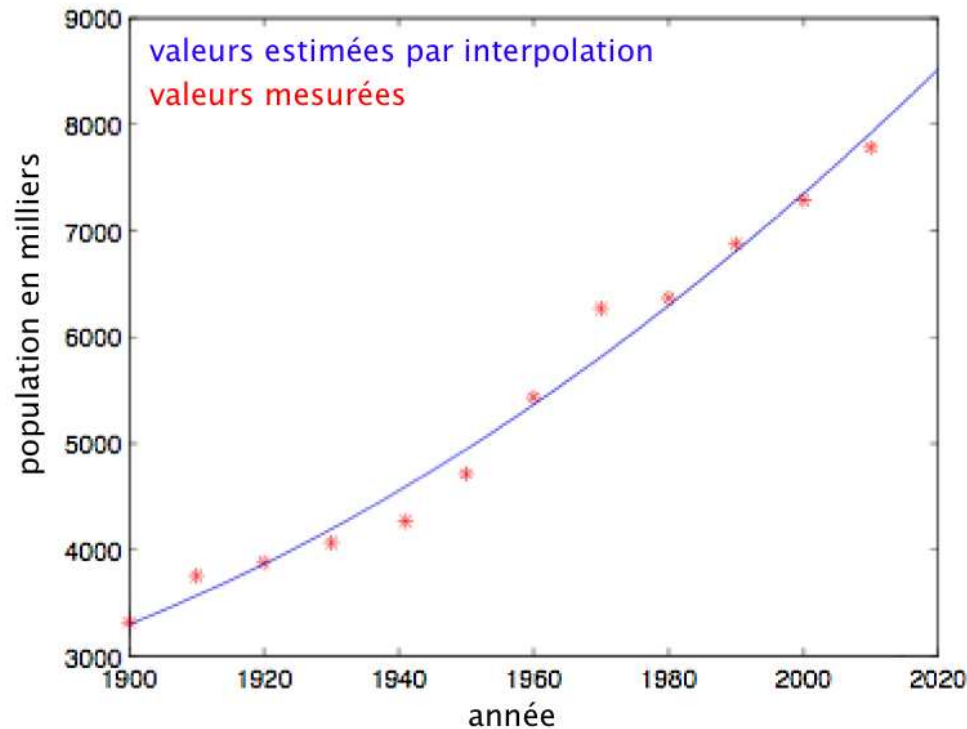
**Example 2.** The result of census of the population of Switzerland between 1900 and 2010 (in thousands):



year	1900	1910	1920	1930	1941	1950
population	3315	3753	3880	4066	4266	4715
year	1960	1970	1980	1990	2000	2010
population	5429	6270	6366	6874	7288	7783

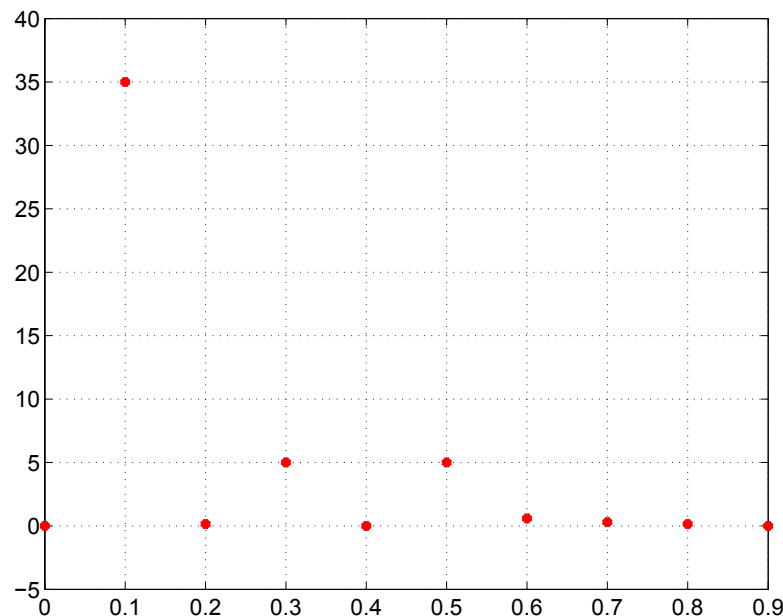
- Is it possible to estimate the number of inhabitants of Switzerland during the year when there has not been census, for example in 1945 and 1975?
- Is it possible to predict the number of inhabitants of Switzerland in 2030?

The polynomial of degree two (parabola) which approximates the data by the method of least squares is  $p(x) = 0.15x^2 - 549.9x + 501600$ .



**Example 3.** Points on the following figure represent the measurements of blood flow in a part of the common carotid artery during a heartbeat. The frequency of data collection is constant and equal to  $10 / T$  where  $T = 1$  sec. is the period of the beat.

We want to deduce from this discrete signal a continuous signal represented by a linear combination of known functions (eg. trigonometric functions - then we can adequately approximate a periodic signal).



Let  $f(t)$  be a signal that we know.  $N = 10$  is a size of a sample  $[f(t_0), \dots, f(t_{N-1})]$ , where  $t_j = jT/N$ . We are looking for  $\{c_k\} \in \mathbb{C}$ ,  $k \in [0, N-1] \subset \mathbb{N}$  such that:

$$f(t_j) = \frac{1}{N} \sum_{k=0}^{N-1} c_k \omega_N^{-kj}, \quad j = 0, \dots, N-1, \quad (1)$$

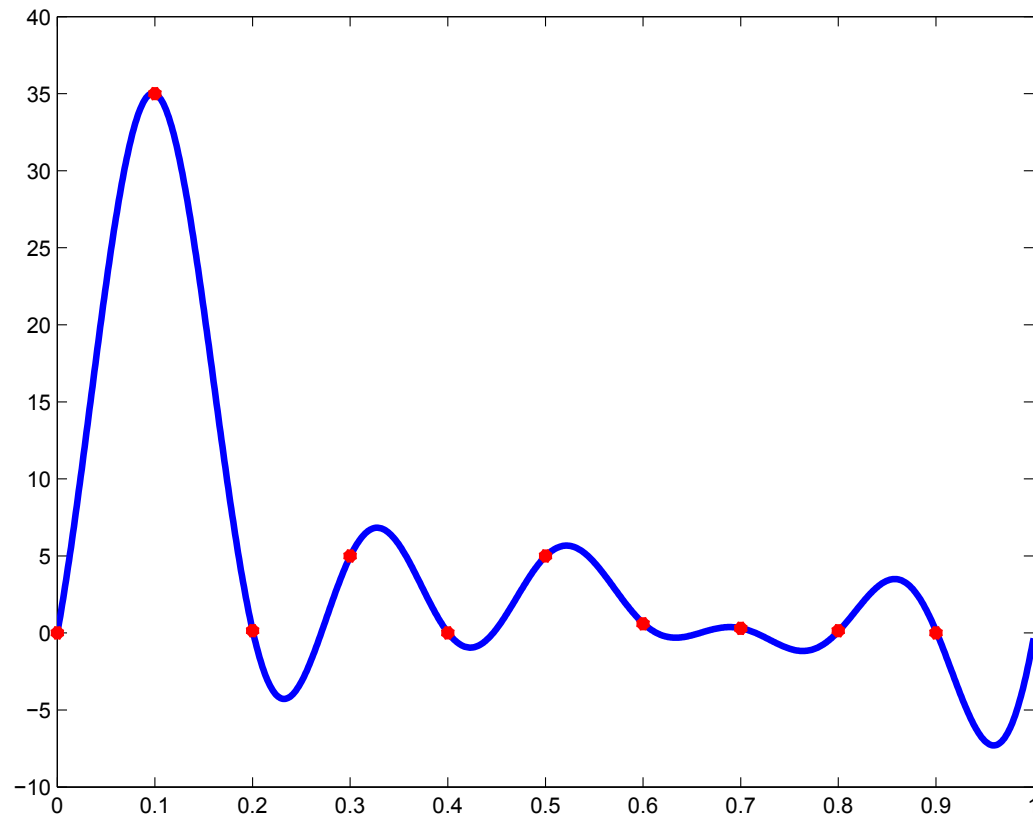
where

$$\omega_N = e^{(-2\pi i)/N} = \cos(2\pi/N) - i \sin(2\pi/N),$$

$i$  is the imaginary unit. We can calculate a vector of the coefficients  $\mathbf{c} = [c_1, \dots, c_{10}]^T$  by the FFT algorithm (Fast Fourier Transform, Cooley & Tuckey, 1965), implemented in Matlab/Octave by the `fft` command.



Using the formula (1), there are several techniques for obtaining the value of  $f$  in each point  $t \in [0, T]$ .  
Using one of techniques we obtain a result that is plotted on a following figure:



# Approximation by the least square method

(Chapt. 3.4 of the book)

Suppose we have  $n + 1$  points  $x_0, x_1, \dots, x_n$  and  $n + 1$  values  $y_0, y_1, \dots, y_n$ . We have seen that if  $n$  is large, the interpolating polynomial may show large oscillations

Instead of interpolating the values, it is possible to define a polynomial of degree  $m < n$  that approximates the data “at best”

**Definition 1.** We call *least squares polynomial approximation of degree  $m$*  the polynomial  $\tilde{f}_m(x)$  of degree  $m$  such that

$$\sum_{i=0}^n |y_i - \tilde{f}_m(x_i)|^2 \leq \sum_{i=0}^n |y_i - p_m(x_i)|^2 \quad \forall p_m(x) \in \mathbb{P}_m$$

**Remark 4.** Where  $y_i = f(x_i)$  ( $f$  is a continuous function) then  $\tilde{f}_m$  is called *the approximation of  $f$  in the least squares sense*.

In other words, the least squares polynomial approximation is the polynomial of degree  $m$  that minimizes the distance to the data.

Let note  $\tilde{f}_m(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$  and define the function

$$\Phi(a_0, a_1, \dots, a_m) = \sum_{i=0}^n |y_i - (a_0 + a_1x_i + a_2x_i^2 + \dots + a_mx_i^m)|^2$$

Then the coefficients of  $\tilde{f}^m$  can be determined by the relation

$$\frac{\partial \Phi}{\partial a_k} = 0, \quad k = 0, \dots, m, \quad (5)$$

i.e.,  $m + 1$  linear equations in with  $m + 1$  unknowns  $a_k, k = 0, \dots, m$ .

**Example 8.** Let  $n = 2$  and  $m = 1$ , nodes  $x_0 = 1, x_1 = 3, x_2 = 4$  with values  $y_0 = 0, y_1 = 2, y_2 = 7$ . We want to compute the (*regression line*), i.e., the the least squares polynomial approximation of degree 1,  $\tilde{f}_1(x) = a_0 + a_1x$ .

We set  $\Phi(a_0, a_1) = \sum_{i=0}^2 [y_i - (a_0 + a_1x_i)]^2$  and impose  $\frac{\partial \Phi}{\partial a_0} = 0$  and  $\frac{\partial \Phi}{\partial a_1} = 0$ :

$$\begin{aligned} \frac{\partial \Phi}{\partial a_0} &= -2 \sum_{i=0}^2 [y_i - (a_0 + a_1x_i)] = -2 \left( \sum_{i=0}^2 y_i - 3a_0 - a_1 \sum_{i=0}^2 x_i \right) \\ &= -2(9 - 3a_0 - 8a_1) \\ \frac{\partial \Phi}{\partial a_1} &= -2 \sum_{i=0}^2 x_i [y_i - (a_0 + a_1x_i)] = -2 \left( \sum_{i=0}^2 x_i y_i - a_0 \sum_{i=0}^2 x_i - a_1 \sum_{i=0}^2 x_i^2 \right) \\ &= -2(34 - 8a_0 - 26a_1) \end{aligned}$$

Hence the coefficients  $a_0$  and  $a_1$  are the solution of the system

$$\begin{cases} 3a_0 + 8a_1 = 9 \\ 8a_0 + 26a_1 = 34 \end{cases}$$

Ideally, for  $\tilde{f}_m(x) = a_0 + a_1x + a_2x^2 + \dots$  we would like to impose  $\tilde{f}_m(x_i) = y_i$  pour  $i = 0, \dots, n$ .

This can be written as a linear system with unknowns  $a_k, k = 0, \dots, m$ :  $B\mathbf{a} = \tilde{\mathbf{y}}$ , where  $B$  is a matrix of dimension  $(n + 1) \times (m + 1)$

$$B = \begin{pmatrix} 1 & x_0 & \dots & x_0^m \\ 1 & x_1 & \dots & x_1^m \\ \vdots & & & \vdots \\ 1 & x_n & \dots & x_n^m \end{pmatrix}$$

Since  $m < n$ , the system is oversized. The solution to (5) is equivalent to the square system (*system of normal equations*)

$$B^T B \mathbf{a} = B^T \tilde{\mathbf{y}}.$$

**Example 9.** We have 8 measures ( $\sigma$ - $\epsilon$ ):

```
>> sigma = [0.00 0.06 0.14 0.25 0.31 0.47 0.50 0.70];
>> epsilon = [0.00 0.08 0.14 0.20 0.22 0.26 0.27 0.29];
```

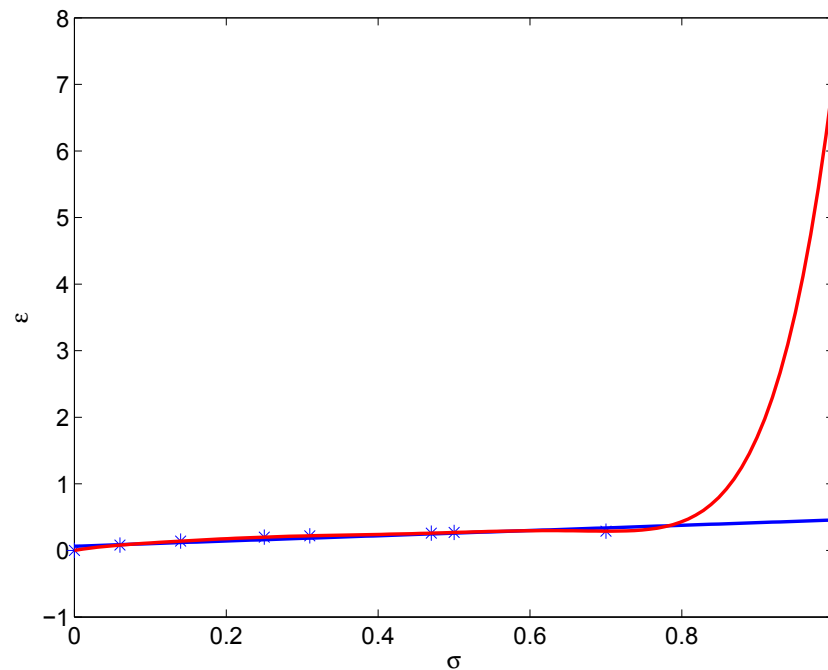
We want to extrapolate the value of  $\epsilon$  for  $\sigma = 0.4$ . We consider two ways:

- compute the interpolating polynomial  $\Pi_7$  of degree 7
- compute the least squares polynomial approximation of degree 1

On peut utiliser les commandes suivantes:

```
>> plot(sigma, epsilon, '*'); hold on; % plotting the known values
>> sigma_sample = linspace(0,1.0,100);
>> p7 = polyfit(sigma, epsilon, 7);
>> pol = polyval(p7, sigma_sample); % interpolating polynomial
>> plot(sigma_sample, pol, 'r');
>> p1 = polyfit(sigma, epsilon, 1);
>> pol_mc = polyval(p1, sigma_sample); % least square
>> plot(sigma_sample, pol_mc, 'g'); hold off;
```

- the interpolating polynomial  $\Pi_7$  of degree 7 is in red
- the least squares polynomial approximation of degree 1 is in blue



For  $\sigma > 0.7$ , the behavior of the two polynomials are very different.



In particular, for  $\sigma = 0.9$  the values of  $\epsilon(\sigma)$  extrapolated with the two methods are

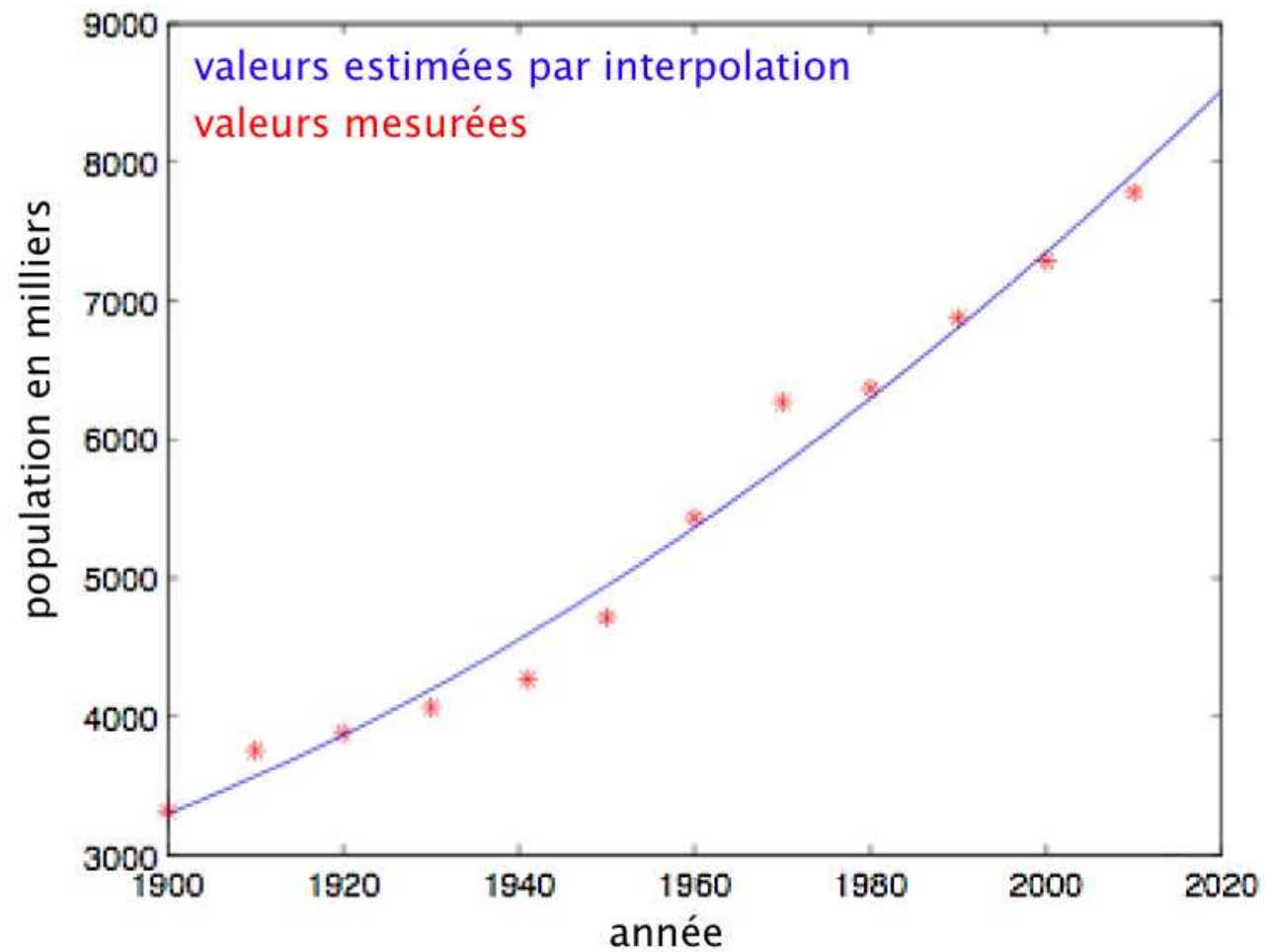
```
>> polyval(p7, 0.9)
ans =
    1.7221
```

```
>> polyval(p1, 0.9)
ans =
    0.4173
```

- The value obtained by  $\Pi_7$  (172.21%) is unrealistic
- On the contrary, the value obtained with the regression line is more appropriate to compute the value at  $\sigma = 0.9$ .

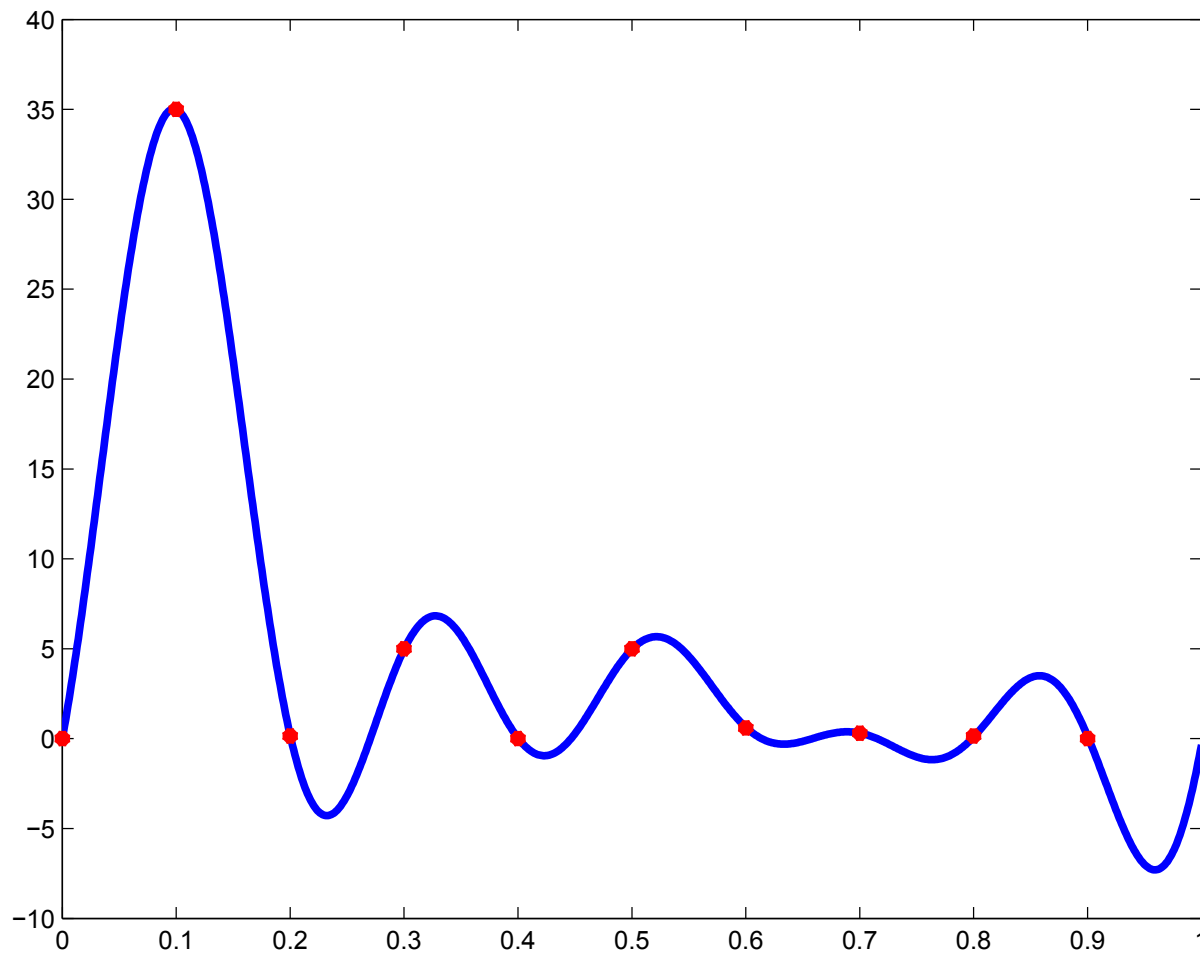
**Example 10.** Swiss population Starting from the population at the 20th century decades, we extrapolate the Swiss population this year. We use a least square polynomial approximation of degree 2

```
>> year = [1900, 1910, 1920, 1930, 1941, 1950,...
           1960, 1970, 1980, 1990, 2000];
>> population = [3315, 3753, 3880, 4066, 4266, 4715,...
                 5429, 6270, 6366, 6874, 7288];
>> p2 = polyfit(year, population, 2);
>> year_sample = [1900:2:2010];
>> vp2 = polyval(p2, year_sample);
>> plot(year, population, '*r', ...
        year_sample, vp2, 'b');
```



**Example. Carotide flow rate** We define 10 couples of data (time vs flowrate) with the vectors **t** and **deb**. With the help of the command **interpft**, we compute the value of the interpolating function at 1000 equidistributed points in the periodicity interval  $[0,1]$ :

```
t = [0 .1 .2 .3 .4 .5 .6 .7 .8 .9];
deb = [0 35 .15 5 0 5 .6 .3 .15 0];
f = interpft(deb, 1000);
plot(linspace(0, 1, 1000), f); hold on;
plot(t, deb, 'r*')
```



# Processing of polynomials

In Matlab/Octave, there are specific commands for doing calculations with polynomials. Let  $x$  be a vector of abscissas,  $y$  be a vector of ordinates and  $p$  (respectively  $p_i$ ) be the vector of coefficients of a polynomial  $P(x)$  (respectively  $P_i$ ); then, we have following commands:

command	action
<code>y=polyval(p,x)</code>	$y = \text{values of } P(x)$
<code>p=polyfit(x,y,n)</code>	$p = \text{coefficients of the interpolating polynomial } \Pi_n$
<code>z=roots(p)</code>	$z = \text{zeros of } P \text{ such that } P(z) = 0$
<code>p=conv(p<sub>1</sub>,p<sub>2</sub>)</code>	$p = \text{coefficients of the polynomial } P_1 P_2$
<code>[q,r]=deconv(p<sub>1</sub>,p<sub>2</sub>)</code>	$q = \text{coefficients of } Q, r = \text{coefficients of } R$ such that $P_1 = QP_2 + R$
<code>y=polyderiv(p)</code>	$y = \text{coefficients of } P'(x)$
<code>y=polyinteg(p)</code>	$y = \text{coefficients of } \int P(x) dx$