

① a)

Solution for \vec{x}_i

Reconstruction weights w_i for \vec{x}_i are found like:

$$w_i = \frac{C^{-1} \mathbf{1}}{\mathbf{1}^T C^{-1} \mathbf{1}} \quad \text{with } C_{jk} = (\vec{x}_i - \vec{\eta}_j)^T (\vec{x}_i - \vec{\eta}_k)$$

$j, k \in N_i \quad N_i \hat{=}$ number of neighbors for each \vec{x}_i

Scaling all data points by α affects the matrix C in the following way:

$$\begin{aligned} C_{jk} &= (\alpha \vec{x}_i - \alpha \vec{\eta}_j)^T (\alpha \vec{x}_i - \alpha \vec{\eta}_k) = (\alpha (\vec{x}_i - \vec{\eta}_j))^T (\alpha (\vec{x}_i - \vec{\eta}_k)) \\ &= \alpha^2 (\vec{x}_i - \vec{\eta}_j)^T (\vec{x}_i - \vec{\eta}_k) = \alpha^2 C_{jk} \end{aligned}$$

So w_i for $\alpha \vec{x}_i$ amounts to

$$w_i = \frac{\left(\frac{1}{\alpha^2} C\right)^{-1} \mathbf{1}}{\mathbf{1}^T \left(\frac{1}{\alpha^2} C\right)^{-1} \mathbf{1}} = \frac{\frac{1}{\alpha^2} C^{-1} \mathbf{1}}{\frac{1}{\alpha^2} \mathbf{1}^T C^{-1} \mathbf{1}}$$

which is the same as before scaling of the data

b) For translation of all data points we can show that the matrix C is unaffected.

$$\begin{aligned} C_{jk} &= (\vec{x}_i + \vec{v}) - (\vec{\eta}_j + \vec{v})^T (\vec{x}_i + \vec{v}) - (\vec{\eta}_k + \vec{v}) \\ &= (\vec{x}_i - \vec{\eta}_j)^T (\vec{x}_i - \vec{\eta}_k) \end{aligned}$$

as w_i depends only on C and C is unaffected by translation w_i isn't either.

c) To show that C is unaffected by rotation of data points we use the Facts: ~~that~~

1. Rotations are described by orthogonal matrices.

2. For an orthogonal matrix U $U^T U = I$ since $U^{-1} = U^T$

$$\begin{aligned} C_{jk} &= (U \vec{x}_i - U \vec{y}_i)^T (U \vec{x}_i - U \vec{y}_i) = (U(\vec{x}_i - \vec{y}_i))^T (U(\vec{x}_i - \vec{y}_i)) \\ &= (\vec{x}_i - \vec{y}_i)^T U^T U (\vec{x}_i - \vec{y}_i) = (\vec{x}_i - \vec{y}_i)^T (\vec{x}_i - \vec{y}_i) \end{aligned}$$

$$\textcircled{2} \text{ a) } \mathcal{E} = \left| \vec{x} - \sum_j^N w_j \vec{y}_j \right|^2 \quad N = \text{number of neighbors}$$

$$\vec{x} = \sum_j w_j \vec{x} \quad \text{iff} \quad \sum_j w_j = 1$$

$$\mathcal{E} = \left| \sum_j^N w_j (\vec{x} - \vec{y}_j) \right|^2 = \left(\sum_j^N w_j (\vec{x} - \vec{y}_j) \right) \left(\sum_k^N w_k (\vec{x} - \vec{y}_k) \right)$$

$$= \sum_{jk} w_j w_k \underbrace{(\vec{x} - \vec{y}_j) (\vec{x} - \vec{y}_k)}_{C_{jk}} = \mathbf{w}^T \mathbf{C} \mathbf{w} = \mathcal{E} \quad \text{given } \mathbf{1}^T \mathbf{w} = 1$$

$C_{jk} \text{ for } \mathbf{C} = (\mathbf{1} \vec{x}^T - \boldsymbol{\eta})(\mathbf{1} \vec{x}^T - \boldsymbol{\eta})^T$

$$\text{b) } \min_{\mathbf{w}} \mathbf{w}^T \mathbf{C} \mathbf{w} \quad \text{subject to } \mathbf{w}^T \mathbf{1} = 1$$

$$\min_{\mathbf{w}} \mathbf{w}^T \mathbf{C} \mathbf{w} - \lambda (\mathbf{w}^T \mathbf{1} - 1)$$

$$\frac{d}{d\mathbf{w}} \left(\mathbf{w}^T \mathbf{C} \mathbf{w} - \lambda (\mathbf{w}^T \mathbf{1} - 1) \right) = 2\mathbf{C} \mathbf{w} - \lambda \mathbf{1} \stackrel{!}{=} 0$$

$$\mathbf{C} \mathbf{w} = \frac{\lambda}{2} \mathbf{1} \quad \Leftrightarrow \quad \mathbf{w} = \underbrace{\frac{\lambda}{2} \mathbf{C}^{-1} \mathbf{1}}_{\substack{\text{constant} \\ \text{scaling factor}}} \quad \Rightarrow \quad \text{direction of } \mathbf{w} = \mathbf{C}^{-1} \mathbf{1}$$

$$\text{Restrict } \mathbf{1}^T \mathbf{w} = 1 \quad \mathbf{w} = \frac{\mathbf{C}^{-1} \mathbf{1}}{a} \quad 1 = \mathbf{1}^T \mathbf{w} = \mathbf{1}^T \frac{\mathbf{C}^{-1} \mathbf{1}}{a}$$

$$\Rightarrow a = \mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}$$

$$\mathbf{w} = \frac{\mathbf{C}^{-1} \mathbf{1}}{\mathbf{1}^T \mathbf{C}^{-1} \mathbf{1}}$$

$$c) \quad 2Cw + h11 \stackrel{!}{=} 0$$

$$Cw = 11 \cdot \left(-\frac{h}{2}\right)$$

$$\left(-\frac{2}{h}\right)Cw = 11 \quad (\Rightarrow) \quad C\left(-\frac{2}{h}\right)w = 11$$

The constant factor in front of w does not change its direction but merely its magnitude. Since we rescale w after solving $Cw = 11$ anyways, the solution stays the same