1a) $$\begin{bmatrix} 0 & C_{xy} \\ C_{yx} & 0 \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \lambda \begin{bmatrix} C_{xx} & 0 \\ 0 & C_{yy} \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\Leftrightarrow (1)\ C_{xy} w_y = \lambda C_{xx} w_x$$
$$(2)\ C_{yx} w_x = \lambda C_{yy} w_y$$

$$(1) \Leftrightarrow w_x^T C_{xy} w_y = \lambda \underbrace{w_x^T C_{xx} w_x}_{=1}$$

Objective: $\max(w_x^T C_{xy} w_y) \Rightarrow$ pick solution (w_x, w_y) that corresponds to largest λ

analogously:

$$(2) \Leftrightarrow C_{yx} w_x = \lambda C_{yy} w_y$$
$$w_y^T C_{yx} w_x = \lambda \underbrace{w_y^T C_{yy} w_y}_{=1}$$
$$w_y^T C_{yx} w_x = w_x^T C_{xy} w_y = \lambda$$

b) (1) $(-w_x)^T C_{xy} (-w_y) = (-1) w_x^T C_{xy} w_y \cdot (-1) = w_x^T C_{xy} w_y$

(2) $-w_x^T C_{xx} (-w_x) = w_x^T C_{xx} w_x$

analogously to (2) for second constraint $w_y^T C_{yy} w_y = 1$

As objective and restrictions have same results for (w_x, w_y) and $(-w_x, -w_y)$, $(-w_x, -w_y)$ is always as good a solution if w_x, w_y is a solution.

2a $$(1)\quad C_{xy} w_y = \lambda C_{xx} w_x$$
$$(2)\quad C_{yx} w_x = \lambda C_{yy} w_y$$

~~mit~~ with $C_{xx} = \frac{1}{N} XX^T \quad C_{yy} = \frac{1}{N} YY^T \quad C_{xy} = \frac{1}{N} XY^T \quad C_{yx} = \frac{1}{N} YX^T$

$$w_x = X\alpha_x \qquad w_y = Y\alpha_y$$

and $X \in \mathbb{R}^{d_1 \times N} \quad Y \in \mathbb{R}^{d_2 \times N} \quad \alpha_x, \alpha_y \in \mathbb{R}^N$

$$(1) \Leftrightarrow \frac{1}{N} XY^T Y\alpha_y = \lambda \frac{1}{N} XX^T X \alpha_x$$

$$(3)\quad K_y \alpha_y = \lambda K_x \alpha_x$$ ~~mit~~ with $K_x = X^T X \in \mathbb{R}^{N \times N}$ and $K_y = Y^T Y \in \mathbb{R}^{N \times N}$

$$(2) \Leftrightarrow \frac{1}{N} YX^T X\alpha_x = \lambda \frac{1}{N} YY^T Y\alpha_y$$
$$\Leftrightarrow K_x \alpha_x = \lambda K_y \alpha_y$$
$$\Leftrightarrow (4)\quad \alpha_y = \frac{K_y^{-1} K_x}{\lambda} \alpha_x$$

(4) in (3)

$$\frac{K_y K_y^{-1} K_x \alpha_x}{\lambda} = \lambda K_x \alpha_x$$

$$\Leftrightarrow I\alpha_x = \lambda^2 \alpha_x \quad \Rightarrow \quad \forall \alpha_x \ \lambda = 1$$

You can pick any α_x and compute the corresponding α_y from α_x: $\boxed{K_y^{-1} K_x \alpha_x = \alpha_y}$

2b (1) on worksheet with $w_x = X\alpha_x$ $w_y Y\alpha_y$

Find $\alpha_x \in \mathbb{R}^N$ $\alpha_y \in \mathbb{R}^N$ maximizing $\alpha_x^T X^T X Y^T Y \alpha_y = \alpha_x^T A \cdot B \alpha_y$

subject to $\alpha_x^T X^T X X^T X \alpha_x = \alpha_x^T A^2 \alpha_x$

$\alpha_y^T Y^T Y Y^T Y \alpha_y = \alpha_y^T B^2 \alpha_y$

$$\mathcal{L} = \alpha_x^T A \cdot B \alpha_y - \lambda_1 (\alpha_x^T A^2 \alpha_x) - \lambda_2 (\alpha_y^T B^2 \alpha_y)$$

(2) $$\frac{\partial L}{\partial \alpha_x} = AB\alpha_y - \lambda_1 A^2 \alpha_x = 0$$

(3) $$\frac{\partial L}{\partial \alpha_y} = \alpha_x^T A B - \lambda_2 B^2 \alpha_y = BA\alpha_x - \lambda_2 B^2 \alpha_y = 0 \quad \boxed{\text{since } A^T = A,\ B^T = B}$$

$\alpha_y^T(3) - \alpha_x^T(2)$: $\cancel{\alpha_y^T B A \alpha_x} - \lambda_2 \underbrace{\alpha_y^T B^2 \alpha_y}_{=1} - \cancel{\alpha_x^T A B \alpha_y} - \lambda_1 \underbrace{\alpha_x^T A^2 \alpha_x}_{=1} = 0$

(4) $-\lambda_2 - \lambda_1 = 0 \Rightarrow \lambda_2 = \lambda_1 = \lambda = \rho$

(4) in (2): $AB\alpha_y = \rho A^2 \alpha_x$

(4) in (3): $BA\alpha_x = \rho B^2 \alpha_y$

$$= \begin{bmatrix} 0 & AB \\ BA & 0 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix} = \rho \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix}$$

c) $A \cdot B \alpha_y = \rho A^2 \alpha_x$

$$X^T X Y^T Y \alpha_y = \rho X^T X X^T X \alpha_x$$

$$\alpha_x^T X^T X Y^T Y \alpha_y = \rho \alpha_x^T X^T X X^T X \alpha_x$$

$$w_x^T C_{xy} w_y = \rho \underbrace{w_x^T C_{xx} w_x}_{=1}$$

Since we want to find $\max(w_x^T C_{xy} w_y)$ we need to find the solution with largest ρ.

d) $w_x = X\alpha_x$ $w_y = Y\alpha_y$

(3) For $X \in \mathbb{R}^{d_1 \times N}$ and $Y \in \mathbb{R}^{d_2 \times N}$ with $d_2 = 1$

from restraint $\frac{w_y^T Y Y^T w_y}{N} = 1$ it follows

$$\Leftrightarrow \frac{w_y^2 \|y\|_2^2}{N} = 1 \Leftrightarrow w_y = \frac{\sqrt{N}}{\|y\|} > 0$$

Simplyfy (1) from worksheet

Find $w_x \in \mathbb{R}^{d_1}$ maximizing $w_x^T C_{xy} = w_x^T X Y^T$

subjectl to $w_x^T C_{xx} w_x = 1 = w_x^T X X^T w_x$

$$L = w_x^T X Y^T - \lambda w_x^T X X^T w_x$$

$$\frac{\partial L}{\partial w_x} = X Y^T - \lambda X X^T w_x = 0 \Leftrightarrow X Y^T = \lambda X X^T w_x$$

$$\Leftrightarrow Y^T = \lambda X^T w_x \quad \Leftrightarrow \frac{1}{\lambda} Y^T = X^T w_x \quad (a)$$

The LSR objective can be rewritten:

$$\|X^T v - y\|^2 = v^T X X^T v - 2 v^T X y^T + \underbrace{\|y\|^2}_{\text{const}}$$

$$\min_v \left(\overbrace{v^T X X^T v - 2 v^T X y}^{D} \right)$$

$$\frac{\partial D}{\partial v} = X X^T v - 2 X y^T \overset{!}{=} 0 \Leftrightarrow X^T v = 2 Y^T \quad (b)$$

From (a) follows $w_x = \frac{1}{\lambda} X^{-T} Y^T$ so with $\lambda = 1/2$ $w_x =$

and from (b) follows $v = 2 X^{-T} Y^T$