Continuous Functions

Definition: Continuity at a Point
A function f is continuous at a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If a function is not continuous at x_0, we say it is discontinuous at x_0.

From the above definition, we can see that in order for a function f to be continuous at a point x_0, f must be defined at x_0, and the limit as x approaches x_0 of f must be equal to the value of the function at that point. One useful way to check if the limit of a function as its argument approaches a point exists is to check if both the left and right-hand limits at that point exist and are equal. If they are not equal, then the limit does not exist, so the function cannot be continuous at that point. Continuity at a point is a point-wise definition, which we would like to extend to the entire domain of the function.

Definition: Continuity
A function f is continuous if it is continuous at every point in its domain.

In our analysis here, when it is not specified, the domain of a function is assumed to be the real numbers $\mathbf{R}$. However, this is not always a sensible assumption. For instance, the natural logarithm $\ln(x)$ is only defined for $x > 0$. This means that the natural logarithm cannot be continuous if its domain is the real numbers, because it is not defined for all real numbers. Nevertheless, if we restrict the domain of $\ln(x)$ to x values for which it is defined, then it is continuous on that domain. Thus, when speaking of continuity, it is very important to be cognizant of what the domain of the function is. Even though a function may not be continuous over the real numbers, it is likely continuous over some restricted domain.

The two most common discontinuities we will encounter are point and jump discontinuities. A point discontinunity occurs when we have a function that would otherwise be continuous, but the definition of it at a single point is changed to make it discontinuous there. A jump discontinuity occurs in piecewise defined functions where the two pieces of the function are not connected. If the limit as a function approaches a point is ∞ or $-\infty$, then the function value is not defined at that point, so it is discontinuous at that point. Similarly, a function is discontinuous at any point where it would have a 0 in its denominator, as it is not defined there either.

Just as we had rules for combining limits, we can also combine continuous functions to create continuous functions.

Properties of Continuous Functions
Suppose $g(x)$ is continuous at x_0

1. If $f(x)$ is continuous at x_0, then $f(x) + g(x)$ is continuous at x_0
2. If $f(x)$ is continuous at x_0, then $f(x) \cdot g(x)$ is continuous at x_0
3. If $f(x)$ is continuous at x_0, $g(x_0) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at x_0
4. If $f(x)$ is continuous at $g(x_0)$, then $f \circ g(x)$ is continuous at x_0

As before, these rules will only be useful if we have some knowledge about basic continuous functions. We will be able to use these functions as building blocks to verify the continuity of more complex functions.

Basic Continuous Functions

1. $f(x) = c$ is continuous
2. $f(x) = x$ is continuous
3. $f(x) = e^x$ is continuous
4. $f(x) = \cos(x)$ is continuous
5. $f(x) = \ln(x)$ is continuous for $x > 0$

Example 1 Is $\sin(\frac{1}{x})$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers? Is it continuous over a restricted domain?
Solution Since there is a division by 0 if $x = 0$, clearly this function is not defined at $x = 0$, so this function is discontinuous over a domain of real numbers. As $x \to 0$ the function oscillates faster and faster between 1 and -1, so it does not approach any single value. Thus, there is no way that we could define $f(0)$ in order to make the function continuous over the domain of real numbers. However, if the domain is restricted to the real numbers minus 0 this function is continuous.

Example 2 Is $x \cdot \sin(\frac{1}{x})$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers? Is it continuous over a restricted domain?
Solution Since there is a division by 0 if $x = 0$, clearly this function is not defined at $x = 0$, so this function is discontinuous over a domain of real numbers. However, as $x \to 0$ despite the fact that the sine function oscillates faster and faster between 1 and -1, it is multiplied by x, so it approaches 0. Thus, if we define $f(0) = 0$, then it will be a continuous function over the domain of reals. Without that redefinition, it is continuous for the domain of all

real numbers minus 0.

Example 3 Is $\tan(x)$ continuous over the real numbers? If not, does it have any discontinuities which can be redefined to make it continuous over the real numbers? Is it continuous over a restricted domain?
Solution If we rewrite the tangent as

$$tan(x) = \frac{sin(x)}{cos(x)}$$

then we can realize immediately that the tangent function is not defined at the zeros of the cosine function, which occur at odd integer multiples of $\frac{\pi}{2}$ (ie $\ldots, \frac{-3\pi}{2}, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$). As the argument of the tangent function approaches one of those values, the tanget function either approaches ∞ or $-\infty$, depending on which side the limit is taken from. Since the limits approach $\pm\infty$, there is no value we can define for the function there to make it continuous over the domain of real numbers. Nevertheless, if we restrict the domain to not include these points, then the tangent function is continuous.

Example 4 Find $\lim_{x\to 0} g(x)$ for

$$g(x) = \frac{e^{x^2-1}}{1+\ln(x+1)}$$

Solution The first step is to check whether the above function is continuous. If it is, then we know the value of the limit will just be the value of the function at the point $x = 0$. In the numerator we have e^{x^2-1} which is a composition of an exponential function and polynomial, which are both continuous for all real numbers. In the denominator, $\ln(x)$ is continuous as long as $x > 0$, which it is in this case. Further, $1 + \ln(1) \neq 0$, so we do not have to worry about a 0 divisor. Since this is a division of continuous functions and the denominator does not equal 0, we can evaluate the limit as

$$\lim_{x\to 0} \frac{e^{x^2-1}}{1+\ln(x+1)} = e^{0^2-1} = e^{-1}$$

Example 5 Recall the dynamical system of the voltage of the atrioventricular node in the heart model, given by

$$V_{t+1} = \begin{cases} cV_t + u & V_t \leq \frac{1}{c}V_c \\ cV_t & V_t > \frac{1}{c}V_c \end{cases}$$

Is this updating function continuous over the domain of $V_t > 0$?
Solution Since we have a piecewise function defined by two linear functions, we know both of the linear functions are continuous. Thus, the only point we are concerned with is $V_t = \frac{1}{c}V_c$. We need to check if the left and right-hand limits at this point are the same. Doing so we find

$$\lim_{V_t \to \frac{1}{c}V_c^-} cV_t + u = c\frac{1}{c}V_c + u = V_c + u$$

and

$$\lim_{V_t \to \frac{1}{c}V_c^+} cV_t = c\frac{1}{c}V_c = V_c$$

Since $u > 0$, it follows that

$$\lim_{V_t \to \frac{1}{c}V_c} cV_t + u \neq \lim_{V_t \to \frac{1}{c}V_c} cV_t$$

so this dynamical system is not continuous at $V_t = \frac{1}{c}V_c$.

Having finished our discussion of continuous functions, we would like to to return to the notion of error tolerance. Consider the following.

Example 6 Recall the discrete-time dynamical system for bacteria population described by $b_{t+1} = 2b_t$. How close must b_0 be to 10^6 for b_1 to be within 10^5 of $2 \cdot 10^6$?
Solution We can solve this using inequalities

$$\begin{aligned} 1.9 \cdot 10^6 &\leq b_1 \leq 2.1 \cdot 10^6 \\ 1.9 \cdot 10^6 &\leq 2b_0 \leq 2.1 \cdot 10^6 \\ 0.95 \cdot 10^6 &\leq b_0 \leq 1.05 \cdot 10^6 \end{aligned}$$

This implies that b_0 must be within $5 \cdot 10^4$ of 10^6 for us to meet this error tolerance.

Now let us compare the above example to a function with a jump discontinuity, such as

$$V_{t+1} = \begin{cases} 3V_t + 1 & V_t \leq 2 \\ 3V_t & V_t > 2 \end{cases}$$

By the above definition if $V_0 = 2$ then $V_1 = 7$. Suppose we want to find values of V_0 near 2 such that V_1 is within 0.5 Volts of 7 Volts. This task is impossible, because any interval around $V_0 = 2$ will contain points from the second branch of the updating function, which will be outside of this error tolerance.