The Dynamics of Solitons

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***Abstract*—The abstract goes here.**

I. Introduction

SOLITONS are waves that propagate at a constant veocity while maintaining their shape. Solitons are governed by the Korteweg de Vries (KdeV) equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

The $u\frac{\partial u}{\partial x}$ term gives the speed of the wave which is dependant on it's amplitude, u, and the $\frac{\partial^3 u}{\partial x^3}$ term gives the dispersion of the wave. Solutions of this equation are in the form,

$$u = 12a^2 sech^2(\alpha(x - 4a^2 t)) \tag{2}$$

whereα is a constant. the wave speed and dispersion terms cancel each other out in this solution. Examples of solitons include EXAMPLES HERE

II. Numerical Methods

A. Discretisation

The time derivative of the KdeV equation can be approximated numerically using a central difference scheme (CDS). An approximation for the first spatial term in Equation 1 can be found by Taylor expanding $u(x+h)$ and $u(x-h)$ giving,

$$\begin{aligned} u(x+h) &= u(x) + u'(x)h + \frac{1}{2}u''(x)h^2 + \frac{1}{6}u'''(x)h^3 \\ u(x-h) &= u(x) + u'(x)h - \frac{1}{2}u''(x)h^2 - \frac{1}{6}u'''(x)h^3 \end{aligned} \tag{3}$$

By subtracting these equations it can be found that,

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - \mathcal{O}(h^2) \tag{4}$$

Similarly a 4^{th} order Taylor expansion can be used to find an approximation for the second spatial term of the KdeV equation[PERHAPS EQUATION HERE.] When combined this gives an approximation for the the time derivative

$$\begin{aligned} \frac{\partial u}{\partial t} \approx \frac{1}{4h}&[(u^n_{i+1})^2 - (u^n_{i-1})^2] \\ &+ \frac{1}{2h^3}[(u^n_{i+2}) - (u^n_{i+1}) + (u^n_{i-1}) - (u^n_{i-2})] \end{aligned} \tag{5}$$

with error $\mathcal{O}(h^2)$. By equating u_{N+1} to u_0 and u_{0-1} to u_N it is simple to create periodic boundary conditions with period N.

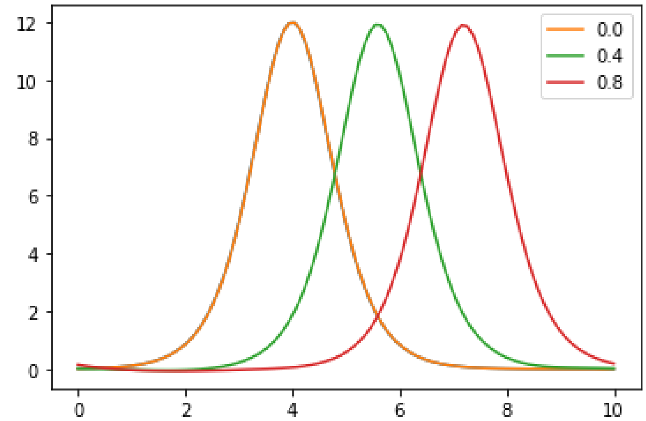

Fig. 1. Dynamics of soliton

B. Time Stepping the Differential Equation

The simplest method for numerically integrating the KdeV equation is the Euler method.

$$u_{n+1} = u_n + f(t_n, u_n)\Delta t \tag{6}$$

COMMENT ON WHY THIS IS UNSTABLE. A more appropriate method to use is the Runge-Kutta method. The Runge-Kutta method is an explicit single-step method that uses MORE HERE.

$$\begin{aligned} f_a &= f(t_n, u_n) \\ f_b &= f(t_n + \Delta t/2, u_n + f_a \Delta t/2) \\ f_c &= f(t_n + \Delta t/2, u_n + f_b \Delta t/2) \\ f_d &= f(t_n + \Delta t, u_n + f_c \Delta t) \\ u_{n+1} &= u_n + \frac{1}{6}(f_a + 2f_b + 2f_c + f_d)\Delta t \end{aligned} \tag{7}$$

III. Validation of Code

IV. Results

A. Dynamics

The solitons maintain their shape as they propagate. The amplitude of the solitons remains constant as shown in FIG2 The taller the solitons the faster they move as shown in FIG3. The relationship between amplitude and speed can be shown in FIG4. This relationship was found by using the assumption that the simulated solitons maintain a constant speed and amplitude and measuring the distance the peaks traveled in 100 time steps.

B. Collisions

When solitons collide they do not maintain a constant amplitude during the collision. Collisions are simulated by

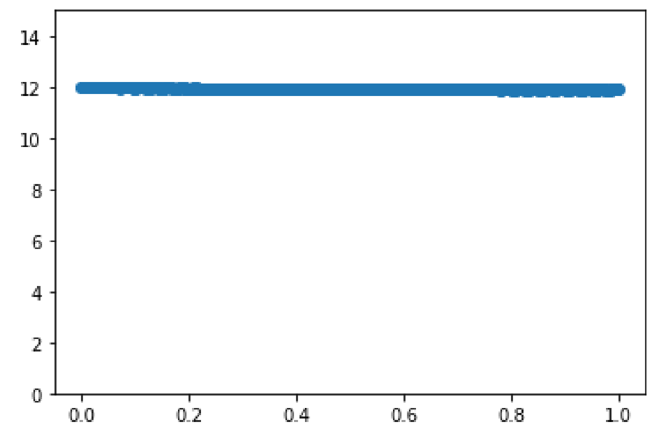

Fig. 2. Dynamics of soliton

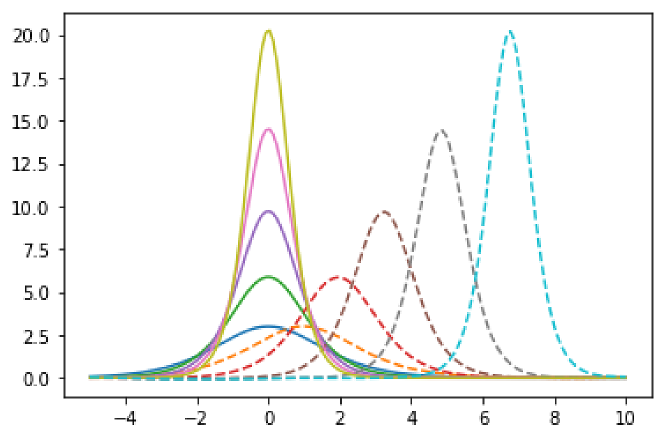

Fig. 3. Dynamics of soliton

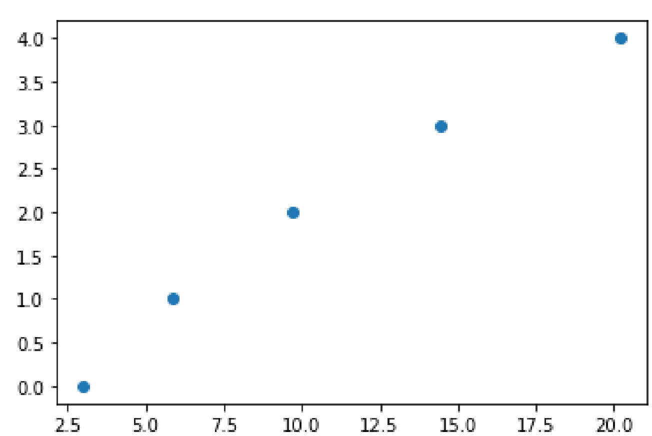

Fig. 4. Dynamics of soliton

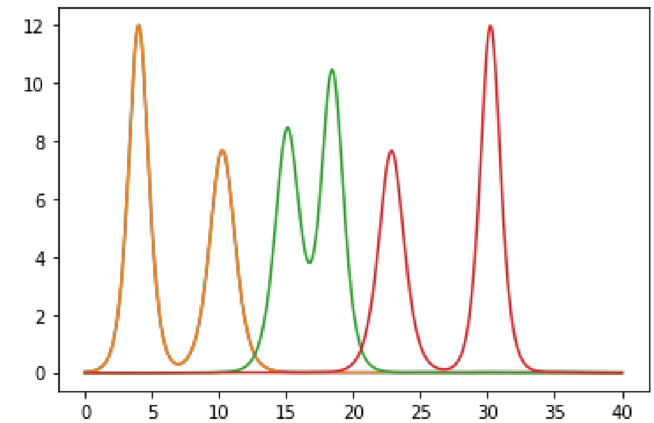

Fig. 5. Dynamics of soliton

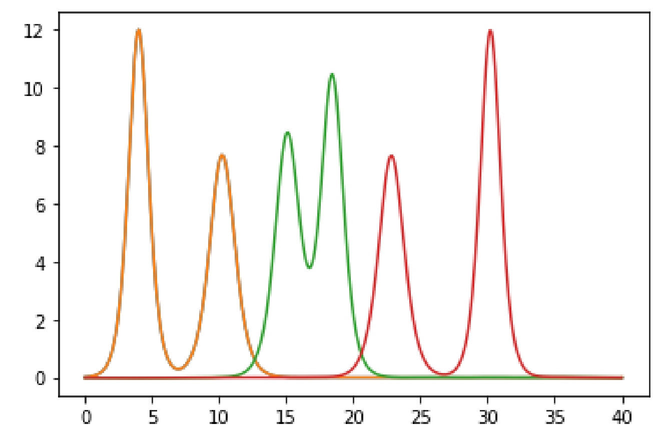

Fig. 6. Dynamics of soliton

using an initial condition made up of two different amplitudes starting at different times. A collision between similar α solitons can be seen in FIG5 and a collision between a solitons with a large difference in amplitude is demonstrated in FIG6

C. *Wave Breaking*

If initial conditions that are not in the form of Equation 5, the wave will break up into solitons. This can be seen by using the positive portion of a sine wave as the in ital condition. The results of this wave propagating through the KdeV equation are shown in FIG7.

D. *Shock Waves*

A shock wave is a wave that travels through a medium with an abrupt change in amplitude. The equation of a shock wave is given by,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{8}$$

However, when time stepped through numerical methods this quickly becomes unstable as it is impossible to deal with the infinite slope of the boundary as shown in FIG8. This can be dealt with by adding a diffusion term to Equation 8 in the

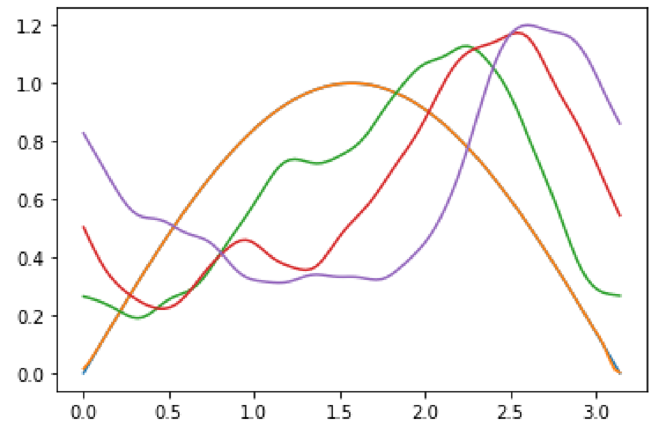

Fig. 7. Dynamics of soliton

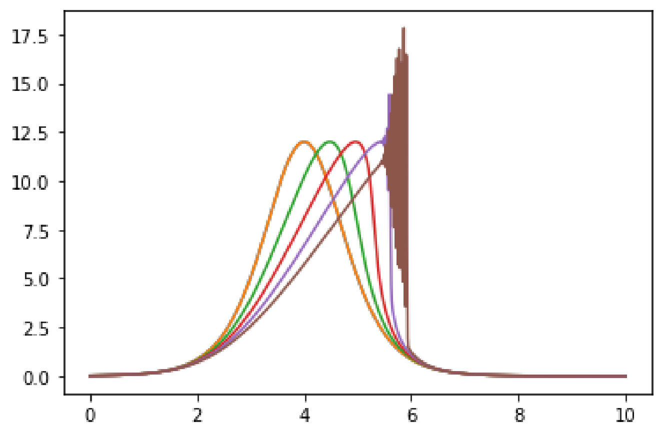

Fig. 8. Dynamics of soliton

form, $-\frac{\partial^2 u}{\partial x^2}$. Using a 2^{nd} order Taylor expansion a CDS for this term can be found (Equation 9).

$$u''(x) = \frac{u(x+h) - u(x) + u(x-h)}{h^2} - \mathcal{O}(h^2) \tag{9}$$

By adding this term the shock wave equation can be integrated in order to propagate the wave through time. Using the same initial conditions as a soliton we get a wave that looks like FIG9

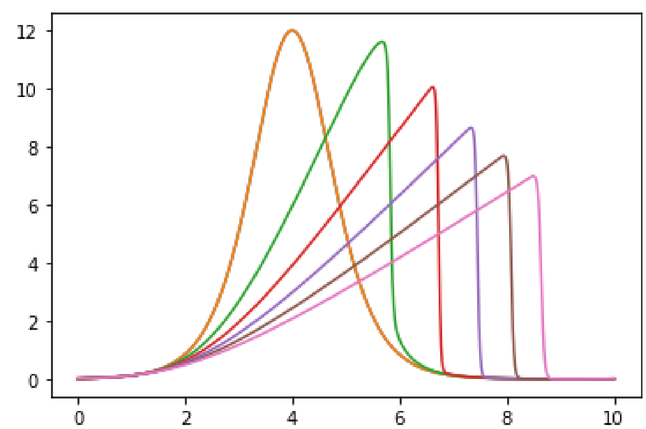

Fig. 9. Dynamics of soliton

V. Conclusions

Appendix A

Acknowledgment