Project 3: The Dynamics of Solitons

Key Topic: PDE Initial Value Problem

1 Background

A soliton is a single pulse that can propagate without change of shape. The earliest observations of solitons were of single water wave pulses in canals and rivers; the Severn Bore is a well-known example. We will be concerned with soliton solutions to the Korteweg de Vries (KdeV) partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where x and t have already been rescaled into dimensionless units. The equation has exact solutions of the form

$$u = 12\alpha^2 \operatorname{sech}^2(\alpha(x - 4\alpha^2 t)). \tag{2}$$

where α is a parameter. Equation (1) differs from the usual advection equation in two important aspects:

- The second term is non-linear, giving a wave speed that depends upon the amplitude of the disturbance, u; this is typical of shock waves.
- The third term introduces dispersive broadening of the waveform.

Whilst both the non-linearity and dispersion on their own cause distortion of the waveform, when combined in the KdeV equation the two effects cancel each other out exactly to allow solutions of the form (2) to propagate without change of shape. In this project you will write code to integrate the KdeV equation forward in time and use it to study the dynamics of single solitons, collisions between solitons, wave breaking and other non-linear wave phenomena.

2 Discretisation

The second and third terms of the KdeV equation can be discretised using centre difference approximations for the spatial derivatives as follows:

$$\frac{1}{4h}\left[\left(u^n_{i+1}\right)^2 - \left(u^n_{i-1}\right)^2\right] + \frac{1}{2h^3}\left[u^n_{i+2} - 2u^n_{i+1} + 2u^n_{i-1} - u^n_{i-2}\right]. \tag{3}$$

Show that this is consistent with the differential equation.

Using the Euler method for the time step, analyse the stability of the difference scheme in the limit of small u and show analytically that it is unstable.

You should choose an alternative method for the time step; various methods will do, but a second-order (or higher) Runge-Kutta method is suitable. You do not need to prove analytically that this alternative method is stable.

3 Dynamics

Write a program to propagate the pulse according to your discretisation of the KdeV equation. It will be convenient to use periodic boundary conditions[1], provided that the total spatial extent is large compared to the width of the pulse. Even with a higher-order time step method, you will need to ensure h and Δt are small enough by trial and error.

Now validate your implementation of the KdeV solver:

[1] One way to do this is with the modulus, so use $(i \pm 1) \bmod N_x$ when calculating the index of the neighbour. Here N_x is the number of spatial points.

- Generate solitons with different values of α and confirm that they are propagated for at least a few transits across the spatial domain without any significant change of shape. Compare them with the analytic solutions.
- Observe how the speed of the soliton is related to its height.
- Investigate how the values of h and Δt needed for stability depend on the soliton parameter α.

3.1 Collisions of solitons

You will have noticed that taller solitons move faster than smaller ones. Therefore a fast soliton should catch up and "collide" with a slower one moving in the same direction. To simulate this, set up two independent solitons of different speed and well-separated initial position. Study how the solitons interact, paying particular attention to the shape of the pulse during and after the collision. Since the waves are non-linear we do not expect height to be conserved (no linear superposition). However, you should find that another property of the wave is conserved during the collision. Study these effects for solitons of similar and very different initial speeds.

3.2 Wave breaking

Solutions of the form (2) may be considered as normal modes of the KdeV equation. In this part of the project, study how a different initial waveform, which is not of the normal mode form, evolves from the KdeV equation. Try an initial waveform such as the positive part of a sine wave. You should find that the wave breaks up into a train of solitons of different sizes. An example of a sine wave breaking is shown in [1]. In the units of the differential equation (1), you should find that you need a long wave period relative to the amplitude of the initial sine wave in order to observe wave breaking.

3.3 Shock waves

The KdeV equation has stable solutions of the form (2) because of the balance of non-linear and dispersive terms. If we remove the final term in equation (1), we are left with a wave equation for a form of shock wave. Try this with your code and see what happens to the shape of the waveform (2) as it progresses. You will find the solution becomes unstable at some point; this may be rectified by introducing a diffusive term (of the form $D\,\partial^2 u/\partial x^2$ where D is the diffusion coefficient) into the new PDE to dampen the waves.

References

[1] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15, 240-243 (1965).