A Brief Review of Inverse Problem using Deep Learning

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1 Inverse problems using GAN priors

For inverse problem, there is a target signal $x \in \mathbb{R}^{\ltimes}$ and $y = Ax \in \mathbb{R}^{\gtrdot}$ denotes the observed measurement. Assume $m < n$, so the inverse problem is ill-posed, because matrix A is not full rank and the equation has infinite solutions. A technique to solve the problem is to transform it into a optimization problem as follows:

$$\begin{aligned} &\hat{x} = argminF(x), \\ &s.t. \quad x \in S \end{aligned} \tag{1}$$

where F is the objective function and $S \subseteq R^n$ is some kind of prior knowledge that x has to obey. When we think about how to define this optimization, we may start in two parts: the objective function F and the prior S.

In these years a family of priors comes from generative model, including generative adversarial network. In GAN, as a example, the nonlinear mapping itself could be seen as a prior, from latent space R^k to the high dimension space R^n.

For compressed sensing, several works explore generative models for reconstruction. In [1], the generative model is a mapping $G : R^k \to R^n(k \ll n)$, so the goal is to optimize the representation $z \in R^k$ that minimize the error $\|AG(z) - y\|^2$. The approach is as follows: in order to recover x under generative model setting, a good measurement matrix A is necessary. In sparse prior, if the matrix A satisfies conditions like Restricted Isometry Property, the sparse vector x can be recovered. In the generative model settings, [1] gives the theoretical analysis that under certain condition(giving bound of m), the measurement matrix A satisfies the S-REC condition with high probability, solving the problem of how to find A. **The key is S-REC condition, which applies to the difference of any two natural vectors comparing with REC condition.** This guarantees the recovery of x, also, the natural vectors can be contained in the range of generative model.

However, [1] does not discuss the algorithm performing minimization process in detail. **The paper [4] considers the gradient descent used in [1] may get stuck in local minima if not initialized correctly**. Based on this, they propose projected gradient descent solving the optimization problem and give the analysis of convergence. Here, [4] did not give support that PGD can alleviate the effect of local minima, and I think using PGD is to make sure x_t is in the range of G(the constraint) in each step, with nearly same complexity with gradient descent.

For phase retrieval problem, the target is to recover a signal $x \in R^n$ give $m \ll n$ phaseless observations in the form $y = |Ax|$ where $|\cdot|$ is to calculate the absolute value entrywise. Same intuition with compressed sensing problem, the phase retrieval problem can be analyzed under a generative prior. The $l2$ minimization form is as follow:

$$\begin{aligned} min\, f(z) &= \frac{1}{2}\|y - |AG(z)|\|^2 \\ s.t. \quad & z \in R^k \end{aligned} \tag{2}$$

The objective function is a non-convex function, and the most difficult part is the absolute value, which makes it non-differentiable.

Therefore, compared with compressed sensing problem, how to transform the original problem is important. [2] approaches the problem by taking in this form:

$$\begin{aligned} min\, f(z) &= \|p_t \odot y - AG(z)\|^2 \\ s.t. \quad & z \in R^k \end{aligned} \tag{3}$$

The key part of this form is $p_t \odot y$, since the derivative of sign function is zero if we restrict $sgn(x) = \{1, -1\}$. Therefore when computing gradient in each step, the sign function part, or to say the non-differential part can be ignored.

2 Inverse problems using Deep Image Prior

In GAN prior's point of view, the distribution of natural image can be captured and learned by GAN, due to the large amount of training data, which represents the idea that learn more, know more. However, the ability to learn can be divided into two different sections: network architecture and training data.

To show the effect of network architecture, [5] proposed Deep Image Prior. The prior information is in the structure itself. They empirically shows that the untrained network prior has good performance with some inverse problems. However, there is an interesting phenomenon. The generator used for deep image prior is over-parameterized, however in the blind restoration tasks, they approach the good performance before overfitting to the input. Also, as [5] points out, in reconstruction tasks, naturally-looking images result in much faster convergence, whereas noise is rejected. There should be a mathematical way to explain these phenomenons.

To give theoretical analysis of untrained network prior, [3] proposes guarantees in the context of compressed sensing and phase retrieval. They presents a special restricted eigenvalue condition holds for measurement matrix A, which depends on the oblivious subspace embedding. Also, to analyze oblivious subspace embedding, [3] use a union of sub-spaces model, making the weights of two-layer neural network a stack of weights with k1k2-dimension and taking a union bound over all possible such networks.

References

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