Advent of Code: Day 6 Solution

Each fish reproduces at the end of their own internal timer, and resets its own timer to 6.

Define $T(L, D)$ be to the number of fish after D days starting with a single fish having an internal timer of L. We have three cases

Case 1. $D = 0, L \in \mathbb{N}$. There are no more days remaining, this fish cannot reproduce. We still have only 1 fish.

Case 2. $L = 0, D > 0$. D is at minimum 1, so this fish can reproduce. This gives us another fish with a timer set to 8 and the original fish's timer is reset to 6. Thus in $D - 1$ days we will have $T(8, D - 1) + T(6, D - 1)$

Case 3. The fish is not yet ready to reproduce, we skip until either the fish timer expires again or the days have been exhausted - whichever comes first. This is $T(L - \min(L, D), D - \min(L, D))$

We get the following recurrence,

$$T(L, D) = \begin{cases} 1 & \text{if } D = 0, L \in \mathbb{N} \\ T(8, D-1) + T(6, D-1) & \text{if } L = 0, D > 0 \\ T(L - \min(L, D), D - \min(L, D)) & \text{otherwise} \end{cases} \tag{1}$$

Base Case. $k = 0, 0 \leq L \leq 8$.
Then $T(L, k) = T(L, 0) = 1$ as wanted.

I.H Suppose by induction that

$$P(n) : T(L, n) \text{ gives the number of fish after } n \text{ days for any } 0 \leq L \leq 8 \tag{2}$$

holds whenever $0 \leq n < k$.

W.T.S $P(k)$ holds. $T(L, k)$ has 3 cases,

(a) $k = 0$. Then $T(L, k) = T(L, 0) = 1$ as wanted (by base case).

(b) $L = 0, k > 0$. $T(L, k) = T(8, k - 1) + T(6, k - 1)$.
Since $k - 1 < k$, by I.H, $T(8, k - 1)$ returns the correct value.
Since $k - 1 < k$, by I.H, $T(6, k - 1)$ returns the correct value.
It is easy to see that number of fish after k days will the be the sum of these two values. Therefore $T(L, k) = T(8, k - 1) + T(6, k - 1)$ as wanted.

(c) $L > 0, k > 0$. Then we have 3 cases,

$L < k$: Then it follows that

$$\begin{align} T(L, k) &= T(L - \min(L, k), k - \min(L, k)) \tag{3} \\ &= T(0, k') \text{ where } k' = k - L \tag{4} \end{align}$$

Since $k' = k - L < k$, $T(0, k')$ returns the correct value. (By I.H)

$L > k$: Then it follows that

$$\begin{align} T(L, k) &= T(L - \min(L, k), k - \min(L, k)) \tag{5} \\ &= T(L', 0) \text{ where } L' = L - k \tag{6} \end{align}$$

Since $0 < k$, $T(L', 0)$ returns the correct value. (By I.H)

$L = k$: Then it follows that

$$\begin{align} T(L, k) &= T(L - \min(L, k), k - \min(L, k)) \tag{7} \\ &= T(0, 0) \tag{8} \\ &= 1 \tag{9} \end{align}$$

Since $0 < k$, $T(0, 0)$ returns the correct value. (By I.H)