Probability and Stochastic Processes (1) Problem Set

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Problem 1

A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

Answer: To simplify this question, let's mark the die with 4 numbers: 1, 2, 3, 4. The sample space should consist of all possible outcomes of the die rolls, which are:

$$2, 4, 12, 14, 32, 34, 112, 114, 312, 314, 132, 134, 332, 334 \cdots$$

More precisely, the sample space could be expressed as

$$\{X_1X_2X_3 \cdots X_n : X_1, X_2, \cdots X_{n-1} \in \{1, 3\}, X_n \in \{2, 4\}, n \in \mathbb{N}^*\}$$

Problem 2

Let $\{A_i : i \in I\}$ be a collection of sets. Prove the "De Morgan's Law":

$$\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c, \qquad \left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c.$$

Answer: To prove the first statement, conside x in $(\bigcup_i A_i)^c$ and show that it is not in any A_i, which means x is in all the complements of A_i at the same time. So x is in $\bigcap_i A_i^c$. Then, if x is in $\bigcap_i A_i^c$, then it is in $A_1^c, A_2^c, \cdots$, which means x is not in the union of A_i. So x is in $(\bigcup_i A_i)^c$. Given x's arbitrariness, we can conclude that $(\bigcup_i A_i)^c = \bigcap_i A_i^c$.

To prove the second statement, we can use the first statement's conclusion. Taking the complement of both sides of the equation yields the second statement.

Problem 3

Let $\mathcal{F}$ be a σ-algebra of subsets of Ω and suppose that $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is a σ-algebra of subsets of B.

Answer: To prove this statement, we have to verify $\mathcal{G}$'s three properties:

1. $\emptyset \in \mathcal{G}$
2. If $A \in \mathcal{G}$, then A^c is in $\mathcal{G}$
3. If $A_1, A_2, \cdots \in \mathcal{G}$, then $\bigcup_i A_i \in \mathcal{G}$

Firstly, since $\mathcal{F}$ is a σ-algebra, $\emptyset \in \mathcal{F}$. $\emptyset \cap B$ is empty, so $\emptyset \in \mathcal{G}$. Secondly, if $A \in \mathcal{G}$, then there exists a set X such that $X \cap B = A$. The complement of $X \cap B$ with the universal set B is $X^c \cap B$. Since $X \in \mathcal{F}$, we have $X^c \in \mathcal{F}$. Therefore, $X^c \cap B \in \mathcal{G}$. Thirdly, if $A_1, A_2, \cdots \in \mathcal{G}$, then there exists $X_1, X2, ...$ such that $X_i \cap B = A_i$. Then $\bigcup_i X_i \cap B = \bigcup_i A_i$. Since $X_i \in \mathcal{F}$, we have $\bigcup_i X_i \in \mathcal{F}$. Therefore, $\bigcup_i X_i \cap B \in \mathcal{G}$, which means $\bigcup_i A_i \in \mathcal{G}$.

Problem 4

Let $\mathcal{F}$ and $\mathcal{G}$ be σ-algebras of subsets of Ω.

(a) Use elementary set operations to show that $\mathcal{F}$ is closed under countable intersubsections; that is, if $A_1, A_2, \cdots$ are in $\mathcal{F}$, then so is $\bigcap_i A_i$.

Answer:
If $A_1, A_2, \cdots$, are in $\mathcal{F}$, then $A_1^c, A_2^c, \cdots$ are also in $\mathcal{F}$. Therefore, $\bigcup_i A_i^c \in \mathcal{F}$. Since $\mathcal{F}$ is closed under complementation, $\bigcap_i A_i \in \mathcal{F}$.

(b) Let $\mathcal{H} = \mathcal{F} \cap \mathcal{G}$ be the collection of subsets of Ω lying in both $\mathcal{F}$ and $\mathcal{G}$. Show that $\mathcal{H}$ is a σ-algebra.

Answer:
To show that $\mathcal{H}$ is a σ-algebra, we have to verify its three properties:

1. $\emptyset \in \mathcal{H}$
2. If $A \in \mathcal{H}$, then $A^c \in \mathcal{H}$
3. If $A_1, A_2, \cdots \in \mathcal{H}$, then $\bigcup_i A_i \in \mathcal{H}$

Firstly, since $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras, $\emptyset \in \mathcal{F}$ and $\emptyset \in \mathcal{G}$. Therefore, $\emptyset \in \mathcal{H}$.
Secondly, if $A \in \mathcal{H}$, then $A = f \cap g$, where $f \in \mathcal{F}$ and $g \in \mathcal{G}$. $A^c = (f \cap g)^c = f^c \cup g^c \in \mathcal{H}$. Therefore, $\mathcal{H}$ is closed under complementation.
Thirdly, if $A_1, A_2, \cdots \in \mathcal{H}$, then $A_1, A_2, \cdots \in \mathcal{F}, \mathcal{G}$. (This is because $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras) Therefore, $\bigcup_i A_i \in \mathcal{F}$. Similarly, $\bigcup_i A_i \in \mathcal{G}$. Therefore, $\bigcup_i A_i \in \mathcal{H}$.

(c) Show that $\mathcal{F} \cup \mathcal{G}$, the collection of subsets of Ω lying in either $\mathcal{F}$ or $\mathcal{G}$, is not necessarily a σ-algebra.

Answer:
It can be proved with a counterexample. Let $\Omega = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$, and $\mathcal{G} = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$. Then $\mathcal{F} \cup \mathcal{G} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$. This is not a σ-algebra, because $\{1\}$ is in $\mathcal{F}$, $\{3\}$ is in $\mathcal{G}$, but$\{1\} \cup \{3\} = \{1, 3\}$ is not in $\mathcal{F} \cup \mathcal{G}$, showing that $\mathcal{F} \cup \mathcal{G}$ is not closed under countable intersections.