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Forecasting stock index price based on M-factors fuzzy time series and particle swarm optimization

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ABSTRACT

In real time, one observation always relies on several observations. To improve the forecasting accuracy, all these observations can be incorporated in forecasting models. Therefore, in this study, we have intended to introduce a new Type-2 fuzzy time series model that can utilize more observations in forecasting. Later, this Type-2 model is enhanced by employing particle swarm optimization (PSO) technique. The main motive behind the utilization of the PSO with the Type-2 model is to adjust the lengths of intervals in the universe of discourse that are employed in forecasting, without increasing the number of intervals. The daily stock index price data set of SBI (State Bank of India) is used to evaluate the performance of the proposed model. The proposed model is also validated by forecasting the daily stock index price of Google. Our experimental results demonstrate the effectiveness and robustness of the proposed model in comparison with existing fuzzy time series models and conventional time series models.

1. Introduction

Based on fuzzy time series concept, first forecasting model was introduced by Song and Chissom [1–3]. They presented the fuzzy time series model by means of fuzzy relational equations involving max–min composition operation, and applied the model to forecast the enrollments in the University of Alabama. In 1996, Chen [4] used simplified arithmetic operations avoiding the complicated max–min operations, and their method produced better results. Later, many studies provided some improvements in Song and Chissom method in terms of the following issues:

1. Effective lengths of intervals [5–8],
2. fuzzification [9–11],
3. fuzzy logical relationships [12–15], and
4. defuzzification techniques [16,17].

To enhance the accuracy in forecasted values, many researchers recently proposed various fuzzy time series models. For example, Wong et al. [18] proposed an adaptive time-variant model that automatically adapts the analysis window size of fuzzy time series based on the predictive accuracy in the training phase and uses heuristic rules to determine forecasting values in the testing phase. To extend the applicability of the univariate models to accommodate multiple variables and to improve forecasting result, Huarng et al. [19] proposed a multivariate heuristic function, which can be integrated with univariate fuzzy time series models to form a multivariate model. Bai et al. [20] proposed a heuristic time-invariant forecasting

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model in which there is a trend predictor for indicating trend of the increase or decrease in time series. To resolve the problem associated with determination of length of intervals and data defuzzification, Singh and Borah [21] proposed two-factors high-order fuzzy time series model. The proposed model is based on the hybridization of ANN with fuzzy time series. Singh and Borah [22] presented a new model based on hybridization of fuzzy time series theory with ANN. In this model, to defuzzify the fuzzified time series values and to obtain the forecasted results, an ANN based architecture is developed, and incorporated in the model.

The application of fuzzy time series in financial forecasting has also attracted many researchers' attention in the recent years. In recent years, many researchers focus on designing the models for TAIEX [6,23–25] and TIFEX [26–29] forecasting. Their applications are limited to deal with either one-factor or two-factors time series data sets. For the stock index forecasting, Huarng and Yu [30] show that the forecasting accuracy can be improved by including more observations (e.g., *close*, *high*, and *low*) in the models.

All the models proposed by researchers above are based on Type-1 fuzzy set concept except the model proposed by Huarng and Yu [30] in 2005, which is based on Type-2 fuzzy set concept. Researchers employ Type-2 fuzzy set concept (which is an extension of Type-1 fuzzy set) in various domains such as control system design and modeling [31,32], because Type-2 fuzzy set systems are much more powerful than Type-1 fuzzy sets systems to represent highly nonlinear and/or uncertain systems [33]. Nowadays, Type-2 fuzzy set concept is successfully applied in time series forecasting [34–37].

In this study, we aim to propose an improved fuzzy time series model by employing M-factors time series data set. To deal with these factors together, we design a model based on Type-2 fuzzy time series concept, which is an improvement over the existing Type-2 model proposed by Huarng and Yu [30]. Later, to enhance the forecasting accuracy, we hybridize the PSO algorithm with the proposed Type-2 model. The daily stock index price data set of SBI is employed for the experimental purpose, which consists of 4-factors, viz., "Open", "High", "Low" and "Close" factors/variables. After that, performance of the hybrid model is evaluated, which demonstrates its effectiveness over conventional fuzzy time series models and non-fuzzy time series models. The proposed model is also validated by forecasting the stock index price of Google.

This paper is organized as follows. In Section 2, we present the problem definitions. In Section 3, we review the theory of fuzzy set with an overview of fuzzy time series. In Section 4, the particle swarm optimization is introduced. In Section 5, we present the algorithm and defuzzification process for the Type-2 model. In Section 6, we explain the proposed Type-2 model. In Section 7, we provide the details of the new proposed hybrid forecasting model. In Section 8, we discuss some statistical parameters, which are used to check the robustness the model. The performance of the model is assessed and presented in Section 9. Finally, the concluding remarks and future works are discussed in Section 10.

2. Problem definitions

From review of literature, it is observed that there are still many unresolved issues that are associated with fuzzy time series models. Determination of effective length intervals and determination of importance of each interval in terms of events, are two most important issues among them. These two issues are explained next.

Issue 1 *(Lengths of intervals).* For fuzzification of time series data set, determination of effective lengths of intervals of the historical time series data set is very important. Previously, most of the existing fuzzy time series models [1,3,4,9,38] maintained the fix lengths of intervals. Later, Huarng [39] shows that the lengths of intervals always affect the results of forecasting. He introduced two methods, viz., *distribution-based length* and *average-based length*, to demonstrate the effectiveness of lengths of intervals. For finding more effective lengths of intervals, recently many researchers applied the PSO algorithm on fuzzy time series models [40,26,28,41,42]. But, their applications are limited to one-factor to two-factors time series data sets. For M-factors time series data set (as in the case of SBI stock index price forecasting), large number of intervals are evolved, which make the determination of effective lengths of intervals very worst. These large number of intervals also affect the accuracy rate of forecasting. Motivated by this critical issue, we incorporate the PSO algorithm with the Type-2 fuzzy time series model. The main role of the PSO algorithm in the Type-2 fuzzy time series model is to improve the forecasting accuracy by adjusting the length of each interval in the universe of discourse and corresponding degree of membership simultaneously.

Issue 2 *(Importance of intervals).* Huarng and Yu [30] have given equal importance to each interval in their proposed M-factors fuzzy time series model. However, this is not an effective way to solve real time problems. In fuzzy time series model, each interval represents various uncertainty exhibited by the events. There are two possible ways to determine the weights of intervals as: (i) Assign weights based on human interpretation, and (ii) Assign weights on intervals based on their frequencies. Assignment of weights based on human knowledge is not an acceptable solution for real world problems as human interpretation varies from one to another. Moreover, human interpretation is still an issue which is not understood by the computational scientists. Therefore, second way is considered, where all the intervals are given importance based on their frequencies. For example, if an interval I_i $(i = 1, 2, \dots, n)$ contains two time series values, then we consider its weight as 2. Singh and Borah [21] also suggested for assigning weights on intervals based on their frequencies. Their forecasting results show that this method is more persuasive than various existing fuzzy time series models. In this study, we adopt this idea for M-factors time series data, and obtain the forecasting results.

3. Fuzzy sets and fuzzy time series

In this section, we will discuss the basics of fuzzy sets and its application in time series forecasting.

Definition 1 *(Fuzzy set).* (See [43].) A fuzzy set is a class with varying degrees of membership in the set. Let U be the universe of discourse, which is discrete and finite, then fuzzy set $\tilde{A}$ can be defined as follows:

$$\tilde{A} = \{\mu_{\tilde{A}(x_1)}/x_1 + \mu_{\tilde{A}(x_2)}/x_2 + \cdots\} = \Sigma_i \mu_{\tilde{A}}(x_i)/x_i, \tag{1}$$

where $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$, $\mu_{\tilde{A}} : U \to [0, 1]$, and $\mu_{\tilde{A}(x_i)}$ is the degree of membership of the element x_i in the fuzzy set $\tilde{A}$. Here, the symbol "+" indicates the operation of union and the symbol "/" indicates the separator rather than the commonly used summation and division in algebra, respectively.

When U is continuous and infinite, then the fuzzy set $\tilde{A}$ of U can be defined as:

$$\tilde{A} = \left\{ \int \mu_{\tilde{A}(x_i)}/x_i \right\}, \quad \forall x_i \in U, \tag{2}$$

where the integral sign stands for the union of the fuzzy singletons, $\mu_{\tilde{A}(x_i)}/x_i$.

Definition 2 *(Fuzzy time series).* (See [1–3].) Let $Y(t)$ $(t = 0, 1, 2, \ldots)$ be a subset of R and the universe of discourse on which fuzzy sets $\mu_i(t)$ $(i = 1, 2, \ldots)$ are defined and let $F(t)$ be a collection of $\mu_i(t)$ $(i = 1, 2, \ldots)$. Then, $F(t)$ is called a fuzzy time series on $Y(t)$ $(t = 0, 1, 2, \ldots)$.

With the help of the following two examples, the concept of fuzzy time series can be explained:

Example 1. The common observations of daily weather condition for certain region can be described using the daily common words "hot", "very hot", "cold", "very cold", "good", "very good", etc. All these words can be represented by fuzzy sets.

Example 2. The common observations of the performance of a group students during the final year of degree examination can be represented using the fuzzy sets "good", "very good", "poor", "bad", "very bad", etc.

Above two examples are dynamic processes, and conventional time series models are not applicable to describe these processes [2]. Therefore, Song and Chissom [2] first time use the fuzzy sets concept in time series forecasting. Later, their proposed method has gained in popularity in scientific community as a "Fuzzy time series forecasting model".

Definition 3 *(Fuzzy logical relationship).* (See [1,3,4].) Assume that $F(t-1) = \tilde{A}_i$ and $F(t) = \tilde{A}_j$. The relationship between $F(t)$ and $F(t-1)$ is referred as a fuzzy logical relationship (FLR), which can be represented as:

$$\tilde{A}_i \to \tilde{A}_j,$$

where $\tilde{A}_i$ and $\tilde{A}_j$ refer to the left-hand side and right-hand side of the FLR, respectively.

Definition 4 *(Fuzzy logical relationship group).* (See [1,3,4].) Assume the following FLRs:

$$\begin{aligned} &\tilde{A}_i \to \tilde{A}_{k1}, \\ &\tilde{A}_i \to \tilde{A}_{k2}, \\ &\quad\vdots \\ &\tilde{A}_i \to \tilde{A}_{km}. \end{aligned}$$

Chen [4] suggested that FLRs having the same fuzzy sets on the left-hand side can be grouped into a same fuzzy logical relationship group (FLRG). So, based on Chen's model [4], these FLRs can be grouped into the same FLRG as:

$$\tilde{A}_i \to \tilde{A}_{k1}, \tilde{A}_{k2}, \ldots, \tilde{A}_{km}.$$

Definition 5 *(M-factors fuzzy time series).* Let fuzzy time series $A(t), B(t), C(t), \ldots, M(t)$ be the factors/observations of the forecasting problems. If we only use $A(t)$ to solve the forecasting problems, then it is called a one-factor fuzzy time series. If we use remaining secondary-factors/secondary-observations $B(t), C(t), \ldots, M(t)$ with $A(t)$ to solve the forecasting problems, then it is called M-factors fuzzy time series.

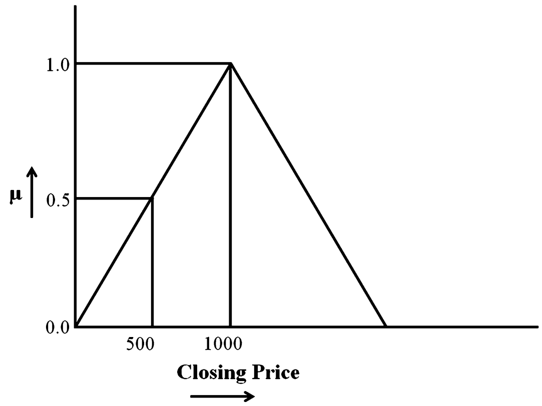

Fig. 1. A Type-1 fuzzy set.

One-factor fuzzy time series model uses only one factor for forecasting, such as *closing* price of the stock index. If some additional information, such as *opening*, *high* and *low* stock indices use with the *closing* price, then it is referred to as M-factors fuzzy time series model. For example, the model proposed by Song and Chissom [2] is based on one-factor, because they simply use the enrollments data to solve the forecasting problems. On the other hand, the model proposed by Huarng and Yu [30] is based on M-factors, because they use *high* and *low* as the secondary-observations to forecast the *closing* price of TAIEX.

Definition 6 *(Type-2 fuzzy set).* (See [44].) Let $\tilde{A}(U)$ be the set of fuzzy sets in U. A Type-2 fuzzy set $\tilde{A}$ in X is fuzzy set whose membership grades are themselves fuzzy. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in U for all x, i.e., $\mu_{\tilde{A}} : X \to \tilde{A}(U)$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid \mu_{\tilde{A}}(x) \in \tilde{A}(U) \forall_x \in X\}. \quad (3)$$

The concept of Type-2 fuzzy set is explained with an example as follows:

Explanation. When we cannot distinguish the degree of membership of an element in a set as 0 or 1, we use Type-1 fuzzy sets. Similarly, when the nature of an event is so fuzzy so that determination of degree of membership as a crisp number in the range $[0, 1]$ is so difficult, then we use Type-2 fuzzy sets [45]. This Type-2 fuzzy sets concept was first introduced by Zadeh [46] in 1975. In Type-1 fuzzy set, the degree of membership is characterized by a crisp value; whereas in Type-2 fuzzy set, the degree of membership is regarded as a fuzzy set [47]. Thus, if there are more uncertainty in the event, and we have difficulty in determining its exact value, then we simply use Type-1 fuzzy sets, rather than crisp sets. But, ideally we have to use some finite-type sets, just like Type-2 fuzzy sets [45]. Based on this explanation, we present an example which is based on article [30] as follows:

Let us consider a fuzzy set for "Closing Price" of stock index, as shown in Fig. 1. Here, we have a crisp degree of membership values 1.0 and 0.5 for the "Closing Price = 1000" and "Closing Price = 500", respectively. Based on the above explanation, the "Closing Price = 1000" can have more than one degree of memberships. For example, in Fig. 2, there are three degrees of memberships (0.4, 0.5 and 0.6) for the "Closing Price = 1000". In other words, there can be multiple degrees of membership for the same "Closing Price = 1000", as shown in Fig. 2. In Fig. 2, the highest degree of membership (0.6) indicates the positive view about the occurrence of event, whereas the lowest degree of membership (0.4) indicates the negative view about the occurrence of event. We can use these positive and negative views together in fuzzy time series modeling approach. In summary, we can use more observations/information from the positive and negative views for forecasting in each time period.

Definition 7 *(Type-2 fuzzy time series model).* (See [30].) A Type-2 fuzzy time series model can be defined as an extension of a Type-1 fuzzy time series model. The Type-2 fuzzy time series model employs the FLRs established by a Type-1 model based on Type-1 observations. Fuzzy operators such as union and intersection are used to establish the new FLRs obtained from Type-1 and Type-2 observations. Then, Type-2 forecasts are obtained from these FLRs.

Following Definition 7, we define two operators, viz., union and intersection, for the set theoretic operations. These two operators are explained as follows [48]:

Observation that is handled by Type-1 fuzzy time series model can be termed as "main-factor/Type-1 observation", whereas observations that are handled by Type-2 fuzzy time series model can be termed as "secondary-factors/Type-2 observations". Due to involvement of Type-2 observations with Type-1 observation, a massive FLRGs are generated in Type-2

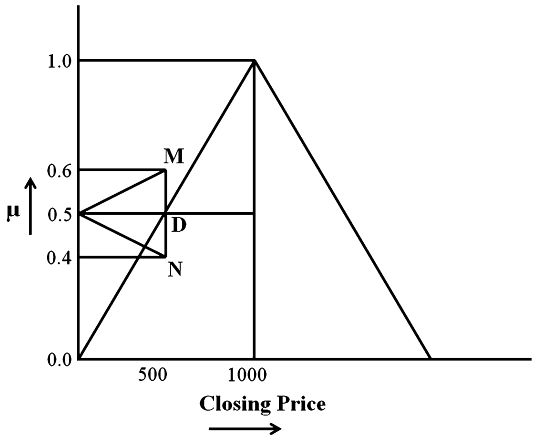

Fig. 2. A Type-2 fuzzy set.

model. To deal with these FLRGs together and establish the relationships among multiple FLRGs, the union (∪) and intersection (∩) operations are employed. Both these operations in terms of handling FLRGs are explained next.

Consider the following FLRGs for Type-1 and Type-2 observations as follows:

$$\text{FLRG for Type-1 observation } \{\tilde{A}_p \to \tilde{A}_{s1}, \tilde{A}_{s2}, \ldots, \tilde{A}_{sm}.$$

$$\text{FLRGs for Type-2 observations } \begin{cases} \tilde{A}_q \to \tilde{A}_{t1}, \tilde{A}_{t2}, \ldots, \tilde{A}_{tm}, \\ \tilde{A}_r \to \tilde{A}_{u1}, \tilde{A}_{u2}, \ldots, \tilde{A}_{um}, \\ \ldots \end{cases}$$

Based on these FLRGs, we define ∪ and ∩ operations as follows:

Definition 8 *(∪ operation for FLRGs).* The operator ∪ is used to establish the relationships between the FLRGs of Type-1 and Type-2 observations by selecting the maximum value from the right-hand side of each of the FLRG as follows:

$$\begin{aligned} \tilde{A}_p, \tilde{A}_q, \tilde{A}_r, \ldots \to & \bigcup(\tilde{A}_{s1}, \tilde{A}_{s2}, \ldots, \tilde{A}_{sm}), \\ & \bigcup(\tilde{A}_{t1}, \tilde{A}_{t2}, \ldots, \tilde{A}_{tm}), \\ & \bigcup(\tilde{A}_{u1}, \tilde{A}_{u2}, \ldots, \tilde{A}_{um}), \\ & \ldots \end{aligned} \tag{4}$$

Following Definition 8, the FLRGs of Type-1 and Type-2 observations are combined as follows:

$$\begin{aligned} \tilde{A}_p, \tilde{A}_q, \tilde{A}_r, \ldots \to & (\tilde{A}_{s1} \vee \tilde{A}_{s2} \vee \cdots \vee \tilde{A}_{sm}), \\ & (\tilde{A}_{t1} \vee \tilde{A}_{t2} \vee \cdots \vee \tilde{A}_{tm}), \\ & (\tilde{A}_{u1} \vee \tilde{A}_{u2} \vee \cdots \vee \tilde{A}_{um}), \\ & \ldots \end{aligned} \tag{5}$$

Definition 9 *(∩ operation for FLRGs).* The operator ∩ is used to establish the relationships between the FLRGs of Type-1 and Type-2 observations by selecting the minimum value from the right-hand side of each of the FLRG as follows:

$$\begin{aligned} \tilde{A}_p, \tilde{A}_q, \tilde{A}_r, \ldots \to & \bigcap(\tilde{A}_{s1}, \tilde{A}_{s2}, \ldots, \tilde{A}_{sm}), \\ & \bigcap(\tilde{A}_{t1}, \tilde{A}_{t2}, \ldots, \tilde{A}_{tm}), \\ & \bigcap(\tilde{A}_{u1}, \tilde{A}_{u2}, \ldots, \tilde{A}_{um}), \\ & \ldots \end{aligned} \tag{6}$$

Following Definition 9, the FLRGs of Type-1 and Type-2 observations are combined as follows:

$$\begin{aligned}\tilde{A}_p, \tilde{A}_q, \tilde{A}_r, \ldots \rightarrow\ & (\tilde{A}_{s1} \wedge \tilde{A}_{s2} \wedge \cdots \wedge \tilde{A}_{sm}),\\ & (\tilde{A}_{t1} \wedge \tilde{A}_{t2} \wedge \cdots \wedge \tilde{A}_{tm}),\\ & (\tilde{A}_{u1} \wedge \tilde{A}_{u2} \wedge \cdots \wedge \tilde{A}_{um}),\\ & \ldots\end{aligned} \tag{7}$$

4. Particle swarm optimization (PSO) algorithm

The PSO algorithm was first developed by Eberhart and Kennedy [49]. It is a population-based evolutionary computation technique, which is inspired by the social behavior of animals such as bird flocking, fish schooling, and swarming theory [50–53]. The PSO can be employed to solve many of the same kinds of problems as genetic algorithms [54]. The PSO algorithm is applied to a set of particles, where each particle has assigned a randomized velocity. Each particle is then allowed to move towards the problem space. At each movement, each particle keeps track of its own best solution (fitness) and the best solution of its neighboring particles. The value of that fitness is called "*pbest*". Then each particle is attracted towards the finding of global best value by keep tracking the overall best value of each particle, and its location [55]. The particle which obtained the global fitness value is called "*gbest*".

At each step of optimization, velocity of each particle is dynamically adjusted according to its own experience and its neighboring particles, which is represented by the following equations:

$$Vel_{id,t} = \alpha \times Vel_{id,t} + M_1 \times R_{and} \times (PB_{id} - CP_{id,t}) + M_2 \times R_{and} \times (PG_{best} - CP_{id,t}). \tag{8}$$

The position of a new particle can be determined by the following equation:

$$CP_{id,t} = CP_{id,t} + Vel_{id,t}, \tag{9}$$

where i represents the ith particle and d represents the dimension of the problem space. In Eq. (8), α represents the inertia weight factor; $CP_{id,t}$ represents the current position of the particle i in iteration t; PB_{id} denotes the previous best position of the particle i that experiences the best fitness value so far (*pbest*); PG_{best} represents the global best fitness value (*gbest*) among all the particles; R_{and} gives the random value in the range of $[0, 1]$; M_1 and M_2 represent the self-confidence coefficient and the social coefficient, respectively; and $Vel_{id,t}$ represents the velocity of the particle i in iteration t. Here, $Vel_{id,t}$ is limited to the range $[-Vel_{max}, Vel_{max}]$, where Vel_{max} is a constant and defined by users.

The steps for the standard PSO are presented in Algorithm 1.

Algorithm 1 Standard PSO algorithm.

Step 1: Initialize all particles with random positions and velocities in the d-dimensional problem space.
Step 2: Evaluate the optimization fitness function of all particles.
Step 3: For each particle, compare its current fitness value with its *pbest*. If current value is better than *pbest*, then update *pbest* value with the current value.
Step 4: For each particle, compare its fitness value with its overall previous best. If the current fitness value is better than *gbest*, then update *gbest* value with the current best particle.
Step 5: For each particle, change the movement (velocity) and location (position) according to Eqs. (8) and (9).
Step 6: Repeat Step 2, until stopping criterion is met, usually a sufficiently *gbest* value is obtained.

5. Algorithm and defuzzification for Type-2 model

In this section, we will first present the algorithm for the proposed Type-2 fuzzy time series model. Then, defuzzification process for the proposed model will be presented in the subsequent subsection.

5.1. Algorithm

In this subsection, we have presented an algorithm for the Type-2 fuzzy time series model, which is based on Huarng and Yu [30] model. Therefore, we first present the algorithm for Type-2 model proposed by [30]. This algorithm is presented as Algorithm 2.

To improve the forecasting accuracy of Type-2 fuzzy time series model, we apply some changes in Algorithm 2. This algorithm is presented as Algorithm 3.

5.2. Defuzzification for Type-2 model

After obtaining the fuzzified forecasting data from the ∪ and ∩ operations, they are defuzzified based on the "Frequency-Weighing Defuzzification Technique", which is the modified version of the defuzzification technique proposed by [21]. In the subsequent section, this technique is discussed. The defuzzified values obtained for the left-hand side and right-hand side

Algorithm 2 Huarng and Yu model for Type-2 fuzzy time series.

Step 1: Choose a Type-1 fuzzy time series model.
Step 2: Pick a variable and Type-1 observations.
Step 3: Apply the Type-1 model to the Type-1 observations and obtain FLRGs.
Step 4: Pick Type-2 observations.
Step 5: Map out-of-sample observations to FLRGs and obtain forecasts.
Step 6: Apply operators to the FLRGs for all the observations.
Step 7: Defuzzify the forecasts.
Step 8: Calculate forecasts for Type-2 model.
Step 9: Evaluate the performance.

Algorithm 3 Proposed Type-2 fuzzy series forecasting model.

Step 1: Select Type-1 and Type-2 observations.
Step 2: Determine the universe of discourse of time series data set and partition it into equal length of intervals.
Step 3: Define linguistic terms for each of the interval.
Step 4: Fuzzify the time series data set of Type-1 and Type-2 observations.
Step 5: Establish the FLRs based on Definition 3.
Step 6: Construct the FLRGs based on Definition 4.
Step 7: Establish the relationships between FLRGs of both Type-1 and Type-2 observations, and mapped-out them to their corresponding day.
Step 8: Based on Definitions 8 and 9, apply $\cup$ and $\cap$ operators on mapped-out FLRGs of Type-1 and Type-2 observations, and obtain the fuzzified forecasting data.
Step 9: Defuzzify the forecasting data based on the "Frequency-Weighing Defuzzification Technique".
Step 10: Compute the forecasted values individually for $\cup$ and $\cap$ operations.

Table 1
Daily stock index price list of SBI (in rupee).

Date (mm-dd-yy)	Open	High	Low	Close
6/4/2012	2064.00	2110.00	2061.00	2082.75
6/5/2012	2100.00	2168.00	2096.00	2158.25
6/6/2012	2183.00	2189.00	2156.55	2167.95
6/7/2012	2151.55	2191.60	2131.50	2179.45
6/10/2012	2200.00	2217.90	2155.50	2164.80
6/11/2012	2155.00	2211.85	2132.60	2206.15
6/12/2012	2210.00	2244.00	2186.00	2226.05
6/13/2012	2212.00	2218.70	2144.10	2150.25
6/14/2012	2168.45	2189.85	2147.00	2183.10
6/17/2012	2216.00	2231.90	2081.60	2087.95
...	...	...	...	...
7/29/2012	1951.25	2040.00	1941.00	2031.75

of the FLRGs can be referred to as points "M" and "N" in Fig. 2, respectively. The forecasted value of the proposed Type-2 model can be derived by taking the average of these defuzzified values. The forecasted value for this Type-2 model can be referred to as point "D" in Fig. 2.

6. Type-2 fuzzy time series forecasting model

In this section, the proposed "Type-2 Fuzzy Time Series Forecasting Model" is presented. To verify the proposed model, the daily stock index price data set of SBI for the period 6/4/2012–7/29/2012 (format: mm-dd-yy), is collected from the website: http://in.finance.yahoo.com. A sample of data set is listed in Table 1. The model consists of ten phases. The functionality of each phase is explained in step-wise as follows:

Step 1. *Select Type-1 and Type-2 observations.*

Explanation. In this study, we select "Actual Price" as the main forecasting objective. To obtain the forecasting results, the "Close" variable of the stock index data set as shown in Table 1, has been selected as Type-1 observation, whereas "Open", "High" and "Low" variables have been selected as Type-2 observations.

Step 2. *Determine the universe of discourse of time series data set and partition it into equal length of intervals.*

Explanation. Define the universe of discourse U of the historical time series data set. Assume that $U = [M_{min} - F_1, M_{max} + F_2]$, where M_{min} and M_{max} are the minimum and maximum values of the historical time series data set. Here, F_1 and F_2 are two positive numbers. These M_{min} and M_{max} values are obtained by considering both Type-1 (Closing price) and Type-2 (Open, High and Low) observations. Based on Table 1, we can see that $M_{min} = 1931.50$ and $M_{max} = 2252.55$. Therefore, we let $F_1 = 2$ and $F_2 = 0.50$. Thus, in this study, the universe of discourse $U = [1929.50, 2253.10]$.

Table 2
Intervals and their corresponding data, mid-points and weights.

Interval	Corresponding data	Mid-value	Weight
$I_1 = (1929.50, 1940.30]$	Open = {Nil} High = {Nil} Low = {1931.50} Close = {Nil}	1934.90	1
$I_2 = (1940.30, 1951.10]$	Open = {Nil} High = {Nil} Low = {1941.00} Close = {1941.00}	1945.7.60	2
$I_3 = (1951.10, 1961.90]$	Open = {1951.25} High = {Nil} Low = {Nil} Close = {Nil}	1956.50	1
...	...	...	...
$I_{26} = (2242.30, 2253.10]$	Open = {Nil} High = {2244.00, 2252.55} Low = {Nil} Close = {Nil}	2247.70	2

Now, divide the universe of discourse U into n equal lengths of intervals (i.e., $a_1, a_2, \dots,$ and a_n). Each interval of time series data set can be defined as follows [16]:

$$a_i = \left[L_B + (i-1)\frac{U_B - L_B}{j}, L_B + i\frac{U_B - L_B}{j}\right] \tag{10}$$

for $i = 1, 2, \dots, n$ and $j = 30$. Here, $L_B = 1929.50$ and $U_B = 2253.10$. Here, j represents the number of intervals which are taken into the consideration.

Based on Eq. (10), the universe of discourse $U = [1929.50, 2253.10]$ is divided into 30 equal lengths of intervals as: $a_1 = (1929.50, 1940.30]$, $a_2 = (1940.30, 1951.10]$, $\dots$, $a_{30} = (2242.30, 2253.10]$. Now, assign the data to their corresponding intervals. Mid-value of each interval is recorded by taking the mean of upper bound and lower bound of the interval. For ease of computation, intervals which do not cover any historical datum is discarded from the list. Intervals which contain historical data are represented as I_i, for $i = 1, 2, \dots, n$; and $n \leqslant 26$. Then, each interval is assigned a weight based on frequency of the interval. For example, in Table 2, interval I_2 has two data with frequency 2. So, we assign weight 2 to the interval I_2. All these intervals, and their corresponding data, mid-values and weights are shown in Table 2.

Step 3. *Define linguistic terms for each of the interval. Assume that the historical time series data set is distributed among n intervals (i.e., $I_1, I_2, \dots,$ and I_n). Therefore, define n linguistic variables $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n$, which can be represented by fuzzy sets, as shown below*:

$$\begin{aligned}
\tilde{A}_1 &= 1/I_1 + 0.5/I_2 + 0/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n,\\
\tilde{A}_2 &= 0.5/I_1 + 1/I_2 + 0.5/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n,\\
\tilde{A}_3 &= 0/I_1 + 0.5/I_2 + 1/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_n,\\
&\vdots\\
\tilde{A}_n &= 0/I_n + 0/I_2 + 0/I_3 + \cdots + 0/I_{n-2} + 0.5/I_{n-1} + 1/I_n.
\end{aligned}$$

Here, maximum degree of membership of the fuzzy set $\tilde{A}_i$ occurs at interval I_i, and $1 \leqslant i \leqslant n$.

Explanation. We define 26 linguistic variables $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_{26}$ for the stock index data set, because the data set is distributed among 26 intervals (i.e., $I_1, I_2, \dots,$ and I_{26}). All these defined linguistic variables are shown as below:

$$\begin{aligned}
\tilde{A}_1 &= 1/I_1 + 0.5/I_2 + 0/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_{26},\\
\tilde{A}_2 &= 0.5/I_1 + 1/I_2 + 0.5/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_{26},\\
\tilde{A}_3 &= 0/I_1 + 0.5/I_2 + 1/I_3 + \cdots + 0/I_{n-2} + 0/I_{n-1} + 0/I_{26},\\
&\vdots\\
\tilde{A}_{26} &= 0/I_1 + 0/I_2 + 0/I_3 + \cdots + 0/I_{n-2} + 0.5/I_{n-1} + 1/I_{26}.
\end{aligned} \tag{11}$$

Table 3
Fuzzified stock index data set.

Date (mm-dd-yy)	Open	Fuzzified Open	High	Fuzzified High	Low	Fuzzified Low	Close	Fuzzified Close
6/4/2012	2064.00	$\tilde{A}_9$	2110.00	$\tilde{A}_{13}$	2061.00	$\tilde{A}_9$	2082.80	$\tilde{A}_{11}$
6/5/2012	2100.00	$\tilde{A}_{12}$	2168.00	$\tilde{A}_{19}$	2096.00	$\tilde{A}_{12}$	2158.30	$\tilde{A}_{18}$
6/6/2012	2183.00	$\tilde{A}_{20}$	2189.00	$\tilde{A}_{21}$	2156.60	$\tilde{A}_{18}$	2167.90	$\tilde{A}_{19}$
6/7/2012	2151.60	$\tilde{A}_{17}$	2191.60	$\tilde{A}_{21}$	2131.50	$\tilde{A}_{15}$	2179.40	$\tilde{A}_{20}$
6/10/2012	2200.00	$\tilde{A}_{22}$	2217.90	$\tilde{A}_{23}$	2155.50	$\tilde{A}_{17}$	2164.80	$\tilde{A}_{18}$
6/11/2012	2155.00	$\tilde{A}_{17}$	2211.80	$\tilde{A}_{23}$	2132.60	$\tilde{A}_{15}$	2206.20	$\tilde{A}_{22}$
6/12/2012	2210.00	$\tilde{A}_{23}$	2244.00	$\tilde{A}_{26}$	2186.00	$\tilde{A}_{20}$	2226.10	$\tilde{A}_{24}$
6/13/2012	2212.00	$\tilde{A}_{23}$	2218.70	$\tilde{A}_{23}$	2144.10	$\tilde{A}_{16}$	2150.30	$\tilde{A}_{17}$
6/14/2012	2168.40	$\tilde{A}_{19}$	2189.80	$\tilde{A}_{21}$	2147.00	$\tilde{A}_{17}$	2183.10	$\tilde{A}_{20}$
6/17/2012	2216.00	$\tilde{A}_{23}$	2231.90	$\tilde{A}_{25}$	2081.60	$\tilde{A}_{11}$	2087.90	$\tilde{A}_{11}$
...	...	...	...	...	...	...	...	...
7/29/2012	1951.30	$\tilde{A}_3$	2040.00	$\tilde{A}_7$	1941.00	$\tilde{A}_2$	2031.80	$\tilde{A}_6$

Table 4
Fuzzy logical relationships (FLRs).

FLRs for Open	FLRs for High	FLRs for Low	FLRs for Close
$\tilde{A}_9 \to \tilde{A}_{12}$	$\tilde{A}_{13} \to \tilde{A}_{19}$	$\tilde{A}_9 \to \tilde{A}_{12}$	$\tilde{A}_{11} \to \tilde{A}_{18}$
$\tilde{A}_{12} \to \tilde{A}_{20}$	$\tilde{A}_{19} \to \tilde{A}_{21}$	$\tilde{A}_{12} \to \tilde{A}_{18}$	$\tilde{A}_{18} \to \tilde{A}_{19}$
...	...	...	...
$A_{12} \to A_{17}$	$A_{20} \to A_{20}$	$A_{11} \to A_{15}$	$A_{20} \to A_{18}$
$A_{12} \to A_{11}$	$A_{14} \to A_{12}$	$A_{10} \to A_9$	$A_{12} \to A_{10}$
...	...	...	...

For ease of computation, the degree of membership values of fuzzy set $\tilde{A}_j$ $(j = 1, 2, \ldots, 26)$ are considered as either 0, 0.5 or 1, and $1 \leqslant j \leqslant 26$. In Eq. (11), for example, $\tilde{A}_1$ represents a linguistic value, which denotes a fuzzy set $= \{I_1, I_2, \ldots, I_{26}\}$. This fuzzy set consists of twenty six members with different degree of membership values $= \{1, 0.5, 0, \ldots, 0\}$. Similarly, the linguistic value $\tilde{A}_2$ denotes the fuzzy set $= \{I_1, I_2, \ldots, I_{26}\}$, which also consists of twenty six members with different degree of membership values $= \{0.5, 1, 0.5, \ldots, 0\}$. The descriptions of remaining linguistic variables, viz., $\tilde{A}_3, \tilde{A}_4, \ldots, \tilde{A}_{26}$, can be provided in a similar manner.

Step 4. *Fuzzify the time series data set. If one day's datum belongs to the interval I_i, then datum is fuzzified into $\tilde{A}_i$, where $1 \leqslant i \leqslant n$.*

Explanation. In Step 3, each fuzzy set contains twenty six intervals, and each interval corresponds to all fuzzy sets with different degree of membership values. For example, interval I_1 corresponds to linguistic variables $\tilde{A}_1$ and $\tilde{A}_2$ with degree of membership values 1 and 0.5, respectively, and remaining fuzzy sets with degree of membership value 0. Similarly, interval I_2 corresponds to linguistic variables $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ with degree of membership values 0.5, 1, and 0.5, respectively, and remaining fuzzy sets with degree of membership value 0. The descriptions of remaining intervals, viz., $I_3, I_4, \ldots, I_{26}$, can be provided in a similar manner.

In order to fuzzify the historical time series data, it is essential to obtain the degree of membership value of each observation belonging to each $\tilde{A}_j$ $(j = 1, 2, \ldots, n)$ for each day. If the maximum membership value of one day's observation occurs at interval I_i and $1 \leqslant i \leqslant n$, then the fuzzified value for that particular day is considered as $\tilde{A}_i$. For example, the stock index price of 6/4/2012 for observation "Open" belongs to the interval I_9 with the highest degree of membership value 1, so it is fuzzified to $\tilde{A}_9$. In this way, we have fuzzified each observation of the historical time series data set. The fuzzified historical time series data set is presented in Table 3.

Step 5. *Establish the FLR between the fuzzified values obtained in Step 4. For example, if the fuzzified values of time $t - 1$ and t are $\tilde{A}_i$ and $\tilde{A}_j$, respectively, then establish the FLR as "$\tilde{A}_i \to \tilde{A}_j$", where "$\tilde{A}_i$" and "$\tilde{A}_j$" are called the previous state and current state of the FLR, respectively.*

Explanation. In Table 3, fuzzified stock index values for "Open" observation for days 6/4/2012 and 6/5/2012 are $\tilde{A}_9$ and $\tilde{A}_{12}$, respectively. So, we can establish an FLR between two consecutive fuzzified values $\tilde{A}_9$ and $\tilde{A}_{12}$ as "$\tilde{A}_9 \to \tilde{A}_{12}$", where "$\tilde{A}_9$" and "$\tilde{A}_{12}$" are called the previous state and current state of the FLR, respectively. In this way, we have obtained all FLRs for both Type-1 and Type-2 fuzzified stock index values, which are presented in Table 4.

Step 6. *Construct the FLRG. Based on the same previous state of the FLRs, the FLRs can be grouped into an FLRG. For example, the FLRG "$\tilde{A}_i \to \tilde{A}_m, \tilde{A}_n$" indicates that there are the following FLRs:*

Table 5
Fuzzy logical relationship groups (FLRGs).

FLRGs for Open	FLRGs for High	FLRGs for Low	FLRGs for Close
$\tilde{A}_8 \to \tilde{A}_3$	$\tilde{A}_9 \to \tilde{A}_7$	$\tilde{A}_1 \to \tilde{A}_2$	$\tilde{A}_2 \to \tilde{A}_6$
$\tilde{A}_9 \to \tilde{A}_{12}$	$\tilde{A}_{11} \to \tilde{A}_9$	$\tilde{A}_4 \to \tilde{A}_1$	$\tilde{A}_5 \to \tilde{A}_2$
$\tilde{A}_{10} \to \tilde{A}_8$	$\tilde{A}_{12} \to \tilde{A}_{11}$	$\tilde{A}_8 \to \tilde{A}_{12}$	$\tilde{A}_{10} \to \tilde{A}_5$
...	...	...	...
$\tilde{A}_{25} \to \tilde{A}_{21}$		$\tilde{A}_{23} \to \tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}$	

Table 6
Mapped-out FLRGs with their corresponding day.

Date (mm-dd-yy)	Mapped-out FLRGs of Type-2 observation (Open)	Mapped-out FLRGs of Type-2 observation (High)	Mapped-out FLRGs of Type-2 observation (Low)	Mapped-out FLRGs of Type-1 observation (Close)
6/4/2012	$\tilde{A}_9 \to \tilde{A}_{12}$	$\tilde{A}_{13} \to \tilde{A}_{19}$	$\tilde{A}_9 \to \tilde{A}_4, \tilde{A}_{12}$	$\tilde{A}_{11} \to \tilde{A}_{12}, \tilde{A}_{18}$
6/5/2012	$\tilde{A}_{12} \to \tilde{A}_{11}, \tilde{A}_{17}, \tilde{A}_{20}$	$\tilde{A}_{19} \to \tilde{A}_{15}, \tilde{A}_{21}$	$\tilde{A}_{12} \to \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{13}, \tilde{A}_{18}$	$\tilde{A}_{18} \to \tilde{A}_{14}, \tilde{A}_{15}, \tilde{A}_{19}, \tilde{A}_{20}, \tilde{A}_{22}$
6/6/2012	$\tilde{A}_{20} \to \tilde{A}_{17}, \tilde{A}_{23}$	$\tilde{A}_{21} \to \tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{25}$	$\tilde{A}_{18} \to \tilde{A}_{15}, \tilde{A}_{17}, \tilde{A}_{19}$	$\tilde{A}_{19} \to \tilde{A}_{20}$
6/7/2012	$\tilde{A}_{17} \to \tilde{A}_{13}, \tilde{A}_{18}, \tilde{A}_{22}, \tilde{A}_{23}$	$\tilde{A}_{21} \to \tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{25}$	$\tilde{A}_{15} \to \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{17}, \tilde{A}_{18}, \tilde{A}_{20}$	$\tilde{A}_{20} \to \tilde{A}_{11}, \tilde{A}_{18}, \tilde{A}_{20}, \tilde{A}_{22}, \tilde{A}_{24}$
...	...	...	...	...
7/29/2012	$\tilde{A}_3 \to ?$	$\tilde{A}_7 \to ?$	$\tilde{A}_2 \to ?$	$\tilde{A}_6 \to ?$

$$\tilde{A}_i \to \tilde{A}_m,$$
$$\tilde{A}_i \to \tilde{A}_n.$$

Explanation. In Table 4, there are 3 FLRs for "Open" observation with the same previous state, $\tilde{A}_{12} \to \tilde{A}_{11}$, $\tilde{A}_{12} \to \tilde{A}_{17}$, and $\tilde{A}_{12} \to \tilde{A}_{20}$. These FLRs are used to form the FLRG, $\tilde{A}_{12} \to \tilde{A}_{11}, \tilde{A}_{17}, \tilde{A}_{20}$. All these FLRGs are shown in Table 5. If the same FLR appears more than once, it is included only once in the group.

Step 7. *Establish the relationships between Type-1 and Type-2 observations. If one day's fuzzified stock index price value for Type-1 observation is $\tilde{A}_i$ with FLRG "$\tilde{A}_i \to \tilde{A}_j$", then FLRG is mapped-out to its corresponding day. For the same day, if fuzzified stock index price values for the three different Type-2 observations are $\tilde{A}_k$, $\tilde{A}_m$ and $\tilde{A}_n$ with FLRGs "$\tilde{A}_k \to \tilde{A}_a$, $\tilde{A}_m \to \tilde{A}_b$, and $\tilde{A}_n \to \tilde{A}_c$" respectively, then all these FLRGs are also mapped-out to that day.*

Explanation. In Table 5, the first three columns represent the FLRGs of Type-2 observations, and the last column represents the FLRGs of Type-1 observation. To establish the relationship between Type-1 and Type-2 observations, FLRGs of both Type-1 and Type-2 observations are mapped-out to their corresponding day. For example, fuzzified stock index price value for Type-1 observation "Close" for day 6/5/2012 is $\tilde{A}_{18}$. FLRG of this fuzzy set is:

$$\text{FLRG for Type-1 observation } \{\tilde{A}_{18} \to \tilde{A}_{14}, \tilde{A}_{15}, \tilde{A}_{19}, \tilde{A}_{20}, \tilde{A}_{22}.$$

Similarly, fuzzified stock index price values for Type-2 observations "Open", "High" and "Low" for day 6/5/2012 are $\tilde{A}_{12}$, $\tilde{A}_{19}$ and $\tilde{A}_{12}$, respectively. FLRGs for these fuzzy sets are shown below:

$$\text{FLRG for Type-2 observations } \begin{cases} \tilde{A}_{12} \to \tilde{A}_{11}, \tilde{A}_{17}, \tilde{A}_{20}, \\ \tilde{A}_{19} \to \tilde{A}_{15}, \tilde{A}_{21}, \\ \tilde{A}_{12} \to \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{13}, \tilde{A}_{18}. \end{cases}$$

FLRGs of both Type-1 and Type-2 observations are now mapped-out together to day 6/5/2012. In this way, we have mapped-out all FLRGs to their corresponding day, which are shown in Table 6.

Step 8. *Obtain the fuzzified forecasting data by applying $\cup$ and $\cap$ operators on mapped-out FLRGs of both Type-1 and Type-2 observations.*

Explanation. Based on Definitions 8 and 9, obtain the forecasting data by applying $\cup$ and $\cap$ operators on mapped-out FLRGs of both Type-1 and Type-2 observations. The mapped-out FLRGs are shown in Table 6. In Table 6, we apply $\cup$ operator to all mapped-out FLRGs of both Type-1 and Type-2 observations. For example, mapped-out FLRG of Type-1 observation for day 6/7/2012 is:

$$\tilde{A}_{20} \to \tilde{A}_{11}, \tilde{A}_{18}, \tilde{A}_{20}, \tilde{A}_{22}, \tilde{A}_{24} \quad \text{(for Close)}$$

and, mapped-out FLRGs of Type-2 observations for day 6/7/2012 are:

Table 7
Fuzzified forecasting data for the ∪ operation.

Date (mm-dd-yy)	Forecasting data—∪ operation
6/4/2012	$\tilde{A}_9, \tilde{A}_{11}, \tilde{A}_{13} \rightarrow \tilde{A}_{12}, \tilde{A}_{18}, \tilde{A}_{19}$
6/5/2012	$\tilde{A}_{12}, \tilde{A}_{18}, \tilde{A}_{19} \rightarrow \tilde{A}_{18}, \tilde{A}_{20}, \tilde{A}_{21}, \tilde{A}_{22}$
6/6/2012	$\tilde{A}_{18}, \tilde{A}_{19}, \tilde{A}_{20}, \tilde{A}_{21} \rightarrow \tilde{A}_{19}, \tilde{A}_{20}, \tilde{A}_{23}, \tilde{A}_{25}$
6/7/2012	$\tilde{A}_{15}, \tilde{A}_{17}, \tilde{A}_{20}, \tilde{A}_{21} \rightarrow \tilde{A}_{20}, \tilde{A}_{23}, \tilde{A}_{24}, \tilde{A}_{25}$
...	...
7/29/2012	$\tilde{A}_2, \tilde{A}_3, \tilde{A}_6, \tilde{A}_7 \rightarrow ?$

$$\begin{aligned}
&\tilde{A}_{17} \rightarrow \tilde{A}_{13}, \tilde{A}_{18}, \tilde{A}_{22}, \tilde{A}_{23} \quad (\text{for Open})\\
&\tilde{A}_{21} \rightarrow \tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{25} \quad (\text{for High})\\
&\tilde{A}_{15} \rightarrow \tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{17}, \tilde{A}_{18}, \tilde{A}_{20} \quad (\text{for Low}).
\end{aligned}$$

Hence, from Definition 8, we have:

$$\begin{aligned}
\tilde{A}_{20}, \tilde{A}_{17}, \tilde{A}_{21}, \tilde{A}_{15} \rightarrow & \bigcup(\tilde{A}_{11}, \tilde{A}_{18}, \tilde{A}_{20}, \tilde{A}_{22}, \tilde{A}_{24}),\\
& \bigcup(\tilde{A}_{13}, \tilde{A}_{18}, \tilde{A}_{22}, \tilde{A}_{23}),\\
& \bigcup(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{25}),\\
& \bigcup(\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{17}, \tilde{A}_{18}, \tilde{A}_{20}).
\end{aligned}$$

Now, based on Eq. (5), above mapped-out FLRGs can be represented in the following form as follows:

$$\begin{aligned}
\tilde{A}_{20}, \tilde{A}_{17}, \tilde{A}_{21}, \tilde{A}_{15} \rightarrow & (\tilde{A}_{11} \vee \tilde{A}_{18} \vee \tilde{A}_{20} \vee \tilde{A}_{22} \vee \tilde{A}_{24}),\\
& (\tilde{A}_{13} \vee \tilde{A}_{18} \vee \tilde{A}_{22} \vee \tilde{A}_{23}),\\
& (\tilde{A}_{21} \vee \tilde{A}_{22} \vee \tilde{A}_{23} \vee \tilde{A}_{25}),\\
& (\tilde{A}_{11} \vee \tilde{A}_{12} \vee \tilde{A}_{17} \vee \tilde{A}_{18} \vee \tilde{A}_{20}),
\end{aligned}$$

i.e., $\tilde{A}_{20}, \tilde{A}_{17}, \tilde{A}_{21}, \tilde{A}_{15} \rightarrow \tilde{A}_{20}, \tilde{A}_{23}, \tilde{A}_{24}, \tilde{A}_{25}$.

Similarly, in Table 6, we apply ∩ operator to mapped-out FLRGs of Type-1 and Type-2 observations for day 6/7/2012. Hence, from Definition 9, we have:

$$\begin{aligned}
\tilde{A}_{20}, \tilde{A}_{17}, \tilde{A}_{21}, \tilde{A}_{15} \rightarrow & \bigcap(\tilde{A}_{11}, \tilde{A}_{18}, \tilde{A}_{20}, \tilde{A}_{22}, \tilde{A}_{24}),\\
& \bigcap(\tilde{A}_{13}, \tilde{A}_{18}, \tilde{A}_{22}, \tilde{A}_{23}),\\
& \bigcap(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{25}),\\
& \bigcap(\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_{17}, \tilde{A}_{18}, \tilde{A}_{20}).
\end{aligned}$$

Now, based on Eq. (7), above mapped-out FLRGs can be represented in the following form as follows:

$$\begin{aligned}
\tilde{A}_{20}, \tilde{A}_{17}, \tilde{A}_{21}, \tilde{A}_{15} \rightarrow & (\tilde{A}_{11} \wedge \tilde{A}_{18} \wedge \tilde{A}_{20} \wedge \tilde{A}_{22} \wedge \tilde{A}_{24}),\\
& (\tilde{A}_{13} \wedge \tilde{A}_{18} \wedge \tilde{A}_{22} \wedge \tilde{A}_{23}),\\
& (\tilde{A}_{21} \wedge \tilde{A}_{22} \wedge \tilde{A}_{23} \wedge \tilde{A}_{25}),\\
& (\tilde{A}_{11} \wedge \tilde{A}_{12} \wedge \tilde{A}_{17} \wedge \tilde{A}_{18} \wedge \tilde{A}_{20}),
\end{aligned}$$

i.e., $\tilde{A}_{15}, \tilde{A}_{17}, \tilde{A}_{20}, \tilde{A}_{21} \rightarrow \tilde{A}_{11}, \tilde{A}_{13}, \tilde{A}_{21}$.

Repeated fuzzy set is discarded from the mapped-out FLRGs. The fuzzified forecasting data obtained after applications of the ∪ and ∩ operators are presented in Tables 7 and 8, respectively.

Step 9. *Defuzzify the forecasting data.*

To defuzzify the fuzzified time series data set and to obtain the forecasted values, defuzzification technique proposed by Singh and Borah [21] is employed here. Based on the application of technique, it is slightly modified and categorized as: *Principle* 1 and *Principle* 2. The procedure for *Principle* 1 is given as follows:

Table 8
Fuzzified forecasting data for the ∩ operation.

Date (mm-dd-yy)	Forecasting data—∩ operation
6/4/2012	$\tilde{A}_9, \tilde{A}_{11}, \tilde{A}_{13} \to \tilde{A}_4, \tilde{A}_{12}, \tilde{A}_{19}$
6/5/2012	$\tilde{A}_{12}, \tilde{A}_{18}, \tilde{A}_{19} \to \tilde{A}_{11}, \tilde{A}_{14}, \tilde{A}_{15}$
6/6/2012	$\tilde{A}_{18}, \tilde{A}_{19}, \tilde{A}_{20}, \tilde{A}_{21} \to \tilde{A}_{15}, \tilde{A}_{17}, \tilde{A}_{20}, \tilde{A}_{21}$
6/7/2012	$\tilde{A}_{15}, \tilde{A}_{17}, \tilde{A}_{20}, \tilde{A}_{21} \to \tilde{A}_{11}, \tilde{A}_{13}, \tilde{A}_{21}$
...	...
7/29/2012	$\tilde{A}_2, \tilde{A}_3, \tilde{A}_6, \tilde{A}_7 \to ?$

- *Principle* 1: The *Principle* 1 is applicable only if there are more than one fuzzified values available in the current state. The steps under *Principle* 1 are explained next.

  Step 1. *Obtain the fuzzified forecasting data for forecasting day $D(t)$, whose previous state is $\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ip}$ $(i = 1, 2, 3, \ldots, n)$, and the current state is $\tilde{A}_{j1}, \tilde{A}_{j2}, \ldots, \tilde{A}_{jp}$ $(j = 1, 2, 3, \ldots, n)$, i.e., FLRG is in the form of $\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ip} \to \tilde{A}_{j1}, \tilde{A}_{j2}, \ldots, \tilde{A}_{jp}$.*

  Step 2. *Obtain the defuzzified forecasting value for the previous state as*:

$$
\begin{aligned}
Defuzz_{prev} = {} & C_{i1} \times \left[\frac{W_{i1}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right] \\
& + C_{i2} \times \left[\frac{W_{i2}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right] \\
& \vdots \\
& + C_{ip} \times \left[\frac{W_{ip}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right],
\end{aligned}
\tag{12}
$$

  where $C_{i1}, C_{i2}, \ldots, C_{ip}$ and $W_{i1}, W_{i2}, \ldots, W_{ip}$ denote mid-values and weights of the intervals $I_{i1}, I_{i2}, \ldots, I_{ip}$ $(i = 1, 2, 3, \ldots, n)$, respectively, and the maximum membership values of $\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ip}$ occur at intervals $I_{i1}, I_{i2}, \ldots, I_{ip}$, respectively.

  Step 3. *Obtain the defuzzified forecasting value for the current state as*:

$$
\begin{aligned}
Defuzz_{curr} = {} & C_{j1} \times \left[\frac{W_{j1}}{\sum_{j=1}^{n} W_{j1} + W_{j2} + \cdots + W_{jp}} \right] \\
& + C_{j2} \times \left[\frac{W_{j2}}{\sum_{j=1}^{n} W_{j1} + W_{j2} + \cdots + W_{jp}} \right] \\
& \vdots \\
& + C_{jp} \times \left[\frac{W_{jp}}{\sum_{j=1}^{n} W_{j1} + W_{j2} + \cdots + W_{jp}} \right],
\end{aligned}
\tag{13}
$$

  where $C_{j1}, C_{j2}, \ldots, C_{jp}$ and $W_{j1}, W_{j2}, \ldots, W_{jp}$ denote mid-values and weights of the intervals $I_{j1}, I_{j2}, \ldots, I_{jp}$ $(j = 1, 2, 3, \ldots, n)$, respectively, and the maximum membership values of $\tilde{A}_{j1}, \tilde{A}_{j2}, \ldots, \tilde{A}_{jp}$ occur at intervals $I_{j1}, I_{j2}, \ldots, I_{jp}$, respectively.

- *Principle* 2: This principle is applicable only if there is an unknown value in the current state. The steps under *Principle* 2 are given as follows:

  Step 1. *Obtain the fuzzified forecasting data for forecasting day $D(t)$, whose the previous state is $\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ip}$ $(i = 1, 2, 3, \ldots, n)$, and the current state is "?" (the symbol "?" represents an unknown value), i.e., FLRG is in the form of $\tilde{A}_{i1}, \tilde{A}_{i2}, \ldots, \tilde{A}_{ip} \to ?$.*

  Step 2. *Obtain the defuzzified forecasting value for the previous state as*:

$$
\begin{aligned}
Defuzz_{prev} = {} & C_{i1} \times \left[\frac{W_{i1}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right] \\
& + C_{i2} \times \left[\frac{W_{i2}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right]
\end{aligned}
$$

$$\vdots$$
$$+ C_{ip} \times \left[\frac{W_{ip}}{\sum_{i=1}^{n} W_{i1} + W_{i2} + \cdots + W_{ip}} \right], \tag{14}$$

where $C_{i1}, C_{i2}, \ldots, C_{ip}$ *and* $W_{i1}, W_{i2}, \ldots, W_{ip}$ *denote mid-values and weights of the intervals* $I_{i1}, I_{i2}, \ldots, I_{ip}$ $(i = 1, 2, 3, \ldots, n)$, *respectively.*

Step 10. *Compute the forecasted value for Type-2 fuzzy time series model.*

If *Principle* 1 is applicable, then forecasted value for day $D(t)$ can be computed as:

$$Forecast_{D(t)} = \frac{Defuzz_{prev} + Defuzz_{curr}}{2}. \tag{15}$$

If *Principle* 2 is applicable, then forecasted value for day $D(t)$ can be computed as:

$$Forecast_{D(t)} = Defuzz_{prev}. \tag{16}$$

In this way, we obtain the forecasted values for the ∪ and ∩ operations individually based on the proposed model. To measure the performance of the model, mean absolute percentage error (MAPE) is used as an evaluation criterion. The MAPE can be defined as follows:

$$\text{MAPE} = \frac{1}{n} \sum_{i=1}^{n} \left| \frac{F_i - A_i}{A_i} \right| \times 100\%. \tag{17}$$

Here, F_i represents the forecasted value at time i, A_i represents the actual value at time i and n represents the total number of days to be forecasted. The MAPE value of the forecasted stock index price is presented in Table 10.

Based on the proposed model, we present here an example to compute the forecasted value of daily stock index price of SBI using the ∪ operation as follows:

Example. Suppose, we want to forecast the stock index price on day, $D(6/4/2012)$. To compute this value, obtain the fuzzified forecasting data for $D(6/4/2012)$ from Table 7, which is $\tilde{A}_9, \tilde{A}_{11}, \tilde{A}_{13} \rightarrow \tilde{A}_{12}, \tilde{A}_{18}, \tilde{A}_{19}$. Now, find the intervals where the maximum membership values for the previous state of FLRG (i.e., $\tilde{A}_9$, $\tilde{A}_{11}$ and $\tilde{A}_{13}$) occur from Table 2, which are (I_9, I_{11} and I_{13}), respectively. The corresponding mid-values and weights for the intervals (I_9, I_{11} and I_{13}) are (2064.30, 2085.90 and 2107.50) and (4, 8 and 3), respectively.

Similarly, find the intervals where the maximum membership values for the current state of FLRG (i.e., $\tilde{A}_{12}$, $\tilde{A}_{18}$ and $\tilde{A}_{19}$) occur from Table 2, which are (I_{12}, I_{18} and I_{19}), respectively. The corresponding mid-values and weights for the intervals (I_{12}, I_{18} and I_{19}) are (2096.70, 2161.40 and 2172.20) and (12, 9 and 8), respectively.

Now, obtain the defuzzified forecasting value for the previous state of the FLRG based on Eq. (12), which is equal to

$$Defuzz_{prev} = 2064.30 \times \left[\frac{4}{4+8+3} \right] + 2085.90 \times \left[\frac{8}{4+8+3} \right] + 2107.50 \times \left[\frac{3}{4+8+3} \right] = 2084.50.$$

Similarly, obtain the defuzzified forecasting value for the current state of the FLRG based on Eq. (13), which is equal to

$$Defuzz_{curr} = 2096.70 \times \left[\frac{12}{12+9+8} \right] + 2161.40 \times \left[\frac{9}{12+9+8} \right] + 2172.20 \times \left[\frac{8}{12+9+8} \right] = 2137.60.$$

Here, *Principle* 1 is applicable to compute the forecasted value, because the current state of the FLRG does not contain any unknown value. Therefore, based on Eq. (15), the forecasted value for $D(6/4/2012)$ is equal to

$$Forecast_{D(6/4/2012)} = \left[\frac{2084.50 + 2137.60}{2} \right] = 2111.10.$$

The forecasted results based on the proposed Type-2 fuzzy time series model are presented in Table 9. The results obtained by both ∪ and ∩ operations are further improved by hybridizing the PSO algorithm with the proposed Type-2 fuzzy time series model. This new hybridized forecasting model is presented in the next Section 7.

7. Improved hybridized forecasting model

The main downside of fuzzy time series forecasting model is that increase in the number of intervals increases accuracy rate of forecasting, and decreases the fuzziness of time series data sets [21]. Kuo et al. [40] show that appropriate selection of intervals also increases the forecasting accuracy of the model. Therefore, in order to get the optimal intervals, they used

Table 9
Forecasted results of the stock index price of SBI (in rupee).

Date (mm-dd-yy)	Actual price (in rupee)	Proposed model	
		(∪ operation)	(∩ operation)
6/4/2012	2079.40	2111.10	2102.90
6/5/2012	2130.60	2161.60	2125.70
6/6/2012	2174.10	2191.00	2173.20
6/7/2012	2163.50	2189.00	2156.80
6/10/2012	2184.60	2176.40	2160.60
6/11/2012	2176.40	2193.90	2166.50
6/12/2012	2216.50	2205.30	2188.00
6/13/2012	2181.30	2191.90	2172.40
...	...	...	...
7/22/2012	2103.90	2124.60	2101.80
7/23/2012	2096.30	2129.90	2093.40
7/24/2012	2078.10	2091.80	2079.30
7/25/2012	2046.30	2047.00	2047.00
7/26/2012	1997.70	2002.80	2002.80
7/29/2012	1991.00	1969.40	1969.40
MAPE		0.68%	0.66%

x_1	x_2	...	x_i	...	x_{n-1}

Fig. 3. The graphical representation of particle.

PSO algorithm in their proposed model [40]. Kuo et al. [40] signify that PSO algorithm is more efficient and powerful than genetic algorithm as applied by Chen and Chung [56] in selection of proper intervals. Therefore, to improve the forecasting accuracy of the proposed Type-2 model, we have hybridized the PSO algorithm with Algorithm 3. The main function of the PSO algorithm in Algorithm 3 is to adjust the length of intervals and membership values simultaneously, without increasing the number of intervals in the model. We have entitled this model as "FTS-PSO". The detailed description of the FTS-PSO model is presented next.

Let n be the number of intervals, x_0 and x_n be the lower and upper bounds of the universe of discourse U on historical time series data set $D(t)$, respectively. A particle is an array consisting of $n-1$ elements such as $x_1, x_2, \ldots, x_i, \ldots, x_{n-2}$ and x_{n-1}, where $1 \leqslant i \leqslant n-1$ and $x_{i-1} < x_i$. Now based on these $n-1$ elements, define the n intervals as $I_1 = (x_0, x_1]$, $I_2 = (x_1, x_2]$, ..., $I_i = (x_{i-1}, x_i]$, ..., $I_{n-1} = (x_{n-2}, x_{n-1}]$ and $I_n = (x_{n-1}, x_n]$, respectively. In case of movement of a particle from one position to another position, the elements of the corresponding new array always require to be adjusted in an ascending order such that $x_1 \leqslant x_2 \leqslant \cdots x_{n-1}$. The graphical representation of particle is shown in Fig. 3.

In this process, the FTS-PSO model allows the particles to move other positions based on Eqs. (8) and (9), and repeats the steps until the stopping criterion is satisfied or the optimal solution is found. If the stopping criterion is satisfied, then employ all the FLRs obtained by the global best position (*gbest*) among all personal best positions (*pbest*) of all particles. Here, the MAPE is used to evaluate the forecasted accuracy of a particle. The complete steps of the FTS-PSO model are presented in Algorithm 4.

Algorithm 4 FTS-PSO algorithm.

initialize all particles' positions and velocities
while the stopping criterion (the optimal solution is found or the maximal moving steps are reached) is not satisfied **do**
  for all particle *id* **do**
    define linguistic terms based on the current position of particle *id*
    fuzzify the time series data set of Type-1 and Type-2 observations according to the linguistic terms defined above
    establish the FLRs based on Definition 3
    construct the FLRGs based on Definition 4
    establish the relationships between FLRGs of both Type-1 and Type-2 observations, and mapped-out them to their corresponding day
      based on Definitions 8 and 9, apply ∪ and ∩ operators on mapped-out FLRGs of Type-1 and Type-2 observations, and obtain the fuzzified forecasting data
    defuzzify the forecasting data based on the "Frequency-Weighing Defuzzification Technique"
    calculate the forecasted values individually for ∪ and ∩ operations
    compute the MAPE value for particle *id* based on Eq. (17)
    update the *pbest* and *gbest* for particle *id* according to the MAPE value mentioned above
  end for
  for all particle *id* **do**
move particle *id* to another position according to Eqs. (8) and (9)
  end for
end while

Table 10
Randomly generated initial positions of all particles.

Particle	x_1	x_2	x_3	x_4	x_5	x_6	...	x_{25}	MAPE
1(∪)	1940.3	1951.1	1961.9	2015.8	2026.6	2037.4	...	2242.3	0.68%
1(∩)	1940.3	1951.1	1961.9	2015.8	2026.6	2037.4	...	2242.3	0.66%
2(∪)	1938.4	1949.2	1960.1	2014.3	2025.2	2036	...	2242.2	0.66%
2(∩)	1938.4	1949.2	1960.1	2014.3	2025.2	2036	...	2242.2	0.67%
3(∪)	1932.6	1943.6	1954.7	2009.9	2021.0	2032.0	...	2242.0	0.64%
3(∩)	1932.6	1943.6	1954.7	2009.9	2021.0	2032.0	...	2242.0	0.67%
4(∪)	1936.4	1947.3	1958.3	2012.8	2023.8	2034.7	...	2242.1	0.67%
4(∩)	1936.4	1947.3	1958.3	2012.8	2023.8	2034.7	...	2242.1	0.69%

Table 11
Randomly generated initial velocities of all particles.

Particle	x_1	x_2	x_3	x_4	x_5	x_6	...	x_{25}
1(∪)	2.04	3.70	1.37	3.71	0.01	4.60	...	2.17
1(∩)	2.04	3.70	1.37	3.71	0.01	4.60	...	2.17
2(∪)	4.45	0.15	4.78	0.70	2.63	2.59	...	4.38
2(∩)	4.45	0.15	4.78	0.70	2.63	2.59	...	4.38
3(∪)	2.60	2.45	2.68	2.84	0.43	2.15	...	2.93
3(∩)	2.60	2.45	2.68	2.84	0.43	2.15	...	2.93
4(∪)	3.19	0.07	4.33	1.25	1.69	1.57	...	2.01
4(∩)	3.19	0.07	4.33	1.25	1.69	1.57	...	2.01

Table 12
The initial *pbest* of all particles.

Particle	x_1	x_2	x_3	x_4	x_5	x_6	...	x_{25}	MAPE
1(∪)	1940.3	1951.1	1961.9	2015.8	2026.6	2037.4	...	2242.3	0.68%
1(∩)	1940.3	1951.1	1961.9	2015.8	2026.6	2037.4	...	2242.3	0.66%
2(∪)	1938.4	1949.2	1960.1	2014.3	2025.2	2036	...	2242.2	0.66%
2(∩)	1938.4	1949.2	1960.1	2014.3	2025.2	2036	...	2242.2	0.67%
3(∪)	1932.6	1943.6	1954.7	2009.9	2021.0	2032.0	...	2242.0	0.64%
3(∩)	1932.6	1943.6	1954.7	2009.9	2021.0	2032.0	...	2242.0	0.67%
4(∪)	1936.4	1947.3	1958.3	2012.8	2023.8	2034.7	...	2242.1	0.67%
4(∩)	1936.4	1947.3	1958.3	2012.8	2023.8	2034.7	...	2242.1	0.69%

The main difference between the existing models [40,28] and the FTS-PSO model is the procedure for handling the intervals based on their importance. The FTS-PSO model also incorporates more information in terms of observations, which are represented in terms of FLRs. These FLRs are later employed for defuzzification based on the technique discussed in Section 6. In the following, an example is presented to demonstrate the whole process of the FTS-PSO model.

Example. The FTS-PSO model employs the PSO to obtain the optimal FLRs of both Type-1 and Type-2 observations by adjusting the length of intervals for the historical data, $D(t)$, where $6/4/2012 \leqslant t \leqslant 7/29/2012$ (see Table 1). In Section 6, we have the universe of discourse $U = [1929.50, 2253.10]$, where lower bound $x_0 = 1929.50$ and upper bound $x_{26} = 2253.10$, respectively. On the universe of discourse, total 30 intervals are defined based on Eq. (10), but the historical data cover only 26 intervals. Therefore, for the representation of particles, we use these 26 intervals. For finding the optimal solution, we define 4 particles. Now, based on $U = [1929.50, 2253.10]$, we define values for the parameters used in Eqs. (8) and (9) as: (a) $CP_{id} = [1929.50, 2253.10]$, (b) $Vel_{id,t} = [-3, 3]$, (c) M_1 and $M_2 = 1.5$, and (d) $\alpha = 1.4$ (where α linearly decreases its value to the lower bound, 0.4, through the whole procedure) respectively. The positions and velocities of all particles are initialized randomly and shown in Tables 10 and Table 11, respectively.

In Table 10, we have shown the 26 intervals for each particle. For example, the intervals for the initial position of particle 1 are as: $I_1 = (1929.50, 1940.30]$, $I_2 = (1940.30, 1951.10]$, $\ldots$, $I_{26} = (2242.30, 2253.10]$, respectively. In Table 10, we consider those intervals for particle 1 that are used in fuzzification of the time series data set, presented in Section 6. In this process, we follow the steps of Algorithm 4 (modified version of Algorithm 3), and obtain the optimal FLRs which are employed for obtaining the forecasted results. The MAPE value for particle 1 is computed based on Eq. (17). The MAPE values for the remaining 3 particles are obtained in a similar manner. Based on the corresponding MAPE value, every particle updates its own *pbest*. For simplicity, the initial *pbests* are considered for the initial positions of all particles [40,28]. The *pbests* of all particles are shown in Table 12. In Table 12, the PG_{best} is obtained by particle 3 (for the ∪ operation) and particle 1 (for the ∩ operation).

Table 13
The second positions of all particles.

Particle	x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$	x_{25}	MAPE
1(∪)	1937.3	1948.1	1958.9	2012.8	2023.6	2034.4	$\cdots$	2239.3	0.65%
1(∩)	1937.3	1948.1	1958.9	2012.8	2023.6	2034.4	$\cdots$	2239.3	0.64%
2(∪)	1935.4	1946.2	1957.1	2011.3	2022.2	2033.0	$\cdots$	2239.2	0.64%
2(∩)	1935.4	1946.2	1957.1	2011.3	2022.2	2033.0	$\cdots$	2239.2	0.66%
3(∪)	1929.6	1940.6	1951.7	2006.9	2018.0	2029.0	$\cdots$	2239.0	0.67%
3(∩)	1929.6	1940.6	1951.7	2006.9	2018.0	2029.0	$\cdots$	2239.0	0.68%
4(∪)	1933.4	1944.3	1955.3	2009.8	2020.8	2031.7	$\cdots$	2239.1	0.63%
4(∩)	1933.4	1944.3	1955.3	2009.8	2020.8	2031.7	$\cdots$	2239.1	0.67%

Table 14
The second *pbest* of all particles.

Particle	x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$	x_{25}	MAPE
1(∪)	1937.3	1948.1	1958.9	2012.8	2023.6	2034.4	$\cdots$	2239.3	0.65%
1(∩)	1937.3	1948.1	1958.9	2012.8	2023.6	2034.4	$\cdots$	2239.3	0.64%
2(∪)	1935.4	1946.2	1957.1	2011.3	2022.2	2033.0	$\cdots$	2239.2	0.64%
2(∩)	1935.4	1946.2	1957.1	2011.3	2022.2	2033.0	$\cdots$	2239.2	0.66%
3(∪)	1929.6	1940.6	1951.7	2006.9	2018.0	2029.0	$\cdots$	2239.0	0.67%
3(∩)	1929.6	1940.6	1951.7	2006.9	2018.0	2029.0	$\cdots$	2239.0	0.68%
4(∪)	1933.4	1944.3	1955.3	2009.8	2020.8	2031.7	$\cdots$	2239.1	0.63%
4(∩)	1933.4	1944.3	1955.3	2009.8	2020.8	2031.7	$\cdots$	2239.1	0.67%

According to Algorithm 4, all particles move towards the second positions based on Eqs. (8) and (9). The second positions for all particles and their corresponding new MAPE values are presented in Table 13. For example, in Table 13, the second position of particle 1 (for the ∪ operation) is obtained by using Eqs. (18) and (19), which are based on Eqs. (8) and (9), respectively.

$$\begin{aligned} Vel_{1,1}(\cup) &= 1.4 \times 2.04 + 1.5 \times Rand \times (1940.3 - 1940.3) + 1.5 \times Rand \\ &\quad \times (1932.6 - 1940.3) = -3, \\ Vel_{1,2}(\cup) &= 1.4 \times 3.70 + 1.5 \times Rand \times (1951.1 - 1951.1) + 1.5 \times Rand \\ &\quad \times (1943.6 - 1951.1) = -3, \\ &\ \vdots \\ Vel_{1,25}(\cup) &= 1.4 \times 2.17 + 1.5 \times Rand \times (2242.3 - 2242.3) + 1.5 \times Rand \\ &\quad \times (2242.0 - 2242.3) = -3, \end{aligned} \tag{18}$$

$$\begin{aligned} CP_{1,1}(\cup) &= 1940.3 + Vel_{1,1}(\cup) = 1940.3 + (-3) = 1937.3, \\ CP_{1,2}(\cup) &= 1951.1 + Vel_{1,2}(\cup) = 1951.1 + (-3) = 1948.1, \\ &\ \vdots \\ CP_{1,25}(\cup) &= 2242.3 + Vel_{1,25}(\cup) = 2242.3 + (-3) = 2239.3. \end{aligned} \tag{19}$$

On comparison of the MAPE values between Tables 12 and 13, it is obvious that particle 1, particle 2 and particle 4 for the ∪ and ∩ operations attained their own *pbest* values so far in Table 13. Therefore, these three particles update their *pbest* values, which are shown in Table 14. In Table 14, the new PG_{best} value is obtained by particle 4 for the ∪ operation and particle 1 for the ∩ operation, because their MAPE values are least among all the particles so far.

The above steps are repeated by the FTS-PSO model until the stopping criterion is satisfied. After execution, a new set of optimal FLRs are obtained by the PG_{best} that the particle 4 (∪ operation) and particle 1 (∩ operation) attain so far, and are further employed for obtaining the final forecasting results.

8. Performance analysis parameters

The robustness of the proposed model is evaluated with the help of mean and standard deviations (SD) of the observed and forecasted values, correlation coefficient (R) and Theil's U statistic. All these parameters can be defined as follows:

Table 15
A comparison of the forecasted accuracy for the FTS-PSO model and the existing models based on the first-order FLRs.

Model	MAPE
Chen [4]	1.34%
Yu [58]	1.29%
FTS-PSO ($\cup$ operation)	0.63%
FTS-PSO ($\cap$ operation)	0.64%

1. The mean can be defined as:

$$\bar{A} = \frac{\sum_{i=1}^{n} A_i}{n}. \tag{20}$$

2. The SD can be defined as:

$$\mathrm{SD} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(A_i - \bar{A})^2}. \tag{21}$$

3. The R can be defined as:

$$R = \frac{n\sum A_i F_i - (\sum A_i)(\sum F_i)}{\sqrt{n(\sum A_i^2) - (\sum A_i)^2}\sqrt{n(\sum F_i^2) - (\sum F_i)^2}}. \tag{22}$$

4. The formula used to calculate Theil's U statistic [57] is:

$$U = \frac{\sqrt{\sum_{i=1}^{n}(A_i - F_i)^2}}{\sqrt{\sum_{i=1}^{n} A_i^2} + \sqrt{\sum_{i=1}^{n} F_i^2}}. \tag{23}$$

Here, each F_i and A_i is the forecasted and actual value of day i respectively, n is the total number of days to be forecasted. In Eqs. (20) and (21), $\{A_1, A_2, \ldots, A_n\}$ are the observed values of the actual time series data set and $\bar{A}$ is the mean value of these observations. Similarly, the mean and SD for the forecasted time series data set are computed. For a good forecasting, the observed means and SDs should be close to the forecasted means and SDs. In Eq. (22), the value of R is such that $-1 < R < +1$. The "+" and "−" indicate the positive linear correlations and negative linear correlations between the forecasted and actual value of time series data set, respectively. A correlation coefficient (R) greater than equal to 0.5 is generally considered the strong. In Eq. (23), U is bound between 0 and 1, with values closer to 0 indicating good forecasting accuracy.

9. Empirical analysis

To illustrate the forecasting performance of the proposed method, the daily stock index price of SBI and Google are used as data sets in verification and validation phases, respectively. The experimental results of the proposed model are compared with different existing models for various orders of FLRs and different intervals.

9.1. Stock index price forecasting of SBI

In this subsection, we present the forecasting results of the FTS-PSO model. The FTS-PSO model is validated using the stock index data set of SBI, as mentioned in Section 6. For forecasting the stock index price, "Open", "High" and "Low" variables are considered as the Type-2 observations, whereas "Close" variable is considered as the Type-1 observation. The "Actual Price" is chosen as the main forecasting objective. Further comparisons on the FTS-PSO model and the other existing models are discussed next.

The FTS-PSO model is trained simultaneously for the $\cup$ and $\cap$ operations, and the best results obtained by the particles are considered to forecast the stock index data. The necessary setting of all the parameters for the FTS-PSO model is discussed in Section 7.

The best forecasted accuracies (i.e., the least MAPE) are made by particle 4 (for the $\cup$ operation) and particle 1 (for the $\cap$ operation). Therefore, results obtained by these particles are used for the empirical analysis. The forecasted results for the $\cup$ and $\cap$ operations are presented in Table 15. In Table 15, the forecasted results for the existing fuzzy time series models [4,58] are also presented. The considered fuzzy time series models including the FTS-PSO model use the first-order FLRs to forecast the stock index data. From Table 15, it is obvious that the FTS-PSO model is more advantageous than the conventional fuzzy time series models [4,58].

Table 16
A comparison of the forecasted accuracy (in terms of MAPE) between the FTS-PSO model and the high-order model with different orders (the number of intervals = 9).

Model	Order						FTS-PSO	
	3	4	5	6	7	8	(∪ operation)	(∩ operation)
Chen [59]	1.39%	1.41%	1.40%	1.41%	1.40%	1.40%	1.07%	1.21%

Table 17
A comparison of the forecasted accuracy (in terms of the MAPE) for the FTS-PSO model based on the different number of intervals.

Model	Number of intervals						
	9	10	11	12	13	14	15
FTS-PSO (∪ operation)	1.07%	1.04%	1.20%	0.88%	0.89%	0.91%	0.82%
FTS-PSO (∩ operation)	1.21%	0.99%	1.22%	1.09%	1.06%	1.04%	1.00%

Table 18
A comparison of the forecasted accuracy (in terms of the MAPE) between the FTS-PSO model and the existing Type-2 model based on the different number of intervals.

Model	Number of intervals				
	16	17	18	19	20
Type-2 model [30]	1.15%	1.15%	1.18%	1.13%	1.13%
FTS-PSO (∪ operation)	0.88%	0.85%	0.79%	0.75%	0.72%
FTS-PSO (∩ operation)	0.91%	0.93%	0.80%	0.78%	0.81%

To verify the superiority of the proposed model under various high-order conditions, existing forecasting model, viz., Chen [59] model, is selected for comparison. A comparison of the forecasted results is shown in Table 16. During simulation, the number of intervals is kept fix (i.e., 9) for the existing model and the FTS-PSO model. At the same intervals, the MAPE values obtained for the existing model are 1.39%, 1.41%, 1.40%, 1.41%, 1.40% and 1.40% for third-order, fourth-order, fifth-order, sixth-order, seventh-order and eight-order FLRs, respectively. On the other hand, at the same intervals, the FTS-PSO model gets the least MAPE values which are 1.07% (for ∪ operation) and 1.21% (for ∩ operation). However, the smallest MAPE value, which is 1.07%, is obtained from the proposed model for the ∪ operation. We can see that the FTS-PSO model outperforms than the existing model under various high-order FLRs at all.

To verify the performance of the proposed model under different number of intervals, the forecasted results are obtained with different intervals ranging from 9 to 15. The forecasted results are listed in Table 17, where the proposed model uses first-order FLRs under different number of intervals. The least MAPE values are 0.82% (for ∪ operation) and 0.99% (for ∩ operation) for the intervals 15 and 10, respectively. However, between intervals 9 and 15, the best forecasted result is obtained from the ∪ operation at interval 15 (MAPE = 0.82%).

To evaluate the performance of the proposed model, it is compared with the existing Type-2 fuzzy time series model [30], under different number of intervals. A comparison of the forecasted results between the proposed model and the existing Type-2 model is shown in Table 18, where both these models use different intervals ranging from 16 to 20. For the existing model, the lowest forecasting error is 1.13%, which is obtained at intervals 19 and 20. For the proposed model, the least MAPE values are 0.72% (for ∪ operation) and 0.78% (for ∩ operation), which are obtained at intervals 20 and 19, respectively. From comparison, it is obvious that the proposed model produces more precise results than existing Type-2 model under different number of intervals.

To verify the superiority of the proposed model in terms of forecasted accuracy, three existing models, viz., Grey model [60], Back-propagation neural network model (BPNN) [61] (with one hidden layer and one output layer) and Hybridized model based on fuzzy time series and ANN [22], are selected for comparison. These three competing models are simulated using MATLAB (version 7.2.0.232 (R2006a)). During the learning process of the BPNN, a number of experiments were carried out to set additional parameters, viz., initial weight, learning rate, epochs, learning radius and activation function to obtain the optimal results, and we have chosen the ones that exhibit the best behavior in terms of accuracy. The determined optimal values of all these parameters are listed in Table 19. The forecasted results for these three models are obtained with different number of input values ranging from 5 to 10. A comparison of the forecasted results is presented in Table 20. The least MAPE values for the Grey model, BPNN model and Hybridized model are 1.05% (for input 5), 1.25% (for input 10) and 1.17% (for input 10), respectively. In our proposed model, the selection of input depends on the establishment of FLRs. For the proposed model, the least MAPE values are 0.63% (for ∪ operation) and 0.64% (for ∩ operation), which are obtained using the first-order FLRs. From comparison, we can see that the proposed model gets a higher forecasting accuracy than the existing three models, viz., Grey model, BPNN model and Hybridized model.

The empirical analysis shows that the proposed model is far better than the existing forecasting models for stock index data set of SBI.

Table 19
Additional parameters and their values during the learning process of the BPNN.

S. No.	Additional parameter	Input value
1	Initial weight	0.3
2	Learning rate	0.5
3	Epochs	10 000
4	Learning radius	3
5	Activation function	Sigmoid

Table 20
A comparison of the forecasted accuracy (in terms of MAPE) between the FTS-PSO model and the existing models (with different number of input values).

Model	Input						FTS-PSO	
	5	6	7	8	9	10	(∪ operation)	(∩ operation)
Grey model [60]	1.05%	1.66%	1.62%	1.95%	2.34%	2.69%	0.63%	0.64%
BPNN model [61]	1.33%	1.47%	1.53%	1.51%	1.58%	1.25%	–	–
Hybridized model [22]	1.23%	1.21%	1.21%	1.21%	1.25%	1.17%	–	–

Table 21
Daily stock index price list of Google (in USD).

Date (mm-dd-yy)	Open	High	Low	Close	Actual price
6/1/2012	571.79	572.65	568.35	570.98	570.94
6/4/2012	570.22	580.49	570.01	578.59	574.83
6/5/2012	575.45	578.13	566.47	570.41	572.62
6/6/2012	576.48	581.97	573.61	580.57	578.16
6/7/2012	587.60	587.89	577.25	578.23	582.74
6/8/2012	575.85	581.00	574.58	580.45	577.97
6/11/2012	584.21	585.32	566.69	568.50	576.18
...	...	...	...	...	...
7/27/2012	618.89	635.00	617.50	634.96	626.59

Table 22
Forecasted results of the stock index price of Google (in USD).

Date (mm-dd-yy)	Actual price	FTS-PSO (∪ operation)	FTS-PSO (∩ operation)
6/1/2012	570.94	574.32	568.01
6/4/2012	574.83	577.21	572.43
6/5/2012	572.62	576.28	569.92
6/6/2012	578.16	580.6	572.68
6/7/2012	582.74	585.32	579.19
6/8/2012	577.97	583.7	574.29
6/11/2012	576.18	574.38	570.32
...	...	...	...
7/27/2012	626.59	622.31	622.31

9.2. Stock index price forecasting of Google

In this subsection, the performance of the proposed model is evaluated with the stock index data set of Google. The data set of stock index price is covered from the period 6/1/2012–7/27/2012, which is shown in Table 21. Here, "Open", "High" and "Low" variables are selected as the Type-2 observations, whereas "Close" variable is selected as the Type-1 observation. The "Actual Price" is chosen as the main forecasting objective. The historical stock index data set of Google is collected from the website: http://in.finance.yahoo.com.

The FTS-PSO model optimizes the forecasting results and obtains the best results (i.e., the least MAPE). The forecasting results are shown in Table 22. The universe of discourse U is considered as $U = [555, 637]$, and is partitioned into 30 intervals. But, the historical data cover only 24 intervals, and the proposed model obtains the forecasted results using these 24 intervals. More detail results on partitions of the universe of the discourse and positions of the particle (i.e., best particle) among these 24 intervals are shown in Table 23.

For comparison studies, various statistical models listed in [15] are simulated using PASW Statistics 18 (http://www.spss.com.hk/statistics/). A comparison of the forecasted accuracy among the conventional statistical models and the proposed model is listed in Table 24. The comparative analysis clearly shows that the proposed model outperforms the considered models.

Table 23
Partitions of the universe of discourse and positions of the particle (i.e., best particle) among these intervals.

Model	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\cdots$	x_{23}	MAPE
FTS-PSO (∪ operation)	557.73	560.47	563.20	565.93	568.67	571.40	574.13	576.87	$\cdots$	620.60	0.6733%
FTS-PSO (∩ operation)	557.73	560.47	563.20	565.93	568.67	571.40	574.13	576.87	$\cdots$	620.60	0.4549%

Table 24
A comparison of the forecasted accuracy (in terms of MAPE) between the FTS-PSO model and the existing statistical models.

Model	MAPE
Logarithmic regression	1.9970%
Inverse regression	2.0472%
Quadratic regression	1.2683%
Cubic regression	1.2212%
Compound regression	1.7123%
Power regression	1.9891%
S-curve regression	2.0370%
Growth regression	1.7123%
Exponential regression	0.8590%
FTS-PSO (∪ operation)	0.6733%
FTS-PSO (∩ operation)	0.4549%

Table 25
Statistics of the FTS-PSO model for the stock index price forecasting of Google.

Statistics	Actual price	Forecasted price (∪ operation)	Forecasted price (∩ operation)
Mean (USD)	580.34	583.43	578.11
SD (USD)	16.12	14.99	16.08
R (USD)	–	0.98	0.99
U (USD)	–	0.0038	0.0028

Robustness. To check the robustness of the proposed model for forecasting the stock index price of Google, various statistical parameters values as mentioned in Section 8, are obtained. The experimental results are shown in Table 25. The values of parameters listed in Table 25 are based on the forecasted results presented in Table 22.

From Table 25, it is clear that the mean of actual price is very close to the mean of forecasted price. The comparison of the SD values between actual price and forecasted price show that predictive skill of our proposed model is good for both the ∪ and ∩ operations. The R values between actual and forecasted values also indicate the efficiency of the proposed model. The U values for both ∪ and ∩ operations are closer to 0, which indicate the effectiveness of the proposed model. Hence, the robustness of the proposed model is strongly convinced with the outstanding performance in case of daily stock index price data set of Google.

10. Conclusions and the way ahead

This paper presents a novel approach combining Type-2 fuzzy time series with the particle swarm optimization (PSO) for building a time series forecasting expert system. The main contributions of this article are presented as follows:

- *First*, the authors show that the problem of stock index price forecasting can be solved using Type-2 fuzzy time series concept. In this work, the authors demonstrate the application of the Type-2 model on 4-factors (i.e., “Low”, “Medium”, “High” and “Close”) time series data set of SBI.
- *Second*, the authors show the application of two fuzzy operators, viz., union and intersection, to establish the relationships among different fuzzy logical relations as obtained from the different observations, i.e., Type-1 and Type-2 observations.
- *Third*, the authors show that how the assignment of weights on intervals based on their frequencies and later utilization of these frequencies in defuzzification process, can improve the forecasting accuracy of the proposed model.
- *Fourth*, the authors show that the accuracy rate of the stock index price forecasting can be improved effectively by hybridizing the PSO algorithm with the Type-2 model.
- *Fifth*, the authors show that the forecasting accuracy of the FTS-PSO model is more precise than the existing fuzzy time series models.
- *Sixth*, the authors show the robustness of the FTS-PSO model by comparing its forecasting accuracy with various statistical models.

Still, there are scopes to apply the model in some other domains in a flexible way as follows:

1. To check the accuracy and performance of the model by forecasting the weather for different regions, and
2. To test the performance of the model for different types of financial, stocks and marketing data set.

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