Objectives

- Wrap up SVM
- Using $f(\alpha)$ to optimize
  - If optimized, we know w and b
- Try optimizing using **direct method**, but this will present an issue
- Try optimizing with **indirect methods** using gradient descent based methods
- Try optimizing with **backpropagation**
  - Backpropagation is the crux of DL (deep learning)
  - Deep learning provides a solution for non-linear data

Optimizing with $f(\alpha)$

Continuing from the previous lecture, we have the equation:

$$f(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j x_i^T x_j \tag{1}$$

The last term in this expression, $x_i^T x_j$, can be separated from the rest of the expression using **kernelization**. This process adds dimensionality that allows non-linearly separable data to be separated by a plane. For now, we will avoid using this method, but it's good to know.

Using purely algebra, we can simplify equation 1 a bit. Let's take a closer look at the second term in our equation. This is what it looks like if we have a vector with three components:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j y_i y_j x_i x_j = \alpha_1 \alpha_1 y_1 y_1 x_1 x_1 + \alpha_1 \alpha_2 y_1 y_2 x_1 x_2 + ... + \alpha_3 \alpha_3 y_3 y_3 x_3 x_3$$

Let's look at the Lagrange multipliers by themselves for this demonstration:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j y_i y_j x_i x_j = \begin{pmatrix} \alpha_1\alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_3 \\ +\alpha_2\alpha_1 + \alpha_2\alpha_2 + \alpha_2\alpha_3 \\ +\alpha_3\alpha_1 + \alpha_3\alpha_2 + \alpha_3\alpha_3 \end{pmatrix}$$

Still using just algebra, we can rearrange and combine terms to rewrite each as α^2. This pattern also applies to the y and x constants. Therefore, equation 1 can be rewritten as:

$$f(\alpha) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \alpha_i^2 y_i^2 x_i^T x_j \tag{2}$$

Let's review a couple definitions before we continue. When we think of α, we should think of it as a vector with components like such:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where most components are equal to zero. With this definition, we can rewrite $f(\alpha)$:

$$f(\alpha) = f\left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}\right)$$

For some component in α, let's call it α_k, we can write $f(\alpha_k)$:

$$f(\alpha_k) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2}\alpha_k^2 y_k^2 x_k^T x_k - \alpha_k y_k \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i y_i x_i^T x_k \tag{3}$$

We do this reformatting because equation 3 will be easier to partially derive than equation 1. Now, our goal is to find $\nabla f'(\alpha) = \frac{\partial f}{\partial \alpha_k}$. Let's rewrite this equation to get something we can use. We are able to separate out a term with all of our variables with subscript k because our summation specifies $i \neq k$ and jumps over α_k, y_k and x_k. Also, when we take the derivative of α^2, we get 2α which cancels out the factor of $\frac{1}{2}$:

$$\nabla f(\alpha_k) = 1 - \alpha_k y_k^2 x_k^T x_k - y_k \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i y_i x_i^T x_k$$

We can take out a common factor of y_k:

$$\nabla f(\alpha_k) = 1 - y_k[\sum_{i=1}^{n} \alpha_i y_i x_i^T x_k]$$

Now, put y_k back into our summation like so:

$$\nabla f(\alpha_k) = 1 - \sum_{i=1}^{n} \alpha_i y_i y_k x_i^T x_k$$

With this definition of $\nabla f(\alpha_k)$, we can rewrite this into vector format where $[1 - \sum_{i=1}^{n} \alpha_i y_i y_k x_i^T x_k]$ is the kth component in $\nabla f(\alpha)$. Then, to take a gradient descent, we set our equation equal to $\vec{0}$. Now we have:

$$\nabla f(\alpha) = \begin{bmatrix} 1 - \sum_{i=1}^{n} \alpha_i y_i y_1 x_i^T x_1 \\ \vdots \\ 1 - \sum_{i=1}^{n} \alpha_i y_i y_k x_i^T x_k \\ \vdots \\ 1 - \sum_{i=1}^{n} \alpha_i y_i y_n x_i^T x_n \end{bmatrix} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

Let's rework this slightly by adding the second term to all components in both vectors; this gives us:

$$\nabla f(\alpha) = \begin{bmatrix} \sum_{i=1}^{n} \alpha_i y_i y_1 x_i^T x_1 \\ \vdots \\ \sum_{i=1}^{n} \alpha_i y_i y_n x_i^T x_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Note that x_i and y_i are known, and α_i is not. We can rewrite these vectors as a matrix of known x and y values multiplied by $\vec{\alpha}$:

$$\begin{bmatrix} y_1 y_1 x_1^T x_1 & y_1 y_2 x_1^T x_2 & \dots & y_1 y_n x_1^T x_n \\ y_2 y_1 x_2^T x_1 & y_2 y_2 x_2^T x_2 & \dots & y_2 y_n x_2^T x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_k y_1 x_k^T x_1 & \dots & \dots & y_k y_n x_k^T x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n y_1 x_n^T x_1 & y_n y_2 x_n^T x_2 & \dots & y_n y_n x_n^T x_n \end{bmatrix} * \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{5}$$

where the first matrix is n x n and both vectors are n x 1. When we do this matrix multiplication, we multiply each row by $\vec{\alpha}$. This is what we get when we multiply the first row of the matrix by $\vec{\alpha}$:

$$\sum_{i=1}^{n} y_1 y_i x_1^T x_i \cdot \alpha_i$$

We can rewrite this abstractly as:

$$Q\alpha = 1 \implies \alpha = Q^{-1} \cdot 1$$

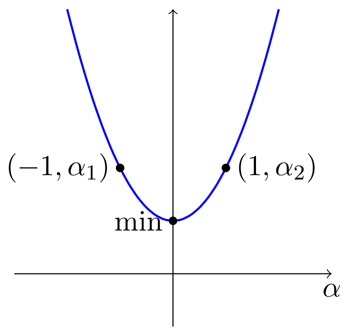
But here's our issue: we don't know what Q is. Additionally, taking the inverse of a matrix is a difficult calculation that's in $O(n^3)$ and the results of such an operation are numerically unstable and untrustworthy. Therefore, this **direct solution** only works theoretically. Let's move on to try the **indirect** or **iterative method**.

The iterative method is defined by the **update rule**:

$$\alpha_{new} = \alpha_{old} - \eta \nabla L(\alpha_{old}) \tag{6}$$

where L is our loss function and η is a hyper-parameter representing the learning rate, often set to 0.1 or 0.01. Recall that our goal is always to minimize loss for maximum accuracy.

The second term, $[\eta \nabla L(\alpha_{old})]$, is subtracted from the equation because we want to step in the direction that will *minimize* the loss function. Let's see an example of a convex quadratic loss function:



The goal is to move closer to the function's minimum, marked clearly above, with each iteration of α_x. The step size will decrease as we go because the gradient approaches 0; we subtract less and less each step as our slope flattens.

Above, we are working with vanilla gradient descent. This can work on a simple function like x^2; however, when we get more complex, wiggly functions, a simple gradient descent like vanilla can get stuck in local minimums.

To get around this, we can use a more robust gradient descent like ADAM. In the next lecture, we will learn more about using gradient descents.