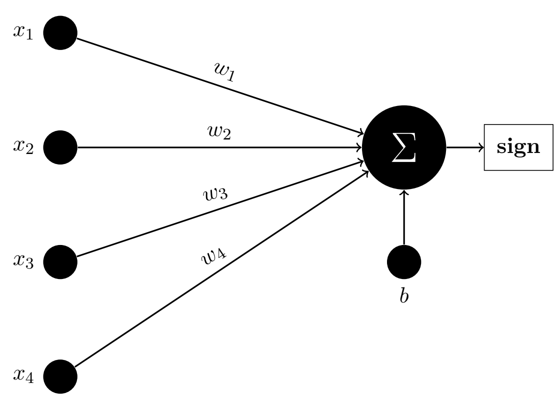
Objectives

- The Perceptron
- Hyperplane
- Perceptron Algorithm
- Definitions and Assumptions for model
- Classifier equation
- Class Code

1 The Perceptron

A perceptron is one of the simplest types of artificial neural network and is a foundational concept in machine learning. It is particularly used for binary classification tasks. It was introduced by Frank Rosenblatt in 1958, one of the earliest neural network models designed to mimic the way the human brain processes information.

The perceptron takes multiple inputs (features), each associated with a weight, and computes a weighted sum. This sum is then passed through an activation function, which produces the final output. To visualize this, consider a perceptron with four inputs x_1, x_2, x_3, x_4, each with corresponding weights w_1, w_2, w_3, w_4, and a bias term b as shown below.



To make the perceptron useful for real-world classification tasks, it must learn the appropriate weights and bias through training. The training process involves adjusting these parameters based on labeled examples.

A training data set consists of multiple examples, where each input x_i is a p-dimensional vector representing features, and each corresponding label y_i is either -1 or 1 (for binary classification). Formally, we define the training data as follows:

$$\text{Training Data} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

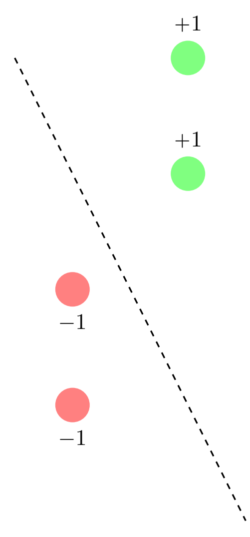
where:

- $x_i \in \mathbb{R}^p$ represents a feature vector with dimensions p.
- $y_i \in \{-1, 1\}$ is the class label that indicates the input category.

This data set helps update the weights of the perceptron using a learning algorithm, ensuring that the model can make accurate predictions on unseen data.

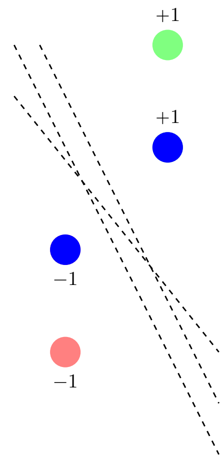
2 Hyperplane

A hyperplane is a decision boundary that separates data points into different classes. In binary classification, the hyperplane divides the space into two regions: one for positive class (+1) and one for negative class (-1). The hyperplane is represented by a slanted decision boundary that separates the two classes as shown below.



Hyperplane separates +1 and -1

When data points are linearly separable, there can be infinitely many hyperplanes that separate the points. These hyperplanes differ in their orientation and position, but all of them separate the points correctly. The ideal hyperplane is the one that maximizes the margin between the two classes.



2.1 For Different Dimensions

- **1D** : The hyperplane is just a single point.
- **2D** : The hyperplane is a line that separates the 2D plane.
- **3D** : The hyperplane is a plane that separates the 3D space.

2.2 Mathematical Definition of a Hyperplane

A hyperplane is a set of vectors $\mathcal{H}$ such that:

$$\mathcal{H} = \{\mathbf{x} \mid \mathbf{w}^T\mathbf{x} + b = 0\}$$

Where: $\mathbf{x} \in \mathbb{R}^p$ is a vector in p-dimensional space, $\mathbf{w}$ is the normal vector to the hyperplane, b is the bias term.

The equation represents all the points $\mathbf{x}$ that lie on the hyperplane.

Thus, there are infinitely many possible hyperplanes that can separate the data points, but the **maximum margin hyperplane** is the one that best generalizes the decision boundary and is selected by techniques such as Support Vector Machines (SVMs).

3 Bundle & Save

In the hyperplane equation:

$$w^T x + b = 0$$

we can bundle the bias term b as part of the input feature vector by augmenting it with a constant value. Specifically, we define the augmented feature vector:

$$x' = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and the extended weight vector:

$$w' = \begin{bmatrix} w \\ b \end{bmatrix}$$

Now the hyperplane equation becomes:

$$w'^T x' = 0$$

4 Assumptions for the Perceptron Model

Data is Linearly Separable For the perceptron model to work, the data must be **linearly separable**. This means that there exists a linear hyperplane that can perfectly separate the data points belonging to different classes.

4.1 What is Linearly Separable?

For a dataset with inputs x_i and corresponding labels y_i, where $y_i \in \{-1, 1\}$, the data is said to be linearly separable if there exists a hyperplane represented by the equation $w^T x + b = 0$ such that for each data point:

$$y_i(w^T x_i + b) > 0$$

If $y_i = +1$ (positive class), then $w^T x_i + b > 0$. If $y_i = -1$ (negative class), then $w^T x_i + b < 0$.

5 Perceptron Algorithm

The goal of training a perceptron is to find the optimal weight vector $\mathbf{w}$ and bias b such that the perceptron can correctly classify the input data.

The perceptron makes predictions using the following equation:

$$\hat{y} = \text{sign}(\mathbf{w}^T \mathbf{x} + b)$$

Where: $\mathbf{w}$ is the weight vector, $\mathbf{x}$ is the input feature vector, b is the bias term, and $\hat{y}$ is the predicted label (+1 or -1).

5.1 Goal of Training

The objective of perceptron training is to find the optimal weight vector $\mathbf{w}$ and bias b that minimize classification errors on the training set.

6 Exercise: Check if $\mathbf{u} = [3, -2, 1]^T$ belongs to the hyperplane

We are given:

$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \quad b = 4, \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

We need to check if $\mathbf{u}$ lies on the hyperplane defined by the equation:

$$\mathbf{w}^T\mathbf{u} + b = 0$$

Substitute the values into the equation:

$$\mathbf{w}^T\mathbf{u} + b = 1(3) + 2(-2) + (-3)(1) + 4$$

$$= 3 - 4 - 3 + 4 = 0$$

Since the result is 0, this means that $\mathbf{u} = [3, -2, 1]^T$ lies on the hyperplane $\mathcal{H}$.

6.1 Why is w Perpendicular?

- For any point $\mathbf{x}_0$ on the hyperplane, it satisfies the equation $\mathbf{w}^T\mathbf{x}_0 + b = 0$.
- Now, if you take any vector $\mathbf{v}$ that lies on the hyperplane (i.e., any vector $\mathbf{v}$ such that the line formed by $\mathbf{x}_0 + \mathbf{v}$ also lies on the hyperplane), the following condition must hold:

  $$\mathbf{w}^T(\mathbf{x}_0 + \mathbf{v}) + b = 0$$

  Since $\mathbf{x}_0$ lies on the hyperplane, $\mathbf{w}^T\mathbf{x}_0 + b = 0$. Therefore, the equation simplifies to:

  $$\mathbf{w}^T\mathbf{v} = 0$$

  This means that $\mathbf{w}$ is perpendicular to $\mathbf{v}$, as $\mathbf{v}$ lies on the hyperplane. Hence, $\mathbf{w}$ is perpendicular to every vector on the hyperplane.

7 Classifier Equation

The general form of the classifier is:

$$h(x) = \text{sign}(\mathbf{w}^T\mathbf{x} + b)$$

Where: $\mathbf{x}$ is the input feature vector, $\mathbf{w}$ is the weight vector, b is the bias term, $\text{sign}(\cdot)$ is the sign function, which outputs $+1$ if the input is positive and -1 if the input is negative.

- **Algorithm Finds $\mathbf{w}$ and b Based on Training Data:** During the training phase, the algorithm adjusts the weight vector $\mathbf{w}$ and bias b based on the labeled training data. The goal is to find values for $\mathbf{w}$ and b that minimize classification errors and effectively separate the data points into their respective classes. The training process involves optimizing the parameters based on the data's features and labels.

- **Classifying New Data:** For new, unseen data, the classifier uses the learned values of $\mathbf{w}$ and b to make a prediction. The new data point $\mathbf{x}$ is classified by evaluating the equation:

$$h(x) = \text{sign}(\mathbf{w}^T\mathbf{x} + b)$$

  - If $\mathbf{w}^T\mathbf{x} + b > 0$, then the classifier predicts class $+1$.
  - If $\mathbf{w}^T\mathbf{x} + b < 0$, then the classifier predicts class -1.

8 Class Code(Perceptron Algorithm Steps)

- **Input:** A dataset $D = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$, where x_i is the feature vector and $y_i \in \{-1, 1\}$ is the label for each data point.
- **Output:** The learned weight vector w and bias b.
- **Initialize:**

$$w = \mathbf{0} \quad \text{(weight vector initialized to zero)}, \quad b = 0 \quad \text{(bias initialized to zero)}.$$

- **Algorithm Steps:**
  1. Set $m = 1$ (initially assuming there are misclassifications).
  2. While $m > 0$ (until no misclassifications):
     (a). Set $m = 0$ (assume no misclassifications for this iteration).
     (b). For each data point $i = 1$ to n:
       - If $y_i(w^T x_i + b) \leqslant 0$ (the point is misclassified):
       - Update the weight vector:

$$w = w + y_i x_i$$

       - Increment the misclassification count: $m = m + 1$