Limits and Continuity

Introduction

1. What are indeterminate forms?
2. What are left hand and right hand limits? What is the necessary condition for a function to exist the limit at a point?
3. Define limit of a function at a point.
4. What is the necessary condition for a function to be continuous at a point?
5. Define continuity of a function at a point.

Short answer questions I

1. A real valued function f is defined by $f(x) = |x-1| = \begin{cases} x-1 & \text{for } x \geq 1 \\ 1-x & \text{for } x < 1 \end{cases}$, does limit of f(x) exists at x = 1?

   (Ans: Yes)

2. A function f(x) is defined as follows: $f(x) = \begin{cases} 2px + 3 & \text{if } x < 1 \\ 1 - px^2 & \text{if } x > 1 \end{cases}$.

   Find the value of p so that f(x) is continuous at x = 1. ***(Ans: p = -2/3)***

3. A function f(x) is defined as follows:

   $$f(x) = \begin{cases} \dfrac{2x^2 - 18}{x - 3} & \text{for } x \neq 3 \\ k & \text{for } x = 3 \end{cases}$$

   Find the value of k so that f(x) is continuous at x = 3. ***(Ans: k = 12)***

4. Find the point of continuous and discontinuous of the following functions.

   a) $f(x) = \dfrac{(x+2)}{(x-1)(x+4)}$ ***(Ans: x = 1, – 4)***

Short answer questions II

1. If $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x \to a} \dfrac{x^3-a^3}{x^2-a^2}$ find a. ***(Ans: $a = \frac{4}{3}$)***

   b) $f(x) = \dfrac{x^2-4}{x-2}$

2. Evaluate the following limits:

   a) $\lim\limits_{x \to 2} \dfrac{(x+2)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x-2}$ ***Ans : $\frac{5}{2}(a+2)^{\frac{3}{2}}$***

   b) $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{x^2+3}-2}$ ***Ans : 2***

   c) $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2+x}\right)$ ***Ans : $\frac{1}{2}$***

d) $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{3}} - (1-x)^{\frac{1}{3}}}{x}$ *Ans* : $\frac{1}{3}$

e) $\lim_{x \to \infty} \frac{5-2x^2}{8x^2+13}$ *Ans* : $-\frac{1}{4}$

f) $\lim_{x \to \infty} \frac{5x+8x^2}{9x^3+12x^2+13}$ *Ans : 0*

g) $\lim_{k \to \infty} \frac{1+2+----+k}{k^2}$ *Ans* : $\frac{1}{2}$

h) $\lim_{k \to \infty} \frac{1^2+2^2+----+k^2}{k^3}$ *Ans* : $\frac{1}{3}$

3. Evaluate following limits.

a) $\lim_{x \to 0} \frac{\tan 3x - x}{5x - \sin x}$ *Ans* : $\frac{1}{2}$

b) $\lim_{x \to \pi} \frac{1-\sin\frac{x}{2}}{(\pi-x)^2}$ *Ans*: $\frac{1}{8}$

c) $\lim_{x \to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2}$ *Ans* : $\frac{1}{2}$

d) $\lim_{x \to y} \frac{\tan y - \tan x}{y-x}$ *Ans* : $\sec^2 y$

e) $\lim_{y \to x} \frac{\sec y - \sec x}{y-x}$ *Ans* : $\sec x\tan x$

f) $\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{\frac{\pi}{4}-x}$ *Ans : 2*

g) $\lim_{x \to c} \frac{\sqrt{x}-\sqrt{c}}{\cos x - \cos c}$ *Ans* : $-\frac{1}{2\sqrt{c}} \operatorname{cosec} c$

h) $\lim_{x \to \theta} \frac{x \operatorname{cosec}\theta - \theta \operatorname{cosec} x}{x-\theta}$

Ans: $\operatorname{cosec}\theta + \theta \operatorname{cosec}\theta \cot\theta$

4. Evaluate following limits.

a) $\lim_{x \to 0} \frac{e^{3x}-1}{e^{5x}-1}$ *Ans* : $\frac{3}{5}$

b) $\lim_{x \to 0} \frac{b^x-1}{a^x-1}$ *Ans* : $\frac{\ln b}{\ln a}$

Continuity (Short questions)

1. Discuss the type of discontinuities of following function with the help of graph.

   $$f(x) = \left.\begin{matrix} x & \text{for } x > 0 \\ 1 & \text{for } x = 0 \\ 2x & \text{for } x < 0 \end{matrix}\right\} \text{at } x = 0.$$

2. Define jump discontinuity.

   Show that $f(x) = \begin{cases} x-2 & \text{for } x \geq 1 \\ 3-2x & \text{for } x < 1 \end{cases}$ has jump discontinuity at x = 1, graphically.

3. Define infinite discontinuity. Show that $y=\frac{1}{x-2}$ has infinite discontinuity at x=2 with the help of graph.

4. Consider a function $f(x) = \frac{x^2-25}{x-5}$. Show that f(5) does not exists. Discuss the type of discontinuity at x = 5. Is it possible to make f(x) continuous at the specified point?

5. A function f(x) is defined as follows: $f(x) = \begin{cases} 2x+1 & \text{for } x < 1 \\ 2 & \text{for } x = 1 \\ 3x & \text{for } x > 1 \end{cases}$

   Calculate the left hand limit and the right hand limit of f(x) at x = 1. Is the function continuous at x = 1? If not, state the type of discontinuity at the point.

6. Evaluate, $\lim\limits_{x\to1^-} f(x)$ and $\lim\limits_{x\to1^+} f(x)$ for $f(x)=\frac{|x-1|}{x-1}$. Also examine the point of continuity and discontinuity for the function.

7. a) Show that $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ geometrically, where θ is measured in radian.

   b) Show that $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$

Multiple Choice Questions (MCQ's)

1. The value of $\lim\limits_{x \to 0} \frac{x^2+2x}{x}$ is

   a) $\frac{0}{0}$ b) 2

   c) ∞ d) 1

2. The value of $\lim\limits_{x \to 0} \frac{e^{\frac{7}{2}x}-1}{x}$ is

   a) $\frac{0}{0}$ b) 1

   c) $\frac{7}{2}$ d) 0

3. $\lim\limits_{x \to 0} \frac{\sin 6x}{\tan 9x}$ is equal to

a) $\frac{2}{3}$ b) 0

c) 1 d) undefined

4. $\lim_{x \to \frac{\pi}{2}^+} \tan x$ is

a) 0 b) 1

c) ∞ d) $-\infty$

5. $\lim_{x \to 2} f(x) = 2$ is equal to

a) 1 b) 2

c) 3 d) Not possible

6. $\lim_{x \to 0} \frac{\sin x}{x}$ is equal to

a) 1 b) $\frac{0}{0}$

c) 0 d) does not exist

7. $\lim_{x \to \infty} \sin\frac{1}{x}$ is equal to

a) 1 b) does not exist

c) $\frac{0}{0}$ d) 0

8. $\lim_{x \to 0} \frac{\sin 2x}{e^x - 1}$ is equal to

a) 0 b) $\frac{0}{0}$

c) 2 d) does not exist

9. $\lim_{x \to 5} \frac{x^n - 5^n}{x - 5} = 500$, then n is

a) $\frac{0}{0}$ b) 0

c) 4 d) does not exist

10. The value of $\lim_{x \to 0} \frac{\sin x^0}{x}$ is equal to

a) $\frac{0}{0}$ b) 0

c) $\frac{180}{\pi}$ d) $\frac{\pi}{180}$

11. $\lim_{x \to a} (a - x)\frac{\tan \pi x}{2a}$ is

a) $\frac{\pi}{2a}$ b) $\frac{2a}{\pi}$

c) 0 d) undefined

12. For what value of k, the following function is continuous at x=π,

$$f(x)=\begin{cases} kx+1 & ,\text{if } x \le \pi \\ \cos x & ,\text{if } x > \pi \end{cases}$$

a) 1 b) π

c) $\frac{2}{\pi}$ d) $-\frac{2}{\pi}$

13. $f(x)=\frac{x+2}{x^3-6x^2+5x}$ is discontinuous at

a) x=0 b) x=1

c) x=5 d) x=0,1,5

14. $\lim\limits_{x\to\infty} \sin\frac{1}{x}$ is equal to

a) 1 b) [-1,1]

c) $\frac{0}{0}$ d) 0

15. $\lim\limits_{x\to 5}\frac{x^n-5^n}{x-5}=500$, then n is

a) $\frac{0}{0}$ b) 0

c) 4 d) does not exist

16. The value of $\lim\limits_{x\to 0}\frac{\sin x^0}{x}$ is equal to

a) $\frac{0}{0}$ b) 0

c) $\frac{180}{\pi}$ d) $\frac{\pi}{180}$

17. For what value of k, the following function is continuous at x=π,

$$f(x)=\begin{cases} kx+1 & ,\text{if } x \le \pi \\ \cos x & ,\text{if } x > \pi \end{cases}$$

a) 1 b) π

c) $\frac{2}{\pi}$ d) $-\frac{2}{\pi}$