# B.Tech. Sem : III (CE-MU)

## **Subject Name:** Probability and Statistics

Marwadi University

Department of CE

Unit no:- 2

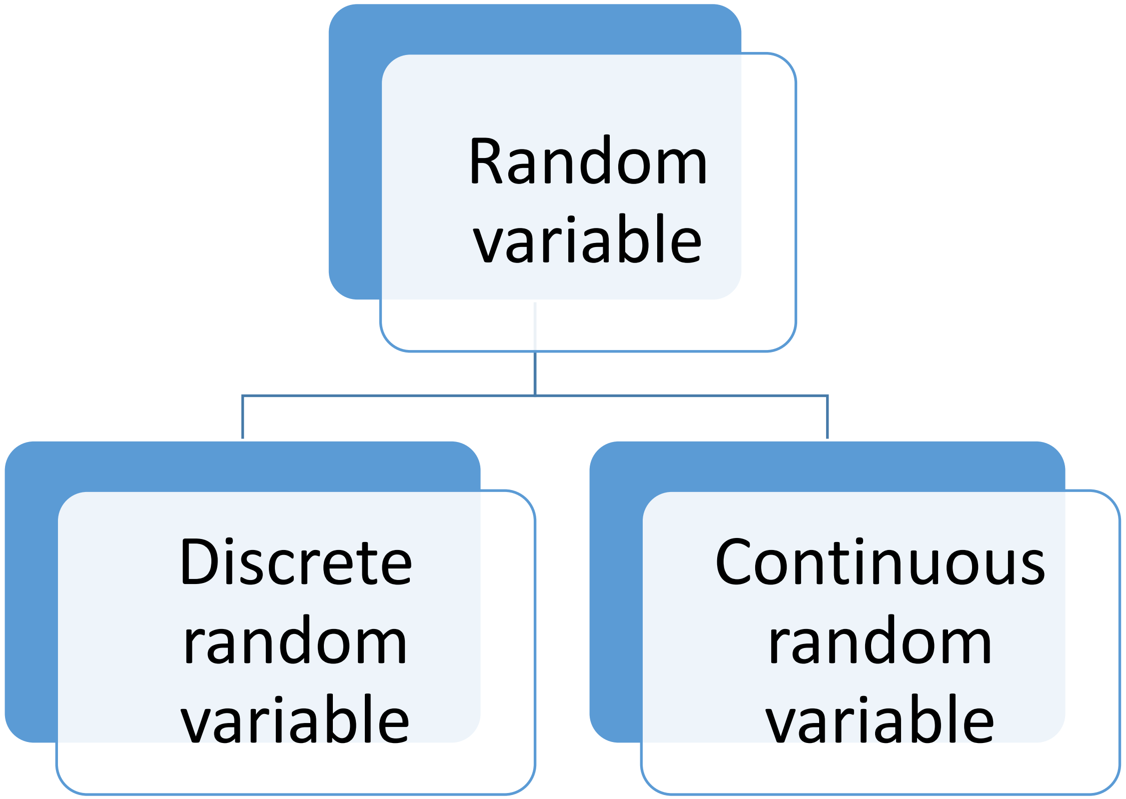
Unit title:

Random variable and probability distributions

Subject name : PS

# Random variable

A random variable usually written as X, is a variable whose possible values are numerical outcomes of a random experiment.



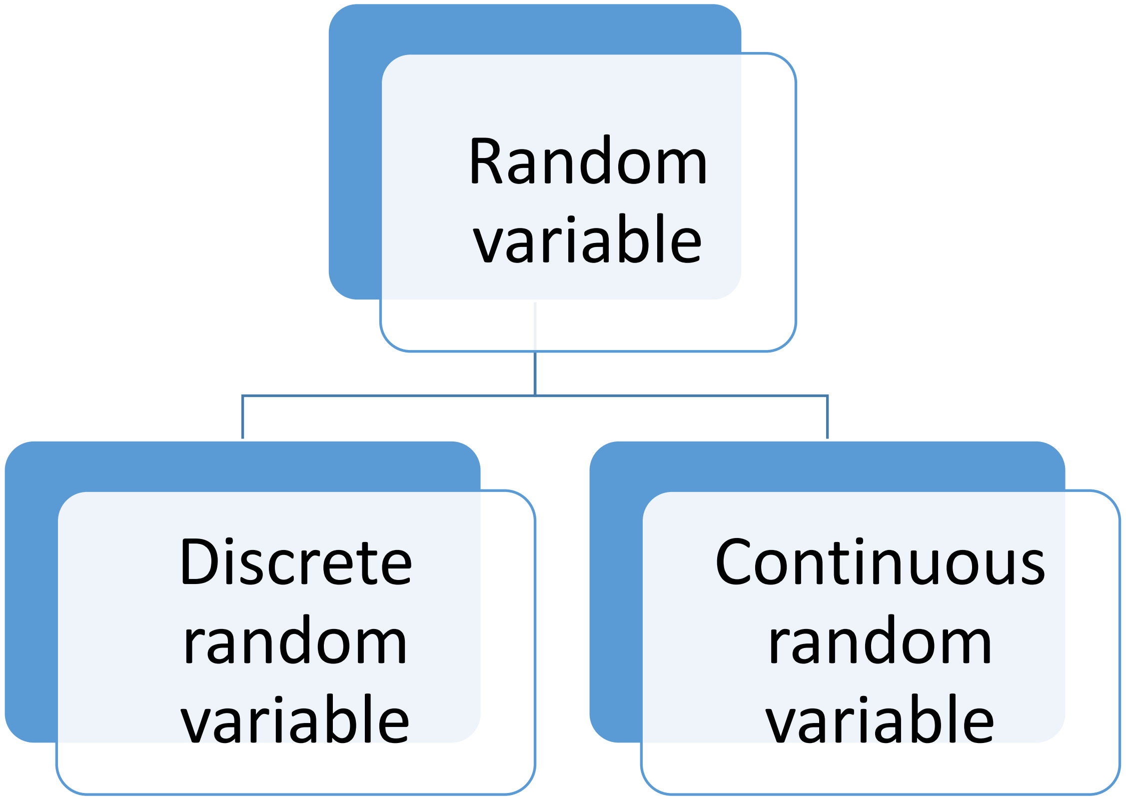

**Discrete** random variable is a variable which can take only countable number of distinct values.

Discrete random variables are usually (not necessarily) counts.

Examples: Dead/alive, number on a die, Children in a family, No. of defective items in a box, etc.

# Random variable

A random variable usually written as X, is a variable whose possible values are numerical outcomes of a random experiment.




**Continuous** random variable is one which takes an infinite number of possible values in a given interval.

Continuous random variables are usually measurements.

Examples: Height, Weight, blood pressure, real numbers between 1 to 5 etc.

# Probability distribution

A graph, table or formula that specifies all possible values that a discrete random variable can take along with associated probabilities then it is called probability distribution.

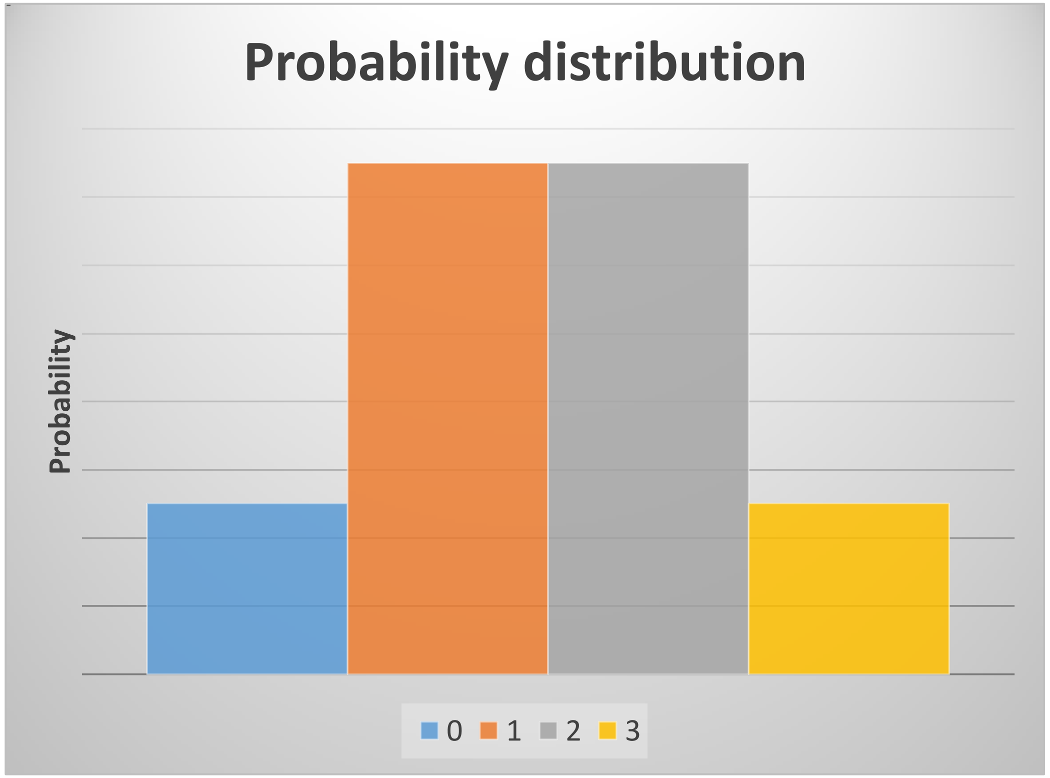
Consider an example of tossing a fair coin 3 times.

Define X: the number of heads obtained

| | |
|---|---|
| X = 0 | TTT |
| X = 1 | HTT, THT, TTH |
| X = 2 | HHT, HTH, THH |
| X = 3 | HHH |

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X) | 1/8 | 3/8 | 3/8 | 1/8 |

Probability distribution



# Probability function

Let X be a random variable then a probability function f(X) maps the possible values of *X* against their respective probabilities of occurrence, *p(X)* .

If f(X) is probability function then

1. $0 \leq f(X_i) \leq 1$, i = 0, 1, 2, $\cdots, n$
2. $\sum_{i=1}^{n} f(X_i) = 1$

Probability function is called probability mass function (pmf) in case of discrete probability distributions and probability density functions (pdf) in case of continuous probability distributions.

# Probability function

Again consider example of tossing a fair coin 3 times.
Define X: the number of heads obtained

| | |
|---|---|
| X = 0 | TTT |
| X = 1 | HTT, THT, TTH |
| X = 2 | HHT, HTH, THH |
| X = 3 | HHH |

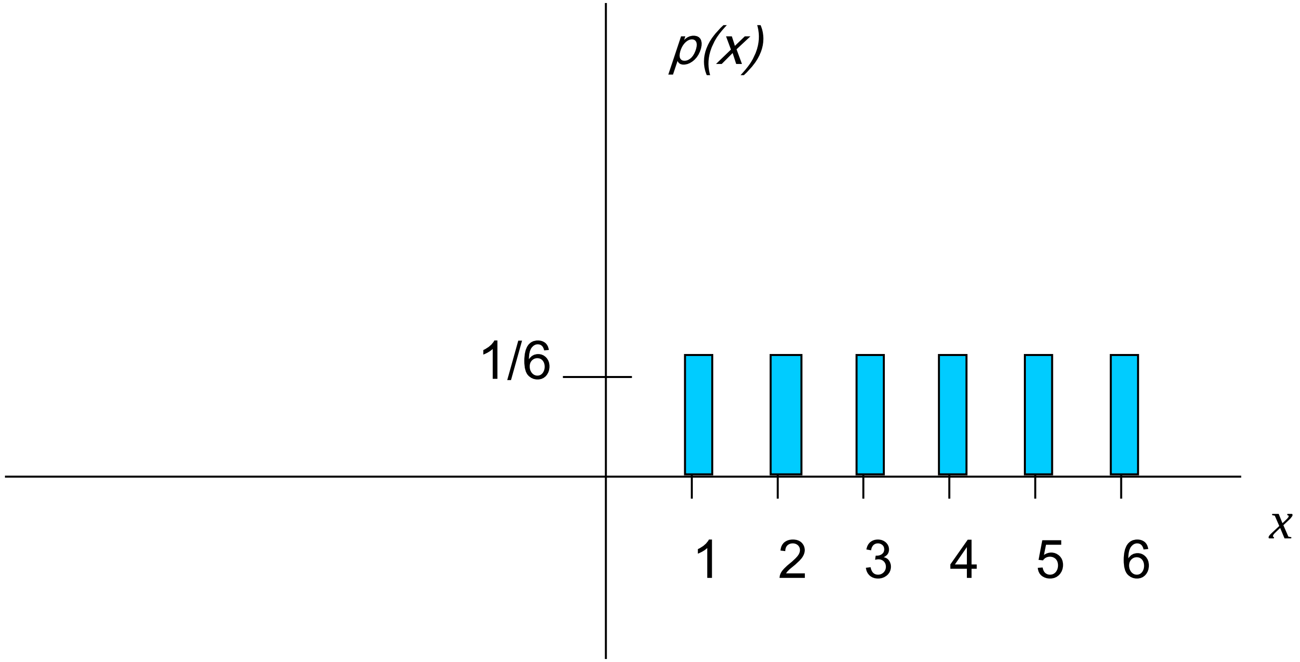
Here $0 \leq P(X) \leq 1$ for every X
and $\sum P(X) = 1$

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X) | 1/8 | 3/8 | 3/8 | 1/8 |

P(X) is Probability function

# Probability function

Consider example of rolling a die.




$$\sum_{\text{all } x} P(x) = 1$$

## Example 1: Check whether $f(X) = X/6$, $X = 0, 1, 2, 3$ is probability mass function or not.

Here, $f(X) = X/6$ ; $X = 0,1,2,3$

For $X=0$, $f(0)= 0$
For $X=1$, $f(1) = 1/6$
For $X=2$, $f(2)= 2/6$
For $X=3$, $f(3) = 3/6$

Here $0 \le f(X) \le 1$ for every X.

Also $\Sigma f(X) = 0 + 1/6 + 2/6 + 3/6 = 1$

Thus, f(X) is a probability mass function.

# Example 2: Check whether function defined below is probability mass function or not.

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| f(X=x) | 0.1 | 0.4 | 0.2 | 0.3 |

Here $0 \leq f(X) \leq 1$ for every X.

Also $\Sigma f(X) = 0.1 + 0.4 + 0.2 + 0.3 = 1$

Thus, f(X) is a probability mass function.

Example 3: A discrete random variable X has following probability distribution.

| X | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(X=x)$ | 0 | k | 0.2 | 2k | 0.3 | 2k |

then find (1) k (2) $P(X<3)$ (3) $P(X\geq 3)$ (4) $P(2\leq X\leq 5)$

Solution:

Here we know that f(X) is a PMF.

Therefore, $\Sigma f(X) = 0 + k + 0.2 + 2k + 0.3 + 2k = 1$

$$5k + 0.5 = 1$$

$$k = 0.1$$

| X | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(X=x) | 0 | 0.1 | 0.2 | 0.2 | 0.3 | 0.2 |

Solution:

(2) $P(X<3) = f(0) + f(1) + f(2) = 0 + 0.1 + 0.2 = 0.3$

(3) $P(X \geq 3) = f(3) + f(4) + f(5) = 0.2 + 0.3 + 0.2 = 0.7$

(4) $P(2 \leq X \leq 5) = f(2) + f(3) + f(4) + f(5) = 0.2 + 0.2 + 0.3 + 0.2 = 0.9$

# Probability density function

- Let X be a continuous random variable, then function f(X) of random variable X is called probability density function if

  $(1) 0 \leq f(x) \leq 1, \; -\infty < x < \infty$

  $(2) \int_{-\infty}^{\infty} f(x)dx = 1$

- Note: $\mathrm{P}(a \leq x \leq b) = \int_{a}^{b} f(x)dx$

# Cumulative distribution function(CDF)

## CDF for Discrete random variable

**Def:** Let X be a discrete random variable which takes the values x1, x2, x3, …. such that x1 < x2 < x3 < ….. then the cumulative distribution function F(x) is defined as

$$F(x_n) = P(X \leq x_n) = \sum_{i=1}^{n} p(x_i)$$
$$= p(x_1) + p(x_2) + \cdots + p(x_n)$$

**Properties**

**(1)** $F(x_n) = p(x_1) + p(x_2) + \cdots + p(x_n)$

**(2)** $\sum_{i=1}^{n} p(x_i) = 1$

**(3)** $0 \leq F(x_i) \leq 1, \quad i = 1,2, \dots, n$

**(4)** $\mathrm{P}(a < X \leq b) = F(b) - F(a)$

## CDF for Continuous random variable

**Def:** Let X be a continuous random variable then the cumulative distribution function F(x) is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

**Properties**

**(1)** $F(-\infty) = \int_{-\infty}^{-\infty} f(t)dt = 0$

**(2)** $F(\infty) = \int_{-\infty}^{\infty} f(t)dt = 1$

**(3)** $0 \leq F(x) \leq 1, \quad -\infty < x < \infty$

**(4)** $\mathrm{P}(a < X \leq b) = F(b) - F(a)$

**Examples 4:** Check whether the following function f(x) is a probability density function or not. If so, find the probability that the variable having this density falls in the interval [1, 2].

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

***Solution:***

$$f(x) \geq 0 \quad \text{in} \quad (0, \infty)$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \left| -e^{-x} \right|_{0}^{\infty}$$

$$= -e^{-\infty} + 1$$

$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(1 \leq X \leq 2) = \int_{1}^{2} f(x)\,dx$$

$$= \int_{1}^{2} e^{-x}\,dx$$

$$= \left| -e^{-x} \right|_{1}^{2}$$

$$= -e^{-2} + e^{-1}$$

$$= 0.233$$

**Examples 5:** For the following PDF of a R.V. X, find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 and 0.2, and (ii) greater than 0.5.

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

***Solution:***

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} k(1-x^2)\,dx + 0 = 1$$

$$k\left|x - \frac{x^3}{3}\right|_0^1 = 1$$

$$k\left(1 - \frac{1}{3}\right) = 1$$

$$k = \frac{3}{2}$$

Hence, $f(x) = \frac{3}{2}(1-x^2) \quad 0 < x < 1$

$= 0 \quad$ otherwise

**Examples 5:** For the following PDF of a R.V. X, find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 and 0.2, and (ii) greater than 0.5.

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

***Solution:***

(i) Probability that the variable will take on a value between 0.1 and 0.2

$$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)\,dx$$

$$= \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= 0.1465$$

**Examples 5:** For the following PDF of a R.V. X, find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 and 0.2, and (ii) greater than 0.5.

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

***Solution:***

(ii) Probability that the variable will take on a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} \frac{3}{2}(1-x^2)\,dx + 0$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1-\frac{1}{3}\right)-\left(0.5-\frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

**Examples 6:** A continuous random variable X has following PDF $f(x)$. Find a and b such that (i) $P(X \leq a) = P(X > a)$ and (ii) $P(X > b) = 0.05$. where, $0 < a, b < 1$.

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

***Solution:***

Since total probability is 1 and $P(X \leq a) = P(X > a)$,

$$P(X \leq a) = \frac{1}{2}$$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2}$$

$$3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$P(X > b) = 0.05$$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$3\left|\frac{x^3}{3}\right|_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = \frac{19}{20}$$

$$b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$$

**Examples 7:** A discrete random variable X takes the values –3, –2, –1, 0, 1, 2, 3, such that P(X = 0) = P(X > 0) = P(X < 0) and P(X = –3) = P(X = –2) = P(X = –1) = P(X = 1) = P(X = 2) = P(X = 3). Obtain the probability distribution and the cumulative distribution function of X.

***Solution:***

Let $P(X=0) = P(X>0) = P(X<0) = k_1$

Since $\sum P(X=x) = 1$

$$k_1 + k_1 + k_1 = 1$$

$$\therefore \quad k_1 = \frac{1}{3}$$

$$P(X=0) = P(X>0) = P(X<0) = \frac{1}{3}$$

Let $P(X=1) = P(X=2) = P(X=3) = k_2$

$$P(X>0) = P(X=1) + P(X=2) + P(X=3)$$

$$\frac{1}{3} = k_2 + k_2 + k_2$$

$$\therefore \quad k_2 = \frac{1}{9}$$

$$P(X=1) = P(X=2) = P(X=3) = \frac{1}{9}$$

Similarly, $P(X=-3) = P(X=-2) = P(X=-1) = \dfrac{1}{9}$

**Examples 7:** A discrete random variable X takes the values –3, –2, –1, 0, 1, 2, 3, such that $P(X = 0) = P(X > 0) = P(X < 0)$ and $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$. Obtain the probability distribution and the cumulative distribution function of X.

***Solution:***

Probability distribution and distribution function

| X | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| P(X = x) | 1/9 | 1/9 | 1/9 | 1/3 | 1/9 | 1/9 | 1/9 |
| F(x) | 1/9 | 2/9 | 3/9 | 6/9 | 7/9 | 8/9 | 1 |

**Examples 8:** Find the constant k such that the following function f(x) is a PDF. Also, find the cumulative distribution function F(x) and P(1 < X ≤ 2).

$$f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & otherwise \end{cases}$$

***Solution:***

Since $f(x)$ is probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{3} kx^2\,dx + 0 = 1$$

$$k\left|\frac{x^3}{3}\right|_0^3 = 1$$

$$k(9-0) = 1$$

$$k = \frac{1}{9}$$

Hence, $f(x) = \frac{1}{9}x^2 \quad 0 < x < 3$

$= 0 \quad$ otherwise

**Examples 8:** Find the constant k such that the following function f(x) is a PDF. Also, find the cumulative distribution function F(x) and P(1 < X ≤ 2).

$$f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & otherwise \end{cases}$$

***Solution:***

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x \frac{1}{9} x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_0^x = \frac{1}{27}x^3$$

Hence, $F(x) = \frac{1}{27}x^3 \quad 0 < x < 3$

$= 0 \quad$ otherwise

$$P(1 < x \le 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$= \frac{1}{27}(8-1) = \frac{7}{27}$$

# Mean or Arithmetic mean or Mathematical expectation

## Mean for Discrete random variable

**Def:** Let p(x) be the probability mass function then the mean or average value (μ) of a discrete random variable X is called as expectation and is denoted by E(X)

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i p(x_i) = \sum xp(x)$$

If $\emptyset(x)$ is a function of discrete random variable X then the expectation of $\emptyset(x)$ is given by

$$E(\emptyset(x)) = \sum_{i=1}^{\infty} \emptyset(x_i)p(x_i) = \sum \emptyset(x)p(x)$$

## Mean for Continuous random variable

**Def:** Let p(x) be the probability density function then the mean or average value (μ) of a continuous random variable X is called as expectation and is denoted by E(X)

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

If $\emptyset(x)$ is a function of continuous random variable X then the expectation of $\emptyset(x)$ is given by

$$E(\emptyset(x)) = \int_{-\infty}^{\infty} \emptyset(x)f(x)dx$$

# Properties of Mean

**(1)** $\boldsymbol{E(k) = k}$

**(2)** $\boldsymbol{E(kX) = kE(X)}$

**(3)** $\boldsymbol{E(aX \pm b) = aE(X) \pm b}$

# Variance

## Variance for Discrete random variable

**Def:** The variance of the probability distribution of a discrete random variable X is given by

$$\begin{aligned}\mathrm{Var(X)} = \sigma^2 &= E(X-\mu)^2 \\ &= E(X^2 - 2X\mu + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu\mu + \mu^2 \\ &= E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2 \\ &= \sum_{i=1}^{\infty} {x_i}^2 p(x_i) - \mu^2\end{aligned}$$

Thus, $\mathrm{Var(X)} = \sigma^2 = E(X-\mu)^2$

$$= \sum_{i=1}^{\infty} {x_i}^2 p(x_i) - \mu^2$$

## Variance for Continuous random variable

**Def:** The variance of the probability distribution of a continuous random variable X is given by

$$\begin{aligned}\mathrm{Var(X)} = \sigma^2 &= E(X-\mu)^2 \\ &= E(X^2) - \mu^2 \\ &\int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2\end{aligned}$$

# Properties of Variance

(1) $Var(k) = 0$

(2) $Var(X + k) = Var(X)$

(3) $Var(kX) = k^2 Var(X)$

(4) $Var(aX \pm b) = a^2 Var(X)$

# Standard deviation

## Standard deviation for Discrete R.V.

**Def:** The standard deviation of the probability distribution of a discrete random variable X is given by

$$\mathrm{SD} = \sqrt{\mathrm{Var(X)}} = \sigma$$

$$= \sqrt{\sum_{i=1}^{\infty} {x_i}^2 p(x_i) - \mu^2}$$

$$= \sqrt{E(X^2) - \mu^2}$$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

## Standard deviation for Continuous R.V.

**Def:** The standard deviation of the probability distribution of a continuous random variable X is given by

$$\mathrm{SD} = \sqrt{\mathrm{Var(X)}} = \sigma$$

$$= \sqrt{\int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2}$$

$$= \sqrt{E(X^2) - \mu^2}$$

$$= \sqrt{E(X^2) - [E(X)]^2}$$

**Examples 9:** The probability distribution of a random variable X is given below. Find (i) E(X), (ii) Var(X), (iii) E(2X–3) and (iv) Var (2X – 3)

| X | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| P(X=x) | 0.2 | 0.1 | 0.3 | 0.3 | 0.1 |

***Solution:***

(i) E(X) = $\sum xp(x)$

= (-2)(0.2)+(-1)(0.1)+0+(1)(0.3)+ (2)(0.1)

= - 0.4 - 0.1 + 0 + 0.3 + 0.2 = 0

(ii) $\mathrm{Var}(X)$

= $\mathrm{E}(X^2) - [E(X)]^2$

= $\sum x^2 p(x) - [E(X)]^2$

= (4)(0.2)+(1)(0.1)+0+(1)(0.3)+ (4)(0.1) - 0

= 1.6

**Examples 9:** The probability distribution of a random variable X is given below. Find (i) E(X), (ii) Var(X), (iii) E(2X–3) and (iv) Var (2X – 3)

| X | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| P(X=x) | 0.2 | 0.1 | 0.3 | 0.3 | 0.1 |

***Solution:***

(iii) E(2X-3) = 2E(X) - 3

= 2(0) – 3

= -3

(iv) $\mathrm{Var}(2X-3)$ = $(2^2)\mathrm{Var}(X)$

= 4(1.6)

= 6.4

**Examples 10:** The monthly demand of a product is known to have the following probability distribution. Find the expected demand for the product. Also, compute the variance.

| Demand (x) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Probability p(x) | 0.08 | 0.12 | 0.19 | 0.24 | 0.16 | 0.1 | 0.07 | 0.04 |

***Solution:***

(i) E(X) = $\sum xp(x)$

= (1)(0.08)+(2)(0.12)+(3)(0.19)+(4)(0.24) + (5)(0.16)+(6)(0.10)+(7)(0.07)+(8)(0.04)

= 0.08 + 0.24 + 0.57 + 0.96 + 0.80 + 0.60 + 0.49 + 0.32

= 4.06

(ii) $\mathrm{Var}(\mathrm{X})$

$= \mathrm{E}(X^2) - [E(X)]^2$

$= \sum x^2 p(x) - [E(X)]^2$

= (1)(0.08)+(4)(0.12)+(9)(0.19)+(16)(0.24) + (25)(0.16)+(36)(0.10)+(49)(0.07) +(64)(0.04) $-(4.06)^2$

= 3.21

**Examples 11:** A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items.

***<u>Solution:</u>***

Let X = number of defective items.
Total number of items = 10
Number of good items = 6
Number of defective items = 4

P(X=0) = P(No defective item)

$$= \frac{C(6,3)}{C(10,3)} = \frac{1}{6}$$

P(X=1) = P(One defective item)

$$= \frac{C(6,2) \cdot C(4,1)}{C(10,3)} = \frac{1}{2}$$

P(X=2) = P(Two defective item)

$$= \frac{C(6,1) \cdot C(4,2)}{C(10,3)} = \frac{3}{10}$$

P(X=3) = P(Three defective item)

$$= \frac{C(4,3)}{C(10,3)} = \frac{1}{30}$$

Therefore, probability distribution is

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(X=x) | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{30}$ |

**Examples 11:** A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items.

***Solution:***

Therefore, expected number of defective items = E(X) = $\sum xp(x)$

$$= 0 + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{3}{10}\right) + (3)\left(\frac{1}{30}\right)$$

= 1.2

**Examples 12:** If the density function of a random variable X is given as below then find (i) value of k (ii) Expectation of X, (iii) Variance, (iv) SD.

$$f(x) = kx(1-x), \qquad 0 < x < 1$$
$$= \quad 0, \qquad \text{otherwise}$$

***Solution:***

Since f(x) is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{1} kx(1-x)\,dx = 1$$

$$k\int_{0}^{1} (x-x^2)\,dx = 1$$

$$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$k\left(\frac{1}{6}\right) = 1 \Rightarrow k = 6$$

(ii) $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$

$$= \int_{0}^{1} x \cdot 6x(1-x)\,dx$$

$$= 6\int_{0}^{1} (x^2 - x^3)\,dx$$

**Examples 12:** If the density function of a random variable X is given as below then find (i) value of k (ii) Expectation of X, (iii) Variance, (iv) SD.

$$f(x) = kx(1-x), \qquad 0 < x < 1$$
$$= \quad 0, \qquad \text{otherwise}$$

***Solution:***

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} = \mu$$

(iii) $Var(x) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$$= \int_0^1 x^2 \cdot 6x(1-x)\, dx - \mu^2$$

$$= 6\int_0^1 (x^3 - x^4)\, dx - \mu^2$$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 - \mu^2$$

$$= 6\left(\frac{1}{20}\right) - \left(\frac{1}{4}\right) = \frac{1}{20}$$

(iv) SD $= \sqrt{Var(x)} = \sqrt{\frac{1}{20}} = \frac{1}{2\sqrt{5}}$

**Examples 13:** Let X be a random variable with E(X) = 10 and Var (X) = 25. Find the positive values of a and b such that Y = aX – b has an expectation of 0 and a variance of 1.

***Solution:***

Here we have, $\mathrm{Y} = (aX - b)$

$$E(Y) = E(aX - b)$$

$$0 = aE(X) - b$$

$$0 = a(10) - b$$

$\therefore 10a - b = 0$ ......Eqn. (1)

$$Var(Y) = Var(aX - b)$$

$$1 = a^2 Var(X)$$

$1 = a^2(25)$ ......Eqn. (2)

On solving Eqn. (1) and Eqn. (2), we have

$$\therefore a = 1/5, \qquad b = 2$$

# Moments

Moments represent a convenient and unifying method for summarizing many of the most commonly used statistical measures such as measures of tendency, variation, skewness and kurtosis.

Types of Moments:
There are two types of moments we calculate

(i) Moments about mean or central moments
(ii) Moments about arbitrary point or raw moments

# Central moment or moment about mean

## Central moment for Discrete random variable

**Def:** The moments about the mean value $\mu = E(X)$ are called central moments and denoted by $\mu_r$ and is defined as

$$\mu_r = E[(x-\mu)^r]$$

$$= \sum_{i=1}^{\infty} (x_i - \mu)^r p(x_i) = \sum (x-\mu)^r p(x)$$

## Central moment for Continuous random variable

**Def:** Central moments or moments about actual mean of the probability distribution of a continuous random variable X is given by

$$\mu_r = E[(x-\mu)^r]$$

$$= \int_{-\infty}^{\infty} \left((x-\mu)^r\right) f(x)dx$$

# Properties of central moments

(i) The zero central moment $\mu_0$ is one i.e. $\mu_0 = 1$

(ii) The first central moment $\mu_1$ is always zero i.e. $\mu_1 = 0$

(iii) The second central moment $\mu_2 = \sigma^2$ or $\sigma = \pm\sqrt{\mu_2}$

(iv) In a symmetric distribution, the odd moments are always zero i.e. $\mu_1 = \mu_3 = \mu_5 = \cdots = 0$

# Raw moment or moment about arbitrary mean

## Raw moment for Discrete random variable

**Def:** The moments about the arbitrary origin are known as raw moments and are denoted by $\mu_r{}'$ and is defined as

$$\mu_r{}' = E[(x-a)^r]$$

$$= \sum_{i=1}^{\infty} (x_i - a)^r p(x_i) = \sum (x-a)^r p(x)$$

## Raw moment for Continuous random variable

**Def:** Central moments or moments about actual mean of the probability distribution of a continuous random variable X is given by

$$\mu_r{}' = E[(x-a)^r]$$

$$= \int_{-\infty}^{\infty} \left((x-a)^r\right) f(x)\,dx$$

# Properties of raw moments

(i) If a = 0 then $\mu_r{}' = E[(x)^r]$ is called simple moment.

(ii) Relation between central moments and raw moments is as below.

$$\mu_r = \mu_r{}' + \sum_{i=1}^{r} (-1)^i \binom{r}{i} (\mu_1{}')^i (\mu_{r-i}{}')$$

$$\mu_r = \mu_r{}' - \binom{r}{1}(\mu_1{}')^1(\mu_{r-1}{}') + \binom{r}{2}(\mu_1{}')^2(\mu_{r-2}{}') - \binom{r}{3}(\mu_1{}')^3(\mu_{r-3}{}') + \cdots.$$

In particular with r = 1,2,3,4 and $\mu_0{}' = 1$

- For r = 1, $\mu_1 = \mu_1{}' - \mu_1{}' = 0$
- For r = 2, $\mu_2 = \mu_2{}' - (\mu_1{}')^2$
- For r = 3, $\mu_3 = \mu_3{}' - 3\,\mu_1{}'\mu_2{}' + 2(\mu_1{}')^3$
- For r = 4, $\mu_4 = \mu_4{}' - 4\,\mu_1{}'\mu_3{}' + 6\mu_2{}'(\mu_1{}')^2 - 3(\mu_1{}')^4$ and so on.

# Properties of raw moments

Similarly, the raw moments can be expressed in terms of central moment.

$$\mu_r{}' = \mu_r + \sum_{i=1}^{r} \binom{r}{i} (\mu_1{}')^i (\mu_{r-i})$$

$$\mu_r{}' = \mu_r + \binom{r}{1}(\mu_1{}')^1(\mu_{r-1}) + \binom{r}{2}(\mu_1{}')^2(\mu_{r-2}) + \binom{r}{3}(\mu_1{}')^3(\mu_{r-3}) + \cdots.$$

In particular with r = 2,3,4 and $\mu_0 = 1$ and $\mu_1 = 0$

- For r = 2, $\mu_2{}' = \mu_2 + (\mu_1{}')^2$
- For r = 3, $\mu_3{}' = \mu_3 + 3\,\mu_1{}'\mu_2 + (\mu_1{}')^3$
- For r = 4, $\mu_4{}' = \mu_4 + 4\,\mu_1{}'\mu_3 + 6(\mu_1{}')^2\mu_2 + (\mu_1{}')^4$

**Examples 14:** The first four moments of distribution about x = 2 are 1, 2.5, 5.5, and 16. Calculate the four moments about mean $\mu$.

***Solution:***

Here,

$$a = 2, \quad \mu_1' = 1, \quad \mu_2' = 2.5, \quad \mu_3' = 5.5, \quad \mu_4' = 16$$

Moments about mean,

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 2.5 - (1)^2 = 1.5$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= 5.5 - 3(2.5)(1) + 2(1)^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 16 - 4(5.5)(1) + 6(2.5)(1)^2 - 3(1)^4 = 6$$

**Examples 15:** Calculate the first 4 central moments for the following probability distribution:

| X | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| p(X) | 0.008 | 0.032 | 0.142 | 0.216 | 0.240 | 0.206 | 0.143 | 0.013 |

***Solution:***

(i) E(X) = $\sum xp(x)$

= (1)(0.008)+(2)(0.032)+(3)(0.142)+(4)(0.216) + (5)(0.240)+(6)(0.206)+(7)(0.143)+(8)(0.013)

= 4.903

Now to calculate Central moments $\mu_1,\ \mu_2, \mu_3, \mu_4$ it is more convenient to calculate raw moments about a = 0 i.e. $\mu_1',\ \mu_2', \mu_3', \mu_4'$ and then use them to calculate central moments.

**Examples 16:** Calculate the first 4 central moments for the following probability distribution:

| X | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| p(X) | 0.008 | 0.032 | 0.142 | 0.216 | 0.240 | 0.206 | 0.143 | 0.013 |

***Solution:***

$\mu_1{}' = \mu = 4.903$

$$\begin{aligned}\mu_2{}' = E(X^2) &= 1(0.008) + 4(0.032) + 9(0.142) + 16(0.216) \\ &\quad +25(0.240) + 36(0.206) + 49(0.143) + 64(0.013) \\ &= 26.125\end{aligned}$$

$$\begin{aligned}\mu_3{}' = E(X^3) &= 1(0.008) + 8(0.032) + 27(0.142) + 64(0.216) \\ &\quad +125(0.240) + 216(0.206) + 343(0.143) + 512(0.013) \\ &= 148.12\end{aligned}$$

$$\begin{aligned}\mu_4{}' = E(X^4) &= 1(0.008) + 16(0.032) + 81(0.142) + 256(0.216) \\ &\quad +625(0.240) + 1296(0.206) + 2401(0.143) + 4096(0.013) \\ &= 880.885\end{aligned}$$

**Examples 16:** Calculate the first 4 central moments for the following probability distribution:

| X | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| p(X) | 0.008 | 0.032 | 0.142 | 0.216 | 0.240 | 0.206 | 0.143 | 0.013 |

***Solution:***

Here,

$a = 0, \quad \mu_1' = 4.903, \qquad \mu_2' = 26.125,$

$\mu_3' = 148.12, \qquad \mu_4' = 880.885$

Moments about mean,

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2 = 26.125 - (4.903)^2 = 2.086$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= 148.12 - 3(26.125)(4.903) + 2(4.903)^3$

$= 148.12 - 384.27 + 235.73 = -0.42$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 880.86 - 4(148.12)(4.903) + 6(26.125)(4.903)^2 - 3(4.903)^4$

$= 10.39$

**Examples 17:** Calculate the first 4 central moments for the following probability distribution:

$f(x) = \frac{1}{2}x^2e^{-x}, \quad 0 < x < \infty$

***Solution:***

Now to calculate Central moments $\mu_1, \mu_2, \mu_3, \mu_4$ it is more convenient to calculate raw moments about a = 0 i.e. $\mu_1', \mu_2', \mu_3', \mu_4'$ and then use them to calculate central moments.

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x)dx$$

$$\mu_r' = \int_0^\infty x^r \frac{1}{2}x^2e^{-x}dx$$

$$\mu_r' = \frac{1}{2}\int_0^\infty x^{r+2}e^{-x}dx \quad = \frac{1}{2}\gamma(r+3) = (r+2)!$$

Recall: Gamma function

$$\gamma(n+1) = \int_0^\infty e^{-x}x^n dx$$

$$\gamma(n+1) = n!$$

**Examples 17:** Calculate the first 4 central moments for the following probability distribution:

$f(x) = \frac{1}{2}x^2e^{-x}, \quad 0 < x < \infty$

***Solution:***

Thus,

$$\mu_r' = \frac{1}{2}(r+2)!$$

$$\mu_1' = \frac{1}{2}(1+2)! = 3$$

$$\mu_2' = \frac{1}{2}(2+2)! = 12$$

$$\mu_3' = \frac{1}{2}(3+2)! = 60$$

$$\mu_4' = \frac{1}{2}(4+2)! = 360$$

Moments about mean,

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 12 - (3)^2 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

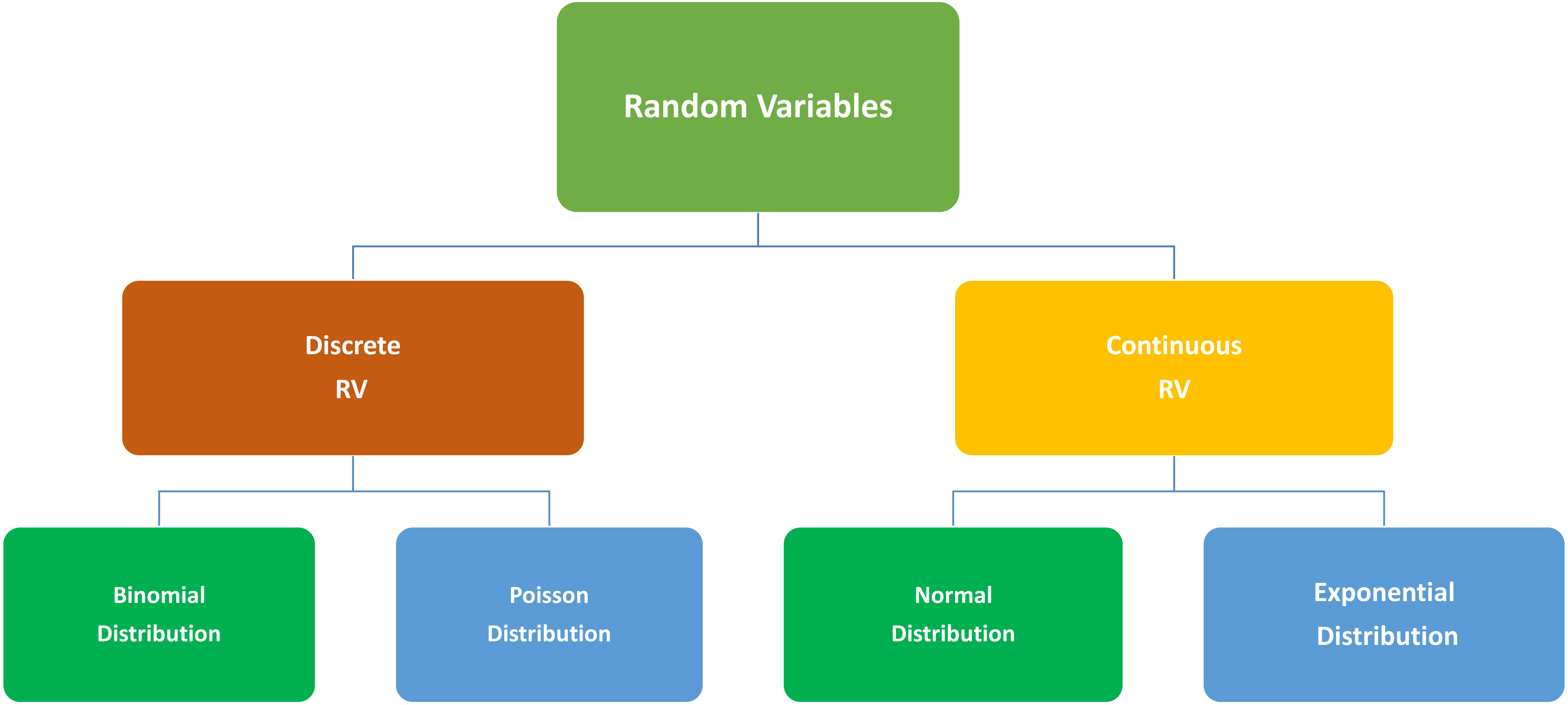
$$= 60 - 3(12)(3) + 2(3)^3 = 6$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= 360 - 4(60)(3) + 6(12)(3)^2 - 3(3)^4$$

$$= 45$$

# ROAD MAP FOR DISTRIBUTIONS



# Binomial Distribution

**Binomial Experiment:**

A binomial experiment is a experiment that satisfies the following conditions.

- ✓ The experiment is repeated for a fixed number of trials, where each trial is independent of other trials.
- ✓ There are only two possible outcomes of interest for each trial. The outcomes can be classified as a success ($S$) or as a failure ($F$).
- ✓ The probability of a success P(success) is the same for each trial.

# Notation for Binomial Experiments

| ***Symbol*** | ***Description*** |
|---|---|
| $n$ | The number of times a trial is repeated |
| $p$ | The probability of success in a single trial |
| $q$ | The probability of failure in a single trial. ($q = 1 – p$) |
| $x$ | The random variable represents a count of the number of successes in $n$ trials: $x = 0, 1, 2, 3, \ldots , n$. |

## Specify the values of *n*, *p*, and *q*, and list the possible values of the random variable *x*.

1) You randomly select a card from a deck of cards, and note if the card is a king. You then put the card back and repeat this process 8 times.

- Yes…This is a binomial experiment. Each of the 8 selections represent an independent trial because the card is replaced before the next one is drawn. There are only two possible outcomes: either the card is a king or not.

$$n = 8$$

$$p = \frac{4}{52} = \frac{1}{13}$$

# Specify the values of *n*, *p*, and *q*, and list the possible values of the random variable *x*.

➢ **You roll a die 10 times and note the number the die lands on.**

- No….. This is not a binomial experiment.  While each trial (roll) is independent, there are more than two possible outcomes: 1, 2, 3, 4, 5, and 6.

So, Binomial Distribution is…

A **discrete** probability distribution that is used for data which can only take one of two values, i.e.

- Pass or fail.
- Yes or no.
- Good or defective.

# Binomial Distribution Formula

In a binomial experiment, the probability of getting exactly *x* successes in *n* trials is :

$$P(X = x) = \binom{n}{x} p^x q^{n-x} \quad = {}^nC_x\, p^x\, q^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} p^x q^{n-x}.$$

Where, *n* = number of trials
*x* = number of success
*p* = probability of success
*q* = probability of failure

- Binomial distribution is fully defined if we know ‘*n*’ and ‘*p*’, so *n* and *p* are called parameters of Binomial distribution.
- ✓ Note that *n* is a discrete parameter whereas *p* is a continuous parameter as $0 < p < 1$.

# Mean of Binomial Distribution

**Mean** (Expected value) *E(X)* for binomial distribution is

So,

$$E(X) = np$$

Similarly, Variance of Binomial distribution *Var(X)* is..

$$\mathbf{Var(X)} = npq$$

***Standard Deviation*** $= \sqrt{var(x)}$

$$\mathbf{SD} = \sqrt{npq}$$

**Ex 18. The mean and standard deviation of a binomial distribution are 5 and 2. Determine the distribution.**

Sol : Here, Mean $\mu = np = 5$

Standard deviation $\sqrt{npq} = 2$

$\therefore\ npq = 4$

$$\frac{npq}{np} = \frac{4}{5}$$

$$\therefore\ q = \frac{4}{5}$$

$$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$$np = 5$$

$$n\left(\frac{1}{5}\right) = 5$$

$$\therefore\ n = 25$$

Hence, the binomial distribution is

$$P(X = x) = {}^{n}C_x\ p^x\ q^{n-x}$$

$$= {}^{25}C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}, \qquad x = 0, 1, 2, \ldots, 25$$

**Ex 19. The mean and variance of a binomial variate are 8 and 6. Find P(X ≥ 2).**

Sol : Here, Mean $\mu = np = 8$

Variance $npq = 6$

$$\frac{npq}{np} = \frac{6}{8} = \frac{3}{4}$$

$$\therefore \quad q = \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$np = 8$$

$$n\left(\frac{1}{4}\right) = 8$$

$$\therefore \quad n = 32$$

$$P(X = x) = {}^{n}C_x \; p^x \; q^{n-x}$$

$$= {}^{32}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x} \quad , \quad x = 0, 1, 2, \ldots, 32$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(X = 0) + P(X = 1)\right]$$

$$= 1 - \sum_{x=0}^{1} P(X = x)$$

$$= 1 - \sum_{x=0}^{1} {}^{32}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}$$

$$= 0.9988$$

# Recurrence Relation for the Binomial Distribution

For binomial distribution:

$$
\begin{aligned}
P(X=x) &= {}^{n}C_{x}\; p^{x}\; q^{n-x} \\
P(X=x+1) &= {}^{n}C_{x+1}\; p^{x+1}\; q^{n-x-1} \\
\frac{P(X=x+1)}{P(X=x)} &= \frac{{}^{n}C_{x+1}\; p^{x+1}\; q^{n-x-1}}{{}^{n}C_{x}\; p^{x}\; q^{n-x}} \\
&= \frac{n!}{(x+1)!\,(n-x-1)!} \times \frac{x!\,(n-x)!}{n!} \cdot \frac{p}{q} \\
&= \frac{(n-x)\,(n-x-1)!\,x!}{(x+1)\,x!\,(n-x-1)!} \cdot \frac{p}{q} \\
&= \frac{n-x}{x+1} \cdot \frac{p}{q}
\end{aligned}
$$

$$P(X=x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X=x)$$

**Ex 20. With the usual notation, find *p* for a binomial distribution if *n* = 6 and *9P(X = 4) = P(X = 2).***

Sol : For a binomial distribution,

$$P(X=x) = {}^nC_x\ p^x\ q^{n-x},\quad x = 0, 1, 2, \ldots, n$$

$$n = 6$$

$$9P(X=4) = P(X=2)$$

$$9\ {}^6C_4\ p^4\ q^2 = {}^6C_2\ p^2 q^4$$

$$9p^2 = q^2 = (1-p)^2$$

$$9p^2 = 1 - 2p + p^2$$

$$8p^2 + 2p - 1 = 0$$

$$p = \frac{-2 \pm \sqrt{4+32}}{2\times 8} = \frac{-2\pm 6}{16} = -\frac{1}{2}, \frac{1}{4}$$

Now, since probability can’t be negative,

***p = 1/4***

**Ex 21 Two dice are thrown five times. Find the probability of getting the sum as 7 (i) at least once, (ii) two times, and (iii) P(1 < X < 5).**

Sol : In a single throw of two dice, a sum of 7 can occur in 6 ways out of 6 × 6 = 36 ways.

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

Let *p* be the probability of getting the sum as 7 in a single throw of a pair of dice.

$$p = \frac{6}{36} = \frac{1}{6}$$

$$So,\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$n = 5$$

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{5}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x};\ x = 0,1,2,3,4,5.$$

Now, Probability of getting the required sum ***x* times in 5 throws** of a pair of dice.

**Ex 21. Two dice are thrown five times. Find the probability of getting the sum as 7 (i) at least once, (ii) two times, and (iii) P(1 < X < 5).**

$$P(X = x) = \binom{5}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{5-x}; \; x = 0,1,2,3,4,5.$$

**i) Probability of getting the sum as 7 at least once in 5 throws of two dice is..**

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \binom{5}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{5-0}$$

$$= 1 - \frac{3125}{7776}$$

$$= \frac{4651}{7776}$$

**Ex 21. Two dice are thrown five times. Find the probability of getting the sum as 7 (i) at least once, (ii) two times, and (iii) P(1 < X < 5).**

$$P(X = x) = \binom{5}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{5-x}; \; x = 0,1,2,3,4,5.$$

**ii) Probability of getting the sum as 7 two times in 5 throws of two dice.**

$$P(X = 2) = \binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{5-2}$$

$$= \frac{625}{3888}$$

**Ex 21. Two dice are thrown five times. Find the probability of getting the sum as 7 (i) at least once, (ii) two times, and (iii) P(1 < X < 5).**

$$P(X=x)=\binom{5}{x}\left(\frac{1}{6}\right)^{x}\left(\frac{5}{6}\right)^{5-x} \;;\; x=0,1,2,3,4,5.$$

**iii) Probability of getting the sum as 7 for P(1 < X < 5) in 5 throws of two dice**

$$P(1<X<5)=\sum_{x=2}^{4}P(X=x)$$

$$=P(X=2)+P(X=3)+P(X=4)$$

$$=\binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{5-2}+\binom{5}{3}\left(\frac{1}{6}\right)^{3}\left(\frac{5}{6}\right)^{5-3}+\binom{5}{4}\left(\frac{1}{6}\right)^{4}\left(\frac{5}{6}\right)^{5-4}$$

$$=\frac{1525}{7776}$$

**Ex 22. The incidence of corona in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random, four or more will suffer from the corona?**

Sol : Let $p$ be the probability of a worker suffering from the corona.

$p = 0.2$

$q = 1 - 0.2 = 0.8$

$n = 6$

According to Binomial distribution…

Probability that $x$ **workers out of 6** will suffer from the corona is…

$$P(X = x) = \binom{6}{x}(0.2)^{x}(0.8)^{6-x};\ x = 0,1,2,3,4,5,6.$$

.

**Ex 22. The incidence of corona in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random, four or more will suffer from the corona?**

$$P(X = x) = \binom{6}{x}(0.2)^x (0.8)^{6-x}\ ;\ x = 0,1,2,3,4,5,6.$$

**Probability that 4 or more workers will suffer from the corona is..**

$$\begin{aligned}
P(X \geq 4) &= \sum_{x=4}^{6} P(X = x) \\
&= P(X = 4) + P(X = 5) + P(X = 6) \\
&= \binom{6}{4}(0.2)^4 (0.8)^2 + \binom{6}{5}(0.2)^5 (0.8)^1 + \binom{6}{6}(0.2)^6 (0.8)^0 \\
&= \frac{53}{3125} \\
&= 0.017
\end{aligned}$$

**Ex 23. If hens of a certain breed lay eggs on 5 days a week on an average, find how many days during a season of 100 days will poultry keeper with 5 hens of this breed expect to receive at least 4 eggs.**

Sol : Let *p* be the probability of hen laying an egg on any day of a week.

$p = 5/7$

$q = 1 - (5/7) = 2/7$

$n = 5$ & $N = 100$

According to Binomial distribution…

Probability of receiving ***x* eggs** on any day of a week

$$P(X = x) = \binom{5}{x}\left(\frac{5}{7}\right)^{x}\left(\frac{2}{7}\right)^{5-x} \; ; \; x = 0,1,2,3,4,5.$$

**Ex 23. If hens of a certain breed lay eggs on 5 days a week on an average, find how many days during a season of 100 days a will poultry keeper with 5 hens of this breed expect to receive at least 4 eggs.**

$$P(X=x)=\binom{5}{x}\left(\frac{5}{7}\right)^{x}\left(\frac{2}{7}\right)^{5-x}\ ;\ x=0,1,2,3,4,5.$$

Now, **Probability of receiving at least 4 eggs on any day of a week is**

$$P(X\geq 4)=\sum_{x=4}^{5}P(X=x)$$

$$=P(X=4)+P(X=5)$$

$$=\binom{5}{4}\left(\frac{5}{7}\right)^{4}\left(\frac{2}{7}\right)^{1}+\binom{5}{5}\left(\frac{5}{7}\right)^{5}\left(\frac{2}{7}\right)^{0}$$

$$=0.5578$$

**Ex 23. If hens of a certain breed lay eggs on 5 days a week on an average, find how many days during a season of 100 days a will poultry keeper with 5 hens of this breed expect to receive at least 4 eggs.**

$$P(X = x) = \binom{5}{x}\left(\frac{5}{7}\right)^x \left(\frac{2}{7}\right)^{5-x} \; ; \; x = 0,1,2,3,4,5.$$

Now, **Probability of receiving at least 4 eggs on any day of a week is**

$$P(X \geq 4) = 0.5578$$

**Expected number of days during a season of 100 days, a poultry keeper with 5 hens of this breed will receive at least 4 eggs = N P(X ≥ 4)**

$$= 100(0.5578)$$
$$= 55.78$$
$$\approx 56 \; days$$

# Poisson Distribution Formula

A random variable X is said to follow Poisson distribution if the probability of $x$ is given by

$$P(X=x)=P(x)=\frac{e^{-\lambda}\lambda^{x}}{x!},\ x=0,1,2,3.....\infty$$

Where $\lambda$ is called the parameter of Poisson distribution.

Actually, Poisson is the limiting case of Binomial distribution.

How….??? Let's see…..

## **Poisson distribution**.

…

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0,1,2,3.....\infty$$

# Conditions of Poisson Approximation

**The Poisson distribution holds under the following conditions:**

(i) The random variable X should be discrete.

(ii) The numbers of trials $n$ is very large.

(iii) The probability of success $p$ is very small (very close to zero).

(iv) $\lambda = np$ is finite.

# Examples of Poisson Approximation

**Following are some examples of Poisson approximation:**

(i) Number of defective bulbs produced by a reputed company.

(ii) Number of printing mistakes per page in a large text.

(iii) Number of persons born blind per year in a large city.

# Mean of the Poisson Distribution

**Mean** (Expected value) ***E(X)*** for **Poisson distribution** is…

$$E(x) = \lambda$$

## Variance of the Poisson Distribution

***Var(X)*** for **Poisson distribution** is…

$$
\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - \mu^2 \\
&= \sum_{x=0}^{\infty} x^2 \; p(x) - \mu^2 \\
&= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} x[(x-1)+1]\frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^{x-2}\lambda^2}{x(x-1)(x-2)\cdots 1} + \lambda - \lambda^2 \\
&= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda - \lambda^2 \\
&= e^{-\lambda}\lambda^2 \left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right) + \lambda - \lambda^2 \\
&= e^{\lambda} e^{-\lambda} \lambda^2 + \lambda - \lambda^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

## Standard deviation of the Poisson Dist.

As we know that,

$$
\begin{aligned}
Standard\ deviation &= \sqrt{variance} \\
&= \sqrt{\lambda}
\end{aligned}
$$

# Recurrence Relation for the Poisson Distribution

As we discussed, for the Poisson distribution:

$$p(x) = \frac{e^{-\lambda}\ \lambda^{x}}{x!}$$

$$p(x+1) = \frac{e^{-\lambda}\ \lambda^{x+1}}{(x+1)!}$$

$$\frac{p(x+1)}{p(x)} = \frac{e^{-\lambda}\ \lambda^{x+1}}{(x+1)!} \cdot \frac{x!}{e^{-\lambda}\ \lambda^{x}}$$

$$= \frac{\lambda}{x+1}$$

$$p(x+1) = \frac{\lambda}{x+1} p(x)$$

Which is known as a Recurrence relation for Poisson distribution.

# Examples of Poisson Distribution

**Ex 24. If the mean of a Poisson variable is 1.8, find *(i) P(X > 1)*, *(ii) P(X = 5)* and *(iii) P(0 < X < 5)*.**

**Sol** : For the Poisson distribution,

$$\lambda = 1.8$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-1.8}1.8^x}{x!} \quad ; x = 0,1,2,3,.....$$

**(i) P(X > 1) :-**

$$P(X > 1) = 1 - P(X \le 1)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \sum_{x=0}^{1} P(X = x)$$

$$= 1 - \sum_{x=0}^{1} \frac{e^{-1.8}1.8^x}{x!}$$

$$= 0.5372$$

**(ii) P(X = 5) :-**

$$P(X = 5) = \frac{e^{-1.8}1.8^5}{5!} = 0.026$$

**(iii) P(0 < X < 5) :-**

$$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \sum_{x=1}^{4} P(X = x)$$

$$= \sum_{x=1}^{4} \frac{e^{-1.8}1.8^x}{x!}$$

$$= 0.7983$$

# Examples of Poisson Distribution

**Ex 25 If a random variable has a Poisson distribution such that *P(X* = 1*)* = *P(X* = 2*)*, find *(i)* the mean of the distribution, *(ii) P(X* ≥ 1*)*, and (iii) *P*(1 < *X* < 4).**

**Sol** : For the Poisson distribution,

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}\ ;x=0,1,2,3,\ldots..$$

$$Now,\ P(X=1)=P(X=2)$$

$$\frac{e^{-\lambda}\lambda^{1}}{1!}=\frac{e^{-\lambda}\lambda^{2}}{2!}$$

$$\lambda=\frac{\lambda^{2}}{2}$$

$$\lambda^{2}-2\lambda=0$$

$$\lambda(\lambda-2)=0$$

$$\therefore \lambda=0\ \&\ \lambda=2$$

$$But,\lambda\neq 0\quad So,\ \lambda=2.$$

$$\therefore P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}=\frac{e^{-2}2^{x}}{x!}\ ;x=0,1,2,\ldots$$

***(ii) P(X* ≥ 1*)* :-**

$$\begin{aligned}P(X\geq 1)&=1-P(X<1)\\&=1-P(X=0)\\&=1-\frac{e^{-2}2^{0}}{0!}\\&=0.8647\end{aligned}$$

***(iii) P(1 < X < 4)* :-**

$$\begin{aligned}P(1<X<4)&=P(X=2)+P(X=3)\\&=\sum_{x=2}^{3}P(X=x)\\&=\sum_{x=2}^{3}\frac{e^{-2}2^{x}}{x!}\\&=0.4511\end{aligned}$$

# Examples of Poisson Distribution

**Ex 26.** **If X is a Poisson variate such that $P(X=0)=P(X=1)$, find $P(X=0)$ and using recurrence relation formula, find the probabilities at x = 1, 2, 3, 4, and 5.**

**Sol :** For a Poisson distribution,

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x=0,1,2,\ldots$$

$$P(X=0)=P(X=1)$$

$$\frac{e^{-\lambda}\lambda^{0}}{0!}=\frac{e^{-\lambda}\lambda^{1}}{1!}$$

$$\lambda=1$$

Hence, $$P(X=x)=\frac{e^{-\lambda}1^{x}}{x!}, \qquad x=0,1,2,\ldots$$

***(i)* For *P*(*X*=0) :**

$$P(X=0)=\frac{e^{-\lambda}\lambda^{0}}{0!}=0.3678$$

***(ii)* As per the recurrence formula:**

$$p(x+1)=\frac{\lambda}{x+1}p(x)$$

$$p(x+1)=\frac{1}{x+1}p(x) \qquad [\because \lambda=1]$$

$$p(1)=p(0)=0.3678$$

$$p(2)=\frac{1}{2}p(1)=\frac{1}{2}(0.3678)=0.1839$$

$$p(3)=\frac{1}{3}p(2)=\frac{1}{3}(0.1839)=0.0613$$

$$p(4)=\frac{1}{4}p(3)=\frac{1}{4}(0.0613)=0.015325$$

$$p(5)=\frac{1}{5}p(4)=\frac{1}{5}(0.015325)=0.003065$$

# Examples of Poisson Distribution

**Ex 27. If X is a Poisson variate such that $P(X=2)=9P(X=4)+90P(X=6)$. Find *(i)* the mean of $X$, *(ii)* the variance of $X$.**

**Sol** : For the Poisson distribution,

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!} \qquad ; x=0,1,2,\ldots$$

More over,

$$P(X=2)=9P(X=4)+90P(X=6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!}=9\frac{e^{-\lambda}\lambda^4}{4!}+90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$=e^{-\lambda}\lambda^2\left(\frac{9\lambda^2}{4!}+\frac{90\lambda^4}{6!}\right)$$

$$\frac{1}{2}=\frac{9\lambda^2}{4!}+\frac{90\lambda^4}{6!}$$

$$\frac{1}{2}=\frac{3\lambda^2}{8}+\frac{\lambda^4}{8}$$

$$\lambda^4+3\lambda^2-4=0$$

$$\lambda^2=-\frac{3\pm\sqrt{9+16}}{2}=\frac{-3\pm5}{2}=1,-4$$

Since, $\lambda>0, \quad \lambda^2\neq-4$

$$\therefore \lambda^2=1 \Rightarrow \lambda=1$$

So, Mean = $\lambda=1$

Variance = $\lambda=1$

# Examples of Poisson Distribution

**Ex 28. An insurance company insured 4000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on an average, 10 persons in 100000 will have car accidents each year that result in this type of injury. What is the probability that more than 3 of the insured will collect on their policy in a given year?**

**Sol** : Let $p$ be the probability of loss of both eyes in a car accident.

$$p = \frac{10}{100000} = 0.0001$$

$$n = 4000$$

Since, $p$ is very small, Poisson dis. is applicable,

$$\lambda = np = 4000(0.0001) = 0.4$$

Let $X$ be the random variable which denotes the number of car accidents in a group of 4000 people.

So, Probability of $x$ car accidents in a group of 4000 people is..

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{e^{-0.4}0.4^{x}}{x!} \quad ; x = 0,1,2,\ldots$$

# Examples of Poisson Distribution

**Ex 28. An insurance company insured 4000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average, 10 persons in 100000 will have car accidents each year that result in this type of injury. What is the probability that more than 3 of the insured will collect on their policy in a given year?**

**Sol** : $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$=\frac{e^{-0.4}0.4^{x}}{x!} \quad ; x=0,1,2,\ldots$$

Now, Probability that more than 3 of the insured will collect on their policy,
i.e., probability of more than 3 car accidents in a group of 4000 people is….

$$\begin{aligned}P(X>3)&=1-P(X\leq 3)\\&=1-\left[P(X=0)+(X=1)+P(X=2)+P(X=3)\right]\\&=1-\sum_{x=0}^{3}P(X=x)\\&=1-\sum_{x=0}^{3}\frac{e^{-0.4}0.4^{x}}{x!}\\&=0.00077\end{aligned}$$

# Examples of Poisson Distribution

**Ex 29. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors?**

**Sol** : Let $p$ be the probability of errors in **a page**.

$$p = \frac{43}{585} = 0.0735$$

$$n = 10$$

Since, $p$ is very small, Poisson dis. is applicable,

$$\lambda = np = 10(0.0735) = 0.735$$

Let $X$ be the random variable which denotes the errors in the pages.

So, Probability of $x$ errors in a page in a book of 585 pages,

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{e^{-0.735}0.735^{x}}{x!} \quad ; x = 0,1,2,\ldots$$

# Examples of Poisson Distribution

**Ex 30. Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors?**

**Sol** :

[illegible]

Now, Probability that a random sample of 10 pages will contain no error…

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad ;let\ \ x = 0$$

$$P(X = 0) = \frac{e^{-0.735}0.735^{0}}{0!}$$

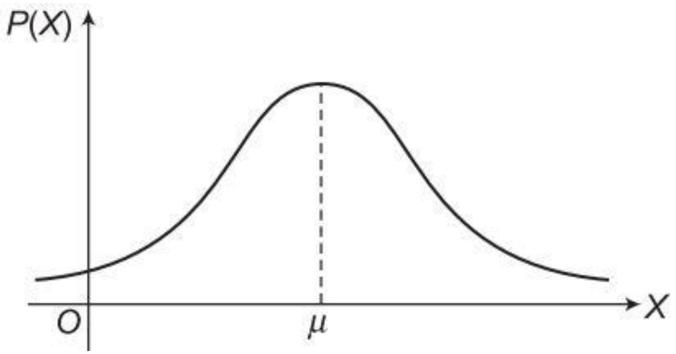
$$= 0.4795$$

# Normal Distribution Function

A continuous random variable $X$ is said to follow **normal distribution** with mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma^2}$, if its probability function is given by..

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad ;where\ -\infty < X < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

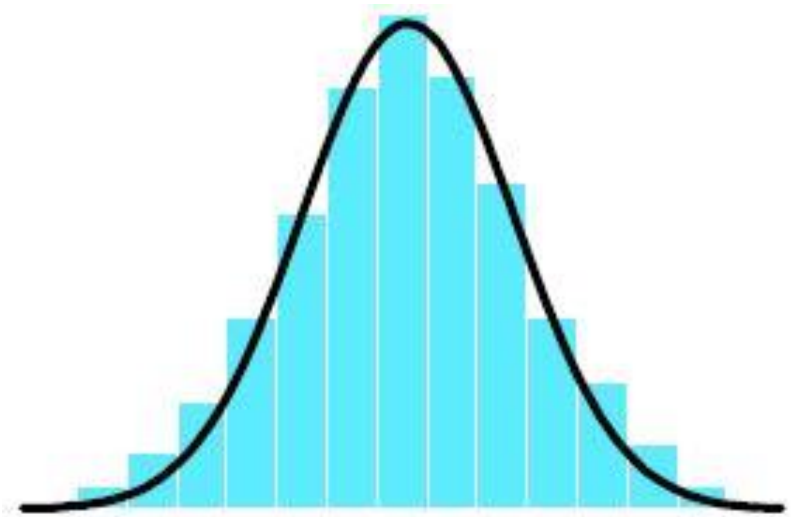
- ✓ Here $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are called **parameters** of the normal distribution.
- ✓ The curve representing the normal distribution is called the **normal curve.**



# Normal Distribution

**Normal Distribution:**

- ➢ A **symmetrical** probability distribution.
- ➢ Most results are located in the **middle** and few are spread on both sides.
- ➢ Has the shape of **“a bell”**.
- ➢ Can entirely be described by its **mean** and **standard deviation**.
- ✓ Normality is an important assumption when conducting statistical analysis so that they can be applied in the right manner.



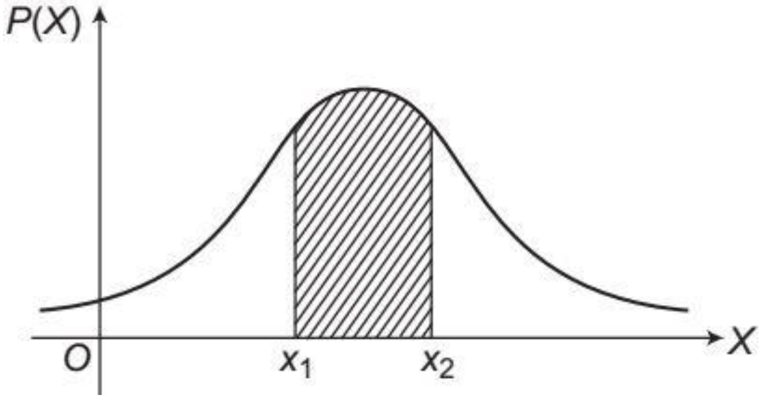
# Normal Distribution Function

**That means…….**

***If X is a normal random variable with mean μ and standard deviation σ. Then probability of X lying in the interval ($x_1$, $x_2$) is given by..***

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)dx$$

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

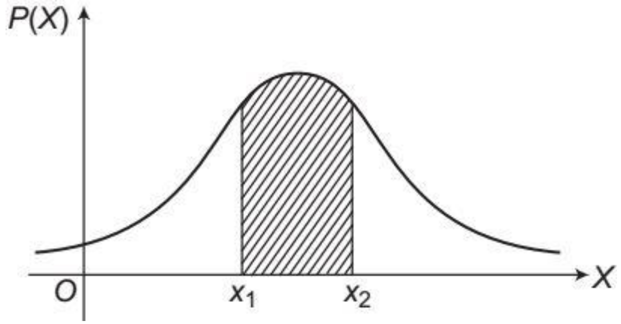



# Probability of a Normal Random Variable in an Interval:

**As we discussed earlier….**

If $X$ is a normal random variable with mean $\mu$ and standard deviation $\sigma$. Then probability of $X$ lying in the interval $(x_1, x_2)$ is given by..

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$




- <u>It looks very difficult to deal with this integration</u>… Isn't it?????

Now, $P(x_1 \le X \le x_2)$ can be evaluated easily by converting a normal random $X$ variable into **another random variable Z**.

$$Let, \quad Z = \frac{x-\mu}{\sigma}$$ be **a new random variable.**

# Probability of a Normal Random Variable in an Interval:

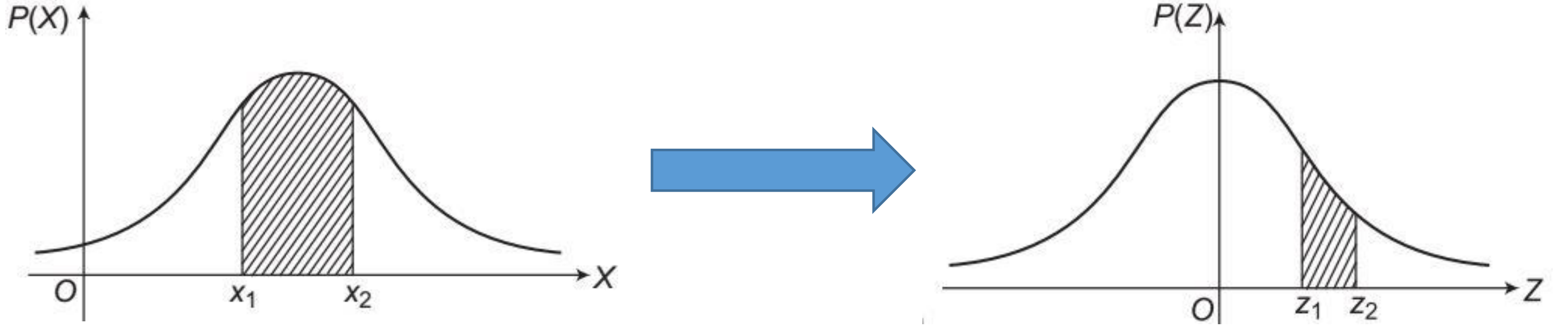
$$P(x_1 \le X \le x_2) = P\left(\frac{x_1-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{x_2-\mu}{\sigma}\right)$$

$$= P(z_1 \le Z \le z_2)$$

Where $z_1 = \frac{x_1-\mu}{\sigma}$ and $z_2 = \frac{x_2-\mu}{\sigma}$

So, the probability $P(x_1 \le X \le x_2)$ is equal to the **area under standard normal curve** between the ordinates at $Z = z_1$ and $Z = z_2$

**Normal Variate *X*** → **Standard Normal Variate *Z***



# Probability of a Normal Random Variable in an Interval:

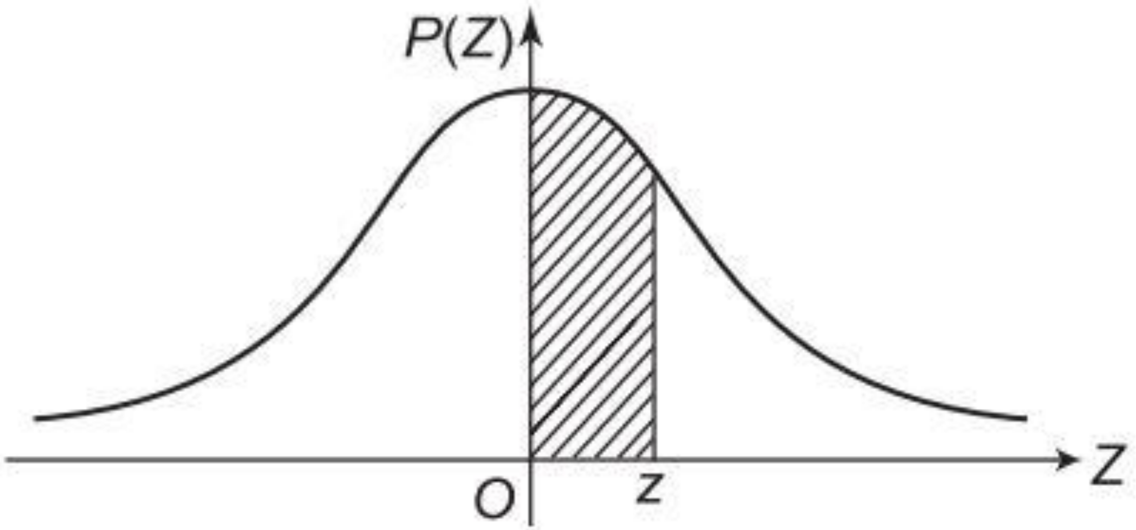
For, $Z = \dfrac{x-\mu}{\sigma}$

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}\left[E(X)-\mu\right] = 0$$

$$\text{Var}(Z) = \text{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\text{Var}(X-\mu) = \frac{1}{\sigma^2}\text{Var}(X) = 1$$

- ✓ The distribution of Z is also **normal**.
- ✓ Thus, if ***X*** is a normal random variable with mean ***μ*** and standard deviation **σ** then ***Z*** is a **normal random variable** with mean **0** and standard deviation **1**.
- ✓ Since the parameters of the distribution of *Z* are **fixed**, it is a known distribution and is termed **standard normal distribution.** Further, *Z* is termed as a standard normal variate.
- ➢ Thus, the distribution of any **normal variate** *X* can **always be transformed** into the distribution of the **standard normal variate** *Z*.

# Standard Normal Z Table



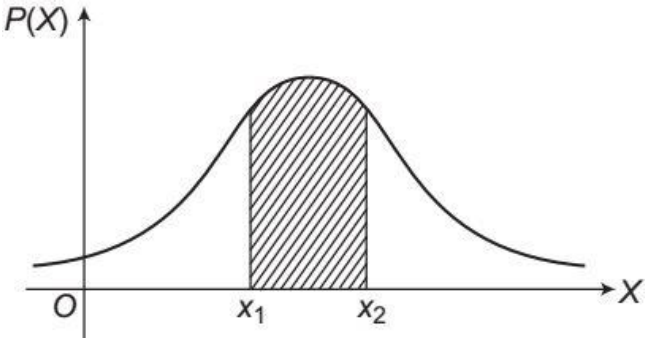

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0753 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2517 | 0.2549 |
| 0.7 | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2995 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3990 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4115 | 0.4131 | 0.4147 | 0.4162 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4429 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4484 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4699 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4756 | 0.4761 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4864 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.4916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 0.4978 | 0.4979 | 0.4979 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4982 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986 |
| 3.0 | 0.4987 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |

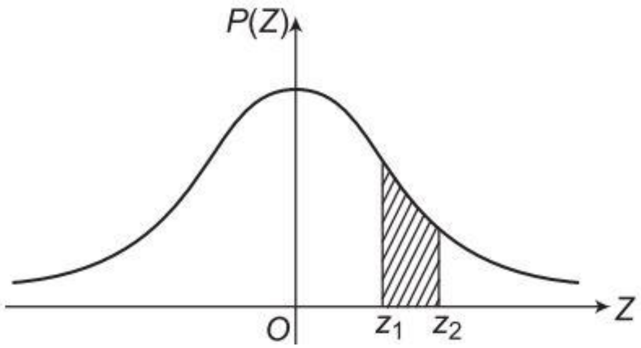
# Dealing with Standard Normal Variate Z

**Case I:- If both $z_1$ and $z_2$ are positive (or both negative)**

$$P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$$
$$= P(0 \le Z \le z_2) - P(0 \le Z \le z_1)$$

$$P(x_1 \le X \le x_2) = (Area\ under\ normal\ curve\ from\ 0\ to\ z_2) - (Area\ under\ normal\ curve\ from\ 0\ to\ z_1)$$



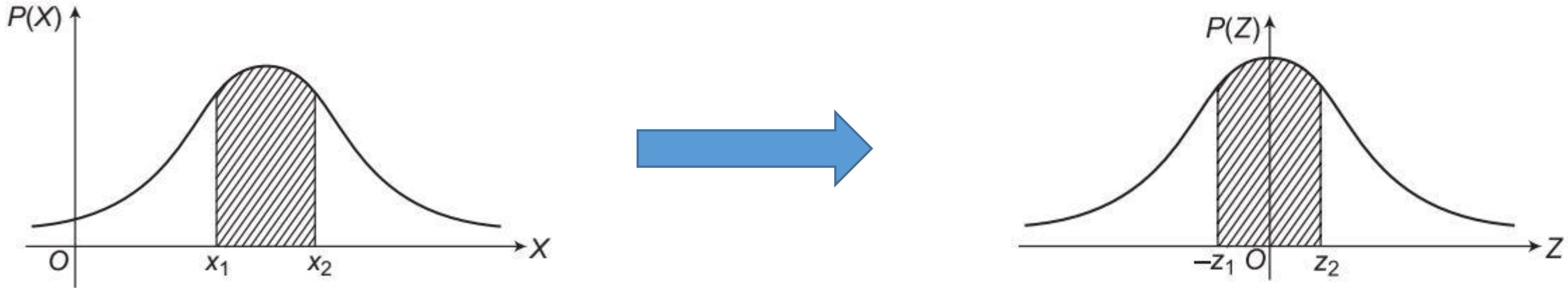



# Dealing with Standard Normal Variate Z

**Case II:- If $z_1 < 0$ and $z_2 > 0$**

$$\begin{aligned} P(x_1 \le X \le x_2) &= P(-z_1 \le Z \le z_2) \\ &= P(-z_1 \le Z \le 0) + P(0 \le Z \le z_2) \\ &= P(0 \le Z \le z_1) + P(0 \le Z \le z_2) \end{aligned}$$

$$\boldsymbol{P(x_1 \le X \le x_2) = (Area\ under\ normal\ curve\ from\ 0\ to\ z_1) + (Area\ under\ normal\ curve\ from\ 0\ to\ z_2)}$$

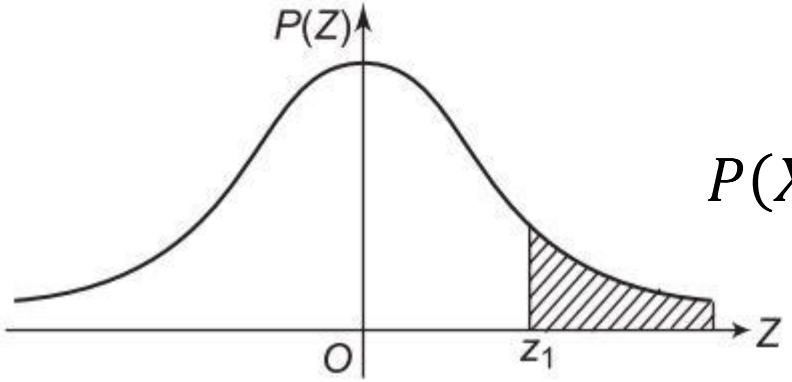


# Dealing with Standard Normal Variate Z

**Some other cases for $\boldsymbol{P(X > x_1)}$:-**

**(I) If $\boldsymbol{z_1 > 0}$**

$$P(X > x_1) = P(Z > z_1)$$
$$= 0.5 - P(0 \le Z \le z_1)$$



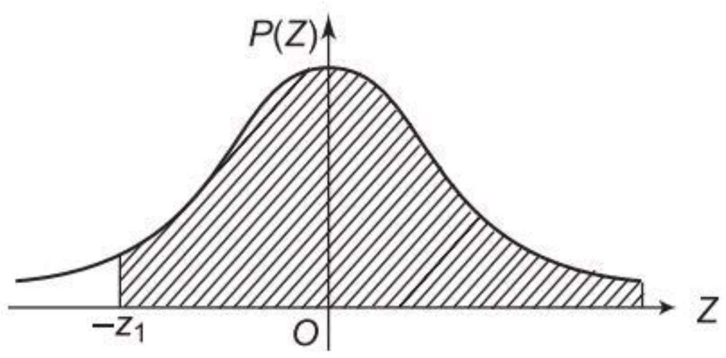

$$P(X > x_1) = 0.5 - (\text{area under the curve from } 0 \text{ to } z_1)$$

# Dealing with Standard Normal Variate Z

**Some other cases for $P(X > x_1)$:-**

**(II) If $z_1 < 0$**

$$
\begin{aligned}
P(X > x_1) &= P(Z > -z_1) \\
&= 0.5 + P(-z_1 < Z < 0) \\
&= 0.5 + P(0 < Z < z_1)
\end{aligned}
$$




$$P(X > x_1) = 0.5 + (\text{area under the curve from 0 to } z_1)$$

**So, we may deal with Z, according to the situation in any other cases also**.

# Properties of the Normal Distribution

A **normal probability curve**, or **normal curve**, has the following properties:

*(i)* It is a **bell-shaped symmetrical curve** about the ordinate $X = \mu$. The ordinate is **maximum** at $X = \mu$.

*(ii)* It is a **unimodal curve** and its tails extend infinitely in both the directions, *i.e.*, the curve is asymptotic to $X - axis$ in both the directions.

*(iii)* All the three measures of central tendency **coincide**, *i.e.*, $mean = median = mode$.

*(iv)* The **total area** under the curve gives the **total probability** of the random variable X taking values between $-\infty$ $to$ $\infty$. Mathematically,

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x = 1$$

*(v)* The ordinate at $X = \mu$ divides the area under the normal curve into **two equal parts,** *i.e.*,

$$\int_{-\infty}^{\mu} f(x)\,\mathrm{d}x = \int_{\mu}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{2}$$

# Properties of the Normal Distribution

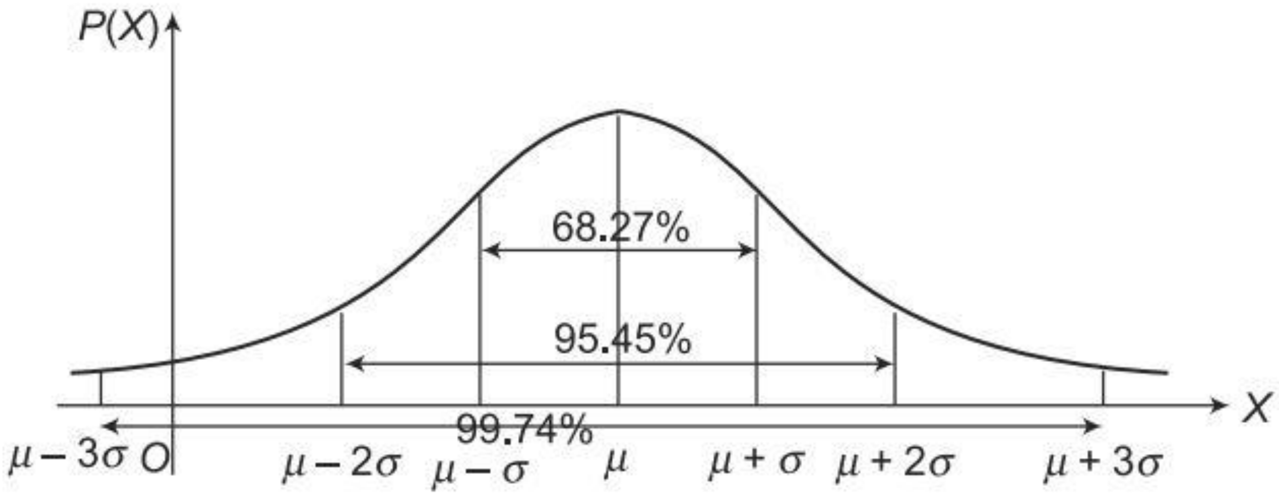
*(vi)* The value of $f(x)$ is always **nonnegative** for all values of $X$, because the whole curve lies above the $X - axis$.

*(vii)* The **area under the normal curve** is distributed as follows:

*(a)* The area between the ordinates at $\mu - \sigma$ and $\mu + \sigma$ is **68.27%**

*(b)* The area between the ordinates at $\mu - 2\sigma$ and $\mu + 2\sigma$ is **95.45%**

*(c)* The area between the ordinates at $\mu - 3\sigma$ and $\mu + 3\sigma$ is **99.74%**



# Parameters of the Normal Distribution

- **Mathematical expectation/Mean** for Normal distribution….

$$E(\mathrm{X}) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \ldots\ldots$$

$$= \ldots.$$

$$E(\mathrm{X}) = \mu$$

Moreover; $\boldsymbol{median} = \boldsymbol{mode} = \boldsymbol{\mu}$

So, Normal distribution is **symmetrical distribution.**

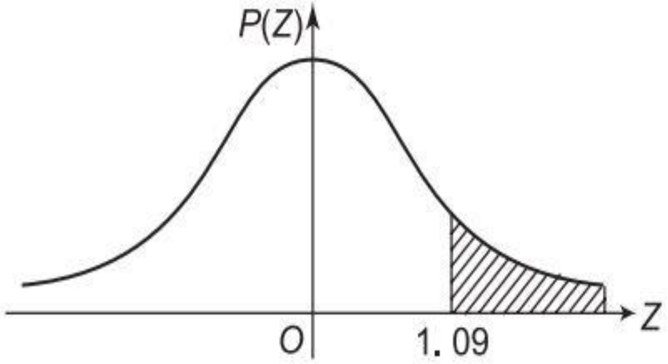
# Examples of Standard Normal Variate:

**Ex 31. What is the probability that a standard normal variate Z will be**
**(i) greater than 1.09? (ii) less than or equal –1.65?**
**(iii) lying between –1 and1.96?(iv) lying between 1.25 and 2.75?**

**Sol** : For standard Normal variate ***Z***,

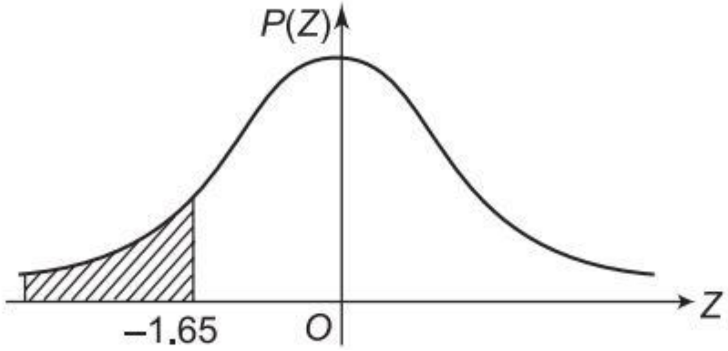
***(i)*** $\boldsymbol{P(\,Z \;>\; 1.09)}$ **:-**

$$\boldsymbol{P(\,Z \;>\; 1.09) = 0.5 - P(\,0 \le Z \;\le\; 1.09)}$$
$$= \;0.5 - 0.3621$$
$$= \;0.1379$$




***(ii)*** $\boldsymbol{P(\,Z \le -1.65)}$ **:-**

$$\boldsymbol{P(\,Z \le -1.65) = 0.5 - P(\,Z > -1.65)}$$
$$= 0.5 - P(0 < Z < 1.65)$$
$$= \;0.5 - 0.4505$$
$$= \;0.0495$$



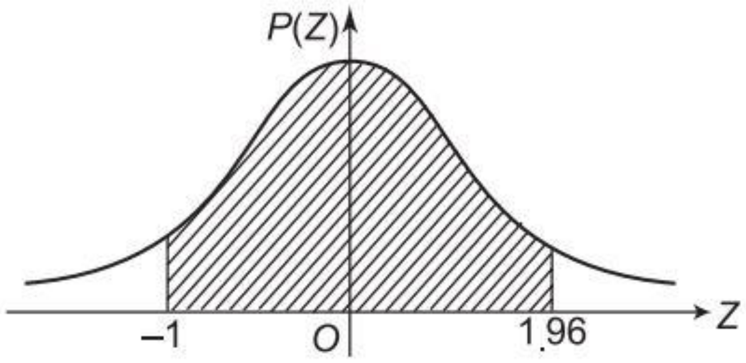
# Examples of Standard Normal Variate:

**Ex 31. What is the probability that a standard normal variate Z will be (i) greater than 1.09? (ii) less than or equal –1.65? (iii) lying between –1 and1.96?(iv) lying between 1.25 and 2.75?**

**Sol** : For standard Normal variate ***Z***,

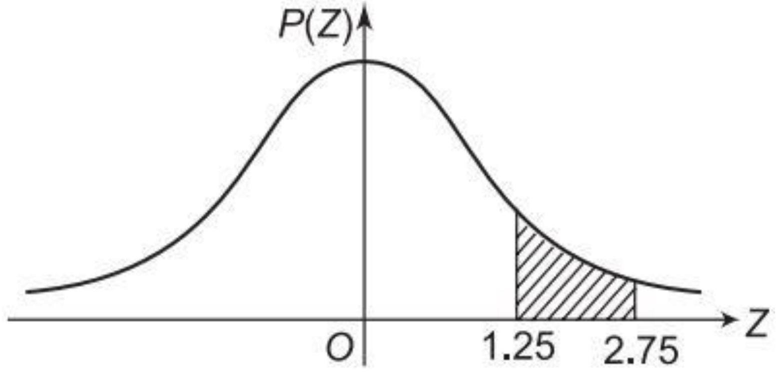
***(iii)*** $\boldsymbol{P(-1 < Z < 1.96)}$ **:-**

$$\boldsymbol{P}(-\mathbf{1} < \boldsymbol{Z} < \mathbf{1.96}) = \boldsymbol{P}(-\mathbf{1} \le \boldsymbol{Z} \le \mathbf{0}) + \boldsymbol{P}(\mathbf{0} \le \boldsymbol{Z} \le \mathbf{1.96})$$
$$= \boldsymbol{P}(\mathbf{0} \le \boldsymbol{Z} \le \mathbf{1}) + \boldsymbol{P}(\mathbf{0} \le \boldsymbol{Z} \le \mathbf{1.96})$$
$$= 0.3413 + 0.4750$$
$$= 0.8163$$




$\boldsymbol{(iv)\ P(1.25 < Z < 2.75)}$ **:-**

$$\boldsymbol{P}(\mathbf{1.25} < \boldsymbol{Z} < \mathbf{2.75}) = \boldsymbol{P}(\mathbf{0} < \boldsymbol{Z} < \mathbf{2.75}) - \boldsymbol{P}(\mathbf{0} < \boldsymbol{Z} < \mathbf{1.25})$$
$$= 0.4970 - 0.3944$$
$$= 0.1026$$



# Examples of Normal Variate:

**Ex 32. If X is a normal variate with a mean of 30 and an SD of 5, find the probabilities that $(i)$ $26 \le X \le 40$, and $(ii)$ $X \ge 45$.**

**Sol** : For Normal variate $\boldsymbol{X}$,

Mean $\mu = 30$

SD $\sigma = 5$

$When\ X = 26$

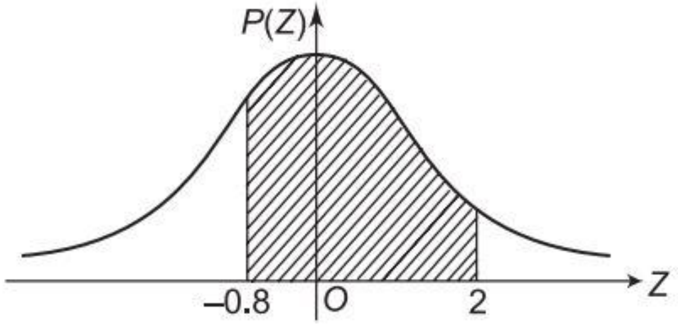
$$Z = \frac{X-\mu}{\sigma} = \frac{26-30}{5} = \frac{-4}{5} = -0.8$$

$When\ X = 40$

$$Z = \frac{X-\mu}{\sigma} = \frac{40-30}{5} = \frac{10}{5} = 2$$

**$(i)$ $P(26 \le X \le 40)$ :-**

$$\begin{aligned} P(26 \le X \le 40) &= P(-0.8 \le Z \le 2) \\ &= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2) \\ &= P(0 \le Z \le 0.8) + P(0 \le Z \le 2) \\ &= 0.2881 + 0.4772 \\ &= 0.7653 \end{aligned}$$



# Examples of Normal Variate:

**Ex 32. If X is a normal variate with a mean of 30 and an SD of 5, find the probabilities that $(i)$ $26 \le X \le 40$, and $(ii)$ $X \ge 45$.**

**Sol** : For Normal variate $\boldsymbol{X}$,

Mean $\mu = 30$

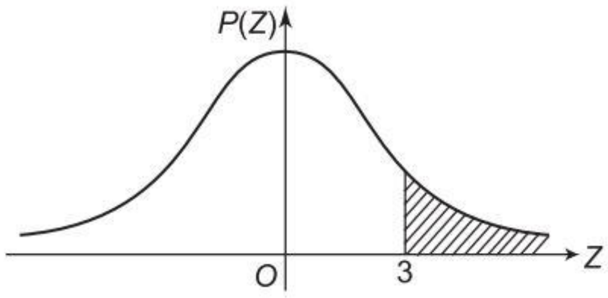
SD $\sigma = 5$

$When\ X = 45$

$$Z = \frac{X-\mu}{\sigma} = \frac{45-30}{5} = \frac{15}{5} = 3$$

**$(ii) P(X \ge 45)$:-**

$$P(X \ge 45) = P(Z \ge 3)$$
$$= \mathbf{0.5} - \boldsymbol{P}(\mathbf{0} < \boldsymbol{Z} < \mathbf{3})$$
$$= 0.5 - 0.4987$$
$$= \mathbf{0.0013}$$



# Examples of Normal Distribution:

**Ex 33. A manufacturer knows from his experience that the resistances of resistors he produces is normal with $mean = 100\ ohms$ and $SD = 2\ ohms$. What percentage of resistors will have resistances between 98 $ohms$ and 102 $ohms$?**

**Sol** : For Normal variate $\boldsymbol{X}$,

Mean $\mu = 100\ ohms$

SD $\sigma = 2\ ohms$

$When\ X = 98$

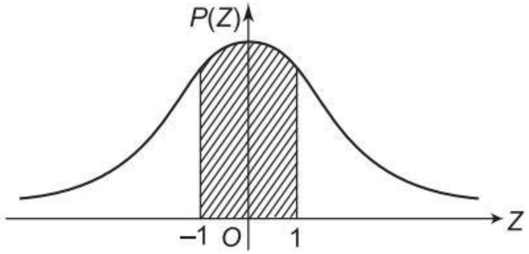
$$Z = \frac{X-\mu}{\sigma} = \frac{98-100}{2} = \frac{-2}{2} = -1$$

$When\ X = 102$

$$Z = \frac{X-\mu}{\sigma} = \frac{102-100}{2} = \frac{2}{2} = 1$$

**Now, $P(98 \le X \le 102)$ :-**

$$\begin{aligned} P(98 \le X \le 102) &= P(-1 \le Z \le 1) \\ &= P(-1 \le Z \le 0) + P(0 \le Z \le 1) \\ &= P(0 \le Z \le 1) + P(0 \le Z \le 1) \\ &= 0.3413 + 0.3413 \\ &= 0.6826 \end{aligned}$$




**Hence, the percentage of resistors have resistances between *98 ohms* and *102 ohms* = 68.26%.**

# Examples of Normal Distribution:

**Ex 34. The average seasonal rainfall in a place is 16 $\boldsymbol{inches}$ with an SD of 4 $\boldsymbol{inches}$. What is the probability that the rainfall in that place will be between 20 and 24 $\boldsymbol{inches}$ in a year?**

**Sol** : For Normal variate $\boldsymbol{X}$,

$$\text{Mean } \mu = 16\, inches$$

$$\text{SD } \sigma = 4\, inches$$

$$When\ X = 20$$

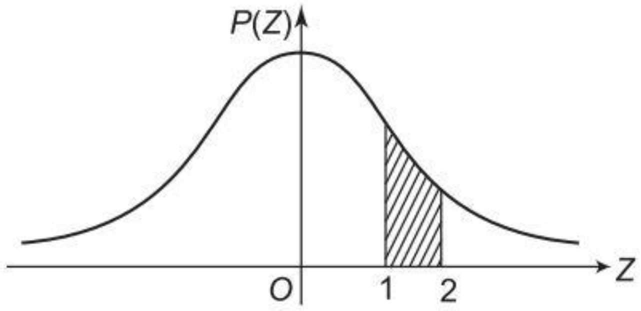
$$Z = \frac{X-\mu}{\sigma} = \frac{20-16}{4} = \frac{4}{4} = 1$$

$$When\ X = 24$$

$$Z = \frac{X-\mu}{\sigma} = \frac{24-16}{2} = \frac{8}{4} = 2$$

**Now, $\boldsymbol{P(\ 20 \le X \le 24)}$ :-**

$$\begin{aligned} P(\ 20 \le X \le 24) &= P(\ 1 \le Z \le 2) \\ &= P(0 \le Z \le 2) - P(0 \le Z \le 1) \\ &= 0.4772 - 0.3413 \\ &= 0.1359 \end{aligned}$$




**Hence, the probability that the rainfall in that place will be between 20 to 24 inches = 0.1359**

# Important Note:

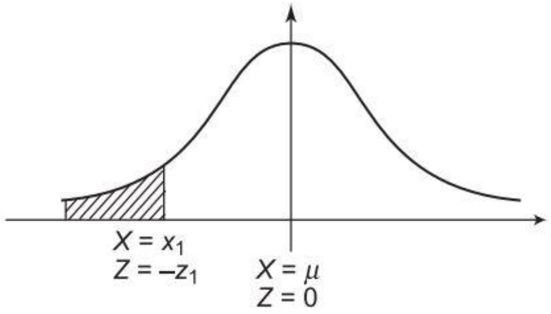
**After dealing with previous examples, lets bag some important information.**

**Here it is…**

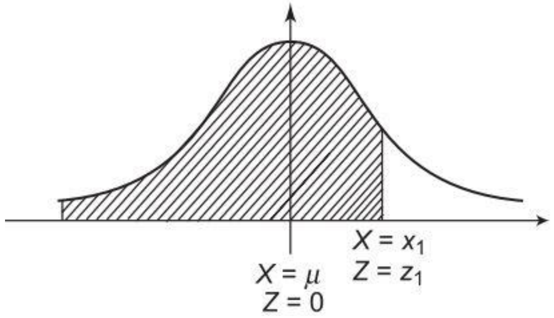
(1) $P(X < x_1) = F(x_1) = \int_{-\infty}^{x_1} f(x)dx$

Hence, $P(X < x_1)$ represent the area under the curve from $X = -\infty$ to $X = x_1$.

(2) If $\boldsymbol{P(X < x_1) < 0.5}$, then the point $x_1$ lies to the **left of $\boldsymbol{X = \mu}$** and the corresponding value of standard normal variate will be **negative**.




(3) If $\boldsymbol{P(X < x_1) > 0.5}$, then the point $x_1$ lies to the **right of $\boldsymbol{X = \mu}$** and the corresponding value of standard normal variate will be **positive**.




We are going to use the information in next examples.

# Examples of Normal Distribution:

**Ex 35. If $X$ is a normal variate with a mean of 120 and a standard deviation of 10, find $c$ such that *(i)* $P(X > c) = 0.02$, and *(ii)* $P(X < c) = 0.05$.**

**Sol** : For Normal variate $X$,

Mean $\mu = 120$

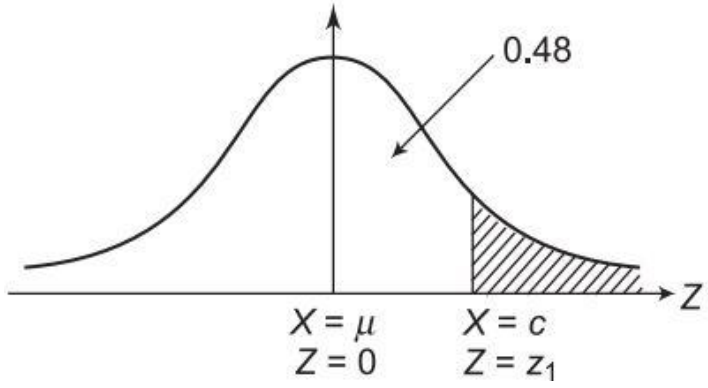
SD $\sigma = 10$

$$Z = \frac{X-\mu}{\sigma}$$

***(i)*** $\boldsymbol{P(X > c) = 0.02}$

$$P(X < c) = 1 - P(X \geq c)$$
$$= 1 - 0.02$$
$$= 0.98 > 0.5$$

Since $\boldsymbol{P(X < c) > 0.5}$, the corresponding value of $\boldsymbol{Z}$ will be **positive**.




$$P(X > c) = P(Z > z_1)$$
$$0.02 = 0.5 - P(0 \leq Z \leq z_1)$$
$$P(0 \leq Z \leq z_1) = 0.48$$
$$\therefore z_1 = 2.05$$ (From Table)

Now, $z_1 = \frac{c-120}{10}$

$$2.05 = \frac{c-120}{10}$$

$$\therefore \boldsymbol{c = 140.05}$$

# Examples of Normal Distribution:

**Ex 35. If $X$ is a normal variate with a mean of 120 and a standard deviation of 10, find $c$ such that *(i)* $P(X > c) = 0.02$, and *(ii)* $P(X < c) = 0.05$.**

**Sol** : For Normal variate $\boldsymbol{X}$,

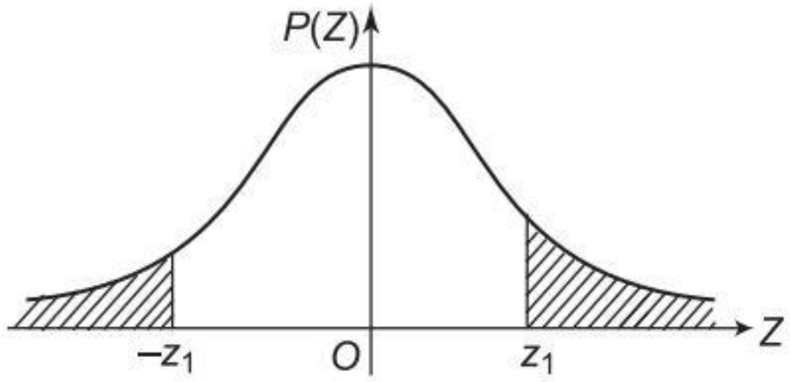
Mean $\mu = 120$

SD $\sigma = 10$

$$Z = \frac{X-\mu}{\sigma}$$

***(ii)*** $\boldsymbol{P(\,X < c) = 0.05}$

Since $\boldsymbol{P(\,X < c) < 0.5}$, the corresponding value of $\boldsymbol{Z}$ will be **negative**.




$$P(\,X < c) = P(\,Z < -z_1)$$
$$0.05 = 1 - P(Z \geq -z_1)$$
$$0.05 = 1 - [0.5 + P(-z_1 \leq Z \leq 0)]$$
$$0.05 = 1 - [0.5 + P(0 \leq Z \leq z_1)]$$
$$0.05 = 0.5 - P(0 \leq Z \leq z_1)$$
$$P(0 \leq Z \leq z_1) = 0.45$$
$$\therefore z_1 = -1.64 \quad \text{(From Table)}$$

Now, $z_1 = \dfrac{c-120}{10}$

$$-1.64 = \frac{c-120}{10} \qquad \therefore \boldsymbol{c = 103.6}$$

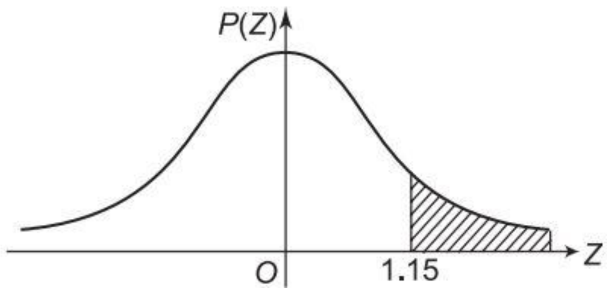
# Examples of Normal Distribution:

**Ex 36. Assume that the mean height of Indian soldiers is $68.22$ inches with a variance of $10.8$ inches. How many soldiers in a regiment of $1000$ would you expect to be over 6 feet tall?**

**Sol** : Let $\boldsymbol{X}$ be the continuous random variable which denotes the height of Indian soldiers.

Mean $\mu = 68.22\ inches$
Variance $\sigma^2 = 10.8$
SD $\sigma = 3.29$
$N = 1000$
$Z = \frac{X-\mu}{\sigma}$

$When\ X = 6\ feet = 72\ inches$
$Z = \frac{X-\mu}{\sigma} = \frac{72-68.22}{3.29} = 1.15$




***Now, We are looking for*** $\boldsymbol{P(X > 72) = ?}$

$$P(X > 72) = P(Z > 1.15)$$
$$= 0.5 - P(0 \leq Z \leq 1.15)$$
$$= 0.5 - 0.3749$$
$$\therefore \boldsymbol{P(X > 72) = 0.1251}$$

Now, Expected number of Indian soldiers having height more than 6 feet is…
$= NP(X > 72)$
$= 1000(0.1251)$
$= 125.1$
$\approx$ **125 soldiers**

# Poisson & Exponential

## Poisson

- **Number of hits** to Marwadi University's website in **one minute**.
- **Number of soldiers** killed by horse-kick **per year**.
- **Number of customers** arriving at first floor's Tea Post **in one hour**.

✓ **So, Events per single unit of time.**


## Exponential

- Number of minutes **between two hits** to Marwadi University's website.
- Number of years **between horse-kick** deaths of soldier.
- Number of hours **between two customers arrive** at first floor's Tea Post.

✓ **So, Time per single event.**

***i.e.*** **Exponential variate is actually time between the events which are in Poisson distribution.**
**(i.e. you may think like that.. 'inverse' of Poisson)**

# Exponential Distribution Function

A continuous random variable ***X*** is said to follow the **Exponential distribution** if its probability function is given by..

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

$$where\ \lambda > 0\ is\ rate\ of\ distribution.$$

**Note:**

- When times between random events follows the **Exponential distribution** with rate $\lambda$, then the total number of events in a time period of length *t* follows the **Poisson distribution** with parameter $\lambda t$.
- Exponential distribution is **memoryless** distribution.(will explain later in this session)

# Parameters of Exponential Distribution

| Mean of Exponential Distribution: | Variance of Exponential Distribution: |
|---|---|

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\mathrm{d}x$$

$$= \int_{0}^{\infty} x\, \lambda e^{-\lambda x}\mathrm{d}x$$

$$= \lambda \left| x \cdot \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right|_0^{\infty}$$

$$= \lambda \cdot \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

$$\mathrm{Var}\,(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x$$

$$= \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\mathrm{d}x$$

$$= \lambda \left| x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{\lambda^2} + 2 \frac{e^{-\lambda x}}{-\lambda^3} \right|_0^{\infty}$$

$$= \lambda \left( \frac{2}{\lambda^3} \right)$$

$$= \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \qquad \left[ \because\ \mu = \frac{1}{\lambda} \right]$$

# Parameters of Exponential Distribution

**S.D. of Exponential Distribution:**

$$\mathrm{SD} = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

# Examples of Exponential Distribution

**Ex 37. Let X be the Exponential random variate with probability density function**

$$f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}}; & x>0\\ 0; & otherwise\end{cases}$$

**Find** $(i)\ P(X>5)\ (ii)P(3\le X\le 6)\ (iii)Mean\ (iv)Variance.$

**Sol:**

$$\lambda=\frac{1}{5}$$

(i) $$P(X>5)=\int_5^{\infty} f(x)\,\mathrm{d}x$$
$$=\int_5^{\infty}\frac{1}{5}e^{-\frac{x}{5}}\,\mathrm{d}x$$
$$=\frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_5^{\infty}$$
$$=-\left|e^{-\frac{x}{5}}\right|_5^{\infty}$$
$$=-(e^{-\infty}-e^{-1})$$
$$=e^{-1}$$
$$=0.3679$$

(ii) $$P(3\le X\le 6)=\int_3^{6} f(x)\,\mathrm{d}x$$
$$=\int_3^{6}\frac{1}{5}e^{-\frac{x}{5}}\,\mathrm{d}x$$
$$=\frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_3^{6}$$
$$=-\left|e^{-\frac{x}{5}}\right|_3^{6}$$
$$=-\left(e^{-\frac{6}{5}}-e^{-\frac{3}{5}}\right)$$
$$=e^{-\frac{3}{5}}-e^{-\frac{6}{5}}$$
$$=0.2476$$

(iii) Mean $$\mu=\frac{1}{\lambda}=\frac{1}{\left(\frac{1}{5}\right)}=5$$

(iv) $$\text{Variance}=\mathrm{Var}(X)=\frac{1}{\lambda^2}=\frac{1}{\left(\frac{1}{5}\right)^2}=25$$

# Examples of Exponential Distribution

**Ex.38 Let X be the Exponential random variate with probability density function,**

$$f(x)=\begin{cases} ce^{-2x}; x>0 \\ 0; otherwise \end{cases}. \quad \text{Find } (i)\ P(X>2)\ (ii) P\left(X<\frac{1}{c}\right)$$

**Sol:** Since, $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} ce^{-2x}\,dx = 1$$

$$\left|\frac{ce^{-2x}}{-2}\right|_0^{\infty} = 1$$

$$-\frac{c}{2}\left|e^{-2x}\right|_0^{\infty} = 1$$

$$-\frac{c}{2}(e^{-\infty}-e^{0}) = 1$$

$$\frac{c}{2} = 1$$

$$c = 2$$

$$\therefore\ f(x) = 2e^{-2x}, \qquad x>0$$

(i)

$$P(X>2) = \int_2^{\infty} f(x)\,dx$$

$$= \int_2^{\infty} 2e^{-2x}\,dx$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_2^{\infty}$$

$$= -\left|e^{-2x}\right|_2^{\infty}$$

$$= -(e^{-\infty}-e^{-4})$$

$$= e^{-4}$$

$$= 0.0183$$

(ii)

$$P\left(X<\frac{1}{c}\right) = P\left(X<\frac{1}{2}\right)$$

$$= \int_0^{\frac{1}{2}} f(x)\,dx$$

$$= \int_0^{\frac{1}{2}} 2e^{-2x}\,dx$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_0^{\frac{1}{2}}$$

$$= -\left|e^{-2x}\right|_0^{\frac{1}{2}}$$

$$= -(e^{-1}-e^{0})$$

$$= -e^{-1}+1$$

$$= 0.6321$$

# Examples of Exponential Distribution

**Ex.39 The mileage which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with *mean 4000 km*. Find the probabilities that one of these tires will last *(i)* at least *2000 km (ii)* at most *3000 km*.**

**Sol:** Let **X be an exponential random variable** which denotes the mileage obtained with the tire.

$$\text{Mean } \mu = \frac{1}{\lambda} = 4000 \text{ km}$$

**So, Prob. density function *f(x)* is..**

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$
$$= \frac{1}{4000} e^{-\frac{1}{4000}x}, \qquad x > 0$$

(i) $$P(X \geq 2000) = \int_{2000}^{\infty} f(x)\,\mathrm{d}x$$
$$= \int_{2000}^{\infty} \frac{1}{4000} e^{-\frac{1}{4000}x}\,\mathrm{d}x$$
$$= \frac{1}{4000} \left| \frac{e^{-\frac{1}{4000}x}}{-\frac{1}{4000}} \right|_{2000}^{\infty}$$
$$= -\left| e^{-\frac{1}{4000}x} \right|_{2000}^{\infty}$$
$$= -(e^{-\infty} - e^{-0.5})$$
$$= e^{-0.5}$$
$$= 0.6065$$

(ii) $$P(X \leq 3000) = \int_{0}^{3000} f(x)\,\mathrm{d}x$$
$$= \int_{0}^{3000} \frac{1}{4000} e^{-\frac{1}{4000}x}\,\mathrm{d}x$$
$$= \frac{1}{4000} \left| \frac{e^{-\frac{1}{4000}x}}{-\frac{1}{4000}} \right|_{0}^{3000}$$
$$= -\left| e^{-\frac{1}{4000}x} \right|_{0}^{3000}$$
$$= -(e^{-0.75} - e^{0})$$
$$= -e^{-0.75} + 1$$
$$= 0.5270$$

# Examples of Exponential Distribution

**Ex.40 The daily consumption of milk in excess of *20000 gallons* is approximately exponentially distributed with *mean = 3000 gallons*. The city has a daily stock of *35000 gallons*. What is the probability that of *2* days selected at random, the stock is insufficient for both the days.**

**Sol:** Let ***Y*** be a random variable which denotes **the daily consumption of milk consumed in a day.**
The random variable ***X = Y – 20000*** has **an exponential distribution.**

$$Mean = 3000\ gallons$$

$$\lambda = \frac{1}{3000}$$

**So, Prob. density function *f(x)* is..**

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$= \frac{1}{3000} e^{-\frac{1}{3000}x}, \qquad x > 0$$

**Now, Probability that stock is insufficient for two days is,**

$$P(Y > 35000) = P(X > 15000)$$

$$= \int_{15000}^{\infty} f(x)\,dx$$

$$= \int_{15000}^{\infty} \frac{1}{3000} e^{-\frac{1}{3000}x}\,dx$$

$$= \frac{1}{3000} \left| \frac{e^{-\frac{1}{3000}x}}{-\frac{1}{3000}} \right|_{15000}^{\infty}$$

$$= -\left| e^{-\frac{1}{3000}x} \right|_{15000}^{\infty}$$

$$= -(e^{-\infty} - e^{-5})$$

$$= e^{-5}$$

$$= 0.0067$$

# Examples of Exponential Distribution

**Ex.41 The time (in hours) required to repair a machine is exponentially distributed with *mean = 2 hours***
***(i)* What is the probability that the repair time exceeds 2 hours?**
***(ii)* What is the probability that a repair takes at least 11 hours given that its direction exceeds 8 hours?**

**Sol:** Let ***X*** be an exponential random variate which denotes the time to repair the machine.

$$Mean = 2\ hours$$

$$\lambda = \frac{1}{2}$$

**So, Prob. density function *f(x)* is..**

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

$$= \frac{1}{2} e^{-\frac{1}{2}x}, \qquad x > 0$$

**Now, Probability that repair time exceeds 2 hours is,**

$$\text{(i)} \quad P(X > 2) = \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x}\,dx$$

$$= \frac{1}{2} \left| \frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}} \right|_2^\infty$$

$$= -\left| e^{-\frac{1}{2}x} \right|_2^\infty$$

$$= -(e^{-\infty} - e^{-1})$$

$$= e^{-1}$$

$$= 0.3679$$

**Now, for question *(ii)* …….**

**We need to deal with one of the property which I mentioned in earlier slide….**
**That is Memory less ness….**

**So lets try to understand, what is Memory-less property???**