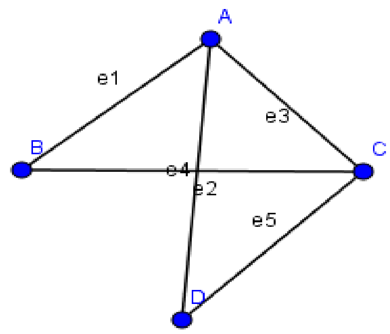
UNIT 5 Representation of Graph using Matrix

Although a pictorial representation of a graph is very convenient for a visual study, other representations are better for computer processing. A matrix is a convenient and useful way of representing a graph to a computer.

In many applications of graph theory, such as in **electrical network analysis** and **operations research**, matrices also turn out to be the natural way of expressing the problem.

In a graph, two vertices are said to be **adjacent, if there is an edge between the two vertices.**

In a graph, two nonparallel edges are said to be adjacent, if there is a common vertex between the two edges.



Here A and B are adjacent vertices, because there is edge e1 between A and B. Vertices B and D are not adjacent vertices.

Edges e1 and e3 are adjacent edges, as there is a common vertex A between them.

Edges e1 and e5 are not adjacent edges, as there is no common vertex.

Adjacency Matrix

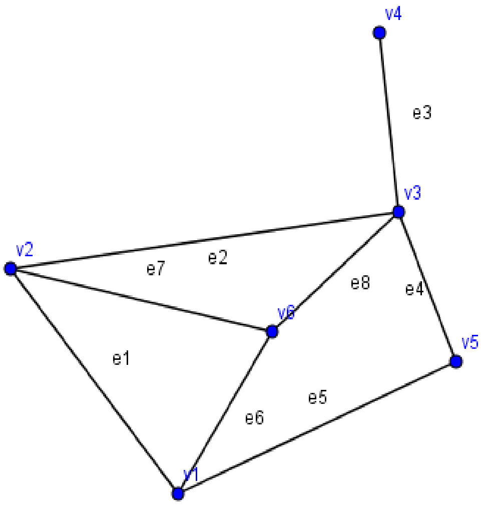
Define matrix $A = [a_{ij}]$ of size n x n,

The adjacency matrix of a graph G with n vertices and no parallel edges is n x n symmetric matrix such that

- The elements of matrix A are

$A = [a_{ij}]$ = 1 if there is edge is between ith and jth vertices.

= 0 otherwise

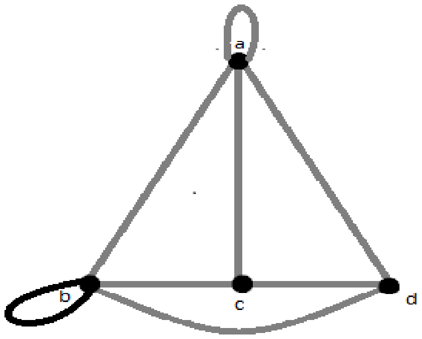


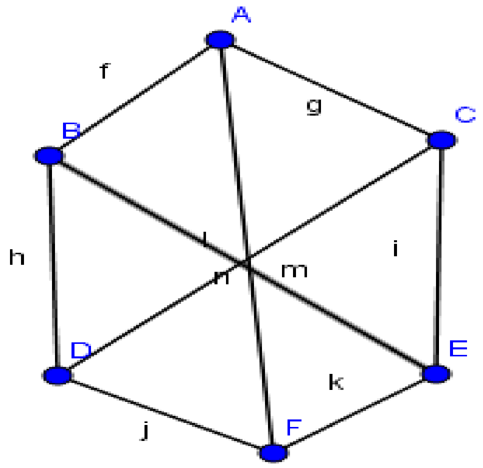
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

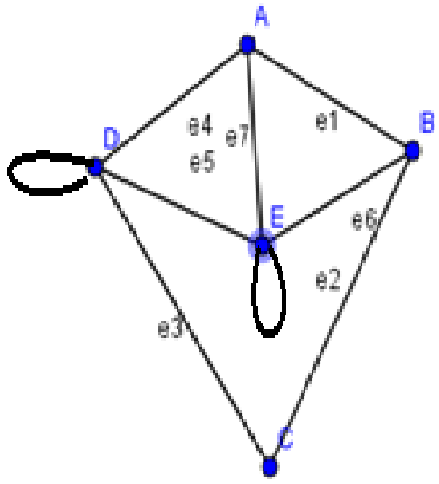
Observations

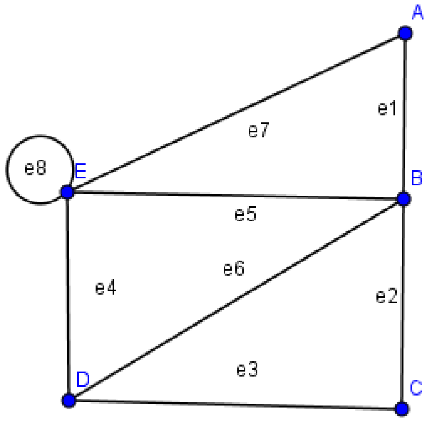
- The entries along the principal diagonal of the adjacency matrix A are 0's if and only if the graph has no self-loops. If a self-loop at the i the vertex then corresponding =1 in the adjacency matrix A
- The definition of adjacency matrix makes no provision for parallel edges.
- If the graph has no self-loop, the degree of the vertex equals the number of 1's in the corresponding row or column of A.

Evaluate an adjacency matrix for the following graph.









Draw Complete graph with K_5 and $K_{3,4}$ and find its adjacency matrix

Draw a graph for following the adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$
\begin{array}{c c}
 & \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

Incidence matrix

Let G be a graph with m vertices, n edges and no self loops,

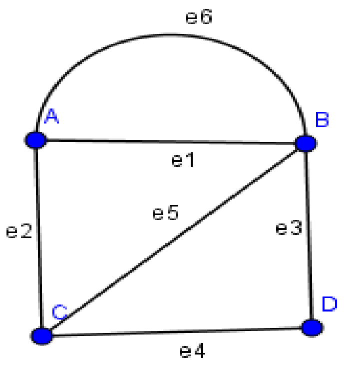
Define matrix $A = [a_{ij}]$ of size m x n,

where m rows correspond to the m vertices and the n columns correspond to the n edges as follows

- The elements of matrix A are

$A = [a_{ij}]$ = 1 if jth edge is incident on ith vertex

= 0 otherwise



Incidence matrix of the given graph is

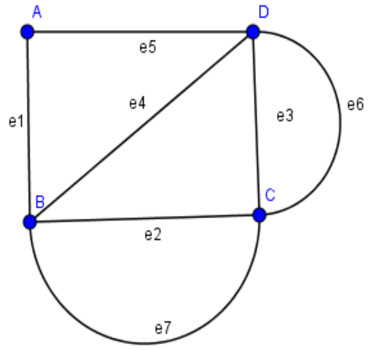
$$\begin{array}{c}
 \\ A \\ B \\ C \\ D
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e1 & e2 & e3 & e4 & e5 & e6 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}
\end{array}$$

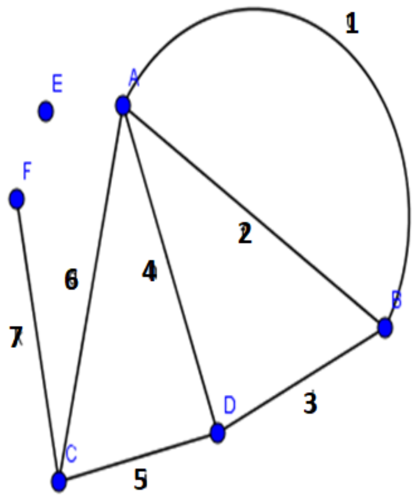
Observations

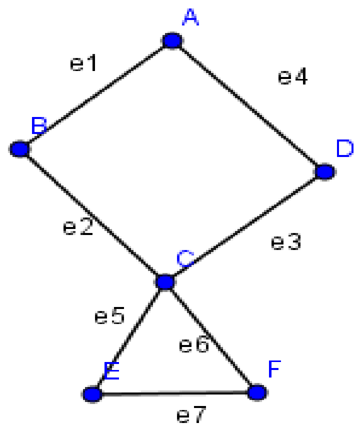
- Since every edge is incident on exactly two vertices, each column of incidence matrix A has exactly two 1.
- The numbers of 1 in each row equal to the degree of the corresponding vertex.
- A row with all 0 elements represents an isolated vertex.
- Parallel edges in a graph produce identical columns in its incidence matrix.

Example

Evaluate the incidence matrix of the following graph.







Draw Complete graph with K_5 and $K_{2,4}$ and find its matrix

Draw a graph for the following incidence matrix.

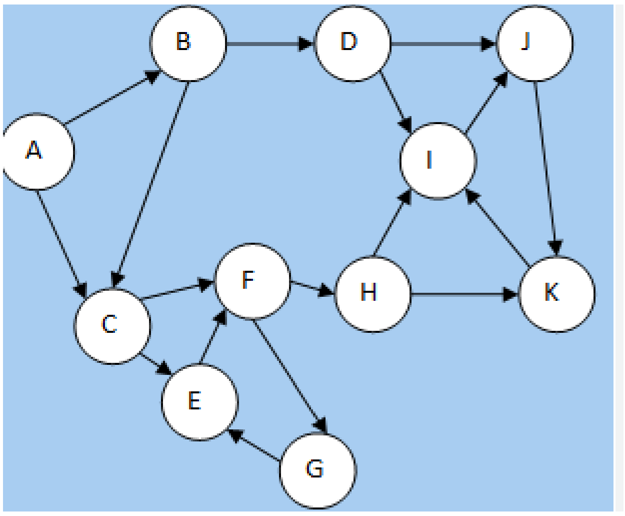
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reachability

A vertex v in a graph G is said to be reachable from the vertex u of graph G, if there exists a path from u to v or from v to u.

Reachable set : The set of vertices which are reachable from a given vertex v is called the reachable set of v and it is denoted by R(v).



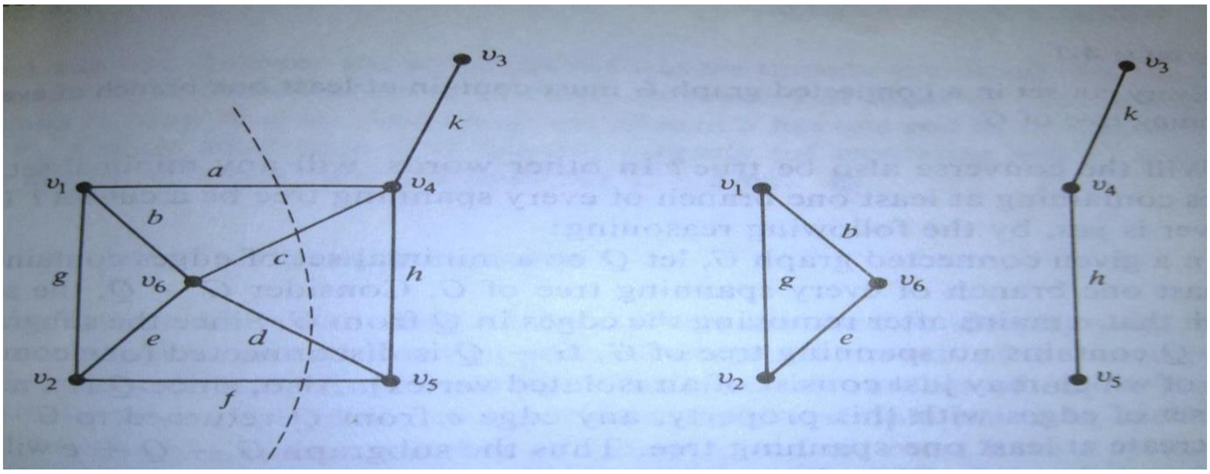
In this graph….

R(A) = {B ,C, D, E, F, G, H,I,J,K, }

R(K) = { I , J }

Cut sets

In a connected graph G, a cut set is a set of edges whose removal from G leaves disconnected, provided removal of no proper subset of these edges disconnects G



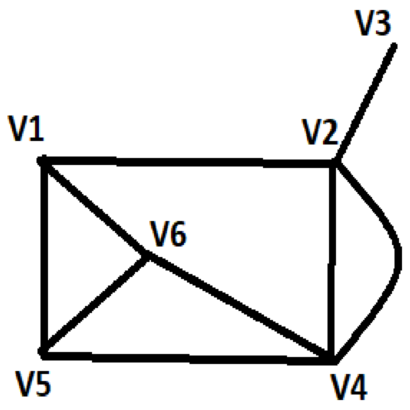
In this figure the set of edge {a, c, d, f } is a cut set

- There are many other cut set such as {a, b, g}, {a, b, e, f}, {d, h, f} and {k}.
- The set of edge {a, c, h, d} is not cut set because one of its proper subset

  {a, c, h} is a cut set.
- The Cut set is also known as co cycle.
- Since removal of any edge from a tree breaks the tree into two parts, so every edge of a tree is a cut set.

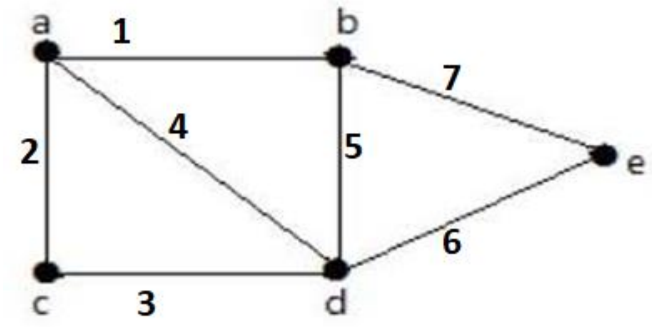
Application of Cut sets

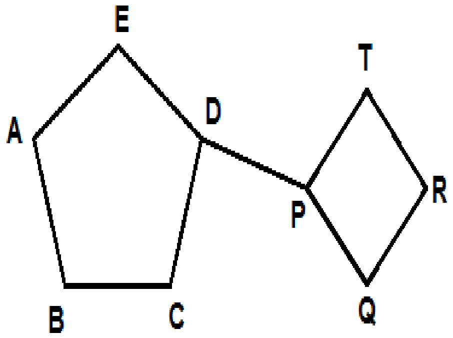
- Cut sets are important in studying properties of communication and transportation networks.
- Suppose there are six vertices in the graph represent six cities connected by telephone lines.
- We wish to find out if there are any weak spots in the network that need strengthening by means of additional telephonic lines .
- We look at all cut set of the graph and the one with the smallest number of edges is the most vulnerable.
- In figure the city represented by vertex v3 can be severed from the rest of the network by the destruction of the just one edge.

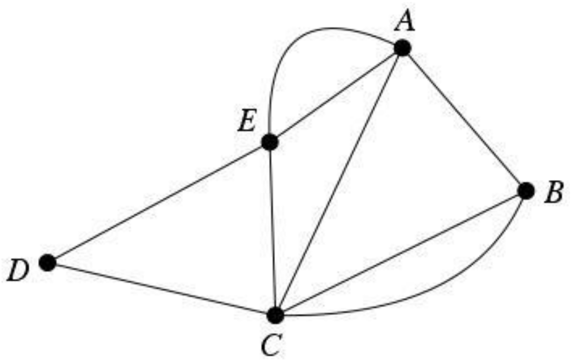


Examples

Find 7 cut sets of the following graphs



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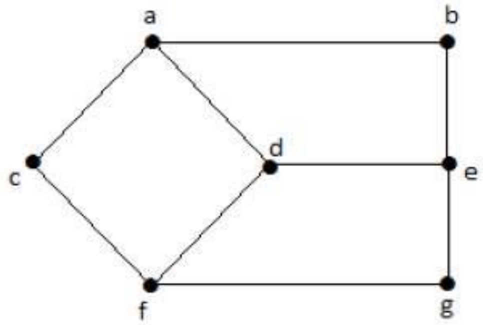
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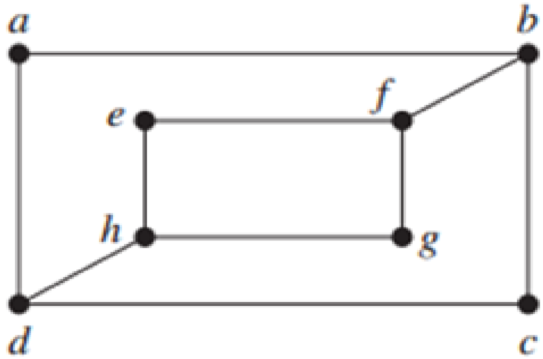
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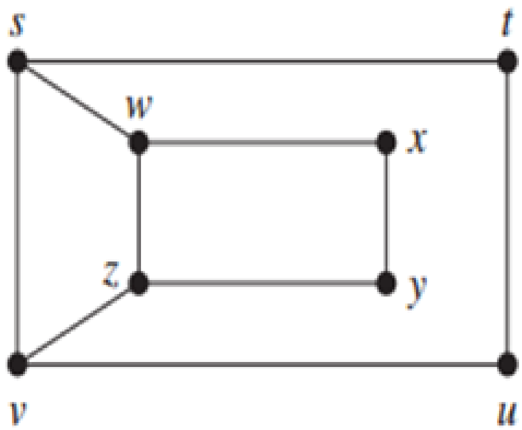
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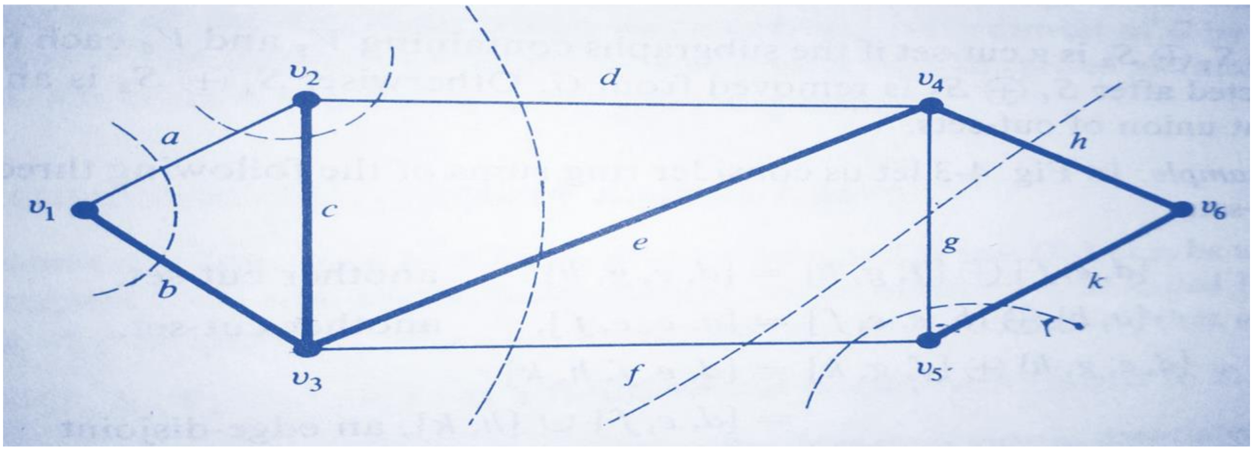
Fundamental cut sets and Fundamental Circuit

- An edge in a spanning tree T is known as a branch and an edge of G that is not in a spanning tree T is known as chord.
- Cut set S of the graph G containing exactly one branch of spanning tree T of G is called Fundamental cut set with respect to T
- Sometimes a Fundamental cut set is also known as basic cut set.
- Consider a spanning tree T of a connected graph G.



- In above graph G, e is a branch of G with respect to spanning tree T
- Consider cut set S={ d, e, f } of graph G
- Cut set S contains only one branch e of T and the rest of the edges in S are chords with respect to T.
- Here S is Fundamental cut set of G
- Consider the spanning tree T of connected graph G, let c be the chord with respect to T and let the circuit R made by chord c and branches b_1, b_2, ..., b_k of G with respect to T then R is called fundamental circuit with respect to T.
- As in above example { a, b, c } , { d, c, e } and { f, e, h, k } are fundamental circuits with respect to T.

Connectivity of graph

1) Edge Connectivity

- Each cut set of a connected graph G consists of a certain number of edges .The number of edges in the smallest cut set is defined as the edge connectivity of G
- The edge connectivity of the tree is one.

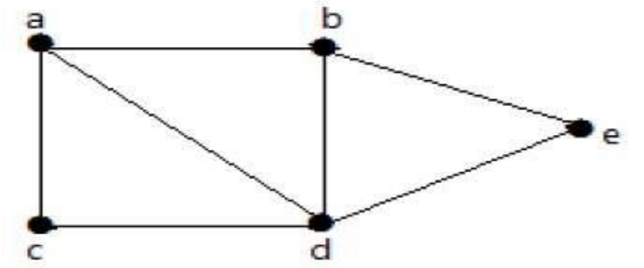
2) Vertex Connectivity

- The vertex connectivity of a connected graph G defined as the minimum number of vertex whose removal from G leaves the remaining graph disconnected.
- The vertex connectivity of a tree is one.

Examples

Find the vertex connectivity and edge connectivity of the following graphs.

1]



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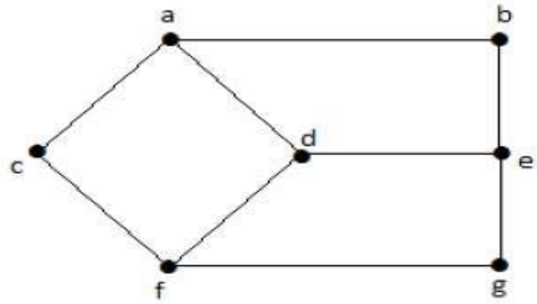

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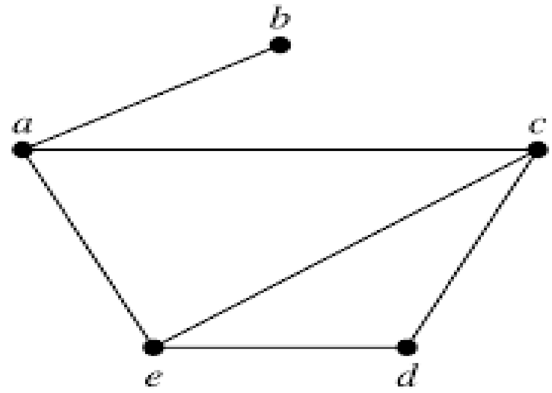

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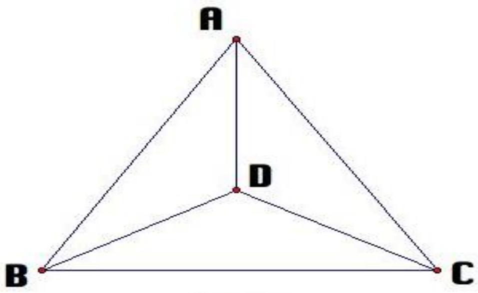

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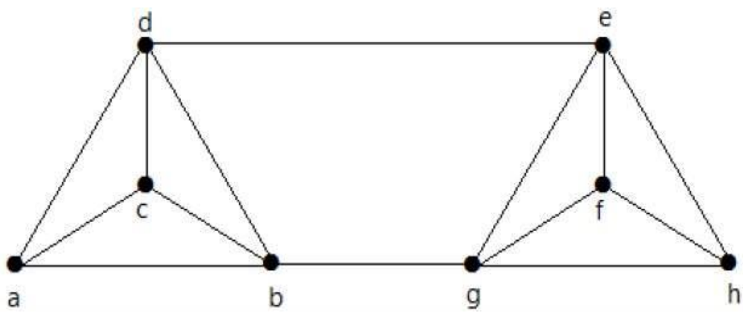

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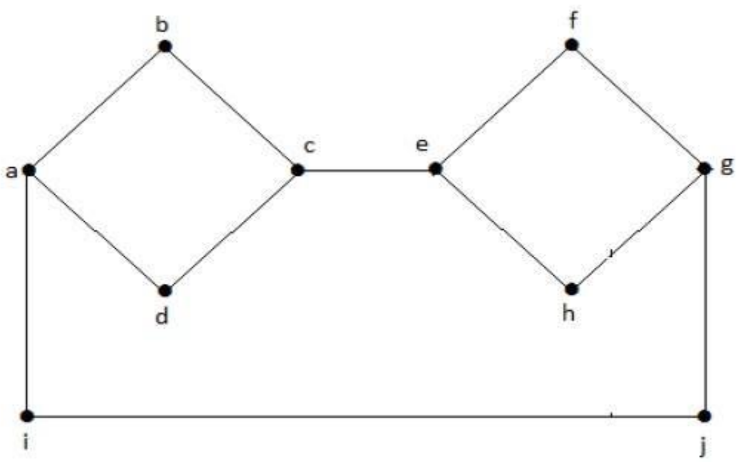

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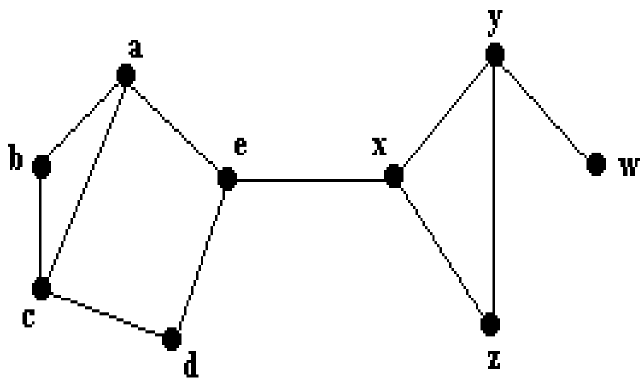
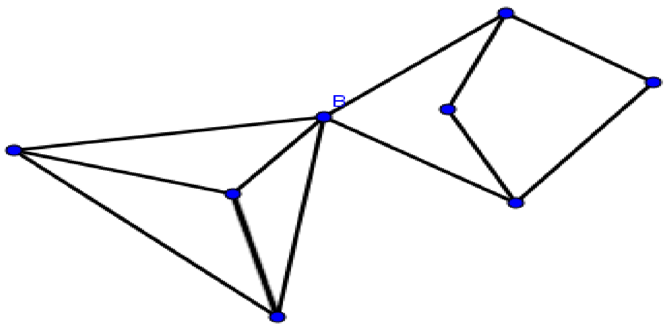
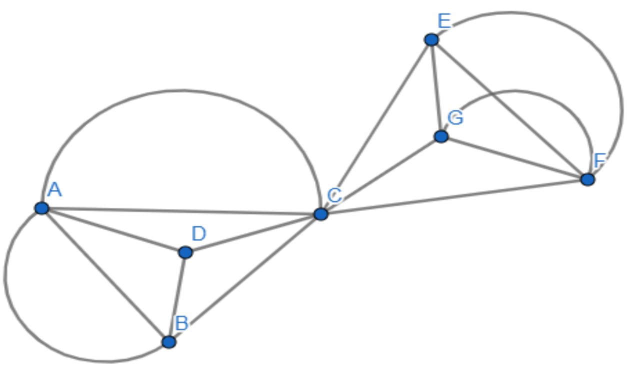


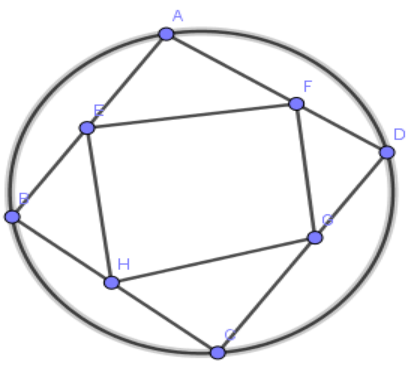
Separable graph

- A connected graph is said to be separable if its vertex connectivity is one.
- In a separable graph a vertex whose removal disconnects the graph is called a cut vertex.
- In tree every vertex with degree greater than one is known as cut vertex



- In this graph cut vertex is B and this is separable graph. It has 1 vertex connectivity and 2 edge connectivity.
- **What is best way of connecting?**





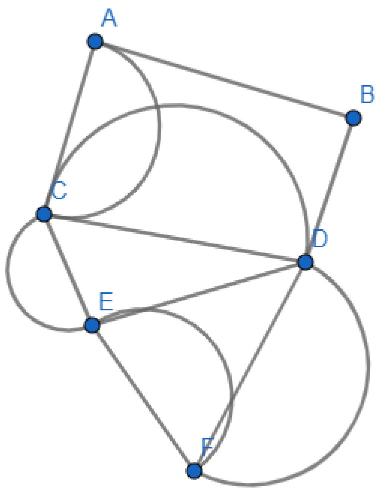
In both graph there are 8 vertices and 16 edges.

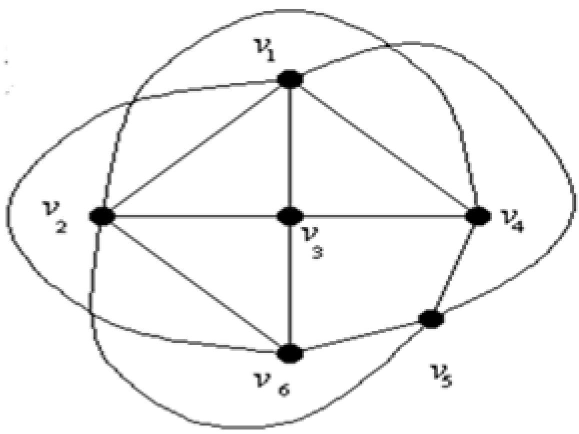
First graph has vertex connectivity is one and edge connectivity is three

While in second graph edge connectivity as well as vertex connectivity is four. Consequently, even after any three stations are bombed or any three line destroyed the remaining stations can still continue to “communicate” with each other.

Thus the second graph is batter connected than that of first graph.

- **Which graph is better connected?**





Answer :

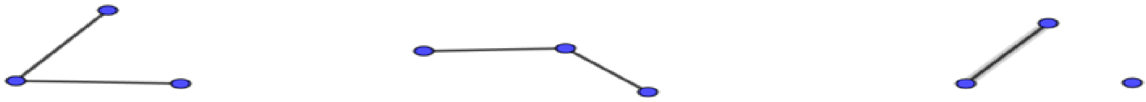
Isomorphic graph

- When we say graphs are “essentially the same” mean that they differ only in the way they are drawn.
- There should be a one to one correspondence between the vertices of the graphs and a one to one correspondence between their edges such that corresponding vertices are incident with corresponding edges. The proper term for essentially the same is isomorphic.
- Definition : Given graphs G1= G1 (V1, E1) and G2 = G2(V2, E2), we say that G1 is isomorphic to G2 if there is a one to one function F : V1 onto V2 such that if there is an edge between u and v in G1 then there is an edge between F(u) and F(v) in G2.

Here F is called isomorphism from G1 to G2

Example 1

- Consider the three graph with number of vertices are three.



G1 G2 G3

- Graph G1 and G2 in figure each consist of two edge incident with a common vertex. they drawn differently ,but the graphs are same. G1 and G2 are isomorphic.
- The graph G3 indicates that this graph has only one edge, G3 is different from G1 and G2 in an essential way, it is not isomorphic to G1 and G2.

G1 G2

- This two graphs are not isomorphic. Each graph consists of four vertices and four edges, but G2 contains a vertex of degree 1 while G1 has no such vertex.

These graphs are isomorphic and they are one to one correspondence graphs with four vertices and four edges and same number of degree.

➢ Suppose two graphs G1 (V1, E1) and G2 (V2, E2) are isomorphic graphs. Then

1. |E1| = |E2| and |V1| = |V2| i. e. Graphs G1 and G2 must have the same number of edges and vertices.

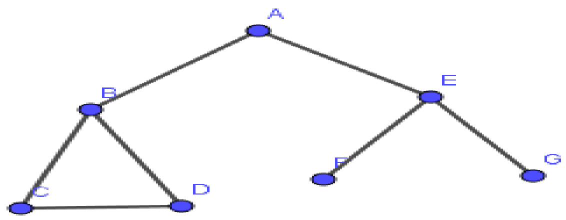
2. Graphs G1 and G2 must have the same number of regions formed by an equal number of edges.

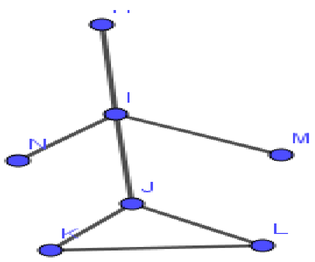
3. Graphs G1 and G2 must have an equal number of loops.

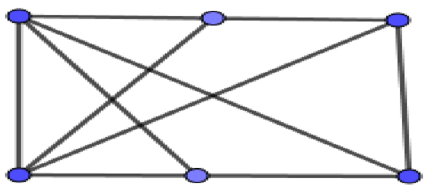
4. Graphs G1 and G2 must have the same number of parallel edges.

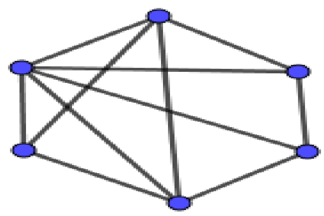
5. Graphs G1 and G2 must have the same number of bridges.

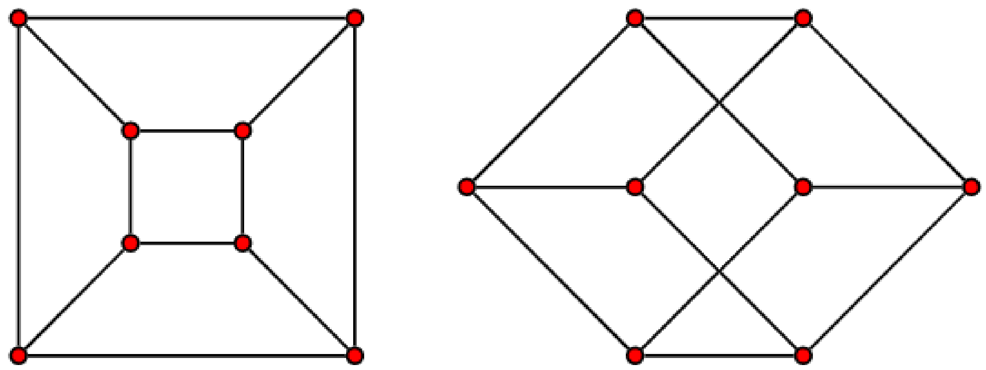
Explain which of the two graphs are isomorphic

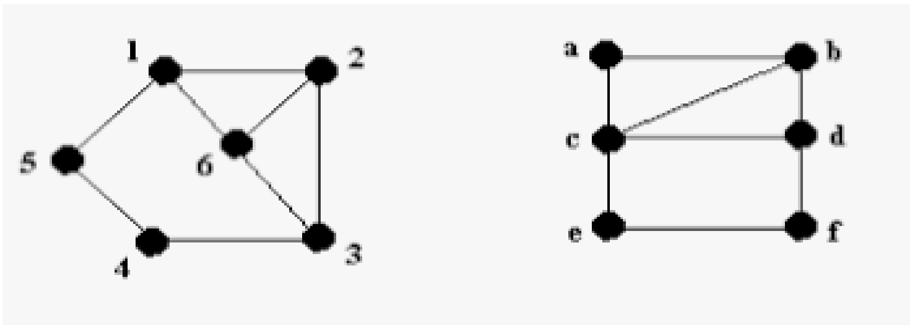








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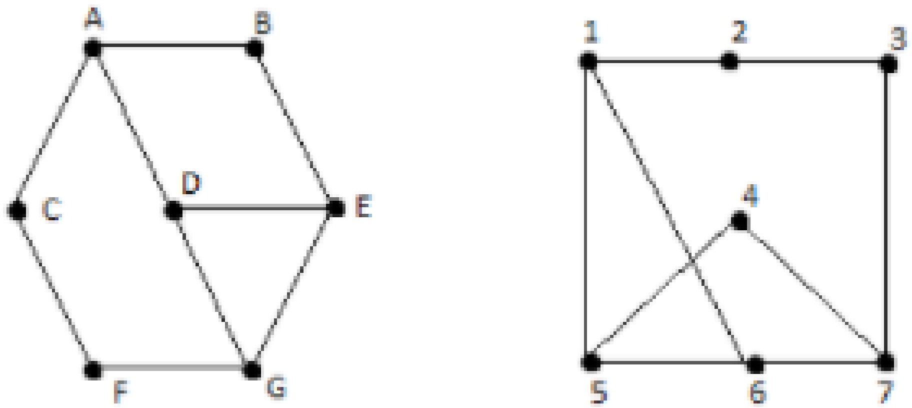
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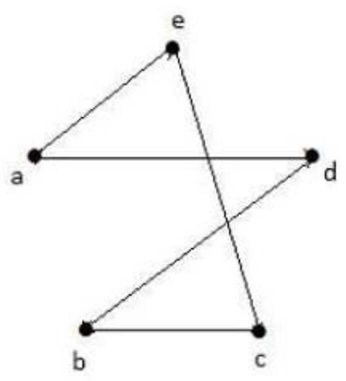
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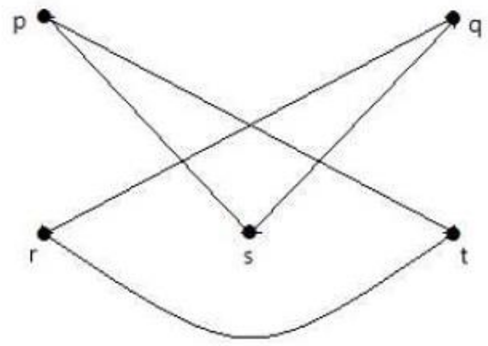
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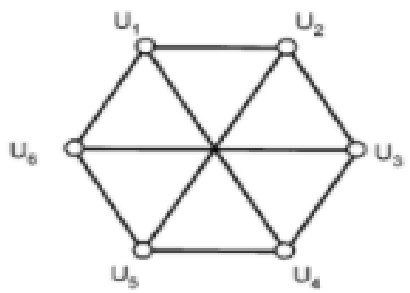
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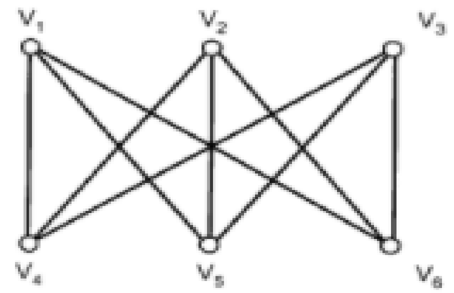
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