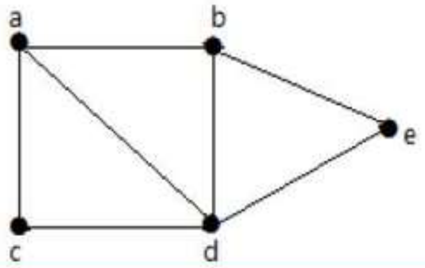
UNIT 6: Planar and non-planar graph

In this unit we will discuss

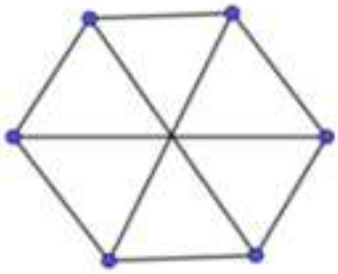
- Planar and Non-planar Graphs
- Graph embedding
- Kuratowski’s first and second graphs
- Euler’s formula
- dual graph
- graph coloring
- Region coloring

Planar graph

A graph G = (V, E) is said to be planar if it can be drawn on a plane so that no two edges cross each other at a non-vertex point; otherwise it is called non-planar graph.



planar graph.



Non Planar graph

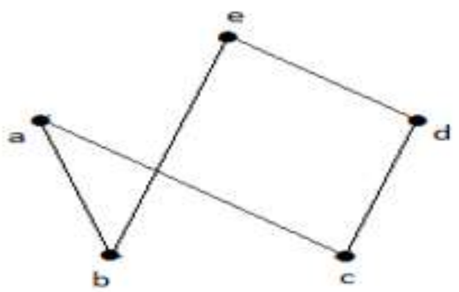
- **Embedding**.

A drawing of a geometric representation of a graph on any surface such that no edge intersects is called embedding.

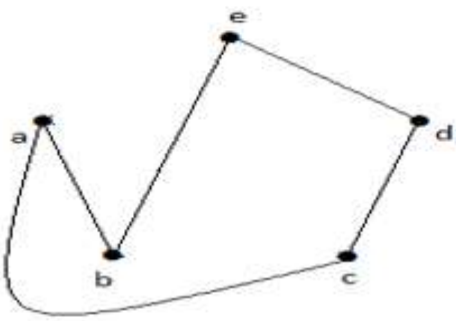
i.e.

When is it possible to draw a graph so that none of the edges cross? If this is possible, we say the graph is planar (since you can draw it on the plane).

An embedding of a planar graph G on a plane is called a plane representation of G.



NON - PLANAR GRAPH



PLANAR GRAPH

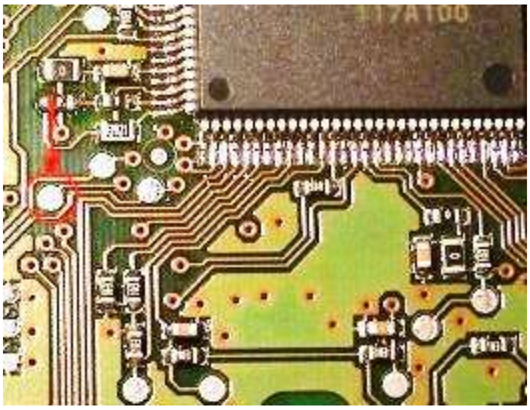
➢ Application of planar graph:

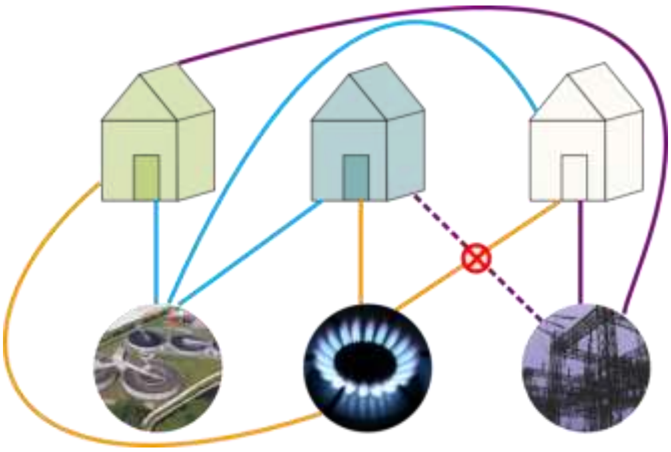
Sometimes, it's really important to be able to draw a graph without crossing edges.

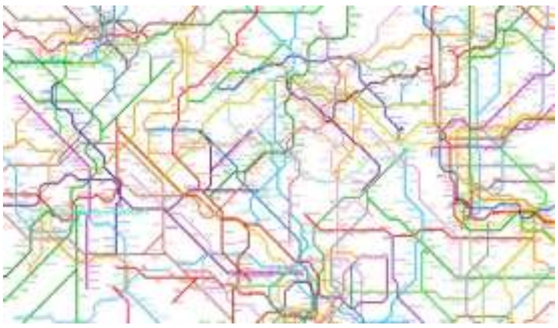
From the text: connecting utilities (electricity, water, natural gas) to houses. If we can keep from crossing those lines, it will be safer and easier to install.

Connecting components on a circuit board: the connections on a circuit board cannot cross. If we can connect them without resorting to another layer of traces, it will be cheaper to produce.

Subway system: if subway lines need to cross, there is needed idea of the planar graph.







Exercise 1 .Apply embedding (if required) and check which of the following graph is planar graph.

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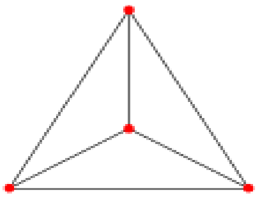

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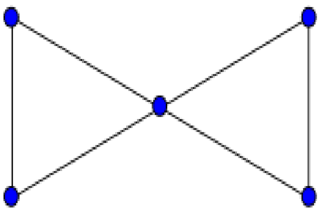

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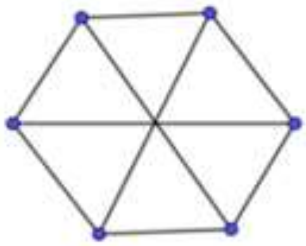

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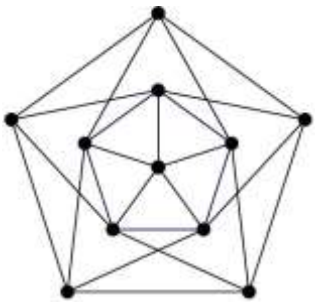

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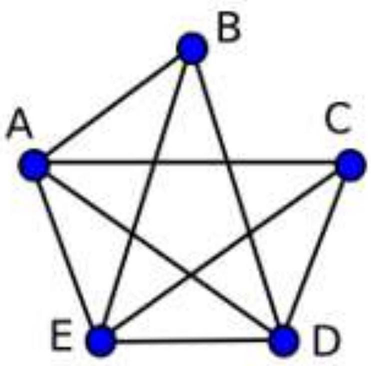

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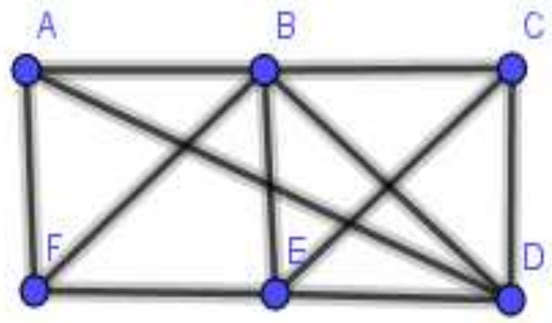

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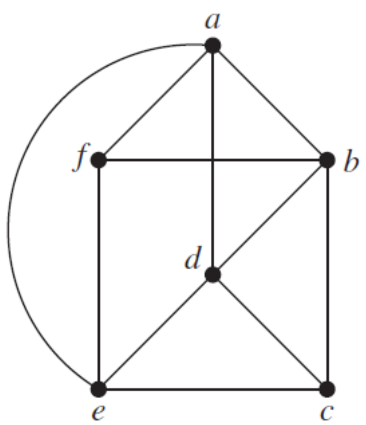
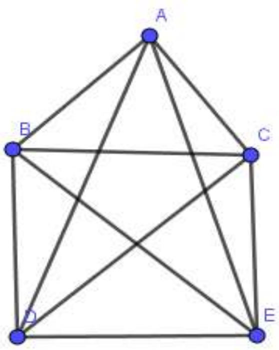
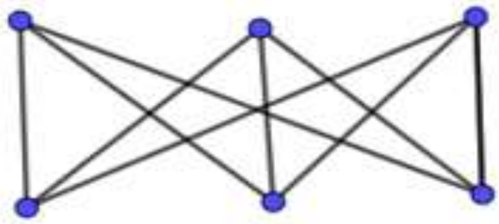


- **Kuratowski's two graphs:**

The graph K_5 and $K_{3,3}$ are known as Kuratowski's graph.

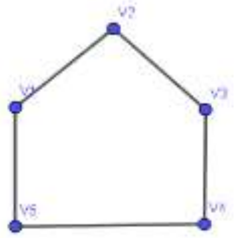




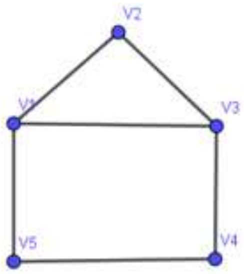
Theorem: The complete graph with five vertices is non-planar.

Proof:

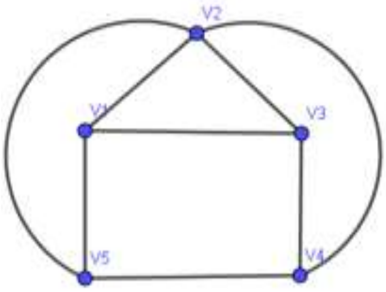
- Let the five vertices in the complete graph be named V_1, V_2 V_3, V_4, and V_5.
- A complete graph as you may recall is a simple graph in which every vertex is joined to every other vertex by means of an edge.
- This being the case, we must have a circuit going from V_1 to V_2 to V_3 to V_4 to V_5 to V_1 that is pentagon. See the first figure. This pentagon must divide the plane of the paper into two regions, one inside and the other outside.



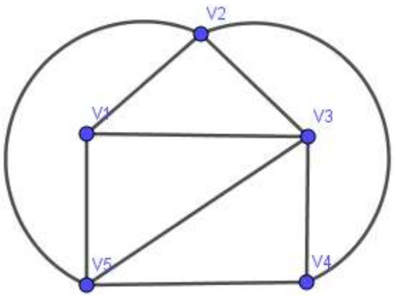
- Since the vertex V_1 is to be connected to V_3 by means of an edge this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously). Suppose that we choose to draw a line from V_1 to V_3 inside the pentagon. See in figure (b) (if we choose outside, we end up with the same argument



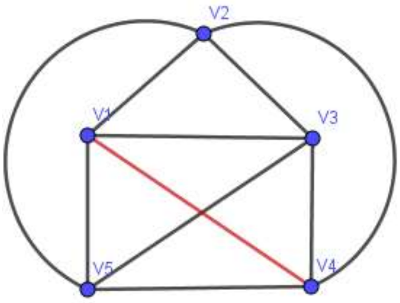
- Now we have to draw an edge from V_2 to V_4 and another from V_2 to V_5. Since neither of these edge can be drawn inside the pentagon without crossing over the edge already drawn, we draw both of these edges outside the pentagon see the figure



- The edge connecting V_3 and V_5 cannot be drawn outside the pentagon without crossing the edge between V_2 and V_4.



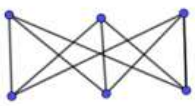
- Now we have yet to draw an edge between V_1 and V_4. This edge cannot be placed inside or outside the pentagon without a crossover .Thus the graph cannot be embedded in a plane.

- 

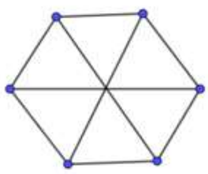
Thus the complete graph of five vertices is nonplanar.

- <u>**Kuratowski's Second graph K $_{3,3}$**</u>

- The second graph of Kuratowski's is a connected graph with six vertices and nine edges, Shown in the two common geometric representation in the following figure (A) and (B)



A



B

- Employing visual geometric arguments similar to those used in proving above theorem, it can be shown that the second graph of Kuratowski's is nonplanar.

- **Properties**

  (1) Both graphs are simple graph

  (2) Both are nonplanar

  (3) Removal of one edge or a vertex each a planar graph.

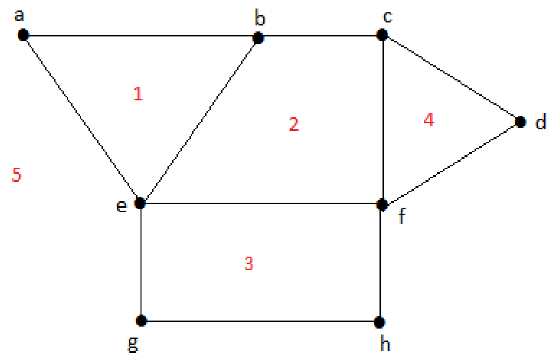
  (4) Kuratowski's first graph is the nonplanar graph with the smallest number of vertices and the Kuratowski's second graph is the nonplanar with the smallest number of edges. Thus both are simplest nonplanar graphs.

  Regions

  Definition

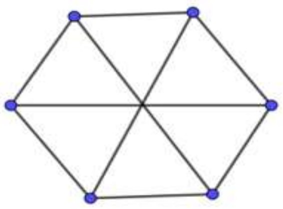
  Every planar graph divides the plane into connected areas called regions.

  See below example

- 

- The regions lying inside the graph (i. e. 1, 2, 3, 4) are called interior faces and the region lying outside the graph on the plane is called exterior face (i. e. 5).

- Note that the region is not defined in non-planar graph.
- For example the following graph has not region.



- **Euler formula:**

- **A connected planar graph with n vertices and e edges and f is the number of faces (region) then $n - e + f = 2$.**

- **Proof:** By induction on the number of edges in the graph.

  Base: For $e = 0$, the graph consists of a single vertex with a single region surrounding it. So we have $1 - 0 + 1 = 2$ which is clearly right.

  Induction:
  Suppose the formula works for all graphs with no more than e edges.
  Let G be a graph with $e + 1$ edge.
  Case 1:
  G is a tree with $e + 1$ edges. Tree has no cycles. So, it will contain only one face which is exterior face. Thus $f = 1$. Also in a tree with $e + 1$ edges, the numbers of vertices are $e+2$. Thus, $n - e + f = e+2 - (e + 1) + 1 = 2$. Thus the formula works for tree with any number of edges.
  Case 2:

  Let G be a graph with $e + 1$ edges and G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G'. Since the cycle separates the plane into two regions, the regions to either side of p must be distinct. When we remove the edge p, we merge these two regions. So G'has one fewer regions than G. Since G' has e edges, the formula works for G'by the induction hypothesis.

  That is $n' - e' + f' = 2$. But $n' = n$, $e' = e$, and $f' = f - 1$.

  Substituting, we find that $n - e + (f - 1) = 2$. So $n - (e + 1) + f = 2$.

  This proves the formula for G.

- **Corollary:**

  Let G be a simple connected and planar graph with n vertices ($n \geq 3$), e edges and f number of faces then $e \leq (3n - 6)$.
- Proof:
  Let G be a simple connected and planar graph with n vertices ($n \geq 3$), e edges and f number of faces.
  Now it can be easily seen that for G, $2e \geq 3f$
  That is $f \leq (2/3)e$,
  but by Euler's formula
  $f = -n + e + 2$
  $-n + e + 2 \leq (2/3)e$
  $e \leq 3n - 6$. Hence proved.
- **This is necessary condition but not sufficient condition.**

Examples: Verify the Euler theorem to the following graph:

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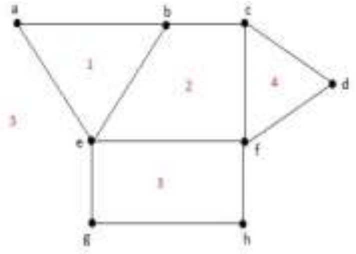

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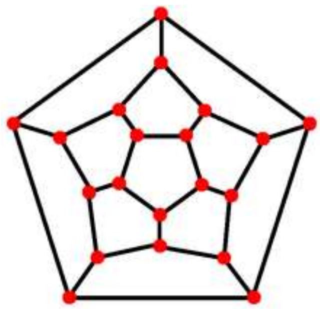

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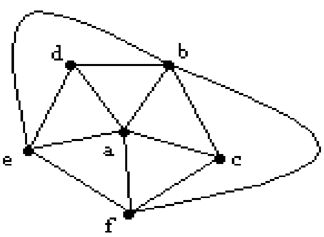


4]

K_4

5]

$K_{3,2}$

Answer the following questions.

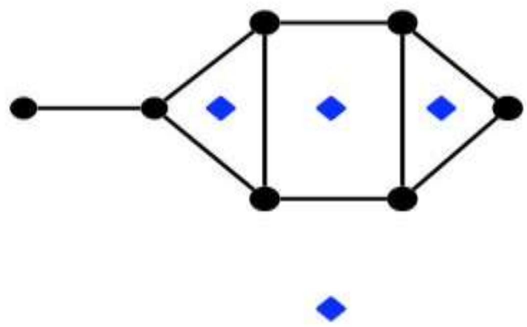
- A planar graph is contains 8 vertices, 9 edges then how many faces that graph contains?

- A planar graph contains 7 faces and 11 edges then how many vertices that graphs contains?

- A planar graph contains 5 vertices and 10 edges then how many faces that graphs contains?

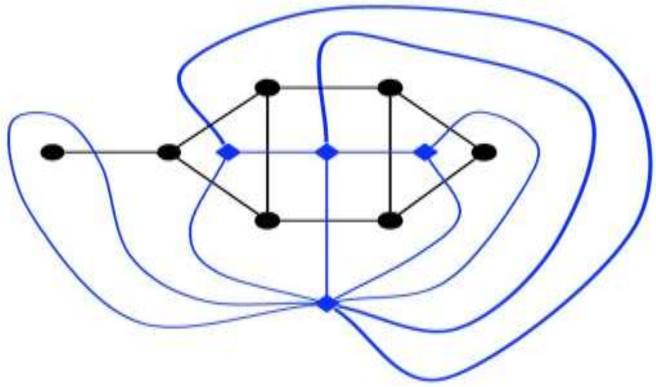
<u>Graph duality or Geometric dual:</u>

- **<u>Definition:</u>**
- Given a planar graph G, its geometric dual G* is constructed by placing a vertex in each region of G (including the exterior region) and, if two regions have an edge e in common, joining the corresponding vertices by an edge e' crossing only e. The dual G* of G is again planar graph.
- Given a planar graph G, its geometric dual G* is constructed by following steps.
- Take a plane drawing of G.
- Choose one point inside each face of the plane drawing. These points are vertices of G*. (As shown in following figure)



- For each edge 'e' of the graph G, draw a line connecting the vertices of G* on each side of 'e'. (This means connect the vertices of two adjacent faces by an edge which intersect the edges which connects both the adjacent faces)

- For an edge ‘e’, lying entirely in one region, draw a self-loop at that point intersecting the edge ‘e’ exactly once.

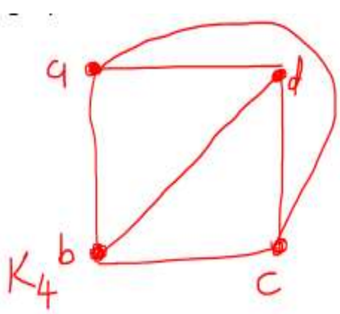


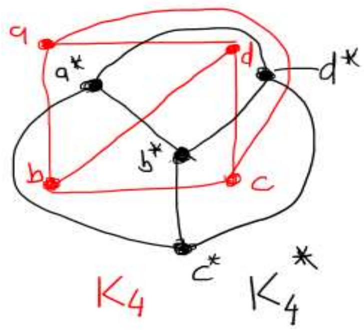
- Some observations about geometric duals
- If G is connected planar than G* is also connected and planar.
- Each plane drawing of G gives rise to only one dual graph G*.
- No. of vertices of G* = No. of Faces of G
- No. of edges of G* = No. of edges of G
- No. of faces of G* = No. of vertices of G

Self-dual graph:

- A Graph G is called self-dual graph if G and G* are isomorphic.

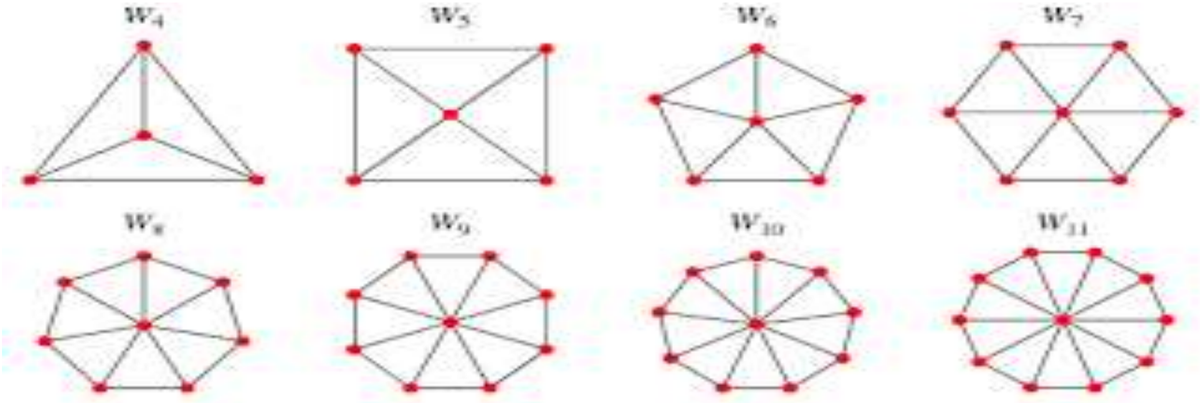
For example K_4 and K_4* are isomorphic graphs. Thus it is self-dual graph.



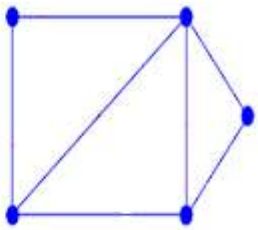


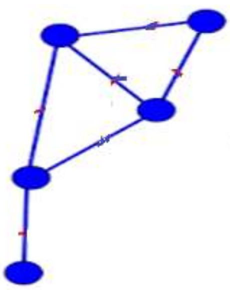
WHEEL GRAPH :

- A **wheel graph** is a graph formed by connecting a single universal vertex to all vertices of a cycle. Its denoted by Wn



Example: Find Geometric duals for following graphs.





Graph coloring:

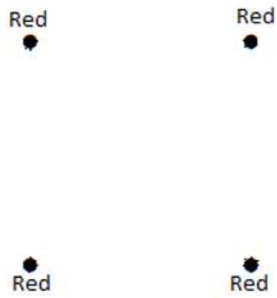
- Graph coloring is nothing but a simple way of labelling graph components such as vertices, edges, and regions under some constraints.

Vertex Coloring

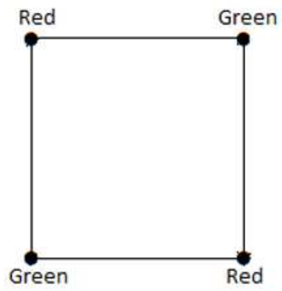
- Vertex coloring is an assignment of colors to the vertices of a graph ‘G’ such that no two adjacent vertices have the same color. Simply put, no two vertices of an edge should be of the same color.

Chromatic Number

- The minimum number of colors required for vertex coloring of graph ‘G’ is called
- as the chromatic number of G, denoted by X(G).
- X(G) = 1 if and only if ‘G’ is a null graph. If ‘G’ is not a null graph, then X(G) ≥ 2.



Null Graph ($\chi(G) = 1$)



Not Null Graph($\chi(G) = 2$)

Note: In the complete graph, each vertex is adjacent to remaining (n – 1) vertices. Hence, each vertex requires a new color. Hence the chromatic number of

Kn = *n*.

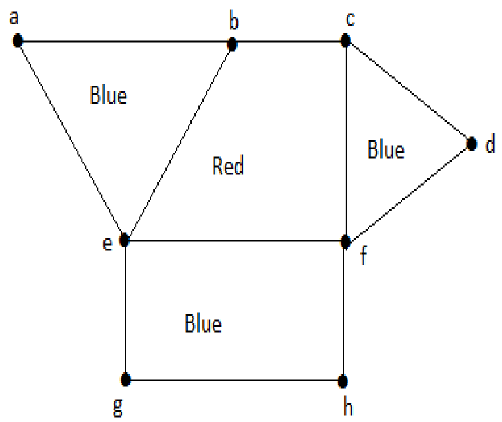
- **Region Coloring**

  Region coloring is an assignment of colors to the regions of a planar graph such that no two adjacent regions have the same color. Two regions are said to be adjacent if they have a common edge.

  Example

  Take a look at the following graph. The regions ‘aeb’ and ‘befc’ are adjacent, as there is a common edge ‘be’ between those two regions.

  Similarly the other regions are also coloured based on the adjacency. This graph is coloured as follows:

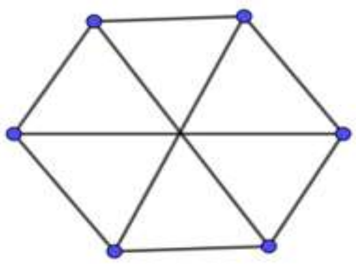


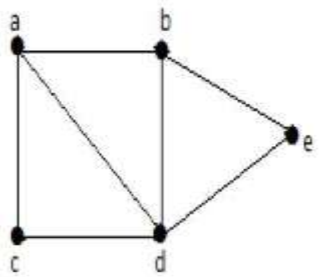
Applications of Graph Coloring

Graph coloring is one of the most important concepts in graph theory. It is used in many real-time applications of computer science such as:

- Clustering
- Data mining
- Image capturing
- Image segmentation
- Networking
- Resource allocation
- Processes scheduling

Example. Find the chromatic number of following graph





Exercise. Apply regions coloring