Topics to be covered

- Set
- Logic
- Function
- Relation
- Proof

Set

Set

- A set is a collection of objects.
- The objects in a set are called elements of the set.

- Examples:
  1. A = {11, 12, 21, 22}
  2. B = {11, 12, 21, 11, 12, 22}

  Roster Notation

  3. C = {x | x is odd integer greater than 1}
  4. D = {x | x Є B and x ≤ 11}

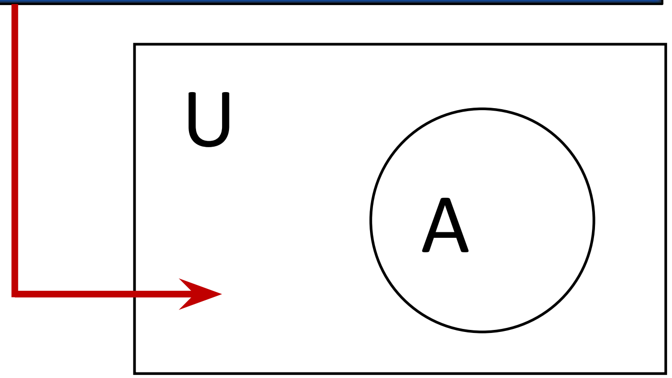
  Set-builder Notation

Operations on Sets

- Operations on the sets are:
  1. Complement
  2. Union
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The complement of a set A is the set A' of everything that is not an element of A from Universal Set U.

$$A' = \{x \in U \mid x \notin A\}$$



- Example:

  U = {1,2,3,4,5}

  A = {1,2}

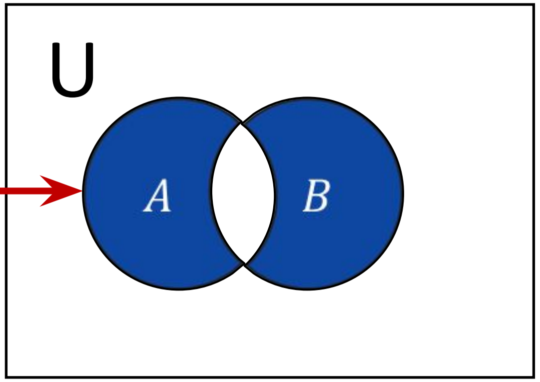
  A' = {3,4,5}

Operations on Sets

- Operations on the sets are:
  1. Complement
  2.
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The Union $(A \cup B)$ is a collection of all distinct elements from both the set A and B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



- Example:

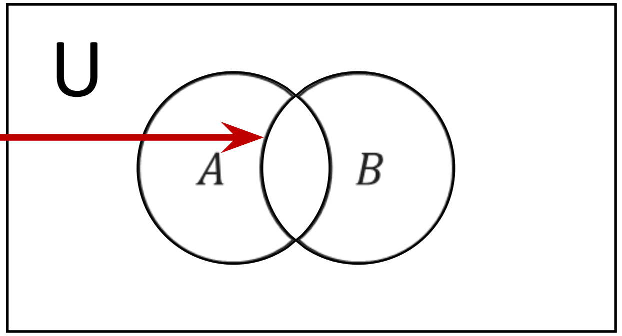
  A = {1, 3, 5, 7, 9}
  B = {1, 2, 3, 4, 5}
  A U B = {1, 2, 3, 4, 5, 7, 9}

Operations on Sets

- Operations on the sets are:
  1. Complement
  2. Union
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The intersection $A \cap B$ of two sets A and B is the set that contains all elements of A that also belong to B, but no other elements.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$



- Example:

  A = {1, 3, 5, 7, 9}

  B = {1, 2, 3, 4, 5}

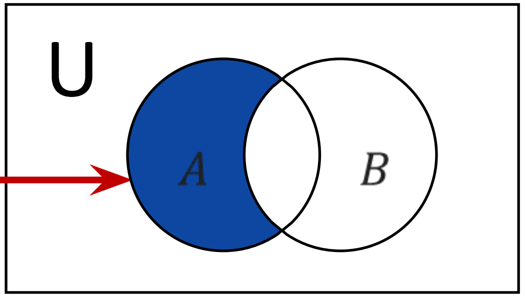
  A ∩ B = {1, 3, 5}

Operations on Sets

- Operations on the sets are:
  1. Complement
  2. Union
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The set difference *A* - *B* of two sets *A* and *B* is the set of everything in *A* but not in *B*.

$$\begin{aligned} A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\ &= \{x \mid x \in A\} \cap \{x \mid x \notin B\} \\ &= A \cap B' \end{aligned}$$



- Example:

  A = {1, 3, 5, 7, 9}

  B = {1, 2, 3, 4, 5}

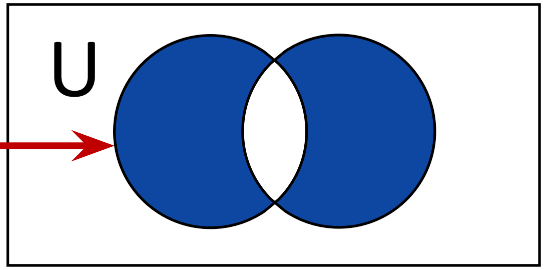
  A - B = {7, 9}

Operations on Sets

- Operations on the sets are:
  1. Complement
  2. Union
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The symmetric difference $A \ominus B$ of two sets A and B is the set of everything in A but not in B or the set of everything in B but not in A.

$$A \ominus B = (A - B) \cup (B - A)$$



- Example:

  A = {1, 3, 5, 7, 9}

  B = {1, 2, 3, 4, 5}

  $A \ominus B$ = {7, 9, 2, 4}

Operations on Sets

- Operations on the sets are:
  1. Complement
  2. Union
  3. Intersection
  4. Set Difference
  5. Symmetric Difference
  6. Cartesian product

- The Cartesian product A x B of two sets A and B is the set of all ordered pairs (a, b) where a $\in$ A and b $\in$ B.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

- Example:

  A = {1, 3, 5}

  B = {2, 4}

  A x B = {(1,2), (1,4), (3,2), (3,4), (5,2), (5,4)}

Set of identities

- Commutative laws

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

- Associative laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- Distributive laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set of identities

- Idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

- Absorptive laws

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

- De Morgan laws

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

Set of identities

- Other complements laws

$$(A')' = A$$
$$A \cap A' = \Phi$$
$$A \cup A' = U$$

- Other empty set laws

$$A \cup \Phi = A$$
$$A \cap \Phi = \Phi$$

- Other universal set laws

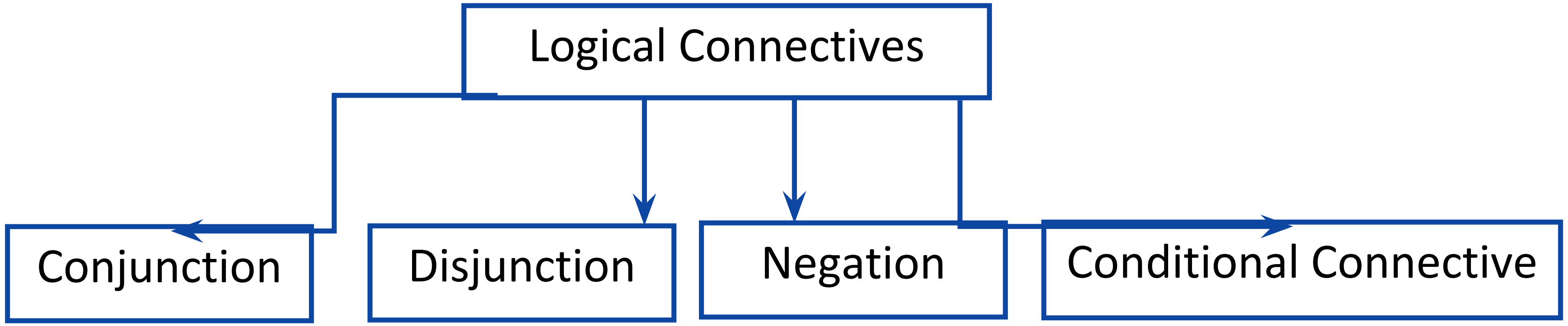
$$A \cup U = U$$
$$A \cap U = A$$

Logic

Propositions

- Declarative statement that is sufficiently objective, meaningful and precise to have a truth value (true or false) is known as proposition.
- Examples:
  1. p : Fourteen is an even integer.
  2. r : 0 = 0
  3. q : Mumbai is the capital city of India.
  4. ~~s : $a^2+b^2=4$~~

Logical Connectives

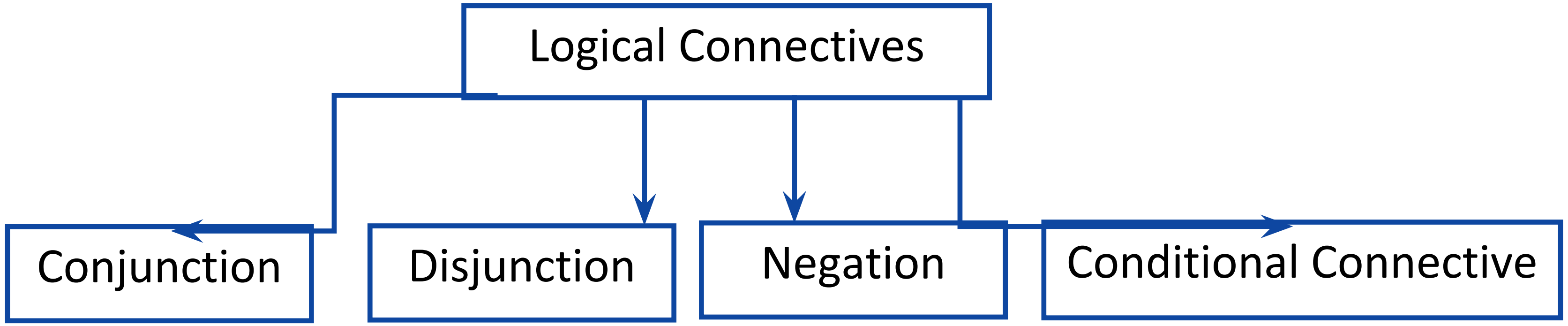


- The logical connective Conjunction (And) is true only when both of the propositions are true.
- Example:
  - p : It is raining
  - q : It is warm
  - r : It is raining **AND** it is warm

Truth table

p	q	r = p ^ q

Logical Connectives

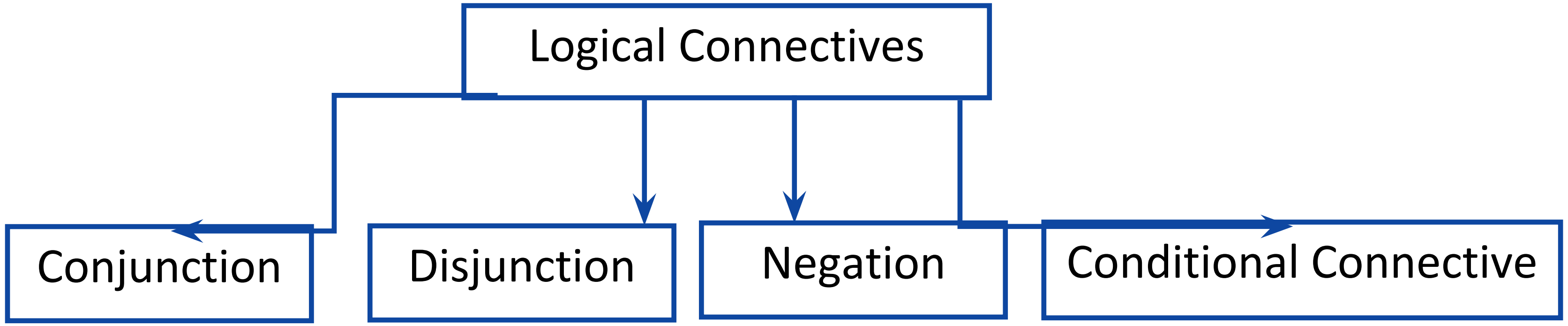


- The logical disjunction, or logical OR, is true if one or both of the propositions are true.
- Example:
  - p : 2 + 2 = 5
  - q : 1 < 2
  - r : 2 + 2 = 5 **OR** 1 < 2

Truth table

p	q	r = p v q

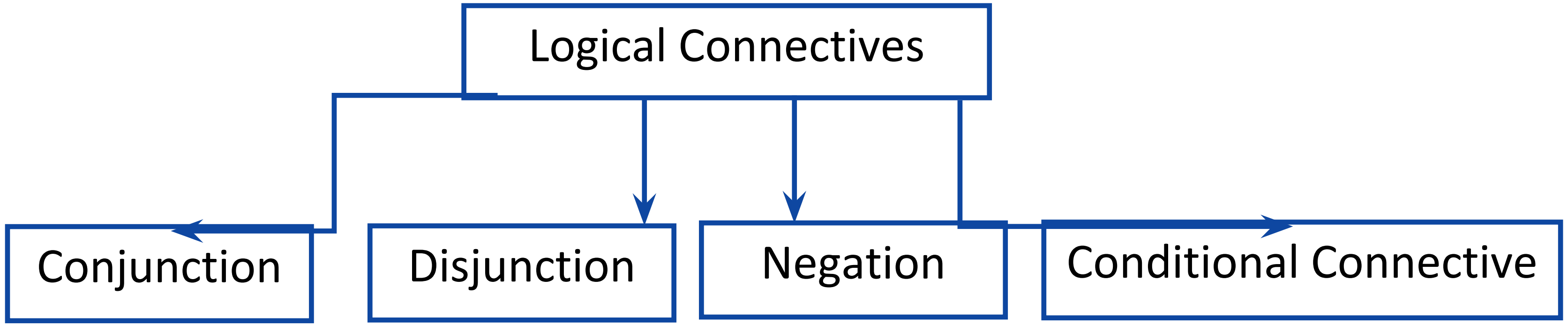
Logical Connectives



- ¬p, the negation of a proposition p, is also a proposition.
- Example: Truth table
  - p : John studies.
  - ¬p : John does **NOT** study.

P	¬P

Logical Connectives



- The proposition p → q is commonly read as “if p then q”.
- Example: P→ I will win the lottery. Q→ I will buy car for you.

I win the lottery	I will buy car for you	Promise kept/ broken

p	q	p→q

Logical Connectives

- The statement $p \rightarrow q$ can be read as following:
    1. "**if** p then q"
    2. "q **if** p"
    3. "p **only if** q"
- Consider following two statements:
    1. "p if q"$(q \rightarrow p)$
    2. "p only if q"$(p \rightarrow q)$
- If we make conjunction of (1) & (2) then,
- $(p \rightarrow q) \wedge (q \rightarrow p) = p \leftrightarrow q$ (biconditional) "p only if q, and p if q"
- Often read as **"p if and only if q"**

Tautology and Contradiction

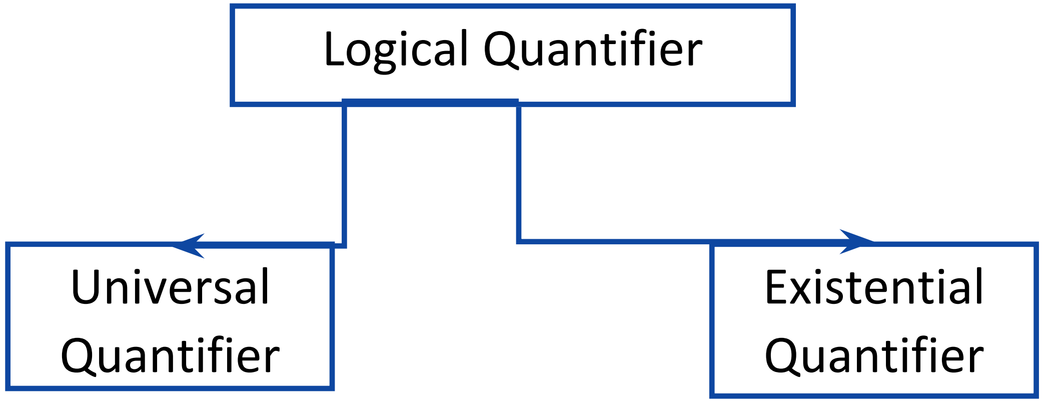
- A Compound proposition is called tautology if it is true in every case.
- Example:

P	¬P	P v ¬P

Tautology

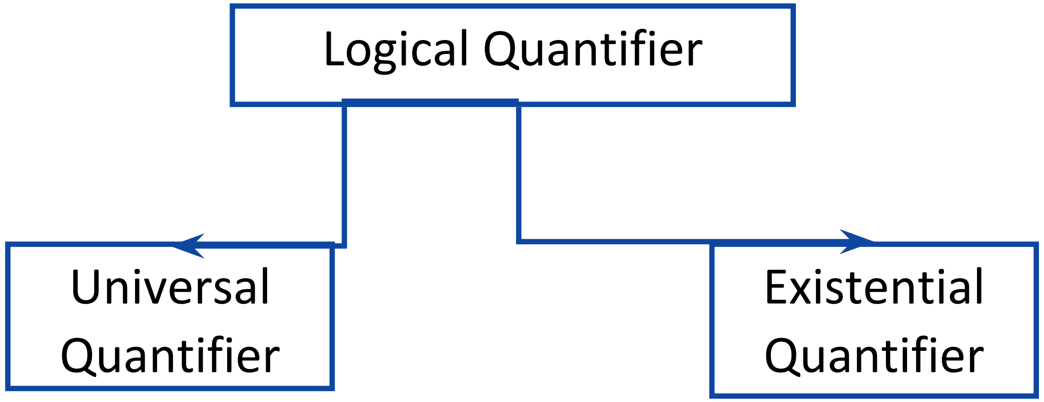
- A contradiction is opposite.
- If p is tautology, ¬P is contradiction.

Logical Quantifiers



- Represented by an upside-down A: $\forall$ (“for all”).
- Example:
  - Let P(x) = x+1 > x, $\forall$x P(x)
  - English translation: “for all values of x, P(x) is true”
  - English translation: “for all values of x, x+1>x is true”

Logical Quantifiers



- Represented by ∃ : ∃ (“for exists”).
- Example:
  - Let P(x) = x+1 > x
  - There is a numerical value for which x+1>x
  - Thus, ∃x P(x) is true

Functions

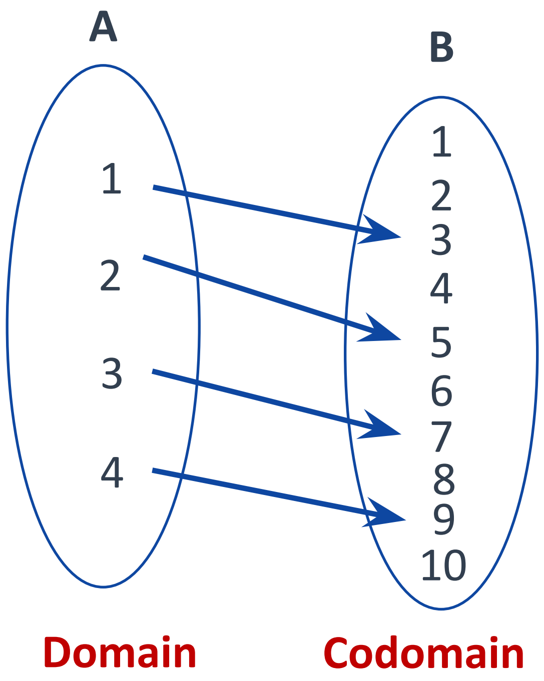
Functions

- **Domain:** What can go into the function is called domain.
- **Codomain:** What may possibly come out from a function is codomain.
- **Range:** What actually come out from a function is range. The range of function is subset of codomain
- **Example:**

$$f: N \rightarrow N, f(x) = 2x + 1$$

f(1)=2(1)+1= 3
f(2)=2(2)+1= 5
f(3)=2(3)+1= 7
f(4)=2(4)+1= 9

Range



- The range of function $f(x) = \{3, 5, 7, 9\}$

Onto Function

- If the range of function and codomain of function are equal or every element of the codomain is actually one of the values of the function, then function is said to be onto or surjective or surjection.

- Example: $f : A \rightarrow B, f(x) = x^2$ where,

  A = {-2,-1,1,2,3,4} and B = {1,4,9,16}

  f(-2) = 4

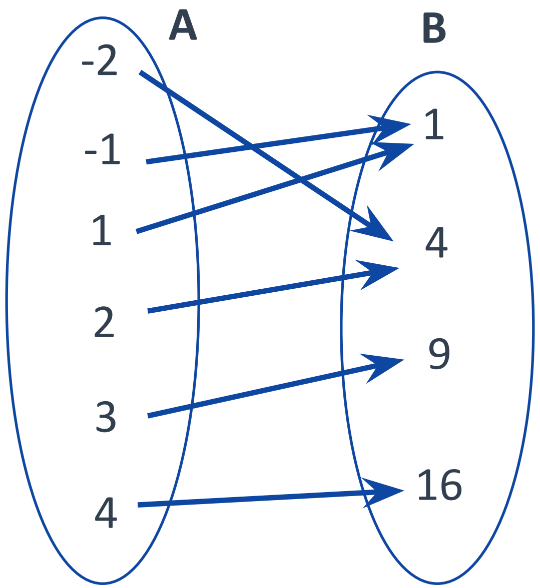
  f(-1) = 1

  f(1) = 1

  f(2) = 4

  f(3) = 9

  f(4) = 16



- The range of function f(A) = {1, 4, 9, 16} = B

One-to-One Function

- A function for which every element of the range of the function corresponds to exactly one element of the domain is known as One-to-One or injective or injection.

- Example: $f : A \rightarrow B, f(x) = x^2$ where,

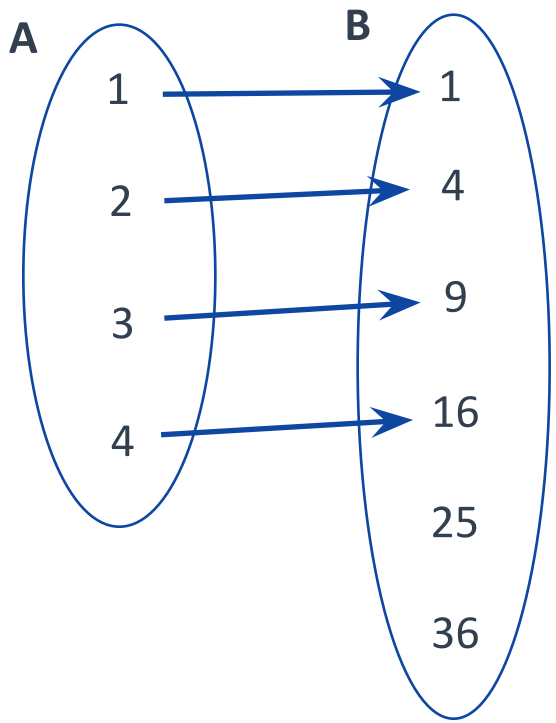
  A = {1,2,3,4} and B = {1,4,9,16,25,36}

  f(1) = 1

  f(2) = 4

  f(3) = 9

  f(4) = 16



Bijection Function

- If function is both one-to-one and onto then function is called Bijection function.
- Example: $f : A \rightarrow B, f(x) = x^2$ where,

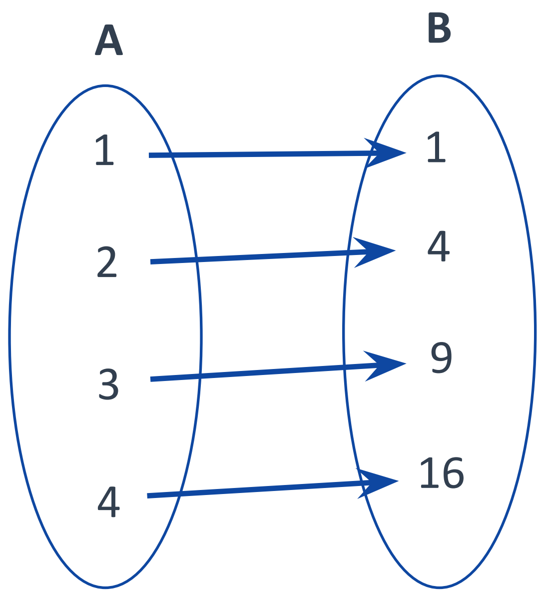
  A = {1,2,3,4} and B = {1,4,9,16}

  f(1) = 1

  f(2) = 4

  f(3) = 9

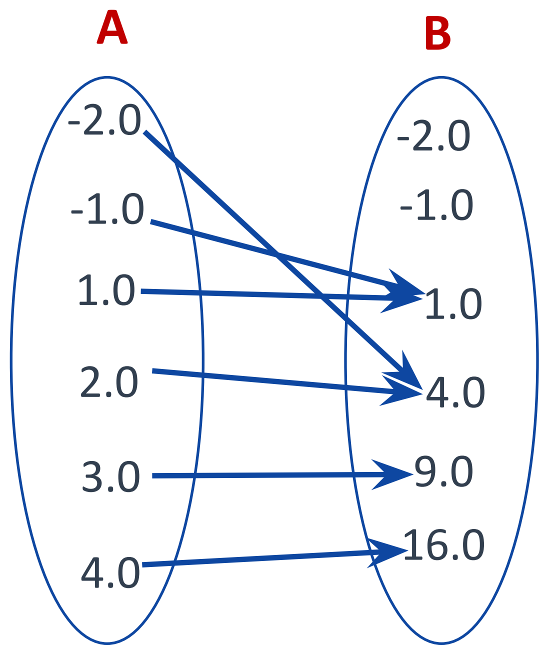
  f(4) = 16



Prove that $f: R \rightarrow R, f(x) = x^2$ is not one-to-one and not onto function

- The range and codomain of $f(x) = x^2$ are not equal, So function f is not onto function.
- The function is not one to one because elements of B are connected with more than one elements of A.

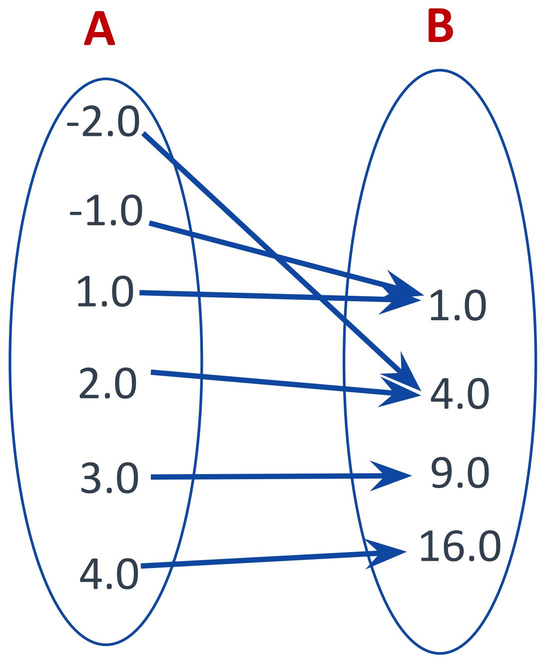
f(-2) = 4
f(-1) = 1
f(1) = 1
f(2) = 4
f(3) = 9
f(4) = 16



Prove that *f: R* → R$^+$, *f*(x) = x^2 is not one-to-one and onto function

- The range and codomain of $f(x) = x^2$ are equal So, function f is onto function.
- The function is not one to one because elements of B are connected with more than one elements of A.

f(-2) = 4
f(-1) = 1
f(1) = 1
f(2) = 4
f(3) = 9
f(4) = 16



Prove that $f: R^+ \rightarrow R$, $f(x) = x^2$ is one-to-one and not onto function

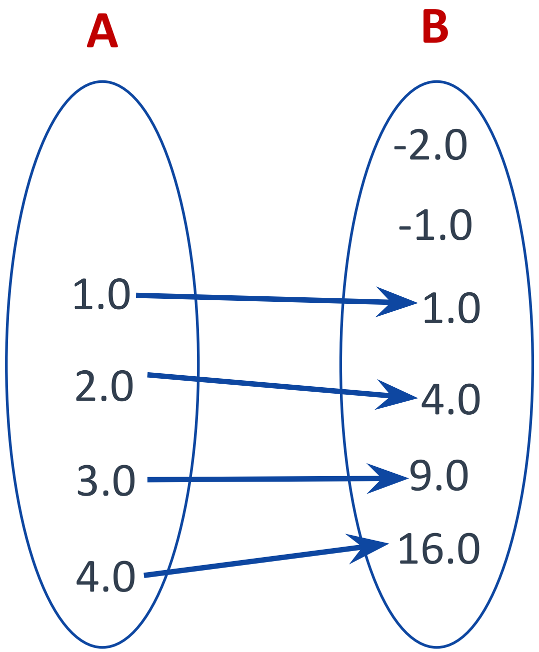
- The range and codomain of $f(x) = x^2$ are not equal So, function f is not onto function.
- The function is one to one because elements of B are connected with single element of A.

  f(1) = 1

  f(2) = 4

  f(3) = 9

  f(4) = 16



Prove that $f: R^+ \rightarrow R^+, f(x) = x^2$ is one-to-one and onto function (bijection)

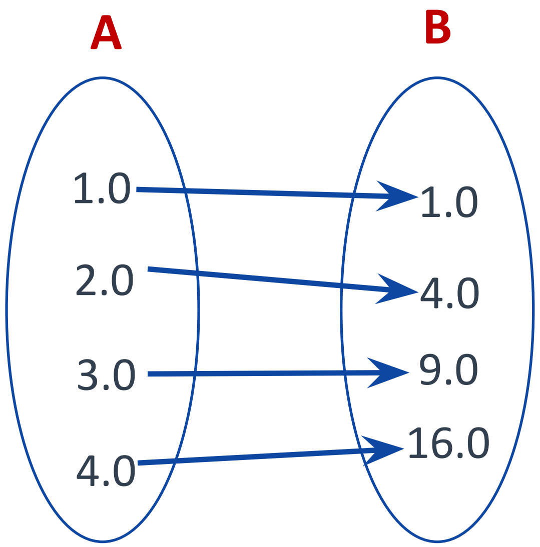
- The range and codomain of $f(x) = x^2$ are equal So, function f is onto function.
- The function is one to one because elements of B are connected with single element of A.
- The function $f(x) = x^2$ is onto function as well as one-to-one function. So, it is called as bijection function.

f(1) = 1

f(2) = 4

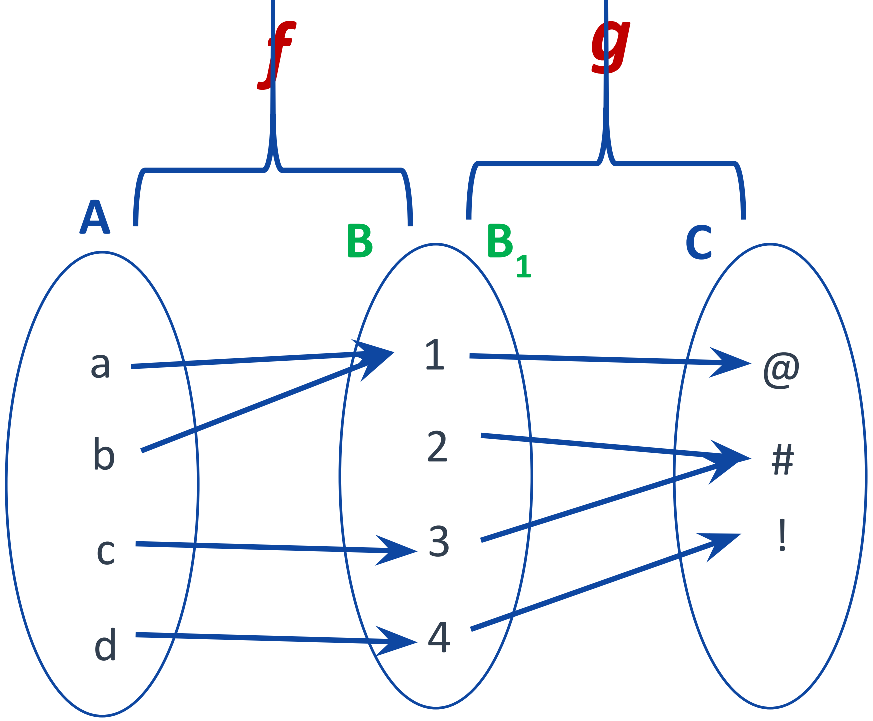
f(3) = 9

f(4) = 16



Compositions of Function

- Let $f : A \rightarrow B$ and $g : B_1 \rightarrow C$, the range of f is a subset of B_1, then $g(f(x))$ makes sense for each $x \in A$ and the function $h : A \rightarrow C$ defined by $h(x) = g(f(x))$ is called the composition of g and f.
- It is written as $h = g \circ f$
- Example:
  - $h(c) = g(f(x)) = \#$

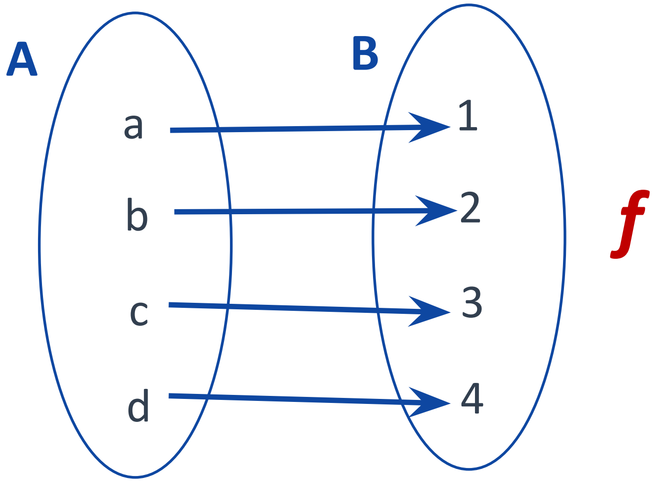


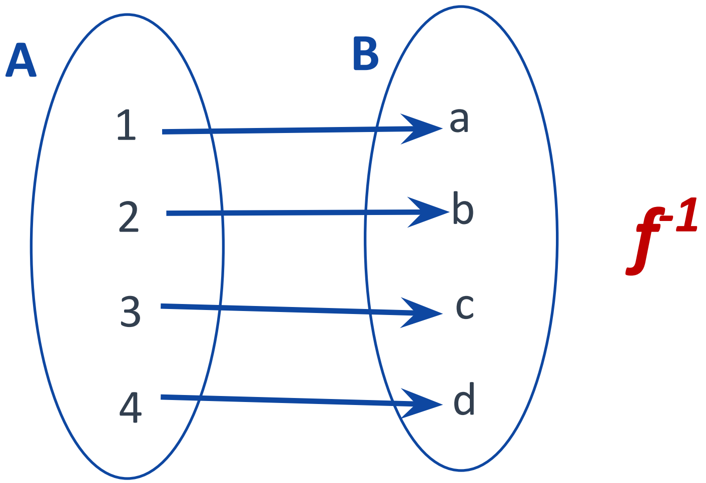
Inverse of Function

- Let f be a function whose domain is the set X, and whose range is the set Y. Then f is invertible if there exists a function g with domain Y and range X, with the property:

$$f(x) = y \Leftrightarrow g(y) = x$$

- To be invertible a function must be both an injection and a surjection.
- Example:





Relations

Relations

- A relation on a set A is defined as subset of $A \times A$.
- The relation R is denoted as aRb where $a, b \in A$ and pair $(a, b) \in R$.
- Example:

$$N = \{1,2,3\}$$
$$N \times N = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

- The '=' relation on $N \times N$ is : {(1,1), (2,2), (3,3)}

  where

  1 = 1

  2 = 2

  3 = 3

Properties of Equivalence Relations

- Assume that R is a relation on a set A, in other words, $R \subseteq A \times A$, where $(x, y) \in R$ to indicate x is related to y via Relation R.
  1. R is reflexive if for every $x \in A$, xRx
  2. R is symmetric if for every x and y in A, if xRy, then yRx
  3. R is transitive if for every x, y and z in A, if xRy and yRz, then xRz.
  4. R is an equivalence relation on A, if R is reflexive, symmetric and transitive.

Example: Equivalence Relation

- A={a, b}, R={(a, a), (b, b), (a, b), (b, a)}
  - Reflexive: {(a, a),(b, b)}
  - Symmetric: {(a, b), (b, a)}
  - Transitive: {(a, a), (a, b), (a, b)

    (b, b), (b, a), (b, a)

    (a, b), (b, b), (a, b)

    (a, b), (b, a), (a, a)

    (b, a), (a, b), (b, b)

    (b, a), (a, a), (b, a)}
- Above relation is **Equivalence relation** because it is Reflexive, symmetric and transitive.

Exercise

1. A={1, 2, 3}, R={(1, 2), (1, 1), (2, 1), (2, 2), (3, 2), (3, 3)} is equivalent relation?
2. A={1, 2, 3, 4}, R={(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)} is equivalent relation?
3. A={0, 1, 2}, R={(0, 0), (1, 1), (2, 2), (1, 0), (2, 1)} is equivalent relation?

Proof

Proof

- A proof of a statement is essentially just a convincing argument that the statement is true.
- A typical step in proof is to derive some statement from:
  1. Assumptions or hypotheses
  2. Statements that have already been derived
  3. Other generally accepted facts
- There are several methods for establishing a proof, some of them are:
  1. Direct proof
  2. By contradiction
  3. By mathematical induction

Rational & Irrational numbers

- A rational number is a number that can be in the form **m/n** where **m** and **n** are integers and **n** is not equal to zero.
- Examples:

1.5 = 3/2

6 = 12/2

5 = 15/3

Rational Numbers

π=22/7= **3.14159265** 358979323846264338327950288419 7.....

Irrational Numbers

Prove: $\sqrt{2}$ is Irrational

- **Definition:** A real number is rational if there are two integers m and n so that $x = m/n$.
- Proof:
- Suppose for the sake of contradiction that $\sqrt{2}$ is rational.
- Then there exists some integers m' and n' such that $\sqrt{2} = m'/n'$.
- By dividing both m' and n' by all the factors that are common to both, we obtain $\sqrt{2} = \mathrm{m}/n$, for some integers m and n having no common factors.
- Since $\sqrt{2} = \mathrm{m}/n$, $\mathrm{m} = n\sqrt{2}$. Squaring both sides of this equation, we obtain $m^2 = 2n^2$, and therefore m^2 is even.
- If a and b are odd, then ab is odd. Since a conditional statement is logically equivalent to its contra positive, we may conclude that for any a and b, if ab is not odd, then either a is not odd *or* b is not odd.

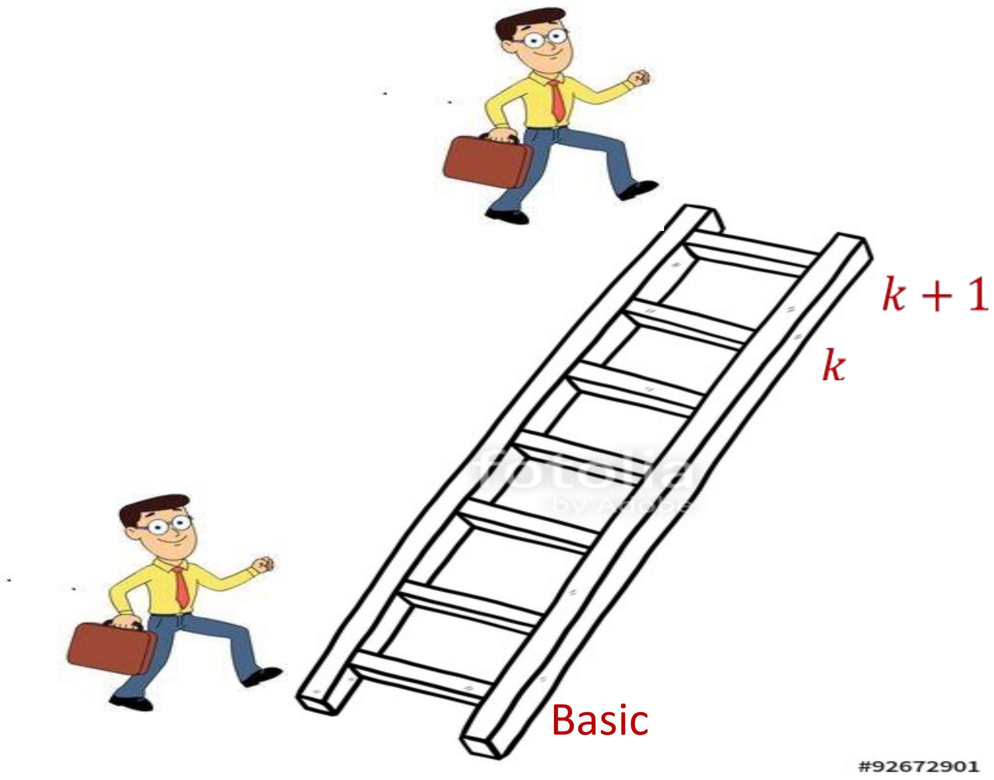
Prove: $\sqrt{2}$ is Irrational

- However, an integer is not odd if and only if it is even, and so for any a and b, if ab is even, a or b is even.
- If we apply this when $a = b = m$, we conclude that since m^2 is even, m must be even.
- This means that for some k, $m = 2k$. Therefore, $(2k)^2 = 2n^2$.
- Simplifying and cancelling 2 from both sides, we obtain $2k^2 = n^2$. Therefore n^2 is even and therefore $n = 2j$ for some j.
- We have shown that m and n are both divisible by 2. This contradicts the previous statement that m and n have no common factor.
- The assumption that $\sqrt{2}$ is rational therefore leads to a contradiction, and the conclusion is that $\sqrt{2}$ is irrational.

Principle of Mathematical Induction

Principle of Mathematical Induction

- Suppose $P(n)$ is a statement involving an integer n. Then to prove that $P(n)$ is true for every $n \geq n_0$, it is sufficient to show these two things:
  1. $\mathrm{P(n_0)}$ is true.
  2. For any $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.



Prove $\sum_{i=1}^{n} i = n(n+1)/2$ using PMI

Step-1: Basic step

We must show that P(1) is true.

P(1) = 1 (L.H.S)

P(1)= $1(1+1)/2$ =1, And this is obviously true.

Step-2: Induction Hypothesis

$k \geq 1$ and $1+2+3+\cdots+k = \mathbf{k(k+1)/2}$

Step-3: Proof of Induction

P(k+1) = 1 + 2 + 3 + …..+ k + (k+1)

= K(K + 1)/2 + (K + 1) (by induction hypothesis)

= (K(K + 1) + 2(K + 1))/2

= (K+1)(K+2)/2

= (K+1) (K+1+1)/2 (Hence Proved)

Prove 1 +3 +5 + … +2n-1 = n^2 using PMI, n>=1

Step-1: Basic step

We must show that P(1) is true.

P(1) = 2(1)-1= 1 (L.H.S)

P(1) = $(1)^2$ = 1 (R.H.S)

And, this is obviously true.

Step-2: Induction Hypothesis

k >= 1 and

p(k) = 1+3+5+…..+(2k-1)=**k^2**

Step-3: Proof of Induction

P(k+1) = 1+3+5+….+(2k-1)+(2(k+1)-1)

= **k^2** + (2(k+1)-1)

= k^2 + (2k+2-1)

= k^2 + 2k+1

=$(k+1)^2$ (Hence Proved)

Prove 7+ 13+19+.....+(6n+1)= n(3n+4) using PMI, n>=1

Step-1: Basic step

We must show that p(1) is true.

P(1)= 6n+1=(6(1)+1)=7

P(1)= n(3n+4) =1(3(1)+4)=7

And, this is obviously true.

Step-2: Induction Hypothesis

k >= 1 and

p(k) = 7+13+19+.....+(6k+1)=**k(3k+4)**

Step-3: Proof of Induction

P(k+1)= 7+13+.....+(6k+1)+(6(k+1)+1)

= **k(3k+4)**+(6(k+1)+1)

= k(3k+4)+(6k+6+1)

= $3k^2$+4k+6k+7

= $3k^2$+10k+7

= $3k^2$+3k+7k+7

= 3k(k+1)+7(k+1)

= (k+1)(3k+7)

= (k+1)(3k+3+4)

= (k+1)(3(k+1)+4) (Hence Proved)

Thank You