Unit 1: Logic and Predicates

In this unit we will discuss

- Logical expressions and Operators,
- Predicates
- Rules of quantifiers,
- Rules of Inference for predicates and propositions.

Application

- Discrete mathematics is the study of mathematics confined to the set of integers
- Cryptography
- Relational Databases
- Logistics
- Computer Algorithms
- Software development
- Programming languages
- Designing password criteria
- An analog clock
- Bankruptcy proceedings

Statement: A sentence is called statement if it can be true or false not both

Examples:

1. **P: 2 is an odd no.**
2. **Q: 3 is less than 5**
3. **R: today weather is sunny**
4. **S : x +10 = 0**

Ans: Here sentence P , Q and R are statement but S in not a statement .All possible value of statement are called truth value its denoted by T AND F

Negation: Let P be the statement then negation of P is denoted by $\sim p$ or $\neg$ p**.** It is defined as p: 2 is an even number

$\sim p$: 2 is not even number

p	**~ p**
T	**F**
F	**T**

Conjunction: The conjunction of the two statement p and q is denoted by

$p \wedge q$ (read as p and q) **.** It is defined as p: 2 is an even number. q: Rajkot is capital of Gujarat **.**

Then $p \wedge q$: 2 is an even number and Rajkot is capital of Gujarat

p	**q**	**p ∧ q**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Dis junction: The Disjunction of the two statement p and q is denoted by

$p \vee q$ (read as p or q)**.** It is defined as p: 2 is an even number. q: Rajkot is capital of Gujarat **.** Then $p \vee q$: 2 is an even number or Rajkot is capital of Gujarat

p	**q**	**p ∨ q**
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

EXAMPLE: Derive truth table:

P	**q**	**~ p**	~q	**~ p** ∨ ~q
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

2] $\sim(p \wedge q) \vee q$

3]$(\sim p \vee q) \wedge r$

4]$(p \vee r) \wedge (q \wedge r)$

5](q ∨ r) ∧ ~p

6](p ∨ q) ∧ (~r ∧ q)

Conditional statement:

Let p and q be the given two statements then conditional statement is denoted by p → q read as p then q

Example : p: the weather is sunny. q: I will take you to the beach

p → q: (I promise that)If the weather is sunny then I will take you to the beach

p	**q**	**p→q**
T	T	T
T	F	**F**
F	T	T
F	F	T

Note

1) the statement q→ p is called converse of p → q

2) the statement $\sim q \rightarrow \sim p$ is called contrapositive of p → q

3) the statement $\sim p \rightarrow \sim q$ is called inverse of p → q

Biconditional statement:

Let p and q be the given two statements then biconditional statement is denoted by $p \leftrightarrow q$ read as p if and only if q

Example: P: the weather is sunny. q: I will take you to the beach

$p \leftrightarrow q$: the weather is sunny if and only if I will take you to the beach

p	**q**	**$p \leftrightarrow q$**
T	T	T
T	F	F
F	T	F
F	F	T

Tautology: A compound statement is always true no matter what the true value of given statements involve in it is called tautology .The compound statement

$p \vee \sim p$ is tautology

p	**$\sim p$**	**$p \vee \sim p$**
T	F	T
F	T	T

Contradiction:
A compound statement is always false no matter what the true value of given statements involve in it is called contradiction**.** The compound statement $p \wedge \sim p$ is contradiction

p	**$\sim p$**	**$p \wedge \sim p$**
T	F	F
F	T	F

Contingency:
A compound statement which is neither tautology nor contradiction is called contingency**.** The compound statement $p \vee q$ is contingency.

p	**q**	**$p \vee q$**
T	T	T
T	F	T
F	T	T
F	F	F

Exercise

Check the following compound statement are tautology or not.

$(p \vee q) \rightarrow (p \wedge q)$

p	**q**	**$(p \vee q)$**	**$(p \wedge q)$**	**$(p \vee q) \rightarrow (p \wedge q)$**
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Ans it is not tautology

2) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

3) $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim q$

4) $(p \vee q) \leftrightarrow (\sim p \vee \sim q)$

5] $(\sim p \wedge q) \rightarrow r$

6] $(p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

Logical equivalent: Let p and q be the two statements then both are logical equivalent if $p \leftrightarrow q$ is tautology

It is denoted by $p \equiv q$ (p logical equivalent to q)

DE 'Morgan's law: Let p and q be two statements then

(1) $\boldsymbol{\sim(p \vee q) \equiv \sim p \wedge \sim q}$

(2) $\boldsymbol{\sim(p \wedge q) \equiv \sim p \vee \sim q}$

Proof: (1) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	**q**	$\boldsymbol{p \vee q}$	$\boldsymbol{\sim(p \vee q)}$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	**A≡B**
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

(2) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	**q**	$\boldsymbol{p \wedge q}$	$\boldsymbol{\sim(p \wedge q)}$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	**A≡B**
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Associative law: Let p , q and r be three statements then

(1) $\boldsymbol{p \vee (q \vee r) \equiv (p \vee q) \vee r}$

(2) $\boldsymbol{p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r}$

Proof: (1) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

p	q	R	$q \vee r$	$p \vee (q \vee r)$	$p \vee q$	$(p \vee q) \vee r$	A≡ B
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T
f	F	F	f	f	F	f	T

(2) $p \wedge (q \wedge \mathrm{r}) \equiv (p \wedge q) \wedge r$

p	q	R	$q \wedge$r	$p \wedge (q \wedge$r)	$p \wedge q$	$(p \wedge q) \wedge$r	A≡ B
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	F	F	F	F	T
T	F	F	F	F	F	F	T
F	T	T	T	F	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
f	F	F	f	F	F	F	T

Absorption law: Let p and q be two statements then

(1) $p \wedge (\mathbf{p} \vee \mathbf{q}) \equiv p$

(2) $p \vee (\mathbf{p} \wedge \mathbf{q}) \equiv p$

Proof:

- (1) $p \wedge (\mathrm{p} \vee \mathrm{q}) \equiv p$

p	Q	p ∨ q	$p \wedge$ (p ∨ q)	$p \wedge$ (p ∨ q) ≡ p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	f	f	F	T

(2) $p \vee (\mathrm{p} \wedge \mathrm{q}) \equiv p$

p	Q	p ∧ q	$p \vee$ (p ∧ q)	$p \vee$ (p ∧ q) ≡ p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

Examples:

3) Show that following statement formula A is logically equivalent to formula B

1. A: $(p \rightarrow q) \rightarrow r$ B: $(p \wedge q) \rightarrow r$

2. A : $(p \rightarrow q) \vee (p \rightarrow r)$ B : $p \rightarrow (q \vee r)$

3. A: $p \leftrightarrow q$ B: $(p \rightarrow q) \vee (q \rightarrow p)$

Validity of Argument:

A finite sequence of statements A1, A2, A3 ,,,,An-1 ,An is called arguments and An is called conclusion **.**Then A1, A2, A3 ,,,,An-1 ,An is called logically validate statements. if (A1 $\wedge$ A2 $\wedge$ A3 $\wedge$,,,, $\wedge$ An-1)$\rightarrow$An is tautology **.**

We write arguments are as forms

A1

A2

.

.

An-1

$\therefore$ **An**

Some well-known arguments

Method of affirming:

A1 $p \rightarrow q$

A2 p

A3 $\therefore$ q is valid arguments

Ans : For checking validity

We prove that $((p \rightarrow q) \wedge p) \rightarrow q$ is tautology

p	**q**	**p →q**	**((p →q) ∧ p**	**((p →q) ∧ p) →q**
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**T**

Method of denying

A1 $\quad p \to q$

A2 $\quad \sim q$

A3 $\therefore \sim p$ is valid arguments

Ans: For checking validity

We prove that $((p \to q) \wedge \sim q) \to \sim p$ is tautology

P	**Q**	**~p**	**~q**	**p →q**	**((p →q) ∧ ~q)**	**A→ B**
T	**T**	**F**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**F**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

3] Dilemma

A1 $\quad p \vee q$

A2 $\quad p \to r$

A3 $\quad q \to r$

A4 $\therefore$ r is valid arguments

Ans: For checking validity

We prove that $(p \vee q) \wedge (p \to r) \wedge (q \to r) \to r$ is tautology

p	**q**	**r**	**p∨q**	**p→ r**	**q→ r**	**x∧y∧z**	**A→B**
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**	**F**	**T**
T	**F**	**T**	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**F**	**T**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**T**	**F**	**T**	**T**	**F**	**T**
F	**F**	**F**	**F**	**T**	**T**	**F**	**T**

4] Hypothetical syllogism

A1 $p\rightarrow q$

A2 $q\rightarrow r$

A3 $\therefore$ $p\rightarrow r$ is valid arguments

Ans: For checking validity

We prove that $(p\rightarrow q) \wedge (q\rightarrow r) \rightarrow (p\rightarrow r)$is tautology

p	**q**	**r**	**$p\rightarrow q$**	**$q\rightarrow r$**	**$p\rightarrow r$**	**$x\wedge y$**	**$A\rightarrow B$**
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**F**	**F**	**T**
T	**F**	**T**	**F**	**T**	**T**	**F**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**	**T**
F	**T**	**T**	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**F**	**F**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**	**T**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**

Disjunction syllogisms:

1) $p \vee q$

$\sim p$

$\therefore$ q is valid arguments

2) $p \vee q$

$\sim q$

$\therefore$ p is valid arguments

Addition

premises: p

conclusion: $p \vee q$

Simplification

premises: $p \wedge q$

conclusion: p

7. Conjunction

premises: $p, \quad q$

conclusion: $p \wedge q$

8. Resolution

premises: $p \vee q, \quad \neg p \vee r$

conclusion: $q \vee r$

Ex1: Prove that the following argument is invalid

$p \to q$

q

$\therefore p$

Ans : we will check

$((p \to q) \wedge q) \to p$ is tautology

Ex 2 Check following arguments valid or not

$\sim p \vee q$

$r \to (\sim q)$

$\therefore p \to (\sim r)$

Ans: we will check

$(\sim p \vee q) \wedge (r \to (\sim q)) \to (p \to (\sim r))$ is tautology

Predicates: A predicates is a declarative sentence whose true false value depends on one or more variables .A predicates with one variable is denoted p(x) .A predicates with two variables is denoted by p(x ,y) And so on …..

1) p(x) : x is divisible by 7 here x is variable and x is divisible by 7 is known as predicates .Predicates p(x) is also known as propositional function of x

2) P(x): x is a prime number

Answer : P(3) : 3 is a prime number . p(3) is true

P(14) : 14 is a prime number . p(14) is false

3) P(x,y): 2x+ y is divisible by y^2

Answer: P(1,-1) : 1 is divisible by 1 . its true thus P(1,-1) is true

P(2,3) : 7 is divisible by 9.It is false : P(2,3) is false

Quantifiers:

1] Universal Quantifier : The universal quantification of p(x) is the statement "p(x) of all x ϵ D " where D is called the domain (domain of discourse). It is denoted by $\forall\, x\, p(x)$

$\forall\, x\, p(x)$ is true if p(x) is true for every x in domain

$\forall\, x\, p(x)$ is false if p(x) is false for at least on value of x in domain

Note : The 'x' for which $\forall\, x\, p(x)$ is false is called counter example of $\forall\, x\, p(x)$

If domain D $= \emptyset\ then\ \forall\, x\, p(x)$ is by default true as there are no counter example

Example:

1] If p(x) : $x^2 > 10$ then what is true /false value of $\forall\, x\, p(x)$ if

1. D = R
2. D ={ 1,2,3,4}
3. D = [11,20]

2] If P(x): x is an even no. then 2x+1 is always an odd no. D = N, then find the truth value of $\forall\, x\, p(x)$

2] Existential Quantifier: The existential quantification of p(x) is the statement "there exist x ϵ D such that p(x) " where D is called the domain

(domain of discourse) .It is denoted by $\exists\, x\, p(x)$

$\exists\, x\, p(x)$ is true if p(x) is true for one or more x ϵ D

$\exists\, x\, p(x)$ is false if p(x) is false for every x ϵ D .

Note : The 'x' for which $\exists\, x\, p(x)$ is true is called 'witness' of $\exists\, x\, p(x)$

If domain D $= \emptyset$ $then \exists\, x\, p(x)$ is by default it false as there are no witness

Examples

If p(x) : x^2 > 10 then what is true /false value of $\exists\ x\ p(x)$ if

1. D = R
2. D ={ 1,2,3,4}
3. D = [1,3]

P(x) : $x \geq x^2$ D = {5,6,9,-9,0} what is true /false value of $\exists\ x\ p(x)$

Example : Let x+1 > 2x and domain of discourse be the set of all integers then find the truth value of following :

1. p(0)
2. p(-1)
3. $\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$
4. $\forall\, x\, p(x)$
5. $\sim\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$
6. $\sim\forall\, x\, p(x)$
7. $\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x}) \rightarrow (p\,(0)\ \vee$ p(-1))
8. $\sim\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x}) \leftrightarrow p(2)$

Solution:

Derive the truth value of $\forall\, x\, p(x)$
(1) p(x) : x > x+ 1 ,D = Z

Derive the truth value of $\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$
(1) p(x) : x+ 1≥ 2x ,D = N

What is the truth value of $\exists \boldsymbol{x}\, \boldsymbol{p}(\boldsymbol{x})$
consider D = R

- P(x) : x+1 = 1
- P(x) : x^2 +1 < x
- P(x) : $x^{1/2}$ = 1