Additional Analysis of PINNs for Black–Scholes Model Generalization, Stability, and European Market Behavior

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1 Generalization Analysis: Interpolation vs Extrapolation

The first analysis investigates how the PINNs model behaves inside and outside the domain on which it was trained.

- **Interpolation** refers to predictions made within the training domain, where the underlying prices lie in the range used during model training.
- **Extrapolation** refers to predictions made beyond this range, where the model has not been explicitly exposed to data.

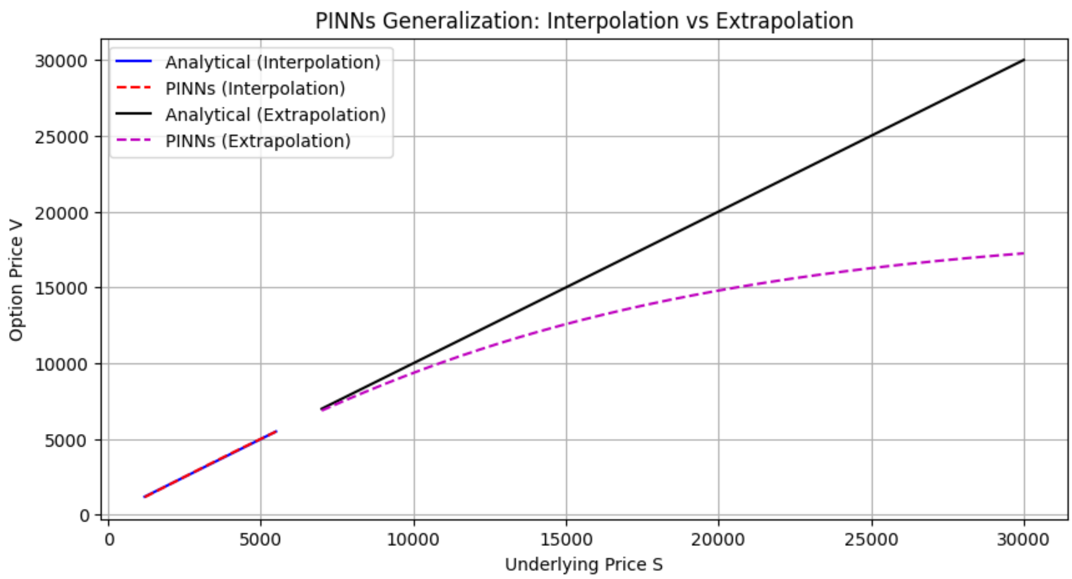

Figure 1: PINNs generalization behavior: comparison of analytical and PINNs solutions under interpolation and extrapolation regimes.

The results clearly show that the PINNs model closely tracks the analytical solution within the trained domain. However, as the underlying price moves beyond the training range, the PINNs predictions begin to deviate, highlighting the sensitivity of neural PDE solvers to extrapolation.

2 Sensitivity to Domain Boundaries

To further analyze the effect of training domain limitations, we examined how the pricing error evolves as the underlying price moves away from the upper boundary of the training domain.

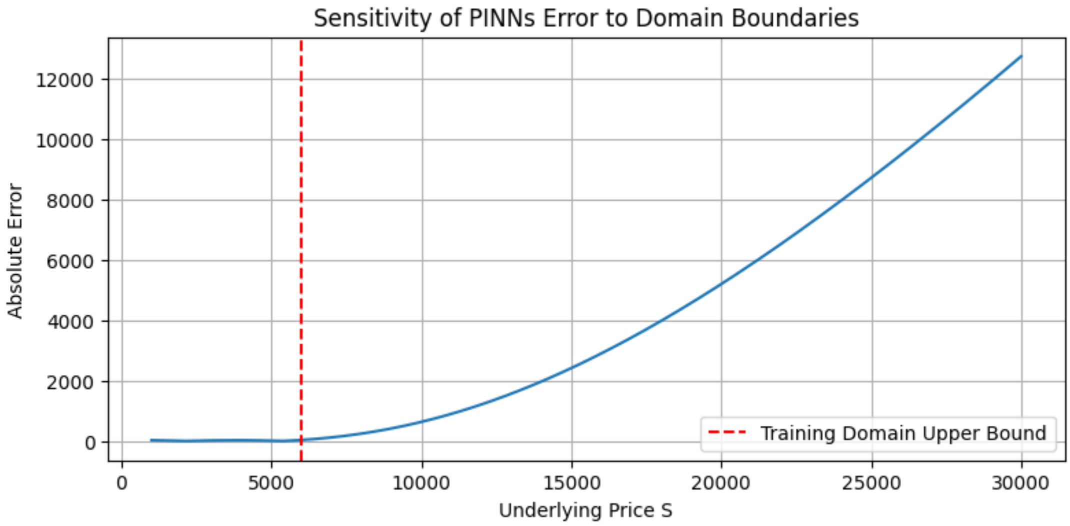

Figure 2: Absolute pricing error of PINNs as a function of underlying price, highlighting sensitivity to the training domain boundary.

The sharp increase in error beyond the training boundary confirms that inadequate exposure to higher price levels directly impacts prediction accuracy. This explains the higher error observed for indices such as DAX and FTSE MIB.

3 Impact of Price Scale Across European Markets

Different European indices operate at significantly different price scales. To understand how this affects PINNs performance, we analyzed the relationship between average underlying price levels and relative error.

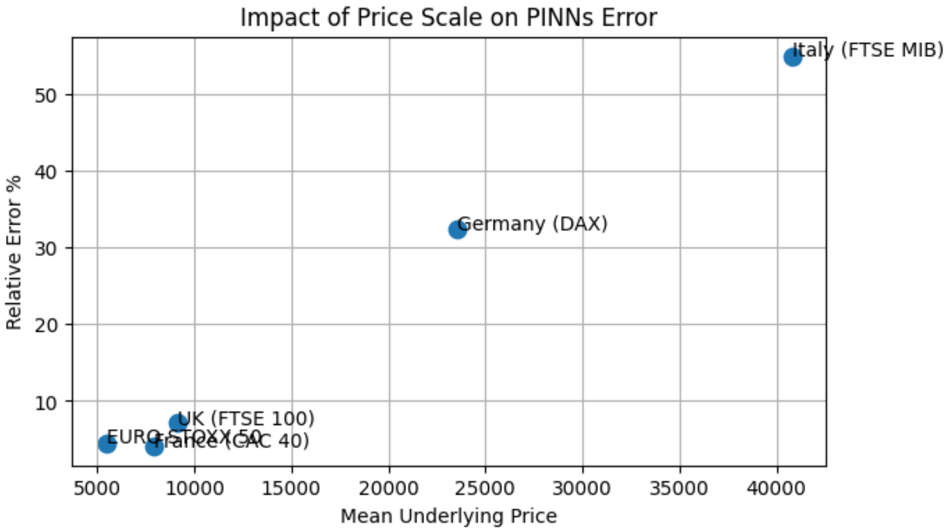

Figure 3: Relationship between mean underlying price and relative PINNs error across European markets.

The results indicate a strong correlation between price scale and prediction error. Markets with price levels well within the training domain exhibit low error, while markets operating far outside the domain show higher deviations. This highlights the importance of domain-aware training for financial PINNs.

4 Theoretical Consistency: Monotonicity Check

According to Black–Scholes theory, the price of a European call option must be monotonically increasing with respect to the underlying price. To verify this property, we computed the numerical derivative of the PINNs prediction with respect to the underlying price.

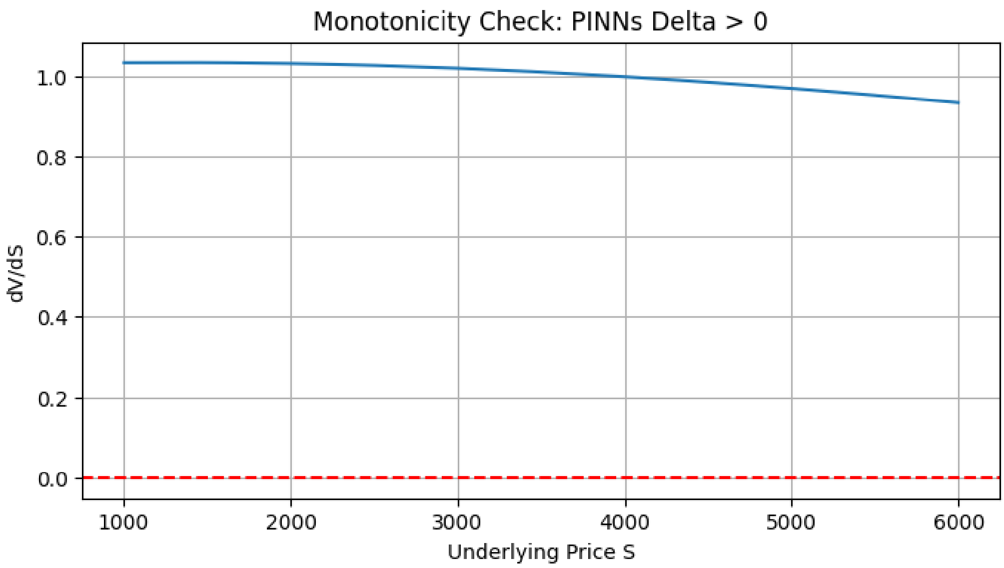

Figure 4: Monotonicity verification: PINNs delta ($\partial V/\partial S$) remains positive across the domain.

The strictly positive derivative confirms that the PINNs solution respects the theoretical monotonicity constraint imposed by the Black–Scholes model.

5 Smoothness Over Time

Smoothness with respect to time is another fundamental property of the Black–Scholes solution. Abrupt oscillations would indicate numerical instability or poor training.

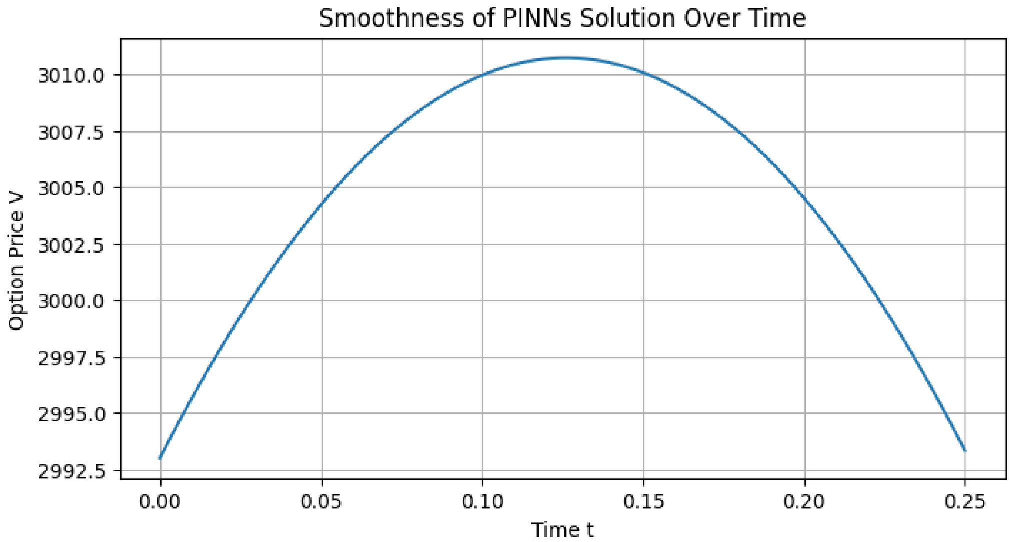

Figure 5: Smooth temporal evolution of PINNs-predicted option price at a fixed underlying price.

The smooth temporal behavior observed confirms the numerical stability of the trained PINNs model.

6 Conclusion

The additional analyses demonstrate that the trained PINNs model:

- Generalizes well within the trained domain but degrades under extrapolation,
- Is highly sensitive to domain boundaries, explaining cross-market error variations,
- Exhibits error trends consistent with underlying price scale differences,
- Respects key theoretical properties of the Black–Scholes solution, including monotonicity and smoothness.

These results reinforce that careful domain selection, scaling, and validation are essential for deploying PINNs in real financial modeling scenarios.