BT-202

B.Tech, I & II Semester

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Mathematics-I

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1) (a) let $u = x+y$, $\frac{du}{dx} = 1 + \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{du}{dx} - 1$

$$\frac{dy}{dx} = \cos(x+y) + \sin(x+y) \quad \text{--- (1)}$$

$$\Rightarrow \frac{du}{dx} - 1 = \cos u + \sin u$$

$$\frac{du}{dx} = \cos u + \sin u + 1$$

$$\Rightarrow \frac{du}{1 + \sin u + \cos u} = dx$$

On integration, we get

$$\int \frac{1}{1 + \cos u + \sin u}\, du = \int dx + c$$

$$\int \frac{1}{\left[1 + \frac{1 - \tan^2(u/2)}{1 + \tan^2(u/2)} + \frac{2\tan(u/2)}{1 + \tan^2(u/2)}\right]}\, du = x + c$$

$$\int \frac{\sec^2(u/2)}{2(1 + \tan(u/2))} = x + c$$

$$\log[1 + \tan(u/2)] = x + c$$

$$\Rightarrow \log\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + c.$$

(2) **1(b)** Sol. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Here, x is alone, so it may be linear d.e. in x,

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2} \quad -(1)$$

which is a linear eq. in x.

Here $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1}y}{1+y^2}$

$\therefore$ I.F. $= e^{\int P dy} = e^{\int \left(\frac{1}{1+y^2}\right)dy} = e^{\tan^{-1}y}$

Hence the required solution is

$$x\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + C$$

$$x\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2}\,e^{\tan^{-1}y}\,dy + C$$

$$= \int t e^{t}\,dt + C \quad \text{where } t = \tan^{-1}y$$

$$x e^{\tan^{-1}y} = t e^{t} - e^{t} + C$$

$$x\,e^{\tan^{-1}y} = \tan^{-1}y\,e^{\tan^{-1}y} - e^{\tan^{-1}y} + C$$

$$x = (\tan^{-1}y - 1) + C e^{-\tan^{-1}y} \quad \text{Ans}$$

2(a) Sol. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x} \quad -(1)$, $P = 1, Q = 0, R = \frac{1}{1+e^x}$

Given second O.D.E. solve by method of V.o.P.

$$(D^2 + D)y = \frac{1}{1+e^x}$$

A.E. is $m^2 + m = 0$, $m(m+1) = 0$, $m = 0, -1$

C.F. $= c_1 + c_2 e^{-x}$

$y_c = c_1 u + c_2 v$ and $u = 1$, $v = e^{-x}$

$$w = \begin{vmatrix} u & v \\ u_1 & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix} = (-e^{-x} - 0) \neq 0, w \neq 0 \quad (3)$$

$$\frac{dA}{dx} = \frac{-vR}{w} \qquad \Bigg| \qquad \frac{dB}{dx} = \frac{uR}{w} = 1 \cdot \frac{1}{1+e^{x}} \cdot -e^{x}$$

$$\frac{dA}{dx} = \frac{-e^{-x}}{1+e^{x} \cdot -e^{-x}} \qquad \Bigg| \qquad \int dB = -\int \frac{e^{-x}}{1+e^{-x}} dx$$

$$\frac{dA}{dx} = \frac{1}{1+e^{x}} \qquad \Bigg| \qquad \boxed{B = -\log(1+e^{-x}) + c_4}$$

$$\int dA = \int \frac{dx}{1+e^{x}} = \int \frac{1}{e^{x}(1+e^{-x})} dx = \int \frac{e^{-x}}{1+e^{-x}} dx$$

$$A = \log(1+e^{-x}) + c_3$$

$$y = [\log(1+e^{-x}) + c_3] + e^{-x} [\log(1+e^{-x}) + c_4$$

$$y = (1+e^{-x}) \log(1+e^{-x}) + c_3 + c_4 \quad \text{Ans.}$$

2(b) $Dx - y = e^{t}$ — (i)

$x + Dy = \sin t$ — (ii)

$D^2x - Dy = De^{t}$

$x + Dy = \sin t$

$D^2x + x = e^{t} + \sin t$

$(D^2+1)x = e^{t} + \sin t$ — (A)

A.E. $m^2+1 = 0, \; m = \pm i$

C.F. $= c_1 \cos t + c_2 \sin t$

(4) $P.I. = \frac{1}{D^2+1}(e^t+\sin t)$

$= \frac{1}{D^2+1}e^t + \frac{1}{D^2+1}\sin t$

$= \frac{1}{1^2+1}e^t + -\frac{t}{2.1}\cos t$

$$\boxed{P.I. = \frac{e^t - t\cos t}{2}}$$

$$\boxed{x = c_1\cos t + c_2\sin t + \frac{e^t}{2} - \frac{t\cos t}{2}}$$

By (1)

$$y = \frac{dx}{dt} - e^t$$

$$y = -c_1\sin t + c_2\cos t + \frac{e^t}{2} - \frac{1}{2}(\cos t - t\sin t) - e^t$$

$$\boxed{y = c_2\cos t - c_1\sin t - \frac{e^t}{2} - \frac{1}{2}(\cos t - t\sin t)}$$

$x(0) = 1$ then

$1 = c_1 + \frac{1}{2}$, $c_1 = -\frac{1}{2}$

$y(0) = 0$

$0 = c_2 - \frac{1}{2} - \frac{1}{2}$, $c_2 = 1$

$\boxed{c_1 = -1/2}$ $\boxed{c_2 = 1}$

$x = \sin t - \frac{1}{2}\cos t + \frac{e^t}{2} - \frac{t\cos t}{2}$

and

$y = \cos t + \frac{1}{2}\sin t - \frac{e^t}{2} - \frac{1}{2}(\cos t - t\sin t)$

3(a) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$

$$(x - x^2)\frac{d^2y}{dx^2} + (2-4x)\frac{dy}{dx} - 2y = 0 \quad \text{--- (1)}$$

Here $P_0(x) = x - x^2$ at $x = 0$ then $\boxed{P_0(x) = 0}$

clearly $x = 0$ is a regular singular point of eq. (1)

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots \quad \text{--- (2)}$$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots \quad \text{--- (3)}$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1} + (m+1)(m+2)a_2x^m + \ldots \quad \text{--- (4)}$$

Putting values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in eq (1), we get

$$-[m(m-1)a_0 x^{m-1}$$
$$+[m(m-1)a_0 x^m + m(m+1)a_1 x^m + (m+2)(m+1)a_2 x^{m+1} + \cdots]$$
$$+2[m(\cancel{m-1})a_0 x^{m-1} + m(m+1)a_1 x^{m+1} + \cdots]$$
$$-4[m a_0 x^m + (m+1)a_1 x^{m+1} + (m+2)a_2 x^{m+2}]$$
$$-2[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots] = 0 \quad \text{--- (2)}$$

Here the lowest power of x is x^{m-1}

$$\Rightarrow -m(m-1)a_0 + 2m\cancel{(m-1)}a_0 = 0 \qquad (m^2 - m + 2m)a_0 = 0$$
$$m(m+1) = 0$$

$$(-m^2 + m + \cancel{2m^2 - 2m})a_0 = 0$$
$$2ma_0$$
$$(\cancel{m^2 - 3m})a_0 = 0, \quad \because a_0 \neq 0, \quad m(m+1) = 0$$
$$\boxed{m = 0, 1} \qquad m = 0, m = -1$$

Now, equating to zero the coefficient of general term (i.e. of x^{m}), we have

$$\Rightarrow -m(m+1)a_1 + \cancel{2}(m+1)a_1 \cancel{+2} - 4ma_0 - 2a_0 = 0$$

$$[-m^2 + m + m + 1]a_1 = (4m+2)a_0$$

$$\boxed{a_1 = \frac{(4m+2)a_0}{(1+2m-m^2)}}$$

$m = 0, 1$

$$\boxed{a_1 = 2a_0}$$

$m = -1$

$$a_1 = \frac{-2a_0}{2} = -a_0$$

$$\boxed{a_1 = -a_0}$$

4(a) We prove that $J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$

we know that Bessel's function of first kind of order n

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma{n+r+1}} \cdot \left(\frac{x}{2}\right)^{n+2r} \quad \text{---} (1)$$

eq (1) put $n = \frac{1}{2}$ in $J_n(x)$, we get

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \cdot \Gamma{r+3/2}} \left(\frac{x}{2}\right)^{\left(\frac{1}{2}+2r\right)}$$

$$\because \Gamma{r+3/2} = \Gamma{\frac{2r+3}{2}} = \frac{2r+1}{2} \cdot \frac{2r-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2}\, \Gamma{1/2}$$

$$= \frac{(2r+1)!}{2^{r+1} \cdot 2^r\, r!} \sqrt{\pi} = \frac{(2r+1)}{2^{2r+1}\, r!}$$

$\boxed{\sqrt{1/2} = \sqrt{\pi}}$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \cdot \frac{2^{2r+1}\, r!}{(2r+1)!\, \sqrt{\pi}} \cdot \left(\frac{x}{2}\right)^{2r+\frac{1}{2}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!} \cdot \left(\frac{2}{\pi x}\right)^{1/2} \cdot x^{2r+1}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2r+1}}{(2r+1)!}$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$\boxed{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x}$$

(8) 4(b) let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ —(1)

Such that $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = p^2+q^2, \frac{\partial f}{\partial z} = -q,$

$$\frac{\partial f}{\partial p} = 2py \text{ and } \frac{\partial f}{\partial q} = 2qy - z$$

Charpit's auxiliary eq. are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} \quad (2)$$

Taking the first two members of (2), we get

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p\,dp + q\,dq = 0$$

$$\int p\,dp + \int q\,dq = 0 \Rightarrow \frac{p^2}{2} + \frac{q^2}{2} = \frac{a^2}{2}$$

where a is constant.

$$\boxed{p^2+q^2 = a^2} \quad (3)$$

Putting $p^2+q^2 = a^2$ in eq. (1), we get

$$a^2y = qz \quad (4)$$

From (3) and (4), solve for p and q,

we get $p = \frac{a}{z}\sqrt{z^2 - a^2y^2}$ and $q = \frac{a^2y}{z}$ (5)

$\therefore$ Complete integral, $dz = p\,dx + q\,dy$ (3)

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\,dx \quad \text{[as separation of variable]}$$

Integrating, we get

$$\int \frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = a\int dx$$

$$\boxed{\sqrt{z^2 - a^2y^2} = ax + b}$$ where b is constant

$$\boxed{z^2 - a^2y^2 = (ax+b)^2}$$ Ans

5(a) Given $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ (1)

Where $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$

The complete solution of eq. (1) is

$$\boxed{z = C.F. + P.I.}$$

To find C.F.:

The A.E. of (1) is $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$, $m = 3, 3$

$m_1 = m_2$

$C.F. = \phi_1(y + mx) + x\,\phi_2(y + mx)$

$C.F. = \phi_1(y + 3x) + x\,\phi_2(y + 3x)$

(10) $$P.I. = \frac{1}{D^2 - 6DD' + 9D'^2}\left[12x^2 + 36xy\right]$$

$$P.I. = \left[\frac{1}{(D-3D')^2}\right]12x^2 + \left[\frac{1}{(D-3D')^2}\right]36xy$$

$$= \frac{12}{D^2}\left[1 - \frac{3D'}{D}\right]^{-2} x^2 + \frac{1}{D^2}\left[1 - \frac{3D'}{D}\right]^{-1} 36xy$$

$$\because (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \frac{12}{D^2}\left[1 + 2\left(\frac{3D'}{D}\right) + 3\left(\frac{3D'}{D}\right)^2\right]x^2 + \frac{1}{D^2}\left[1 + 2\left(\frac{3D'}{D}\right) + 3\left(\frac{3D'}{D}\right)^2\right] 36xy$$

$$= \frac{12}{D^2}\left[x^2\right] + \frac{1}{D^2}\left[36xy + \frac{6}{D}x\right]$$

$$= \frac{12}{D}\cdot\left(\frac{x^3}{3}\right) + \frac{1}{D}\left[36\frac{x^2}{2}\cdot y + \frac{6}{D}\frac{x^2}{2}\right)$$

$$= x^4 + \left[18\frac{x^3}{3}\cdot y + \frac{3}{D}x^3\right]$$

$$= x^4 + 6x^3y + \frac{3x^4}{4}$$

$$= x^4 + 6x^3y + \frac{3}{4}x^4 = \frac{7x^4}{4} + 6x^3y$$

$$z = \phi_1(y+3x) + x\,\phi_2(y+3x) + \frac{7}{4}x^4 + 6x^3y \quad \text{Ans}$$

5(b) Let $f(z) = u + iv$ be ~~an~~ an analytic function with constant modulus. Then

$$|f(z)| = |u + iv| = \text{constant}$$

$$\Rightarrow \sqrt{u^2 + v^2} = \text{constant} = k \text{ (say)}$$

Squaring both sides, we get

$$u^2 + v^2 = k^2 \quad \text{——— (2)}$$

Differentiating eq (1) Partially w.r.to 'x' we get (11)

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0 \Rightarrow \boxed{u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0} \quad (2)$$

Again, P.D w.r.to 'y'; we get

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0 \Rightarrow \boxed{u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0} \quad (3)$$

$$\Rightarrow u\left(-\frac{\partial v}{\partial x}\right)+v\left(\frac{\partial u}{\partial x}\right)=0 \quad (4) \quad \left[\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \;\&\; \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}\right.$$

By C-R eq.]

Squaring and adding eq. (2) and (4)

$$(u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2=0 \quad [\because u^2+v^2=c^2\neq 0$$

$$\Rightarrow |f'(z)|^2=0 \quad \left[\because f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right.$$

$$\Rightarrow |f'(z)|=0 \quad \left. |f'(z)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right]$$

$\Rightarrow f(z)$ is constant

6(a) Let $f(z)=\dfrac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}$, $|z|=3$

we know that $\int_C f(z) = 2\pi i\,[\sum \text{Res. of } f(z) \text{ at each pole in C.}]$ — (1)

For the poles of $f(z)$: Put $(z-1)(z-2)=0$

$\Rightarrow z=1,2$ are simple poles.

At pole $z=1$, then $|z|=|1|=1<3$, inside the circle C.

At pole $z=2$, then $|z|=|2|=2<3$, inside the circle C.

(12) Hence both simple poles are lie inside of C.

(i) $[\text{Res}. f(z)]_{z=1} = \lim_{z\to 1} (z-1) \cdot \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$

$= \lim_{z\to 1} \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}$

$= \frac{\sin\pi + \cos\pi}{(1-2)} = \frac{0-1}{-1} = 1$

(ii) $[\text{Res}, f(z)]_{z=2} = \lim_{z\to 2} \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}$

$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)} = \frac{0+1}{1} = 1$

Hence (1) becomes:

$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = 2\pi i [1+1] = 4\pi i$

6(b) let $I = \int_0^{2\pi} \frac{\cos 4\theta}{5+4\cos\theta} d\theta$

Put $z = e^{i\theta}$

$\Rightarrow dz = i e^{i\theta} d\theta$

$\Rightarrow d\theta = \frac{dz}{e^{i\theta} . i}$

$\boxed{d\theta = \frac{dz}{iz}}$

since $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\cancel{\cos 4\theta = \frac{e^{i4\theta} + e^{-i4\theta}}{2}}$

$\cancel{= \frac{1}{2}\left(z^4 + 1/z^4\right)}$

$\cancel{= \frac{1}{2}\left(\frac{z^8+1}{z^4}\right)}$

$$I = \int_0^{2\pi} \frac{\cos 4\theta}{5+4\cos\theta}\, d\theta = \int_C \frac{\frac{1}{2}\cdot\left(\frac{z^8+1}{z^4}\right)}{5+4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_C \frac{z^8+1}{z^4} \times \frac{z^2}{5z+2z^2+2} \cdot \frac{dz}{z}$$

$$= \frac{1}{2i}\int_C \frac{z^2(z^8+1)}{z^3(2z^2+5z+2)}\, dz$$

$$I = \frac{1}{2i}\int_C \frac{z(z^8+1)}{z^3(2z^2+5z+2)}\, dz, \text{ where } C \text{ is unit circle } |z|=1$$

Let $I = \int_0^{2\pi} \frac{\cos 4\theta}{5+4\cos\theta}\, d\theta$

$$= \text{Real Part of } \int_0^{2\pi} \frac{\cos 4\theta + i\sin 4\theta}{5+4\cos\theta}\, d\theta$$

$$= \text{R.P. of } \int_0^{2\pi} \frac{e^{i4\theta}}{5+4\cos\theta}\, d\theta \quad \text{--- (1)}$$

We know that for unit circle,

$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}, \quad d\theta = \frac{dz}{iz}$

$$\therefore I = \text{R.P. of } \int_C \frac{z^4}{5+2\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \text{R.P. of } \frac{1}{i}\int_C \frac{z^4}{5z+2z^2+2}\, dz \quad \text{--- (2)}$$

Let $f(z) = \dfrac{z^4}{2z^2+5z+2} = \dfrac{z^4}{2z^2+5z+2}$

(14) The Poles are given by,

$$2z^2+5z+2=0 \Rightarrow 2z^2+5z+2=0$$

$$(2z+1)(z+2)=0 \quad , \quad z=-\tfrac{1}{2},-2$$

Now $|z|=\left|\frac{1}{2}\right|=\frac{1}{2}<1$ and $|z_2|=|2|=2>1$

Clearly $z=\frac{1}{2}$ lies inside the unit circle $|z|=1$.

$$\text{Residue}\left(z=\tfrac{1}{2}\right)=\lim_{z\to\frac{1}{2}}\left(z+\tfrac{1}{2}\right)f(z)$$

$$=\frac{1}{2}\lim_{z\to 1/2}(z+1/2)\frac{z^4}{2z^2+5z+2}$$

$$=\frac{1}{2}\lim_{z\to 1/2}(z+1/2)\frac{z^3}{(z+1/2)(z+2)}$$

$$=\frac{1}{2}\lim_{z\to 1/2}\frac{z^4}{(z+2)}=$$

$$=\frac{1}{2}\frac{(1/2)^4}{(1/2+2)}=\frac{1}{2}\times\frac{1}{16}\times\frac{2}{5}=\frac{1}{80}$$

By Cauchy's residue theorem,

$$\int_C \frac{z^4}{2z^2+5z+2}\,dz = 2\pi i\,[\text{Sum of residue inside in unit circle}]$$

$$=2\pi i\left(\frac{1}{80}\right)=\frac{\pi i}{40}$$

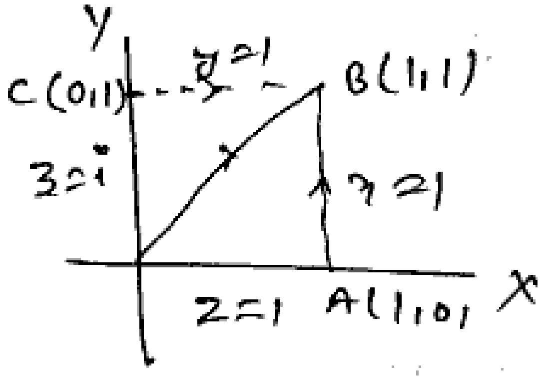
By (2)

$$\int_0^{2\pi}\frac{\cos 4\theta}{5+4\cos\theta}\,d\theta = \text{R.P. of}\left[\frac{1}{i}\left(\frac{\pi i}{40}\right)\right]=\frac{\pi}{40}$$

7(a) Since $z = x+iy$ so that $dz = dx + i\,dy$

$$\therefore \int_0^{1+i} (x-y+ix^2)\,dz = \int_0^{1+i} (x-y+ix^2)(dx+i\,dy) \quad \text{--- (1)}$$

(i) Along the straight line OB from $z=0$ to $z=1+i$



$\therefore$ The eq. of line joining the Point O(0,0) and B(1,1) is $y = x \Rightarrow dy = dx$

and x varies from 0 to 1

From eq (1), we get

$$\int_{OB} (x-y+ix^2)\,dz = \int_0^1 (x-x+ix^2)(dx+i\,dy)$$
$$= i(1+i)\int_0^1 x^2\,dx = \frac{-1+i}{3}$$

$$\boxed{\int_{OB} (x-y+ix^2)\,dz = \frac{-1+i}{3}}$$

(ii) Along the path OAB where A is $z=1$

$$\therefore \int_{OAB} (x-y+ix^2)\,dz = \int_{OA} (x-y+ix^2)\,dz + \int_{AB} (x-y+ix^2)\,dz \quad \text{--- (2)}$$

Now, along the line OA,

Here $y=0$, $dy=0$, so that $dz = dx$ and x varies from 0 to 1.

From eq (1); we get

$$\int_{OA} (x-y+ix^2)\,dz = \int_0^1 (x+ix^2)\,dx = \left(\frac{x^2}{2} + \frac{ix^3}{3}\right)_0^1 = \frac{1}{2} + \frac{i}{3}$$

(15) Again along the line AB,

here $x=1$, $dx=0$, so that $dz = i\,dy$ and

y varies from 0 to 1.

From eq. (1), we get

$$\int_{AB}(x-y+ix^2)\,dz = \int_0^1 (1-y+i)\,i\,dy$$

$$= i\left[y-\frac{y^2}{2}+iy\right]_0^1 = i\left(1-\frac{1}{2}+i\right)$$

$$= -1+\frac{i}{2}$$

By eq. (2)

$$\int_{OAB}(x-y+ix^2)\,dz = \frac{1}{2}+\frac{i}{3}-1+\frac{i}{2} = \frac{-1}{2}+\frac{5i}{6}$$

$$= \frac{-1}{2}+\frac{5i}{6}$$

7(b) We prove that $\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$

Since $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

We have $\nabla^2 f(r) = \frac{\partial^2}{\partial x^2}f(r)+\frac{\partial^2}{\partial y^2}f(r)+\frac{\partial^2}{\partial z^2}f(r)$ — (1)

Now, $\frac{\partial^2}{\partial x^2}f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}f(r)\right]$

$= \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{\partial r}{\partial x}\right]$

| $r^2 = x^2+y^2+z^2$

| $\frac{\partial r}{\partial x} = \frac{x}{r}$

$= \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$

$$= \frac{\frac{\partial}{\partial x}[x f'(r)]\cdot r - x f'(r)\frac{\partial r}{\partial x}}{r^2}$$

(12)

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{r\left[x f''(r)\cdot \frac{\partial r}{\partial x} + f'(r)\cdot\right] - x f'(r)\cdot \frac{x}{r}}{r^2}$$

$$= \frac{r x f''(r)\cdot \frac{x}{r} + r\cdot f'(r) - \frac{x^2}{r} f'(r)}{r^2}$$

$$\therefore \frac{\partial^2}{\partial x^2} f(r) = \frac{f''(r)}{r^2} x^2 + \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) \quad (2)$$

Similarly, we have

$$\frac{\partial^2}{\partial y^2} f(r) = \frac{F''(r)}{r^2} y^2 + \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) \quad (3)$$

$$\frac{\partial^2}{\partial z^2} f(r) = \frac{F''(r)}{r^2} z^2 + \frac{1}{r} f'(r) - \frac{z^2}{r^3} f'(r) \quad (4)$$

Hence eq (1) becomes,

$$\nabla^2 f(r) = \frac{3}{r} F'(r) + \frac{F''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3}{r} f'(r) + \frac{f''(r)}{r^2} \cdot r^2 - \frac{f'(r)}{r^3} \cdot r^2$$

$$\boxed{\because r^2 = x^2+y^2+z^2}$$

$$\nabla^2 f(r) = \frac{3}{r} f'(r) + f''(r) - \frac{1}{r} f'(r)$$

$$\boxed{\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

(18) 8(a) Given the scalar function is,

$$f = e^{2x}\cos yz$$

Now, $\text{grad } f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(e^{2x}\cos yz)$

$\text{grad } f = (2e^{2x}\cos yz)\hat{i} - (ze^{2x}\sin yz)\hat{j} - (ye^{2x}\sin yz)\hat{k}$

$\Rightarrow \text{grad } f = 2\hat{i}$ at $(0,0,0)$.

$\therefore$ Tangent to the given curve is

$$\frac{d\vec{r}}{dt} = a\cos t\,\hat{i} - a\sin t\,\hat{j} + a\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}; \text{ at } t = \pi/4$$

$$\therefore \hat{n} = \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{-a}{\sqrt{2}}\right) + a^2}} = \frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{1}{\sqrt{2}}\hat{k}$$

The D.D. of scalar function f at $(0,0,0)$ in the the direction of tangent at $t = \pi/4$ is,

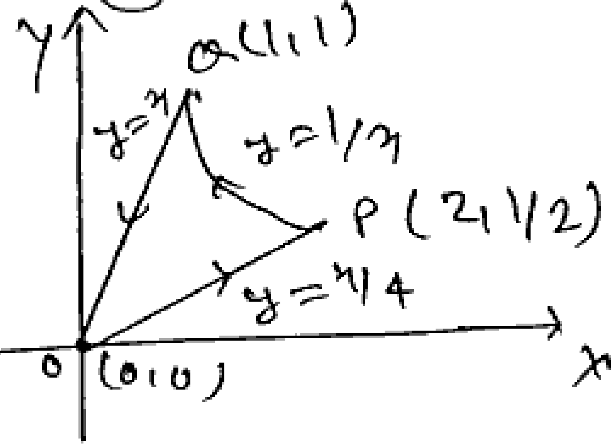
$$\text{D.D.} = \hat{n}\,\text{grad } f$$
$$= \left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right)(2\hat{i}) = \frac{1}{2}(2) = 1$$

$$\boxed{\text{D.D.} = 1}$$

8(b) Using Green's theorem, find the area of the region in the first quadrant, given by $\frac{1}{2}\int_c (x\,dy - y\,dx)$ and bounded by the curves $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$. (19)

Sol. According to Green's theorem, area A of the region bounded by closed curve $C\left(y = x,\ y = \frac{1}{x},\ y = \frac{x}{4}\right)$ is given by

$$A = \frac{1}{2}\int_c (x\,dy - y\,dx) \quad \text{—(1)}$$



Clearly c bounded by three lines c_1, c_2 and c_3.

So that

$$A = \frac{1}{2}\int_c = \frac{1}{2}\left[\int_{c_1} + \int_{c_2} + \int_{c_3}\right] \quad \text{—(2)}$$

(i) Along curve C_1: $y = \frac{x}{4} \Rightarrow dy = \frac{1}{4}\,dx$ and $x \to 0$ to 2

$$\int_{c_1} = \int_c (x\,dy - y\,dx) = \int_0^2 \left(x \cdot \frac{1}{4}\,dx - \frac{x}{4}\,dx\right) = 0$$

(ii) Along curve C_2: $y = \frac{1}{x} \Rightarrow dy = -\frac{1}{x^2}\,dx$ and $x \to 2$ to 1

$$\therefore \int_{c_2} = \int_{c_2} (x\,dy - y\,dx) = \int_2^1 x\left(-\frac{1}{x^2}\right)dx - \frac{1}{x}\,dx$$

$$= \int_2^1 -\frac{2}{x}\,dx = -2\left(\log x\right)_2^1 = -2(0 - \log 2)$$

$$= 2\log_e 2.$$

(20) Along curve C_3: $y = x \Rightarrow dy = dx$ and $x \to 1$ to 0

$$\int_{C_3} = \int_{C_3} (x\,dy - y\,dx) = \int_1^0 (x\,dx - x\,dx) = 0$$

Hence (2) becomes

$$A = \frac{1}{2} \cdot 2\log_e 2 = \log_e 2.$$