MATH 2568: LINEAR ALGEBRA

CURTIS WENDLANDT

Abstract. These are lecture notes for Math 2568 (Linear algebra) at The Ohio State University. The notes are mostly based on the course textbook [JRA].

Contents

1. Systems of linear equations and matrices

1.1. **Introduction to linear systems and matrices.** Throughout this course, the standard notation $\mathbb{R}$ will be used to denote the set of *real numbers*.

The goal of this section is twofold. Firstly, we will introduce a class of equations called *linear equations* and motivate the rather fundamental problem of solving several of them simultaneously. Secondly, we will introduce the language of *matrices*, primarily as a tool for making this problem more manageable, and even solving it.

1.1.1. *Linear equations and linear systems.*

Definition 1.1.

(1) A $\mathbb{R}$-*linear equation* in n unknowns $x_1, \dots, x_n$ is an equation which can put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1.1}$$

where $b, a_1, \dots, a_n$ are fixed real numbers.

(2) A *solution* to (1.1) is a sequence of real numbers $s_1, \dots, s_n \in \mathbb{R}$, often written as a n-tuple $(s_1, \dots, s_n)$, satisfying

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n = b.$$

Remark 1.2. More generally, one can define linear equations over the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$, or over any mathematical structure called a *ring* (which includes $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$). As we shall be working almost exclusively with $\mathbb{R}$, we will drop the $\mathbb{R}$ from "$\mathbb{R}$-linear".

The terminology "linear" can be explained by the fact that such equations generalize the equation of a line - we will come back to this below. Linear equations are among the simplest and most well understood equations in mathematics. Let's begin with a few examples:

Example 1.3. (a) The equation $0 = 0x_1 + \cdots + 0x_n = 0$ is linear. It is called the *trivial* linear equation. Every sequence $s_1, \dots, s_n \in \mathbb{R}$ is a solution of it.

(b) For any $m, b \in \mathbb{R}$, $y = mx + b$ is linear. Note that this is just the equation of a line with slope m and y-intercept b. We will come back to this.

(c) $x_1 + 2x_2 + \ldots + 100x_{100} = 5050$ is linear, with a solution given by

$$1 = x_1 = x_2 = \cdots = x_{100}.$$

(d) $x^3 + 10x = 0$ and $x_1x_2 + x_3 + x_4 = 0$ are **not** linear equations.

Our focus in this class will not just be on linear equations, but rather systems of linear equations:

Definition 1.4.

(1) An $m \times n$ *system of linear equations* is a set of m linear equations in n unknowns $x_1, \dots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{1.2}$$

where $b_1, \dots, b_m$ and each a_{ij} are fixed real numbers.

(2) A *solution* to the system (1.2) is a sequence $s_1, \dots, s_n \in \mathbb{R}$, often written as $(s_1, \dots, s_n)$, which is a solution to each of the m equations in the system.

(3) If the system (1.2) admits at least one solution, it is said to be *consistent*. Otherwise, it is said to *inconsistent*.

Let's illustrate this definition with a few examples.

Example 1.5. (a) The linear equation (1.1) is itself a $1 \times n$ linear system. It is inconsistent exactly when

$$(a_1, \dots, a_n, b) = (0, \dots, 0, b) \quad \text{for some} \;\; b \neq 0.$$

Exercise. Prove this assertion.

(b) The following is a 2×3 linear system:

$$\begin{aligned} x + 2y + z &= 1 \\ x + z &= 1 \end{aligned}$$

Exercise. Show that every solution of this system takes the form

$$\begin{aligned} x &= s \\ y &= 0 \\ z &= 1 - s \end{aligned}$$

where s is an *independent* (or *free*) variable, which can take arbitrary values in $\mathbb{R}$. The above is called the *general solution* to the system, as it describes all of its solutions.

(c) An example of a 4×4 system is

$$\begin{aligned} x_1 + x_2 + 6x_4 &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 2x_3 + 5x_4 &= 0 \\ x_1 + 2x_2 + x_3 + 6x_4 &= 0 \end{aligned}$$

It has one solution given by $0 = x_1 = x_2 = x_3 = x_4$. However, at this point it is not clear how the general solution to a larger system like this can be found (and $n = m = 4$ is still very small).

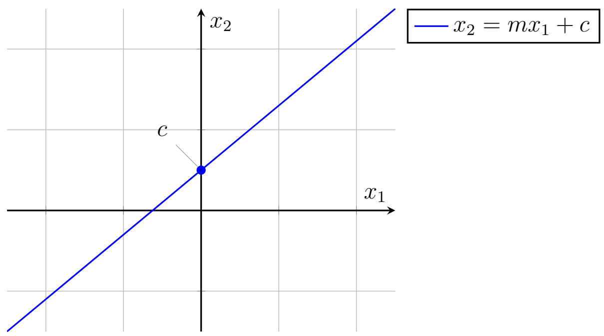
1.1.2. *Geometric analysis of $m\times 2$ systems.* Before continuing with our study of linear systems in general, let's focus in on the special case where $n = 2$. In this setting, the corresponding systems and their solutions can be analyzed geometrically on the plane $\mathbb{R}^2$. Indeed, each equation in such a system is either inconsistent, trivial, or of the form

$$a_1x_1 + a_2x_2 = b,$$

with a_1 and a_2 not both zero. If $a_2 \neq 0$ the above is equivalent to

$$x_2 = mx_1 + c \quad \text{where } m = -\frac{a_1}{a_2},\ c = \frac{b}{a_2}.$$

This is the equation of a line with slope m and x_2-intercept c:



If instead $a_2 = 0$, the above equation is the vertical line $x_1 = b/a_1$. Note that geometrically, the trivial equation $0x_1 + 0x_2 = 0$ corresponds to the entire plane.

Remark 1.6. As hinted at earlier, this geometric description of a linear equation in 2 unknowns is one reason why a linear equation is called "linear".

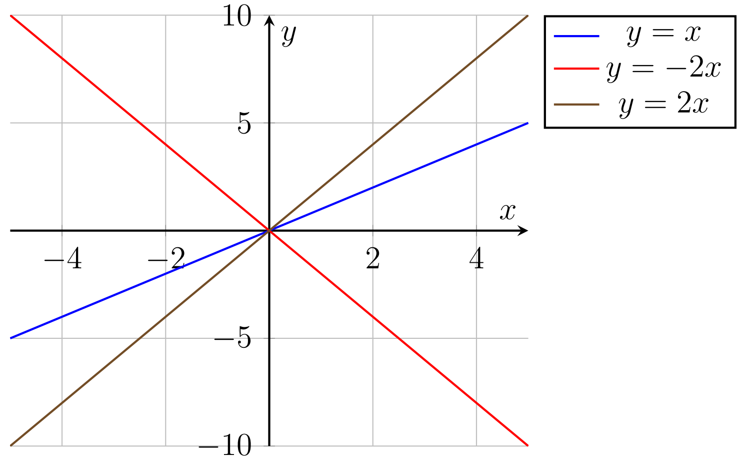
Next, assuming we are given a $m\times 2$ system of consistent equations, a solution to the system is precisely a point (x_1, x_2) on the plane where the associated lines intersect[1]. In particular, given a graph of the lines, one may solve the system geometrically by identifying the points of intersection.

Example 1.7. (a) The system

$$\begin{aligned} x - y &= 0 \\ 2x + y &= 0 \\ -2x + y &= 0 \end{aligned}$$

corresponds to the three lines $y = x$, $y = -2x$ and $y = 2x$. Geometrically, we have:

[1]Note that each trivial equation can be safely deleted from the system; only the lines play a role here.

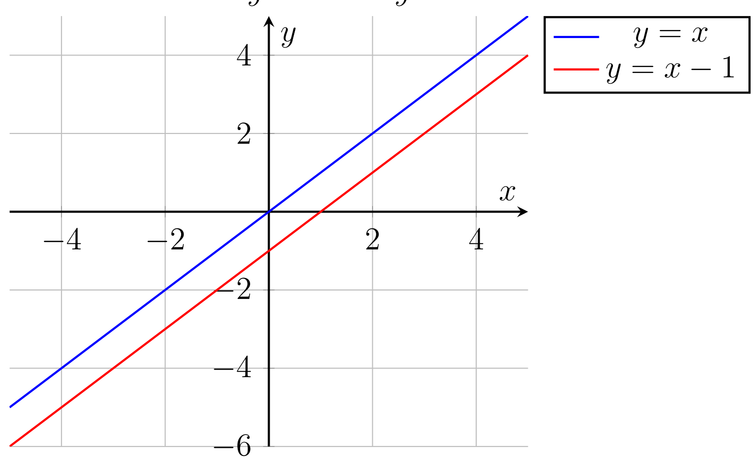


It is clear from this graph that there is a unique solution, which occurs at $(x, y) = (0, 0)$.

(b) The 2×2 linear system

$$\begin{aligned} x - y &= 0 \\ 2x - 2y &= 2 \end{aligned}$$

corresponds to the two lines $y = x$ and $y = x - 1$:



Since these lines are parallel, the system has *no solution* and is thus inconsistent.

(c) The 2×2 system

$$\begin{aligned} x - y &= 0, \\ 3x - 3y &= 0. \end{aligned}$$

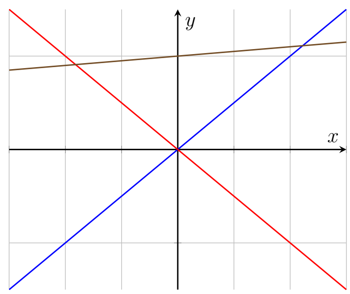
As both equations correspond to the line $y = x$, there are infinitely many points of intersection (every point (s, s) is on both lines, and hence a solution to both equations). Therefore, the system has *infinitely many solutions*.

One can see that, more generally, a $m \times 2$ system either has a unique solution, infinitely many solutions, or no solution, corresponding to the geometric configuration of the associated set of lines. The following result spells this out.

Proposition 1.8. *Let $m \geq 2$. A system of m consistent, non-trivial, linear equations in 2 unknowns has*

(1) *A unique solution precisely when the associated lines intersect at a single point.*
(2) *Infinitely many solutions precisely when all m lines coincide.*
(3) *No solution when either*
    (a) *There are two parallel lines, or*
    (b) *There are two lines which intersect a third line at two distinct points.*

The situation (3b) is illustrated in the following diagram:



Note that the red and blue lines meet the brown line at two different places, so all three lines have no mutual intersection.

Remark 1.9.

(1) One can perform a similar analysis when $n = 3$, with "lines" in $\mathbb{R}^2$ replaced by "planes" in three dimensional space $\mathbb{R}^3$. The end result is the same: an $m \times 3$ system has either a unique solution or infinitely many solutions when it is consistent (I encourage you to think about this as an exercise when $m = 2$ and $m = 3$).
(2) In fact, we will soon see that this is true for any $m \times n$ system: If the system is consistent, it has either *exactly one solution* or *infinitely many solutions*.

Note. End of Lecture 1 (01/06/2020)

1.1.3. *Equivalent systems and elementary operations.* When $n > 3$ (or even $n = 3$), it is typically near impossible to try to solve a $m \times n$ linear system geometrically. Moreover, our solve-by-inspection algebraic techniques are far from efficient. These observations motivate our first main goal of this course, which is to solve the following problem:

Problem 1.10. Given an arbitrary $m \times n$ system of linear equations,

Q1) Determine if the system is consistent.

Q2) If the system is consistent, systematically find its general solution.

Strategy (version 0)**.** Our strategy for solving this problem is as follows:

Step 1. *Transform* the linear system to an *equivalent*, far simpler, system.
Step 2. Solve Q1) and Q2) for the simpler system.

For now, this is vague and not the most helpful. To make it more precise, let's define the keywords "transform" and "equivalent" carefully.

Definition 1.11. Two systems of linear equations in n unknowns are said to be *equivalent* provided they have the same set of solutions.

The *transformations* we should apply to our system are then exactly those which produce an equivalent system. The following theorem gives us a toolbox of three such operations. Remarkably, these are the only operations we will need.

Given an arbitrary $m \times n$ system as in (1.2), let E_i denote the i^{th}-equation:

$$E_i : \quad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

Theorem 1.12. *Let $1 \leq i, j \leq m$. Then each of the following operations produces an equivalent system to* (1.2)*:*

(1) *Interchanging the two equations E_i and E_j:*
$$E_i \longleftrightarrow E_j.$$
(2) *Multiplying the equation E_i by a nonzero scalar $k \in \mathbb{R}$:*
$$E_i \mapsto kE_i.$$
(3) *Replacing E_i by the sum $E_i + kE_j$, where $k \in \mathbb{R}$:*
$$E_i \mapsto E_i + kE_j.$$

We will call these operations *elementary operations*.

Remark 1.13. It should be more or less clear that (1) and (2) do not change the solution set of the system. You will demonstrate that this is also the case for (3) when $n = 2$ on your first homework. This generalizes quite readily to arbitrary n.

Example 1.14. Consider the 3×3 system

$$\begin{aligned} x + y &= 0 \\ 2x + 2y + 2z &= 0 \\ 2x + 2y + z &= 0 \end{aligned}$$

Performing the operations $E_2 \mapsto \frac{1}{2}E_2$ and $E_3 \mapsto E_3 - 2E_1$, we obtain

$$\begin{aligned} x + y &= 0 \\ x + y + z &= 0 \\ z &= 0 \end{aligned}$$

After applying $E_2 \mapsto E_2 - E_1$, we find that the original system is equivalent to the very simple 2×3 system

$$\begin{aligned} x + y &= 0 \\ z &= 0 \end{aligned}$$

Therefore, the original system has general solution given by

$$\begin{aligned} x &= -s \\ y &= s \\ z &= 0 \end{aligned}$$

where s is a free variable. Equivalently, this can be written as

$$\begin{aligned} x &= -y \\ y &\ \text{ is free} \\ z &= 0 \end{aligned}$$

1.1.4. *Matrices and elementary row operations.* To bring even more power to the above technique for solving systems, we will will learn to encode any given linear system in a *matrix*.

Definition 1.15. An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns, which takes the general form

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \tag{1.3}$$

where each a_{ij} is a fixed real number.

Example 1.16. The following are all matrices:

$$\begin{pmatrix} 1 & 2 & 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & \pi \\ 4 & 2 & 17 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & \pi & 1 \\ 4 & 2 & 17 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 1.17.

(1) The *coefficient matrix* of the system (1.2) is the $m \times n$ matrix $A = (a_{ij})$ from (1.3).
(2) The *augmented matrix* of the system (1.2) is the $m \times (n+1)$ matrix

$$(A|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right), \quad \text{where } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Note that the augmented matrix of a system completetely determines it; one can easily go back and forth between the system and the matrix.

Example 1.18. The system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 4x_2 + 6x_3 &= 2 \\ 3x_1 + 4x_2 + 5x_3 &= 3 \end{aligned}$$

has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 3 & 4 & 5 & 3 \end{array}\right).$$

On the other hand, the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

corresponds to the system

$$\begin{aligned} x_1 + 2x_3 &= 0, \\ x_2 + x_3 &= 1, \\ x_4 &= 0. \end{aligned}$$

Given an arbitrary $m \times n$ matrix A as in (1.3), let R_i denotes its i^{th} row:

$$R_i : \quad \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}.$$

Then, in the language of matrices, elementary operations correspond to elementary row operations, which we now define.

Definition 1.19.

(1) Each of the following operations, applied to A, is called an *elementary row operation*:
   (a) Interchanging the two rows R_i and R_j:
   $$R_i \longleftrightarrow R_j.$$
   (b) Multiplying the row R_i by a nonzero scalar $k \in \mathbb{R}$:
   $$R_i \mapsto kR_i.$$
   (c) Replacing R_i by the sum $R_i + kR_j$, where $k \in \mathbb{R}$:
   $$R_i \mapsto R_i + kR_j.$$
(2) Two $m \times n$ matrices A and B are *row equivalent* if one can be obtained from the other by a sequence of elementary row operations.

Remark 1.20.

(1) In practice, we will write kR_i for $R_i \mapsto kR_i$, and $R_i + kR_j$ for $R_i \mapsto R_i + kR_j$.

(2) You should think of the vertical line which appears in an augmented matrix as imaginary. The last column should be treated as any ordinary column in the matrix when it comes to applying row operations.

As a Corollary to Theorem 1.12, we immediately obtain the following.

Corollary 1.21. *If $(A|\mathbf{b})$ and $(C|\mathbf{d})$ are row equivalent $m \times (n+1)$ augmented matrices, then they represent equivalent systems of linear equations.*

We can now revise version 0 of our strategy as follows:

Strategy (version 1)**.** To solve Q1) and Q2) for a given $m \times n$ system, we

Step 1. Find the augmented matrix $(A|\mathbf{b})$ of the system.
Step 2. Apply elementary row operations to $(A|\mathbf{b})$ to obtain a simpler, row equivalent matrix $(C|\mathbf{d})$.
Step 3. Write down the system of equations represented by $(C|\mathbf{d})$, and solve Q1) and Q2) for this simpler system.

Example 1.22. Suppose we want to solve the 3×4 system

$$\begin{aligned} 5x_1 + 5x_3 + 5x_4 &= 0 \\ 2x_1 + x_2 + x_3 + x_4 &= 2 \\ 2x_1 + 2x_3 + x_4 &= 1 \end{aligned}$$

We then translate this to an augmented matrix and apply row operations:

$$\left(\begin{array}{cccc|c} 5 & 0 & 5 & 5 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{5}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R_2+R_3]{R_1-R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

From this matrix, we get the equivalent linear system

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 - x_3 &= 1 \\ x_4 &= -1 \end{aligned}$$

This is consistent, with a general solution given by

$$\begin{aligned} &x_1 = 1 - x_3 \\ &x_2 = 1 + x_3 \\ &x_3 \text{ is free} \\ &x_4 = -1 \end{aligned}$$

Hence, by Corollary 1.21, our original system is consistent and has the above general solution.

Note. End of Lecture 2 (01/08/2020)

1.2. **Echelon form and Gauss-Jordan elimination.** We now have a semi-precise strategy for solving an arbitrary linear system. However, version 1 of our strategy still does not specify

(1) If there is an ideal form the simpler matrix $(C|\mathbf{d})$ should take, and
(2) If there is a systematic procedure for passing from $(A|\mathbf{b})$ to $(C|\mathbf{d})$.

In this subsection, we will introduce the so-called *reduced echelon form* of a matrix, which gives us our ideal choice of $(C|\mathbf{d})$. The desired systematic procedure for obtaining it is called *Gauss-Jordan elimination* or *row reduction*.

1.2.1. *Echelon form.* Recall that, given a $m \times n$ matrix A, the notation R_i is used to denotes the i^{th} row of A, for each $1 \leq i \leq m$. If in addition R_i is nonzero, we call its leftmost nonzero entry its *leading entry*.

Definition 1.23. An $m \times n$ matrix A is in *echelon form* (EF) if

EF.1) All zero rows appear below all nonzero rows.
EF.2) The leading entry of each nonzero row is 1.
EF.3) If R_i and R_ℓ are nonzero rows with $i > \ell$, then the leading entry of R_i appears to the right of the leading entry of R_ℓ.

Example 1.24. The following matrices are in echelon form

$$\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here the leading entries of nonzero rows have been highlighted in red (note the staircase like pattern they form), and the zero rows have been highlighted in blue.

The following result tells us that every matrix is row equivalent to a matrix in echelon form, and provides a precise algorithm for obtaining it.

Proposition 1.25. *Let A be a $m \times n$ matrix. Then the following sequence of steps transforms A to a row equivalent matrix in echelon form:*

Step *1.* *Locate the leftmost nonzero column* $\mathbf{c}$ *of A.*
Step *2.* *Apply $R_1 \leftrightarrow R_j$ if needed to ensure the first entry of* $\mathbf{c}$ *is nonzero.*
Step *3.* *Apply $R_1 \mapsto kR_1$ if needed to ensure the first entry of* $\mathbf{c}$ *is* 1*. We call this entry the pivot.*
Step *4.* *Kill all nonzero entries directly below the pivot by applying operations of the form $R_j \mapsto R_j + kR_1$ for $j = 2, \ldots, m$.*
Step *5.* *Leave R_1 fixed, and repeat* Step *1.*–Step *4.* *on the $(m-1) \times n$ matrix obtained from A by removing R_1. Keep repeating the process until the resulting matrix is in echelon form.*

Let's do an example to illustrate how this algorithm works in practice.

Example 1.26. Let A be the 3×4 matrix

$$\begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \end{pmatrix}.$$

Step 1: The 2^{nd} column (highlighted in blue) is the first nonzero column.

Step 2: We apply $R_1 \leftrightarrow R_2$ to make the first nonzero entry of the column nonzero.

$$\begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 4 & 0 & 4 \end{pmatrix}$$

Step 3: Not needed, the leading entry (our pivot) of R_1 is already 1. It is highlighted in red above.

Step 4: We just need to apply $R_3 \mapsto R_3 - 4R_1$.

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow{R_3 - 4R_1} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

Step 5. Now we ignore the first row, and repeat. Our next pivot will be where the highlighted 2 is.

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -4 \end{pmatrix} \xrightarrow[\text{Step 3}]{\frac{1}{2}R_2} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{pmatrix} \xrightarrow[\text{Step 4}]{R_3 + 4R_2} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

And finally, we repat once again. Since we only have one row left, we just have to rescale it so that it has a 1 in its leading entry.

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow[\text{Step 3}]{\frac{1}{4}R_4} \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1.2.2. *Reduced echelon form.* From the point of view of solving linear systems, the echelon form of an augmented matrix is already very simple. However, it is not unique and can in fact be reduced further to give us a *reduced echelon form*, from which a general solution can be easily read.

Definition 1.27. An $m \times n$ matrix A is in *reduced echelon form* (REF) if

REF.1) It is in echelon form.
REF.2) Each leading entry of a nonzero row is the only nonzero entry in its column.

Example 1.28. The following matrices are in reduced echelon form

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here each column has been highlighted which contains a leading entry.

We can obtain the reduced echelon form of a matrix by adding one more step to our previous algorithm. In fact, we have the following stronger assertion:

Theorem 1.29. *Let A be a $m \times n$ matrix. Then A is row equivalent to a unique $m \times n$ matrix B in reduced echelon form, which can be obtained as follows:*

Step *1. Use* Step *1.* – Step *5. to put A in echelon form.*
Step *2. Kill the entries above each leading 1 using operations of the type $R_i \mapsto R_i + kR_j$.*

Remark 1.30. The algorithm used to obtain B in Theorem 1.29 is called *row reduction* or *Gauss-Jordan elimination*.

Example 1.31. Let's continue with our last example. We already transformed our matrix to echelon form:

$$\begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then proceed as follows

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 - 2R_3]{R_2 - 2R_3} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This last matrix is in reduced echelon form.

Remark 1.32. Though the above algorithm always works, in practice there is often more efficient routes to obtaining the reduced echelon form of a matrix, that involve skipping or combining several steps.

1.2.3. *Application to $m \times n$ linear systems.* So what does reduced echelon form have to do with solving linear systems? As hinted at earlier, the point is that the system encoded by an augmented matrix in reduced echelon form is in a very simple form; so simple that its consistency and general solution can be immediately read. Let's illustrate this with a few examples.

Example 1.33. (1) The REF augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

represents a system with general solution

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$$

In particular, the system has a unique solution $(1,2)$.

(2) The REF augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Corresponds to the system

$$\begin{aligned} x_1 + 2x_2 + 3x_4 &= 1 \\ x_3 + x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

Rearranging these equations gives the general solution

$$\begin{aligned} & x_1 = 1 - 2x_2 - 3x_4 \\ & x_2 \text{ is free} \\ & x_3 = 1 - x_4 \\ & x_4 \text{ is free} \end{aligned}$$

This means that an arbitrary solution takes the form

$$(1 - 2x_2 - 3x_4, x_2, 1 - x_4, x_4),$$

where we are free to choose x_2 and x_4 to be any real numbers (so there are infinitely many solutions). For instance, a *particular solution*, obtained by taking $x_2 = 0$ and $x_4 = 1$ is

$$(-2, 0, 0, 1).$$

(3) The REF augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

has last row which represents the (inconsistent!) equation

$$0x_1 + 0x_2 + 0x_3 = 1.$$

The system represented by the above matrix is therefore inconsistent.

These examples serve as excellent motivation for stating the final version of our strategy for solving systems of linear equations:

Strategy (version 2)**.** To solve a given $m \times n$ linear system, we

Step 1. Find the augmented matrix $(A|\mathbf{b})$ of the system.

Step 2. Row reduce $(A|\mathbf{b})$ to obtain a row equivalent matrix $(C|\mathbf{d})$ in reduced echelon form.

Step 3. Write down the system of equations represented by $(C|\mathbf{d})$, and either conclude the system is inconsistent or find its general solution.

We will further exploit reduced echelon form as a powerful tool for studying linear systems and their solutions in the next subsection. In particular, we will study Step 3 in signficant detail to see how and why it always works.

Let's conclude this section with a few complete examples which make use of our refined strategy[2].

Example 1.34. (1) Suppose we wish to solve the 3×4 system

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 + 4x_4 &= 4 \\ 3x_1 + x_2 + 9x_3 + 3x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2 \end{aligned}$$

The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 3 & 1 & 9 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{array}\right)$$

According to our strategy, we should now row reduce (that is, apply Gauss-Jordan elimination):

$$\xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 3 & 1 & 9 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-3R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & -5 & 0 & -3 & -5 \\ 0 & -2 & -4 & -2 & -2 \end{array}\right)$$

Here we have applied Step 1. - Step 4. of the algorithm from Proposition 1.25, with the highlighted 1 serving as our pivot. We now repeat, ignoring the first row:

$$\xrightarrow{-\frac{1}{5}R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 0 & \frac{3}{5} & 1 \\ 0 & -2 & -4 & -2 & -2 \end{array}\right) \xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 0 & \frac{3}{5} & 1 \\ 0 & 0 & -4 & -\frac{4}{5} & 0 \end{array}\right)$$

We now rescale the third row to get the echelon form

$$\xrightarrow{-\frac{1}{4}R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 0 & \frac{3}{5} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & 0 \end{array}\right)$$

To get to reduced echelon form, we work upwards and use our leading 1's to kill the entries above it:

$$\xrightarrow{R_1-3R_3} \left(\begin{array}{cccc|c} 1 & 2 & 0 & \frac{7}{5} & 2 \\ 0 & 1 & 0 & \frac{3}{5} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & 0 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & \frac{3}{5} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & 0 \end{array}\right)$$

[2] These examples are supplementary: they have not been given in class.

This matrix is in reduced echelon form. At each step I have highlighted the entry we are using to kill the entries above or below it. The corresponding system is

$$\begin{aligned} x_1 + \tfrac{1}{5}x_4 &= 0 \\ x_2 + \tfrac{3}{5}x_4 &= 1 \\ x_3 + \tfrac{1}{5}x_4 &= 0 \end{aligned}$$

Rearranging, we get the general solution

$$\begin{aligned} x_1 &= -\tfrac{1}{5}x_4 \\ x_2 &= -\tfrac{3}{5}x_4 + 1 \\ x_3 &= -\tfrac{1}{5}x_4 \\ x_4 &\ \text{ is free} \end{aligned}$$

Since there is a free variable (namely, x_4), the above means there are infinitely many solutions, each of which takes the form

$$(-\frac{1}{5}x_4, -\frac{3}{5}x_4 + 1, -\frac{1}{5}x_4, x_4)$$

for some real number x_4. For example, setting $x_4 = 0$ we get the particular solution $(0, 1, 0, 0)$.

(2) Consider now the 3×3 system

$$\begin{aligned} 2x_2 + 2x_3 &= 4 \\ 4x_1 + 2x_2 + 2x_3 &= 8 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

We begin by forming the augmented matrix and row reducing to echelon form:

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 4 \\ 4 & 2 & 2 & 8 \\ 1 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 2 & 8 \\ 0 & 2 & 2 & 4 \end{array}\right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & \text{-2} & -2 & 4 \\ 0 & 2 & 2 & 4 \end{array}\right)$$

At this point we can skip a step; rather than rescale the second row so that its leading entry is 1, we note that we can already kill the 2 below it by applying $R_3 \mapsto R_3 + R_2$:

$$\xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

There is no reason to proceed further; the last row represents the inconsistent equation

$$0x_1 + 0x_2 + 0x_3 = 8.$$

Therefore, our original system is inconsistent.

Note. End of Lecture 3 (01/10/2020).

1.3. **Reduced echelon form and linear systems.** We now turn towards carrying out a deeper analysis of solutions to linear systems, using our new found tool: the reduced echelon form of a matrix[3].

1.3.1. *A note on column operations.* It will be useful for us to understand what effect interchanging columns has on the system encoded by an arbitrary augmented matrix.

Lemma 1.35. *Let $\big(A \mid \mathbf{b}\big)$ be an augmented $m \times (n+1)$ matrix, with j^{th} column $\mathbf{c}_j$ $(1 \le j \le n)$. Then interchanging $\mathbf{c}_i$ and $\mathbf{c}_j$ $(\mathbf{c}_i \leftrightarrow \mathbf{c}_j)$ produces the matrix $\big(C \mid \mathbf{b}\big)$ which represents the system obtained by interchanging x_i and x_j.*

Though the solutions of the two systems are not the same (the systems are not equivalent!), they are identical up to a *change of variables*; all one has done is re-label the indices of the unknowns $x_1, \ldots, x_n$.

Example 1.36. The column operation

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right) \xrightarrow{\mathbf{c}_1 \leftrightarrow \mathbf{c}_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 0 & 2 & 1 \end{array}\right)$$

corresponds to interchanging x_1 and x_3:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_3 &= 1 \end{aligned} \xrightarrow{x_1 \leftrightarrow x_3} \begin{aligned} x_3 + 2x_2 + x_1 &= 3 \\ 2x_3 + x_1 &= 1 \end{aligned}$$

One can return to the orginal system by making the change of variables

$$y_1 = x_3, \quad y_2 = x_2, \quad y_3 = x_1.$$

1.3.2. *Analysis of systems via reduced echelon form.* Let us now assume that $\big(A \mid \mathbf{b}\big)$ is an arbitrary $m \times (n+1)$ augmented matrix in reduced echelon form. We begin with a general remark which has already played a role in some of our previous examples.

Remark 1.37. The following three statements are equivalent:

(1) $\big(A \mid \mathbf{b}\big)$ contains a row of the form $\big(0 \ \cdots \ 0 \mid 1\big)$.
(2) The last nonzero row of $\big(A \mid \mathbf{b}\big)$ is of the form $\big(0 \ \cdots \ 0 \mid 1\big)$.
(3) The column $\mathbf{b}$ contains a leading entry of $\big(A \mid \mathbf{b}\big)$.

Moreover, if any of these statements hold, then the $m \times n$ system represented by $\big(A \mid \mathbf{b}\big)$ is inconsistent, as it contains the equation

$$0x_1 + \cdots + 0x_n = 1.$$

Having dealt with this situation, we henceforth assume $\big(A \mid \mathbf{b}\big)$ does not contain a row of the form $\big(0 \ \cdots \ 0 \mid 1\big)$.

[3] In [JRA], this subsection is titled "Consistent systems of linear equations".

Let's now take advantage of what we know about column operations. Define

$$\begin{aligned} r &= \text{number of nonzero rows of } \left(A \mid \mathbf{b}\right) \\ &= \text{number of leading entries of } \left(A \mid \mathbf{b}\right). \end{aligned} \tag{1.4}$$

By interchanging the first n columns of $\left(A \mid \mathbf{b}\right)$ as needed, we can move all the columns with leading 1's to the left, to obtain another reduced echelon matrix of the form

$$\left(C \mid \mathbf{b}\right) = \left(\begin{array}{cccccccc|c} 1 & 0 & \cdots & 0 & c_{1,r+1} & \cdots & c_{1,n} & b_1 \\ 0 & 1 & \cdots & 0 & c_{2,r+1} & \cdots & c_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{r,r+1} & \cdots & c_{r,n} & b_r \\ \hline 0 & \cdots & & \cdots & & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & & \cdots & & \cdots & 0 & 0 \end{array}\right)$$

Here the r leading 1's are highlighted in red, and the $(m-r)$ zero rows are highlighted in blue. Note that we must have $r \leq n$, since each leading entry occupies its own column.

The system of equations this matrix encodes is

$$\begin{aligned} y_1 &= b_1 - c_{1,r+1}y_{r+1} - \cdots - c_{1,n}y_n \\ y_2 &= b_2 - c_{2,r+1}y_{r+1} - \cdots - c_{2,n}y_n \\ &\vdots \\ y_r &= b_r - c_{r,r+1}y_{r+1} - \cdots - c_{r,n}y_n \end{aligned}$$

This is already in the form of a general solution! In particular, the system is consistent with

(1) A *unique solution* when $r = n$. Indeed, in this case the above becomes
$$y_1 = b_1, \quad y_2 = b_2, \quad \cdots, \quad y_n = b_n.$$
(2) *Infinitely many solutions* when $r < n$, with $(n-r)$ independent (free) variables $y_{r+1}, \ldots, y_n$ and dependent variables $y_1, \ldots, y_r$.

Moreover, by Lemma 1.35, going back to the system represented by $\left(A \mid \mathbf{b}\right)$ requires only making a change of variables. In this relabeling of variables, $y_1, \ldots, y_r$ correspond to those variables associated to a leading entry in $\left(A \mid \mathbf{b}\right)$.

The above analysis provides a proof of the following theorem:

Theorem 1.38. *Let $\left(A \mid \mathbf{b}\right)$ be an $m \times (n+1)$ augmented matrix in reduced echelon form, with r nonzero rows. Then the associated $m \times n$ linear system is inconsistent if and only if the last nonzero row of $\left(A \mid \mathbf{b}\right)$ is*

$$\left(0 \quad \cdots \quad 0 \mid 1\right).$$

Otherwise, the system is consistent with $r \leq n$ and

(1) *A unique solution when $r = n$.*
(2) *Infinitely many solutions with $(n - r)$ free variables when $r < n$.*

In these two cases, the system admits a general solution in which the r dependent variables correspond to the r columns with leading entries.

We have already seen several examples of this theorem in action: see Example 1.33 and 1.34. Here is another example:

Example 1.39. Consider the augmented REF matrix

$$\left(\begin{array}{cccccc|c} 1 & 4 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right)$$

By Theorem 1.38, the system represented by this matrix is consistent with $n - r = 6 - 3 = 3$ free variables, and dependent variables x_1, x_3 and x_5 corresponding to the three highlighted columns with leading entries. Explicitly, it has general solution

$$\begin{aligned} & x_1 = 1 - 4x_2 - 2x_4 - x_6 \\ & x_2 \text{ is free} \\ & x_3 = 2 - 3x_4 \\ & x_4 \text{ is free} \\ & x_5 = 3 - x_6 \\ & x_6 \text{ is free} . \end{aligned}$$

Since every matrix is row equivalent to a matrix in reduced echelon form (by Theorem 1.29), the above theorem implies the second part of Remark 1.9:

Corollary 1.40. *If a $m \times n$ system of linear equations is consistent, then it either has a unique solution or infinitely many solutions.*

Another useful corollary to Theorem 1.38 is the following result.

Corollary 1.41. *Suppose that $m < n$. Then any $m \times n$ consistent system of linear equations has infinitely many solutions.*

Proof. The augmented matrix $\big(A \mid \mathbf{b}\big)$ of the system is row equivalent to a unique matrix $\big(C \mid \mathbf{d}\big)$ in reduced echelon form. The number r of nonzero rows of the matrix $\big(C \mid \mathbf{d}\big)$ satisfies $r \leq m < n$. By Theorem 1.38, it follows that the system it represents has infinitely many solutions. As this system is equivalent to the original system, the proof is complete. □

For example, a 9×10 linear system cannot possibly have a unique solution. Neither can a single linear equation in $n > 1$ unknowns. However, it is certainly possible for a 10×9 system to have a unique solution.

1.3.3. *Homogeneous systems.* There is a special class of linear systems which are ubiquitous in mathematics and are always consistent. These are the so-called *homogeneous systems.*

Definition 1.42. A $m \times n$ linear system is called *homogeneous* if it is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned} \tag{1.5}$$

The *trivial solution* of such a system is the solution

$$(x_1, \dots, x_n) = (0, \dots, 0).$$

Any other solution of (1.5) is said to be *nontrivial.*

Remark 1.43. If a homogeneous system as in (1.5) admits a single nontrivial solution, then it automatically has infinitely many solutions.

Since a homogeneous system is always consistent, Corollary 1.41 gives the following useful result.

Corollary 1.44. *Suppose that $m < n$. Then any $m \times n$ homogeneous linear system has infinitely many solutions.*

Note. End of Lecture 4 (01/13/2019).

1.3.4. *Practical simplifications.*

Remark 1.45. In practice, one does not need to row reduce an augmented matrix all the way to reduced reduced echelon form to determine how many solutions the system it represents has.

(1) If at any point in the reduction process one encounters a row of the form

$$\left(\begin{array}{ccc|c} 0 & \cdots & 0 & b \end{array}\right) \qquad \text{for } b \neq 0,$$

then one can stop the process and conclude the corresponding system is inconsistent.

(2) Similarly, from non-reduced echelon form (even with the requirement the leading entries are 1 omitted), one can conlude using Theorem 1.38 how many solutions the system has; the number of nonzero rows will not change after this point.

We conclude this subsection by illustrating the above remark with an example.

Example 1.46. Let $\left(A \mid \mathbf{b}\right)$ be the matrix

$$\left(A \mid \mathbf{b}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 2 & 1 & 2 & b \\ 5 & 1 & 8 & 2b^2+1 \end{array}\right).$$

For what values of b does $\big(A \mid \mathbf{b}\big)$ represent a consistent system? For these values, how many solutions does the system have?

Solution. We start by applying Gauss-Jordan elimination:

$$\xrightarrow[R_3-5R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 3 & -6 & b-2 \\ 0 & 6 & -12 & 2b^2-4 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 3 & -6 & b-2 \\ 0 & 0 & 0 & 2b^2-2b \end{array}\right)$$

At this point there is no reason to continue: The system will be consistent provided

$$b^2 - b = b(b-1) = 0.$$

That is, the system is consistent exactly when $b = 0$ or $b = 1$.

For these values of b, we can see from the above matrix that the associated reduced echelon form will have exactly 2 nonzero rows, and hence the system will have infinitely many solutions with $3 - 2 = 1$ free variables.

1.4. **Matrix operations.** [4]

So far, we have thought of matrices as merely a convenient way to encode linear systems. Technically, we could have done everything up until this point without a matrix at all (perhaps a bit uncomfortably). We now begin the process of slowly changing this viewpoint, and shifting our focus from linear systems to matrices themselves. In this section, we make the first step in this direction by introducing certain fundamental operations on matrices.

1.4.1. *Matrix equality, addition and scalar multiplication.* Recall that, given an $m \times n$ matrix A, we use the notation $A = (a_{ij})$ to indicate that the number in the i-th row and j-th column of A (called its (i, j)-*th entry*) is denoted a_{ij}.

We begin by addressing what it means for two matrices to be equal:

Definition 1.47. Let $A = (a_{ij})$ be a $m \times n$ matrix and $B = (b_{ij})$ be a $k \times \ell$ matrix. Then A and B are *equal* (denoted $A = B$) if $(m, k) = (n, \ell)$ and

$$a_{ij} = b_{ij} \quad \text{for all} \quad 1 \leq i \leq m,\ 1 \leq j \leq n.$$

In words: two matrices are equal if they have the same size and identical entries.

Definition 1.48. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices, and let $k \in \mathbb{R}$. Then:

(1) The *sum* $A + B$ is the $m \times n$ matrix with (i, j)-th entry

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$

(2) The *product* kA is the $m \times n$ matrix with (i, j)-th entry

$$(kA)_{ij} = ka_{ij}.$$

[4] This is §1.5 in the course textbook.

Example 1.49. Let A and B be the 3×3 matrices

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 10 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A + (-B) = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 9 & 2 \\ 0 & 1 & -1 \end{pmatrix},$$

$$2A = \begin{pmatrix} 4 & 8 & 8 \\ 2 & 20 & 6 \\ 0 & 2 & 0 \end{pmatrix}, \quad 3B = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{pmatrix}, \quad 2A + 3B = \begin{pmatrix} 7 & 11 & 11 \\ 2 & 23 & 9 \\ 0 & 2 & 3 \end{pmatrix}$$

1.4.2. *The n-dimensional real space $\mathbb{R}^n$.*

Definition 1.50. For any positive integer n, a $n \times 1$ matrix is called a n-dimensional *column* vector while a $1 \times n$ matrix is called a n-dimensional *row* vector.

By definition, n-dimensional column and row vectors take the general form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix},$$

respectively, with the x_i taking arbitrary real number values.

Remark 1.51. Generally, we will use square brackets for row and column vectors, and round brackets otherwise. This is simply a stylistic choice, with no mathematical substance, and need not be followed.

In linear algebra, the set of all n-dimensional column vectors plays a special role and is given the distinction of having its own name:

Definition 1.52. The *n-dimensional real space*[5] $\mathbb{R}^n$ is the set of all n-dimensional column vectors:

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}.$$

[5] $\mathbb{R}^n$ is often also called *Euclidean n-space*, or *n-dimensional Euclidean space*.

Its elements (n-dimensional column vectors) are often called *n-dimensional vectors*, and will be denoted using boldface font[6]:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Note that, since n-dimensional vectors are just $n \times 1$ matrices, they can be added together and multiplied by real numbers.

Remark 1.53.

(1) The name "n-dimensional real space" puts emphasis on the fact that $\mathbb{R}^n$ provides the most natural example of a *n-dimensional real vector space*. This is a type of mathematical structure which will play a fundamental role later in this course.
(2) You may have seen points in $\mathbb{R}^n$ represented as n-tuples $(x_1, \ldots, x_n)$ elsewhere. Both notations are perfectly acceptable. However, the column notation will have added practical value in this course.

1.4.3. *The vector form of a general solution.* Henceforth, we will think of a solution $s_1, \ldots, s_n$ of a $m \times n$ linear system as a column vector in $\mathbb{R}^n$:

$$\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^n.$$

Definition 1.54. The *vector form* of a general solution to a $m \times n$ linear system is an expression for it as a column vector in $\mathbb{R}^n$, of the form

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1}\boldsymbol{v}_1 + \cdots + x_{i_k}\boldsymbol{v}_k + \boldsymbol{v}_{k+1},$$

where $x_{i_1}, \ldots, x_{i_k}$ are distinct free variables and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} \in \mathbb{R}^n$ are fixed. The term $\boldsymbol{v}_{k+1}$ is called the *constant term*.

Let's dissect this definition by going through a few examples.

Example 1.55. (1) The system encoded by the REF augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

has general solution

$$\begin{aligned} &x_1 = -2x_2 + x_4 + 2 \\ &x_2 \ \text{ is free} \end{aligned}$$

[6] In class, we will use the underlined notation $\underline{x}$.

$$x_3 = x_4 + 1$$
$$x_4 \text{ is free.}$$

We may write this as a column vector as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_4 + 2 \\ x_2 \\ x_4 + 1 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

This last expression is our vector form of the general solution.

(2) The augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

represents a homogeneous system with general solution

$$x_1 = 2x_3$$
$$x_2 = x_3$$
$$x_3 \text{ is free.}$$
$$x_4 = 0$$

The vector form of this general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Remark 1.56. Note that the vector form in the second example above has *no constant term*. You will show in the second homework that this is always the case when one starts with a homogeneous system.

Remark 1.57. At this point, it might seem rather mysterious why we would bother with writing the general solution of a system in vector form. This will be a lot clearer later when we study vector subspaces of $\mathbb{R}^n$.

Note. End of Lecture 5 (01/15/2020).

1.4.4. *Towards matrix multiplication.* So far, we have learned that matrices of the same size can be added, and that a matrix can be multiplied by any scalar (that is, any real number). We now work towards defining a type of *multiplication* for any two matrices subject to certain size restrictions.

Our first step is to define how to left multiply a n-dimensional vector $\boldsymbol{x}$ by a $m \times n$ matrix A. Let us denote the j-th column of A by $\mathbf{A}_j$, thought of as the vector

$$\mathbf{A}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m.$$

With this understanding, we may write

$$A = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_n \end{pmatrix}.$$

Definition 1.58. Let A be as a above, and

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Then the product $A\boldsymbol{x} \in \mathbb{R}^m$ is the m-dimensional vector

$$\begin{aligned} A\boldsymbol{x} &= x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n \\ &= \sum_{j=1}^{n} x_j \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix}. \end{aligned} \tag{1.6}$$

Remark 1.59. By definition, the entry in the i^{th} row of $A\boldsymbol{x}$ is[7]

$$\begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_{j=1}^{n} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n.$$

Some of you may recognize this as the *dot product* $\boldsymbol{a}_i \cdot \boldsymbol{x}$ of the vectors

$$\boldsymbol{a}_i = \begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 1.60. Let A and $\boldsymbol{x}$ be given by

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 2 & 0 \end{pmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

[7]We always identify the 1×1 matrix $[a]$ with the number a.

Then $A\boldsymbol{x} \in \mathbb{R}^3$ is given by

$$A\boldsymbol{x} = 0\begin{bmatrix}1\\0\\2\end{bmatrix} + 1\begin{bmatrix}0\\1\\3\end{bmatrix} + 2\begin{bmatrix}2\\0\\2\end{bmatrix} + 0\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}4\\1\\7\end{bmatrix}.$$

One nice application of this definition of product is that it provides a very elegant way of encoding a $m \times n$ linear system as a *matrix equation.* Indeed, suppose were are given a $m \times n$ sytem with augmented matrix $\left(A \mid \mathbf{b}\right)$. Then, by (1.6), we have

$$A\boldsymbol{x} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n\end{bmatrix}.$$

It follows that the underlying system is equivalent to the *matrix equation*

$$A\boldsymbol{x} = \boldsymbol{b}, \quad \text{where} \quad \boldsymbol{b} = \begin{bmatrix}b_1\\ \vdots \\ b_m\end{bmatrix},$$

and $\boldsymbol{x}$ is treated as a vector of n-unknowns.

Example 1.61. Find all values of $x_1, x_2, x_3 \in \mathbb{R}$ which solve the vector equation

$$x_1\begin{bmatrix}1\\0\\1\end{bmatrix} + x_2\begin{bmatrix}2\\1\\2\end{bmatrix} + x_3\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{1.7}$$

Solution. We can rewrite this as

$$\begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_2 + x_3 \\ x_1 + 2x_2 \end{bmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Hence, we are back to the usual problem of solving a system of equations, which we are well equipped to handle. We row reduce the associated augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[R_1+R_3]{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Therefore, there is a unique choice of x_1, x_2 and x_3 solving (1.7), given by

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3\\-1\\1\end{bmatrix}.$$

1.4.5. *Matrix multiplication.* We can now use Definition 1.58 to define what it means to multiply two matrices of appropriate sizes.

Definition 1.62. Let A be a $m \times n$ matrix and B be a $n \times \ell$ matrix. Then the *product* AB is defined to be the $m \times \ell$ matrix

$$AB = \begin{pmatrix} A\mathbf{B}_1 & A\mathbf{B}_2 & \cdots & A\mathbf{B}_\ell \end{pmatrix},$$

where $\mathbf{B}_k$ is the k-th column of B.

By (1.6), this is equivalent to defining the (i,j)-th entry of AB to be

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \begin{bmatrix} a_{i1} & \cdots a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq \ell$.

Example 1.63. Let A, B and C be the matrices

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1(1)+0(2)+2(0) & 1(0)+0(1)+2(4) \\ 2(1)+3(2)+1(0) & 2(0)+3(1)+1(4) \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 8 \\ 8 & 7 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 3 & 5 \\ 8 & 12 & 4 \end{pmatrix}, \\ (AB)C &= \begin{pmatrix} 1 & 8 \\ 8 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 8 & 15 \end{pmatrix}, \\ C(AB) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 8 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 8 & 7 \end{pmatrix}. \end{aligned}$$

Remark 1.64.

(1) The above example illustrates a very important fact: Matrix multiplication is *non-commutative*. This means that, even when AB and BA are defined and of the same size, it is generally the case that they are not equal:
$$AB \neq BA.$$

(2) Not all matrices can be multiplied! For instance, in the above example CB is undefined. For AB to be defined, the number of columns of A must match the number of rows of B.

1.5. **Algebraic properties of matrix operations.** [8]

We have already seen in the last subsection that, in contrast to ordinary multiplication of real numbers, matrix multiplication is a non-commutative operation. What other properties fail for matrix multiplication?

Another such property is the *cancellation law* for real numbers:

$$ab = ac \implies b = c \quad \text{for all} \quad a, b, c \in \mathbb{R} \;\text{ with }\; a \neq 0.$$

For instance, consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}.$$

Then, we have $B \neq C$ and yet

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = AC.$$

The first goal of this section is to show that, despite these observations, matrix addition and multiplication do have many of the algebraic properties that ordinary addition and multiplication have.

We begin by introducing some convenient notation.

Definition 1.65. Fix positive integers m and n. We then define $\mathrm{M}_{m,n}(\mathbb{R})$ to be the set of all $m \times n$ matrices:

$$\mathrm{M}_{m,n}(\mathbb{R}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \; : \; a_{ij} \in \mathbb{R} \right\}.$$

When $m = n$, we will simply write $\mathrm{M}_n(\mathbb{R})$ for $\mathrm{M}_{m,n}(\mathbb{R})$.

Note. End of Lecture 6 (01/17/2020).

Theorem 1.66. *Addition and scalar multiplication in $\mathrm{M}_{m,n}(\mathbb{R})$ have the following properties, for any $A, B, C \in \mathrm{M}_{m,n}(\mathbb{R})$ and $r, s \in \mathbb{R}$:*

Properties of addition

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. *There is a unique matrix $O_{m,n} \in \mathrm{M}_{m,n}(\mathbb{R})$ such that*
$$A + O_{m,n} = A \quad \textit{for all} \quad A \in \mathrm{M}_{m,n}(\mathbb{R}).$$
4. *For each $A \in \mathrm{M}_{m,n}(\mathbb{R})$, there is a unique matrix $P \in \mathrm{M}_{m,n}(\mathbb{R})$ such that*
$$A + P = O_{m,n}.$$

Properties of scalar multiplication

[8] This is §1.6 in [JRA].

(5) $r(sA) = (rs)A$.
(6) $r(A + B) = rA + rB$ *and* $(r + s)A = rA + sA$.
(7) $1A = A$.

Most of the above properties are not difficult to prove and follow readily from the definition of addition and scalar multiplication. We leave the bulk of the proof of the above theorem as an exercise to the reader:

Exercise. Use the definition of addition and scalar multiplication to establish Parts (1), (2) and (5) - (7) of the theorem.

The matrix $O_{m,n}$ from (3), called the *zero matrix*, is given explicitly by

$$O_{m,n} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

That is, $O_{m,n}$ is the $m \times n$ matrix with all zero entries. When m and n are clear from context, we simply write O for $O_{m,n}$. It is easy to see that O satisfies (3):

$$A + O = A \quad \text{for all } \ A \in \mathrm{M}_{m,n}(\mathbb{R}).$$

Why is it the unique matrix with this property? If O' was another such matrix, then

$$\begin{aligned} O' &= O' + O && \text{by } (3) \\ &= O + O' && \text{by } (1) \\ &= O && \text{by } (3) \end{aligned}$$

Similarly, the matrix P from (4), called the *additive inverse* of A, is just the matrix $-A = -1A$. To show it is unique, suppose that P' is another such matrix. Then

$$\begin{aligned} P' &= (A + P) + P' && \text{by } (4) \text{ and } (3) \\ &= A + (P + P') && \text{by } (2) \\ &= A + (P' + P) && \text{by } (1) \\ &= (A + P') + P && \text{by } (2) \\ &= P && \text{by } (4), (1) \text{ and } (3). \end{aligned}$$

Remark 1.67. The theorem tells us exactly that $\mathrm{M}_{m,n}(\mathbb{R})$ is a *vector space*. We will return to this later in the course.

We now turn towards establishing the analogue of the above theorem for matrix multiplication:

Theorem 1.68.

(1) *Matrix multiplication is associative:*

$$(AB)C = A(BC)$$

for all $A \in \mathrm{M}_{m,n}(\mathbb{R})$, $B \in \mathrm{M}_{n,\ell}(\mathbb{R})$ *and* $C \in \mathrm{M}_{\ell,p}(\mathbb{R})$.

(2) *Matrix multiplication is bilinear:*
  (a) $r(AB) = (rA)B = A(rB)$
  (b) $(A + D)B = AB + DB$
  (c) $A(B + E) = AB + AE$

  for all $A, D \in \mathrm{M}_{m,n}(\mathbb{R})$, $B, E \in \mathrm{M}_{n,\ell}(\mathbb{R})$ *and* $r \in \mathbb{R}$.

(3) *There is a unique matrix* $I_n \in \mathrm{M}_n(\mathbb{R})$, *such that*
$$I_n A = AI_n = A \quad \text{for all} \;\; A \in \mathrm{M}_n(\mathbb{R}).$$
Moreover, I_n *satisfies*
$$I_n B = B \quad \text{and} \quad CI_n = C \quad \text{for all} \;\; B \in \mathrm{M}_{n,m}(\mathbb{R}),\, C \in \mathrm{M}_{m,n}(\mathbb{R}).$$

Proof of (1). Let us see why (1) is true. We need to show the (i, j)-th entries of $A(BC)$ and $(AB)C$ are identical. By definition, we have
$$\begin{aligned}(A(BC))_{ij} = \sum_{k=1}^{n} a_{ik}(BC)_{kj} &= \sum_{k=1}^{n} a_{ik}\left(\sum_{q=1}^{\ell} b_{kq}c_{qj}\right) \\ &= \sum_{q=1}^{\ell}\left(\sum_{k=1}^{n} a_{ik}b_{kq}\right)c_{qj} = \sum_{q=1}^{\ell}(AB)_{iq}c_{qj} = ((AB)C)_{ij},\end{aligned}$$
as desired. □

The proof of (2) is very similar, and left as an exercise:

Exercise. Use the definitions of matrix multiplication, addition, and scalar multiplication to prove (2) of Theorem 1.68.

The matrix I_n from (3) is called the $n \times n$ *identity matrix*, and is given by
$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$
Equivalently, the (i, j)-th entry of I_n is
$$(I_n)_{ij} = \begin{cases} 1 & \text{if } \; i = j, \\ 0 & \text{if } \; i \neq j. \end{cases}$$
When n is clear from context, we will just write $I = I_n$.

The following exercise will then complete the proof of Theorem 1.68.

Exercise. Prove (3) as follows:

(1) Show that there is at most one matrix I_n satisfying (3).
(2) Show that the above specific choice of I_n satisfies (3).

Remark 1.69. Matrix multiplication is a *binary* operation; it takes as input only two matrices and outputs a third matrix. However, associativity (1) allows us to define the product of several (> 2) matrices at once without needing to specify which two matrices to multiply first. For instance, we can define

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}},$$

for any $A \in \mathrm{M}_n(\mathbb{R})$. This called the k-th *power* of A.

1.5.1. *Matrix transpose and symmetric matrices.* We now define an additional matrix operation, called matrix *transposition* or *transpose*, which takes a $m \times n$ matrix to a $n \times m$ matrix.

Definition 1.70. Let $A = (a_{ij}) \in \mathrm{M}_{m,n}(\mathbb{R})$. Then the *transpose* of A is the matrix $A^T \in \mathrm{M}_{n,m}(\mathbb{R})$ with (i,j)-th entry

$$(A^T)_{ij} = a_{ji}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In words, the transpose of A is the $n \times m$ matrix obtained from A by interchanging its rows and columns. Let's do a few examples to make this clear.

Example 1.71. Let A, B and $\boldsymbol{x}$ be given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Then, we have

$$A^T = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{pmatrix}, \quad \boldsymbol{x}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}.$$

Theorem 1.72. *Let $A, B \in \mathrm{M}_{m,n}(\mathbb{R})$, $C \in \mathrm{M}_{n,\ell}(\mathbb{R})$ and $r, s \in \mathbb{R}$. Then*

(1) *Transposition is a linear operation:*

$$(rA + sB)^T = r(A^T) + s(B^T).$$

(2) $(AC)^T = C^T A^T$.
(3) $(A^T)^T = A$.

Let us prove the more difficult assertion (Part (2)), leaving (1) and (3) as an exercise to the reader.

Proof of (2). By definition, the (i,j)-th entry of $(AC)^T$ is

$$(AC)_{ji} = \sum_{k=1}^{n} a_{jk} c_{ki} = \sum_{k=1}^{n} c_{ki} a_{jk} = \sum_{k=1}^{n} (C^T)_{ik} (A^T)_{kj} = (C^T A^T)_{ij}. \qquad \square$$

There is a special class of matrices which are *invariant* with respect to transposition. These are the so-called *symmetric matrices*:

Definition 1.73. A matrix A is said to be *symmetric* if $A = A^T$.

Example 1.74. The following matrices are symmetric:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 \\ 5 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Remark 1.75. For a matrix to be symmetric, it must be a *square matrix.* That is, it must belong to $\mathrm{M}_n(\mathbb{R})$. Such matrices are called square because they have the same number of rows and columns, and thus consist of a square array.

We can identify visually when a matrix is symmetric by identifying if it is equal to its reflection about its *main diagonal*:

Definition 1.76. The *main diagonal* of $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ consists of its entries $a_{11}, a_{22}, \dots, a_{nn}$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Proposition 1.77. *For any $A \in \mathrm{M}_n(\mathbb{R})$, the matrix $A^T A$ is symmetric.*

Proof. Using (2) and (3) of Theorem 1.72, we obtain

$$(A^T A)^T = A^T (A^T)^T = A^T A. \qquad \square$$

1.5.2. *Euclidean norm.* We conclude this section with a geometric application of matrix multiplication and matrix transposition.

We have already seen that matrix multiplication gives us a way to multiply a column vector by a row vector to obtain a real number (see Remark 1.59). This gives rise to the *dot product* of two vectors, as well as the *Euclidean distance* and *norm*:

Definition 1.78.

(1) The *dot product* $\boldsymbol{x} \cdot \boldsymbol{y}$ of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined to be the real number

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \quad \text{where} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

(2) The *Euclidean norm* $\|\boldsymbol{x}\|$ of $\boldsymbol{x} \in \mathbb{R}^n$ is the non-negative number

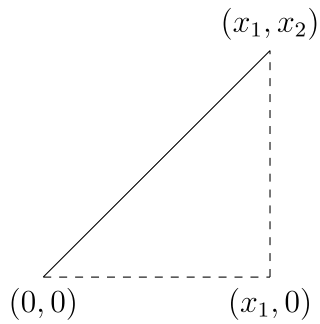
$$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\boldsymbol{x}^T \boldsymbol{x}} = \sqrt{x_1^2 + \cdots + x_n^2}$$

(3) The *Euclidean distance* between vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$ is the non-negative number

$$\|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$$

Each of the above has a *geometric interpretation*, which allows us to do things like define the length of a vector, the angle between any two vectors, and the distance between two vectors. Time permitting, we will come back to this in the coming weeks.

For the moment, you should think about what $\|\boldsymbol{x}\|$ and $\|\boldsymbol{x}-\boldsymbol{y}\|$ mean for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. For instance, plotting a generic point $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in the plane, we obtain a triangle



By the Pythagorean Theorem, the length of the hypotenuse of this triangle, which is precisely the distance from $\boldsymbol{x}$ to $\mathbf{0}$, is

$$\|x\| = \sqrt{x_1^2+x_2^2},$$

so the norm of a vector computes its length, i.e. *its distance from the origin.* This also tells us that $\|\boldsymbol{x}-\boldsymbol{y}\|$ computes the distance from $\boldsymbol{x}-\boldsymbol{y}$ to $\mathbf{0}$, which is precisely the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Note. End of Lecture 7 (01/22/2020).

1.6. **Linear independence and nonsingular matrices.** [9]

1.6.1. *Linear combinations.*

Definition 1.79. A vector $\boldsymbol{b} \in \mathbb{R}^m$ is a *linear combination* of $\boldsymbol{v}_1, \boldsymbol{v}_2 \ldots, \boldsymbol{v}_n \in \mathbb{R}^m$ if it can be expressed in the form

$$\boldsymbol{b} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n$$

for some $x_1, \ldots, x_n \in \mathbb{R}$.

Example 1.80.

[9] This is §1.7 in [JRA].

(1) A vector $\boldsymbol{v} \in \mathbb{R}^n$ is a linear combination of a single vector $\boldsymbol{w} \in \mathbb{R}^n$ if and only if it is a scalar multiple of it:

$$\boldsymbol{v} = k\boldsymbol{w} \quad \text{for some} \;\; k \in \mathbb{R}.$$

(2) Every vector $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and is thus a linear combination of the vectors $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(3) More generally, every vector $\boldsymbol{x} \in \mathbb{R}^n$ can be written as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and is thus a linear combination of the vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ defined by

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \; \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

These are called the standard *unit vectors* in $\mathbb{R}^n$.

This is not an unfamiliar concept to us; Definition 1.79 says exactly that $\boldsymbol{b}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ if the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1.8}$$

has a solution $\boldsymbol{x} \in \mathbb{R}^n$, where A is the $m \times n$ matrix $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$. We are thus back to the problem of determining when a $m \times n$ system of linear equations is consistent.

Example 1.81. Write $\boldsymbol{b}$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, where

$$\boldsymbol{b} = \begin{bmatrix} 27 \\ 17 \\ 22 \end{bmatrix}, \; \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \; \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; \boldsymbol{v}_3 = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}.$$

Solution. This is equivalent to solving the matrix equation

$$\begin{pmatrix} 2 & 1 & 8 \\ 1 & 1 & 5 \\ 2 & 0 & 7 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 17 \\ 22 \end{bmatrix},$$

which is nothing but a 3×3 system of linear equations. To solve it, we row reduce the associated augmented matrix, as usual:

$$\left(\begin{array}{ccc|c} 2 & 1 & 8 & 27 \\ 1 & 1 & 5 & 17 \\ 2 & 0 & 7 & 22 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Therefore, we may write $\boldsymbol{b}$ as the linear combination

$$\begin{bmatrix} 27 \\ 17 \\ 22 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}.$$

1.6.2. *Linear independence.* Let $\mathbf{0} = O_{m,1} \in \mathbb{R}^m$ denote the zero vector:

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^m.$$

Definition 1.82. A set of m-dimensional vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subset \mathbb{R}^m$ is *linearly independent* if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}$$

has only the trivial solution

$$0 = x_1 = x_2 = \cdots = x_n.$$

If $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is not linearly independent, it is said to be *linearly dependent.*

Example 1.83. The unit vectors $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n} \in \mathbb{R}^n$ form a linearly independent set. Indeed, we have

$$x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e_n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

which is equal to $\mathbf{0}$ if and only if each x_i is zero.

The problem of determining when $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} \subset \mathbb{R}^m$ is linearly independent is not a new problem for us either: It is equivalent to determining when the matrix equation

$$A\boldsymbol{x} = \mathbf{0} \quad \text{where } A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix},$$

has a *unique solution.* This amounts to solving a homogeneous $m \times n$ system of linear equations.

Example 1.84. Determine if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset \mathbb{R}^3$ is linearly independent, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}.$$

Solution. This amounts to solving the matrix equation

$$A\boldsymbol{x} = \begin{pmatrix} 2 & 1 & 8 \\ 1 & 1 & 5 \\ 2 & 0 & 7 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve this, we apply Gauss-Jordan elimination to $\left(A \mid \mathbf{0}\right)$, or just to A as the last column consisting of all zeroes will not change. This has already been done in Example 1.81: A has reduced echelon form

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $\left(I_3 \mid \mathbf{0}\right)$ represents a system with a unique solution (the trivial solution), the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

We now prove a theorem which gives two criteria for determining when a set of vectors is linearly dependent.

Theorem 1.85.

(1) *Suppose* $\boldsymbol{v}_1 \in \mathbb{R}^m$ *is a linear combination of vectors* $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n \in \mathbb{R}^m$*. Then* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ *is linearly dependent.*
(2) *Suppose that* $m < n$*. Then any set of* n *vectors* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ *in* $\mathbb{R}^m$ *is linearly dependent.*

Proof of (1)*.* By assumption, there are $x_2, \ldots, x_n \in \mathbb{R}$ such that

$$\boldsymbol{v}_1 = x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n.$$

Consequently, $(y_1, y_2, \ldots, y_n) = (-1, x_2, \ldots, x_n)$ is a nontrival solution of

$$y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 + \cdots + y_n\boldsymbol{v}_n = \mathbf{0},$$

so $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is linearly dependent.

Proof of (2)*.* [10] The vector equation

$$x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n = \mathbf{0}$$

is equivalent to a $m \times n$ homogeneous system of equations. Since $m < n$, Corollary 1.44 implies that this system has infinitely many solutions, and hence a nontrivial solution

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \neq \mathbf{0}.$$

Therefore, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly dependent. □

[10] This proof was not presented in class.

1.6.3. *Nonsingular matrices.* For the rest of §1.6, we narrow our focus to square matrices and sets of n vectors in $\mathbb{R}^n$.

Definition 1.86. $A \in \mathrm{M}_n(\mathbb{R})$ is called *singular* if the matrix equation

$$A\boldsymbol{x} = \mathbf{0}$$

has a nontrivial solution $\boldsymbol{x} \neq \mathbf{0} \in \mathbb{R}^n$. If A is not singular, it is called *nonsingular.*

The definition says that A is nonsingular exactly when the matrix equation

$$A\boldsymbol{x} = \mathbf{0}$$

has a unique solution (namely, the trivial solution $\boldsymbol{x} = \mathbf{0}$). We can rephrase this in terms of linear independence as follows:

Let $\mathbf{A}_j \in \mathbb{R}^n$ denote the j^{th} column of A. Then

$$A\boldsymbol{x} = \begin{pmatrix} \mathbf{A}_1 & \cdots & \mathbf{A}_n \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n.$$

Thus, A is nonsingular precisely when its set of columns $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ is a linearly independent set in $\mathbb{R}^n$:

Proposition 1.87. *The matrix* $A = \begin{pmatrix} \mathbf{A}_1 & \cdots & \mathbf{A}_n \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$ *is nonsingular if and only if* $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ *is linearly independent in* $\mathbb{R}^n$*.*

Remark 1.88. This means that "nonsingular" is just another way of saying a set of n vectors in $\mathbb{R}^n$, organized into a matrix A, are linearly independent. In the next section, we will learn that this is also equivalent to the matrix A being *invertible*.

Proposition 1.89. *If* $A \in \mathrm{M}_n(\mathbb{R})$ *is nonsingular, then it is row equivalent to the identity matrix* I_n*.*

Proof. [11] By Theorem 1.38, the equation $A\boldsymbol{x} = \mathbf{0}$ has a unique solution exactly when the augmented matrix $\left(A \mid \mathbf{0}\right)$ has reduced echelon form $\left(C \mid \mathbf{0}\right)$, where C has $r = n$ nonzero rows. This means that C is an $n \times n$ reduced echelon form matrix with a leading entry in each row and each column. There is only one such matrix, namely

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n. \qquad \square$$

Remark 1.90. Since the identity matrix I_n is in reduced echelon form, the proposition implies that I_n is the unique reduced echelon form matrix row equivalent to any nonsingular matrix $A \in \mathrm{M}_n(\mathbb{R})$.

[11] This proof was not presented in class.

Our last result of this subsection illustrates that nonsingular matrices always lead to consistent systems with unique solutions:

Theorem 1.91. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for every $\boldsymbol{b} \in \mathbb{R}^n$ if and only if A is nonsingular.*

Proof. [12] If $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for each $\boldsymbol{b} \in \mathbb{R}^n$, then it has a unique solution when $\boldsymbol{b} = \boldsymbol{0}$. Hence, A is nonsingular by definition.

Conversely, suppose that A is nonsingular and fix $\boldsymbol{b} \in \mathbb{R}^n$. We need to show that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution. By Proposition 1.89, the unique reduced echelon form matrix row equivalent to A is the identity matrix I_n. This means that the augmented matrix $\big(A \mid \boldsymbol{b}\big)$ has reduced echelon form $\big(I_n \mid \boldsymbol{c}\big)$, for some vector $\boldsymbol{c} \in \mathbb{R}^n$ depending on $\boldsymbol{b}$.

Since $\big(I_n \mid \boldsymbol{c}\big)$ does not have a row of the form $\big(0 \ \ \cdots \ \ 0 \mid 1\big)$ and has n nonzero rows, Theorem 1.38 tells us that it represents a consistent system with a unique solution. As this system is equivalent to $A\boldsymbol{x} = \boldsymbol{b}$, we are done. $\square$

Note. End of Lecture 8 (01/24/2020).

1.7. **Invertible matrices.** [13]

If a is a nonzero real number, then there exists a unique $b \in \mathbb{R}$ satisfying

$$ab = 1 = ba.$$

The number b is called the *multiplicative inverse* of a, and is nothing but the reciprocal $b = \frac{1}{a}$. For a square matrix $A \in \mathrm{M}_n(\mathbb{R})$, the analogue of the above equation is

$$AB = I = BA,$$

where $I = I_n$ is the $n \times n$ identity matrix. In this section, we study this equation in detail. We will see that in general there does not exist a $B \in \mathrm{M}_n(\mathbb{R})$ satisfying this equation. In fact, we will show that B exists exactly when A is nonsingular.

1.7.1. *The inverse of a matrix.*

Definition 1.92. Let $A \in \mathrm{M}_n(\mathbb{R})$. Then A is called *invertible* if there is $B \in \mathrm{M}_n(\mathbb{R})$ such that

$$AB = I = BA, \tag{1.9}$$

where $I = I_n$ is the $n \times n$ identity matrix. The matrix B is called an *inverse* of A.

The following lemma says that if such a matrix B exists, then it is unique.

Lemma 1.93. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then there is at most one matrix $B \in \mathrm{M}_n(\mathbb{R})$ satisfying (1.9).*

[12] This proof was not presented in class.

[13] This is §1.9 in [JRA].

Proof. Suppose that B and C both satisfy (1.9):
$$AB = BA = I = AC = CA.$$
We then have
$$(B - C)A = BA - CA = I - I = O.$$
Right multiplying this equation by B and using that $OB = O$, we obtain
$$(B - C)AB = O.$$
Since $AB = I$, the left hand side is $(B - C)I = B - C$, so the above says exactly that $B = C$. $\square$

This shows any matrix has at most one inverse, and hence it makes sense to talk about *the inverse* of an invertible matrix A, which we denote by A^{-1}.

Example 1.94.

(1) The identity matrix $I = I_n$ has inverse $I^{-1} = I$.
(2) The 3×3 diagonal matrix
$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$
is invertible exactly when a, b, c are all nonzero. In this case,
$$D^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}.$$

The general situation is much more complicated; the inverse of an arbitrary invertible matrix A is not obtained by just inverting each entry of A.

Our current goal is to determine when A^{-1} exists, while simultaneously uncovering an algorithm for computing it.

1.7.2. *The equation $AB = I$.* Let $A \in \mathrm{M}_n(\mathbb{R})$. Since $I = \begin{pmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{pmatrix}$, the equation $AB = I$ can be written equivalently as
$$AB = \begin{pmatrix} A\boldsymbol{b}_1 & \cdots & A\mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{pmatrix},$$
where $\boldsymbol{b}_j \in \mathbb{R}^n$ is the j^{th} column of B. This means that solving $AB = I$ for $B \in \mathrm{M}_n(\mathbb{R})$ is equivalent to solving the n equations
$$A\boldsymbol{b}_1 = \boldsymbol{e}_1, \quad \ldots, \quad A\boldsymbol{b}_n = \boldsymbol{e}_n, \tag{1.10}$$
for $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \in \mathbb{R}^n$. In particular, if these equations can't be solved, A is not invertible.

Remark 1.95. This analysis doesn't tell us that a solution B of $AB = I$ is necessarily the unique inverse of A, as we haven't shown it also satisfies $BA = I$. The next proposition says that this is indeed the case.

Proposition 1.96.

(1) *Suppose* $A, B \in \mathrm{M}_n(\mathbb{R})$ *satisfy* $AB = I$. *Then* A *and* B *are nonsingular.*
(2) *If* A *is nonsingular, then there is a unique matrix* B *such that* $AB = I$.
(3) *If* $A, B \in \mathrm{M}_n(\mathbb{R})$ *satisfy* $AB = I$, *then* A *is invertible with inverse* B*:*

$$AB = I = BA.$$

Proof of (1). *Step 1.* First, note that I is nonsingular since $I\boldsymbol{x} = \mathbf{0}$ implies $\boldsymbol{x} = \mathbf{0}$ (as $I\boldsymbol{x} = \boldsymbol{x}$).

Step 2. We now show B is nonsingular. Suppose that $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $B\boldsymbol{x} = \mathbf{0}$. We need to show that $\boldsymbol{x} = \mathbf{0}$. We have

$$\boldsymbol{x} = I\boldsymbol{x} = A(B\boldsymbol{x}) = A\mathbf{0} = \mathbf{0}.$$

This proves B is nonsingular.

Step 3. Let us now argue A is nonsingular. Let $\boldsymbol{x} \in \mathbb{R}^n$ satisfy $A\boldsymbol{x} = 0$. As before, we must show that $\boldsymbol{x} = \mathbf{0}$. Since B is nonsingular, Theorem 1.91 implies that there is a unique $\boldsymbol{y} \in \mathbb{R}^n$ satisfying $B\boldsymbol{y} = \boldsymbol{x}$. We then have

$$\boldsymbol{y} = I\boldsymbol{y} = A(B\boldsymbol{y}) = A\boldsymbol{x} = \mathbf{0}$$

and $\boldsymbol{x} = B\boldsymbol{y} = B\mathbf{0} = \mathbf{0}$. This shows A is nonsingular.

Proof of (2). Suppose that A is nonsingular. By Theorem 1.91, each of the n equations (1.10) has a unique solution. This means exactly that there is a unique matrix $B \in \mathrm{M}_n(\mathbb{R})$ such that $AB = I$.

Proof of (3). We are given $AB = I$, and need to show that $BA = I$. By Part (1), B is nonsingular. Part (2) then implies that there is a unique matrix $C \in \mathrm{M}_n(\mathbb{R})$ such that $BC = I$. Since

$$A = AI = A(BC) = (AB)C = IC = C,$$

we can conclude that $BA = I$, and thus that A is invertible with $A^{-1} = B$. $\square$

Combining the proposition with our previous analysis, we obtain the following theorem.

Theorem 1.97. *Let* $A \in \mathrm{M}_n(\mathbb{R})$. *Then the following statements are equivalent:*

(1) A *is invertible.*
(2) A *is nonsingular.*
(3) *There is* $B \in \mathrm{M}_n(\mathbb{R})$ *satisfying* $AB = I$.
(4) *For each* $1 \leq j \leq n$, *the equation* $A\boldsymbol{x} = \boldsymbol{e}_j$ *has a solution.*
(5) *For each* $1 \leq j \leq n$, *the equation* $A\boldsymbol{x} = \boldsymbol{e}_j$ *has a unique solution.*

1.7.3. *Computing the inverse.* The above theorem gives us a concrete method for both determining when A^{-1} exists and computing it. Namely, we proceed as follows:

Step 1. Solve the equation
$$A\boldsymbol{x} = \boldsymbol{e}_j \quad \text{for each } \ 1 \le j \le n.$$
This is done as usual, by row reducing $\left(A \mid \boldsymbol{e}_j\right)$ to reduced echelon form.

Step 2. If any of these equations are inconsistent, A is not invertible. Otherwise, A is invertible and has reduced echelon form I (because it is nonsingular), so the above step will take us to a matrix of the form $\left(I \mid \boldsymbol{b}_j\right)$.

Step 3. A^{-1} is then equal to the matrix
$$A^{-1} = \begin{pmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{pmatrix}.$$

Note. End of Lecture 9 (01/27/2020)

The following remark provides a particularly useful improvement upon our algorithm for computing A^{-1}.

Remark 1.98. We can solve all the equations $A\boldsymbol{x} = \boldsymbol{e}_j$ at the same time by forming the $n \times 2n$ matrix
$$\left(A \mid \boldsymbol{e}_1 \quad \cdots \quad \boldsymbol{e_n}\right) = \left(A \mid I\right)$$
and applying Gauss-Jordan elimination until we obtain a matrix of the form
$$\left(C \mid B\right)$$
with C the unique reduced echelon form matrix row equivalent to A. If $C = I$, A is invertible with $A^{-1} = B$. Otherwise, A is not invertible.

Example 1.99.

(1) Find A^{-1} if it exists, where
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution. We row reduce $\left(A \mid I_3\right)$:
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$$\xrightarrow[R_2 - R_2]{R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

This shows A is invertible with
$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(2) Find A^{-1} if it exists, where

$$A = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}.$$

Solution. We row reduce $\left(A \mid I_2\right)$:

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 6 & 7 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow[-\frac{1}{2}R_2]{\frac{1}{2}R_1} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$
$$\xrightarrow{R_1 - \frac{3}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} - \frac{9}{4} & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & -\frac{7}{4} & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right).$$

This shows A is invertible with

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -7 & 3 \\ 6 & -2 \end{pmatrix}.$$

(3) Find A^{-1} if it exists, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 2 & 4 & 6 \end{pmatrix}.$$

Solution. We row reduce $\left(A \mid I_3\right)$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 2 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & -2 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right).$$

At this point there is no need to continue. The last row represents the three inconsistent equations

$$0 = -2, \quad 0 = -1 \quad \text{and} \quad 0 = 1.$$

Therefore, A is *not invertible.*

1.7.4. *Solving systems using inverses.* Suppose we are given a $n \times n$ system of linear equations with augmented matrix $\left(A \mid \boldsymbol{b}\right)$. The system is then equivalent to the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Suppose we are given the additional information that A is invertible. Since A is nonsingular, we know the above equation has a unique solution. Left multiplying by A^{-1}, we find that it is given by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} \in \mathbb{R}^n.$$

In the case where A^{-1} is already known, this observation gives us a new technique for solving the underlying system.

Example 1.100. Consider the 2×2 system

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 6x_1 + 7x_2 &= 0 \end{aligned}$$

This is equivalent to the matrix equation

$$A\boldsymbol{x} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

We computed A^{-1} in (2) of Example 1.99. The above 2×2 system therefore has unique solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\boldsymbol{b} = \frac{1}{4} \begin{pmatrix} -7 & 3 \\ 6 & -2 \end{pmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{pmatrix} -7 & 3 \\ 6 & -2 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \end{bmatrix}.$$

1.7.5. *General formula for* 2×2 *inverses.* There is a general formula for the inverse of a matrix A in terms of the so-called *determinant* of A, which is an extremely useful polynomial in the entries of A which will play an important role later in this course. In this brief subsection, we present this formula for 2×2 matrices.

Let $A \in \mathrm{M}_2(\mathbb{R})$ be an arbitrary 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Definition 1.101.

(1) The *determinant* $\det(A)$ of A is the real number
$$\det(A) = ad - bc \in \mathbb{R}.$$

(2) The *adjoint matrix* $\mathrm{adj}(A) \in \mathrm{M}_2(\mathbb{R})$ of A is the matrix
$$\mathrm{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We then have the following proposition

Proposition 1.102. $A \in \mathrm{M}_2(\mathbb{R})$ *is invertible if and only if* $\det(A) \neq 0$, *in which case*

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof. [14] There are two things to show: *Step 1.* If $\det(A) = 0$ then A is not invertible, and *Step 2.* If $\det(A) \neq 0$ then the above formula gives the inverse of A (in particular, A is invertible).

[14] Only the main ideas behind the proof were sketched in class.

Step 1. Suppose that $\det(A) = ad - bc = 0$. Consider first the case where d and c are not both zero. Then $\boldsymbol{x} = \begin{bmatrix} d \\ -c \end{bmatrix}$ is nonzero, and

$$A\boldsymbol{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad - bc \\ cd - dc \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This means that A is singular, and hence cannot be invertible (by Theorem 1.97, for instance). If instead $c = d = 0$, then A has a row of zeroes:

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

and hence cannot possibly be row equivalent to the identity matrix. Therefore, by Proposition 1.89, A is singular, and therefore not invertible.

Step 2. Suppose that $\det(A) \neq 0$. We must verify that

$$A\left(\frac{1}{\det(A)}\operatorname{adj}(A)\right) = I = \left(\frac{1}{\det(A)}\operatorname{adj}(A)\right)A.$$

We will check the equivalent relation

$$A \cdot \operatorname{adj}(A) = \det(A)I = \operatorname{adj}(A) \cdot A.$$

We have

$$A \cdot \operatorname{adj}(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & -cb + da \end{pmatrix} = \det(A)I.$$

The verification of the relation $\operatorname{adj}(A) \cdot A = \det(A)I$ is very similar, and left as an exercise to the reader. □

1.7.6. *Properties of matrix inverse.* We end this section with a study of some of the algebraic properties satisfied by the inverse of a matrix and, additionally, provide a final characterization of invertible matrices.

Proposition 1.103. *Let $A, B \in \mathrm{M}_n(\mathbb{R})$ be invertible. Then*

(1) *A^{-1} is invertible with $(A^{-1})^{-1} = A$.*
(2) *AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$.*
(3) *A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$*

Part (1) is immediate. Let us prove Parts (2) and (3):

Proof of (2). We have

$$(AB)(B^{-1}A^{-1}) = A((BB^{-1})A^{-1}) = A(IA^{-1}) = AA^{-1} = I,$$
$$(B^{-1}A^{-1})(AB) = B^{-1}((A^{-1}A)B) = B^{-1}(IB) = B^{-1}B = I.$$

This proves that AB is invertible with inverse $B^{-1}A^{-1}$.

Proof of (3). Since $(AB)^T = B^T A^T$ (by Theorem 1.72) and $I^T = I$, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I,$$
$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

This shows that A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$. □

Our final result of this section summarizes the many different synonyms for "invertible" and "nonsingular" we have seen in this section. It is a variant of Theorem 1.97.

Theorem 1.104. *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the following are equivalent:*

(1) *A is invertible.*
(2) *A is nonsingular.*
(3) *$A\boldsymbol{x} = \mathbf{0}$ has a unique solution (namely $\boldsymbol{x} = \mathbf{0} \in \mathbb{R}^n$).*
(4) *For each $\boldsymbol{b} \in \mathbb{R}^n$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} \in \mathbb{R}^n$.*
(5) *The column vectors of A form a linearly independent set.*
(6) *A is row equivalent to I.*

Proof. It suffices to show each of (1) and (3) - (6) is equivalent to (2). Part (3) is just the definition of nonsingular, so it is equivalent to (2). The equivalence of (2) and (4) is the content of Theorem 1.91. The equivalence of (2) and (5) is the content of Proposition 1.87, while the equivalence of (1) and (2) is due to Theorem 1.97.

That (2) implies (4) is precisely the statement of Proposition 1.89. We are thus left to show that (4) implies (2). Suppose that A is row equivalent to I and consider the matrix equation

$$A\boldsymbol{x} = \mathbf{0}.$$

We must show that this equation has a unique solution, as this means A is nonsingular by definition. We repeat the argument given in the proof of Theorem 1.91: By assumption, the reduced echelon form of $\big(A \mid \mathbf{0}\big)$ is $\big(I \mid \mathbf{0}\big)$. As this augmented matrix represents the equation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

we can conlude that $A\boldsymbol{x} = \mathbf{0}$ has a unique solution (namely, the trivial solution). □

Note. End of Lecture 10 (01/29/2020)

Note. Lecture 11 is Midterm I review (not to be typed).

2. Vector spaces

We now begin the second part of our course, where we will study the theory of vector spaces, with emphasis on $\mathbb{R}^n$. We will develop fundamental notions, such as *dimension*, *bases* and *orthogonality*, which are important throughout mathematics, physics, and beyond.

2.1. Geometric properties of vectors in $\mathbb{R}^n$.

Before getting into the heart of the theory of vector spaces, we begin with a gentle review of the geometric properties of $\mathbb{R}^n$, with emphasis on $n = 2$ and $n = 3$. In particular, we will see how the *Euclidean norm*, *Euclidean distance*, and *dot product* can be interpreted as concrete geometric notions.

2.1.1. *Vectors in $\mathbb{R}^n$ and the Euclidean norm.* For us, an n-dimensional vector is a $n \times 1$ matrix

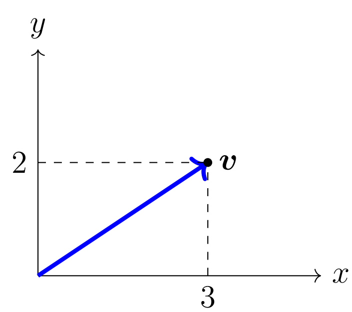
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

When $n = 2$ or $n = 3$, we may identify this with a point in 2-dimensional or 3-dimensional space using the standard xy or xyz rectangular coordinates.

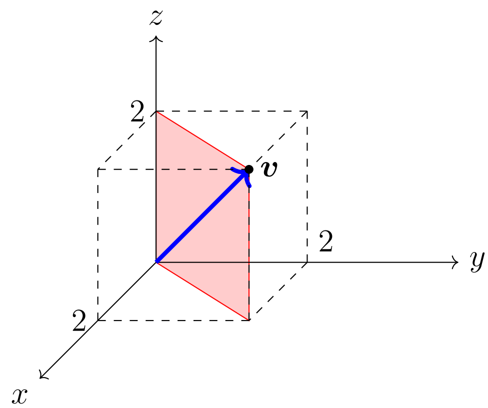
In addition, we draw an arrow from the *origin* (i.e. the vector $\mathbf{0}$) to this point, which allows us to visualize its *length* and *direction*.

Example 2.1.

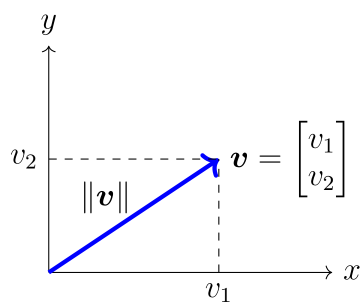
(1) Consider $\boldsymbol{v} = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$. On the xy-plane, it is represented as



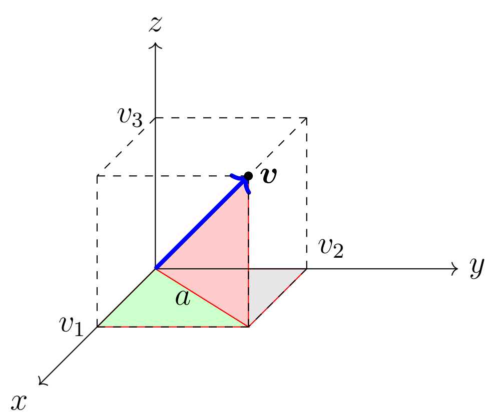
(2) The vector $\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}^T \in \mathbb{R}^3$ is represented as



Now let us turn to a generic vector $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T \in \mathbb{R}^2$. The length of the arrow representing $\boldsymbol{v}$ is precisely the straight-line distance from $\boldsymbol{v}$ to $\mathbf{0}$. This is exactly the *Euclidean norm* $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}$:



Indeed, this is a simple consequence of Pythagoras' Theorem, which has already been demonstrated at the end of Section 1.5. This statement is also true for a vector $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \in \mathbb{R}^3$, one must just apply Pythagoras' Theorem twice:



First, it is applied to the green triangle to compute a^2: $a^2 = v_1^2 + v_2^2$. Then, it is applied to the red triangle to compute the length of the blue arrow, say ℓ:

$$\ell = \sqrt{a^2 + v_3^2} = \sqrt{v_1^2 + v_2^2 + v_3^2} = \|\boldsymbol{v}\|.$$

More generally, as suggested in Section 1.5, the Euclidean norm $\|\boldsymbol{v}\|$ of $\boldsymbol{v} \in \mathbb{R}^n$ should be understood as computing the length of of $\boldsymbol{v}$, i.e. *the straight-line distance* from $\mathbf{0}$ to $\boldsymbol{v}$:

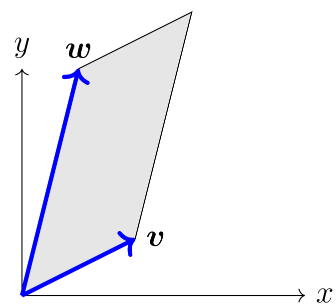
Proposition 2.2. *Let $\boldsymbol{v} \in \mathbb{R}^n$. Then the Euclidean norm $\|v\|$ is equal to the straight-line distance from $\mathbf{0}$ to $\boldsymbol{v}$.*

Remark 2.3. When $n > 3$, this reads as a proposition-definition, as we have not formally defined a notion of straight-line distance in $\mathbb{R}^n$. However, every vector $\boldsymbol{v} \in \mathbb{R}^n$ lives on infinitely many planes sitting in $\mathbb{R}^n$, each of which can be identified with a copy of $\mathbb{R}^2$. The straight-line distance from $\mathbf{0}$ to $\boldsymbol{v}$ is well-defined on any such plane, and the proposition states that it is equal to $\|v\|$.

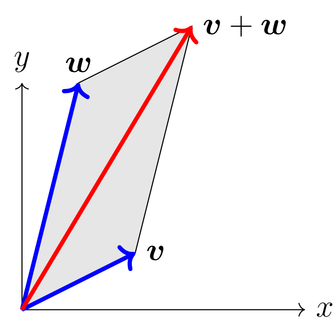
2.1.2. *Vectors operations and the Euclidean distance.* Vector addition in $\mathbb{R}^n$ can be understood geometrically with the use of *parallelograms*. Any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ define a parallelogram, namely $P_{\boldsymbol{v},\boldsymbol{w}}$ given by

$$P_{\boldsymbol{v},\boldsymbol{w}} = \{s\boldsymbol{v} + t\boldsymbol{w} \,:\, 0 \leq s, t \leq 1\}.$$

Let us once again turn to thinking about this concretely in $\mathbb{R}^2$. Take for example $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ is the entire shaded region below:



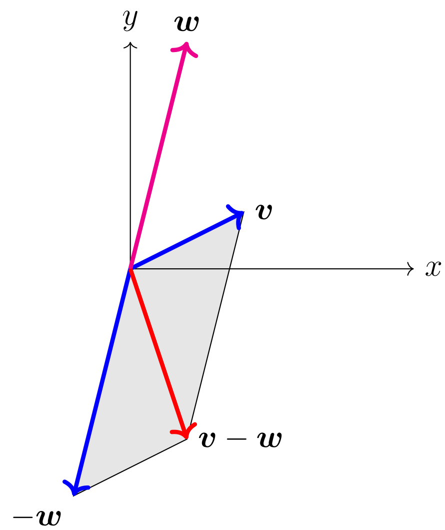
The vector $\boldsymbol{v} + \boldsymbol{w}$ is given by the fourth vertex of the parallelogram:



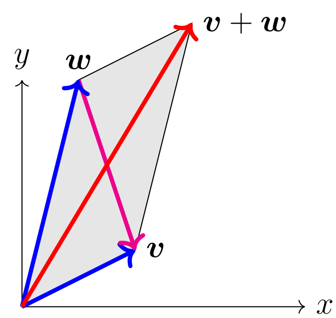
More generally, one obtains $\boldsymbol{v}+\boldsymbol{w}$ geometrically by

a) Forming the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ (the side lengths are $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$).
b) The sum $\boldsymbol{v}+\boldsymbol{w}$ is then the fourth vertex of $P_{\boldsymbol{v},\boldsymbol{w}}$, the other three being $\mathbf{0}, \boldsymbol{v}$ and $\boldsymbol{w}$.

As for subtraction, say $\boldsymbol{v}-\boldsymbol{w}$ for $\boldsymbol{v}, \boldsymbol{w}$ as above, it is just the addition of an additive inverse: $\boldsymbol{v}+(-\boldsymbol{w})$. Geometrically, we have the following picture:



Note that the above parallelogram is the same one as before, just translated down by $\boldsymbol{w}$. Hence, we summarize the geometric interpretation of $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ in the single parallelogram below:



Here the magenta arrow represents $\boldsymbol{v}-\boldsymbol{w}$, translated so as to form the diagonal of the parallelogram. Since it has length $\|\boldsymbol{v}-\boldsymbol{w}\|$, this illustrates that the *Euclidean distance* does in fact compute the straight-line distance from $\boldsymbol{w}$ to $\boldsymbol{v}$:

Proposition 2.4. *Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Then the Euclidean distance $\|\boldsymbol{v}-\boldsymbol{w}\|$ computes the straight-line distance from $\boldsymbol{w}$ to $\boldsymbol{v}$.*

Remark 2.5. For $n \geq 3$, the above discussion should be understood as follows. Assuming $\boldsymbol{v}$ is not of the form $\boldsymbol{v} = k\boldsymbol{w}$, the vectors $\boldsymbol{v}, \boldsymbol{w}$ define a unique plane in $\mathbb{R}^n$:

$$\{s\boldsymbol{v} + t\boldsymbol{w} : s, t \in \mathbb{R}\},$$

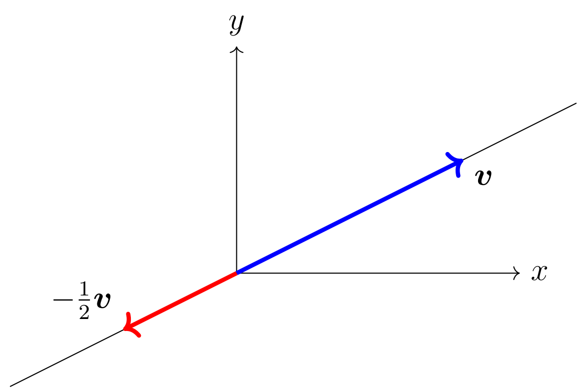
which can be identified with $\mathbb{R}^2$. The parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ resides on this plane, so the entire above analysis is still valid. If instead $\boldsymbol{v} = k\boldsymbol{w}$, then the situation is even simpler: $\boldsymbol{v}$ and $\boldsymbol{w}$ define a line in $\mathbb{R}^n$ (i.e. a copy of $\mathbb{R}$), and $P_{\boldsymbol{v},\boldsymbol{w}}$ is just a line segment on this copy of $\mathbb{R}$.

Before moving on, let us briefly comment on scalar multiplication. Let $\boldsymbol{v} \in \mathbb{R}^n$ be nonzero. Then $\boldsymbol{v}$ defines a unique line in $\mathbb{R}^n$, namely

$$L_{\boldsymbol{v}} = \{s\boldsymbol{v} : s \in \mathbb{R}\},$$

which can be identified with the real number line $\mathbb{R}$. For any $k \in \mathbb{R}$, the vector $k\boldsymbol{v} \in \mathbb{R}^n$ is then obtained geometrically as follows:

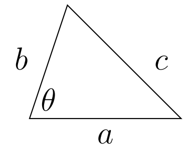
a) If $k \geq 0$, then $k\boldsymbol{v}$ is the unique vector on $L_{\boldsymbol{v}}$ with arrow pointing in the same direction as the arrow for $\boldsymbol{v}$ and length $k\|\boldsymbol{v}\|$.
b) If $k < 0$, then $k\boldsymbol{v}$ is the unique vector on $L_{\boldsymbol{v}}$ with arrow pointing in the same direction as the arrow for $-\boldsymbol{v}$ and length $|k|\|\boldsymbol{v}\|$.



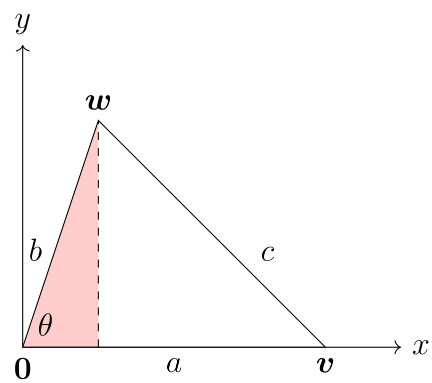
2.1.3. *The angle between two vectors in* $\mathbb{R}^n$. We now turn to the problem of computing the angle between any two nonzero vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. As before, this question makes sense for all n because any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ sit on a copy of $\mathbb{R}^2 \subset \mathbb{R}^n$. WIth this in mind, we will focus on $\mathbb{R}^2$.

We begin by recalling the *law of cosines*:

Lemma 2.6 (Law of cosines)**.** *Consider a triangle with side lengths* a, b, c *and angle* θ *as below. Then* $c^2 = a^2 + b^2 - 2ab\cos(\theta)$.



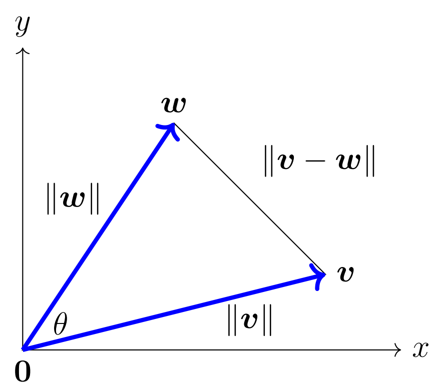
Proof. We place the triangle in $\mathbb{R}^2$ so the bottom vertices are at $\mathbf{0} = (0, 0)$ and $\boldsymbol{v} = (a, 0)$:



Using the shaded triangle, one finds that the top vertex $\boldsymbol{w}$ is then $(b\cos\theta, b\sin\theta)$. We then have

$$\begin{aligned} c^2 &= \|\boldsymbol{v} - \boldsymbol{w}\|^2 \\ &= (b\cos\theta - a)^2 + b^2\sin^2\theta \\ &= b^2(\cos^2\theta + \sin^2\theta) + a^2 - 2ab\cos\theta = b^2 + a^2 - 2ab\cos\theta. \end{aligned}$$ □

Now let us use this to compute the angle θ between any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$. These vectors form a triangle with side lengths $\|\boldsymbol{v}\|$, $\|\boldsymbol{w}\|$ and $\|\boldsymbol{v} - \boldsymbol{w}\|$:



By the law of cosines, we have

$$\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta.$$

Writing $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ and $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ and expanding, we obtain an identity for the *dot product*:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T\boldsymbol{w} = v_1w_1 + v_2w_2 = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta.$$

This formula more generally holds for two vectors in $\mathbb{R}^n$, and allows us to express the angle between them in terms of the dot product.

Proposition 2.7. *Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ be nonzero vectors. Then the angle $0 \leq \theta \leq \pi$ between $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by*

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|}.$$

Note. End of Lecture 12 (02/03/2020)

2.1.4. *Orthogonality and projections.* Now that we have a notion of the angle between two vectors, it makes sense to talk about when two vectors are *perpendicular* or *orthogonal*.

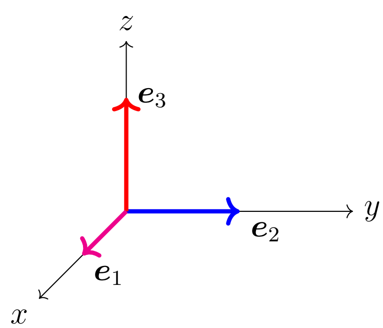
Definition 2.8. Two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ are said to be *orthogonal* if

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = 0.$$

By Proposition 2.7, two nonzero vectors are orthogonal exactly when the angle between them is $\pi/2$. Let's further reinforce this with a few examples.

Example 2.9.

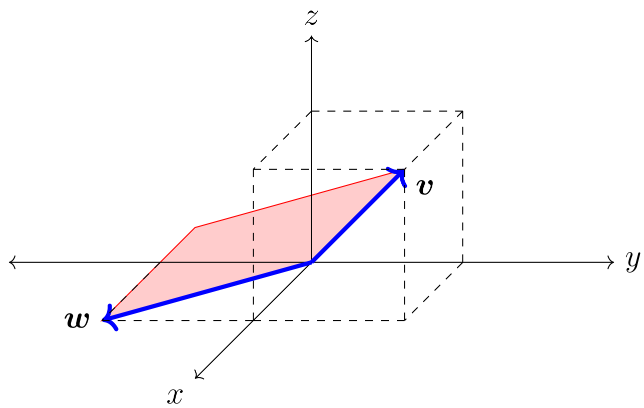
(1) The standard unit vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ are pairwise mutually orthogonal. That is, the vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ are orthogonal for any $i \neq j$. These vectors give rise to the standard rectangular coordinates. For instance, in $\mathbb{R}^3$ we have



(2) The vectors $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ are orthogonal:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 - 1 + 0 = 0.$$

Geometrically, we have:

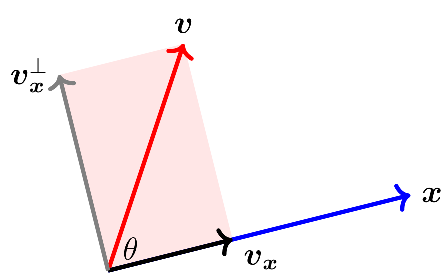


The red shaded region is the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$, which is a rectangle (just tilted), because the angle between the two vectors is $\pi/2$.

Now let us fix a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$. In practice, it often very useful to be able to decompose any given vector $\boldsymbol{v} \in \mathbb{R}^n$ as

$$\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{x}}^{\perp} + \boldsymbol{v}_{\boldsymbol{x}},$$

where $\boldsymbol{v}_{\boldsymbol{x}} \in \mathbb{R}^n$ is parallel to $\boldsymbol{x}$, and $\boldsymbol{v}_{\boldsymbol{x}}^{\perp} \in \mathbb{R}^n$ is orthogonal to $\boldsymbol{x}$. Obtaining such a decomposition is our goal for the rest of this subsection. The problem is summarized by the following diagram:



It is enough to find $\boldsymbol{v}_{\boldsymbol{x}}$, since we then have $\boldsymbol{v}_{\boldsymbol{x}}^{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\boldsymbol{x}}$. We know that

$$\boldsymbol{v}_{\boldsymbol{x}} = k\boldsymbol{x} \qquad \text{for some } \ k > 0.$$

We can solve for k using what we know about θ. Namely, the above picture shows that

$$k\|\boldsymbol{x}\| = \|\boldsymbol{v}_{\boldsymbol{x}}\| = \|\boldsymbol{v}\| \cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{x}\|},$$

where the third equality is due to Proposition 2.7. Hence,

$$k = \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\boldsymbol{x}\cdot\boldsymbol{x}} \qquad \text{and} \qquad \boldsymbol{v}_{\boldsymbol{x}} = \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{x}\|^2}\boldsymbol{x}.$$

Let's verify that $\boldsymbol{v}_{\boldsymbol{x}}^{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\boldsymbol{x}}$ and $\boldsymbol{x}$ are indeed orthogonal:

$$\boldsymbol{x}\cdot\boldsymbol{v}_{\boldsymbol{x}}^{\perp} = \boldsymbol{x}\cdot\left(\boldsymbol{v} - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{x}\|^2}\boldsymbol{x}\right) = \boldsymbol{x}\cdot\boldsymbol{v} - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{x}\|^2}\|\boldsymbol{x}\|^2 = 0.$$

We have thus proven the following result:

Proposition 2.10. *Let $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{R}^n$, with $\boldsymbol{x}$ nonzero. Define*

$$\boldsymbol{v}_{\boldsymbol{x}} = \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{\|\boldsymbol{x}\|^2}\boldsymbol{x} \quad \text{and} \quad \boldsymbol{v}_{\boldsymbol{x}}^{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\boldsymbol{x}}.$$

Then $\boldsymbol{v}_{\boldsymbol{x}}$ and $\boldsymbol{x}$ are parallel, $\boldsymbol{v}_{\boldsymbol{x}}^{\perp}$ and $\boldsymbol{x}$ are orthogonal, and

$$\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{x}}^{\perp} + \boldsymbol{v}_{\boldsymbol{x}}.$$

The vector $\boldsymbol{v}_{\boldsymbol{x}}$ played a special role in our analysis, and is often given its own name and suggestive notation, as illustrated in the next definition.

Definition 2.11. Let $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{R}^n$, with $\boldsymbol{x}$ nonzero. Then *the projection of $\boldsymbol{v}$ onto $\boldsymbol{x}$* is the vector

$$\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{v}) = \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{\|\boldsymbol{x}\|^2}\boldsymbol{x} = \boldsymbol{v}_{\boldsymbol{x}}.$$

Example 2.12. Consider the vector $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{R}^3$ given by

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}.$$

Then the projection of $\boldsymbol{v}$ onto $\boldsymbol{x}$ is

$$\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{v}) = \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{\|\boldsymbol{x}\|^2}\boldsymbol{x} = \frac{3(5) + 5(0) + 2(5)}{5^2 + 0^2 + 5^2}\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \frac{25}{50}\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ \frac{5}{2} \end{bmatrix}.$$

By Proposition 2.10, the vector

$$\boldsymbol{v}_{\boldsymbol{x}}^{\perp} = \boldsymbol{v} - \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{v}) = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{5}{2} \\ 0 \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 5 \\ -\frac{1}{2} \end{bmatrix}$$

is orthogonal to $\boldsymbol{x}$, and we have $\boldsymbol{v} = \boldsymbol{v}_{\boldsymbol{x}}^{\perp} + \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{v})$.

2.1.5. *The cross product.* In this subsection, we introduce an operation on $\mathbb{R}^3$ called the *cross product*. Our motivation behind this new operation is to solve the problem of constructing a vector $\boldsymbol{x} \in \mathbb{R}^3$ orthogonal to a pair of fixed vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.

To begin, let us rephrase the above problem in more familiar terms. Suppose we are given two fixed vectors in $\mathbb{R}^3$, say

$$\boldsymbol{v} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}.$$

We then want to construct $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in \mathbb{R}^3$ satisfying

$$\boldsymbol{v} \cdot \boldsymbol{x} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0.$$
$$\boldsymbol{w} \cdot \boldsymbol{x} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0.$$

This is a 2×3 homogeneous system, and thus has infinitely many solutions. It is not difficult to verify that one such solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{12}a_{23} - a_{13}a_{22} \\ a_{13}a_{21} - a_{11}a_{23} \\ a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} \det A_1 \\ -\det A_2 \\ \det A_3 \end{bmatrix}, \tag{2.1}$$

where $A_j \in \mathrm{M}_2(\mathbb{R})$ is the 2×2 matrix obtained from the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

by deleting the j-th column. This particular choice is the so-called cross product of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Definition 2.13. Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ be as above. Then the *cross product* $\boldsymbol{v} \times \boldsymbol{w} \in \mathbb{R}^3$ of $\boldsymbol{v}$ and $\boldsymbol{w}$ is the vector introduced in (2.1).

Remark 2.14. The formula for the cross product is most easily remembered as the symbolic determinant of a 3×3 matrix. Since we have not yet developed the theory of determinants beyond the 2×2 case, we postpone any such discussion.

Example 2.15.

(1) The cross product $\boldsymbol{e}_1 \times \boldsymbol{e}_2$, $\boldsymbol{e}_1 \times \boldsymbol{e}_3$ and $\boldsymbol{e}_2 \times \boldsymbol{e}_3$ are given by

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{e}_3, \quad \boldsymbol{e}_1 \times \boldsymbol{e}_3 = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -\boldsymbol{e}_2, \quad \boldsymbol{e}_2 \times \boldsymbol{e}_3 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{e}_1$$

Exercise. Verify that $\boldsymbol{e}_2 \times \boldsymbol{e}_1 = -\boldsymbol{e}_3$, $\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$ and $\boldsymbol{e}_3 \times \boldsymbol{e}_2 = -\boldsymbol{e}_1$. Note in particular that the cross product is *not* a commutative operation!

(2) Let $\boldsymbol{v}$ and $\boldsymbol{x}$ be as in Example 2.12:

$$\boldsymbol{v} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}.$$

Then $\boldsymbol{v} \times \boldsymbol{x} = \begin{bmatrix} \det A_1 & -\det A_2 & \det A_3 \end{bmatrix}^T$, where

$$A = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 0 & 5 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 5 & 2 \\ 0 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 2 \\ 5 & 5 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & 5 \\ 5 & 0 \end{pmatrix}.$$

We then have

$$\boldsymbol{v} \times \boldsymbol{x} = \begin{bmatrix} \det A_1 \\ -\det A_2 \\ \det A_3 \end{bmatrix} = \begin{bmatrix} 25 \\ -5 \\ -25 \end{bmatrix}.$$

By construction, the cross product has a geometric interpretation related to orthogonality. In the rest of this subsection, we will demonstrate that it is also directly related to the area of the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ defined by any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$. We begin with the following lemma:

Lemma 2.16. *Let θ be the angle between two fixed vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$. Then the length of $\boldsymbol{v} \times \boldsymbol{w}$ is given by*

$$\|\boldsymbol{v} \times \boldsymbol{w}\| = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta.$$

Proof. One first checks by a direct, but tedious, computation that

$$\|\boldsymbol{v} \times \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2.$$

Since $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta$, we then have

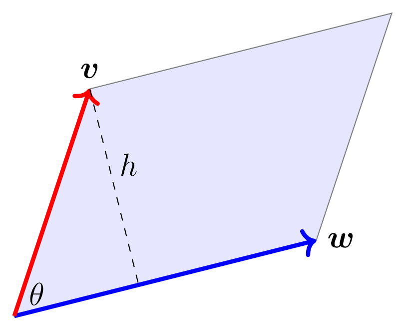
$$\begin{aligned}\|\boldsymbol{v} \times \boldsymbol{w}\|^2 &= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 \cos^2(\theta)\\ &= \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2(1 - \cos^2(\theta)) = \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 \sin^2(\theta),\end{aligned}$$

which gives the desired result. □

Now recall that the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ defined by two vectors in $\mathbb{R}^3$ (more generally, in $\mathbb{R}^n$) is defined algebraically by

$$P_{\boldsymbol{v},\boldsymbol{w}} = \{s\boldsymbol{v} + t\boldsymbol{w} : 0 \leq s, t \leq 1\}.$$

Since $\boldsymbol{v}$ and $\boldsymbol{w}$ live in a common plane in $\mathbb{R}^3$, we can view them on a 2-dimensional surface, as in the below diagram:



Here the shaded region is the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$. Since $h = \|\boldsymbol{v}\| \sin\theta$, we obtain

$$\text{Area}(P_{\boldsymbol{v},\boldsymbol{w}}) = \|\boldsymbol{w}\|h = \|\boldsymbol{w}\|\|\boldsymbol{v}\| \sin\theta = \|\boldsymbol{v} \times \boldsymbol{w}\|.$$

Note. End of Lecture 13 (02/07/2020)

This immediately gives the first part of the following theorem.

Theorem 2.17.

(1) *Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$. Then the area of the parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$ is given by*

$$\text{Area}(P_{\boldsymbol{v},\boldsymbol{w}}) = \|\boldsymbol{v} \times \boldsymbol{w}\|.$$

(2) *Let* $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ *and* $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ *be two vectors in* $\mathbb{R}^2$. *Then*

$$\text{Area}(P_{\boldsymbol{v},\boldsymbol{w}}) = \left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right|.$$

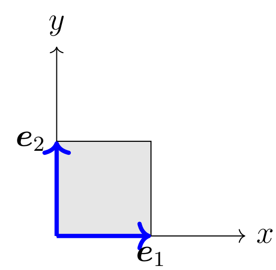
Proof. Let us prove Part (2). We may view $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ in $\mathbb{R}^3$ via the natural identification

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix}$$

By Part (1) and the definition of the cross product, we then have

$$\text{Area}(P_{\boldsymbol{v},\boldsymbol{w}}) = \|\boldsymbol{v} \times \boldsymbol{w}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \end{bmatrix} \right\| = \left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right|. \qquad \square$$

Example 2.18. Let $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^2$ be the standard unit vectors. Then $P_{\boldsymbol{e}_1,\boldsymbol{e}_2}$ is the unit square of area 1:

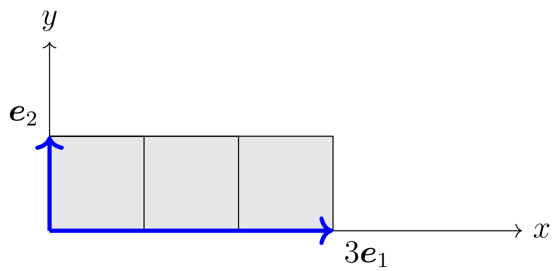


We can confirm this algebraically using the determinant:

$$\text{Area}(P_{\boldsymbol{e}_1,\boldsymbol{e}_2}) = \left|\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1.$$

What happens to the area of the parallelogram when we perform a row operation of the form $R_i \mapsto kR_i$ on the 2×2 identity matrix (appearing above)?

For example, suppose $k = 3$ and $i = 1$. This amounts to replacing $\boldsymbol{e}_1$ by $3\boldsymbol{e}_1$, which increases the length of one side of the square by 3, and hence increases the overall area by a factor of 3:



We can confirm this algebraically using our theorem:

$$\text{Area}(P_{3\boldsymbol{e}_1,\boldsymbol{e}_2}) = \left|\det\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\right| = 3.$$

More generally, since

$$\det\begin{pmatrix} kv_1 & kv_2 \\ w_1 & w_2 \end{pmatrix} = k(v_1w_2 - v_2w_1) = k\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix},$$

we have

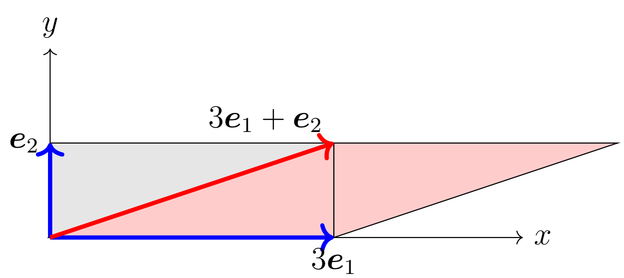
$$\text{Area}(P_{k\boldsymbol{v},\boldsymbol{w}}) = |k|\left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right| = \text{Area}(P_{\boldsymbol{v},\boldsymbol{w}}).$$

The same logic applies if the roles of $\boldsymbol{v}$ and $\boldsymbol{w}$ are interchanged.

Next, what happens if we perform a row operation of the type $R_2 \mapsto R_2 + kR_1$ on the matrix

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

appearing above? If $k = 1$, then this amounts to replacing $P_{3\boldsymbol{e}_1,\boldsymbol{e}_2}$ by $P_{3\boldsymbol{e}_1,3\boldsymbol{e}_1+\boldsymbol{e}_2}$:



As the above diagram illustrates, the area has not changed; the top triangle of the original rectangle has just been translated right. We can confirm this algebraically:

$$\text{Area}(P_{3\boldsymbol{e}_1,3\boldsymbol{e}_1+\boldsymbol{e}_2}) = \left|\det\begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}\right| = 3.$$

More generally, row operations of the type $R_i \mapsto R_i + kR_j$ applied to a matrix

$$\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

will not change the area of the underlying parallelogram $P_{\boldsymbol{v},\boldsymbol{w}}$. For $i = 2$ and $j = 1$, this is due to the fact that

$$\begin{aligned}\det\begin{pmatrix} v_1 & v_2 \\ w_1 + kv_1 & w_2 + kv_2 \end{pmatrix} &= v_1(w_2 + kv_2) - v_2(w_1 + kv_1) \\ &= v_1w_2 - v_2w_1 = \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}.\end{aligned}$$

The same logic applies if the roles of the two rows are reversed.

2.2. **Subspaces in $\mathbb{R}^n$.** We now shift our attention from the geometric study of vectors in $\mathbb{R}^n$, to the algebraic theory of vector spaces, with $\mathbb{R}^n$ as our working example. We begin by defining what it means to be a vector space *inside* $\mathbb{R}^n$, which is called a subspace of $\mathbb{R}^n$.

Definition 2.19. A subset $V \subset \mathbb{R}^n$ is called a *subspace* of $\mathbb{R}^n$ if it satisfies the following properties:

Closure properties. For each $\boldsymbol{v}, \boldsymbol{w} \in V$ and $r \in \mathbb{R}$, we have

(1) $\boldsymbol{v} + \boldsymbol{w} \in V$.
(2) $r\boldsymbol{v} \in V$.

Properties of addition. For each $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in V$, we have

(3) $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$.
(4) $\boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{x}) = (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{x}$.
(5) V contains the zero vector $\mathbf{0} \in \mathbb{R}^n$, which satisfies $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$.
(6) The vector $-\boldsymbol{v} \in \mathbb{R}^n$ belongs to V, and satisfies $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$.

Properties of scalar multiplication For each $\boldsymbol{v}, \boldsymbol{w} \in V$ and $r, s \in \mathbb{R}$, we have

(7) $r(s\boldsymbol{v}) = (rs)\boldsymbol{v}$.
(8) $r(\boldsymbol{v} + \boldsymbol{w}) = r\boldsymbol{v} + r\boldsymbol{w}$ and $(r + s)\boldsymbol{v} = r\boldsymbol{v} + s\boldsymbol{v}$.
(9) $1\boldsymbol{v} = \boldsymbol{v}$.

Many of these properties are redundant, as they are inherited automatically from $\mathbb{R}^n$. We will see this below, but first let us give a few examples.

Example 2.20.

(1) The set $V = \{\mathbf{0}\} \subset \mathbb{R}^n$, consisting of just the zero vector, is a subspace. It is called the *trivial subspace* of $\mathbb{R}^n$.
(2) The whole space $V = \mathbb{R}^n \subset \mathbb{R}^n$ is itself a subspace. This statement is a special case of Theorem 1.66, stated earlier in the course.

(3) Each space $\mathbb{R}^n$ can naturally be viewed as a subspace of $\mathbb{R}^{n+1}$ by identifying $\mathbb{R}^n$ with $V \subset \mathbb{R}^{n+1}$ defined by

$$V = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{bmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\} \subset \mathbb{R}^{n+1}.$$

When $n = 2$, this reduces to the familiar way we geometrically identify $\mathbb{R}^2$ inside $\mathbb{R}^3$ as the xy plane.

The next theorem drastically simplifies the task of identifying a subspace of $\mathbb{R}^n$.

Theorem 2.21. *A subset $V \subset \mathbb{R}^n$ is a subspace if and only if it is nonempty and satisfies the following two conditions:*

(1) $\boldsymbol{v} + \boldsymbol{w} \in V$ *for each* $\boldsymbol{v}, \boldsymbol{w} \in V$.
(2) $k\boldsymbol{v} \in \boldsymbol{V}$ *for each* $k \in \mathbb{R}$ *and* $v \in V$.

Proof. If $V \subset \mathbb{R}^n$ is a subspace then $\mathbf{0} \in \mathbb{R}^n$, so V is nonempty. Moreover, the properties (1) and (2) of Definition 2.19 imply that the conditions of the theorem hold.

To prove the converse, suppose that $V \subset \mathbb{R}^n$ satisfies the conditions of the theorem. We must argue that V satisfies all the conditions of Definition 2.19.

The properties (3), (4), (7) – (9) hold automatically, since they hold for any vectors in $\mathbb{R}^n$ (by Theorem 1.66). Moreover, we are given that the closure properties (1) and (2) are satisfied. We are left to show that

a) $\mathbf{0} \in V$.
b) $-\boldsymbol{v} \in V$ for each $v \in V$.

Since V is nonempty, it contains a vector $v \in V$. By the condition (2) with $k = 0$, we have $\mathbf{0} = 0\boldsymbol{v} = \mathbf{0} \in V$, which proves a). Now, for any $v \in V$, (2) with $k = -1$ gives $-\boldsymbol{v} = (-1)\boldsymbol{v} \in V$, proving b). □

Note. End of Lecture 14 (02/10/2020)

Let's see how Theorem 2.21 is applied in practice.

Example 2.22.

(1) Determine if $V \subset \mathbb{R}^3$ is a subspace, where

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ 3v_1 + v_2 \end{bmatrix} : v_1, v_2 \in \mathbb{R} \right\}.$$

Solution. Step 1. V can be described equivalently as

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_3 = 3v_1 + v_2,\ v_1, v_2 \in \mathbb{R} \right\}.$$

So an element $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T \in \mathbb{R}^3$ belongs to V exactly when

$$v_3 = 3v_1 + v_2.$$

Using this precise description, we now check V satisfies the conditions of Theorem 2.21.

Step 2. Since $0 = 3(0) + 0$, the zero vector $\mathbf{0} \in \mathbb{R}^3$ belongs to V. Therefore V is nonempty.

Step 3. Suppose $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$ and $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T$ belong to V. Then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

and we have

$$v_3 + w_3 = (3v_1 + v_2) + (3w_1 + w_2) = 3(v_1 + w_1) + (v_2 + w_2),$$

and hence $\boldsymbol{v} + \boldsymbol{w} \in V$.

Step 4. Now let $k \in \mathbb{R}$ and $\boldsymbol{v} \in V$, as above. We then have

$$k\boldsymbol{v} = \begin{bmatrix} kv_1 \\ kv_2 \\ kv_3 \end{bmatrix},$$

and hence $kv_3 = k(3v_1 + v_2) = 3(kv_1) + (kv_2)$, which shows $k\boldsymbol{v} \in V$. We may thus conclude that V is a subspace of $\mathbb{R}^3$.

(2) Determine if $V \subset \mathbb{R}^4$ is a subspace, where

$$V = \left\{ \begin{bmatrix} x \\ x + 2y \\ y \\ x - y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Solution. Step 1. Note that

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : v_2 = v_1 + 2v_3,\ v_4 = v_1 - v_3,\ v_1, v_3 \in \mathbb{R} \right\}$$

This means a vector $\boldsymbol{v} \in \mathbb{R}^4$ belongs to V if and only if its components satisfy

$$v_2 = v_1 + 2v_3 \quad \text{and} \quad v_4 = v_1 - v_3.$$

As before, we use this precise description to check the conditions of Theorem 2.21 are satisfied.

Step 2. V is nonempty since $\mathbf{0} \in V$ (as $0 = 0 + 20$ and $0 = 0 - 0$).

Step 3. If $\boldsymbol{v}, \boldsymbol{w} \in V$, then the components of $\boldsymbol{v} + \boldsymbol{w}$ satisfy

$$v_2 + w_2 = (v_1 + 2v_3) + (w_1 + 2w_3) = (v_1 + w_1) + 2(v_3 + w_3),$$
$$v_4 + w_4 = (v_1 - v_3) + (w_1 - w_3) = (v_1 + w_1) - (v_3 + w_3),$$

which shows that $\boldsymbol{v} + \boldsymbol{w} \in V$.

Step 4. If $\boldsymbol{v} \in V$ and $k \in \mathbb{R}$, then the components of $k\boldsymbol{v}$ satisfy

$$kv_2 = k(v_1 + 2v_3) = kv_1 + 2(kv_3) \quad \text{and} \quad kv_4 = k(v_1 - v_3) = kv_1 - kv_3,$$

which shows that $k\boldsymbol{v} \in V$. Since we have show the conditions of Theorem 2.21 are satisfied, V is a subspace of $\mathbb{R}^4$.

Remark 2.23. In general, the above steps can be followed to determine if a specified subset $V \subset \mathbb{R}^n$ is a subspace: After first clarifying exactly when a given vector $\boldsymbol{v} \in \mathbb{R}^n$ belongs to V, one proceeds to check that all the conditions of Theorem 2.21 are satisfied.

If a given subset $V \subset \mathbb{R}^n$ is *not* a subspace, then one of the properties of Theorem 2.21 will fail, and one can prove this by finding a concrete example, as demonstrated in the following example.

Example 2.24. Determine if $V \subset \mathbb{R}^2$ is a subspace, where

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} : v_1 v_2 = 0, \ v_1, v_2 \in \mathbb{R} \right\}.$$

Solution. It is easy to show that $\mathbf{0} \in V$ (so V is nonempty) and $k\boldsymbol{v} \in V$ for each $k \in \mathbb{R}$ and $v \in V$. However, it is not true that $\boldsymbol{v} + \boldsymbol{w} \in V$ for each $\boldsymbol{v}, \boldsymbol{w} \in V$. Indeed, taking $\boldsymbol{v} = \boldsymbol{e}_1 \in V$ and $\boldsymbol{w} = \boldsymbol{e}_2 \in V$, we obtain

$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $1(1) = 1 \neq 0$, this vector does not belong to V. Hence, we can conclude V is not a subspace of $\mathbb{R}^2$.

2.3. **Examples of subspaces in $\mathbb{R}^n$.**

We now turn towards introducing certain fundamental examples of subspaces in $\mathbb{R}^n$, the first of which is closely related to the concept of a *linear combination* introduced earlier.

2.3.1. *Span of a set S.*

Recall that $\boldsymbol{v} \in \mathbb{R}^n$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in \mathbb{R}^n$ if there are $x_1, \ldots, x_\ell \in \mathbb{R}$ such that

$$\boldsymbol{v} = x_1 \boldsymbol{v}_1 + \cdots + x_\ell \boldsymbol{v}_\ell.$$

Definition 2.25. Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset \mathbb{R}^n$. Then the *span* of S, denoted $\operatorname{span}(S)$, is the subset of $\mathbb{R}^n$ consisting of all linear combinations of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell$:

$$\operatorname{span}(S) = \{x_1\boldsymbol{v}_1 + \cdots + x_\ell \boldsymbol{v}_\ell : x_1, \dots, x_\ell \in \mathbb{R}\}.$$

Proposition 2.26. *Let $S \subset \mathbb{R}^n$ be as above. Then $\operatorname{span}(S)$ is a subspace of $\mathbb{R}^n$.*

Proof. First, $\operatorname{span}(S)$ is nonempty since $\mathbf{0} = 0\boldsymbol{v}_1 + \cdots + 0\boldsymbol{v}_\ell \in V$. Next, suppose that $\boldsymbol{v} = x_1\boldsymbol{v}_1 + \cdots + x_\ell\boldsymbol{v}_\ell$ and $\boldsymbol{w} = y_1\boldsymbol{v}_1 + \cdots + y_\ell\boldsymbol{v}_\ell$ are two vectors in $\operatorname{span}(S)$. Then

$$\boldsymbol{v} + \boldsymbol{w} = (x_1 + y_1)\boldsymbol{v}_1 + \cdots + (x_\ell + y_\ell)\boldsymbol{v}_\ell \in \operatorname{span}(S).$$

Finally, if $\boldsymbol{v} \in \operatorname{span}(S)$ is as above and $k \in \mathbb{R}$, then

$$k\boldsymbol{v} = (kx_1)\boldsymbol{v}_1 + \cdots + (kx_\ell)\boldsymbol{v}_\ell \in \operatorname{span}(S).$$

Therefore, by Theorem 2.21, $\operatorname{span}(S)$ is a subspace of $\mathbb{R}^n$. □

Example 2.27. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} \subset \mathbb{R}^3$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find sufficient and necessary conditions for a vector $\boldsymbol{b} \in \mathbb{R}^3$ to belong to $\operatorname{span}(S)$.

Solution. By definition, $\boldsymbol{b} \in \operatorname{span}(S)$ if and only if there are $x_1, x_2 \in \mathbb{R}$ such that

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{b}.$$

Hence we are back to the problem of determining when a system is consistent, namely the 3×2 system with matrix form

$$A\boldsymbol{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}.$$

We row reduce the corresponding augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 1 & b_2 \\ 0 & 1 & b_3 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 1 & b_3 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right).$$

Therefore, the system is consistent precisely when $b_3 + b_2 - 2b_1 = 0$. This allows us to conclude that $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T \in \mathbb{R}^3$ belongs to $\operatorname{span}(S)$ if and only if

$$b_3 = 2b_1 - b_2.$$

Equivalently,

$$\operatorname{Span}(S) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_3 = 2b_1 - b_2,\ b_1, b_2 \in \mathbb{R} \right\}.$$

2.3.2. *The null space of a matrix.* Each of the remaining examples of subspaces we introduce are associated to a given $m \times n$ matrix A. The first is the *null space* of A, which is closely related to the question of whether or not the columns of A are linearly independent.

Definition 2.28. Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then the *null space* of A, denoted $\mathcal{N}(A)$, is the subset of $\mathbb{R}^n$ defined by

$$\mathcal{N}(A) = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \mathbf{0}\}.$$

Proposition 2.29. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then the null space $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.*

Proof. Since $A\mathbf{0} = \mathbf{0}$, the zero vector $\mathbf{0} \in \mathbb{R}^n$ belongs to $\mathcal{N}(A)$, and so $\mathcal{N}(A)$ is nonempty. If $\boldsymbol{x}, \boldsymbol{v} \in \mathcal{N}(A)$, then

$$A(\boldsymbol{x} + \boldsymbol{v}) = A\boldsymbol{x} + A\boldsymbol{v} = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

and so $\boldsymbol{x} + \boldsymbol{v} \in \mathcal{N}(A)$. Similarly, if $\boldsymbol{x} \in \mathcal{N}(A)$ and $k \in \mathbb{R}$, then

$$A(k\boldsymbol{x}) = kA\boldsymbol{x} = k\mathbf{0} = \mathbf{0},$$

which shows $k\boldsymbol{x} \in \mathcal{N}(A)$. This completes the proof that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Note. End of Lecture 15 (02/12/2020).

Remark 2.30. If the columns of A form a linearly independent set, then the equation $A\boldsymbol{x} = \mathbf{0}$ only has the trivial solution $\boldsymbol{x} = \mathbf{0}$. In this case, we always have $\mathcal{N}(A) = \{\mathbf{0}\}$.

Describing the null space of a matrix amounts to finding all solutions $\boldsymbol{x} \in \mathbb{R}^n$ of the matrix equation $A\boldsymbol{x} = \mathbf{0}$, as illustrated in the following example.

Example 2.31. Determine the null space of the matrix A, expressed as the span of a set S, where

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

Solution. We must find all solutions $\boldsymbol{x} \in \mathbb{R}^4$ of $A\boldsymbol{x} = \mathbf{0}$. This is done as usual, by applying Gauss-Jordan elimination to the augmented matrix $\left(A \mid \mathbf{0}\right)$, or just A itself as the last column, consisting of all zeroes, will remain fixed. We have:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow[-R_2]{R_1 + 2R_2} \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

Therefore, the general solution to $A\boldsymbol{x} = \mathbf{0}$ is

$$x_1 = -4x_3 - 2x_4$$
$$x_2 = 2x_3 + x_4$$

with x_3 and x_4 free variables. Using the vector form of the general solution, we deduce that a vector $\boldsymbol{x}$ belongs to $\mathcal{N}(A)$ exactly when it takes the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

In particular, $\boldsymbol{x} \in \mathbb{R}^4$ belongs to $\mathcal{N}(A)$ if and only if it is a linear combination of the above two vectors. That is,

$$\mathcal{N}(A) = \operatorname{span}\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2.3.3. *The range and column space of a matrix.*

Definition 2.32. Let $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} \in \mathrm{M}_{m,n}(\mathbb{R})$. Then the *range* of A, denoted $\mathcal{R}(A)$, is the subset of $\mathbb{R}^m$ given by

$$\mathcal{R}(A) = \{\boldsymbol{b} \in \mathbb{R}^m : A\boldsymbol{x} = \boldsymbol{b} \text{ for some } \boldsymbol{x} \in \mathbb{R}^n\}.$$

The *column space* of A is the span of the set of columns of A:

$$\mathrm{Col}(A) = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \subset \mathbb{R}^m.$$

Since the column space is defined as the span of a set of vectors, it is a subspace of $\mathbb{R}^m$. Moreover, a vector $\boldsymbol{b} \in \mathbb{R}^m$ belongs to $\mathcal{R}(A)$ if and only if it can be expressed as

$$\boldsymbol{b} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n \quad \text{for some } x_1, \dots, x_n \in \mathbb{R}.$$

This means exactly that $\boldsymbol{b} \in \mathrm{Col}(A)$. Hence, the range and column space of a matrix describe the same thing:

Proposition 2.33. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then $\mathcal{R}(A) = \mathrm{Col}(A)$. In particular, $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$.*

Remark 2.34. By definition, $\boldsymbol{b} \in \mathbb{R}^m$ belongs to $\mathcal{R}(A)$ exactly when the $m \times n$ linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. A description of $\mathcal{R}(A)$ may therefore be obtained by reducing $\left(A \mid \boldsymbol{b}\right)$ to echelon form and determining when the resulting (equivalent) system is consistent.

Example 2.35. Let's take another look at Example 2.27, where we had

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

There, we computed $\mathcal{R}(A)$ explicitly, without at the time knowing it. To do this, we row reduced $\left(A \mid \boldsymbol{b}\right)$ to

$$\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_2 - 2b_1 \end{array}\right), \qquad \text{where} \qquad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

from which we concluded $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if the components of $\boldsymbol{b}$ satisfy

$$b_3 = 2b_1 - b_2.$$

This tells us the range of A (equivalently, $\mathrm{Col}(A)$) is given by

$$\mathcal{R}(A) = \mathrm{Col}(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_3 = 2b_1 - b_2,\ b_1, b_2 \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

2.3.4. *The row space of a matrix.* The last special type of subspace we introduce in this section is called the *row space* of a matrix.

Suppose that $A \in \mathrm{M}_{m,n}(\mathbb{R})$. We may then write

$$A = \begin{pmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_m^T \end{pmatrix}, \qquad \text{with} \qquad \boldsymbol{u}_j \in \mathbb{R}^n.$$

Hence, we can view the rows of A as vectors in $\mathbb{R}^n$ by identifying the j^{th} row with the vector $\boldsymbol{u}_j \in \mathbb{R}^n$.

Definition 2.36. Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$ be as above. Then the *row space* of A, denoted $\mathrm{Row}(A)$, is the span of its rows in $\mathbb{R}^n$:

$$\mathrm{Row}(A) = \mathrm{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\} \subset \mathbb{R}^n.$$

Similarly, to $\mathrm{Col}(A)$, $\mathrm{Row}(A)$ is a subspace by Proposition 2.26, as it is defined as the span of a set. In fact, these two types of subspaces are related by the transpose operation, as the first part of the next theorem demonstrates.

Theorem 2.37. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then*

(1) $\mathcal{R}(A) = \mathrm{Col}(A) = \mathrm{Row}(A^T)$.
(2) *If A is row equivalent to B, then* $\mathrm{Row}(A) = \mathrm{Row}(B)$.

Proof. The first part of the theorem follows from the fact that the columns of A are the rows of A^T. For the second part, we provide only a sketch of the proof, leaving the details as an exercise for the reader[15].

[15]This sketch was not presented in class.

It suffices to consider the case where A and B are related by a single row operation. Since each such operation involves only two rows, it is enough to consider the case where $m = 2$, so A has two rows, identified with the vectors $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ in $\mathbb{R}^n$. If the operation is of the type $R_i \leftrightarrow R_j$ or $R_i \mapsto kR_i$ for $k \neq 0$, then the statement is straightforward. If instead the operation is of the type $R_i \mapsto R_i + kR_j$ for some $k \in \mathbb{R}$, then

$$\mathrm{Row}(A) = \mathrm{span}\{\boldsymbol{u}_i, \boldsymbol{u}_j\} \quad \text{and} \quad \mathrm{Row}(B) = \mathrm{span}\{\boldsymbol{u}_i + k\boldsymbol{u}_j, \boldsymbol{u}_j\}.$$

If $\boldsymbol{v} \in \mathrm{Row}(A)$, then $\boldsymbol{v} = x_i\boldsymbol{u}_i + x_j\boldsymbol{u}_j$ for some $x_i, x_j \in \mathbb{R}$ and

$$\boldsymbol{v} = x_i\boldsymbol{u}_i + x_j\boldsymbol{u}_j = x_i(\boldsymbol{u}_i + k\boldsymbol{u}_j) + (x_j - kx_i)\boldsymbol{u}_j \in \mathrm{Row}(B).$$

Conversely, if $\boldsymbol{v} = x_i(\boldsymbol{u}_i + k\boldsymbol{u}_j) + x_j\boldsymbol{u}_j \in \mathrm{Row}(B)$, then

$$\boldsymbol{v} = x_i(\boldsymbol{u}_i + k\boldsymbol{u}_j) + x_j\boldsymbol{u}_j = x_i\boldsymbol{u}_i + (x_j + kx_i)\boldsymbol{u}_j \in \mathrm{Row}(A).$$

This shows that $\mathrm{Row}(A) = \mathrm{Row}(B)$. □

As an application, we will see in the next section that this theorem can be used to construct a minimal set S, called a *basis*, such that $\mathcal{R}(A) = \mathrm{span}(S)$.

2.4. **Bases for subspaces in $\mathbb{R}^n$.** Consider the set $S = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ of standard unit vectors. This set of vectors has some particularly nice properties, two of which are the following:

a) Every vector $\boldsymbol{v} \in \mathbb{R}^n$ is a linear combination of the vectors in S. That is,

$$\mathbb{R}^n = \mathrm{span}(S).$$

More precisely, if $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$, then

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n.$$

b) The vector $\boldsymbol{x}$ can be written in exactly one way as a linear combination of $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$. That is, the coefficients $x_1, \dots, x_n$ in the above decomposition uniquely determine the vector $\boldsymbol{x}$.

These properties characterize S as what is called a *basis* of $\mathbb{R}^n$. In this section, we will define what it means to be a basis of any subspace $V \subset \mathbb{R}^n$. We will then learn techniques for computing bases of the various special classes of subspaces introduced in the previous section.

2.4.1. *Spanning sets.* Our first task is to generalize the property *a)* above to the setting of any subspace $V \subset \mathbb{R}^n$. The proper generalization is given by the notion of a spanning set.

Definition 2.38. Let $V \subset \mathbb{R}^n$ be a subspace and $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset V$ be a set of vectors in V. We say S is a *spanning set* for V (or simply S *spans* V) if every vector $\boldsymbol{v} \in V$ is a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell$. That is, if

$$V = \mathrm{span}(S).$$

Example 2.39.

(1) $S = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ spans $\mathbb{R}^n$.
(2) Let $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} \in \mathrm{M}_{m,n}(\mathbb{R})$. Then $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a spanning set for $\mathcal{R}(A)$.
(3) In Example 2.31 we found a spanning set for the null space of $\mathcal{N}(A)$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

Namely, by finding the vector form of the general solution to $A\boldsymbol{x} = \boldsymbol{0}$, we showed that

$$\mathcal{N}(A) = \operatorname{span}\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(4) In Example 2.22. we showed that $V \subset \mathbb{R}^4$ is a subspace, where

$$V = \left\{ \begin{bmatrix} x \\ x+2y \\ y \\ x-y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Since

$$\begin{bmatrix} x \\ x+2y \\ y \\ x-y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix},$$

a spanning set for V is

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Note. End of Lecture 16 (02/14/2020).

2.4.2. *Bases.* In general, a spanning set S for a subspace $V \subset \mathbb{R}^n$ can contain redundant information. In particular, this happens when the set S is linearly dependent, as the next proposition illustrates.

Proposition 2.40. *Suppose $\boldsymbol{v} \in \mathbb{R}^n$ is a linear combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in \mathbb{R}^n$. Then*

$$\operatorname{span}\{\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}.$$

Proof. Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. By definition, we may rewrite the left-hand side as

$$\operatorname{span}\{\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} = \{k\boldsymbol{v} + \boldsymbol{w} : k \in \mathbb{R},\ \boldsymbol{w} \in \operatorname{span}(S)\}.$$

By assumption, $\boldsymbol{v} \in \operatorname{span}(S)$ and therefore any vector $k\boldsymbol{v}+\boldsymbol{w}$ in the above set also belongs to $\operatorname{span}(S)$, as it is a subspace. Conversely, by taking $k=0$ we see that every vector in $\operatorname{span}(S)$ can be written in the form $k\boldsymbol{v}+\boldsymbol{w}$. Therefore,

$$\operatorname{span}\{\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} = \{k\boldsymbol{v}+\boldsymbol{w} : k \in \mathbb{R},\ \boldsymbol{w} \in \operatorname{span}(S)\} = \operatorname{span}(S). \qquad \square$$

If we impose the additional assumption that a spanning set S is linearly independent, this type of redundancy never arises. This is exactly what we do to obtain the definition of a basis.

Definition 2.41. Let $V \subset \mathbb{R}^n$ be a subspace. Then a set $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} \subset V$ is called a *basis* for V if

(1) B is a spanning set for V: $V = \operatorname{span}(B)$.
(2) B is a linearly independent set.

Example 2.42.

(1) $B = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ is a basis for $V = \mathbb{R}^n$, called the *natural basis* of $\mathbb{R}^n$.
(2) Let $\boldsymbol{v} \in \mathbb{R}^n$ be a nonzero vector. Set

$$V = \operatorname{span}\{\boldsymbol{v}\} = \{k\boldsymbol{v} : k \in \mathbb{R}\}.$$

Then $B = \{\boldsymbol{v}\}$ is a basis for V. Indeed, B spans V by definition and is linearly independent since the equation

$$x\boldsymbol{v} = \boldsymbol{0}$$

only has the trivial solution $x = 0 \in \mathbb{R}$.
(3) $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Proof. Let A be the 3×3 matrix

$$A = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

To show that $\mathbb{R}^3 = \operatorname{span}(B)$, we need to show that the 3×3 system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for any $\boldsymbol{b} \in \mathbb{R}^3$, as this means exactly that

$$\boldsymbol{b} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 \in \operatorname{span}(B)$$

for some $x_1, x_2, x_3 \in \mathbb{R}$. We do this as usual, by row reducing $\big(A \mid \boldsymbol{b}\big)$. However, this matrix is already in the shape of echelon form:

$$\big(A \mid \boldsymbol{b}\big) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 2 & 2 & b_2 \\ 0 & 0 & 3 & b_3 \end{array}\right), \quad \text{where } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This represents a consistent system for any $\boldsymbol{b} \in \mathbb{R}^3$, and therefore $\mathbb{R}^3 = \operatorname{span}(B)$, as desired.

In fact, the form of $\left(A \mid \boldsymbol{b}\right)$ shows that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b} \in \mathbb{R}^3$. In particular, $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution, so B is linearly independent. $\square$

The linear independence criteria is exactly what we need to generalize the property b) satisfied by $S = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$, as demonstrated by the next proposition.

Proposition 2.43. *Let $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} \subset V$ be a basis for a subspace $V \subset \mathbb{R}^n$. Then every vector $\boldsymbol{v} \in V$ admits a unique decomposition*

$$\boldsymbol{v} = x_1\boldsymbol{v}_1 + \cdots + x_\ell \boldsymbol{v}_\ell \quad \text{with } x_1, \ldots, x_\ell \in \mathbb{R}. \tag{2.2}$$

Proof. Since $V = \operatorname{span}(B)$, there exists a decomposition for $\boldsymbol{v}$ as in (2.2). To see that it is unique, let

$$\boldsymbol{v} = y_1\boldsymbol{v}_1 + \cdots + y_\ell \boldsymbol{v}_\ell$$

be another such decomposition. We must show that $x_i = y_i$ for each $1 \le i \le \ell$. To do this, note that

$$\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = (x_1 - y_1)\boldsymbol{v}_1 + \cdots + (x_\ell - y_\ell)\boldsymbol{v}_\ell.$$

Since B is a linearly independent set, this is only possible if $x_i - y_i = 0$ for each $1 \le i \le \ell$, which gives the desired result. $\square$

Remark 2.44. The coefficients $x_1, \ldots, x_\ell$ in the decomposition (2.2) are called the *coordinates* of $\boldsymbol{v}$ with respect to B. They allow us to identify $\boldsymbol{v}$ with a vector in $\mathbb{R}^\ell$ with coordinates $x_1, \ldots, x_\ell$. We will come back to this a bit later in the course.

2.4.3. *Bases for $\mathcal{R}(A)$ and* $\operatorname{Row}(A)$. We now turn to developing strategies for computing bases of the special types of subspaces introduced in §2.3.

Theorem 2.45. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then*

(1) *If A has reduced echelon form C, then the set $S \subset \mathbb{R}^n$ of nonzero rows of C is a basis for* $\operatorname{Row}(A)$.
(2) *If A^T has reduced echelon form D, then the set $S \subset \mathbb{R}^m$ of nonzero rows of D is a basis for $\mathcal{R}(A)$.*

Proof. Part (2) follows from Part (1) and the equatlity $\mathcal{R}(A) = \operatorname{Row}(A^T)$, established in Theorem 2.37.

Consider now Part (1). Since A and C are row equivalent matrices, Theorem 2.37 implies that $\operatorname{Row}(A) = \operatorname{Row}(C)$. Since $\operatorname{Row}(C)$ is spanned by the set S of its nonzero rows, we need only check that S is linearly independent.

In fact, it is always the case that the set of nonzero rows of a matrix in echelon form is a linearly independent set. We won't prove this here, but you should think about why it is true. $\square$

Example 2.46. Find a basis for the range of A, where

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 5 & 3 & 4 \end{pmatrix}.$$

Solution. By Theorem 2.45, this can be accomplished by row reducing A^T. We have

$$A^T = \begin{pmatrix} 3 & 6 & 5 \\ 1 & 2 & 3 \\ 2 & 4 & 4 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 5 \\ 1 & 2 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$
$$\xrightarrow[-\frac{1}{4}R_2]{R_1 + 3R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the theorem, the range $\mathcal{R}(A)$ of A then has basis

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2.4.4. *Bases for $\mathcal{N}(A)$.* Let us now shift our attention to computing a basis for the null space of a matrix.

Theorem 2.47. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$, and suppose that the homogeneous $m \times n$ linear system $A\boldsymbol{x} = \boldsymbol{0}$ has general solution given in vector form by*

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1}\boldsymbol{v}_1 + x_{i_2}\boldsymbol{v}_2 + \cdots + x_{i_k}\boldsymbol{v}_k, \tag{2.3}$$

where $x_{i_1}, \ldots, x_{i_k} \in \{x_1, \ldots, x_n\}$ are distinct free variables and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ are nonzero vectors. Then

$$B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$$

is a basis for the null space $\mathcal{N}(A)$ of A.

Proof. The vector form of the general solution (2.3) tells us exactly that every solution $\boldsymbol{x} \in \mathbb{R}^n$ of $A\boldsymbol{x} = \boldsymbol{0}$ is a linear combination of the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. Hence, we have

$$\mathcal{N}(A) = \operatorname{span}(B).$$

It therefore suffices to prove that B is linearly independent. By definition, we must show that the equation

$$\boldsymbol{0} = y_1\boldsymbol{v}_1 + \cdots + y_k\boldsymbol{v}_k$$

has only the trivial solution, given by setting each y_j to zero. However, the above equation is just (2.3) with $\boldsymbol{x} = \boldsymbol{0}$ and $y_j = x_{i_j}$ for each $1 \leq j \leq k$. Since the x_{i_j} are components of $\boldsymbol{x}$ and we have set $\boldsymbol{x} = \boldsymbol{0}$, they must themselves be zero. This shows B is linearly independent. $\square$

Example 2.48. Find a basis for the null space of A, where

$$A = \begin{pmatrix} 3 & -6 & 0 & -3 \\ -2 & 4 & 1 & 2 \\ 1 & -2 & 1 & -1 \end{pmatrix}$$

Solution. The above theorem tells us that we can find such a basis by finding the vector form of the general solution to $A\boldsymbol{x} = \mathbf{0}$. We begin by row reducing A to reduced echelon form:

$$\begin{pmatrix} 3 & -6 & 0 & -3 \\ -2 & 4 & 1 & 2 \\ 1 & -2 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 1 & -1 \\ -2 & 4 & 1 & 2 \\ 3 & -6 & 0 & -3 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{3}R_2]{R_3 + R_2} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the general solution to $A\boldsymbol{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 &= 2x_2 + x_4 \\ x_2 &\ \text{ is free} \\ x_3 &= 0 \\ x_4 &\ \text{ is free.} \end{aligned}$$

From this, we find the vector form of the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2.4}$$

Using Theorem 2.47, we can conclude that $\mathcal{N}(A)$ has basis

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note. End of Lecture 17 (02/17/2020).

2.4.5. *From spanning sets to bases.* We conclude this section by determining a general strategy for computing a basis for $V = \operatorname{span}(S)$, where $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset \mathbb{R}^n$ is any set of vectors in $\mathbb{R}^n$. We motivate our approach with an example:

Example 2.49. Let $V = \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_4\}$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v}_3 = \begin{bmatrix} -3 \\ 6 \\ -9 \end{bmatrix}, \ \boldsymbol{v}_4 = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}.$$

Find a subset $B \subset \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_4\}$ such that B is a basis for V.

Solution. Let's first determine if $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_4\}$ is linearly independent. If it is, then it is a basis for V and we are done. We solve the equation $A\boldsymbol{x} = \mathbf{0}$, where

$$A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_4 \end{pmatrix},$$

by applying Gauss-Jordan elimination to A:

$$\begin{pmatrix} 1 & 0 & -3 & -1 \\ -1 & -1 & 6 & 2 \\ 2 & 1 & -9 & -3 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2+R_1} \begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & -3 & -1 \end{pmatrix} \xrightarrow[R_2\leftrightarrow R_3]{R_2+R_3} \begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $A\boldsymbol{x} = \mathbf{0}$ has general solution

$$\begin{aligned} x_1 &= 3x_3 + x_4 \\ x_2 &= 3x_3 + x_4 \\ x_3, x_4 &\ \text{ free.} \end{aligned}$$

As there are two free variables, $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions, so $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_4\}$ is *not* a basis for V.

Hence, we must delete vectors from $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_4\}$. By taking $(x_3, x_4) = (1, 0)$ in the above general solution and then $(x_3, x_4) = (0, 1)$, we get the two relations

$$3\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0} \quad \text{and} \quad \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_4 = \mathbf{0}.$$

We can rewrite this as

$$\boldsymbol{v}_3 = -3\boldsymbol{v}_1 - 3\boldsymbol{v}_2 \quad \text{and} \quad \boldsymbol{v}_4 = -\boldsymbol{v}_1 - \boldsymbol{v}_2,$$

so $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ are both linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2$. By Proposition 2.40, we then have

$$V = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

Therefore, $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ will be a basis for V provided it is linearly independent. This is indeed the case: every solution to $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \mathbf{0}$ is obtained from the above general solution by setting $x_3 = 0 = x_4$, and this automatically gives $x_1 = 0 = x_2$.

The procedure carried out in the above example can be generalized as follows.

Algorithm. Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset \mathbb{R}^n$. Then we can compute a basis $B \subset S$ of $V = \operatorname{span}(S)$ as follows:

Step 1. Find the general solution of $A\boldsymbol{x} = \mathbf{0}$, where $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell \end{pmatrix}$, by row reducing A to reduced echelon form.

Step 2. If there are no free variables, $B = S$ is itself a basis for V, and we are done. Otherwise, S is linearly dependent.

Step 3. If x_i is a free variable, delete the vector $\boldsymbol{v}_i$ from S. The remaining set B is then a basis for V.

2.5. **The dimension of a subspace in $\mathbb{R}^n$.** There is a geometric notion of dimension that you may be familiar with: a point has dimension zero, a line has dimension one, a plane has dimension two, and so on. In this section, we develop an algebraic notion of dimension for subspaces consistent with its geometric counterpart.

We begin with a few examples which illustrate a connection between the geometric dimension of a subspace and the size of a basis.

Example 2.50.

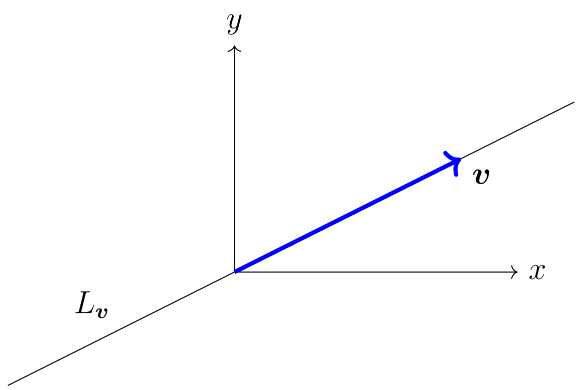
(1) $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ have dimensions 1, 2 and 3, respectively. They have bases

$$\{1\}, \quad \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\},$$

respectively. Note that these bases contain 1, 2 and 3 vectors, respectively.

(2) Let $\boldsymbol{v} = \begin{bmatrix}4\\2\end{bmatrix} \in \mathbb{R}^2$. Recall that the line defined by $\boldsymbol{v}$ is

$$L_{\boldsymbol{v}} = \left\{k\begin{bmatrix}4\\2\end{bmatrix} : k \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{bmatrix}4\\2\end{bmatrix}\right\}.$$

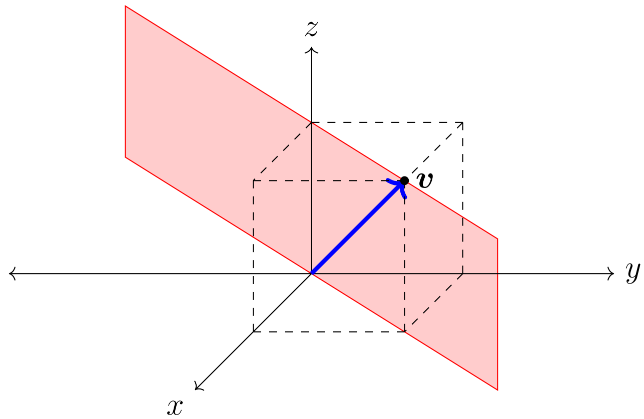


$L_{\boldsymbol{v}}$ has geometric dimension 1. Moreover, it has basis $\{\boldsymbol{v}\}$, consisting of one vector.

(3) Consider the subspace $V \subset \mathbb{R}^3$ defined by

$$V = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x = y\right\}.$$

Geometrically, V is a plane, as the following diagram illustrates.



The shaded red region is V, except that it should extend infinitely outwards in each direction. The vector $\boldsymbol{v}$ is one vector in V. The (geometric) dimension of V is 2, as it is plane.

On the other hand, it is not hard to show that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for V. Note that it contains two vectors.

The above examples suggest we define the dimension of V to be the number of vectors in a basis for V. However, we don't yet know this is well-defined, as we don't know that any two bases of V have the same size. Our first goal in this section is to show this is indeed the case.

Theorem 2.51. *Let $V \subset \mathbb{R}^n$ be a subspace with spanning set $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. Then any set of $m > \ell$ vectors in V is linearly dependent.*

Proof. Let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be a set of m vectors in V, with $m > \ell$. Introduce matrices

$$C = \begin{pmatrix} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \end{pmatrix} \in \mathrm{M}_{n,m}(\mathbb{R}), \quad D = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell \end{pmatrix} \in \mathrm{M}_{n,\ell}(\mathbb{R}).$$

We need to show the equation $C\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions, as this means $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ is linearly dependent. Since $V = \mathrm{span}(S)$, there are $a_{ij} \in \mathbb{R}$ such that

$$\boldsymbol{w}_j = a_{1j}\boldsymbol{v}_1 + \cdots + a_{\ell j}\boldsymbol{v}_\ell \quad \text{for each } \ 1 \leq j \leq m.$$

We can write each of these equations equivalently as

$$\boldsymbol{w}_j = D\mathbf{A}_j, \quad \text{where} \quad \mathbf{A}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{\ell j} \end{bmatrix}.$$

The above equation implies that

$$C = \begin{pmatrix} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_m \end{pmatrix} = \begin{pmatrix} D\mathbf{A}_1 & \cdots & D\mathbf{A}_m \end{pmatrix} = DA,$$

$$\text{where} \quad A = \begin{pmatrix} \mathbf{A}_1 & \cdots & \mathbf{A}_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{\ell 1} & \cdots & a_{\ell m} \end{pmatrix}.$$

As $\ell < m$, Corollary 1.44 implies that the homogeneous system $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions. Since any solution $\boldsymbol{x} \in \mathbb{R}^m$ of this equation is a solution to

$$\mathbf{0} = C\boldsymbol{x} = DA\boldsymbol{x},$$

we can conclude that $C\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions, as desired. □

Note. End of Lecture 18 (02/19/2020).

Corollary 2.52. *Let $V \subset \mathbb{R}^n$ be a subspace with basis $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. Then every basis of V contains ℓ vectors.*

Proof. Let $\mathcal{Q} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be an arbitrary basis for V. We need to show $m = \ell$. Since $\mathcal{Q}$ is linearly independent and B is a spanning set, Theorem 2.51 implies that $m \leq \ell$. Conversely, since B is linearly independent and $\mathcal{Q}$ is a spanning set, Theorem 2.51 implies that $\ell \leq m$. We can thus conclude that $m = \ell$. □

As a consequence of the above corollary, we can define the dimension of a subspace in the manner alluded to earlier.

Definition 2.53. Let $V \subset \mathbb{R}^n$ be a subspace. Then the *dimension* of V, denoted $\dim V$, is equal to the number of vectors in any basis for V.

Let's consider a few more quick examples before studying dimension in more detail.

Example 2.54.

(1) Since $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ is a basis, $\dim \mathbb{R}^n = n$, as expected.
(2) Let $V \subset \mathbb{R}^3$ be the subspace

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - 2y - z = 0 \right\}.$$

Geometrically, this is a plane, so we should expect $\dim V = 2$. To see this algebraically, note that $V = \mathcal{N}(A)$, where $A = \begin{bmatrix} 1 & -2 & -1 \end{bmatrix}$. The vector form of the general solution to $A\boldsymbol{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors on the right-hand side therefore form a basis of V, and we can conclude that $\dim V = 2$.

2.5.1. *Properties of dimension.* The dimension of a subspace plays an essentially identical role to that played by n for $\mathbb{R}^n$. The following theorem helps to illustrate why this is indeed the case.

Theorem 2.55. *Let $V \subset \mathbb{R}^n$ be a subspace with $\dim V = \ell$. Then:*

(1) *Any set S of $m > \ell$ vectors in V is linearly dependent.*
(2) *Any set S of $m < \ell$ vectors in V is not a spanning set.*
(3) *Any set S of ℓ linearly independent vectors in V is a basis.*
(4) *Any set S of ℓ vectors in V with $V = \operatorname{span}(S)$ is a basis.*

Proof. Since $\dim V = \ell$, V has a basis $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$ of ℓ vectors. Part (1) is a consequence of Theorem 2.51, as B is a spanning set.

Proof of (2). Suppose $S = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ *is* a spanning set for V with $m < \ell$ vectors. Since B is a set of $\ell > m$ vectors in V, Theorem 2.51 implies that B is linearly dependent. This is impossible, as B is a basis. Therefore, no such spanning set S can exist.

Proof of (3). Let $S = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\ell\} \subset V$ be a linearly independent set with ℓ vectors. We need to show that $V = \operatorname{span}(S)$, as this implies S is a basis for V. Let $\boldsymbol{v} \in V$. Then the set

$$\{\boldsymbol{v}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_\ell\}$$

is linearly dependent by Part (1), as it has more than ℓ vectors. Therefore, there are $k, k_1, \ldots, k_\ell \in \mathbb{R}$, not all zero, such that

$$k\boldsymbol{v} + k_1\boldsymbol{w}_1 + \cdots + k_\ell \boldsymbol{w}_\ell = \boldsymbol{0}.$$

Note that $k \neq 0$, as otherwise we could conclude that all $k_i's$ are zero by the linear independence of S. Rearanging, we obtain

$$\boldsymbol{v} = \left(-\frac{k_1}{k}\right)\boldsymbol{w}_1 + \cdots + \left(-\frac{k_\ell}{k}\right)\boldsymbol{w}_\ell \in \operatorname{span}(S).$$

Since $\boldsymbol{v} \in V$ was arbitrary, this shows that every vector in V is in $\operatorname{span}(S)$, and thus that $V = \operatorname{span}(S)$.

Proof of (4). Let $S = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\ell\} \subset V$ be a set of ℓ vectors with $V = \operatorname{span}(S)$. If S is linearly dependent, we can delete at least one vector $\boldsymbol{w}_i$ from S to form S' with $\ell - 1$ vectors and $V = \operatorname{span}(S')$. This contradicts Part (2) as $m = \ell - 1 < \ell$. Therefore, S must be linearly independent. □

The next result shows that every subspace has a basis, and thus a dimension. It also gives a useful equivalent characterization of dimension.

Corollary 2.56. *Let $V \subset \mathbb{R}^n$ be a subspace. Then*

(1) *V admits a basis and $0 \leq \dim V \leq n$.*

(2) $\dim V$ *is equal to the maximum number of vectors in any linearly independent set in* V.

Proof. Let ℓ denote the maximum number of vectors in any linearly independent set in V, and let $B = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} \subset V$ be a linearly independent set with ℓ vectors.

Since $V \subset \mathbb{R}^n$ and $\dim \mathbb{R}^n = n$, we must have $\ell \leq n$ as otherwise there is a set of more than n linearly independent vectors in $\mathbb{R}^n$, which is impossible by Part (1) of Theorem 2.55.

To complete the proof, it suffices to show that B spans V, as this implies it is a basis for V and that $\dim V = \ell$. Let $\boldsymbol{v} \in V$. Since the set

$$\{\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$$

contains $\ell + 1$ vectors, it must be linearly dependent, by maximality of ℓ. One now concludes exactly as in in the proof of (3) of Theorem 2.55 that $\boldsymbol{v} \in \operatorname{span}(B)$. Therefore, B is indeed a basis for V. □

Remark 2.57. In particular, the trivial subspace $V = \{\mathbf{0}\} \subset \mathbb{R}^n$ has dimension zero (not one!). This means that a basis for V must be a set consisting of no vectors. This set is called the *empty set*, and denoted $\emptyset$.

This observation is consistent with the fact that the geometric dimension of a single point is zero.

Note. End of Lecture 19 (02/24/2020).

2.5.2. *Rank and nullity of a matrix.* The dimension of the range and null space of a matrix show up in many different contexts and are deserving of their own names:

Definition 2.58. Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then

(1) The *rank* of A, denoted $\operatorname{rank}(A)$, is the dimension of the range of A:
$$\operatorname{rank}(A) = \dim \mathcal{R}(A).$$
(2) The *nullity* of A, denoted $\operatorname{nullity}(A)$, is the dimension of the null space of A:
$$\operatorname{nullity}(A) = \dim \mathcal{N}(A).$$

Example 2.59. Suppose that $A \in \mathrm{M}_{m,n}(\mathbb{R})$ is in reduced echelon form with r nonzero rows. Then

$$\operatorname{rank}(A) = \dim \operatorname{Row}(A) = \operatorname{rank}(A^T) = r,$$
$$\operatorname{nullity}(A) = n - r.$$

For instance, consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the columns of A contain the unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$, we must have $\mathcal{R}(A) = \mathrm{Col}(A) = \mathbb{R}^3$. Hence, $\mathrm{rank}(A) = 3$. Similarly,

$$\mathcal{R}(A^T) = \mathrm{Row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since these three vectors are linearly independent, $\mathrm{rank}(A^T) = \dim \mathrm{Row}(A) = 3$. Finally, since A has 3 nonzero rows, the general solution to the system $A\boldsymbol{x} = \boldsymbol{0}$ has $4 - 3 = 1$ free variables. By Theorem 2.47, there is thus a basis for $\mathcal{N}(A)$ with one vector. This shows that $\mathrm{nullity}(A) = n - r = 4 - 3 = 1$.

The following theorem provides a vast generalization of the previous example.

Theorem 2.60 (Rank-Nullity Theorem)**.** *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$. Then*

(1) *We have*

$$\mathrm{rank}(A) = \dim \mathrm{Row}(A) = \mathrm{rank}(A^T).$$

(2) *We have*

$$\mathrm{rank}(A) = r \quad \text{and} \quad \mathrm{nullity}(A) = n - r,$$

where r is the number of nonzero rows in the reduced echelon form of A. In particular,

$$n = \mathrm{rank}(A) + \mathrm{nullity}(A).$$

Proof. We will not give a proof of Part (1) here. Let's see why Part (2) is true, assuming Part (1) holds. By Theorem 2.47, $\mathrm{nullity}(A)$ is given by the number of free variables in a general solution to $A\boldsymbol{x} = \boldsymbol{0}$, which is equal to $n - r$ by Theorem 1.38.

To see that $\mathrm{rank}(A) = r$, we use that $\mathrm{rank}(A) = \dim \mathrm{Row}(A)$. By Theorem 2.45, $\mathrm{Row}(A)$ has a basis given by the r nonzero rows in the reduced echelon form of A. Therefore, we have $\mathrm{rank}(A) = r$, as claimed. Finally, we have

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = r + (n - r) = n. \qquad \square$$

Example 2.61. Find $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$, where

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 4 & 8 & 4 & 5 \end{pmatrix}$$

Solution. We row reduce A to echelon form:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 4 & 8 & 4 & 5 \end{pmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -3 \end{pmatrix} \xrightarrow[-\frac{1}{3}R_2]{R_3 - R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The number of nonzero rows will not change after this point. Hence, by Theorem 2.60, we have $\mathrm{rank}(A) = 2$ and $\mathrm{nullity}(A) = 2$.

We conclude this section by giving a characterization of nonsingularity and consistency in terms of rank.

Proposition 2.62.

(1) *Let* $A \in \mathrm{M}_n(\mathbb{R})$. *Then* A *is nonsingular if and only if*
$$\operatorname{rank}(A) = n.$$
(2) *Let* $A \in \mathrm{M}_{m,n}(\mathbb{R})$ *and* $\boldsymbol{b} \in \mathbb{R}^m$. *Then the system* $A\boldsymbol{x} = \boldsymbol{b}$ *is consistent if and only if*
$$\operatorname{rank}(A) = \operatorname{rank}(\begin{pmatrix} A \mid \boldsymbol{b} \end{pmatrix}).$$

Proof. Consider first Part (1). The matrix A is nonsingular if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has unique solution $\boldsymbol{x} = \mathbf{0}$. That is, if and only if $\mathcal{N}(A) = \{\mathbf{0}\}$. This happens exactly when $\operatorname{nullity}(A) = 0$. By Part (2) of Theorem 2.60, this is equivalent to $\operatorname{rank}(A) = n$.

Consider now Part (2). Let us write $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$. If $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, then $\boldsymbol{b} \in \mathcal{R}(A) = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$. Consequently,
$$\operatorname{rank}(A) = \dim \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} = \dim \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{b}\} = \operatorname{rank}(\begin{pmatrix} A \mid \boldsymbol{b} \end{pmatrix}).$$

Conversely, if we are given that $\operatorname{rank}(A) = \operatorname{rank}(\begin{pmatrix} A \mid \boldsymbol{b} \end{pmatrix})$, then we must have
$$\dim \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} = \dim \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n, \boldsymbol{b}\}.$$
This implies that $\boldsymbol{b} \in \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\} = \mathcal{R}(A)$. Hence, $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. □

2.6. **Orthogonal bases for subspaces in** $\mathbb{R}^n$**.** Recall from Definition 2.8 that two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ are *orthogonal* if
$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = 0.$$
In this section, we will learn how to construct a basis for any subspace $V \subset \mathbb{R}^n$ consisting of orthogonal vectors. In addition, we will study some of the wonderful properties such bases have.

2.6.1. *Orthogonal sets and bases.* We begin by defining what it means to be an orthogonal set of vectors.

Definition 2.63. $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\} \subset \mathbb{R}^n$ is called an *orthogonal set* if $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal for each $1 \leq i \neq j \leq \ell$.

Let's take a look at a few examples.

Example 2.64.

(1) $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} \subset \mathbb{R}^n$ is an orthogonal set.
(2) For any fixed angle θ, the following is an orthogonal set:
$$\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\} \subset \mathbb{R}^2.$$

(3) The following subset of $\mathbb{R}^3$ is easily seen to be an orthogonal set:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

One very strong property orthogonal sets possess is that they are linearly independent, provided they do not contain a zero vector.

Proposition 2.65. *Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset \mathbb{R}^n$ be an orthogonal set of nonzero vectors. Then S is linearly independent.*

Proof. Suppose that $k_1, \dots, k_n \in \mathbb{R}$ satisfy

$$k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell = \mathbf{0}$$

We must argue that $k_i = 0$ for each $1 \le i \le \ell$. To show this, left multiply the equation by $\boldsymbol{v}_i^T$, where $1 \le i \le \ell$ is fixed. The right-hand side becomes 0, whereas the left-hand side becomes

$$\begin{aligned} \boldsymbol{v}_i^T(k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell) &= k_1(\boldsymbol{v}_i^T\boldsymbol{v}_1) + \cdots + k_i(\boldsymbol{v}_i^T\boldsymbol{v}_i) + \cdots + k_\ell(\boldsymbol{v}_i^T\boldsymbol{v}_\ell) \\ &= k_i(\boldsymbol{v}_i^T\boldsymbol{v}_i) = k_i\|\boldsymbol{v}_i\|^2, \end{aligned}$$

where we have used that $\boldsymbol{v}_i^T\boldsymbol{v}_j = 0$ for $i \neq j$ in the second equality. We thus obtain

$$k_i\|\boldsymbol{v}_i\|^2 = 0 \quad \text{for each} \quad 1 \le i \le \ell.$$

Since $\|\boldsymbol{v}_i\| \neq 0$, this means that $k_i = 0$ for each i, as desired. □

Note. End of Lecture 20 (02/26/2020).

The above result demonstrates that orthogonal sets are intimately related to bases. Namely, if such a set S doesn't contain the zero vector, then it will be a basis for $V = \operatorname{span}(S)$. Continuing in this spirit, we now introduce the notion of an orthogonal basis.

Definition 2.66. Let $V \subset \mathbb{R}^n$ be a subspace and $B = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ be a basis for V. Then

(1) B is called an *orthogonal basis* if it is an orthogonal set.
(2) B is called an *orthonormal basis* if it is an orthogonal basis and

$$\|\boldsymbol{v}_i\| = 1 \quad \text{for each} \quad 1 \le i \le \ell.$$

Example 2.67. Reconsidering the previous examples, we find that

$$\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\} \subset \mathbb{R}^n \quad \text{and} \quad \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\} \subset \mathbb{R}^2$$

are orthonormal bases of $\mathbb{R}^n$ and $\mathbb{R}^2$, respectively. On the other hand,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\} \subset \mathbb{R}^3$$

is not an orthonormal basis of $\mathbb{R}^3$. It is, however, an orthogonal basis.

The second part of the below corollary shows that it is a very simple task to transform an orthogonal basis to an orthonormal basis.

Corollary 2.68. *Let $V \subset \mathbb{R}^n$ be a subspace of dimension ℓ. Then*

(1) *Any orthogonal set of ℓ nonzero vectors in V is a basis.*
(2) *If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ is an orthogonal basis, then*
$$\left\{\frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1, \dots, \frac{1}{\|\boldsymbol{v}_\ell\|}\boldsymbol{v}_\ell\right\}$$
is an orthonormal basis for V.

Proof. Part (1) follows from the previous proposition and the fact that any set of ℓ linearly independent vectors in V is a basis (by Theorem 2.55).

Part (2) is a consequence of the following relation, which holds for any nonzero $\boldsymbol{v} \in \mathbb{R}^n$:
$$\left\|\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}\right\|^2 = \left(\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}\right)^T \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{1}{\|\boldsymbol{v}\|^2}\boldsymbol{v}^T\boldsymbol{v} = 1. \qquad \square$$

Next, let $B = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ be an orthogonal basis for a subspace V of $\mathbb{R}^n$. Since B is a basis, any $\boldsymbol{v} \in V$ can be written uniquely as a linear combination
$$\boldsymbol{v} = k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell.$$
Due to the fact B is orthogonal, we can determine the coefficients k_i explicitly. Indeed, left-multiplying the above by $\boldsymbol{v}_i^T$, we obtain
$$\boldsymbol{v}_i^T\boldsymbol{v} = k_1\boldsymbol{v}_i^T\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_i^T\boldsymbol{v}_\ell = k_i\boldsymbol{v}_i^T\boldsymbol{v}_i = k_i\|\boldsymbol{v}_i\|^2.$$
Rearranging, this gives the identity
$$k_i = \frac{\boldsymbol{v}_i^T\boldsymbol{v}}{\|\boldsymbol{v}_i\|^2} \quad \text{for each } \ 1 \le i \le \ell.$$
In particular, $k\boldsymbol{v}_i$ is equal to the projection of $\boldsymbol{v}$ onto $\boldsymbol{v}_i$, defined in Definition 2.11:
$$k_i\boldsymbol{v}_i = \frac{\boldsymbol{v}_i^T\boldsymbol{v}}{\|\boldsymbol{v}_i\|^2}\boldsymbol{v}_i = \operatorname{proj}_{\boldsymbol{v}_i}(\boldsymbol{v}).$$
We have thus proven the following corollary:

Corollary 2.69. *Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ be an orthogonal basis for a subspace $V \subset \mathbb{R}^n$. Then any $\boldsymbol{v} \in V$ decomposes as*
$$\boldsymbol{v} = \operatorname{proj}_{\boldsymbol{v}_1}(\boldsymbol{v}) + \cdots + \operatorname{proj}_{\boldsymbol{v}_\ell}(\boldsymbol{v}), \quad \textit{where} \ \operatorname{proj}_{\boldsymbol{w}}(\boldsymbol{v}) = \frac{\boldsymbol{v}^T\boldsymbol{w}}{\|\boldsymbol{w}\|^2}\boldsymbol{w}.$$

Example 2.70. Consider the vectors $\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ defined by

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

One can show that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthogonal basis of $\mathbb{R}^3$. Moreover, we have

$$\begin{aligned} \mathrm{proj}_{\boldsymbol{v}_1}(\boldsymbol{v}) &= \frac{\boldsymbol{v}^T \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2} \boldsymbol{v}_1 = \frac{8}{6} \boldsymbol{v}_1 = \frac{4}{3} \boldsymbol{v}_1, \\ \mathrm{proj}_{\boldsymbol{v}_2}(\boldsymbol{v}) &= -\frac{2}{2} \boldsymbol{v}_2 = -\boldsymbol{v}_2, \\ \mathrm{proj}_{\boldsymbol{v}_3}(\boldsymbol{v}) &= \frac{2}{3} \boldsymbol{v}_3. \end{aligned}$$

Therefore, the above corollary yields

$$\boldsymbol{v} = \frac{4}{3} \boldsymbol{v}_1 - \boldsymbol{v}_2 + \frac{2}{3} \boldsymbol{v}_3.$$

2.6.2. *Constructing orthogonal bases.* Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ be linearly independent vectors, so that $\{\boldsymbol{v}, \boldsymbol{w}\}$ is a basis for $V = \mathrm{span}\{\boldsymbol{v}, \boldsymbol{w}\}$. Recall from Proposition 2.10 that the vector

$$\boldsymbol{v}_{\boldsymbol{w}}^{\perp} = \boldsymbol{v} - \mathrm{proj}_{\boldsymbol{w}}(\boldsymbol{v}) = \boldsymbol{v} - \frac{\boldsymbol{v}^T \boldsymbol{w}}{\|\boldsymbol{w}\|^2} \boldsymbol{w} \in V$$

is orthogonal to $\boldsymbol{w}$. It follows that $\{\boldsymbol{w}, \boldsymbol{v}_{\boldsymbol{w}}^{\perp}\}$ is an orthogonal basis for V.

Therefore, we have a method for constructing an orthogonal basis of any two dimensional subspace, starting from any given basis. There is a generalization of this procedure which works for any subspace of $\mathbb{R}^n$, called the *Gram-Schmidt* process. It is given by the following theorem.

Theorem 2.71 (Gram-Schmidt)**.** *Let $V \subset \mathbb{R}^n$ be a subspace with basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. Then $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\ell\}$ is an orthogonal basis, where*

$$\begin{aligned} \boldsymbol{w}_1 &= \boldsymbol{v}_1 \\ \boldsymbol{w}_2 &= \boldsymbol{v}_2 - \mathrm{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) \\ \boldsymbol{w}_3 &= \boldsymbol{v}_3 - \mathrm{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3) - \mathrm{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3) \\ &\vdots \\ \boldsymbol{w}_\ell &= \boldsymbol{v}_\ell - \mathrm{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_\ell) - \cdots - \mathrm{proj}_{\boldsymbol{w}_{\ell-1}}(\boldsymbol{v}_\ell) \end{aligned}$$

That is, for each $1 \le i \le \ell$, the vector $\boldsymbol{w}_i$ is given by

$$\boldsymbol{w}_i = \boldsymbol{v}_i - \sum_{k=1}^{i-1} \mathrm{proj}_{\boldsymbol{w}_k}(\boldsymbol{v}_i).$$

Though the proof of the theorem is not outside our grasp, it is a bit technical. For this reason, we will not reproduce it here.

Example 2.72. Find an orthogonal basis for $V = \operatorname{span}(S)$, where $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

Solution. Step 1. To apply Gram-Schmidt, we must input a basis of V. It is easy to see that S is linearly independent, and thus is itself a basis. Indeed:

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 = \begin{bmatrix} x_2 \\ x_1 + 2x_2 + 2x_3 \\ x_3 \\ x_1 \end{bmatrix}$$

which is the zero vector if and only if $x_1 = x_2 = x_3 = 0$.

Step 2. We now apply Gram-Schmidt to S to obtain an orthogonal basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ of V. First, we set $\boldsymbol{w}_1 = \boldsymbol{v}_1$. To compute $\boldsymbol{w}_2$, note that

$$\operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \frac{\boldsymbol{v}_2^T\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = \frac{2}{2}\boldsymbol{v}_1 = \boldsymbol{v}_1.$$

Hence, we have

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

Next, to compute $\boldsymbol{w}_3$, note that

$$\operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3) = \frac{2}{2}\boldsymbol{v}_1 = \boldsymbol{v}_1, \quad \operatorname{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3) = \frac{2}{3}\boldsymbol{w}_2.$$

We thus find that $\boldsymbol{w}_3$ is given by

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \boldsymbol{v}_1 - \frac{2}{3}\boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

By the Gram-Schmidt Theorem, we can conclude $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthogonal basis for V.

Example 2.73. [16] Find an orthogonal basis for $V = \operatorname{span}(S)$, where $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is given by

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

[16] This example was not presented in class.

Solution. Step 1. Again, we must first check that S is actually linearly independent. Set $A = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}$; we solve $A\boldsymbol{x} = \boldsymbol{0}$ by row reducing A:

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \\ 4 & 3 & 3 \end{pmatrix} \xrightarrow[R_4 - 2R_1]{R_3 - R_1} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_2 - 2R_3]{R_4 - R_3} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\frac{1}{2}R_1, -R_3]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since this echelon form has 3 nonzero rows, $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution. This shows S is linearly independent, and thus a basis for V.

Step 2. We now apply Gram-Schmidt to S to obtain an orthogonal basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ of V. First, we set $\boldsymbol{w}_1 = \boldsymbol{v}_1$.

To compute $\boldsymbol{w}_2$, note that

$$\operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \frac{\boldsymbol{v}_2^T \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = \frac{2+4+12}{4+4+16}\boldsymbol{v}_1 = \frac{18}{24}\boldsymbol{v}_1 = \frac{3}{4}\boldsymbol{v}_1.$$

Hence, we have

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_2) = \boldsymbol{v}_2 - \frac{3}{4}\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}.$$

Similarly, to find $\boldsymbol{w}_3$ we must compute $\operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3)$ and $\operatorname{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3)$:

$$\operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3) = \frac{\boldsymbol{v}_1^T \boldsymbol{v}_3}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_3 = \frac{3}{4}\boldsymbol{v}_1, \quad \operatorname{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3) = 2\frac{5}{18}\boldsymbol{w}_2 = \frac{5}{9}\boldsymbol{w}_2$$

Hence, we have

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \operatorname{proj}_{\boldsymbol{w}_1}(\boldsymbol{v}_3) - \operatorname{proj}_{\boldsymbol{w}_2}(\boldsymbol{v}_3) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix} - \frac{5}{18}\begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

By the Gram-Schmidt Theorem, we can conclude that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, defined above, is an orthogonal basis for V.

Note. End of Lecture 21 (02/28/2020).

Note. Lecture 22 (03/02/2020) is Midterm II review (not to be typed).

2.7. **Linear transformations between subspaces.** We have learned that a subspace of $\mathbb{R}^n$ is a subset $V \subset \mathbb{R}^n$ with some additional structure which allows one to do vector operations inside of V. Linear transformations are special kinds of functions from one subspace to another which "keep track" of information about these underlying vector operations.

Before formalizing this into a definition, we briefly recall some basic notions related to functions.

2.7.1. *Basics on functions.* Let V and W be sets. Recall that a function f from V to W, written $f : V \to W$, is a rule which assigns to each $x \in V$ a point $f(x) \in W$.

That is, f takes as *input* points in V, and *outputs* points in W. Moreover:

- V is called the *domain* of f.
- W is called the *codomain* of f.

The notation $f : V \to W$ always specifies exactly what the domain and codomain are, which is important since this is part of the data necessary to define a function.

As an example, consider $f(x) = \cos(x)$. If the domain of f were taken to be $V = \mathbb{R}$, then f would not be a one-to-one function[17] (since $\cos(x)$ is periodic). However, if V were instead taken to be $[0, \frac{\pi}{2}]$, then f would be one-to-one. Consequently, we see that both of these choices lead to a very different picture, and neither choice should be assumed as the default. This is one reason why the domain V must be specified in the definition of a function.

Let's look at a few more examples of functions, some familiar and some new.

Example 2.74.

(1) $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = e^x$ (the *exponential* function).

Note that f is one-to-one ($e^{x_1} = e^{x_2}$ implies $x_1 = x_2$), but it is *not* onto, as $e^x > 0$ for all x (so -1 cannot be written as e^x for any x, for instance).

If we view f instead as a function

$$f : \mathbb{R} \to \mathbb{R}_{>0}, \qquad \text{where } \mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\},$$

then it is both one-to-one and onto with inverse $f^{-1} : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f^{-1}(x) = \log(x)$.

This example illustrates the importance of specifying the codomain in the definition of a function.

(2) $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2.$$

(3) $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, given by $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$.

(4) $f : \mathbb{R}^2 \to \mathbb{R}^3$, given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_1 + 2x_2 \\ x_2 \end{bmatrix}.$$

(5) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n + 1$. Here $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of *natural numbers.*

[17] We will review what it means to be a one-to-one, onto, and invertible function a bit later.

(6) $f : \mathrm{M}_2(\mathbb{R}) \to \mathbb{R}$, $f(A) = \det(A)$ (the *determinant* function). This function tells us exactly when a 2×2 matrix is invertible; A is invertible if and only if $f(A) \neq 0$.

(7) $f : \mathrm{M}_{m,n}(\mathbb{R}) \to \mathrm{M}_{n,m}$, $f(A) = A^T$ (the *transpose* function).

2.7.2. *Introduction to linear transformations.* As indicated above, linear transformations are special kinds of functions between subspaces which preserve certain information about vector operations. Let's motivate the formal definition with an example.

Example 2.75. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 + x_3 \end{bmatrix}.$$

Note that we can rewrite f equivalently as

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\boldsymbol{x}.$$

Hence, f is of the form $f(\boldsymbol{x}) = A\boldsymbol{x}$ for a matrix A. One interesting property this function has is that

$$f(\boldsymbol{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad f(\boldsymbol{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In other words $f(\boldsymbol{e}_j)$ is the j-th column of A. Two other important properties are that

$$\begin{gathered} f(\boldsymbol{x} + \boldsymbol{y}) = A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = f(\boldsymbol{x}) + f(\boldsymbol{y}), \\ f(k\boldsymbol{x}) = A(k\boldsymbol{x}) = kA\boldsymbol{x} = kf(\boldsymbol{x}) \end{gathered}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ and $k \in \mathbb{R}$. These last two properties capture exactly what it means to be a linear transformation.

Definition 2.76. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be subspaces. A *linear transformation* f from V to W is a function $f : V \to W$ satisfying

$$\begin{gathered} f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y}), \\ f(k\boldsymbol{x}) = kf(\boldsymbol{x}), \end{gathered}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in V$ and $k \in \mathbb{R}$.

Example 2.77. Let's look at a few more examples of functions which are, and aren't, linear transformations:

(1) As indicated in the previous example, if $A \in \mathrm{M}_{m,n}(\mathbb{R})$, then

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad f(\boldsymbol{x}) = A\boldsymbol{x}$$

is a linear transformation. We will soon see that *every* linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ is of this form.

(2) Let $V \subset \mathbb{R}^n$ be a subspace. Then

$$\mathrm{id}_V : V \to V, \quad \mathrm{id}_V(\boldsymbol{v}) = \boldsymbol{v} \quad \text{for all} \quad \boldsymbol{v} \in V$$

is a linear transformation, called the *identity transformation* on V. Similarly,

$$f : V \to \{\mathbf{0}\}, \quad f(\boldsymbol{v}) = \mathbf{0} \quad \text{for all} \quad \boldsymbol{v} \in V$$

is a linear transformation, called the *zero transformation* on V.

(3) $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 1 \\ x_2 \end{bmatrix}$$

is *not* a linear transformation. Indeed, we have

$$f(\mathbf{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f(\boldsymbol{e}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

and

$$f(\mathbf{0} + \boldsymbol{e}_1) = f(\boldsymbol{e}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = f(\mathbf{0}) + f(\boldsymbol{e}_1).$$

(4) Let $V \subset \mathbb{R}^n$ be a subspace and $\boldsymbol{w} \in V$ be a nonzero vector. Set $W = \mathrm{span}\{\boldsymbol{w}\}$. Then

$$\mathrm{proj}_{\boldsymbol{w}} : V \to W, \quad \mathrm{proj}_{\boldsymbol{w}}(\boldsymbol{v}) = \frac{\boldsymbol{v}^T \boldsymbol{w}}{\|\boldsymbol{w}\|^2} \boldsymbol{w}$$

is a linear transformation.

2.7.3. *The matrix of a linear transformation.* Our current goal is to prove that every linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ is of the form $f(\boldsymbol{x}) = A\boldsymbol{x}$ for some matrix $A \in \mathrm{M}_{m,n}(\mathbb{R})$. The main ingredient we will need is the following simple lemma.

Lemma 2.78. *Let $f : V \to W$ be a linear transformation, and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in V$ and $k_1, \ldots, k_\ell \in \mathbb{R}$. Then*

$$f(k_1 \boldsymbol{v}_1 + \cdots + k_\ell \boldsymbol{v}_\ell) = k_1 f(\boldsymbol{v}_1) + \cdots + k_\ell f(\boldsymbol{v}_\ell).$$

Proof. The lemma follows from repeated application of the two properties

$$f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y}) \quad \text{and} \quad f(k\boldsymbol{x}) = kf(\boldsymbol{x}).$$

Indeed, using the first property, we obtain

$$\begin{aligned} f(k_1 \boldsymbol{v}_1 + k_2 \boldsymbol{v}_2 \cdots + k_\ell \boldsymbol{v}_\ell) &= f(k_1 \boldsymbol{v}_1) + f(k_2 \boldsymbol{v}_2 + \cdots + k_\ell \boldsymbol{v}_\ell) \\ &= f(k_1 \boldsymbol{v}_1) + f(k_2 \boldsymbol{v}_2) + f(k_3 \boldsymbol{v}_3 + \cdots + k_\ell \boldsymbol{v}_\ell) \\ &\;\;\vdots \\ &= f(k_1 \boldsymbol{v}_1) + f(k_2 \boldsymbol{v}_2) + \cdots + f(k_\ell \boldsymbol{v}_\ell). \end{aligned}$$

Applying the second property to each term, we obtain the desired relation. $\square$

Theorem 2.79. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then*

$$f(\boldsymbol{x}) = A\boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n,$$

where $A \in \mathrm{M}_{m,n}(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} f(\boldsymbol{e}_1) & \cdots & f(\boldsymbol{e}_n) \end{pmatrix}.$$

Proof. Let $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$. Then

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + \cdots + x_n\boldsymbol{e}_n$$

so the previous lemma implies that

$$f(\boldsymbol{x}) = f(x_1\boldsymbol{e}_1 + \cdots x_n\boldsymbol{e}_n) = x_1 f(\boldsymbol{e}_1) + \cdots + x_n f(\boldsymbol{e}_n) = A\boldsymbol{x},$$

where A is an in the statement of the theorem. □

Definition 2.80. The matrix A defined above is called the *matrix associated to the linear transformation* f, with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Remark 2.81. The above theorem establishes an exact correspondence

$$\{\text{Linear transformations } f : \mathbb{R}^n \to \mathbb{R}^m\} \longleftrightarrow \mathrm{M}_{m,n}(\mathbb{R}).$$

One goes from left to right by sending f to the matrix A associated to it, and one goes from right to left by sending A to the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = A\boldsymbol{x}$. These two assignments are mutual inverses to each other. Consequently, one can (and should) think of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ as the exact same thing as a $m \times n$ matrix.

Note. End of Lecture 23 (03/04/2020).

Let's see the above theorem in action.

Example 2.82.

(1) Let $f = \mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$. Since $f(\boldsymbol{e}_i) = \boldsymbol{e}_i$, the matrix associated to f is

$$\begin{pmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \end{pmatrix} = I,$$

the $n \times n$ identity matrix. Indeed, $f(\boldsymbol{x}) = \boldsymbol{x} = I\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

(2) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ 2x_1 + 2x_3 \end{bmatrix}$$

is a linear transformation. Moreover, we have

$$f(\boldsymbol{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad f(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad f(\boldsymbol{e}_3) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Hence, the matrix associated to f is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

By the theorem, $f(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^3$. Indeed, we have

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ 2x_1 + 2x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(3) Let $f = \mathrm{proj}_{\boldsymbol{e}_1} : \mathbb{R}^n \to \mathbb{R}^n$, so that

$$f(\boldsymbol{x}) = (\boldsymbol{x}^T \boldsymbol{e}_1)\boldsymbol{e}_1 = x_1 \boldsymbol{e}_1 \quad \text{for all } \ v \in \mathbb{R}^n.$$

Since $f(\boldsymbol{e}_1) = \boldsymbol{e}_1$ and $f(\boldsymbol{e}_j) = \boldsymbol{0}$ for $j \neq 1$, the matrix associated to f is

$$A = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

You can now check that $A\boldsymbol{x} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \end{bmatrix}^T = x_1 \boldsymbol{e}_1 = f(\boldsymbol{x})$.

2.7.4. *Operations on linear transformations.* Vector operations provide us a way to obtain new vectors starting from some finite number of vectors. Similarly, there are operations we can apply to linear transformations to obtain new linear transformations.

Proposition 2.83. *Let $k \in \mathbb{R}$, and suppose we are given linear transformations $f : V \to W$, $g : V \to W$, and $h : U \to V$. Then each of the following is a linear transformation:*

(1) *The sum $f + g : V \to W$, defined by*

$$(f+g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}) \quad \textit{for all} \quad \boldsymbol{x} \in V.$$

(2) *The scalar multiple $kf : V \to W$, defined by*

$$(kf)(\boldsymbol{x}) = kf(\boldsymbol{x}) \quad \textit{for all} \quad \boldsymbol{x} \in V.$$

(3) *The composition $f \circ h : U \to W$, defined by*

$$(f \circ h)(\boldsymbol{x}) = f(h(\boldsymbol{x})) \quad \textit{for all} \quad \boldsymbol{x} \in U.$$

Proof. The proof that $f + g$ and kf are linear transformations is left as an exercise to the reader. Let's show that the composition $f \circ h$ is a linear transformation. Let $\boldsymbol{x}, \boldsymbol{y} \in U$. Then

$$\begin{aligned} (f \circ h)(\boldsymbol{x} + \boldsymbol{y}) &= f(h(\boldsymbol{x} + \boldsymbol{y})) \\ &= f(h(\boldsymbol{x}) + h(\boldsymbol{y})) \\ &= f(h(\boldsymbol{x})) + f(h(\boldsymbol{x})) = (f \circ h)(\boldsymbol{x}) + (f \circ h)(\boldsymbol{y}), \end{aligned}$$

where the second and third equalities are due to h and f being linear transformations, respectively. Similarly, if $k \in \mathbb{R}$ and $\boldsymbol{x} \in U$, then

$$(f \circ h)(k\boldsymbol{x}) = f(h(k\boldsymbol{x})) = f(kh(\boldsymbol{x})) = kf(h(\boldsymbol{x})) = k(f \circ h)(\boldsymbol{x}). \qquad \square$$

Example 2.84. Let f and g be the linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ with matrices A and B, respectively, where

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, we have

$$(f+g)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A+B)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
$$(f \circ g)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\left(B\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise. Let g^k be the linear transformation obtained by composing g with itself k times:

$$g^k = \underbrace{g \circ \cdots \circ g}_{k \text{ times}}.$$

Show that $g^{2k} = \mathrm{id}_{\mathbb{R}^2}$ and $g^{2k+1} = g$ for all natural numbers k.

In the above example, we say that the matrices associated to $f+g$ and $f \circ g$ were $A+B$ and AB, respectively. This is true more generally, and can be established the same way.

Proposition 2.85. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^\ell \to \mathbb{R}^n$ be linear transformations with matrices A, B and C, respectively. Then:*

(1) *The matrix associated to kf is kA.*
(2) *The matrix associated to $f+g$ is $A+B$.*
(3) *The matrix associated to $f \circ h$ is AC.*

2.7.5. *Coordinate transformations.* Now let us fix a subspace $V \subset \mathbb{R}^n$ with basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. Since $\mathcal{B}$ is a basis, any vector $\boldsymbol{v} \in V$ can be written uniquely as

$$\boldsymbol{v} = k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell \quad \text{with} \quad k_i \in \mathbb{R}.$$

Definition 2.86. The *coordinate vector* $[\boldsymbol{v}]_\mathcal{B}$ of $\boldsymbol{v}$ with respect to $\mathcal{B}$ is

$$[\boldsymbol{v}]_\mathcal{B} = \begin{bmatrix} k_1 \\ \vdots \\ k_\ell \end{bmatrix} \in \mathbb{R}^\ell.$$

The numbers $k_1, \ldots, k_\ell$ are called the *coordinates* of $\boldsymbol{v}$ with respect to $\mathcal{B}$.

In fact, the terminology "coordinates" was introduced all the way back in Remark 2.44. Note that $[\boldsymbol{v}]_{\mathcal{B}}$ can be equivalently defined as the *unique* solution of the system

$$A\boldsymbol{x} = \boldsymbol{v}, \quad \text{where} \quad A = \begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell\end{pmatrix} \in \mathrm{M}_{n,\ell}(\mathbb{R}). \tag{2.5}$$

In particular, if $\ell = n$ then $V = \mathbb{R}^n$, A is invertible and

$$[\boldsymbol{v}]_{\mathcal{B}} = A^{-1}\boldsymbol{v}. \tag{2.6}$$

Let's take a look at a few examples.

Example 2.87.

(1) If $V = \mathbb{R}^n$ and $\mathcal{B} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, then $[\boldsymbol{v}]_{\mathcal{B}} = \boldsymbol{v}$ and the coordinates of $\boldsymbol{v}$ are $v_1, \dots, v_n$ as usual, assuming $\boldsymbol{v} = \begin{bmatrix}v_1 & \cdots & v_n\end{bmatrix}^T$.

(2) Fix $0 \leq \theta \leq \pi$ and consider the basis $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of $\mathbb{R}^2$, where

$$\boldsymbol{v}_1 = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}.$$

Since $\mathcal{B}$ is an orthonormal set, the matrix $A = \begin{pmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2\end{pmatrix}$ is orthogonal: $A^{-1} = A^T$. By (2.6), we have

$$\begin{bmatrix}x_1 \\ x_2\end{bmatrix}_{\mathcal{B}} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}x_1\cos\theta + x_2\sin\theta \\ -x_1\sin\theta + x_2\cos\theta\end{bmatrix}.$$

(3) Let $V \subset \mathbb{R}^n$ be a subspace with an orthogonal basis $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$. Then

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix}k_1 \\ \vdots \\ k_\ell\end{bmatrix}, \quad \text{where} \quad k_i = \frac{\boldsymbol{x}^T\boldsymbol{v}_i}{\|\boldsymbol{v}_i\|^2}.$$

Indeed, we have

$$\boldsymbol{x} = \operatorname{proj}_{\boldsymbol{v}_1}(\boldsymbol{x}) + \cdots + \operatorname{proj}_{\boldsymbol{v}_\ell}(\boldsymbol{x}) = k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell,$$

with k_i defined as above.

Coordinate vectors provide us with another very valuable source of linear transformations, as the below lemma illustrates.

Lemma 2.88. *Let $V \subset \mathbb{R}^n$ be a subspace with basis $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$. Then the function $T_{\mathcal{B}} : V \to \mathbb{R}^\ell$ given by*

$$T_{\mathcal{B}}(\boldsymbol{v}) = [\boldsymbol{v}]_{\mathcal{B}} \quad \textit{for all} \quad \boldsymbol{v} \in V$$

is a linear transformation.

Proof. We need to show that, for any $k \in \mathbb{R}$ and $\boldsymbol{v}, \boldsymbol{w} \in V$, we have

$$[\boldsymbol{v}]_{\mathcal{B}} + [\boldsymbol{w}]_{\mathcal{B}} = [\boldsymbol{v} + \boldsymbol{w}]_{\mathcal{B}} \quad \text{and} \quad k[\boldsymbol{v}]_{\mathcal{B}} = [k\boldsymbol{v}]_{\mathcal{B}}.$$

Let $A = \begin{pmatrix}\boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell\end{pmatrix}$, and consider the first equality. Since

$$A([\boldsymbol{v}]_{\mathcal{B}} + [\boldsymbol{w}]_{\mathcal{B}}) = A[\boldsymbol{v}]_{\mathcal{B}} + A[\boldsymbol{w}]_{\mathcal{B}} = \boldsymbol{v} + \boldsymbol{w},$$

the vector $[\boldsymbol{v}]_{\mathcal{B}} + [\boldsymbol{w}]_{\mathcal{B}}$ is a solution of $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. By (2.5), $[\boldsymbol{v} + \boldsymbol{w}]_{\mathcal{B}}$ is the unique solution of this equation, and therefore

$$[\boldsymbol{v}]_{\mathcal{B}} + [\boldsymbol{w}]_{\mathcal{B}} = [\boldsymbol{v} + \boldsymbol{w}]_{\mathcal{B}}.$$

Similarly, both $k[\boldsymbol{v}]_{\mathcal{B}}$ and $[k\boldsymbol{v}]_{\mathcal{B}}$ are solutions to $A\boldsymbol{x} = k\boldsymbol{v}$, and therefore are equal by uniqueness. □

Definition 2.89. The transformation $T_{\mathcal{B}} : V \to \mathbb{R}^\ell$ defined above is called the *coordinate transformation* of V with respect to $\mathcal{B}$.

Remark 2.90. The coordinate transformation $T_{\mathcal{B}}$ allows us to think of vectors in V as ℓ-dimensional vectors, by identifying $\boldsymbol{v} \in V$ with the coordinate vector $[\boldsymbol{v}]_{\mathcal{B}} \in \mathbb{R}^\ell$. In this way, we really can think of a basis $\mathcal{B}$ for V as giving us a *coordinate system* for vectors in V. We will later see that $T_{\mathcal{B}}$ is invertible, which implies that V and $\mathbb{R}^\ell$ are essentially identical as subspaces.

Using coordinate transformations and composition, we can build new linear transformations. Indeed, let $V$and $\mathcal{B}$ be as above, and suppose $W \subset \mathbb{R}^m$ is a subspace. Fix ℓ vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_\ell \in W$, and define $f : V \to W$ by

$$f(\boldsymbol{v}) = A[\boldsymbol{v}]_{\mathcal{B}} \quad \text{for all } \ \boldsymbol{v} \in V,$$

where $A = \begin{pmatrix} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_\ell \end{pmatrix} \in \mathrm{M}_{m,\ell}(\mathbb{R})$. Then f is a linear transformation. In fact, it is just the composition

$$f = S \circ T_{\mathcal{B}}$$

where $S : \mathbb{R}^\ell \to \mathbb{R}^m$ is the linear transformation $S(\boldsymbol{x}) = A\boldsymbol{x}$.

Remarkably, every linear transformation $f : V \to W$ is of this form, as the following generalization of Theorem 2.79 demonstrates.

Theorem 2.91. *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be subspaces, $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ be a basis for V, and $f : V \to W$ be any linear transformation. Then*

$$f(\boldsymbol{v}) = A[\boldsymbol{v}]_{\mathcal{B}} \quad \textit{for all } \ \boldsymbol{v} \in V,$$

where $A \in \mathrm{M}_{m,\ell}(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} f(\boldsymbol{v}_1) & \cdots & f(\boldsymbol{v}_\ell) \end{pmatrix}.$$

Proof. We apply the same argument as used in the proof of Theorem 2.79. Let $\boldsymbol{v} \in V$ and expand $\boldsymbol{v} = k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell$, so that the k_i are the coordinates of $\boldsymbol{v}$ with respect to $\mathcal{B}$. Then

$$f(\boldsymbol{v}) = k_1 f(\boldsymbol{v}_1) + \cdots + k_\ell f(\boldsymbol{v}_\ell) = \begin{pmatrix} f(\boldsymbol{v}_1) & \cdots & f(\boldsymbol{v}_\ell) \end{pmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_\ell \end{bmatrix} = A[\boldsymbol{v}]_{\mathcal{B}}.$$ □

Definition 2.92. The matrix A defined above is called the *matrix associated to the linear transformation* $f : V \to W$, *with respect to* the basis $\mathcal{B}$.

Note that when $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ and $\mathcal{B} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, this is just the matrix associated to $f : \mathbb{R}^n \to \mathbb{R}^m$, as in Definition 2.80.

Note. End of Lecture 24 (03/23/20).

2.7.6. *Switching coordinates.* The previous theorem provides one way of using $T_{\mathcal{B}} : V \to \mathbb{R}^\ell$ to identify any linear transformation $f : V \to W$ with a concrete matrix A, whose j-th column is just the vector $f(\boldsymbol{v}_j)$, where $\boldsymbol{v}_j$ is the j-th vector in $\mathcal{B}$.

However, this is not actually the optimal way of identifying f with a matrix, as it does not encode the dimension of W, or any information about a basis for W. For instance, A is $m \times \ell$, where ℓ is the dimension of V, but m is in general larger than the dimension of W, as $W \subset \mathbb{R}^m$. We are forgetting W and just replacing the codomain of f by $\mathbb{R}^m$.

The next theorem, which further generalizes Theorem 2.79, provides a remedy to this issue.

Theorem 2.93. *Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be subspaces with bases $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$ and $\mathcal{C} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_p\}$, respectively. Let $f : V \to W$ be any linear transformation. Then*

$$[f(\boldsymbol{v})]_{\mathcal{C}} = A[\boldsymbol{v}]_{\mathcal{B}} \quad \text{for all } \boldsymbol{v} \in V,$$

where $A \in \mathrm{M}_{p,\ell}(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} [f(\boldsymbol{v}_1)]_{\mathcal{C}} & \cdots & [f(\boldsymbol{v}_\ell)]_{\mathcal{C}} \end{pmatrix}.$$

Proof. Let $\boldsymbol{v} \in V$, and $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} k_1 & \cdots & k_\ell \end{bmatrix}^T$, so that k_i is the i-th coordinate of $\boldsymbol{v}$ with respect to $\mathcal{B}$. By the previous theorem,

$$f(\boldsymbol{v}) = \begin{pmatrix} f(\boldsymbol{v}_1) & \cdots & f(\boldsymbol{v}_\ell) \end{pmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_\ell \end{bmatrix} = k_1 f(\boldsymbol{v}_1) + \cdots + k_\ell f(\boldsymbol{v}_\ell).$$

Applying the coordinate transformation $T_{\mathcal{C}} : W \to \mathbb{R}^p$ to both sides, we obtain

$$[f(\boldsymbol{v})]_{\mathcal{C}} = k_1 [f(\boldsymbol{v}_1)]_{\mathcal{C}} + \cdots + k_\ell [f(\boldsymbol{v}_\ell)]_{\mathcal{C}} = A \begin{bmatrix} k_1 \\ \vdots \\ k_\ell \end{bmatrix} = A[\boldsymbol{v}]_{\mathcal{B}},$$

where A is as in the statement of the theorem. □

Definition 2.94. The matrix A from the above theorem is called *the matrix associated to f with respect to the bases $\mathcal{B}$ and $\mathcal{C}$*. We will sometimes denote it by $[f]_{\mathcal{B},\mathcal{C}}$.

Remark 2.95. When $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, with $\mathcal{B}$ and $\mathcal{C}$ taken to be the standard bases, $[f]_{\mathcal{B},\mathcal{C}}$ is just the ordinary matrix associated to $f : \mathbb{R}^n \to \mathbb{R}^m$, and the above theorem reduces to Theorem 2.79. More generally, the above theorem is equivalent to the equality

$$T_{\mathcal{C}} \circ f = S \circ T_{\mathcal{B}},$$

where $S : \mathbb{R}^\ell \to \mathbb{R}^p$ is the linear transformation with matrix $[f]_{\mathcal{B},\mathcal{C}}$. We can write this equality diagrammatically:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "T_{\mathcal{B}}"'] & W \arrow[d, "T_{\mathcal{C}}"] \\ \mathbb{R}^\ell \arrow[r, "S"'] & \mathbb{R}^p \end{tikzcd}$$

If we start at V, it doesn't matter which of the two paths ($T_{\mathcal{C}} \circ f$ or $S \circ T_{\mathcal{B}}$) we take to get to $\mathbb{R}^p$.

The following example illustrates how the the above theorem provides new insights even when one starts with a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$, as it allows you to "rewrite" the matrix associated to f in terms of different bases.

Example 2.96. Consider the basis $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ of $\mathbb{R}^3$, where

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

You showed on your second midterm that $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$. Now consider the linear transformation

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(\boldsymbol{x}) = \mathrm{proj}_{\boldsymbol{v}_1}(\boldsymbol{x}) + 2\mathrm{proj}_{\boldsymbol{v}_2}(\boldsymbol{x}) + 3\mathrm{proj}_{\boldsymbol{v}_3}(\boldsymbol{x}).$$

We then have $f(\boldsymbol{v}_1) = \boldsymbol{v}_1$, $f(\boldsymbol{v}_2) = 2\boldsymbol{v}_2$ and $f(\boldsymbol{v}_3) = 3\boldsymbol{v}_3$. Hence

$$[f]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

This means that f is determined by

$$[f(\boldsymbol{x})]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} [\boldsymbol{x}]_{\mathcal{B}} \quad \forall \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{2.7}$$

The above matrix is a very simple diagonal matrix. We can use it to easily compute powers of f:

$$[(f \circ f)(\boldsymbol{x})]_{\mathcal{B}} = [f(f(\boldsymbol{x}))]_{\mathcal{B}} = [f]_{\mathcal{B},\mathcal{B}}[f(\boldsymbol{x})]_{\mathcal{B}} = [f]^2_{\mathcal{B},\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}},$$

which shows that the matrix $[f \circ f]_{\mathcal{B},\mathcal{B}}$ is equal to

$$[f]^2_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

More generally, we find that the matrix $[f^k]_{\mathcal{B},\mathcal{B}}$, where $f^k = \underbrace{f \circ \cdots \circ f}_{k \text{ times}}$, is

$$[f]^k_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix}.$$

These calculations would be much more difficult if we had instead used the ordinary matrix A associated to f (i.e. with $\mathcal{B} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$). In this case, a very tedious computation would give

$$A = \begin{pmatrix} f(\boldsymbol{e}_1) & f(\boldsymbol{e}_2) & f(\boldsymbol{e}_3) \end{pmatrix} = \begin{pmatrix} \frac{7}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{31}{15} & -\frac{7}{15} \\ -\frac{1}{6} & -\frac{7}{15} & \frac{83}{30} \end{pmatrix}.$$

However, we can translate between $[f]_{\mathcal{B},\mathcal{B}}$ and A using what we know about coordinate vectors. By (2.6), we have

$$[\boldsymbol{x}]_{\mathcal{B}} = P^{-1}\boldsymbol{x} \quad \text{where} \quad P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}.$$

Plugging this into (2.7), and left-multiplying by P, we get

$$f(\boldsymbol{x}) = P[f]_{\mathcal{B},\mathcal{B}}P^{-1}\boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^3.$$

This implies that $A = P[f]_{\mathcal{B},\mathcal{B}}P^{-1}$! In particular, we have

$$A^2 = (P[f]_{\mathcal{B},\mathcal{B}}P^{-1})(P[f]_{\mathcal{B},\mathcal{B}}P^{-1}) = P[f]^2_{\mathcal{B},\mathcal{B}}P^{-1}.$$

More generally, we can compute powers of A by the formula

$$A^k = P[f]^k_{\mathcal{B},\mathcal{B}}P^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix} P^{-1}.$$

We will learn more about this type of relationship later in the course. It allows us to deduce fundamental properties for the more complicated matrix A from the very simple diagonal matrix $[f]_{\mathcal{B},\mathcal{B}}$ using that both represent the same linear transformation, just with respect to different bases.

We conclude this subsection with a corollary of Theorem 2.93, which explains how to switch between coordinate vectors associated to different bases. You should think of a basis as giving you a coordinate system via the coordinate transformation $T_{\mathcal{B}}$, as in Remark 2.90, and then this result explains how to switch from one coordinate system to another.

Corollary 2.97. *Let $V \subset \mathbb{R}^n$ be a subspace with bases $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$ and $\mathcal{C} = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\ell\}$. Then*

$$[\boldsymbol{v}]_{\mathcal{C}} = A[\boldsymbol{v}]_{\mathcal{B}} \quad \text{for all } \boldsymbol{v} \in V,$$

where $A \in \mathrm{M}_\ell(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} [\boldsymbol{v}_1]_{\mathcal{C}} & \cdots & [\boldsymbol{v}_\ell]_{\mathcal{C}} \end{pmatrix}.$$

Proof. Take $f = \mathrm{id}_V$ in the previous theorem. $\square$

2.7.7. *Isomorphisms.* A linear transformation $f : V \to W$ gives us a way of transferring information about the subspace structure of V to W. A natural question to ask is, under what circumstances does f encode *all* of the information about V? More specifically, what conditions on f are necessary and sufficient for us to think of V and W as the *same* subspace, up to a relabeling of vectors given by f?

This is the question at the heart of this subsection, and the answer is that f should be a *isomorphism.* We begin by introducing the relevant terminology.

Definition 2.98. Let $f : V \to W$ be a linear transformation. Then

(1) f is *one-to-one* if, for all $\boldsymbol{x}, \boldsymbol{v} \in V$, $f(\boldsymbol{x}) = f(\boldsymbol{v})$ implies $\boldsymbol{x} = \boldsymbol{v}$.
(2) f is *onto* if for each $\boldsymbol{w} \in W$, there is $\boldsymbol{v} \in V$ with $f(\boldsymbol{v}) = \boldsymbol{w}$.
(3) f is an *isomorphism* if it is both one-to-one and onto. In this case, V and W are said to be *isomorphic* and we write $V \cong W$.

Remark 2.99. In words, f is one-to-one if distinct input points $\boldsymbol{x} \neq \boldsymbol{v}$ in V have distinct outputs $f(\boldsymbol{x}) \neq f(\boldsymbol{v})$ in W. The function f is onto if each point $\boldsymbol{w}$ in W is the output of at least one point $\boldsymbol{v}$ in V.

Lemma 2.100. *A linear transformation $f : V \to W$ is an isomorphism if and only if there is a function $f^{-1} : W \to V$ satisfying*

$$f^{-1} \circ f = \mathrm{id}_V \quad \textit{and} \quad f \circ f^{-1} = \mathrm{id}_W \tag{2.8}$$

Proof. Let's first show that if there is f^{-1} satisfying (2.8), then f is an isomorphism. Let $\boldsymbol{x}, \boldsymbol{v} \in V$, and suppose $f(\boldsymbol{x}) = f(\boldsymbol{v})$. Applying f^{-1} to both sides of this equality gives

$$\boldsymbol{x} = f^{-1}(f(\boldsymbol{x})) = f^{-1}(f(\boldsymbol{v})) = \boldsymbol{v},$$

which proves that f is one-to-one.

To show that f is onto, let $\boldsymbol{w} \in W$. We must prove there is $\boldsymbol{v} \in V$ such that $f(\boldsymbol{v}) = \boldsymbol{w}$. Setting $\boldsymbol{v} = f^{-1}(\boldsymbol{w})$, we have $f(\boldsymbol{v}) = f(f^{-1}(\boldsymbol{w})) = \boldsymbol{w}$, as desired. This proves that f is an isomorphism.

Conversely, suppose that f is an isomorphism. Since f is onto, any $\boldsymbol{w} \in W$ can be written as $\boldsymbol{w} = f(\boldsymbol{v})$ for some $\boldsymbol{v} \in V$. Since f is one-to-one, there is only one such $\boldsymbol{v}$. We may thus define a function $f^{-1} : W \to V$ by

$$f^{-1}(\boldsymbol{w}) = \boldsymbol{v}, \tag{2.9}$$

where $\boldsymbol{v} \in V$ is the unique vector satisfying $f(\boldsymbol{v}) = \boldsymbol{w}$. It is now a straightforward exercise, left to the reader, to verify that f^{-1} satisfies the relations (2.8). $\square$

Definition 2.101. The function $f^{-1} : W \to V$ defined by (2.9) is called the *inverse* function of f.

It is not entirely clear from the above definition that f^{-1} will itself be a linear transformation. This is indeed the case:

Proposition 2.102. *Let $f : V \to W$ be an isomorphism with inverse $f^{-1} : W \to V$. Then f^{-1} is itself a linear transformation.*

Note. End of Lecture 25 (03/25/20).

An isomorphism $f : V \to W$ establishes an exact pairing between vectors in V and W. You can think of V and W as being the same sets: the only difference is that their vectors are perhaps labeled differently. The function f tells you exactly how this labeling works, i.e. the point $\boldsymbol{v}$ in V corresponds to $f(\boldsymbol{v})$ in W, and one can "decode" this correspondence by applying f^{-1}. By the end of this section, we will see that V and W also have all the same properties as subspaces, and not just as sets.

In addition, we develop a "matrix criterion" for determining exactly when any linear transformation is invertible. We begin with an example central to this goal.

Example 2.103. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with matrix $A \in \mathrm{M}_n(\mathbb{R})$. By the above, f is an isomorphism if and only if there is a linear transformation $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$f \circ f^{-1} = \mathrm{id}_{\mathbb{R}^n} = f \circ f^{-1}.$$

Let B be the matrix associated to f^{-1}. Since $f \circ f^{-1}$, $\mathrm{id}_{\mathbb{R}^n}$, and $f \circ f^{-1}$ have matrices AB, I and BA, the matrix form of the above relation is

$$AB = I = BA$$

Hence, we see that f is an isomorphism if and only its associated matrix is invertible, in which case f^{-1} is given by

$$f^{-1}(\boldsymbol{x}) = A^{-1}\boldsymbol{x} \quad \text{ for all } \ \boldsymbol{x} \in \mathbb{R}^n.$$

Another important family of isomorphisms is given by coordinate transformations.

Theorem 2.104. *Let $V \subset \mathbb{R}^n$ be a subspace with basis $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$. Then the coordinate transformation $T_{\mathcal{B}} : V \to \mathbb{R}^\ell$ is an isomorphism with inverse*

$$T_{\mathcal{B}}^{-1} : \mathbb{R}^\ell \to V, \qquad T_{\mathcal{B}}^{-1}(\boldsymbol{x}) = A\boldsymbol{x} \quad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^\ell,$$

where $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell \end{pmatrix}$. In particular, V is isomorphic to $\mathbb{R}^\ell$:

$$V \cong \mathbb{R}^\ell.$$

Proof. By Lemma 2.100, it suffices to show that $T_{\mathcal{B}}^{-1}$ defined above satisfies

$$T_{\mathcal{B}} \circ T_{\mathcal{B}}^{-1} = \mathrm{id}_{\mathbb{R}^\ell} \quad \text{ and } \quad T_{\mathcal{B}}^{-1} \circ T_{\mathcal{B}} = \mathrm{id}_V.$$

Since $T_{\mathcal{B}}^{-1} \circ T_{\mathcal{B}}$ and $T_{\mathcal{B}} \circ T_{\mathcal{B}}^{-1}$ are linear transformations, it suffices to check these equalities on the vectors in the bases $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\}$ and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_\ell\}$.

As $T_{\mathcal{B}}^{-1}(\boldsymbol{e}_i) = A\boldsymbol{e}_i$ is the i-th column of A, we have

$$T_{\mathcal{B}}^{-1}(\boldsymbol{e}_i) = \boldsymbol{v}_i \quad \text{ and } \quad T_{\mathcal{B}}(\boldsymbol{v}_i) = [\boldsymbol{v}_i]_{\mathcal{B}} = \boldsymbol{e}_i.$$

Hence, $T_{\mathcal{B}}^{-1}(T_{\mathcal{B}}(\boldsymbol{v}_i)) = T_{\mathcal{B}}^{-1}(\boldsymbol{e}_i) = \boldsymbol{v}_i$ and $T_{\mathcal{B}}(T_{\mathcal{B}}^{-1}(\boldsymbol{e}_i)) = T_{\mathcal{B}}(\boldsymbol{v}_i) = \boldsymbol{e}_i$ for all i, as desired. $\square$

Remark 2.105. The theorem implies that *every* ℓ-dimensional subspace V of $\mathbb{R}^n$ (for any n) is isomorphic $\mathbb{R}^\ell$. This means that you can think of any ℓ-dimensional subspace as just a copy of $\mathbb{R}^\ell$, up to a relabeling of vectors.

Henceforth, we assume that $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are fixed subspaces with

$$\dim V = \ell \quad \text{and} \quad \dim W = p.$$

Since we now know that every coordinate transformation is an isomorphism, we can rewrite the statement of Theorem 2.93 using Remark 2.95 as follows.

Corollary 2.106. *Let $f : V \to W$ be a linear transformation, and suppose $\mathcal{B} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_\ell\} \subset V$ and $\mathcal{C} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_p\} \subset W$ are bases. Then*

$$f = T_{\mathcal{C}}^{-1} \circ S \circ T_{\mathcal{B}},$$

where $S : \mathbb{R}^\ell \to \mathbb{R}^p$ is given by $S(\boldsymbol{x}) = [f]_{\mathcal{B},\mathcal{C}}\boldsymbol{x}$.

Proof. By Remark 2.95, we have $T_{\mathcal{C}} \circ f = S \circ T_{\mathcal{B}}$. Composing with $T_{\mathcal{C}}^{-1}$ on the left, we obtain $f = T_{\mathcal{C}}^{-1} \circ S \circ T_{\mathcal{B}}$. □

Remark 2.107. By Theorem 2.104, $T_{\mathcal{C}}^{-1}(\boldsymbol{x}) = A\boldsymbol{x}$, where $A = \begin{pmatrix} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_p \end{pmatrix}$. The above corollary therefore reads as

$$f(\boldsymbol{v}) = A[f]_{\mathcal{B},\mathcal{C}}[\boldsymbol{v}]_{\mathcal{B}} \quad \text{for all } \boldsymbol{v} \in V.$$

Since $A[\boldsymbol{w}]_{\mathcal{C}} = \boldsymbol{w}$ for each $\boldsymbol{w} \in W$, we have

$$A[f]_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} A[f(\boldsymbol{v}_1)]_{\mathcal{C}} & \cdots & A[f(\boldsymbol{v}_\ell)]_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} f(\boldsymbol{v}_1) & \cdots & f(\boldsymbol{v}_\ell) \end{pmatrix}.$$

Therefore, $A[f]_{\mathcal{B},\mathcal{C}}$ is just the matrix associated to f with respect to $\mathcal{B}$, as in Definition 2.92, and the above theorem recovers Theorem 2.91.

We now come to one of our main results of this section, which gives us a precise matrix criteria for determining when any linear transformation is an isomorphism.

Theorem 2.108. *Let $f : V \to W$ be a linear transformation, and suppose that $\mathcal{B} \subset V$ and $\mathcal{C} \subset W$ are bases. Then:*

(1) *f is one-to-one if and only if* $\text{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$.
(2) *f is onto if an only if* $\text{rank}([f]_{\mathcal{B},\mathcal{C}}) = p$.
(3) *f is an isomorphism if and only if $[f]_{\mathcal{B},\mathcal{C}}$ is invertible. In this case, we have*

$$[f^{-1}]_{\mathcal{C},\mathcal{B}} = [f]_{\mathcal{B},\mathcal{C}}^{-1}.$$

Proof. Let us sketch the main ideas behind the proof[18]. The equality

$$f = T_{\mathcal{C}}^{-1} \circ S \circ T_{\mathcal{B}}$$

of Corollary 2.106 implies that the statement of the theorem holds for f if and only if it holds for the transformation $S : \mathbb{R}^\ell \to \mathbb{R}^p$. It therefore suffices to prove the Theorem with f replaced by S.

[18] A less complete sketch was given in class.

Proof of (1) *for* S. Suppose that S is one-to-one, and let $\boldsymbol{x} \in \mathcal{N}([f]_{\mathcal{B},\mathcal{C}})$. We need to show $\boldsymbol{x} = \mathbf{0}$, as this implies $\mathcal{N}([f]_{\mathcal{B},\mathcal{C}}) = \{\mathbf{0}\}$, and thus $\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$. Since S is one-to-one, this follows immediately from the equality

$$S(\boldsymbol{x}) = [f]_{\mathcal{B},\mathcal{C}}\boldsymbol{x} = \mathbf{0} = [f]_{\mathcal{B},\mathcal{C}}\mathbf{0} = S(\mathbf{0}).$$

Conversely, suppose that $\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$, and assume that $S(\boldsymbol{x}) = S(\boldsymbol{y})$. This can be rewritten equivalently as

$$[f]_{\mathcal{B},\mathcal{C}}(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{0},$$

which implies that $\boldsymbol{x} - \boldsymbol{y} \in \mathcal{N}([f]_{\mathcal{B},\mathcal{C}})$. Since $\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$, we can conclude that $\boldsymbol{x} = \boldsymbol{y}$, and thus that S is one-to-one. This completes the proof of (1) for S.

Proof of (2) *for* S. The transformation S is onto if and only if

$$[f]_{\mathcal{B},\mathcal{C}}\boldsymbol{x} = \boldsymbol{b}$$

has a solution for each $\boldsymbol{b} \in \mathbb{R}^p$. That is, if and only if $\mathcal{R}([f]_{\mathcal{B},\mathcal{C}}) = \mathbb{R}^p$. This means exactly that $\mathrm{range}([f]_{\mathcal{B},\mathcal{C}}) = p$.

Proof of (3) *for* S. S is an isomorphism if and only if it is both one-to-one and onto. By (2) and (3), this occurs exactly when $\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$ and $\mathrm{rank}([f]_{\mathcal{B},\mathcal{C}}) = p$. By the Rank-Nullity Theorem, we have

$$\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) + \mathrm{rank}([f]_{\mathcal{B},\mathcal{C}}) = \ell.$$

Hence, S is an isomorphism if and only if $\mathrm{nullity}([f]_{\mathcal{B},\mathcal{C}}) = 0$, $\mathrm{rank}([f]_{\mathcal{B},\mathcal{C}}) = p$ and $p = \ell$. This is one of the many equivalent ways of saying $[f]_{\mathcal{B},\mathcal{C}} \in \mathrm{M}_p(\mathbb{R})$ is an invertible matrix.

Finally, the formula $[f^{-1}]_{\mathcal{C},\mathcal{B}} = [f]_{\mathcal{B},\mathcal{C}}^{-1}$ follows from Example 2.103, together with the fact that the inverse of $f = T_{\mathcal{C}}^{-1} \circ S \circ T_{\mathcal{B}}$ is $T_{\mathcal{B}}^{-1} \circ S^{-1} \circ T_{\mathcal{C}}$. □

We have already seen the above theorem in action in Example 2.103, which treated the special case where $V = \mathbb{R}^n = W$ and $\mathcal{B} = \mathcal{C} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. Let's take a look at another way this result can be applied.

Example 2.109. In Example 2.96, we considered the linear transformation

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(\boldsymbol{x}) = \mathrm{proj}_{\boldsymbol{v}_1}(\boldsymbol{x}) + 2\mathrm{proj}_{\boldsymbol{v}_2}(\boldsymbol{x}) + 3\mathrm{proj}_{\boldsymbol{v}_3}(\boldsymbol{x}), \quad \text{where}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

The ordinary matrix associated to f was found to be

$$A = \begin{pmatrix} \frac{7}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{31}{15} & -\frac{7}{15} \\ -\frac{1}{6} & -\frac{7}{15} & \frac{83}{30} \end{pmatrix}.$$

It would be an unpleasant exercise to use A to determine if f is an isomorphism. Instead, let us make use of the orthogonal basis $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. We computed in Example 2.96 that

$$[f]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

This matrix is clearly invertible with

$$[f]_{\mathcal{B},\mathcal{B}}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Hence, we can conclude from the above theorem that f is an isomorphism, with inverse determined by

$$[f^{-1}(\boldsymbol{x})]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} [\boldsymbol{x}]_{\mathcal{B}} \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^3.$$

Additionally, we found that $A = P[f]_{\mathcal{B},\mathcal{B}}P^{-1}$, where $P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{pmatrix}$. We thus have

$$\begin{aligned} A^{-1} = (P[f]_{\mathcal{B},\mathcal{B}}P^{-1})^{-1} = ([f]_{\mathcal{B},\mathcal{B}}P^{-1})^{-1}P^{-1} = P[f]_{\mathcal{B},\mathcal{B}}^{-1}P^{-1} \\ = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} P^{-1}. \end{aligned}$$

Finally, we come to the last result of this section, which should be understood as saying that isomorphic subspaces $V \cong W$ have all the same properties as subspaces, and should be thought of as being essentially identical.

Theorem 2.110. *Suppose that $f : V \to W$ is an isomorphism. Fix a subset $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subset V$ and set*

$$f(\mathcal{B}) = \{f(\boldsymbol{v}_1), \ldots, f(\boldsymbol{v}_k)\} \subset W \quad \textit{and} \quad f(A) = \begin{pmatrix} f(\boldsymbol{v}_1) & \cdots & f(\boldsymbol{v}_k) \end{pmatrix} \in \mathrm{M}_{m,k}(\mathbb{R}),$$

where $A = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix} \in \mathrm{M}_{n,k}(\mathbb{R})$. Then

1. *For any $\boldsymbol{b} \in V$, $\boldsymbol{x} \in \mathbb{R}^k$ is a solution to $A\boldsymbol{x} = \boldsymbol{b}$ if and only if it is a solution of $f(A)\boldsymbol{x} = f(\boldsymbol{b})$.*
2. *$\mathcal{B}$ spans V if and only $f(\mathcal{B})$ spans W.*
3. *$\mathcal{B}$ is linearly independent if and only if $f(\mathcal{B})$ is linearly independent.*
4. *$\mathcal{B}$ is a basis for V if and only if $f(\mathcal{B})$ is a basis for W.*
5. *$\dim V = \dim W$.*

Proof. Part (5) is an immediate consequence of Part (4), and Part (4) follows directly from (2) and (3).

Let us now prove that (1) holds. Writing $\boldsymbol{x} = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}^T$, we find that $A\boldsymbol{x} = \boldsymbol{b}$ and $f(A)\boldsymbol{x} = f(\boldsymbol{b})$ can be rewritten as

$$\boldsymbol{x}_1\boldsymbol{v}_1 + \cdots + x_k\boldsymbol{v}_k = \boldsymbol{b} \quad \text{and} \quad \boldsymbol{x}_1 f(\boldsymbol{v}_1) + \cdots + x_k f(\boldsymbol{v}_k) = f(\boldsymbol{b}).$$

If the left equation holds, then applying f leads to the equation on the right. Conversely, applying f^{-1} to the equation on the right returns the equation on the left. This proves (1).

We now argue that both (2) and (3) follow from Part (1).

By (1), $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b} \in V$ if and only if $f(A)\boldsymbol{x} = \boldsymbol{w}$ has a solution for each $\boldsymbol{w} \in W$. This is because f is onto, so every $\boldsymbol{w} \in W$ can be written in the form $\boldsymbol{w} = f(\boldsymbol{b})$ for some $\boldsymbol{b} \in V$. Equivalently,

$$\mathcal{R}(A) = V \quad \iff \quad \mathcal{R}(f(A)) = W.$$

Since $\mathcal{R}(A) = \operatorname{span}(\mathcal{B})$ and $\mathcal{R}(f(A)) = \operatorname{span}(f(\mathcal{B}))$, the above is precisely the statement of (2).

Now let us show that (3) follows from (1). Since f is a linear transformation, it satisfies $f(\mathbf{0}) = \mathbf{0}$. Indeed,

$$f(\mathbf{0}) = f(\mathbf{0} - \mathbf{0}) = f(\mathbf{0}) - f(\mathbf{0}) = \mathbf{0}.$$

Hence, by (1), $\boldsymbol{x}$ is a solution of $A\boldsymbol{x} = \mathbf{0}$ if and only if it is a solution of $f(A)\boldsymbol{x} = \mathbf{0}$. In particular, $\mathcal{N}(A) = \mathcal{N}(f(A))$. The set $\mathcal{B}$ is linearly independent if and only if the left-hand side is $\{\mathbf{0}\}$, and $f(\mathcal{B})$ is linearly independent if and only if the right-hand side is $\{\mathbf{0}\}$. This implies the equivalence (3). □

Note. End of Lecture 26 (03/27/20).

3. Diagonalization, determinants and eigenvalues

3.1. **Motivation.** When studying linear transformations $f : \mathbb{R}^n \to \mathbb{R}^n$, we saw that it was particularly useful when we could find a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ for $\mathbb{R}^n$ such that $[f]_{\mathcal{B},\mathcal{B}}$ is a *diagonal matrix* D:

$$[f]_{\mathcal{B},\mathcal{B}} = D = \begin{pmatrix} \lambda_1\boldsymbol{e}_1 & \cdots & \lambda_n\boldsymbol{e}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \text{for } \lambda_i \in \mathbb{R}.$$

Let's determine what it means for the above to be true in terms of the matrix $A \in \mathrm{M}_n(\mathbb{R})$ associated to f. By Corollary 2.106, f satisfies

$$f = T_{\mathcal{B}}^{-1} \circ S \circ T_{\mathcal{B}}, \tag{3.1}$$

where $S : \mathbb{R}^n \to \mathbb{R}^n$ is given by $S(\boldsymbol{x}) = D\boldsymbol{x}$. Moreover, by (2.6), $T_{\mathcal{B}} : \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation with matrix P^{-1}, where

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}.$$

The equation (3.1) is therefore equivalent to the matrix identity

$$A = PDP^{-1} = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}. \tag{3.2}$$

When A admits a decomposition of this form, there are many interesting conclusions we can draw for A (and thus f) using the much simpler diagonal matrix D. A few are given by the following list:

(1) For any $k \geq 1$, we have
$$A^k = PD^kP^{-1} = P \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} P^{-1}.$$
Indeed, this can be shown using the same reasoning as given in Example 2.96.

(2) A is invertible if and only if $\lambda_i \neq 0$ for each $1 \leq i \leq n$. In this case,
$$A^{-1} = PD^{-1}P^{-1} = P \begin{pmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{pmatrix} P^{-1}$$
This genearal fact is established exactly as in Example 2.109.

(3) You have seen in calculus that the exponential function e^x can be represented as a Taylor series:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots.$$
One can use this series to define the *matrix exponential* $\exp(B)$ of $B \in \mathrm{M}_n(\mathbb{R})$:
$$\exp(B) = \sum_{n=0}^{\infty} \frac{B^n}{n!}.$$
It is certainly not clear that the right-hand side converges to a matrix in $\mathrm{M}_n(\mathbb{R})$, though this is indeed the case. For our matrix A it is easy to show this, and we can even compute $\exp(A)$:
$$\begin{aligned} \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{PD^nP^{-1}}{n!} &= P \left(\sum_{n=0}^{\infty} \frac{D^n}{n!} \right) P^{-1} \\ &= P \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix} P^{-1} \in \mathrm{M}_n(\mathbb{R}). \end{aligned}$$
The matrix exponential $y(t) = e^{tA}$ is the unique solution of the differential equation
$$\frac{d}{dt} y(t) = Ay(t), \quad \text{with } \ y(0) = I,$$

so you should be able to believe that our decomposition for A has important applications in the theory of differential equations.

The above points barely scratch the surface when it comes to illustrating the value of a decomposition of the form (3.2) with D a diagonal matrix. They will, however, suffice to motivate our main goal of this section, which is determine when and how one can find a basis $\mathcal{B}$ such that $D = [f]_{\mathcal{B},\mathcal{B}}$ is diagonal, as above.

3.2. **Definitions and general theory.** For a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$, the equality

$$[f]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} \lambda_1 \boldsymbol{e}_1 & \cdots & \lambda_n \boldsymbol{e}_n \end{pmatrix}$$

is equivalent to $[f(\boldsymbol{v}_i)]_{\mathcal{B}} = \lambda_i \boldsymbol{e}_i$ for all $1 \leq i \leq n$. This means exactly that

$$f(\boldsymbol{v}_i) = \lambda_i \boldsymbol{v}_i \quad \text{for all} \quad 1 \leq i \leq n.$$

If $A \in \mathrm{M}_n(\mathbb{R})$ is the matrix associated to f, then we can rewrite the above as

$$A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \quad \text{for all} \quad 1 \leq i \leq n.$$

This motivates our first definitions of this section:

Definition 3.1. $\lambda \in \mathbb{R}$ is called an *eigenvalue* of $A \in \mathrm{M}_n(\mathbb{R})$ if there is a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ with

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

In this case, $\boldsymbol{v}$ is called an *eigenvector* for λ, or simply a λ-*eigenvector*.

Eigenvalues and eigenvectors can also be characterized in terms of concepts we are very familiar with; namely, singular matrices and null spaces. The following lemma makes this precise.

Lemma 3.2. *Let $A \in \mathrm{M}_n(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$ be a nonzero vector. Then:*

(1) *λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ is singular.*
(2) *$\boldsymbol{v}$ is a λ-eigenvector if and only if $\boldsymbol{v} \in \mathcal{N}(A - \lambda I)$.*

Proof. By definition, $A - \lambda I$ is singular if and only if there is a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ with

$$(A - \lambda I)\boldsymbol{v} = A\boldsymbol{v} - \lambda \boldsymbol{v} = \mathbf{0}.$$

The statement of the lemma therefore follows from the observation that the above equality is equivalent to $A\boldsymbol{v} = \lambda \boldsymbol{v}$. □

The null space which appears in the above lemma plays an important role, and deserves its own name.

Definition 3.3. Given an eigenvalue λ of $A \in \mathrm{M}_n(\mathbb{R})$, the subspace

$$E_\lambda = \mathcal{N}(A - \lambda I)$$

is called the *eigenspace* for λ, or λ-*eigenspace*.

Note that, by the previous lemma, E_λ is precisely the set of all λ-eigenvectors together with the vector $\mathbf{0}$. Now let us take a look at a few examples.

Example 3.4.

(1) Let $A \in \mathrm{M}_n(\mathbb{R})$ be the diagonal matrix
$$A = \begin{pmatrix} \lambda_1 \boldsymbol{e}_1 & \cdots & \lambda_n \boldsymbol{e}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$
Then each λ_i is an eigenvalue for A with eigenvector $\boldsymbol{e}_i$:
$$A\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i \quad \text{for all } \ 1 \le i \le n.$$

(2) The 2×2 matrix
$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$
has eigenvalue $\lambda = 2$. This is because $A - 2I$ is singular:
$$A - 2I = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$
which has linearly dependent columns, and is thus singular, as claimed. One can check that the vector form of the general solution to $(A - 2I)\boldsymbol{x} = \mathbf{0}$ is
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
so the eigenspace E_2 is spanned by the single vector $\boldsymbol{v} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. In particular, this vector is an eigenvector for the eigenvalue 2:
$$A\boldsymbol{v} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\boldsymbol{v}.$$

Remark 3.5. Since the eigenvectors for an eigenvalue λ of A are just the nonzero vectors in $E_\lambda = \mathcal{N}(A - \lambda I)$, we can find all of them by computing a basis for this null space, using our usual methods. At this point, the more difficult problem for us is to find a way of determining all eigenvalues λ for A. To do this, we will develop the theory of *determinants* later in this section.

We now introduce terminology relevant to the decomposition (3.2) at the heart of our main goal.

Definition 3.6. $A \in \mathrm{M}_n(\mathbb{R})$ is said to be *diagonalizable* if there is a diagonal matrix
$$D = \begin{pmatrix} \lambda_1 \boldsymbol{e}_1 & \cdots & \lambda_n \boldsymbol{e}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
and an invertible matrix $P = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$ such that
$$A = PDP^{-1}. \tag{3.3}$$

Remark 3.7. If A is diagonalizable as in (3.3), then λ_i is an eigenvalue for A with eigenvector $\boldsymbol{v}_i$. To see why this is the case, let us rewrite (3.3) as

$$AP = PD.$$

By definition of matrix multiplication, the left-hand side is

$$AP = \begin{pmatrix} A\boldsymbol{v}_1 & \cdots & A\boldsymbol{v}_n \end{pmatrix},$$

while the right-hand side is

$$PD = \begin{pmatrix} \lambda_1 P\boldsymbol{e}_1 & \cdots & \lambda_n P\boldsymbol{e}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \boldsymbol{v}_1 & \cdots & \lambda_n \boldsymbol{v}_n \end{pmatrix}.$$

Comparing both sides, we obtain the desired equalities

$$A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \quad \text{for all } \ 1 \leq i \leq n.$$

We now come to our first main theorem of this section, which gives us both sufficient and necessary conditions for a matrix to be diagonalizable.

Theorem 3.8. *$A \in \mathrm{M}_n(\mathbb{R})$ is diagonalizable if and only if there is a basis $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ consisting of eigenvectors for A. In this case, we have*

$$A = PDP^{-1},$$

where $P = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 \boldsymbol{e}_1 & \cdots & \lambda_n \boldsymbol{e}_n \end{pmatrix}$, with λ_i the eigenvalue corresponding to $\boldsymbol{v}_i$.

Proof. If A is diagonalizable, as in (3.3), then $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ a basis for $\mathbb{R}^n$ (since P is invertible), which by the above remark consists of eigenvectors for A.

Conversely, assume that $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis of eigenvectors for A, so $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for each $1 \leq i \leq n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with matrix A. Then $f(\boldsymbol{v}_i) = A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for each i, so

$$[f]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} [f(\boldsymbol{v}_1)]_{\mathcal{B}} & \cdots & [f(\boldsymbol{v}_n)]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \boldsymbol{v}_1 & \cdots & \lambda_n \boldsymbol{v}_n \end{pmatrix} = D.$$

It then follows as in (3.2) that A is diagonalizable with $A = PDP^{-1}$. □

By the theorem, showing a matrix A is diagonalizable amounts to finding a basis for $\mathbb{R}^n$ consisting of eigenvectors for A. But how do we construct such a basis? The next lemma provides a first glimpse into how this could be done, provided we have some additional information about the eigenvalues of A.

Lemma 3.9. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in \mathbb{R}^n$ be eigenvectors for distinct eigenvalues $\lambda_1, \ldots, \lambda_\ell$, respectively, of $A \in \mathrm{M}_n(\mathbb{R})$. Then*

$$\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$$

is a linearly independent set.

Proof. Suppose that $k_1, \ldots, k_n \in \mathbb{R}$ satisfy the relation

$$\mathbf{0} = k_1\boldsymbol{v}_1 + \cdots + k_\ell \boldsymbol{v}_\ell.$$

We need to show that $k_i = 0$ for all $1 \le i \le \ell$. We begin by noting that

$$(A - \lambda_j I)\boldsymbol{v}_i = A\boldsymbol{v}_i - \lambda_j \boldsymbol{v}_i = (\lambda_i - \lambda_j)\boldsymbol{v}_i,$$

and that, since $\lambda_i \neq \lambda_j$ for $i \neq j$, the coefficient $(\lambda_i - \lambda_j)$ is nonzero if and only if $i \neq j$. Consider now the product

$$B_i = \prod_{a \neq i}(A - \lambda_a I) = (A - \lambda_1 I)\cdots(A - \lambda_{i-1} I)(A - \lambda_{i+1} I)\cdots(A - \lambda_n I).$$

By the above observation,

$$B_i \boldsymbol{v}_j = \prod_{a \neq i}(\lambda_j - \lambda_a)\boldsymbol{v}_j \quad \text{for all } \ 1 \le j \le \ell.$$

If $j \neq i$, then the product $\prod_{a \neq i}(\lambda_j - \lambda_a)$ contains the factor $\lambda_j - \lambda_j = 0$, and hence is zero. Therefore, left-multiplying (3.2) by B_i yields

$$\mathbf{0} = B_i(k_1\boldsymbol{v}_1 + \cdots + k_\ell\boldsymbol{v}_\ell) = k_i B_i \boldsymbol{v}_i = k_i \prod_{a \neq i}(\lambda_i - \lambda_a)\boldsymbol{v}_i.$$

As $\prod_{a \neq i}(\lambda_i - \lambda_a) \neq 0$ and $\boldsymbol{v}_i \neq \mathbf{0}$, this forces $k_i = 0$, as desired. $\square$

One nice application of this lemma is the below sufficient condition for $A \in \mathrm{M}_n(\mathbb{R})$ to be diagonalizable.

Corollary 3.10. *Let $A \in \mathrm{M}_n(\mathbb{R})$ be a matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then A is diagonalizable.*

Proof. For each $1 \le i \le n$, let $\boldsymbol{v}_i \in E_{\lambda_i}$ be an eigenvector. By Lemma 3.9, $\mathcal{B} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent, and thus a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. By Theorem 3.8, A is diagonalizable. $\square$

Note. End of Lecture 27 (03/30/2020).

Example 3.11.

(1) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The only eigenvalue of this matrix is $\lambda = 1$. Indeed, we have

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix},$$

which is singular if and only if $\lambda = 1$ (otherwise $A - \lambda I$ has rank 2 and is therefore invertible).

By Theorem 3.8, A is diagonalizable if and only if there is a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ for $\mathbb{R}^2$ with $\boldsymbol{v}_1, \boldsymbol{v}_2$ both eigenvectors for A. Since A has the single eigenvalue

$\lambda = 1$, this happens if and only if $\dim E_1 = \text{nullity}(A - I) = 2$. However, we have

$$A - I = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

which has nullity 1. Therefore, A is *not* diagonalizable.

(2) Consider now the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

This matrix has eigenvalues 1 and 3, because

$$A - I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \quad A - 3I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$$

are both singular matrices. Since A is 2×2 and has 2 distinct eigenvalues, it *is* diagonalizable by Corollary 3.10.

To find P and D such that $A = PDP^{-1}$, we need a basis $\mathcal{B}$ for $\mathbb{R}^2$ consisting of eigenvectors for A. We will do this by finding a basis $\mathcal{B}_1$ for E_1, a basis $\mathcal{B}_3$ for E_3, and then gluing them together.

a) Basis $\mathcal{B}_1$ for E_1:

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \xrightarrow[\frac{1}{2}R_1]{R_2 - R_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

From this, we deduce that the vector form of the general solution to $(A - I)\boldsymbol{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, we may take $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

b) Basis $\mathcal{B}_3$ for E_3:

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

From this reduced echelon form, one readily deduces that $\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for E_3.

We then obtain a basis $\mathcal{B}$ of $\mathbb{R}^2$ by taking the union of $\mathcal{B}_1$ and $\mathcal{B}_3$:

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

In this particular case, the linear independene of $\mathcal{B}$ is clear. More generally, it would follow from Lemma 3.9, which tells us that eigenvectors for different eigenvalues are linearly independent.

By Theorem 3.8, we then have $A = PDP^{-1}$ for

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Let's check this directly, using that

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We then have

$$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = A.$$

Our current goal is to formalize the previous example into a concrete algorithm for diagonalizing a matrix, given its set of eigenvalues. We begin with the below lemma.

Lemma 3.12. *Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of $A \in \mathrm{M}_n(\mathbb{R})$, and let $\mathcal{B}_{\lambda_i}$ be a basis for the eigenspace E_{λ_i}. Then the set*

$$\mathcal{B} = \mathcal{B}_{\lambda_1} \cup \cdots \cup \mathcal{B}_{\lambda_k} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in \mathcal{B}_{\lambda_i} \ \textit{for some} \ 1 \leq i \leq k\}$$

is linearly independent. In particular, if

$$\sum_{i=1}^{k} \dim E_{\lambda_i} = n,$$

then A is diagonalizable.

Proof. The main idea behind the proof is as follows. If $\mathcal{B}$ were not linearly independent then, using the linear independence of each $\mathcal{B}_{\lambda_i}$, one deduces that there would be a relation of the form

$$\mathbf{0} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_k$$

with $\boldsymbol{v}_i \in E_{\lambda_i}$, not all equal to zero. By Lemma 3.9, this is impossible because any set of eigenvectors corresponding to distinct eigenvalues is linearly independent. Therefore $\mathcal{B}$ must be linearly independent.

The sum $\sum_{i=1}^{k} \dim E_{\lambda_i}$ is the number of vectors in the set $\mathcal{B}$. If this number is n, then $\mathcal{B}$ is a linearly independent set of size n in $\mathbb{R}^n$, and is therefore a basis for $\mathbb{R}^n$. By Theorem 3.8, we can then conclude that A is diagonalizable. □

Definition 3.13. Given an eigenvalue λ of $A \in \mathrm{M}_n(\mathbb{R})$ the number $\dim E_\lambda$ is called the *geometric multiplicity* of λ, and denoted m_λ:

$$m_\lambda = \dim E_\lambda = \operatorname{nullity}(A - \lambda I).$$

The converse to the above lemma also holds, and this is precisely the statement of the following theorem, which provides a very useful refinement of Theorem 3.8.

Theorem 3.14. *Let $\lambda_1, \ldots, \lambda_k$ be all the distinct eigenvalues of $A \in \mathrm{M}_n(\mathbb{R})$, and let $\mathcal{B}_{\lambda_i} \subset E_{\lambda_i}$ be bases, as above. Then A is diagonalizable if and only if*

$$\sum_{i=1}^{k} m_{\lambda_i} = n.$$

In this case, $\mathcal{B} = \mathcal{B}_{\lambda_1} \cup \cdots \cup \mathcal{B}_{\lambda_k}$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors for A.

Half of the proof of the theorem is immediate from Lemma 3.12. Though a full proof is within our grasp, we won't produce one here, and instead leave it to the reader to think about some of the remaining subtleties.

Algorithm. Suppose $A \in \mathrm{M}_n(\mathbb{R})$ has eigenvalues $\lambda_1, \ldots, \lambda_k$. To diagonalize A, we proceed as follows:

Step 1. Find the geometric multiplicities m_{λ_i}. If $\sum_{i=1}^{k} m_{\lambda_i} \neq n$, A is not diagonalizable, and we can stop. Otherwise it is, and we proceed to *Step* 2.

Step 2. For each $1 \leq i \leq k$, find a basis $\mathcal{B}_{\lambda_i}$ for the eigenspace E_{λ_i}. The set

$$\mathcal{B} = \mathcal{B}_{\lambda_1} \cup \cdots \cup \mathcal{B}_{\lambda_k}$$

is then a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Step 3. By Theorem 3.8, we then have $A = PDP^{-1}$, where the columns of P are given by the vectors in $\mathcal{B}$, taken in order, and D is the block diagonal matrix

$$D = \begin{pmatrix} \boxed{\lambda_1 I_{m_{\lambda_1}}} & & \\ & \ddots & \\ & & \boxed{\lambda_k I_{m_{\lambda_k}}} \end{pmatrix},$$

so the eigenvalue λ_i appears m_{λ_i} times on the diagonal. For instance, one has

$$\begin{pmatrix} \boxed{7I_2} & & \\ & \boxed{2I_1} & \\ & & \boxed{4I_1} \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Example 3.15. The matrix

$$A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

has eigenvalues $\lambda = 2$ and $\lambda = 4$.

Step 1. We have

$$A - 2I = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad A - 4I = \begin{pmatrix} -2 & 0 & 2 & 4 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices both have nullity 2, so we can conclude that $m_2 = 2 = m_4$. Since $m_2 + m_4 = 4 = n$, A is diagonalizable.

Step 2. We now proceed to find bases $\mathcal{B}_2$ and $\mathcal{B}_4$ of E_2 and E_4, respectively.

a) Basis $\mathcal{B}_2$ of E_2: row reducing $A - 2I$ to reduced echelon form we obtain

$$\begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we readily find that a basis $\mathcal{B}_2$ for $E_2 = \mathcal{N}(A - 2I)$ is given by

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

b) Basis $\mathcal{B}_4$ of E_4: row reducing $A - 4I$ to reduced echelon form we obtain

$$\begin{pmatrix} -2 & 0 & 2 & 4 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-\frac{1}{2}R_2]{-\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the vector form of the general solution to $(A - 4I)\boldsymbol{x} = \mathbf{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors on the right-hand side give us a basis $\mathcal{B}_4$ of E_4.

Step 3. Hence, we may conclude that $A = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 4 & \\ & & & 4 \end{pmatrix}.$$

Note. End of Lecture 28 (04/01/2020).

3.3. **Determinants.** We now know exactly how to diagonalize a matrix, given a complete list of its eigenvalues. However, we do not yet have a very good method for finding these eigenvalues in the first place. Our main goal in this section is to resolve this issue, and we will do this by developing the theory of *determinants*.

3.3.1. *Motivation: the $n = 2$ case.* Suppose we are given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We want to find all the eigenvalues of this matrix. By Lemma 3.2, λ is an eigenvalue of A if and only if $A - \lambda I$ is a singular matrix. By Proposition 1.102 this happens if and only if

$$\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

This reduces the problem of finding eigenvalues of A to finding the roots of a quadratic polynomial.

Definition 3.16. The polynomial $p_A(t) = \det(A - tI)$ is called the *characteristic polynomial* of $A \in \mathrm{M}_2(\mathbb{R})$.

Let's rephrase the above observation armed with our new terminology:

Proposition 3.17. *Let $A \in \mathrm{M}_2(\mathbb{R})$, then λ is an eigenvalue for A if and only if it is root of the characteristic polynomial of A: $p_A(\lambda) = 0$.*

The polynomial $p_A(t)$ is a quadratic polynomial of the form

$$p_A(t) = t^2 + \beta t + \kappa.$$

It therefore factors as $p_A(t) = (t - \lambda_1)(t - \lambda_2)$, where the λ_i are given by the quadratic formula

$$\lambda_1 = \frac{-\beta + \sqrt{\beta^2 - 4\kappa}}{2} \quad \text{and} \quad \lambda_1 = \frac{-\beta - \sqrt{\beta^2 - 4\kappa}}{2}.$$

Hence, the above proposition gives an entirely complete method for finding the eigenvalues of any matrix $A \in \mathrm{M}_2(\mathbb{R})$.

Remark 3.18. We note, however, that since we are currently interested in eigenvalues which take values in $\mathbb{R}$, it is not necessarily the case that a given matrix $A \in \mathrm{M}_2(\mathbb{R})$ will have an eigenvalue. If we were instead working over the complex numbers $\mathbb{C}$, this would not be an issue. We will come back to this point later.

Example 3.19.

(1) Consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It's characteristic polynomial is given by

$$p_A(t) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1 = (x - i)(x + i).$$

Hence, A has no eigenvalues in $\mathbb{R}$, and therefore is not diagonalizable (over $\mathbb{R}$).

(2) Consider now the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}.$$

We then have

$$p_A(t) = (1-t)(-2-t) - 4 = t^2 + t - 6 = (t+3)(t-2).$$

Hence, A has eigenvalues $\lambda = -3$ and $\lambda = 2$. As A has two distinct eigenvalues, it is diagonalizable by Corollary 3.10.

Next, let us find a bases for $E_{-3} = \mathcal{N}(A+3I)$ and $E_2 = \mathcal{N}(A-2I)$:

a) Basis $\mathcal{B}_{-3}$ of E_{-3}:

$$A + 3I = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \xrightarrow[\frac{1}{4}R_1]{R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}.$$

Hence, the general solution to $(A+3I)\boldsymbol{x} = \mathbf{0}$ and its vector form are given by

$$\begin{aligned} x_1 &= -\tfrac{1}{4}x_2 \\ x_2 &\text{ is free} \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}.$$

We may thus take $\mathcal{B}_{-3} = \left\{ \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} \right\}$.

b) Basis $\mathcal{B}_2$ for E_2:

$$A - 2I = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \xrightarrow[-R_1]{R_2 + 4R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

from which we find $\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

It follows from the above that

$$A = P \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}, \quad \text{where} \quad P = \begin{pmatrix} -\frac{1}{4} & 1 \\ 1 & 1 \end{pmatrix}.$$

3.3.2. *Alternating forms and determinant functions.* We now want to construct a function $\det : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ which generalizes the determinant function on $\mathrm{M}_2(\mathbb{R})$.

Given $A \in \mathrm{M}_n(\mathbb{R})$, we will write

$$A_{R_i \leftrightarrow R_j}, \quad A_{kR_i} \quad \text{and} \quad A_{R_i + kR_j}$$

for the matrice obtained from A by applying the row operations $R_i \leftrightarrow R_j$, $R_i \to kR_i$ and $R_i \mapsto R_i + kR_j$, respectively.

Definition 3.20. An *alternating form* f on $\mathrm{M}_n(\mathbb{R})$ is a function $f : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ satisfying the three conditions

(1) $f(A_{R_i \leftrightarrow R_j}) = -f(A)$,
(2) $f(A_{kR_i}) = kf(A)$,

(3) $f(A_{R_i+kR_j}) = f(A)$,

for $1 \leq i, j \leq n$ and nonzero $k \in \mathbb{R}$. An alternating form f is called a *determinant function* if in addition it satisfies

$$f(I) = 1.$$

Note that the determinant $\det : \mathrm{M}_2(\mathbb{R}) \to \mathbb{R}$ is indeed a determinant function, as has essentially been verified in the course of Example 2.18. The above properties are imposed so as to generalize how this 2×2 determinant behaves with respect to row operations.

The definition of alternating form is already quite strong, and the next lemma demonstrates a few important conclusions that can be drawn from it with little effort.

Lemma 3.21. *Suppose that f is an alternating form, and let $A \in \mathrm{M}_n(\mathbb{R})$ with reduced echelon form C.*

(1) *If A has a row of zeros, then $f(A) = 0$.*
(2) *If $C \neq I$, then $f(C) = 0$.*
(3) *There is a a nonzero number $n_A \in \mathbb{R}$ with*

$$f(A) = n_A f(C).$$

(4) *If A is singular, then $f(A) = 0$.*

Proof. Suppose that the i-th row R_i of A is zero. Then, for each nonzero $k \in \mathbb{R}$, we have $A = A_{kR_i}$, and therefore (2) yields

$$f(A) = f(A_{kR_i}) = kf(A).$$

This is only possible if $f(A) = 0$, which proves (1).

An $n \times n$ matrix C in reduced echelon form is either equal to I or has less than n leading entries, and thus a row of all zeros. Therefore, Part (2) follows from Part (1).

By Definition 3.20, any sequence of row operations applied to A will only change $f(A)$ by a nonzero scalar factor. This observation implies Part (3).

Finally, to prove Part (4), note that if A is singular then $C \neq I$, so combining (2) and (3) gives

$$f(A) = n_A f(C) = 0. \qquad \square$$

Using this lemma, we can prove that if a determinant function exists, then it is unique and necessarily satisfies the correct generalization of Proposition 1.102.

Theorem 3.22.

(1) *If f is a determinant function, then $A \in \mathrm{M}_n(\mathbb{R})$ is nonsingular if and only if $f(A) \neq 0$.*
(2) *There is at most one determinant function.*

Proof. The matrix A is nonsingular if and only if has reduced form equal to I. Since $f(I) = 1$, Parts (2) and (3) of Lemma 3.21 imply that this happens exactly when $f(A) \neq 0$. This proves Part (1).

Suppose now that f and g are two determinant functions. Consider the function

$$h = f - g : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}.$$

Since f and g are alternating forms, so is h. If $A \in \mathrm{M}_n(\mathbb{R})$ is singular, then $h(A) = 0$ by Part (4) of Lemma 3.21. If A is nonsingular, Part (3) of the lemma instead gives

$$h(A) = n_A h(I) = n_A(f(I) - g(I)) = n_A(1 - 1) = 0,$$

for some nonzero number n_A. It follows that h is the zero function, so $f = g$. $\square$

Note. End of Lecture 29 (04/03/2020).

Note that Theorem 3.22 does not allow us to conclude that a determinant function actually exists. Such a function does indeed exist, and we will return to this shortly. For the moment, we will take this for granted, which allows us to make the following definition.

Definition 3.23. The unique determinant function $\det : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is called the *determinant* on $\mathrm{M}_n(\mathbb{R})$.

Even though we haven't given a concrete definition for $\det(A)$, we can use row reduction and the properties of Definition 3.20 to compute its value. This is illustrated in the following set of examples.

Example 3.24.

(1) Let's first return to the $n = 2$ case. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

We then have

$$\det(A) \xlongequal{R_2 - 3R_1} \det \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xlongequal{R_1 + R_2} \det \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \xlongequal{-\frac{1}{2}R_2} -2 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -2.$$

(2) Consider now the 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 3 & 6 & 4 \end{pmatrix}.$$

Using the properties of Definition 3.20, we obtain

$$\det(A) \xlongequal{R_3 - 3R_2} \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xlongequal[R_2-2R_1]{R_2-R_3} \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xlongequal{R_1 \leftrightarrow R_2} -\det(I) = -1.$$

We now introduce a special class of matrices whose determinants can be computed particularly easily.

Definition 3.25. A matrix $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ is called *upper triangular* if $a_{ij} = 0$ for all $i > j$. That is, if A takes the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

Proposition 3.26. *Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ be an upper triangular matrix as above. Then*

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11}a_{22}\cdots a_{nn}.$$

Proof. Suppose first that there is an index $1 \leq i \leq n$ such that $a_{ii} = 0$. In this case, the reduced echelon form of A will have at most $n-1$ leading entries. Therefore A is singular and the first part of Theorem 3.22 gives $\det(A) = 0$, as desired.

Suppose instead that each a_{ii} is nonzero. Setting $B = (b_{ij})$, where $b_{ij} = \frac{1}{a_{ii}} a_{ij}$ for each i, j, we have

$$\begin{aligned} \det(A) &\xlongequal[\text{for each } i]{\frac{1}{a_{ii}} R_i} a_{11}a_{22}\cdots a_{nn} \det(B) \\ &= a_{11}a_{22}\cdots a_{nn} \det\begin{pmatrix} 1 & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \end{aligned}$$

The matrix B is nonsingular and can be reduced to I by only applying operations of the type $R_i \mapsto R_i + kR_j$. As these operations do not change the value of the determinant, we have

$$\det(A) = a_{11}a_{22}\cdots a_{nn} \det(I) = a_{11}a_{22}\cdots a_{nn}. \qquad \square$$

The above proposition has several useful consequences. For instance, it tells us exactly what the eigenvalues of any upper triangular matrix are, as we shall make precise later on. It can also be applied to simplify computations of determinants using Gaussian elimination, as illustrated in the next example.

Example 3.27. Consider the 4×4 matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

Using row reduction, we obtain

$$\det(A) \xlongequal[R_4-R_1]{R_2+2R_1, R_3+R_1} \det \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \xlongequal[R_4-R_2]{R_3+2R_2} \det \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & -2 & -5 \end{pmatrix}$$

$$\xlongequal[R_3 \leftrightarrow R_4]{R_3+3R_4} - \det \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & -2 & -5 \\ 0 & 0 & 0 & -7 \end{pmatrix} = -(-1)(-1)(-2)(-7) = -14.$$

3.3.3. *Existence and cofactor expansion.* So far, we have developed the theory of determinants without knowing that a determinant function actually exists, only using the properties of Definition 3.20 and their consequences. In this subsection, we settle the issue of existence. This is done by providing a formula for $\det(A)$, which can be shown to satisfy the properties of Definition 3.20.

We begin by introducing some auxiliary notation.

Definition 3.28. Let $A \in \mathrm{M}_n(\mathbb{R})$, and fix $1 \leq i, j \leq n$. Then the (i,j)-th *minor matrix* of A is the matrix $A_{ij} \in \mathrm{M}_{n-1}(\mathbb{R})$ obtained from A by deleting the i-th row and j-th column.

So, for instance, if $A \in \mathrm{M}_3(\mathbb{R})$ is the matrix

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix},$$

then the minor matrices A_{11}, A_{23} and A_{31} are given by

$$A_{12} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 2 & 4 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad A_{32} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

Theorem 3.29. *There exists a unique determinant function*

$$\det : \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}.$$

Explicitly, for each $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$, $\det(A)$ is defined by the recursive formula

$$\det(A) = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(A_{ij}) \tag{3.4}$$

for any fixed index $1 \leq j \leq n$.

To prove Theorem 3.29, it suffices to prove that, for any $1 \leq j \leq n$, the recursive formula (3.4) defines a determinant function, as in Definition 3.20. This would show that a determinant function exists, and by uniqueness is given by (3.4) for any $1 \leq j \leq n$. This verification can be accomplished using an inductive argument, but it is perhaps a bit outside the scope of our course, and is therefore omitted.

Definition 3.30. The number

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

is called the (i, j)-th *cofactor* of A, and the expansion (3.4) is called the *cofactor expansion* of $\det(A)$ along the j-th column.

Let's take a look at how cofactor expansions along columns can be used to compute determinants of matrices.

Example 3.31.

(1) Consider an arbirary 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Let's compute the cofactor expansion of $\det(A)$ along the j-th column with $j = 1$. We have

$$\det(A) = (-1)^2 a_{11} \det(A_{11}) + (-1)^3 a_{21} \det(A_{21}) = a_{11}a_{22} - a_{21}a_{12},$$

where we have used that $\det(A_{11}) = \det(a_{22}) = a_{22} \det(1) = a_{22}$, and similarly $\det(A_{21}) = a_{12}$.

(2) Consider again the matrix A appearing above Theorem 3.29:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}.$$

To compute $\det(A)$ using the cofactor expansion (3.4), it is easiest to use the $j = 2$ column, as it has the most zero entries. We have

$$\begin{aligned} \det(A) &= a_{12}(-1)^3 \det(A_{12}) + a_{22}(-1)^4 \det(A_{22}) + a_{32}(-1)^5 \det(A_{32}) \\ &= -\det \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = -(4-4) = 0. \end{aligned}$$

(3) Consider now 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 4 & 6 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Using the first column, we obtain

$$\det(A) = -\det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Using again the first column to compute this determinant, we find that

$$\det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - 2\det \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} = 1 - 2(-2) = 5.$$

Therefore, we can conclude that $\det(A) = -5$.

Note. End of Lecture 30 (04/06/2020).

References

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Department of Mathematics, The Ohio State University.

Email address: `Wendlandt.4@osu.edu`