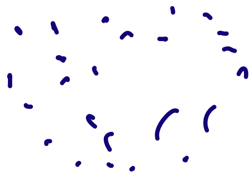
Simplicial complexes from a point cloud & The Nerve Theorem



$\longrightarrow$ simplicial complex

- should somehow preserve the "shape" of the cloud
- some stability

(Vietoris) Rips Complex

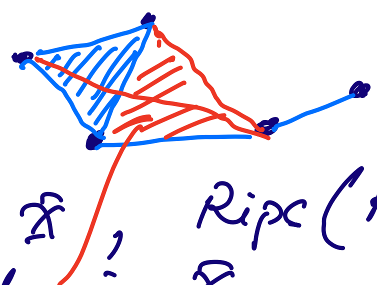
Let F be a finite subset of metric space X, and $r \geq 0$ be a scale. The Rips complex $Rips(F, r)$ is an abstract simplicial complex defined as:

- The vertex set is F
- $\sigma \subset F$ is a simplex iff $\mathrm{diam}(\sigma) \leq r$

Recall: an abstract scx K on a finite set F is a set of non-empty subsets of F s.t. if $\sigma \in K$ & $\phi \neq \tau \subset \sigma$ then $\tau \in K$

Remark. ⓪ clearly $Rips(F, r)$ is a scx ✓

① easy to visualize & compute



as $r \nearrow$

② If $F \subset X$, $Rips(F, r)$ may not be embeddable in X

③ if $r < \min\limits_{x,y \in F} d(x,y)$, $Rips(F, r) = F$

If $r \geq \mathrm{diam}(F)$, $Rips(F, r) = (|F| - 1)$ simplex

(4) $r < r' \Rightarrow \mathrm{Rips}(F,r) \subset \mathrm{Rips}(F,r')$

(5) we have a filtration based on F: $\{\mathrm{Rips}(F,r)\}_{r\geq 0}$ w/ $\mathrm{Rips}(F,r) \xhookrightarrow{i_{r,r'}} \mathrm{Rips}(F,r')$
$r<r'$

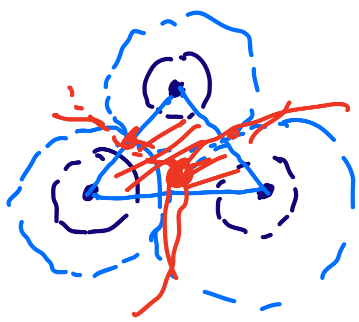
filtration of $(|F|-1)$ scx

Q: What does $\{\mathrm{Rips}(F,r)\}$ preserve of F?

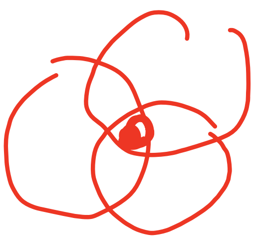
Čech Complex:

Once again, $F \subset \hat{X}$ finite, & $r \geq 0$. The Čech cx is an abstract scx defined as:

- vertex set $= F$
- $\sigma \subset F$ is a sx $\iff \bigcap_{x\in\sigma} B(x,r) \neq \emptyset$



Q: at which r? How?



- not so simple to compute ← ! [Miniball algo]
- if $r < \min_{x,y\in F} \frac{1}{2} d(x,y)$, $\check{C}\mathrm{ech}(F(x,r)) = F$

  if $r \geq 2 \max_{x,y\in F} d(x,y)$, $\check{C}\mathrm{ech}(F(x,r)) = (|F|-1)$ simplex
- once again, $\{\check{C}\mathrm{ech}(F,r)\}_{r\geq 0}$ is a filtration w/ inclusion maps.

Further remarks:

(1) observe $\check{C}ech(F, r) \subset Rips(F, 2r)$ } HW(?)

(2) " $Rips(F, r) \subset \check{C}ech(F, r)$ }

* (3) prove $Rips(F, \sqrt{2}r) \subset \check{C}ech(F, r)$,

challenging on Euclidean space $\mathbb{R}^n$. [If stuck, look up Jung's thm]

Q: What does $\{\check{C}ech(F, r)\}$ preserve of F?

Algebraic Topology → the nerve construction.

(Alexandrov ~1930)

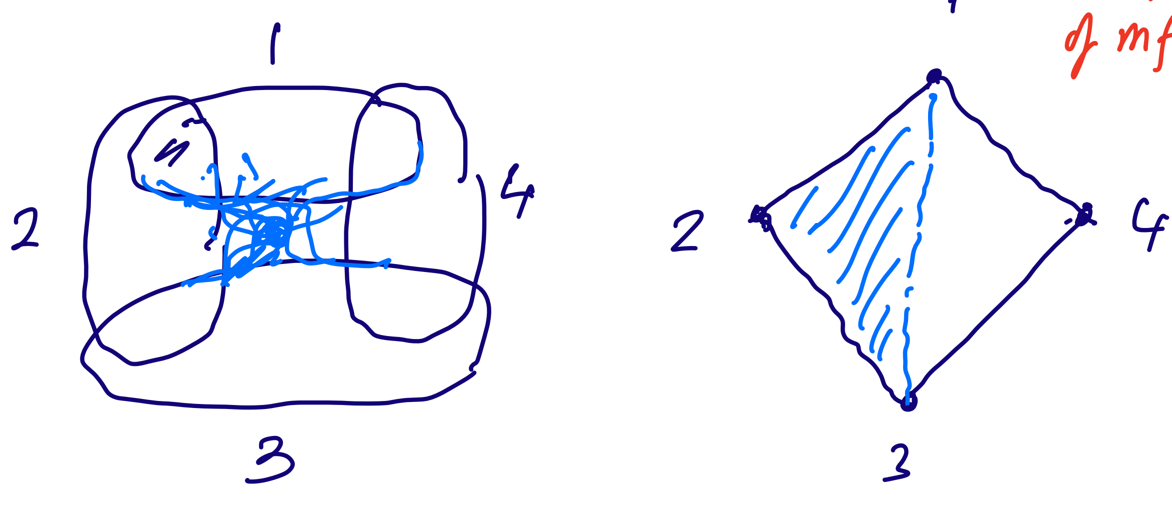
construction: given a family $\{U_\alpha\}_{\alpha \in I}$ of subsets of a topological space X, the nerve complex of $\{U_\alpha\}$ is an abstract simplicial complex s.t.

- the set of vertices is $\{U_\alpha\}$ ← i.e. indexed by I
- a subset $\sigma \subset I$ is a simplex iff $\bigcap_{\alpha \in \sigma} U_\alpha \neq \emptyset$

note lots of appls

ex covers of mfld

Nerve



Immediate Observation:

$$\check{C}ech(F, r) = \mathcal{N}\left(\{B(x,r)\}_{x\in F}\right)$$

$$\boxed{\text{(also), Delaunay triangulation} = \mathcal{N}(\text{Voronoi Sets})}$$

(HW). Sketch an union of 5 sets of $\mathbb{R}^2$ whose nerve is not homotopy eqv. to $\bigcup_{i=1}^{5} \mathcal{U}_i$

Nerve Thm. → various researchers Weil, Borsuk, Leray, ~1950 Edelsbrunner - Harer.

given a cover $\mathcal{U}$ of a topological space

Hypothesis	Conclusion
if. X nice (topological cnd) • $\mathcal{U}$ nice (finite intersection of elements are AE)	$X \simeq \mathcal{N}(\mathcal{U})$

• Weil: $X \times X \times [0,1]$ normal

$\boxed{\text{def. } Y \text{ is an } AE(X) \text{ if for any closed } A \subset X \text{ any cts fcn } f: A \to Y, \text{ extends to cts } F: X \to Y}$

✓

• $\mathcal{U}$ locally finite & $\mathcal{U}_S$ are $AE(C)$

- Borsuk
  - X finite dim, compact
  - $\mathcal{U}$ cover by closed sets AE

- (Leray - Edelsbrunner - Harer)

Nerve theorem

- X Euclidean
- $\mathcal{U}$ finite collection of closed & convex sets

$\bigcup_{i=1}^{N} U_i \simeq N(\mathcal{U})$

Slogan: (Nerve thm) A cover of a topological space w/ elements that are AE(e) provide structure similar to a triangulation.

Immediate Observation:

$$\check{C}ech(F,r) = N\left(\{B(x,r)\}_{x\in F}\right) \overset{\text{Nerve thm}}{\simeq} \bigcup_{x\in F} B(x,r)$$

$\therefore$ $\check{C}ech(F,r)$ preserves the shape of the cloud F, if you take shape to be the homotopy type of the ~~red~~ r-neighbourhood of F.

One concern:

As discussed earlier, (for finite $F \subset \mathbb{X}$), both VR complex & Čech complex may be high dimensional, & definitely not embedded in $\mathbb{X}$.

Enters alpha-complex:

If $F \subset \mathbb{R}^n$ a finite collection of pts s.t. they are in "general position" [no $(n+2)$-pts lie on the same $(n-1)$-sphere]. For $x \in F$, let V_x be the Voronoi cell for x. Then the alpha complex @ scale r is the nerve $\mathcal{N}\left(\{B(x,r) \cap V_x\}_{x \in F}\right)$

H.E.

$$\text{alpha complex}(r) \simeq \check{C}ech(r)$$

$\simeq$ Nerve thm $\simeq$

$$\bigcup \left(B(x,r) \cap V_x\right) = \bigcup_{x \in F} B(x,r)$$

HW

∴ Alpha complex capture the homotopy type of Čech but lives in $\mathbb{R}^n$ (ambient space)