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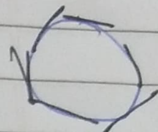
Envelopes :-

The Envelopes of a family of curves is a curve which touches each member of this family and at each point is touched by some other member of family.

Eg :- $y = mx + a\sqrt{1+m^2}$ is tangent to the curve $x^2+y^2=a^2$
(straight line) (circle)



∴ The path formed by multiple tangents → Envelope
to form a circle/ellipse/etc......

Methods of finding Envelope :-

① Let $f(x,y,\alpha) = 0$ be family of curves and $\alpha \rightarrow$ parameters ①
∴ α value is changing.

Differiate $f(x,y,\alpha)$ partially w.r.t α :-

$$f'(x,y,\alpha) = 0 \quad \text{—— ②}$$

Eliminate α from eq^s ① & ② after solving ① & ②
The eq^n obtained after α elimination → Envelope of given curve

② If the family of the curve is given by :-

$$\cancel{A\alpha^2 + B\alpha + C = 0}$$

$$A\alpha^2 + B\alpha + C = 0$$

then eq^n at envelope is given by :-

$$B^2 - 4AC = 0$$

Q1 Find the envelope of the family of straight line $y = mx + \frac{a}{m}$

Solⁿ:- $f(x,y,m) \Rightarrow y = mx + \frac{a}{m}$ —— (1)

Differentiate w.r.t m

$$\frac{d(y)}{dm} = \frac{d}{dm}(mx) + \frac{d}{dm}\left(\frac{a}{m}\right)$$

$$\Rightarrow 0 = x + a\left(\frac{-1}{m^2}\right)$$

$$\Rightarrow m^2 = \frac{a}{x} \qquad \therefore \boxed{m = \sqrt{\frac{a}{x}}} \longrightarrow \text{Put in eq}^n \text{ (1)}$$

$$\therefore (1) \Rightarrow y = \sqrt{\frac{a}{x}} \cdot x + a \cdot \sqrt{\frac{x}{a}}$$

$$\therefore y = \sqrt{a}\sqrt{x} + \sqrt{a}\sqrt{x}$$

$$= 2\sqrt{a}\sqrt{x}$$

$$\Rightarrow \boxed{y^2 = 4ax} \longrightarrow \text{Eq}^n \text{ of Envelope (Parabola)}$$

(*) Q2. Find the equation of the envelope of the family of curve represented by equation $x^2\sin\alpha + y^2\cos\alpha = a^2$

Solⁿ:- $x^2\sin\alpha + y^2\cos\alpha = a^2$ —— (1)

Differentiate w.r.t α

$$\frac{d}{d\alpha}(x^2\sin\alpha) + \frac{d}{d\alpha}(y^2\cos\alpha) = \frac{d}{d\alpha}(a^2)$$

$$\Rightarrow x^2\cos\alpha - y^2\sin\alpha = 0$$

$$\Rightarrow x^2\cos\alpha = y^2\sin\alpha \quad \cancel{\Rightarrow \tan\alpha =}$$

$$\Rightarrow \tan\alpha = \frac{x^2}{y^2}$$

But we need to eliminate α (Method 1)

$\therefore x^2 \cos\alpha = y^2 \sin\alpha$

$\Rightarrow \dfrac{x^2}{\sin\alpha} = \dfrac{y^2}{\cos\alpha}$

Squaring both sides :-

$\Rightarrow \dfrac{x^4}{\sin^2\alpha} = \dfrac{y^4}{\cos^2\alpha}$

Now, $\dfrac{1}{2} = \dfrac{2}{4} = \dfrac{1+2}{2+4} = \dfrac{3}{6} = \dfrac{1}{2}$

Similarly,

$$\frac{x^4}{\sin^2\alpha} = \frac{y^4}{\cos^2\alpha}$$

$$= \frac{x^4+y^4}{\sin^2\alpha+\cos^2\alpha}$$

$$= x^4+y^4$$

$\therefore \dfrac{x^4}{\sin^2\alpha} = \dfrac{y^4}{\cos^2\alpha} = x^4+y^4$ —— (2)

Hence, From (2) :-

$\dfrac{x^4}{\sin^2\alpha} = x^4+y^4$

$\Rightarrow \sin^2\alpha = \dfrac{x^4}{x^4+y^4}$

$\therefore \sin\alpha = \dfrac{x^2}{\sqrt{x^4+y^4}}$

$\dfrac{y^4}{\cos^2\alpha} = x^4+y^4$

$\Rightarrow \cos^2\alpha = \dfrac{y^4}{x^4+y^4}$

$\therefore \cos\alpha = \dfrac{y^2}{\sqrt{x^2+y^2}}$

Putting them in eqⁿ (1) we get :-

$$x^2\left(\frac{x^2}{\sqrt{x^4+y^4}}\right) + y^2\left(\frac{y^2}{\sqrt{x^2+y^2}}\right) = a^2$$

$$\Rightarrow \frac{x^4+y^4}{\sqrt{x^4+y^4}} = a^2$$

$$\Rightarrow \sqrt{x^4+y^4} = a^2$$

(or)

$$\boxed{x^4+y^4 = a^4}$$ $\longrightarrow$ Eqⁿ of Envelopes (circle)

Q3. Find the envelope of family of curve given by :-

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2-\alpha^2} = 1 \text{, where } \alpha \rightarrow \text{parameter}$$

Sol^n :-

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2-\alpha^2} = 1$$

$$\Rightarrow \frac{x^2(k^2-\alpha^2) + y^2\alpha^2}{\alpha^2(k^2-\alpha^2)} = 1$$

$$\Rightarrow x^2k^2 - x^2\alpha^2 + y^2\alpha^2 = k^2\alpha^2 - \alpha^4 \quad \text{——— (1)}$$

(or)

$$\alpha^4 - \alpha^2(x^2 - y^2 + k^2) + x^2k^2 = 0 \rightarrow \text{Quadratic form (Method 2)}$$

Here,

- $A = 1$
- $B = -(x^2+y^2-k^2)$
- $C = x^2k^2$

To find Envelope by 2nd Method :-

$B^2 - 4AC = 0$

envelope eq^n

$\therefore B^2 - 4AC = 0$

$$\Rightarrow [-(x^2 - y^2 + k^2)]^2 - 4(1)(x^2k^2) = 0$$

$$\Rightarrow (x^2 - y^2 + k^2)^2 - \underline{4x^2k^2} = 0 \qquad [(a^2 - b^2) = (a+b)(a-b)]$$

$(2xk)$

$$\therefore (x^2 - y^2 + k^2 + 2xk)(x^2 - y^2 + k^2 - 2xk) = 0$$

$$\Rightarrow [(x^2+k^2+2xk) - y^2][(x^2+k^2-2xk) - y^2] = 0$$

$$\Rightarrow [(x+k)^2 - y^2][(x-k)^2 - y^2]$$

$$\Rightarrow (x+k+y)(x+k-y)(x-k+y)(x-k-y) = 0$$

$\therefore$ Envelope eq^n(s) are :-

- $x+k+y = 0$
- $x+k-y = 0$
- $x-k+y = 0$
- $x-k-y = 0$

4 eq^n(s) of envelope of curve (V)