Evolute :- The locus of centre of curvature
(or)
The evolute of a curve is the envelope at the normal of the curve

Method to calculate Evolute :-

If α and β are centre of curvature of curve at (x,y)

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2} \qquad \beta = y + \frac{1+y_1^2}{y_2}$$

eliminate x & y from here

Q1 Find eqⁿ of the evolute of the curve $y^2 = 4ax$ [Method 1]

Solⁿ :- $y^2 = 4ax$

y_1 :- Differentiate both side w.r.t x

$$2y \cdot \frac{dy}{dx} = 4a$$

$$\Rightarrow y_1 = \frac{2a}{y}$$

y_2 :- Differentiate y_1 w.r.t x

$$y_2 = 2a \cdot \left(-\frac{1}{y^2}\right) \cdot \frac{dy}{dx}$$

$$= \frac{-2a}{y^2}\left(\frac{2a}{y}\right)$$

$$\Rightarrow y_2 = \frac{-4a^2}{y^3}$$

$$\therefore \alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= x - \frac{\left(\frac{2a}{y}\right)\left(1+\frac{4a^2}{y^2}\right)}{\frac{-4a^2}{y^3}}$$

$$\Rightarrow \alpha = x + \frac{\left(\frac{2a}{y}\right)\left(\frac{y^2+4a^2}{y^2}\right)}{\left(\frac{4a^2}{y^3}\right)}$$

$$\beta = y + \frac{1+y_1^2}{y_2}$$

$$= y + \frac{1+\frac{4a^2}{y^2}}{\frac{-4a^2}{y^3}}$$

$$\Rightarrow \beta = y - \frac{\frac{y^2+4a^2}{y^2}}{\frac{4a^2}{y^3}}$$

$$\alpha = x + \left(\frac{2a}{y}\right)\left(\frac{y^2+4a^2}{y^2}\right) \times \frac{y^3}{4a^2}$$

$$= x + \frac{y^2+4a^2}{2a}$$

But $y^2 = 4ax$:-

$$\alpha = x + \frac{4ax+4a^2}{2a}$$

$$\boxed{\alpha = x+2x+2a}$$

or

$$\boxed{\alpha = 3x+2a} \quad \text{—— (1)}$$

$$\beta = y - \frac{y^2+4a^2}{y^2} \times \frac{y^3}{4a^2}$$

$$= y - \frac{y}{4a^2}\left(4ax+4a^2\right)$$

$$\beta = -2x^{3/2}a^{-1/2}$$

$$\Rightarrow \beta a^{1/2} = -2x^{3/2}$$

or

$$\boxed{\beta^2 a = 4x^3} \quad \text{—— (2)}$$

(Eliminate x/y)

$$(2) \Rightarrow x^3 = \frac{\beta^2 a}{4}$$

$$\cancel{(1) \Rightarrow \alpha^3 = 27x}$$

$$(1) \Rightarrow \alpha = 3x+2a$$

$$\Rightarrow (\alpha - 2a) = 3x$$

Cube both sides :-

$$\Rightarrow (\alpha-2a)^3 = 27x^3$$

$$= 27\left(\frac{\beta^2 a}{4}\right)$$

$$\Rightarrow 4(\alpha-2a)^3 = 27\beta^2 a$$

Now, we write $\alpha \to \bar{x}$ & $\beta \to \bar{y}$

$\therefore$ Evolute of $y^2 = 4ax$:-

$$27\bar{y}^2 a = 4(\bar{x}-2a)^3$$

Q2. Find Evolute of parabola $y^2 = 4ax$ [Method 2]

Sol:- Eqⁿ of any normal to the Parabola given by:-

$$y = mx - 2am - am^3$$

Differentiate w.r.t m:-

$$\Rightarrow 0 = x - 2a - 3m^2 a$$

$$\Rightarrow m = \left(\frac{x-2a}{3a}\right)^{1/2}$$

$$\therefore y = \left(\frac{x-2a}{3a}\right)^{1/2} \cdot x - 2a\left(\frac{x-2a}{3a}\right)^{1/2} - a\left(\frac{x-2a}{3a}\right)^{3/2}$$

$$= \left(\frac{x-2a}{3a}\right)^{1/2}\left[x - 2a - \cancel{a}\left(\frac{x-2a}{3\cancel{a}}\right)\right]$$

$$\Rightarrow \left(\frac{3a}{x-2a}\right)^{1/2} \cdot y = \frac{3x - \cancel{6}a - x + 2a}{3}$$

$$\Rightarrow \cancel{3x} \qquad = \left(\frac{2x-4a}{3}\right)$$

Squaring both sides:-

$$\Rightarrow \left(\frac{3a}{x-2a}\right) y = \left(\frac{2x-4a}{3}\right)^2$$

$$\Rightarrow 27a^2 y = 4(x-2a)^3$$