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Q1. Find the extreme values of function $f(x,y) = x^3+y^3-3x-12y+20$.

Sol^n:- $f(x,y) = x^3+y^3-3x-12y+20$

$f_x = 3x^2-3$ & $f_y = 3y^2-12$

$f_x = 0$	$f_y = 0$
$\Rightarrow 3x^2-3=0$	$\Rightarrow 3y^2-12=0$
$\Rightarrow x^2-1=0$	$\Rightarrow y^2-4=0$
$\Rightarrow x^2=1$	$\Rightarrow y^2=4$
$\Rightarrow x=\sqrt{1}$	$\Rightarrow y=\sqrt{4}$

$\therefore x = 1, -1$

$y = 2, -2$

The stationary points are:- $(1,2), (1,-2), (-1,2), (-1,-2)$

$r = f_{xx} = \frac{d}{dx}(3x^2-3) \Rightarrow r = 6x$ | $s = f_{xy} = \frac{d}{dx}(3y^2-12) \Rightarrow s = 0$ | $t = f_{yy} = \frac{d}{dy}(3y^2-12) \Rightarrow t = 6y$

- For $(1,2)$:- $r = 6(1) = 6$ & $s = 0$ & $t = 6(2) = 12$
  $\therefore rt - s^2 = 6(12) - 0^2 = 72$
  $\because rt-s^2 > 0$ & $r > 0 \longrightarrow$ Minima / minimum point is $(1,2)$

- For $(1,-2)$:- $r = 6(1) = 6$ & $s = 0$ & $t = 6(-2) = -12$
  $\therefore rt - s^2 = 6(-12) - 0^2 = -72$
  $\because rt-s^2 < 0 \longrightarrow$ Neither maxima / minima $\therefore$ It's a saddle point

- For $(-1,2)$:- $r = 6(-1) = -6$ & $s = 0$ & $t = 6(2) = 12$
  $\therefore rt - s^2 = (-6)(12) - 0^2 = -72$
  $\because rt-s^2 < 0 \longrightarrow$ Saddle point

- For $(-1,-2)$:- $r = 6(-1) = -6$ & $s = 0$ & $t = 6(-2) = -12$
  $\therefore rt - s^2 = (-6)(-12) - 0^2 = 72$
  $\because rt-s^2 > 0$ & $r < 0 \longrightarrow$ Maxima / maximum point is $(-1,-2)$

Hence:-

Maximum value $= f(-1,-2) = (-1)^3+(-2)^3-3(-1)-12(-2)+20 = 38$

Minimum value $= f(1,2) = (1)^3+(2)^3-3(1)-12(2)+20 = 2$

Q2. Examine the following function for extreme values:-

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Sol^n:- $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$f_x = 4x^3 - 4x + 4y$ & $f_y = 4y^3 + 4x - 4y$

$f_x = 0$	$f_y = 0$
$\Rightarrow 4x^3 - 4x + 4y = 0$	$\Rightarrow 4y^3 + 4x - 4y = 0$
$\Rightarrow x^3 - x + y = 0$ —— (1)	$\Rightarrow y^3 + x - y = 0$ —— (2)

(1) + (2) $\Rightarrow x^3 + y^3 = 0$

$\Rightarrow y^3 = -x^3 \Rightarrow y = \sqrt{-x^3} = \sqrt{(-1)^3 x^3} \Rightarrow y = -x$ —— (3)

For (1) (3) in (1):-

$x^3 - x + (-x) = 0$

$\Rightarrow x(x^2 - 2) = 0 \qquad \therefore x = 0, \sqrt{2}, -\sqrt{2}$

For $x = 0$:- $0^3 - 0 + y = 0 \Rightarrow y = 0 \qquad \therefore (0,0)$

For $x = \sqrt{2}$:- $(\sqrt{2})^3 - \sqrt{2} + y = 0 \Rightarrow y = -\sqrt{2} \qquad \therefore (\sqrt{2}, -\sqrt{2})$

For $x = -\sqrt{2}$:- $(-\sqrt{2})^3 - (-\sqrt{2}) + y = 0 \Rightarrow -2\sqrt{2} + \sqrt{2} + y = 0 \Rightarrow y = +\sqrt{2} \quad \therefore (-\sqrt{2}, \sqrt{2})$

$\therefore$ Stationary points are:- $(0,0)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$

For (2) We get same point as we evaluating same $(y = -x)$ here

- $r = f_{xx} = \frac{d}{dx}(4x^3 - 4x + 4y)$

  $\Rightarrow r = 12x^2 - 4$

- $s = f_{xy} = \frac{d}{dx}(4y^3 + 4x - 4y)$

  $\Rightarrow s = 4$

- $t = f_{yy} = \cancel{\frac{d}{dx}}$

  $= \frac{d}{dy}(4y^3 + 4x - 4y)$

  $\Rightarrow t = 12y^2 - 4$

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- For $(0,0)$:- $r = 12(0)^2 - 4$ & $s = 4$ & $t = 12(0)^2 - 4$
$\Rightarrow r = -4$ $\qquad$ $\Rightarrow t = -4$

$rt - s^2 = (-4)(-4) - (4)^2 \Rightarrow rt - s^2 = 0$

$\therefore rt - s^2 = 0 \longrightarrow$ Inconclusive point

- For $(\sqrt{2}, -\sqrt{2})$:- $r = 12(\sqrt{2})^2 - 4$ & $s = 4$ & $t = 12(-\sqrt{2})^2 - 4$
$\Rightarrow r = 20$ $\qquad$ $\Rightarrow t = 20$

$rt - s^2 = (20)(20) - (4)^2 \Rightarrow rt - s^2 = 384$

$\therefore rt - s^2 > 0$ & $r > 0 \longrightarrow$ Minima/Minimum point is $(\sqrt{2}, -\sqrt{2})$

- For $(-\sqrt{2}, \sqrt{2})$:- $r = 12(-\sqrt{2})^2 - 4$ & $s = 4$ & $t = 12(\sqrt{2})^2 - 4$
$\Rightarrow r = 20$ $\qquad$ $\Rightarrow t = 20$

$rt - s^2 = (20)(20) - (4)^2 \Rightarrow rt - s^2 = 384$

$\therefore rt - s^2 > 0$ & $r > 0 \longrightarrow$ Minima/Minimum point is $(-\sqrt{2}, \sqrt{2})$

Note :- Here, there is no Maxima
It's not necessary for $f(x,y)$ to have both maxima & minima

Minimum values of $f(x,y)$ are :-

- $f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$

$~~= 4 + 4 - 4 - 8 - 8~~$

$= 4 + 4 - 4 - 8 - 4$

$\Rightarrow f(\sqrt{2}, -\sqrt{2}) = -8$

- $f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$

$= 4 + 4 - 4 - 8 - 4$

$\Rightarrow f(-\sqrt{2}, \sqrt{2}) = -8$

Note :- For both minima points $(\sqrt{2}, -\sqrt{2})$ & $(-\sqrt{2}, \sqrt{2})$:-
Minimum value is same

$\therefore$ If there are more than one maxima/minima points, the maximum/minimum value for that function will remain the same

(*)

Q 3. Find the maximum & minimum values of x^2-xy+y^2-2x+y

Sol^n:- $f(x,y) = x^2-xy+y^2-2x+y$

$f_x = 2x-y-2$ & $f_y = -x+2y+1$

$f_x = 0$

$\Rightarrow 2x-y-2=0$ ——— (1)

$f_y = 0$

$\Rightarrow -x+2y+1=0$ ——— (2)

Solving (1) & (2):-

(1) + (2)×2 $\Rightarrow \cancel{2x}-y-\cancel{2}-\cancel{2x}+4y+\cancel{2}=0$

~~$\Rightarrow -3y = 4$~~

$\Rightarrow 3y = 0$ $\therefore y = 0$

(*) Put the values you get in (1) & (2) [$f_x=0$] [$f_y=0$]

Put y=0 in (1)

(1) $\Rightarrow 2x-0-2=0$

$\Rightarrow x-1=0$ $\therefore x=1$ & Stationary point $\rightarrow (1,0)$

Put y=0 in (2)

(2) $\Rightarrow -x+2(0)+1=0$

$\Rightarrow x-1=0$ $\therefore x=1$ Stationary point $\rightarrow (1,0)$

$\therefore$ Stationary points are:- $(1,0)$ (only one stationary point here)

- $r = f_{xx} = \frac{d}{dx}(2x-y-2)$ $\Rightarrow \boxed{r=2}$
- $s = f_{xy} = \frac{d}{dx}(-x+2y+1)$ $\Rightarrow \boxed{s=-1}$
- $t = f_{yy} = \frac{d}{dy}(-x+2y+1)$ $\Rightarrow \boxed{t=2}$

For $(1,0)$:- $r=2$ $s=-1$ $t=2$

$rt-s^2 = (2)(2)-(-1)^2 = 3$

$\therefore rt-s^2>0$ & $r>0$ $\longrightarrow$ Minima/minimum point is $(1,0)$

$\therefore$ There is no maxima point & no maximum value of $f(x,y)$

~~Maxi~~ Minimum value of $f(x,y)$ $= f(1,0)$

$= (1)^2-(1)(0)+0^2-2(1)+0$

$= -1$

$\therefore$ No maximum value

Min value $= -1$

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Q4. Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$

Sol:- $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ & $f_y = 2x^3y - 2x^4y - 3x^3y^2$

$f_x = 0$

$\Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0$

∴ We have:

- $x^2 = 0 \rightarrow x = 0$
- $y^2 = 0 \rightarrow y = 0$
- $3-4x-3y = 0$ —— ①

$f_y = 0$

$\Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(2-2x-3y) = 0$

- $x^3 = 0 \rightarrow x^3 = 0$
- $y = 0 \rightarrow y = 0$
- $2-2x-3y = 0$ —— ②

∴ We put $x=0, y=0$ in ① & ②:- (Nope we do not do that)
(seperately)

$x = 0,\ y = 0 \rightarrow (0,0)$ is a stationary point

Solve ① & ② and the values you get is used in ① & ② to get the remaining stationary point.

① − ② × 2 ⇒ $3 - \cancel{4x} - 3y - 4 + \cancel{4x} + 6y = 0$

$\Rightarrow 3y = 1$ ∴ $y = \frac{1}{3}$

Put $y = 1/3$ in ①: ① $\Rightarrow 3 - 4x - 3\left(\frac{1}{3}\right) = 0 \Rightarrow x = \frac{+2}{+4_2}$ ∴ $x = \frac{1}{2}$

∴ Stationary point → $\left(\frac{1}{2}, \frac{1}{3}\right)$

Put $y = 1/3$ in ②: ② $\Rightarrow 2 - 2x - 3\left(\frac{1}{3}\right) = 0 \Rightarrow x = \frac{+1}{+2}$ ∴ $x = \frac{1}{2}$

∴ Stationary point → $\left(\frac{1}{2}, \frac{1}{3}\right)$

Hence, Stationary points are:-

$(0,0)$ & $\left(\frac{1}{2}, \frac{1}{3}\right)$

- $r = f_{xx} = \frac{d}{dx}\left(3x^2y^1 - 4x^3y^2 - 3x^2y^3\right)$

$\Rightarrow r = 6xy^2 - 12x^2y^2 - 6xy^3$

- $s = f_{xy} = \frac{d}{dx}\left(2x^3y - 2x^4y - 3x^3y^2\right)$

$\Rightarrow s = 6x^2y - 8x^3y - 9x^2y^2$

- $t = f_{yy} = \frac{d}{dy}\left(2x^3y - 2x^4y - 3x^3y^2\right)$

$\Rightarrow s = 2x^3 - 2x^4 - 6x^3y$

Now :-

- For $(0,0)$:- $r = 6(0)(0)^2 - 12(0)^2(0)^2 - 6(0)(0)^3$ Similary, $s = 0$

$\Rightarrow r = 0$ $\quad t = 0$

$rt - s^2 = 0 \longrightarrow$ Inconclusive point

- for $\left(\frac{1}{2}, \frac{1}{3}\right)$:-

- $r = 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3$

$= \frac{1}{3} - \frac{1}{3} - \frac{1}{9} \Rightarrow r = -\frac{1}{9}$

- $s = 6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right) - 9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \Rightarrow s = -\frac{1}{12}$

- $t = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)$

$= \frac{1}{4} - \frac{1}{8} \Rightarrow t = -\frac{1}{8}$

$rt - s^2$

$= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$

$\Rightarrow rt - s^2 = \frac{1}{144}$

$= 0.....$

$\therefore rt - s^2 > 0$

$\& r < 0$

(Maxima point)

~~$rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right)$~~

$\therefore$ Maximum value of $f(x,y)$:-

Max value $= f(1/2, 1/3) = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left[1 - \frac{1}{2} - \frac{1}{3}\right]$

$\therefore$ Max value $= \frac{1}{432}$

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Q5. Examine $x^3y^2(12-x-y)$ for extreme values. [Similar Q to 4]

Sol^n:- $f(x,y) = x^3y^2(12-x-y) = 12x^3y^2 - x^4y^2 - x^3y^3$

$f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$ & $f_y = 24x^3y - 2x^4y - 3x^3y^2$

$f_x = 0$	$f_y = 0$
$\Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$	$\Rightarrow 24x^3y - 2x^4y - 3x^3y^2 = 0$
$\Rightarrow x^2y^2(36 - 4x - 3y) = 0$	$\Rightarrow x^3y(24 - 2x - 3y) = 0$
We have: • $x = 0$, • $y = 0$ } $(0,0)$	• $x = 0$, • $y = 0$ } $(0,0)$
$\therefore (0,0)$ is a stationary point	
• $36 - 4x - 3y = 0$ —— (1)	• $24 - 2x - 3y = 0$ —— (2)

Solve (1) & (2):-

(1) − (2) $\Rightarrow 36 - 4x - 3y - 24 + 2x + 3y = 0$
$\Rightarrow 12 - 2x = 0 \quad \therefore x = 6$

Put $x=6$ in (1): (1) $\Rightarrow 36 - 4(6) - 3y = 0$
$\Rightarrow 3y = 12 \quad \therefore y = 4 \quad \therefore (6,4)$ is a stationary point

Put $x=6$ in (2): (2) $\Rightarrow 24 - 2(6) - 3y = 0$
$\Rightarrow 3y = 12 \quad \therefore y = 4 \quad \therefore (6,4)$ is a " "

Hence, stationary points of $f(x,y)$ are:-

$$(0,0) \quad \& \quad (6,4)$$

• $r = f_{xx} = \frac{d}{dx}\left(36x^2y^2 - 4x^3y^2 - 3x^2y^3\right) \Rightarrow r = 72xy^2 - 12x^2y^2 - 6xy^3$

• $s = f_{xy} = \frac{d}{dx}\left(24x^3y - 2x^4y - 3x^3y^2\right) \Rightarrow s = 72x^2y - 8x^3y - 9x^2y^2$

• $t = f_{yy} = \frac{d}{dy}\left(24x^3y - 2x^4y - 3x^3y^2\right) \Rightarrow t = 24x^3 - 2x^4 - 6x^3y$

Now :-

- For $(0,0)$:- $r = 0$, $s = 0$, $t = 0$ } $\therefore rt - s^2 = 0$
  $\therefore rt - s^2 = 0 \longrightarrow$ Inconclusive point

- For $(6,4)$:- $r = 72(6)(4)^2 - 12(6)^2(4)^2 - 6(6)(4)^3$
  $\Rightarrow r = -2304$

  $s = 72(6)^2(4) - 8(6)^3(4) - 9(6)^2(4)^2$
  $\Rightarrow s = 1728$

  $t = 24(6)^3 - 2(6)^4 - 6(6)^3(4)$
  $\Rightarrow t = -2592$

$\therefore rt - s^2 = (-2304)(-2592) - (1728)^2 = 2985984$

$\therefore rt - s^2 > 0$ & $r < 0 \longrightarrow$ Maxima/maximum point at $(6,4)$

Maximum value of $f(x,y) = f(6,4)$

$= (6)^3(4)^2(12 - 6 - 4)$

$= 216 \times 16 \times 2$

$\therefore$ Max value $= 6912$