Q7. Solve: $(D^2-4D+4)y = e^{2x}$

Solⁿ:- A.E:- $\cancel{m^2-4D+4}\; m^2-4m+4=0$

$\Rightarrow (m-2)(m-2)=0 \quad \therefore m=2,2$

CF:- $CF = (C_1+C_2x)e^{2x} = C_1e^{2x}+C_2xe^{2x}$

Here, $y_1 = e^{2x}$ $\qquad y_2 = xe^{2x}$

$y_1' = 2e^{2x}$ $\qquad y_2' = e^{2x}+2xe^{2x}$

$X = e^{2x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{2x}(e^{2x}+2xe^{2x}) - (2e^{2x})(xe^{2x})$

$\Rightarrow W = e^{4x}+2xe^{4x}-2xe^{4x} \quad \therefore W = e^{4x}$

- $P = -\int \frac{y_2 X}{W}dx = -\int \frac{xe^{2x}\cdot e^{2x}}{e^{4x}}dx = -\int x\,dx$

$\Rightarrow P = \frac{-x^2}{2}$

- $Q = \int \frac{y_1 X}{W}dx = \int \frac{e^{2x}\cdot e^{2x}}{e^{4x}}dx = \int dx$

$\Rightarrow Q = x$

P.I:- $PI = Py_1 + Qy_2$

$= \left(\frac{-x^2}{2}\right)(e^{2x}) + (x)(xe^{2x})$

$\Rightarrow PI = \frac{-x^2e^{2x}}{2} + x^2e^{2x}$

$= \frac{x^2}{2}e^{2x}$

$\therefore y = CF + PI$

$= C_1e^{2x}+C_2xe^{2x}+\frac{x^2}{2}e^{2x}$

Q8. Solve : $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Soln:- $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

A.E:- $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)(m-3) = 0 \quad \therefore m = 3, 3$

C.F:- $CF = (C_1 + C_2 x)e^{3x} = C_1 e^{3x} + C_2 x e^{3x}$

Here, $y_1 = e^{3x}$ $\qquad y_2 = xe^{3x}$

$y_1' = 3e^{3x}$ $\qquad y_2' = e^{3x} + 3xe^{3x}$

$X = \frac{e^{3x}}{x^2}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{3x}(e^{3x} + 3xe^{3x}) - (3e^{3x})(xe^{3x})$

$\Rightarrow W = e^{6x} + \cancel{3xe^{6x}} - \cancel{3xe^{6x}} \quad \therefore W = e^{6x}$

- $P = -\int \frac{y_2 X}{W} dx = -\int \frac{\cancel{x}\cancel{e^{3x}}}{\cancel{e^{6x}}} \cdot \frac{\cancel{e^{3x}}}{x^{\cancel{2}}} dx = -\int \frac{1}{x} dx \quad \therefore P = -\ln(x)$
- $Q = \int \frac{y_1 X}{W} dx = \int \frac{\cancel{e^{3x}}}{\cancel{e^{6x}}} \frac{\cancel{e^{3x}}}{x^2} dx = \int \frac{1}{x^2} dx = \frac{x^{-2+1}}{-2+1} \quad \therefore Q = -\frac{1}{x}$

P.I:- $P.I = Py_1 + Qy_2$

$= -\ln(x) \cdot e^{3x} + \left(-\frac{1}{\cancel{x}}\right)\cancel{x}e^{3x}$

$\Rightarrow P.I = -e^{3x}\ln(x) - e^{3x}$

$\therefore y = CF + P.I$

$= C_1 e^{3x} + C_2 xe^{3x} - e^{3x}\ln(x) - e^{3x}$

Q9. Solve $(D^2+2D+5)y = e^{-x}\tan x$

Sol:- A.E:- $m^2+2m+5=0$

Here:- $a=1, b=2, c=5$

$\therefore m = \frac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)} = \frac{-2\pm\sqrt{-16}}{2}$

$= \frac{-2\pm 4i}{2}$ $\therefore m = -1\pm 2i$

$\alpha=-1, \beta=2$

CF:- $CF = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

$= e^{-x}(C_1\cos 2x + C_2\sin 2x)$

$\Rightarrow CF = C_1\cos 2x\, e^{-x} + C_2\sin 2x\, e^{-x}$

Here,

$y_1 = \cos 2x\, e^{-x}$ $\qquad y_2 = \sin 2x\, e^{-x}$

$y_1' = -2e^{-x}\sin 2x - e^{-x}\cos 2x$ $\qquad y_2' = 2e^{-x}\cos 2x - e^{-x}\sin 2x$

$X = e^{-x}\tan x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x\, e^{-x} & \sin 2x\, e^{-x} \\ -2e^{-x}\sin 2x - e^{-x}\cos 2x & 2e^{-x}\cos 2x - e^{-x}\sin 2x \end{vmatrix}$

$= 2e^{-2x}\cos^2 2x - \cancel{e^{-2x}\sin 2x\cos 2x} + 2e^{-2x}\sin^2 2x + \cancel{e^{-2x}\sin 2x\cos 2x}$

$= 2e^{-2x}(\cos^2 2x + \sin^2 2x)$ $\qquad \therefore W = 2e^{-2x}$

$P = -\int \frac{y_2 X\,dx}{W} = -\int \frac{(\sin 2x\cdot \cancel{e^{-x}})\cdot(\cancel{e^{-x}}\tan x)}{2\cancel{e^{-2x}}}\,dx$

$= -\int \frac{\cancel{2}\sin x\cos x}{\cancel{2}}\cdot\left(\frac{\sin x}{\cos x}\right)dx$

$= -\int \sin^2 x\,dx$

$= -\int\left(\frac{1-\cos 2x}{2}\right)dx$

$= -\int \frac{1}{2}dx + \int \frac{\cos 2x}{2}dx$

$\Rightarrow P = -\frac{x}{2} + \frac{\sin 2x}{4}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1-2\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$

& $\int \cos 2\theta\,d\theta = \frac{\sin 2\theta}{\frac{d}{d\theta}(2\theta)}$

$\Rightarrow \int \cos 2\theta\,d\theta = \frac{\sin 2\theta}{2}$

$$Q = \int \frac{y_1 X}{W}\,dx = \int \frac{(\cos 2x\, e^{-x})(e^{-x}\tan x)}{2e^{-2x}}\,dx$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$= \int \frac{\cos 2x}{2} \cdot \frac{\sin x}{\cos x}\,dx = \frac{1}{2}\int (2\cos^2 x - 1)\frac{\sin x}{\cos x}\,dx$$

$$= \frac{1}{2}\left[\int 2\sin x\cos x\,dx - \int \frac{\sin x}{\cos x}\,dx\right]$$

$2\sin\theta\cos\theta = \sin 2\theta$

& let $t = \cos x$

$\frac{dt}{dx} = -\sin x$

$\Rightarrow dx = \frac{-dt}{\sin x}$

$$= \frac{1}{2}\left[\int \sin 2x\,dx - \int \frac{\cancel{\sin x}}{t}\left(\frac{-dt}{\cancel{\sin x}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{\frac{d}{dx}(2x)} + \int \frac{1}{t}\,dt\right]$$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2} + \log t\right] \Rightarrow Q = \cancel{\frac{1}{2}}\left[\frac{-\cos 2x}{4} + \frac{\log(\cos x)}{2}\right]$$

P.I:- $P.I = Py_1 + Qy_2$

$$= \left[-\frac{x}{2} + \frac{\sin 2x}{4}\right](\cos 2x\, e^{-x}) + \left[\frac{-\cos 2x}{4} + \frac{\log(\cos x)}{2}\right](\sin 2x\, e^{-x})$$

$$\therefore P.I = \frac{-x\cos 2x\, e^{-x}}{2} + \cancel{\frac{\sin 2x\cos 2x\, e^{-x}}{4}} - \cancel{\frac{\sin 2x\cos 2x\, e^{-x}}{4}} + \frac{\log(\cos x)\cdot\sin 2x\, e^{-x}}{2}$$

$$\Rightarrow P.I = \frac{-x\cos 2x\, e^{-x}}{2} + \frac{\log(\cos x)\sin 2x\, e^{-x}}{2}$$

$\therefore y = CF + PI$

$$= C_1\cos 2x\, e^{-x} + C_2\sin 2x\, e^{-x} - \frac{x\cos 2x\, e^{-x}}{2} + \frac{\log(\cos x)\sin 2x\, e^{-x}}{2}$$