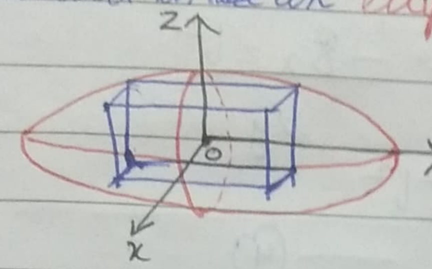
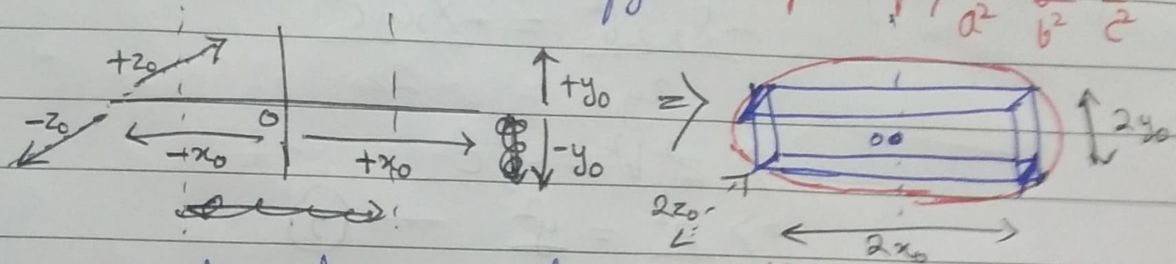
Q5. Find the maximum volume of the largest rectangular parallelepiped that can be inscribed in ~~the~~ an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Solⁿ:-



$\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$ geometric formula of ellipsoid (3D ellipse)

For a rectangular parallelepiped to be inscribed in ~~the~~ an ellipsoid, The coordinates of its vertices must satisfy the ellipsoid eqⁿ $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



$x_0, y_0, z_0 \rightarrow$ vertices along x, y, z plane in ellipsoid

- Length(l) $= 2x$, Breadth(b) $= 2y$, Height $= 2z$ (for rectangular parallelepiped)

We have :- $V(x,y,z) = (2x)(2y)(2z) = 8xyz$

$$E(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

$$\therefore F(x,y,z) = V(x,y,z) + \lambda E(x,y,z)$$

$$= 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

- $F_x = 8yz + \lambda\left(\frac{2x}{a^2}\right)$ • $F_y = 8xz + \lambda\left(\frac{2y}{b^2}\right)$ • $F_z = 8xy + \lambda\left(\frac{2z}{c^2}\right)$

Now:-

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow \lambda\left(\frac{2x}{a^2}\right) = -8yz$	$\Rightarrow \lambda\left(\frac{2y}{b^2}\right) = -8xz$	$\Rightarrow \lambda\left(\frac{2z}{c^2}\right) = -8xy$
$\Rightarrow \lambda = \frac{-8yza^2}{2x}$ — (1)	$\Rightarrow \lambda = \frac{-8xzb^2}{2y}$ — (2)	$\Rightarrow \lambda = \frac{-8xyc^2}{2z}$ — (3)

Equating ① & ②:-

$$\frac{-\cancel{8}y\cancel{z}a^2}{\cancel{2}x} = \frac{-\cancel{8}x\cancel{z}b^2}{\cancel{2}y}$$

$$\Rightarrow y^2a^2 = x^2b^2$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \quad \text{--- ④}$$

Equating ② & ③:-

$$\frac{-\cancel{8}\cancel{x}zb^2}{\cancel{2}y} = \frac{-\cancel{8}\cancel{x}yc^2}{\cancel{2}z}$$

$$\Rightarrow z^2b^2 = y^2c^2$$

$$\Rightarrow \frac{z^2}{c^2} = \frac{y^2}{b^2} \quad \text{--- ⑤}$$

From ④ & ⑤, $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$ (Put in $E(x,y,z) = 0$)

$$\therefore E(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\Rightarrow 3\frac{x^2}{a^2} = 1$$

$$\Rightarrow x^2 = \frac{a^2}{3} \quad \therefore x = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$ & $z = \frac{c}{\sqrt{3}}$

$$\therefore V(x,y,z) = 8xyz$$

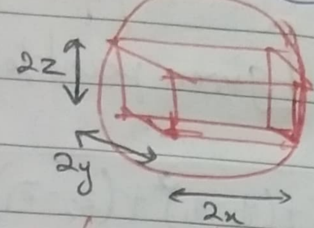
$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}} \text{ unit}^3 \text{ (Max volume)}$$

Q6 Prove that the rectangular solid of maximum volume which can be inscribed in a sphere $\underbrace{x^2+y^2+z^2=a^2}$ is a cube.

$\rightarrow$ geometric formula for sphere

Solⁿ :-



We have :- $V(x,y,z) = (2x)(2y)(2z) = 8xyz$

$S(x,y,z) = x^2+y^2+z^2-a^2$

$$F(x,y,z) = V(x,y,z) + \lambda S(x,y,z)$$
$$= 8xyz + \lambda(x^2+y^2+z^2-a^2)$$

- $F_x = 8yz + 2x\lambda$
  $F_x = 0$
  $\Rightarrow 8yz + 2x\lambda = 0$
  $\Rightarrow \lambda = \dfrac{-8yz}{2x}$ — ①

- $F_y = 8xz + 2y\lambda$
  $F_y = 0$
  $\Rightarrow 8xz + 2y\lambda = 0$
  $\Rightarrow \lambda = \dfrac{-8xz}{2y}$ — ②

- $F_z = 8xy + 2z\lambda$
  $F_z = 0$
  $\Rightarrow 8xy + 2z\lambda = 0$
  $\Rightarrow \lambda = \dfrac{-8xy}{2z}$ — ③

Equating ① & ② :-

$$\frac{-8yz}{2x} = \frac{-8xz}{2y}$$
$$\Rightarrow y^2 = x^2$$

Equating ② & ③ :-

$$\frac{-8xz}{2y} = \frac{-8xy}{2z}$$
$$\Rightarrow z^2 = y^2$$

$\therefore x^2 = y^2 = z^2$

(Put in $S(x,y,z) = 0$)

$\therefore S(x,y,z) = x^2+y^2+z^2 = a^2$

$\Rightarrow x^2 + x^2 + x^2 = a^2$

$\Rightarrow 3x^2 = a^2 \qquad \therefore x = \sqrt{\dfrac{a^2}{3}} = \dfrac{a}{\sqrt{3}}$

Similarly, $y = \dfrac{b}{\sqrt{3}}$ & $z = \dfrac{c}{\sqrt{3}}$

i.e. $\therefore x = y = z$

$\Rightarrow 2x = 2y = 2z$

$\therefore$ Length = Breadth = Height $\rightarrow$ It's a cube (All sides equal)

Q7. Find the longest and the shortest distances from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$

(in general formula of sphere)

Sol:- Let (x,y,z) be point on sphere [given point to be calculated from is $(1,2,-1)$]

(x_2, y_2, z_2) (x_1, y_1, z_1)

Distance formula :- $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

$\Rightarrow d^2 = (x-1)^2+(y-2)^2+(z+1)^2$

We have:- $D(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2$

$S(x,y,z) = x^2+y^2+z^2-24$

$F(x,y,z) = D(x,y,z) + \lambda S(x,y,z)$

$\Rightarrow F(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$

- $F_x = 2(x-1)+\lambda(2x)$ • $F_y = 2(y-2)+\lambda(2y)$ • $F_z = 2(z+1)+\lambda(2z)$

$F_x = 0$ $F_y = 0$ $F_z = 0$

$\Rightarrow 2(x-1)+\lambda(2x)=0$ $\Rightarrow 2(y-2)+\lambda(2y)=0$ $\Rightarrow 2(z+1)+\lambda(2z)=0$

$\Rightarrow -\cancel{2}\lambda x = \cancel{2}(x-1)$ $\Rightarrow -\cancel{2}\lambda y = \cancel{2}(y-2)$ $\Rightarrow -\cancel{2}\lambda z = \cancel{2}(z+1)$

$\Rightarrow \lambda = \frac{-(x-1)}{x}$ — ① $\Rightarrow \lambda = \frac{-(y-2)}{y}$ — ② $\Rightarrow \lambda = \frac{-(z+1)}{z}$ — ③

Equating ① & ② :-

$\frac{\cancel{-}(x-1)}{x} = \frac{\cancel{-}(y-2)}{y}$

$\Rightarrow \cancel{xy} - y = \cancel{xy} - 2x$

$\Rightarrow x = \frac{y}{2}$

Equating ② & ③ :-

$\frac{\cancel{-}(y-2)}{y} = \frac{\cancel{-}(z+1)}{z}$

$\Rightarrow \cancel{yz} - 2z = \cancel{yz} + y$

$\Rightarrow z = \frac{-y}{2}$

$\therefore y = 2x$ & $z = -\left(\frac{y}{2}\right) = -x$

$\Rightarrow z = -x$

$\therefore S(x,y,z) = x^2+y^2+z^2 - 24 = 0$

$\Rightarrow x^2 + (2x)^2 + (-x)^2 = 24$

$\Rightarrow x^2 + 4x^2 + x^2 = 24$

$\Rightarrow 6x^2 = 24$

$\Rightarrow x^2 = 4$ $\therefore x = \pm 2$ or $+2, -2$

If $x = +2$:-

$y = 2(2) = 4$

$z = -(2) = -2$

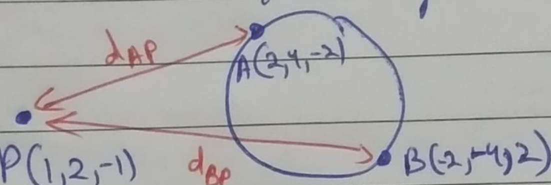
If $x = -2$:-

$y = 2(-2) = -4$

$z = -(-2) = 2$

$\therefore$ Stationary points are :- $A(2,4,-2)$ & $B(-2,-4,2)$

These two points are on the sphere



$\therefore AP = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2}$

$= \sqrt{1+4+1}$

$= \sqrt{6}$ units

$BP = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$

$= \sqrt{9+36+9}$

$= \sqrt{54}$ ~~units~~

$= 3\sqrt{6}$ units

$\because AP < BP$

$\therefore$ Shortest distance $= \sqrt{6}$ units

Longest distance $= 3\sqrt{6}$ units