classmate
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Both Gamma & Beta Functions → Improper function
↳ $\int_{-\infty}^{\infty}$ (or) $\int_{0}^{\infty}$

Gamma Function v/s Beta Function

Gamma Function	Beta Function
1) It is a function of single variable	1) It is a function of double variables
2) $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$ $(x>0)$	2) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ $(m>0, n>0)$
3) It is a recursion formula:- $\Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)\ldots 3\times 2\cdot\Gamma(1)$, $\Gamma(1)=1$ $= n!$ & $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$	3) It uses gamma function:- $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ where m, n is from:- $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Gamma :-

Q1. Evaluate $\int_0^{\infty} x^4 e^{-x}\,dx$

Solⁿ:- $\int_0^{\infty} x^4 e^{-x}\,dx = \int_0^{\infty} x^{5-1} e^{-x}\,dx$ ∴ $n=5$

$\therefore \Gamma(n) = \Gamma(5)$
$= (5-1)!$
$= 4! = 24$

Beta :-

Q2. Evaluate :-

(i) $B(3,5)$ here, $m=3$ & $n=5$

Solⁿ: $\therefore B(3,5) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \frac{\Gamma(3)\,\Gamma(5)}{\Gamma(8)}$
$= \frac{2!\,4!}{7!} = \frac{1}{105}$

(ii) $B\left(\frac{3}{2}, 2\right)$ here, $m = \frac{3}{2}$ & $n = 2$

Sol^n:- $B\left(\frac{3}{2}, 2\right) = \dfrac{\Gamma\frac{3}{2}\,\Gamma 2}{\Gamma\left(\frac{3}{2}+2\right)}$ $\left(\text{Use } \Gamma(n+1) = n\Gamma n,\ \Gamma\frac{1}{2} = \sqrt{\pi}\right)$

$= \dfrac{\Gamma\left(\frac{1}{2}+1\right)\, 2!}{\Gamma\frac{7}{2}} = \dfrac{\frac{1}{\cancel{2}}\Gamma\frac{1}{2}\cdot\cancel{2}}{\Gamma\left(\frac{5}{2}+1\right)}$

$= \dfrac{\cancel{\frac{1}{2}}\sqrt{\pi}}{\frac{5}{2}\times\frac{3}{2}\times\cancel{\frac{1}{2}}\,\Gamma\frac{1}{2}} = \dfrac{\cancel{\sqrt{\pi}}}{\frac{15}{\cancel{8}\,4}\cancel{\sqrt{\pi}}} = \dfrac{\cancel{8}\,4}{15}$

$\frac{7}{2}-1$, $\frac{5}{2}-1$, $\frac{3}{2}-1$, $\frac{1}{2}$

Q3. Evaluate $\int_0^1 x^4 (1-x)^3\,dx$

Sol^n:- $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}$

$\therefore m-1 = 4 \Rightarrow m = 5$ $\quad n-1 = 3 \Rightarrow n = 4$ $\quad \therefore B(5, 4)$

$B(5,4) = \dfrac{\Gamma 5\,\Gamma 4}{\Gamma 9} = \dfrac{4!\,3!}{8!} = \dfrac{1}{280}$

(*) **2nd property of Beta Function :-**

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta$$

Q4. Evaluate $\int_0^{\pi/2} \sin^6\theta\,d\theta$

Sol^n:- $\int_0^{\pi/2} \sin^6\theta\cos^0\theta\,d\theta$ $\quad \therefore p = 6,\ q = 0$

$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = B\left(\frac{6+1}{2}, \frac{0+1}{2}\right) = B\left(\frac{7}{2}, \frac{1}{2}\right)$

$2\int_0^{\pi/2} \sin^6\theta\cos^0\theta\,d\theta = B\left(\frac{7}{2}, \frac{1}{2}\right)$

$\Rightarrow \int_0^{\pi/2} \sin^6\theta\cos^0\theta\,d\theta = \frac{1}{2} B\left(\frac{7}{2}, \frac{1}{2}\right)$

Now,

$$\frac{7}{2}-1=\frac{5}{2}-1=\frac{3}{2}-1=\frac{1}{2}$$

$$\frac{1}{2}B\left(\frac{7}{2},\frac{1}{2}\right)=\frac{1}{2}\times\frac{\Gamma\frac{7}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)}$$

$$=\frac{1}{2}\times\frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{4!}$$

$$=\frac{5\pi}{32}$$

Q5. Evaluate $\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta$

Sol^n:- $p=4,\ \cancel{a}q=5$

$$2\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta=\cancel{B(4,5)}\,B\left(\frac{4+1}{2},\frac{5+1}{2}\right)$$

$$\Rightarrow\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta=\frac{1}{2}B\left(\frac{5}{2},3\right)$$

Now,

$$\cancel{\frac{14}{3}-1=\frac{11}{3}-1=\frac{8}{3}-1=\frac{5}{3}-1=\frac{2}{3}}$$

$$\frac{1}{2}B\left(\frac{5}{2},3\right)=\frac{1}{2}\times\frac{\Gamma\frac{5}{2}\,\Gamma 3}{\Gamma\left(\frac{5}{2}+3\right)}$$

$$=\frac{1}{2}\times\frac{\Gamma\frac{5}{2}\,\Gamma 3}{\Gamma\frac{11}{2}}$$

$$=\frac{8}{315}$$