Particular Integral (Type 5 $\rightarrow X = x \cdot V$)

$\hookrightarrow$ V is Trigonometry/Exponential function

For this type,

$$P.I = \frac{1}{f(D)} xV = \left\{ x - \frac{1}{f(D)} \cdot f'(D) \right\} \times \frac{1}{f(D)} \cdot V$$

Ques $(D^2+4) = x \sin x \rightarrow$ Type 5 because $x \rightarrow$ a trigo/exponential function

Solⁿ:- For A.E:- $D^2+4=0$

$\Rightarrow D = \sqrt{-4} \quad \therefore D = \underbrace{+2i, -2i}_{0 \pm 2i} \rightarrow \alpha = 0, \beta = 2$

For C.F:- $CF = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$\Rightarrow CF = c_1 \cos 2x + c_2 \sin 2x$

For P.I:- $P.I = \frac{1}{f(D)} x \sin x$

$= \left\{ x - \frac{1}{f(D)} f'(D) \right\} \underbrace{\frac{1}{f(D)} \cdot \sin x}_{\text{Type 2}}$ [Here, V = sin x]

Here, $f(D) = D^2+4$ & $f'(D) = 2D$

$a = 1, \therefore D^2 = -a^2 = -1$

$\therefore P.I = \left\{ x - \frac{2D}{D^2+4} \right\} \cdot \frac{1}{(-1)+4} \cdot \sin x$

$= \left\{ x - \frac{2D}{D^2+4} \right\} \frac{\sin x}{3}$

$= \frac{x \sin x}{3} - \frac{2}{3} \cdot \frac{1}{(D^2+4)} \cdot D(\sin x)$ $\rightarrow \frac{d}{dx}(\sin x) = \cos x$

$= \frac{x \sin x}{3} - \frac{2}{3} \cos x \cdot \frac{1}{(-1)+4}$

$\Rightarrow P.I = \frac{x \sin x}{3} - \frac{2}{9} \cos x$

General solⁿ given by:- $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x \sin x}{3} - \frac{2}{9} \cos x$

Q2. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

Sol^n:- $(D^2 - 2D + 1)y = x e^x \sin x$

Whenever there is 'e^x' after 'x', we write it as $e^x x$

$\therefore (D^2 - 2D + 1)y = e^x \cdot x \sin x$

For A.E:- $D^2 - 2D + 1 = 0$

$\Rightarrow (D-1)^2 = 0 \quad \therefore D = 1, 1$ (repeated)

For C.F:- $C.F = (c_1 + c_2 x)e^x$

For P.I:- $P.I = \frac{1}{f(D)} e^x \cdot x \sin x$

Here, $e^x \cdot (x \sin x)$ is of type $e^{ax} \cdot V$ (Type 4)

$\cancel{\therefore P.I = e^x \frac{1}{f(D)}}$ Here, $a = 1 \quad \therefore D \longrightarrow D + a = D + 1$

$\therefore P.I = e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \cdot (x \sin x)$

$= e^x \cdot \frac{1}{D^2 + 1 + 2D - 2D - 2 + 1} \cdot (x \sin x)$

$= e^x \left(\frac{1}{D^2} \cdot x \sin x \right)$

Here, $x \sin x$ is of type $x \cdot V$ (Type 5)

Let $g(D) = D^2 \longrightarrow \frac{1}{g(D)} \cdot x \sin x$

$g'(D) = 2D$

$\therefore P.I = e^x \cdot \left\{ x - \frac{g'(D)}{g(D)} \right\} \cdot \frac{1}{g(D)} \cdot \sin x$

$= e^x \left\{ x - \frac{2D}{D^2} \right\}$ Type 2

Here, $a = 1 \quad \therefore D^2 = -a^2 = -1$

$$\therefore P.I = e^x \left\{ x - \frac{2D}{D^2} \right\} \cdot \frac{1}{D^2} \sin x$$

$$= e^x \left\{ x - \frac{2}{D} \right\} \cdot \frac{1}{(-1)} \sin x$$

$$= e^x \left\{ -x \sin x + 2 \int \sin x \, dx \right\}$$

$$= e^x \left(-x \sin x + 2 \right($$

Don't make (✗) $\frac{D}{D^2} = \frac{1}{D}$

Leave denominator as it is and multiply as it is

$$\therefore P.I = e^x \left\{ x - \frac{2D}{D^2} \right\} \frac{1}{(-1)} \sin x$$

$$= -e^x \left\{ x \sin x - \frac{2D(\sin x)}{D^2} \right\}$$

Here, $D^2 = -1$ [$\because$ Type 2]

$$\therefore P.I = -e^x \left\{ x \sin x - \frac{2 \cos x}{(-1)} \right\}$$

$$= -e^x \{ x \sin x + 2 \cos x \}$$

$\therefore$ General solⁿ : $y = (c_1 + c_2 x) e^x - e^x (x \sin x + 2 \cos x)$

For mix types like these, you have to identify the types in the expression.

Solve it via breakdown of expression

Here, Type 4 $\longrightarrow$ Type 5 $\longrightarrow$ Type 2