Lagrange's Method (3 variables)

~~steps~~ 1) Let $f(x,y,z) =$

$\phi(x,y,z) =$

2) $F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$

3) $dF = \left[\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right] + \lambda\left[\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz\right]$

4) $dF = 0 \Rightarrow \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz + \left(\frac{\partial \phi}{\partial x}\right)\lambda dx + \left(\frac{\partial \phi}{\partial y}\right)\lambda dy + \left(\frac{\partial \phi}{\partial z}\right)\lambda dz = 0$

$\Rightarrow dx\left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial x}\right) + dy\left(\frac{\partial f}{\partial y} + \frac{\partial \phi}{\partial y}\right) + dz\left(\frac{\partial f}{\partial z} + \frac{\partial \phi}{\partial z}\right) = 0$

$\Rightarrow dx\left(\frac{\partial F}{\partial x}\right) + dy\left(\frac{\partial F}{\partial y}\right) + dz\left(\frac{\partial F}{\partial z}\right) = 0$

5) $\frac{\partial F}{\partial x} = 0$ & $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial z} = 0$ } can skip (4) and do (5)

From these, find $\rightarrow x, y, z$ (in terms of λ)

$\downarrow$ put values in $\phi(x,y,z)$

find $\rightarrow \lambda =$

$\downarrow$ put λ values in x, y, z

find $\rightarrow x, y, z$

6) Using the various x, y, z you get from Step 5 :-

Calculate $f(x,y,z)$ for each λ value $\rightarrow x, y, z$ ~~value~~

Q1. If $u = x^2+y^2+z^2$ where $\phi = ax+by+cz-p = 0$
Find the stationary value of u.

Sol:- Let $f(x,y,z) = x^2+y^2+z^2$ & $\phi(x,y,z) = ax+by+cz-p$

$\therefore F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$\Rightarrow F(x,y,z) = x^2+y^2+z^2+\lambda(ax+by+cz-p)$

Now, we find $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

- $\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + a\lambda = 0$
  $\Rightarrow x = \frac{-a\lambda}{2}$
- $\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + b\lambda = 0$
  $\Rightarrow y = \frac{-b\lambda}{2}$
- $\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + c\lambda = 0$
  $\Rightarrow z = \frac{-c\lambda}{2}$

Put x, y, z in $\phi(x,y,z) = 0$

$\therefore \phi(x,y,z) = 0$

$\Rightarrow a\left(\frac{-a\lambda}{2}\right) + b\left(\frac{-b\lambda}{2}\right) + c\left(\frac{-c\lambda}{2}\right) - p = 0$

$\Rightarrow \frac{-a^2\lambda - b^2\lambda - c^2\lambda}{2} = p$

$\Rightarrow \lambda(-a^2-b^2-c^2) = 2p$

$\Rightarrow \lambda = \frac{2p}{-a^2-b^2-c^2}$

$= \frac{2p}{-(a^2+b^2+c^2)}$

$\rightarrow$ Put λ in x, y, z

- $x = \frac{-a\lambda}{2} = \frac{\cancel{+}a}{\cancel{2}}\left(\frac{\cancel{+}\cancel{2}p}{a^2+b^2+c^2}\right) = \frac{ap}{a^2+b^2+c^2}$
- $y = \frac{-b\lambda}{2} = \frac{\cancel{+}b}{\cancel{2}}\left(\frac{\cancel{+}\cancel{2}p}{a^2+b^2+c^2}\right) = \frac{bp}{a^2+b^2+c^2}$
- $z = \frac{-c\lambda}{2} = \frac{\cancel{+}c}{\cancel{2}}\left(\frac{\cancel{+}\cancel{2}p}{a^2+b^2+c^2}\right) = \frac{cp}{a^2+b^2+c^2}$

$$\therefore f(x,y,z) = x^2+y^2+z^2$$
$$= \left(\frac{ap}{a^2+b^2+c^2}\right)^2 + \left(\frac{bp}{a^2+b^2+c^2}\right)^2 + \left(\frac{cp}{a^2+b^2+c^2}\right)^2$$
$$= \frac{p^2\,(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}$$
$$\Rightarrow f(x,y,z) = \frac{p^2}{(a^2+b^2+c^2)}$$

$\hookrightarrow$ stationary value of u

If you are asked to find maxima & minima :-

Sub conditions :- $f_{xx} = A$, $f_{xy} = B$, $f_{yy} = C$

i) $A > 0$ & $AC - B^2 > 0 \longrightarrow f$ is minimum

ii) $A < 0$ & $AC - B^2 > 0 \longrightarrow f$ is maximum