Legendre Type :-

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = X$$

[k -> constant]

This time :-

Let $(ax+b) = e^z$

$\Rightarrow \log(ax+b) = z$

$\therefore$ We have :-

- $(ax+b) D y = a D' y$
- $(ax+b)^2 D^2 y = a^2 D'(D'-1) y$
- $(ax+b)^3 D^3 y = a^3 D'(D'-1)(D'-2) y$

Use these to update the eq^n in terms of z

By Legendre :-

$$(ax+b)^n D^n y = a^n D'(D'-1)(D'-2) \dots (D'-n+1)$$

Eg :- ~~$(x^3 D^3 + 3x^2 D^2 + x D + 8) y = 65 \cos(\log x)$~~

Eg :- $(3x+2)^2 \frac{d^2 y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36 y = 3x^2 + 4x + 1$

Here,

$3x+2 = e^z \Rightarrow z = \log(3x+2)$; $x = \frac{e^z - 2}{3}$

Here, $a = 3$ & $b = 2$

$\therefore$ We have :-

- $(3x+2) D = 3D'$
- $(3x+2)^2 D^2 = 3^2 D'(D'-1) = 9D'(D'-1)$

$$\therefore [9D'(D'-1) + 3(3D') - 36] y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

Q. Solve: $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

Sol^n:- Let $D = \frac{d}{dx}$

$\therefore \left[(2x+1)^2D^2 - 2(2x+1)D - 12\right]y = 6x$

Let $2x+1 = e^z$ —— ① $\Rightarrow z = \log(2x+1)$

(here, $a=2$ & $b=1$)

$\therefore$ We have:-
- $(2x+1)D = 2D'$
- $(2x+1)^2D^2 = 2^2D'(D'-1) = 4D'(D'-1)$

From ① $\Rightarrow x = \frac{e^z-1}{2}$

$\therefore \left[4D'(D'-1) - 2(2D') - 12\right]y = \cancel{6}^{\,3}\left(\frac{e^z-1}{\cancel{2}}\right)$

$\Rightarrow \left[4D'^2 - 4D' - 4D' - 12\right]y = 3(e^z-1)$

$\Rightarrow \left[4D'^2 - 8D' - 12\right]y = 3(e^z-1)$

$\Rightarrow (D'^2 - 2D' - 3)y = \frac{3}{4}(e^z-1)$

A.E:- $m^2 - 2m - 3 = 0$

$\Rightarrow (m-3)(m+1) = 0 \quad \therefore m = -1, 3$

C.F:- $CF = C_1e^{m_1z} + C_2e^{m_2z}$

$= C_1e^{-z} + C_2e^{3z}$

But:- $e^z = 2x+1$

$\therefore CF = C_1(2x+1)^{-1} + C_2(2x+1)^3$

$= \frac{C_1}{2x+1} + C_2(2x+1)^3$

P.I :- $PI = \dfrac{1}{D'^2 - 2D' - 3} \cdot \dfrac{3}{4}(e^z - 1)$

$$= \frac{3}{4}\left[\frac{e^z}{\underset{D'=1}{D'^2-2D'-3}} - \frac{e^{0z}}{\underset{D'=0}{D'^2-2D'-3}}\right] \quad \text{(Type 1)}$$

$$= \frac{3}{4}\left[\frac{e^z}{(1)^2-2(1)-3} - \frac{e^{0z}}{0-2(0)-3}\right]$$

$$= \frac{3}{4}\left[\frac{e^z}{-4} + \frac{1}{3}\right]$$

But, $e^z = 2x+1$

$$\therefore PI = \frac{3}{4}\left[\frac{(2x+1)}{-4} + \frac{1}{3}\right]$$

$$\therefore y = CF + PI$$

$$= \frac{C_1}{2x+1} + C_2(2x+1)^3 + \frac{3}{4}\left[-\frac{(2x+1)}{4} + \frac{1}{3}\right]$$