Test for Convergence :-

- If $\lim S_n = l$ and l is finite Then:- $\sum u_n$ converges
- If $\lim S_n = +\infty$ or $-\infty$ then :- $\sum u_n$ diverges

where, $u_n = \langle S_n \rangle$

Q1. The series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges

Soln:- Let $u_n = \frac{1}{3^n}$

$$\langle S_n \rangle = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$= \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$$

$$= 1 + \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots$$

$$= a + ar + ar^2 + ar^3 + \ldots \quad \left[a = 1, r = \frac{1}{3}\right]$$

$$\Rightarrow \langle S_n \rangle = \frac{a(1 - r^{n+1})}{1 - r}$$

→ GP :-

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad [1 > r]$$

Put $a = 1$ & $r = \frac{1}{3}$:-

$$\langle S_n \rangle = \frac{1\left(1-(1/3)^{n+1}\right)}{1-(1/3)}$$

$$= \frac{1-\left(\frac{1}{3}\right)^{n+1}}{2/3}$$

Now, find limit :- $\left[\lim_{n\to\infty} S_n = l\right]$

$$\therefore \lim_{n\to\infty} \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right)$$

$$= \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^{\infty+1}\right]$$

$$= \frac{3}{2}\left[1-0\right] \Rightarrow l = \frac{3}{2}$$

$\therefore l$ is finite $\left(\frac{3}{2}\right)$ $\therefore \sum u_n$ converges $\therefore \sum_{n=0}^{\infty} \frac{1}{3^n}$ is convergent sequence

Q2. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges

Sol:- Let $u_n = \frac{1}{n(n+1)}$

$$\Rightarrow u_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\langle S_n \rangle = u_1 + u_2 + u_3 + \dots$$

$$= \left(\frac{1}{1} - \frac{1}{1+1}\right) + \left(\frac{1}{2} - \frac{1}{2+1}\right) + \left(\frac{1}{3} - \frac{1}{3+1}\right) + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$\Rightarrow \langle S_n \rangle = 1 - \frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty}(1) - \lim_{n\to\infty}\frac{1}{n+1}$$

$$= 1 - \cancel{\frac{1}{\infty+1}}^{\,0}$$

$$\Rightarrow l = 1$$

$\because l$ is finite (1), $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent sequence

Q3. The series $1+2+3+\dots+n$ diverges to ∞

Sol^n:- $S_n = 1+2+3+\dots+n$

$$= \frac{n(n+1)}{2}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n(n+1)}{2}$$

$$= \frac{\infty(\infty+1)}{2}$$

$$\Rightarrow l = \infty$$

$\because l$ is infinite (∞), $\sum S_n$ is divergent sequence