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Q11. Find the minimum value of x^2yz^3 where $2x+y+3z=a$

Sol^n <u>We have</u>:- $f(x,y,z) = x^2yz^3$
$\phi(x,y,z) = 2x+y+3z-a$

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$
$$= x^2yz^3 + \lambda(2x+y+3z-a)$$

- $F_x = 2xyz^3 + 2\lambda$
$F_x = 0$
$\Rightarrow 2xyz^3 + 2\lambda = 0$
$\Rightarrow \lambda = \dfrac{-\cancel{2}xyz^3}{\cancel{2}}$
$= -xyz^3$ —— (1)

- $F_y = x^2z^3 + \lambda$
$F_y = 0$
$\Rightarrow x^2z^3 + \lambda = 0$
$\Rightarrow \lambda = -x^2z^3$
$= -x^2z^3$ —— (2)

- $F_z = 3x^2yz^2 + 3\lambda$
$F_z = 0$
$\Rightarrow 3x^2yz^2 + 3\lambda = 0$
$\Rightarrow \lambda = \dfrac{-\cancel{3}x^2yz^2}{\cancel{3}}$
$= -x^2yz$ —— (3)

<u>Equating (1) & (2)</u>:-

$\cancel{-}xy\cancel{z^3} = \cancel{-}x^{\cancel{2}}\cancel{z^3}$
$\Rightarrow x = y$

$\therefore x = y = z$

<u>Equating (2) & (3)</u>:-

$\cancel{-}\cancel{x^2}z^3 = \cancel{-}\cancel{x^2}yz^2$
$\Rightarrow y = z$

$\therefore \phi(x,y,z) = 0 \Rightarrow 2x + \cancel{x}y + 3z = a$
$\Rightarrow 2x + x + 3x = a$
$\Rightarrow 6x = a$

$\therefore x = \dfrac{a}{6}, y = \dfrac{a}{6}, z = \dfrac{a}{6}$

$\therefore$ Min value of $f(x,y,z)$ is at $\left(\dfrac{a}{6}, \dfrac{a}{6}, \dfrac{a}{6}\right)$

$$\therefore f\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right) = \left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3$$
$$= \left(\frac{a}{6}\right)^6$$

Q12. Find the maximum and minimum value of $x^2+y^2+z^2$, where $x+y+z=3a$

Sol^n:- We have:- $f(x,y,z) = x^2+y^2+z^2$
$\phi(x,y,z) = x+y+z-3a$

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$
$= x^2+y^2+z^2+\lambda(x+y+z-3a)$

- $F_x = 2x+\lambda$
$F_x = 0$
$\Rightarrow 2x+\lambda = 0$
$\Rightarrow \lambda = -2x$ —— ①

- $F_y = 2y+\lambda$
$F_y = 0$
$\Rightarrow 2y+\lambda = 0$
$\Rightarrow \lambda = -2y$ —— ②

- $F_z = 2z+\lambda$
$F_z = 0$
$\Rightarrow 2z+\lambda = 0$
$\Rightarrow \lambda = -2z$ —— ③

Equating ① & ②:-

$\cancel{-2}x = \cancel{-2}y$
$\Rightarrow x = y$

Equating ② & ③:-

$\cancel{-2}y = \cancel{-2}z$
$\Rightarrow y = z$

$\therefore x = y = z$

$\therefore \phi(x,y,z) = 0 \Rightarrow x+y+z = 3a$
$\Rightarrow x+x+x = 3a$
$\Rightarrow 3x = 3a$ $\therefore x = a, y = a, z = a$

$\therefore$ Stationary point at (a,a,a)

∵ We don't have a way to determine 'a''s value, ~~we~~
we use $rt - s^2$ method :-

From $\phi(x,y,z) = 0$, $z = 3a - x - y$

$$\therefore f(x,y,z) = x^2 + y^2 + (3a-x-y)^2$$

- $f_x = \frac{d}{dx}\left(x^2+y^2+(3a-x-y)^2\right) = 2x - 2(3a-x-y)$
- $f_y = \frac{d}{dy}\left(x^2+y^2+(3a-x-y)^2\right) = 2y - 2(3a-x-y)$

- $f_{xx} = 2 - 2(-1)$ $\Rightarrow r = 4$
- $f_{xy} = 0 - 2(-1)$ $\Rightarrow s = 2$
- $f_{yy} = 2 - 2(-1)$ $\Rightarrow t = 4$

Here,

$$rt - s^2 = (4)(4) - (2)^2 = 16 - 4 = 12 > 0 \quad \& \quad r = 4 > 0$$

∴ Minima at (a, a, a) & No Maxima

∴ Minimum value $= f(a,a,a)$

$$\cancel{= a^2 + a^2 +}$$
$$= x^2 + y^2 + z^2$$
$$= a^2 + a^2 + a^2$$
$$= 3a^2$$

Q13. Find the point on the plane $ax+by+cz=p$ at which $f=x^2+y^2+z^2$ has a stationary ~~point~~ value and find the stationary value of f.

Solⁿ:- We have:-
$$f(x,y,z)=x^2+y^2+z^2$$
$$\phi(x,y,z)=ax+by+cz-p$$

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$
$$=x^2+y^2+z^2+\lambda(ax+by+cz)$$

- $F_x=2x+a\lambda$

$F_x=0$

$\Rightarrow 2x+a\lambda=0$

$\Rightarrow \lambda=\frac{-2x}{a}$ — ①

- $F_y=2y+b\lambda$

$F_y=0$

$\Rightarrow 2y+b\lambda=0$

$\Rightarrow \lambda=\frac{-2y}{b}$ — ②

- $F_z=2z+c\lambda$

$F_z=0$

$\Rightarrow 2z+c\lambda=0$

$\Rightarrow \lambda=\frac{-2z}{c}$ — ③

Equating ① & ②:-
$$\frac{\cancel{-2}x}{a}=\frac{\cancel{-2}y}{b}$$
$$\Rightarrow \frac{x}{a}=\frac{y}{b}$$

Equating ② & ③:-
$$\frac{\cancel{-2}y}{b}=\frac{\cancel{-2}z}{c}$$
$$\Rightarrow \frac{y}{b}=\frac{z}{c}$$

$$\therefore \frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

$$\phi(x,y,z)=0 \Rightarrow ax+by+cz=p$$
$$\Rightarrow ax+b\left(\frac{xb}{a}\right)+c\left(\frac{xc}{a}\right)=p$$
$$\Rightarrow a^2x+b^2x+c^2x=ap$$
$$\Rightarrow x(a^2+b^2+c^2)=ap$$
$$\Rightarrow x=\frac{ap}{a^2+b^2+c^2}$$

$$\therefore x=\frac{ap}{a^2+b^2+c^2}$$

$$y=\frac{bx}{a}=\frac{b}{\cancel{a}}\left(\frac{\cancel{a}p}{a^2+b^2+c^2}\right)\Rightarrow y=\frac{bp}{a^2+b^2+c^2}$$

$$z=\frac{cx}{a}=\frac{c}{\cancel{a}}\left(\frac{\cancel{a}p}{a^2+b^2+c^2}\right)\Rightarrow z=\frac{cp}{a^2+b^2+c^2}$$

Stationary point is $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$

~~Stat~~

Stationary value of f :-

$$f\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$$

$$= x^2+y^2+z^2$$

$$= \left(\frac{ap}{a^2+b^2+c^2}\right)^2 + \left(\frac{bp}{a^2+b^2+c^2}\right)^2 + \left(\frac{cp}{a^2+b^2+c^2}\right)^2$$

$$= \frac{p^2}{\cancel{(a^2+b^2+c^2)}}\cancel{\left[a^2+b^2+c^2\right]}$$

$$= p^2$$

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Q14. Using Lagrange method, show that the stationary value of $a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ occur at:-

$$x=\frac{a+b+c}{a}, \quad y=\frac{a+b+c}{b}, \quad z=\frac{a+b+c}{c}$$

Sol: We have:- $f(x,y,z)=a^3x^2+b^3y^2+c^3z^2$

$$\phi(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$
$$=a^3x^2+b^3y^2+c^3z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

- $F_x=2xa^3-\frac{\lambda}{x^2}$ • $F_y=2yb^3-\frac{\lambda}{y^2}$ • $F_z=2zc^3-\frac{\lambda}{z^2}$

$F_x=0 \Rightarrow 2xa^3-\frac{\lambda}{x^2}=0 \Rightarrow \lambda=2a^3x^3$ — ①

$F_y=0 \Rightarrow 2yb^3-\frac{\lambda}{y^2}=0 \Rightarrow \lambda=2b^3y^3$ — ②

$F_z=0 \Rightarrow 2zc^3-\frac{\lambda}{z^2}=0 \Rightarrow \lambda=2c^3z^3$ — ③

Equating ① & ②:-

$$\cancel{2}a^3x^3=\cancel{2}b^3y^3$$
$$\Rightarrow ax=by$$

Equating ② & ③:-

$$\cancel{2}b^3y^3=\cancel{2}c^3z^3$$
$$\Rightarrow by=cz$$

$ax=by=cz$

$$\phi(x,y,z)=0 \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0$$
$$\Rightarrow \cancel{\frac{yz+xz+xy-xyz}{xyz}=0}$$
$$\Rightarrow \frac{1}{x}+\frac{1}{\frac{ax}{b}}+\frac{1}{\frac{ax}{c}}=1$$
$$\Rightarrow \frac{1}{x}+\frac{b}{ax}+\frac{c}{ax}=1$$
$$\Rightarrow \frac{1}{x}\left(1+\frac{b}{a}+\frac{c}{a}\right)=1$$
$$\Rightarrow \frac{1}{x}\left(\frac{a+b+c}{a}\right)=1$$

$$\Rightarrow x = \frac{a+b+c}{a}$$

$$\therefore y = \frac{ax}{b}$$

$$= \cancel{a}\left(\frac{a+b+c}{\cancel{a}}\right)\frac{1}{b}$$

$$\Rightarrow y = \frac{a+b+c}{b}$$

$$z = \frac{ax}{c}$$

$$= \cancel{a}\left(\frac{a+b+c}{\cancel{a}}\right)\cdot\frac{1}{c}$$

$$\Rightarrow z = \frac{a+b+c}{c}$$

$\therefore$ Stationary value of $f(x,y,z)$ occurs at :-

$$\left(\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}\right)$$