Q3. Solve : $x^2y'' - 4xy' + 6y = x^2 + \log x$

Sol:- $(x^2D^2 - 4xD + 6)y = 0$

Let $x = e^z \Rightarrow z = \log x$

$\therefore$ We have :- ~~x^2D^2~~ $xD = D'$

$x^2D^2 = D'(D'-1)$

$\therefore [D'(D'-1) - 4D' + 6]y = e^{2z} + z$

$\Rightarrow [D'^2 - D' - 4D' + 6]y = e^{2z} + z$

$\Rightarrow [D'^2 - 5D' + 6]y = e^{2z} + z$

AE :- $m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 2m - 3m + 6 = 0$

$\Rightarrow m(m-2) - 3(m-2) = 0$

$\Rightarrow (m-3)(m-2) = 0$ $\quad \therefore m = 2, 3$

CF :- $CF = C_1e^{2z} + C_2e^{3z}$ (in terms of z)

$= C_1e^{2\log x} + C_2e^{3\log x}$

$\Rightarrow CF =$ ~~$C_1 \log$~~ $C_1e^{\log x^2} + C_2e^{\log x^3}$

For PI :-

• $PI_1 = \dfrac{e^{2z}}{D'^2 - 5D' + 6}$ (here, $D' = 2$) [Type 1]

$D'^2 - 5D' + 6 = 2^2 - 5(2) + 6 = 0$

$\therefore$ We differentiate denominator w.r.t D' ~~& multi~~

& multiply numerator by z

$\Rightarrow PI_1 = \dfrac{e^{2z} \cdot z}{2D' - 5}$

$\bullet\ PI_1 = \frac{z e^{2z}}{2(2)-5}$

$= -z e^{2z}$] (or) $-z(e^z)^2$

But $z = \log x$ & $e^z = x$

$\Rightarrow PI_1 = -\log x \ e^{2\log x}$

$= -\log x \ e^{\log x^2}$] (or) $PI_1 = -\log x \cdot x^2$

$= -x^2 \log x$

$\bullet\ PI_2 = \frac{z}{D'^2 - 5D' + 6}$ [Type 3]

$= \frac{z}{6\left[\frac{D'^2 - 5D'}{6} + 1\right]}$

$= \frac{1}{6}\left[1 + \frac{D'^2 - 5D'}{6}\right]^{-1} z$

We know,

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$

$\Rightarrow PI_2 = \frac{1}{6}\left[1 - \left(\frac{D'^2 - 5D'}{6}\right)\right] z$ (other higher powers = 0)

$= \frac{1}{6}\left[z - \cancel{\frac{D'^2(z)}{6}} + \frac{5D'(z)}{6}\right]$ $D' = \frac{d}{dz}$ & $D'^2 = \frac{d^2}{dz^2}$

$= \frac{1}{6}\left[z + \frac{5}{6}\right]$

$\bullet\ PI_2 = \frac{z}{6} + \frac{5}{36} = \frac{\log x}{6} + \frac{5}{36}$

$\therefore PI = PI_1 + PI_2$

$= -x^2 \log x + \frac{\log x}{6} + \frac{5}{36}$

$\therefore y = CF + PI$

$= C_1 e^{\log x^2} + C_2 e^{\log x^3} - x^2 \log x + \frac{\log x}{6} + \frac{5}{36}$

Q4 Solve: $(x^2D^2+4xD+2)y=\log x$ when $x=1$, $y=0$ & $\frac{dy}{dx}=0$

$\rightarrow$ find C_1 & C_2

Sol:- $(x^2D^2+4xD+2)y=\log x$

Let $x=e^z \Rightarrow z=\log x$

$\therefore$ We have:- $xD=D'$

$x^2D^2=D'(D'-1)$

$\therefore [D'(D'-1)+4D'+2]\,y=\cancel{\log x}\ z$

$\Rightarrow [D'^2-D'+4D'+2]\,y=z$

$\Rightarrow (D'^2+3D'+2)\,y=z$

A.E:- $m^2+3m+2=0$

$\Rightarrow m^2+m+2m+2=0$

$\Rightarrow m(m+1)+2(m+1)=0$

$\Rightarrow (m+2)(m+1)$ $\therefore m=-1,-2$

C.F:- $CF=C_1e^{-z}+C_2e^{-2z}$ (in terms of z)

$=C_1\cdot\frac{1}{e^z}+C_2\frac{1}{(e^z)^2}$

~~P.I:- P.I =~~

$\Rightarrow C.F=\frac{C_1}{x}+\frac{C_2}{x^2}$

P.I:- $PI=\frac{z}{D'^2+3D'+2}$ [Type 3]

$=\frac{z}{2\left(\frac{D'^2+3D'}{2}+1\right)}$

$=\frac{1}{2}\left[1+\frac{D'^2+3D'}{2}\right]^{-1}z$

$\cancel{=\frac{1}{2}\left[1+\frac{D'^2}{2}+\right.}$

We know:-

$$(1+x)^{-1} = 1 - x + x^2 + \ldots$$

• ~~P.I =~~

• $P.I = \frac{1}{2}\left[1 - \left(\frac{D'^2 + 3D'}{2}\right)\right] 2$ $\left[D' = \frac{d}{dz} \text{ \& } D'^2 = \frac{d^2}{dz^2}\right]$

$$= \frac{1}{2}\left[2 - \frac{D'^2(2)}{2} - \frac{3D'(2)}{2}\right]$$

$$= \frac{1}{2}\left[2 - \frac{3(1)}{2}\right]$$

$$= \frac{2}{2} - \frac{3}{4}$$

$$\Rightarrow P.I = \frac{\log x}{2} - \frac{3}{4}$$

$\therefore y = CF + PI$

$$\Rightarrow y = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{\log x}{2} - \frac{3}{4} \quad \text{—— } (*)$$

• A/Q, $x = 1$ & $y = 0$

$$\Rightarrow 0 = \frac{C_1}{1} + \frac{C_2}{1^2} + \frac{\log(1)}{2} - \frac{3}{4}$$

$$\Rightarrow C_1 + C_2 = \frac{3}{4} \quad \text{—— } (1)$$

Differentiate $(*)$ w.r.t x

$$\Rightarrow \frac{dy}{dx} = \frac{-C_1}{x^2} - \frac{2C_2}{x^3} + \frac{1}{2x}$$

• A/Q, $x = 1$ & $dy/dx = 0$

$$\Rightarrow 0 = \frac{-C_1}{1^2} - \frac{2C_2}{1^3} + \frac{1}{2(1)}$$

$$\Rightarrow C_1 + 2C_2 = \frac{1}{2} \quad \text{—— } (2)$$

Solving (1) & (2) :-

$(1) - (2) \Rightarrow C_2 - 2C_2 = \frac{3}{4} - \frac{1}{2}$

$\Rightarrow -C_2 = \frac{3-2}{4}$

$\Rightarrow C_2 = -\frac{1}{4}$

$(2) \Rightarrow C_1 + 2\left(-\frac{1}{4}\right) = \frac{1}{2}$

$\Rightarrow C_1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow C_1 = 1$

$$\therefore C_1 = -\frac{1}{4} \text{ \& } C_2 = 1$$

Q. Solve: $(x^2D^2+3xD+5)y = x\cos(\log x)$

Sol: $(x^2D^2+3xD+5)y = x\cos(\log x)$

Let $x = e^z \Rightarrow \log x = z$

$\therefore$ We have: $xD = D'$

$x^2D^2 = D'(D'-1)$

$\left(D' = \dfrac{d}{dz} \;\therefore \text{In terms of } z\right)$

$\therefore [D'(D'-1)+3D'+5]y = e^z\cos(z)$

$\Rightarrow [D'^2-D'+3D'+5]y = e^z\cos z$

$\Rightarrow [D'^2+2D'+5]y = e^z\cos z$

A.E:- $m^2+2m+5=0$

Here, $a=1$, $b=2$, $c=5$

$\therefore m = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}$

$= \dfrac{-2\pm\sqrt{4-20}}{2} = \dfrac{-2\pm\sqrt{-16}}{2}$

$\Rightarrow m = \dfrac{-2\pm 4i}{2} = -1\pm 2i$ $\quad [\alpha = -1 \;\&\; \beta = 2]$

CF:- $CF = e^{\alpha z}(C_1\cos\beta z + C_2\sin\beta z)$

~~$= e^{-1z}(C_1\cos 2$~~

$= e^{-z}(C_1\cos 2z + C_2\sin 2z)$

$= \dfrac{1}{e^z}(C_1\cos 2z + C_2\sin 2z)$

But, $e^z = x$ & $z = \log x$

$\Rightarrow CF = \dfrac{1}{x}\left(C_1\cos 2(\log x) + C_2\sin 2(\log x)\right)$

$= \dfrac{1}{x}\left[C_1\cos(2\log x) + C_2\sin(2\log x)\right]$

$$P.I = \frac{e^{z}\cos z}{D'^2+2D'+5} \xrightarrow{\text{for this}} [D' \to D'+1] \quad (\text{Type } 4)$$

$$= \frac{e^z \cos z}{(D'+1)^2+2(D'+1)+5}$$

$$= \frac{e^z \cos z}{D'^2+2D'+1+2D'+2+5}$$

$$= \frac{e^z \cos z}{D'^2+4D'+8} \xrightarrow{\text{for this}} \left[\begin{array}{l} D'^2 = -a^2 = -1^2 \\ \Rightarrow D'^2 = -1 \end{array}\right] \quad (\text{Type } 2)$$

$$= \frac{e^z \cos z}{(-1)+4D'+8} = \frac{e^z \cos z}{4D'+7}$$

Now, we find its conjugate :-

$$P.I = \frac{e^z \cos z}{(4D'+7)} \frac{(4D'-7)}{(4D'-7)}$$

$$= \frac{(4D'-7)\, e^z \cos z}{16D'^2-49} \quad \left[D' = \frac{d}{dz}\right]$$

Only multiply $\cos z$ with $(4D'-7)$ and not e^z

$$\Rightarrow P.I = \frac{e^z \cdot (4D'\cos z - 7\cos z)}{16(-1)-49} \quad \left[D'\cos z = \frac{d}{dz}(\cos z) = -\sin z\right]$$

$$= \frac{e^z(-4\sin z - 7\cos z)}{-65}$$

$$P.I = \frac{e^z(4\sin z + 7\cos z)}{65}$$

$$= \frac{x}{65}\left[4\sin(\log x) + 7\cos(\log x)\right]$$

$$y = CF + PI$$

$$= \frac{1}{x}\left[C_1\cos(2\log x) + C_2\sin(2\log x)\right] + \frac{x}{65}\left[4\sin(\log x) + 7\cos(\log x)\right]$$