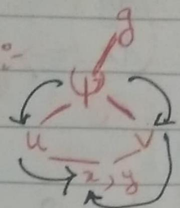
Partial Differentiation of Implicit functions

Q1. If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$, $v = 2xy$

~~Pro~~

Prove that :- $\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\dfrac{\partial^2 \psi}{\partial u^2} + \dfrac{\partial^2 \psi}{\partial v^2}\right]$

Soln :- Here, The chain(multi) :-



Here, we have to find $\dfrac{\partial g}{\partial x}$ and not $\dfrac{dg}{dx}$

A/c, $\dfrac{\partial g}{\partial x} = \dfrac{\partial \psi}{\partial x}$ $\left[g(x,y) = \psi(u,v)\right]$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \text{— (1)}$$

Here, $\dfrac{\partial \psi}{\partial u}$ & $\dfrac{\partial \psi}{\partial v}$ will remain as it is

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}(x^2 - y^2) = 2x$
- $\dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}(2xy) = 2y$

$$(1) \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(2y)$$

Hence;

$$\frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{\partial g}{\partial x} = \frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(2y) \quad \text{— (2)}$$

Now, To get $\dfrac{\partial^2 g}{\partial x^2}$:- ~~g we~~

We differentiate both sides w.r.t x in (2)

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(2y)\right]$$

Now, let's analyse:-

$$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right)$$

here, $\frac{\partial g}{\partial x} = \frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y)$

Since there is no 'g', we remove the g & φ

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial u}(2x) + \frac{\partial}{\partial v}(2y)$$

Do not ~~d~~ differentiate $\frac{\partial}{\partial u}(2x)$ & $\frac{\partial}{\partial v}(2y)$] keep them seperate

$$\therefore \frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) \quad \left[\begin{array}{l} \frac{\partial}{\partial x} = \frac{\partial}{\partial u}(2x) + \frac{\partial}{\partial v}(2y) \\ \frac{\partial g}{\partial x} = \frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y) \end{array}\right]$$

$$= \left[\frac{\partial}{\partial u}(2x) + \frac{\partial}{\partial v}(2y)\right]\left[\frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y)\right]$$

Here, • $\frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial u}\right) = \frac{\partial^2 \varphi}{\partial u^2}$ • $\frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial v}\right) = \frac{\partial^2 \varphi}{\partial u \partial v}$

• $\frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial u}\right) = \frac{\partial^2 \varphi}{\partial v \partial u}$ • $\frac{\partial}{\partial v}\left(\frac{\partial \varphi}{\partial v}\right) = \frac{\partial^2 \varphi}{\partial v^2}$

$$\therefore \frac{\partial^2 g}{\partial x^2} = 4x^2 \frac{\partial^2 \varphi}{\partial u^2} + 4xy \frac{\partial^2 \varphi}{\partial u \partial v} + 4xy \frac{\partial^2 \varphi}{\partial v \partial u} + 4y^2 \frac{\partial^2 \varphi}{\partial v^2}$$

Therefore, we got 1st expression $\left(\frac{\partial^2 g}{\partial x^2}\right)$

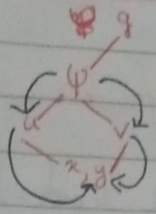
Key points:- 1) $\frac{\partial}{\partial x}(2x)$ do not write it as $\frac{\partial(2x)}{\partial x} = 2\frac{\partial x}{\partial x} = 2$ ✗

Leave $\frac{\partial}{\partial x}$ as it is

2) $\frac{\partial}{\partial u}\left(\frac{\partial \varphi}{\partial u}\right)$ can be written as $\frac{\partial^2 \varphi}{\partial u^2}$

Any derivative w.r.t something can be multiplied by $\frac{\partial}{\partial u}$. Do not multiply non-derivative (eg. 2x) with $\frac{\partial}{\partial u}$ (Point 1)

Similarly, for $\frac{\partial g}{\partial y} = \frac{\partial \varphi}{\partial y}$



$\therefore \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial y}$

Here, $\frac{\partial \varphi}{\partial u}$ & $\frac{\partial \varphi}{\partial v}$ will remain as it is

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$

$\therefore \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial u}(-2y) + \frac{\partial \varphi}{\partial v}(2x)$

Hence,

$$\frac{\partial g}{\partial y} = \frac{\partial \varphi}{\partial y}$$

$$\Rightarrow \frac{\partial g}{\partial y} = \frac{\partial u}{\partial y}(-2y) + \frac{\partial v}{\partial y}(2x)$$

To get $\frac{\partial^2 g}{\partial y^2}$, we differentiate $\frac{\partial g}{\partial y}$ w.r.t $\frac{\partial}{\partial y}$

Now, let's analyse

$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right)$

$\rightarrow$ here, $\frac{\partial g}{\partial y} = \frac{\partial \varphi}{\partial u}(-2y) + \frac{\partial \varphi}{\partial v}(2x)$

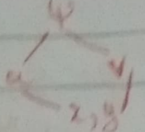
$\downarrow$ Remove 'g' & φ

$\therefore \frac{\partial}{\partial y} = \frac{\partial}{\partial u} \cdot (-2y) + \frac{\partial}{\partial v}(2x)$

Do not write it as $\left\{ \frac{\partial(-2y)}{\partial u} = -2\frac{\partial y}{\partial u} \times \quad \& \quad \frac{\partial}{\partial v}(2y) = 2\frac{\partial y}{\partial v} \times \right\}$

Dont do it

(*) The reason for this is simple:-



Here, u depends on x & y $\rightarrow \frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

But, x & y does not depend on u

Hence, we cannot write $\frac{\partial y}{\partial u}$ or $\frac{\partial x}{\partial u}$

∴ We have:-

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial u}(-2y) + \frac{\partial}{\partial v}(2y)$$

$$\& \quad \frac{\partial g}{\partial y} = \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}(2y)$$

$$\therefore \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \left[\frac{\partial}{\partial u}(-2y) + \frac{\partial}{\partial v}(2x)\right]\left[\frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}(2x)\right]$$

$$\therefore \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 \psi}{\partial u^2}(4y^2) - 4xy\frac{\partial^2 \psi}{\partial u \partial v} - 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4x^2\frac{\partial^2 \psi}{\partial v^2}$$

Therefore, we got 2^{nd} expression $\frac{\partial^2 g}{\partial y^2}$

$$\therefore LHS = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$

$$= 4x^2\frac{\partial^2 \psi}{\partial u^2} + 4xy\frac{\partial^2 \psi}{\partial u \partial v} + 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4y^2\frac{\partial^2 \psi}{\partial v^2}$$

$$+ 4y^2\frac{\partial^2 \psi}{\partial u^2} - 4xy\frac{\partial^2 \psi}{\partial u \partial v} - 4xy\frac{\partial^2 \psi}{\partial v \partial u} + 4x^2\frac{\partial^2 \psi}{\partial v^2}$$

Here, $4x^2\frac{\partial^2 \psi}{\partial u^2}$ & $4x^2\frac{\partial^2 \psi}{\partial v^2}$ are two different term, take common & dont add them as $8x^2$ ....

$$= \frac{\partial^2 \psi}{\partial u^2}(4x^2+4y^2) + \frac{\partial^2 \psi}{\partial v^2}(4x^2+4y^2)$$

$$= (4x^2+4y^2)\left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right)$$

$$= 4(x^2+y^2)\left(\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right)$$

$= RHS$ Hence proved

Q2. If $u = e^x \cos y$; $v = e^x \sin y$

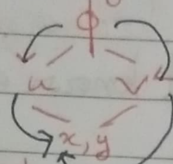
ϕ is a function of u & v and also x & y

Prove that:- $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\dfrac{\partial^2\phi}{\partial u^2} + \dfrac{\partial^2\phi}{\partial v^2}\right)$

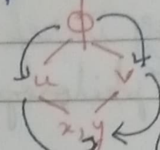
Solⁿ:- $\phi(x, y) = \phi(u, v)$ [A/c to question]

Also, $u = e^x \cos y$ & $v = e^x \sin y$ $\therefore$ u & v depends on x & y

By multi-chain rule:-



(For $\partial\phi/\partial x$)



(For $\partial\phi/\partial y$)

For 1st expression:-

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

We leave $\partial\phi/\partial u$ & $\partial\phi/\partial v$ as it is.

$\therefore \cancel{\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial u}}$ $\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}\left(e^x \cos y\right) = e^x \cos y$

& $\dfrac{\partial v}{\partial x} = \dfrac{\partial}{\partial x}\left(e^x \sin y\right) = e^x \sin y$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}(e^x \cos y) + \frac{\partial\phi}{\partial v}(e^x \sin y) \quad \text{——— (1)}$$

Now to find $\dfrac{\partial^2\phi}{\partial x^2}$:- $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\phi}{\partial x}\right)$

From (1), $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial u}(e^x \cos y) + \dfrac{\partial}{\partial v}(e^x \sin y)$

$$\therefore \frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \left[\frac{\partial}{\partial u}(e^x \cos y) + \frac{\partial}{\partial v}(e^x \sin y)\right]\left[\frac{\partial\phi}{\partial u}(e^x \cos y) + \frac{\partial\phi}{\partial v}(e^x \sin y)\right]$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial u^2}\left(e^{2x}\cos^2 y\right) + \left(e^{2x}\sin y \cos y\right)\frac{\partial^2\phi}{\partial u\,\partial v} + \left(e^{2x}\sin y \cos y\right)\frac{\partial^2\phi}{\partial u\,\partial v} + \left(e^{2x}\sin^2 y\right)\frac{\partial^2\phi}{\partial v^2} \quad \text{——— (I)}$$

$\therefore$ We get 1st expression

$$\cancel{\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial y}\cdot\frac{\partial u}{\partial y}} + \quad \text{For } 2^{nd} \text{ expression:-}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(e^x \cos y\right) = e^x(-\sin y)$$

$$\& \ \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin y\right) = e^x \cos y$$

$$\therefore \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\left(-e^x \sin y\right) + \frac{\partial \phi}{\partial v}\left(e^x \cos y\right) \quad \text{—— (2)}$$

Now to find $\frac{\partial^2 \phi}{\partial y^2}$:- $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$

From (1), $\frac{\partial}{\partial y} = \frac{\partial}{\partial u}\left(-e^x \sin y\right) + \frac{\partial}{\partial v}\left(e^x \cos y\right)$

$$\therefore \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \cancel{\left[\frac{\partial}{\partial u}\left(-e^x \sin y\right) + \frac{\partial}{\partial v}\right.}$$

$$= \left[\frac{\partial}{\partial u}\left(-e^x \sin y\right) + \frac{\partial}{\partial v}\left(e^x \cos y\right)\right]\left[\frac{\partial \phi}{\partial u}\left(-e^x \sin y\right) + \frac{\partial \phi}{\partial v}\left(e^x \cos y\right)\right]$$

$$= e^{2x}\sin^2 y \frac{\partial^2 \phi}{\partial u^2} - \left(e^{2x}\sin y \cos y\right)\frac{\partial^2 \phi}{\partial u \partial v} - \left(e^{2x}\sin y \cos y\right)\frac{\partial^2 \phi}{\partial v \partial u} + \left(e^{2x}\cos^2 y\right)\frac{\partial^2 \phi}{\partial v^2} \quad \text{—— (II)}$$

$$\text{(I)} + \text{(II)} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial u^2}\left[e^{2x}\cos^2 y + e^{2x}\sin^2 y\right] + \frac{\partial^2 \phi}{\partial v^2}\left[e^{2x}\sin^2 y + e^{2x}\cos^2 y\right]$$

A/c, $u = e^x \cos y$, $v = e^x \sin y$

$\therefore e^{2x}\cos^2 y = (e^x \cos y)^2 = u^2$

$\therefore e^{2x}\sin^2 y = (e^x \sin y)^2 = v^2$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial u^2}\left(u^2+v^2\right) + \frac{\partial^2 \phi}{\partial v^2}\left(u^2+v^2\right)$$

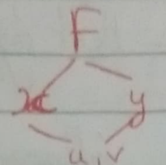
$$= \left(u^2+v^2\right)\left(\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right)$$

Hence proved.

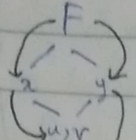
Q3. $F(x,y) = F(u,v)$ where $x = e^u \sin v$, $y = e^u \cos v$

Show that :- $\dfrac{\partial^2 F}{\partial u^2} + \dfrac{\partial^2 F}{\partial v^2} = \underbrace{(x^2+y^2)}_{e^{2u}} \left[\dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2}\right]$

Sol^n:- Here, by multichain we have :-



1. For $\dfrac{\partial^2 F}{\partial u^2}$:-



$\cancel{\dfrac{\partial F}{\partial u}}$ $\left[\because \begin{array}{l} x \xrightarrow{\text{depends on}} u \\ y \xrightarrow{\text{"}} v \end{array}\right]$

$$\therefore \frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} \quad \text{—— (1)}$$

Here, $\dfrac{\partial F}{\partial x}$ & $\dfrac{\partial F}{\partial y}$ will remain as it is

- $\dfrac{\partial x}{\partial u} = \dfrac{\partial}{\partial u}\left(e^u \sin v\right) = e^u \sin v$ • $\dfrac{\partial y}{\partial u} = \dfrac{\partial}{\partial u}\left(e^u \cos v\right) = e^u \cos v$

$$(1) \Rightarrow \frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\left(e^u \sin v\right) + \frac{\partial F}{\partial y}\left(e^u \cos v\right)$$

But, $x = e^u \sin v$ & $y = e^u \cos v$

$$\Rightarrow \frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} \cdot x + \frac{\partial F}{\partial y} \cdot y \quad \text{—— (2)}$$

To get $\dfrac{\partial^2 F}{\partial u^2}$, we differentiate ~~both sides~~ w.r.t. u in $\dfrac{\partial F}{\partial u}$

$$\therefore \frac{\partial^2 F}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right)$$

↳ To find $\dfrac{\partial}{\partial u}$, we eliminate the following :-

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) \quad [F \to \text{eliminated}]$$

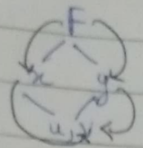
$\therefore$ We have :-
- $\dfrac{\partial F}{\partial u} = \dfrac{\partial F}{\partial x}(x) + \dfrac{\partial F}{\partial y}(y)$
- $\dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y)$

$$\therefore \frac{\partial^2 F}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right)$$

$$= \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)\right]\left[\frac{\partial F}{\partial x}(x) + \frac{\partial F}{\partial y}(y)\right]$$

$$\Rightarrow \frac{\partial^2 F}{\partial u^2} = \frac{\partial^2 F}{\partial x^2}(x^2) + \frac{\partial^2 F}{\partial x \partial y}(xy) + \frac{\partial^2 F}{\partial y \partial x}(xy) + \frac{\partial^2 F}{\partial y^2}(y^2) \quad \text{— (I)}$$

2. For $\frac{\partial^2 F}{\partial v^2}$:-



$$\therefore \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} \quad \text{—— (3)}$$

- $\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(e^u \sin v) = e^u \cos v$ • $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(e^u \cos v) = -e^u \sin v$

$$(3) \Rightarrow \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}(e^u \cos v) + \frac{\partial F}{\partial y}(-e^u \sin v)$$

But, $x = e^u \sin v$ & $y = e^u \cos v$

$$\Rightarrow \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}(y) - \frac{\partial F}{\partial y}(x) \quad \text{—— (4)}$$

To get $\frac{\partial^2 F}{\partial v^2}$, we differentiate $\frac{\partial F}{\partial u}$ w.r.t v

$$\therefore \frac{\partial^2 F}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial v}\right)$$

$$\rightarrow \frac{\partial}{\partial v} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)$$

$$\Rightarrow \frac{\partial^2 F}{\partial v^2} = \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]\left[\frac{\partial F}{\partial x}(y) - \frac{\partial F}{\partial y}(x)\right]$$

$$\Rightarrow \frac{\partial^2 F}{\partial v^2} = \frac{\partial^2 F}{\partial x^2}(y^2) - \frac{\partial^2 F}{\partial x \partial y}(xy) - \frac{\partial^2 F}{\partial y \partial x}(xy) + \frac{\partial^2 F}{\partial y^2}(x^2) \quad \text{— (II)}$$

$\therefore$ (I) + (II) :-

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = \frac{\partial^2 F}{\partial x^2}(x^2) + \frac{\partial^2 F}{\partial x^2}(y^2) + \frac{\partial^2 F}{\partial y^2}(y^2) + \frac{\partial^2 F}{\partial y^2}(x^2)$$

$$= \frac{\partial^2 F}{\partial x^2}(x^2 + y^2) + \frac{\partial^2 F}{\partial y^2}(x^2 + y^2)$$

$$\Rightarrow \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = (x^2 + y^2)\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right]$$

Hence proved