Q1. Find the radius of curvature at θ of cycloid:-
$x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

Sol:- Given parametric form:- • $x = a(\theta + \sin\theta)$
• $y = a(1 - \cos\theta)$

Differentiate w.r.t θ:-
• $x_1 = a(1 + \cos\theta)$
• $x_2 = a(-\sin\theta)$
• $y_1 = a(-(-\sin\theta)) = a\sin\theta$
• $y_2 = a\cos\theta$

Radius of curvature is given by:-

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$$

$$\therefore \rho = \frac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a(1+\cos\theta)\,a\cos\theta - a\sin\theta \cdot a(-\sin\theta)}$$

$$= \frac{[a^2(1+\cos^2\theta + 2\cos\theta) + a^2\sin^2\theta]^{3/2}}{a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{[a^2 + a^2(\cos^2\theta + \sin^2\theta) + 2a^2\cos\theta]^{3/2}}{a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{[2a^2(1+\cos\theta)]^{3/2}}{a^2(1+\cos\theta)}$$

$$= \frac{2^{3/2} a^{2\times 3/2} (1+\cos\theta)^{3/2}}{a^2(1+\cos\theta)}$$

$$= 2^{3/2} a (1+\cos\theta)^{3/2 - 1}$$

$$= 2^{3/2} a (1+\cos\theta)^{1/2}$$

$1 + \cos\theta = 1 + \cos 2\left(\frac{\theta}{2}\right)$
$= \cancel{1} + 2\cos^2\frac{\theta}{2} - \cancel{1}$

$\Rightarrow 1 + \cos\theta = 2\cos^2(\theta/2)$

$$\therefore \rho = 2^{3/2} a \left[2\cos^2(\theta/2)\right]^{1/2}$$

$$= 2^{3/2 + 1/2} a \cos\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \rho = 4a\cos(\theta/2)$$

Q2 For the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$
Prove that radius of curvature is $a\theta$

Sol:- By method 1 :- $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Given Parametric form :- $\cdot\ x = a(\cos\theta + \theta\sin\theta)$
$\cdot\ y = a(\sin\theta - \theta\cos\theta)$

- Find $\dfrac{dy}{dx}$:- $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$

- $\dfrac{dy}{d\theta} = a\left[\cos\theta - \{\theta(-\sin\theta) + \cos\theta\}\right]$

$= a[\cancel{\cos\theta} + \theta\sin\theta - \cancel{\cos\theta}]$ $\therefore \dfrac{dy}{d\theta} = a\theta\sin\theta$

- $\dfrac{dx}{d\theta} = a\left[\sin\theta - \{\theta(\cos\theta) + \sin\theta\}\right]$

$= a[\cancel{\sin\theta} - \theta\cos\theta - \cancel{\sin\theta}]$ $\therefore \dfrac{dx}{d\theta} = -a\theta\cos\theta$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx} = \dfrac{\cancel{a\theta}\sin\theta}{-\cancel{a\theta}\cos\theta} \Rightarrow \dfrac{dy}{dx} = -\tan\theta$

- Find $\dfrac{d^2y}{dx^2}$:- $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) \times \dfrac{d\theta}{dx}$

- $\dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{d\theta}\left(-\tan\theta\right)$ $\therefore \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) = -\sec^2\theta$

- $\dfrac{d\theta}{dx} = -\dfrac{1}{a\theta\cos\theta}$

$\therefore \dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right) \times \dfrac{d\theta}{dx}$

$= \cancel{-}\sec^2\theta \times \dfrac{1}{\cancel{-}a\theta\cos\theta}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{1}{a\theta\cos^3\theta}$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{\{1+(-\tan\theta)^2\}^{3/2}}{\frac{1}{a\theta\cos^3\theta}}$$

$$= \frac{(1+\tan^2\theta)^{3/2}}{} \times a\theta\cos^3\theta$$

$$= (\sec^2\theta)^{3/2} \times a\theta \frac{1}{\sec^3\theta}$$

$$= \cancel{\sec^3\theta} \times a\theta \frac{1}{\cancel{\sec^3\theta}}$$

$$\Rightarrow \rho = a\theta$$

Hence proved

Q3. Find the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$

Sol:- Given curve :- $x^{2/3}+y^{2/3}=a^{2/3}$

Differentiate both sides w.r.t x :-

- $\frac{dy}{dx}$:- $\frac{2}{3}x^{2/3-1} + \frac{2}{3}y^{2/3-1}\frac{dy}{dx} = 0$

$$\Rightarrow \frac{2}{3}\left[x^{-1/3} + y^{-1/3}\frac{dy}{dx}\right] = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}}$$

$$\therefore y_1 = \frac{y^{1/3}}{x^{1/3}}$$

At $(a\cos^3\theta, a\sin^3\theta)$:- $y_1 = \frac{(a\sin^3\theta)^{1/3}}{(a\cos^3\theta)^{1/3}}$

$$= \frac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}$$

$$\Rightarrow y_1 = \tan\theta$$

$\frac{d^2y}{dx^2}$:- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\tan\theta\right)$

$= \sec^2\theta \cdot \frac{d\theta}{dx}$

At $(a\cos^3\theta, a\sin^3\theta)$

$\quad\downarrow x \qquad \downarrow y$

$\therefore x = a\cos^3\theta$

$\Rightarrow \frac{dx}{d\theta} = a\,3\cos^2\theta\,(-\sin\theta)$

$\cancel{\therefore \frac{d^2y}{dx^2} = \sec^2\theta \times a(-3)\cos^2\theta\sin\theta}$

$\cancel{= \frac{-3a}{\cos^2\theta} \times \cos^2\theta\sin\theta}$

$\cancel{\Rightarrow \frac{d^2y}{dx^2} =}$

$\therefore \frac{d\theta}{dx} = \frac{1}{-3a\cos^2\theta\sin\theta}$

Here, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{\left[1+\tan^2\theta\right]^{3/2}}{\left(\frac{-1}{3a\cos^4\theta\sin\theta}\right)}$

$= \frac{(\sec^2\theta)^{3/2}}{\left(\frac{-1}{3a\cos^4\theta\sin\theta}\right)}$

$\therefore \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{-3a\cos^2\theta\sin\theta}$

$= \frac{1}{\cos^2\theta} \times \frac{1}{-3a\cos^2\theta\sin\theta}$

$= \frac{-1}{3a\cos^4\theta\sin\theta}$

$\Rightarrow \therefore \rho = \sec^3\theta \times -3a\cos^4\theta\sin\theta$

$= \frac{1}{\cos^3\theta} \times -3a\cos^4\theta\sin\theta$

$= -3a\cos\theta\sin\theta$

can't be -ve

$\therefore \rho = 3a\cos\theta\sin\theta$