Raabe's Test :-

The series Σu_n of positive terms is :-

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

(i) Convergence if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$

(ii) Divergence if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$

(iii) Test fails if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$

Q1. Test the convergence of series :- ~~$1 + \frac{3x}{7} + \frac{3.6}{7.10} + 3.6$~~

$$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \ldots$$

Soln :- In the series, $1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \ldots$

We can omit as we are concerned with the general form of u_n

n starts from $2 \to \infty$

$$u_n = \frac{3.6.9\ldots(3n-3)}{7.10.13\ldots(3n+1)} \cdot x^{n-1}$$

$$\therefore u_{n+1} = \frac{3.6.9\ldots(3n-3)(3n)}{7.10.13\ldots(3n+1)(3n+4)} \cdot x^n$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{3.6.9\ldots(3n-3)(3n)\cdot \cancel{x^n}}{7.10.13\ldots(3n+1)(3n+4)} \cdot \frac{7.10.13\ldots(3n+1)}{3.6.9\ldots(3n-3)\,\cancel{x^n}\cdot x^{-1}}$$

$$= \frac{3n}{(3n+4)x^{-1}}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{3n}{3n+4} \cdot x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n}{3n+4} \cdot x$$

$$= \lim_{n\to\infty} \cdot \frac{3\cancel{n}}{\cancel{n}\left(\frac{3}{\cancel{n}}+\frac{4}{n}\right)} \cdot x$$

$$= \frac{3x}{\left(3+\frac{4}{\infty}\right)}$$

$$= \frac{\cancel{3}x}{\cancel{3}}$$

$$\Rightarrow l = x$$

- By ~~Raabe~~ Ratio's Test, $x < 1 \rightarrow$ convergent
  $x > 1 \rightarrow$ divergent

For Raabe's Test :-

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{3n+4}{3n\cdot x} - 1\right)$$

$$= \cancel{n}\left(\frac{3n+4-3nx}{3\cancel{n}x}\right)$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{\cancel{3n}+4-\cancel{3n}}{3x}$$

$$= \lim_{n\to\infty} \frac{\cancel{n\left(3+\frac{4}{n}-3\right)}\,4}{3x}$$

$$\Rightarrow l = \frac{4}{3x}$$

- By Raabe's Test, $x > 1$, $l \rightarrow +ve$:- convergent
  $x < 1$, $l \rightarrow -ve$:- divergent

Do 1) Ratio test (1st)
2) Raabe's test (2nd)

* Q2. Find the convergence of the series :- (By Raabe's Test)

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot4\cdot5\cdot6}x^6 + \dots\dots \infty$$

Sol:- In this series, $x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot4\cdot5\cdot6}x^6 + \dots\dots$

omitted

$\cancel{u_n = (2n)^2}$ n starts from $1 \to \infty$

$$\therefore\ u_n = \underbrace{\frac{2^2\cdot 4^2\cdots\cdots(2n)^2}{3\cdot5\cdots(2n+1)\times 4\cdot 6\cdots(2n+2)}}_{\text{multiplicative series}}\ \underbrace{x^{2n+2}}_{\text{summation series}}$$

$$u_{n+1} = \frac{2^2\cdot4^2\cdots(2n)^2(2n+2)^2}{3\cdot5\cdots(2n+1)(2n+3)\times 4\cdot6\cdots(2n+2)(2n+4)}\ x^{2n+4}$$

term

previous terms

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2^2\cdot4^2\cdots(2n)^2}{3\cdot5\cdots(2n+1)\times4\cdot6\cdots(2n+2)}\times x^{2n+2} \times \frac{3\cdot5\cdots(2n+1)(2n+3)\times4\cdot6\cdots(2n+2)(2n+4)}{2^2\cdot4^2\cdots(2n)^2(2n+2)^2\times x^{2n+2}\cdot x^2}$$

$$= \frac{(2n+3)(2n+4)}{(2n+2)^2 \cdot x^2}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2\cdot x^2} - 1\right]$$

Lt $x = 1$

$$= n\left[\frac{4n^2+14n+12-4n^2-8n-4}{4n^2+8n+4}\right]$$

$$= n\left[\frac{14n+8}{4n^2+8n+8}\right]$$

Take n common :-

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n\cdot n\left(14+\frac{8}{n}\right)}{n^2\left(4+\frac{8}{n}+\frac{8}{n^2}\right)}$$

$$= \frac{14+(8/n)}{4+\frac{8}{n}+\frac{8}{n^2}}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\frac{\left(14+\frac{8}{n}\right)}{\left(4+\frac{8}{n}+\frac{8}{n^2}\right)} = \frac{\left(14+\frac{8}{\infty}\right)}{\left(4+\frac{8}{\infty}+\frac{8}{\infty^2}\right)}$$

$$= \frac{14}{4} > 1$$

→ convergent if $x^2 > 1$
divergent if $x^2 < 1$

Q3. Test the convergence of :-

$$1 + a + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \ldots \infty$$

Sol:- In this series, 1 + a + $\frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \ldots$

omitted

n starts from $1 \to \infty$

$$\therefore u_n = \frac{a\cdot(a+1)(a+2)\ldots(a+n-1)}{1\cdot 2\cdot 3\ldots n}$$

$$\therefore u_{n+1} = \frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{1\cdot 2\cdot 3\ldots n(n+1)}$$

$$\cancel{\therefore \frac{u_n}{u_{n+1}} = 2}$$

• $1^{st} \to$ Test for Ratio Test :-

$$\frac{u_{n+1}}{u_n} = \frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{1\cdot 2\cdot 3\ldots n(n+1)} \times \frac{1\cdot 2\cdot 3\ldots n}{a(a+1)(a+2)\ldots(a+n-1)}$$

$$= \frac{a+n}{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{a+n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n\left(\frac{a}{n}+1\right)}{n\left(1+\frac{1}{n}\right)}$$

$$= \frac{\left(\frac{a}{\infty}+1\right)}{\left(1+\frac{1}{\infty}\right)}$$

$$\Rightarrow l = 1$$

$\because l = 1$, Ratio Test fails

$2^{nd} \rightarrow$ Test for Raabe's Test

$$\frac{u_{n+1}}{u_n} = \frac{a+n}{n+1} \quad \therefore \boxed{\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n}}$$

$$\therefore n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{n+1}{a+n} - 1\right]$$

$$\cancel{= n\left[\frac{n+1-a-1}{a+1}\right]}$$

$$= n\left[\frac{\cancel{n}+1-a-\cancel{n}}{a+n}\right]$$

$$= \frac{\cancel{n}(1-a)}{\cancel{n}\left(\frac{a}{n}-1\right)}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{(1-a)}{\left(\frac{a}{n}-1\right)}$$

$$= \frac{1-a}{\left(\frac{a}{\infty}-1\right)}$$

$$\Rightarrow l = 1-a$$

By Raabe's Test, series is :-

- Convergent if $l > 1$
  $\Rightarrow \cancel{1} - a > \cancel{1}$
  $\Rightarrow a < 0$ (or) $\boxed{a \leq 0}$
- Divergent if $l < 1$
  $\Rightarrow \cancel{1} - a < \cancel{1}$
  $\Rightarrow a > 0 \Rightarrow \boxed{a > 0}$