Q1. Find the circle of curvature for the curve $xy = c^2$ at (c, c)

Sol:- For Circle of curvature :-

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

where :-

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \qquad \bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

Given curve :- $xy = c^2$

Differentiate both sides w.r.t x :-

$\cancel{\frac{dy}{dx}}$

$$\frac{d}{dx}(x \cdot y) = \frac{d}{dx}(c^2)$$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \boxed{y_1 = -\frac{y}{x}}$$

At (c,c): $y_1 = -\frac{\cancel{c}}{\cancel{c}}$

$$\Rightarrow y_1 = -1 \quad / \quad \frac{dy}{dx} = 1$$

$$y_2 = \frac{d}{dx}\left(\frac{-y}{x}\right)$$

$$= -\left[\frac{x\cdot\frac{dy}{dx} - y\cdot\frac{dx}{dx}}{x^2}\right]$$

$$= -\left[\frac{x(-1) - y\cdot(1)}{x^2}\right]$$

$$= -\left(\frac{-x-y}{x^2}\right) = \frac{x+y}{x^2}$$

At (c, c):- $\quad y_2 = \dfrac{c+c}{c^2}$

$$= \frac{2c}{c^2} \Rightarrow y_2 = \frac{2}{c}$$

- $\therefore \rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+(-1))^{3/2}}{2/c}$

$$= 2^{3/2-1} \times c$$

$$\Rightarrow \rho = \sqrt{2}\, c$$

- $\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$$= c - \frac{(-1)(1+(-1)^2)}{2/c}$$

$$= c + \frac{\cancel{2}c}{\cancel{2}}$$

$$\Rightarrow \bar{x} = 2c$$

- $\bar{y} = y + \dfrac{(1+y_1^2)}{y_2}$

$$= c + \frac{(1+(-1)^2)}{2/c}$$

$$= c + \frac{\cancel{2}c}{\cancel{2}}$$

$$\Rightarrow \bar{y} = 2c$$

Eqn of Circle of curvature at (c, c):-

$$(x-2c)^2 + (y-2c)^2 = (\sqrt{2}\,c)^2$$

$$\Rightarrow (x-2c)^2 + (y-2c)^2 = 2c^2$$

Q2. Find the circle of curvature to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left(\frac{a}{4},\frac{a}{4}\right)$

Sol:- Given curve :- $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Differentiate both sides w.r.t x :-

- $$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\cancel{2}\sqrt{x}}\times\cancel{2}\sqrt{y}$$

$$\Rightarrow y_1=-\frac{\sqrt{y}}{\sqrt{x}}$$

~~y_2~~ At $\left(\frac{a}{4},\frac{a}{4}\right)$:- $y_1=-\frac{\cancel{\sqrt{a/4}}}{\cancel{\sqrt{a/4}}}$ $\Rightarrow y_1=-1$

- $$y_2=-\frac{d}{dx}\left(\frac{\sqrt{y}}{\sqrt{x}}\right)$$

$$=\frac{\sqrt{x}\times\frac{d}{dx}(\sqrt{y})-\sqrt{y}\cdot\times\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2}$$

$$=\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot\cancel{\frac{dy}{dx}}^{-1}-\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}$$

$$=\frac{-\frac{\sqrt{x}}{2\sqrt{y}}-\frac{\sqrt{y}}{2\sqrt{x}}}{x}\qquad=\frac{-1}{2}\left(\frac{x+y}{\sqrt{xy}}\right)\times\frac{1}{x}$$

$=\cancel{\frac{-1}{2/4}}$ At $\left(\frac{a}{4},\frac{a}{4}\right)$:- $y_2=\frac{-1}{2}\left(\frac{\frac{a}{4}+\frac{a}{4}}{\sqrt{\frac{a}{4}\cdot\frac{a}{4}}}\right)\cdot\frac{1}{a/4}$

$$=-\frac{1}{\cancel{2}}\times\frac{\cancel{2}a}{\cancel{4}}\times\frac{\cancel{4}}{\cancel{a}}\times\frac{1}{a/4}$$

$$\Rightarrow y_2=\frac{4}{a}$$

Now, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+(-1)^2)^{3/2}}{4/a}$$

$$= \frac{(1+1)^{3/2}}{\cancel{4}\,2^2} \times a$$

$$= 2^{3/2-2} \times a = 2^{-1/2} a$$

$$\Rightarrow \rho = \frac{a}{\sqrt{2}}$$

- $\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$$= \frac{a}{4} - \frac{(-1)(1+(-1)^2)}{4/a}$$

$$= \frac{a}{4} + \frac{2a}{4}\cancel{\times a}$$

$$\Rightarrow \bar{x} = \frac{3a}{4}$$

- $\bar{y} = y + \dfrac{(1+y_1^2)}{y_2}$

$$= \frac{a}{4} + \frac{(1+(-1)^2)}{4/a}$$

$$= \frac{a}{4} + \frac{2a}{4}$$

$$\Rightarrow \bar{y} = \frac{3a}{4}$$

∴ ~~Rad~~ Circle of curvature given by :-

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\Rightarrow \left(x - \frac{3a}{4}\right)^2 + \left(y - \frac{3a}{4}\right)^2 = \frac{a^2}{2}$$

Q:- Find the circle of curvature of curve $y^2 = 12x$ at $(3,6)$

Sol:- ~~Given curve~~ $y^2 = 12x$

Differentiate w.r.t x :-

$$2y\frac{dy}{dx} = 12$$

$$\Rightarrow \frac{dy}{dx} = \frac{12}{2y}$$

$$\Rightarrow y_1 = \frac{6}{y}$$

At $(3,6)$:-

$$y_1 = \frac{6}{6}$$

$$\Rightarrow y_1 = 1$$

$$y_2 = \frac{d}{dx}\left(\frac{6}{y}\right)$$

$$= \frac{-6}{y^2}\cdot\cancel{\frac{dy}{dx}}^{1} = \frac{-6}{y^2}$$

At $(3,6)$:-

$$y_2 = \frac{-6}{6^2}$$

$$\Rightarrow y_2 = \frac{-1}{6}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+1^2]^{3/2}}{(-1/6)} = 2^{3/2}\times 2\times 3$$

$$\cancel{\Rightarrow \rho = \frac{\sqrt{2}}{3}}$$

$$\Rightarrow \rho = 2^1 \times 2^{1/2} \times 6$$

$$= 12\sqrt{2}$$

$$\therefore \boxed{\rho = 12\sqrt{2}}$$

- $$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 3 - \frac{(1)(1+1^2)}{(-1/6)}$$

$$= 3 + 2\times 6$$

$$\Rightarrow \bar{x} = 15$$

- $$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= 6 + \frac{(1+(1)^2)}{-1/6}$$

$$= 6 + 2\times 6$$

$$\Rightarrow \bar{y} = 18$$

The circle of curvature given by:- at $(3,6)$

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$(12\sqrt{2})^2 = 144\times 2$$

$$\Rightarrow (x-15)^2 + (y-18)^2 = 288$$