Q8 Find the length of the shortest line from the point $\left(0,0,\frac{25}{9}\right)$ to the surface $z = xy$

Sol^n:- Let (x,y,z) be point on surface $z = xy$

We have :- $\cdot A\left(0,0,\frac{25}{9}\right)$ $\cdot B(x,y,z)$

$$d = \sqrt{(x-0)^2 + (y-0)^2 + \left(z-\frac{25}{9}\right)^2}$$

$$\Rightarrow d^2 = x^2 + y^2 + \left(z - \frac{25}{9}\right)^2$$

We have :- $D(x,y,z) = x^2 + y^2 + \left(z-\frac{25}{9}\right)^2$

$$L(x,y,z) = z - xy$$

$$\therefore F(x,y,z) = D(x,y,z) + \lambda L(x,y,z)$$
$$= x^2 + y^2 + \left(z-\frac{25}{9}\right)^2 + \lambda(z - xy)$$

$\cdot\ F_x = 2x + \lambda(-y)$ $\quad \cdot\ F_y = 2y + \lambda(-x)$ $\quad \cdot\ F_z = 2\left(z - \frac{25}{9}\right) + \lambda$

$\cancel{F_x = 0}$ $\quad \cancel{F_y = 0}$

$F_x = 0$ $\quad F_y = 0$ $\quad F_z = 0$

$\Rightarrow \cancel{-}\lambda xy = \cancel{-}2x$ $\quad \Rightarrow \cancel{-}\lambda x = \cancel{-}2y$ $\quad \Rightarrow \lambda = -2\left(z - \frac{25}{9}\right)$

$\Rightarrow \lambda = \frac{2x}{y}$ — ① $\quad \Rightarrow \lambda = \frac{2y}{x}$ — ② $\quad \Rightarrow \lambda = -2\left(z-\frac{25}{9}\right)$ — ③

Equating ① & ② :-

$$\frac{\cancel{2}x}{y} = \frac{\cancel{2}y}{x}$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow y = \pm x \quad \text{— ④}$$

Equating ② & ③ :-

$$\frac{\cancel{2}y}{x} = -\cancel{2}\left(z - \frac{25}{9}\right)$$

$$\Rightarrow y = -x\left(z - \frac{25}{9}\right) \quad \text{— ⑤}$$

~~If $y = +x$~~ $\quad$ ~~If $y = x$~~

If $y = +x$

$(5) \Rightarrow \cancel{x} = \cancel{x}\left(z - \frac{25}{9}\right)$

$\Rightarrow -z + \frac{25}{9} = 1$

$\Rightarrow z = \frac{25}{9} - 1 = \frac{25-9}{9}$

$\Rightarrow z = \frac{16}{9}$

For $y=x$ $\hookrightarrow z = \frac{16}{9}$

If $y = -x$

$(5) \Rightarrow -\cancel{x} = -\cancel{x}\left(z - \frac{25}{9}\right)$

$\Rightarrow z - \frac{25}{9} = 1$

$\Rightarrow z = 1 + \frac{25}{9} = \frac{9+25}{9}$

$\Rightarrow z = \frac{34}{9}$

For $y=-x$ $\rightarrow z = \frac{34}{9}$

1) Using $y = x$ & $z = \frac{16}{9}$ in $z = xy$:-

$\frac{16}{9} = x(x) \Rightarrow x^2 = \frac{16}{9}$ $\therefore x = \pm\frac{4}{3}$

2) Using $y = -x$ & $z = \frac{34}{9}$ in $z = xy$:-

$\frac{34}{9} = x(-x) \Rightarrow -x^2 = \frac{34}{9}$ $\therefore x = \sqrt{\frac{-34}{9}}$ (not real value)

$\therefore x = \pm\frac{4}{3}$, $y = \pm\frac{4}{3}$ (since $y=x$), $z = \frac{16}{9}$

$\rightarrow y = x$ $\therefore$ If $+x, y = +x$; $-x, y = -x$

Stationary points are:- $A\left(\frac{4}{3}, \frac{4}{3}, \frac{16}{9}\right)$ & $B\left(-\frac{4}{3}, -\frac{4}{3}, \frac{16}{9}\right)$

$P\left(0, 0, \frac{25}{9}\right)$

- $AP = \sqrt{\left(\frac{4}{3}-0\right)^2 + \left(\frac{4}{3}-0\right)^2 + \left(\frac{16}{9}-\frac{25}{9}\right)^2} = \sqrt{\frac{16}{9} + \frac{16}{9} + \left(\frac{-9}{9}\right)^2} = \sqrt{\frac{41}{9}} = \frac{\sqrt{41}}{3}$ units

- $BP = \sqrt{\left(\frac{-4}{3}-0\right)^2 + \left(\frac{-4}{3}-0\right)^2 + \left(\frac{16}{9}-\frac{25}{9}\right)^2} = \sqrt{\frac{16}{9} + \frac{16}{9} + \left(\frac{9}{9}\right)^2} = \frac{\sqrt{41}}{3}$ units

$\therefore$ Length of shortest line from point $\left(0, 0, \frac{25}{9}\right)$ to surface $z = xy$ is $\frac{\sqrt{41}}{3}$ units

Q9. The temperature $u(x,y,z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2=1$

Sol:- We have:- $u(x,y,z) = 400xyz^2$
$S(x,y,z) = x^2+y^2+z^2-1$

$$F(x,y,z) = u(x,y,z) + \lambda S(x,y,z)$$
$$= 400xyz^2 + \lambda(x^2+y^2+z^2-1)$$

- $F_x = 400yz^2 + 2\lambda x$
$F_x = 0$
$\Rightarrow 400yz^2 + 2\lambda x = 0$
$\Rightarrow \lambda = \dfrac{-400yz^2}{2x}$ ——— (1)

- $F_y = 400xz^2 + 2\lambda y$
$F_y = 0$
$\Rightarrow 400xz^2 + 2\lambda y = 0$
$\Rightarrow \lambda = \dfrac{-400xz^2}{2y}$ ——— (2)

- $F_z = 800xyz + 2\lambda z$
$F_z = 0$
$\Rightarrow 800xyz + 2\lambda z = 0$
$\Rightarrow \lambda = \dfrac{-800xyz}{2z}$ ——— (3)

Equating (1) & (2) :-
$$\frac{-400yz^2}{2x} = \frac{-400xz^2}{2y}$$
$\Rightarrow y^2 = x^2$
$\therefore y = \pm x$

Equating (2) & (3) :-
$$\frac{-400xz^2}{2y} = \frac{-800xyz}{2z}$$
$\Rightarrow z^2 = 2y^2 = 2x^2$
$\therefore z = \pm\sqrt{2}\,y$ ——— (4)

$\therefore S(x,y,z) = 0 \Rightarrow x^2+y^2+z^2 = 1$
$\Rightarrow x^2 + x^2 + 2x^2 = 1$
$\Rightarrow 4x^2 = 1$

$\therefore x^2 = \dfrac{1}{4} \Rightarrow x = \pm\dfrac{1}{2}$

$\therefore x = \pm\dfrac{1}{2}$, $y = \pm\dfrac{1}{2}$, $z = \pm\sqrt{2}\cdot\dfrac{1}{2} = \pm\dfrac{\sqrt{2}}{2}$

Possible cases :-

1) $x = \dfrac{1}{2}$, $y = \dfrac{1}{2}$ (4) $\Rightarrow z = \pm\sqrt{2}\left(\dfrac{1}{2}\right) = \pm\dfrac{\sqrt{2}}{2}$, $z = \pm\dfrac{\sqrt{2}}{2} \rightarrow \dfrac{\sqrt{2}}{2}$ or $-\dfrac{\sqrt{2}}{2}$

2) $x = \dfrac{1}{2}$, $y = -\dfrac{1}{2}$ (4) $\Rightarrow z = \pm\sqrt{2}\left(-\dfrac{1}{2}\right) = \pm\dfrac{\sqrt{2}}{2}$, $z = \pm\dfrac{\sqrt{2}}{2} \rightarrow \dfrac{\sqrt{2}}{2}$ or $-\dfrac{\sqrt{2}}{2}$

3) $x = -\dfrac{1}{2}$, $y = \dfrac{1}{2}$ (4) $\Rightarrow z = \pm\sqrt{2}\left(\dfrac{1}{2}\right) = \pm\dfrac{\sqrt{2}}{2}$, $z = \pm\dfrac{\sqrt{2}}{2} \rightarrow \dfrac{\sqrt{2}}{2}$ or $-\dfrac{\sqrt{2}}{2}$

4) $x = -\dfrac{1}{2}$, $y = -\dfrac{1}{2}$ (4) $\Rightarrow z = \pm\sqrt{2}\left(-\dfrac{1}{2}\right) = \pm\dfrac{\sqrt{2}}{2}$, $z = \pm\dfrac{\sqrt{2}}{2} \rightarrow \dfrac{\sqrt{2}}{2}$ or $-\dfrac{\sqrt{2}}{2}$

The stationary points are :-

$\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{-\sqrt{2}}{2}\right)$

$\left(\frac{1}{2}, \frac{-1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}, \frac{-\sqrt{2}}{2}\right)$

$\left(\frac{-1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{-1}{2}, \frac{1}{2}, \frac{-\sqrt{2}}{2}\right)$

$\left(\frac{-1}{2}, \frac{-1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-\sqrt{2}}{2}\right)$

These 4 points give ~~max~~ non-negative value

$u = 400xyz^2$

[∵ Substituting any one of them will give max value]

$\therefore u\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right) = \cancel{400}^{100} \times \left(\frac{1}{\cancel{2}}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{\cancel{2}}\right)^2$

$= 100 \cdot \frac{\cancel{2}}{\cancel{4}\,2}$

$\Rightarrow$ Max temperature $= 50$

Q10. Find the maximum value of $x^m y^n z^p$ where $x+y+z=a$

Sol^n:- We have:- $f(x,y,z) = x^m y^n z^p$

$g(x,y,z) = x+y+z-a$

$$F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$$
$$= x^m y^n z^p + \lambda(x+y+z-a)$$

• $F_x = mx^{m-1} \cdot y^n z^p + \lambda$ • $F_y = ny^{n-1} \cdot x^m z^p + \lambda$ • $F_z = pz^{p-1} \cdot x^m y^n + \lambda$

$F_x = 0$ $F_y = 0$ $F_z = 0$

$\Rightarrow mx^{m-1} y^n z^p + \lambda = 0$ $\Rightarrow nx^m y^{n-1} z^p + \lambda = 0$ $\Rightarrow px^m y^n z^{p-1} + \lambda = 0$

$\Rightarrow \lambda = -mx^{m-1} y^n z^p$ — (1) $\Rightarrow \lambda = -nx^m y^{n-1} z^p$ — (2) $\Rightarrow \lambda = -px^m y^n z^{p-1}$ — (3)

Equating (1) & (2):-

$\cancel{-}mx^{m-1} y^n \cancel{z^p} = \cancel{-}n x^m y^{n-1} \cancel{z^p}$

$\Rightarrow \dfrac{mx^{m-1}}{x^m} = \dfrac{n y^{n-1}}{y^n}$

$\Rightarrow mx^{-1} = ny^{-1} \therefore \dfrac{m}{x} = \dfrac{n}{y}$

Equating (2) & (3):-

$\cancel{-}n\cancel{x^m} y^{n-1} z^p = \cancel{-}p\cancel{x^m} y^n z^{p-1}$

$\Rightarrow \dfrac{n y^{n-1}}{y^n} = \dfrac{p z^{p-1}}{z^p}$

$\Rightarrow n \cdot y^{-1} = pz^{-1} \therefore \dfrac{n}{y} = \dfrac{p}{z}$

$\therefore \dfrac{m}{x} = \dfrac{n}{y} = \dfrac{p}{z}$:- 1) $\dfrac{m}{x} = \dfrac{n}{y} \Rightarrow y = \dfrac{nx}{m}$

2) $\dfrac{n}{y} = \dfrac{p}{z} \Rightarrow z = \dfrac{p}{n} y = \dfrac{p}{\cancel{n}}\left(\dfrac{\cancel{n}x}{m}\right) = \dfrac{px}{m}$

$\therefore g(x,y,z) = 0 \Rightarrow x+y+z = a$

$\Rightarrow x + \left(\dfrac{nx}{m}\right) + \left(\dfrac{px}{m}\right) = a$

$\Rightarrow \dfrac{mx + nx + px}{m} = a$

$\Rightarrow x\left(\dfrac{m+n+p}{m}\right) = a$

$\Rightarrow x = \dfrac{am}{m+n+p}$

$\therefore x = \dfrac{am}{m+n+p}$, $y = \dfrac{n}{\cancel{m}} \cdot \dfrac{a\cancel{m}}{m+n+p}$, $z = \dfrac{p}{\cancel{m}} \cdot \dfrac{a\cancel{m}}{m+n+p}$

$\Rightarrow y = \dfrac{an}{m+n+p}$ $\Rightarrow z = \dfrac{ap}{m+n+p}$

$\therefore$ Stationary point is $\left(\dfrac{am}{m+n+p}, \dfrac{an}{m+n+p}, \dfrac{ap}{m+n+p}\right)$

$\therefore$ Maximum value is given by :-

$$f\left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right) = \left[\frac{am}{m+n+p}\right]^m \left[\frac{an}{m+n+p}\right]^n \left[\frac{ap}{m+n+p}\right]^p$$

$$= \frac{a^m m^m}{(m+n+p)^m} \cdot \frac{a^n n^n}{(m+n+p)^n} \cdot \frac{a^p \cdot p^p}{(m+n+p)^p}$$

$$= \frac{a^{m+n+p} \cdot m^m n^n p^p}{(m+n+p)^{m+n+p}}$$