Maclaurin Series Expansion :-

If $f(x)$ is a function ~~at the~~ such that it can be expanded in ascending order/power of 'x' & this function is differentiable at any no. of times

Then :- $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots \frac{x^n}{n!} f^n(0)$

Eg :- Expand e^x :- Let $y = e^x$ or $f(x) = e^x$

$\therefore y = e^x \longrightarrow y(0) = e^0 = 1$

$y' = e^x \longrightarrow y'(0) = e^0 = 1$

$\vdots$ (so on)

$y_n(0) = 1$

$\therefore e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \ldots\ldots \frac{x^n}{n!}(1)$

Eg :- Expand $\sin x$:- Let $y = \sin x \longrightarrow y(0) = \sin(0) = 0$

$y_1 = \cos x \longrightarrow y_1(0) = \cos(0) = 1$

$y_2 = -\sin x \longrightarrow y_2(0) = -\sin(0) = 0$

$y_3 = -\cos x \longrightarrow y_3(0) = -\cos(0) = -1$

$\vdots$ (so on)

$\therefore \sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \ldots$

Proof :- $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots\ldots$

Here, $f(0) = A_0 + 0 + \ldots = A_0$

$f'(0) = A_1 + 2A_2 x + 3A_3 x^2 + \ldots\ldots = A_1$

$f''(0) = 2A_2 + 6A_3 x + \ldots\ldots = 2A_2$

$f'''(0) = 6A_3 + 12A_4 x = 6A_3$

$\therefore A_0 = f(0) \qquad A_2 = \frac{f''(0)}{2!} \qquad A_3 = \frac{f'''(0)}{3!}$

$A_1 = \frac{f'(0)}{1!}$

$\therefore f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$

Taylor Series Expansion :- (Function of 1 variable)

If $(a+h)$ be a function of the variable h such that it can be expanded in ascending powers of 'h' & this expansion is differentiable at any number of times

Then :- $f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots$

Proof :- $f(a+h) = A_0 + hA_1 + h^2A_2 + h^3A_3 + \ldots\ldots$

Here, $f(a+h) = A_0 + hA_1 + h^2A_2 + h^3A_3 + \ldots\ldots \longrightarrow A_0$

$f'(a+h) = A_1 + 2hA_2 + 3h^2A_3 + \ldots \longrightarrow A_1$

(w.r.t. h)

$f''(a+h) = 2A_2 + 6hA_3 + \ldots \longrightarrow 2A_2 = \frac{f''(a)}{2!}$

$f'''(a+h) = 6A_3 + \ldots \longrightarrow 6A_3 \; \frac{f'''(a)}{3!}$

(at $h=0$)

$\therefore A_0 = f(a) \quad A_2 = \frac{h^2 f''(a)}{2!} \quad A_3 = \frac{h^3 f'''(a)}{3!}$

$A_1 = hf'(a)$

Special case :- For $a+h = x$ (let)

$\Rightarrow h = x-a$

(Taylor series) $\therefore f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots$

to

if $\boxed{a=0}$

(Maclaurin series) $f(h) = f(0) + h f'(0) + h^2 \frac{f''(0)}{2!} + h^3 \frac{f'''(0)}{3!} + \ldots\ldots$

Taylor Series for Functions of 2 variables :-

A function $f(x,y)$ can be expanded [about a point (a,b) (or) in powers of $(x-a)$ & $(y-b)$]

Formula

$$f(x,y) = f(a,b) + \frac{1}{1!}\Big[\underbrace{(x-a)f_x(a,b)}_{a} + \underbrace{(y-b)f_y(a,b)}_{b}\Big]$$

(treat as a & b)

$$+ \frac{1}{2!}\Big[\underbrace{(x-a)^2}_{a^2} f_{xx}(a,b) + \underbrace{2(x-a)(y-b)}_{2ab} f_{xy}(a,b) + \underbrace{(y-b)^2}_{b^2} f_{yy}(a,b)\Big]$$

[in the form of $(a+b)^2$]

$$+ \frac{1}{3!}\Big[\underbrace{(x-a)^3}_{a^3} f_{xxx}(a,b) + \underbrace{3(x-a)^2(y-b)}_{3a^2b} f_{xxy}(a,b) + \underbrace{3(x-a)(y-b)^2}_{3ab^2} f_{xyy}(a,b) + \underbrace{(y-b)^3}_{b^3} f_{yyy}(a,b)\Big] + \frac{1}{4!} \cdots\cdots$$

[so on]

Pascal's Triangle :-

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

: (so on)

(Coefficients) → of $(x+y)^n$ expansion

Eg:- $(x-2)^3 = 1(x)^3 \cdot (-2)^0 + 3(x)^2(-2)^1 + 3(x)(-2)^2 + 1(x)^0(-2)^3$

Here,

$x \to 3$ to 0 & $-2 \to 0$ to 3

$\therefore (x+y)^n = \sum_{r=0}^{n} nC_r \cdot x^{n-r} \cdot y^r$ [where $nC_r = \frac{n!}{(n-r)!\,r!}$]

This is called Binomial Expansion

Using this, you can calculate $(x-a)^n$ [$n \to$ power of expression]

Eg:- $(x-a)^5 = 5C_0\, x^{5-0} y^0 + 5C_1\, x^{5-1} y^1 + 5C_2\, x^{5-2} y^2 + \ldots\ldots 5C_5\, x^{5-5} y^5$

$(n=5)$ $= x^5 + 5x^4 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$