Beta Function:-

For a positive value of m and n,

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

where $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

Here, Γ is the gamma symbol

Q1. Find value of $\int_0^1 x^7(1-x)^{10}\,dx$

Sol:- Here, $x^7 \longrightarrow m-1 = 7$

$\Rightarrow m = 8$

$(1-x)^{10} \longrightarrow n-1 = 10$

$\Rightarrow n = 11$

$$\therefore \int_0^1 x^7(1-x)^{10}\,dx = \beta(m,n)$$
$$= \beta(8,11)$$

But, we know, $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$$\therefore \beta(8,11) = \frac{\Gamma 8\,\Gamma 11}{\Gamma 8+11}$$
$$= \frac{\Gamma 8\,\Gamma 11}{\Gamma 19}$$

But we know, $\Gamma n = (n-1)!$

$$\therefore \frac{\Gamma 8\,\Gamma 11}{\Gamma 19} = \frac{7!\,10!}{18!}$$

Q2. Find value of $\int_0^1 x^{7/2}(1-x)^{5/2}\,dx$

Soln:- Here, $x^{7/2} \longrightarrow m-1 = \frac{7}{2}$

$$\Rightarrow m = \frac{9}{2}$$

$(1-x)^{5/2} \longrightarrow n-1 = \frac{5}{2}$

$$\Rightarrow n = \frac{7}{2}$$

$$\therefore \int_0^1 x^{7/2}(1-x)^{5/2}\,dx = \beta(m,n)$$
$$= \beta\left(\frac{9}{2}, \frac{7}{2}\right)$$

We know, $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

$$\therefore \beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{\Gamma\frac{9}{2}\,\Gamma\frac{7}{2}}{\Gamma\left(\frac{9}{2}+\frac{7}{2}\right)} = \frac{\Gamma\frac{9}{2}\,\Gamma\frac{7}{2}}{\Gamma 8}$$

Find:- (i) $\Gamma\frac{9}{2} = \frac{7}{2}\Gamma\frac{7}{2}$

$$= \frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2}$$
$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$$
$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$$
$$\Rightarrow \Gamma\frac{9}{2} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

(ii) $\Gamma\frac{7}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

(iii) $\Gamma 8 = 7! = 7\times6\times5\times4\times3\times2\times1$

$$\therefore \int_0^1 x^{7/2}(1-x)^{5/2}\,dx = \frac{\left[\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right]\left[\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\right]}{7\times6\times5\times4\times3\times2\times1}$$

R.W

(1) $\frac{9}{2}-1 = \frac{7}{2}$

(2) $\frac{7}{2}-1 = \frac{5}{2}$

(3) $\frac{5}{2}-1 = \frac{3}{2}$

(4) $\frac{3}{2}-1 = \frac{1}{2}$

(5) $\Gamma\frac{1}{2} = \sqrt{\pi}$

Another Representation of Beta Function

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Q. Find value of $\int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx$

Solⁿ:- Here, $x^7 \rightarrow n-1 = 7$

$\Rightarrow n = 8$

$(1+x)^{12} \rightarrow m+n = 12$

$\Rightarrow m+8 = 12$

$\Rightarrow m = 4$

$$\therefore \int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx = \beta(4,8)$$

We know, $\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

$$\therefore \beta(4,8) = \frac{\Gamma 4\,\Gamma 8}{\Gamma 4+8} = \frac{\Gamma 4\,\Gamma 8}{\Gamma 12}$$

We know, $\Gamma n = (n-1)!$

$$\therefore \frac{\Gamma 4\,\Gamma 8}{\Gamma 12} = \frac{3!\,7!}{11!}$$

Hence, $\int_0^\infty \frac{x^7}{(1+x)^{12}}\,dx = \frac{3!\,7!}{11!}$

Same thing can be done by :- $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ Representation

Q2. Find value of $\int_0^\infty \frac{x^2}{(1+x)^{10}} dx$

Sol^n:- Here, $x^2 \longrightarrow m-1=2$
$\Rightarrow m=3$
$(1+x)^{10} \longrightarrow m+n=10$
$\Rightarrow 3+n=10$
$\Rightarrow n=7$

$$\therefore \int_0^\infty \frac{x^2}{(1+x)^{10}} dx = \beta(m,n) = \beta(3,7)$$
$$= \frac{\Gamma 3\ \Gamma 7}{\Gamma 3+7}$$
$$\Rightarrow \int_0^\infty \frac{x^2}{(1+x)^{10}} dx = \frac{2!\ 6!}{9!} = \frac{1}{252}$$

(*) Q3. Find value of $\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^9} dx$

Sol^n:- $\int_0^\infty \frac{x^2(1+x^4)}{(1+x)^9} = \int_0^\infty \frac{x^2+x^6}{(1+x)^9}$

Seperate them into :-

$\int_0^\infty \frac{x^2}{(1+x)^9} dx - \int_0^\infty \frac{x^6}{(1+x)^9} dx$ $\left[\text{in terms of } \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} = \beta(m,n)\right]$

$\downarrow$ $\beta(6,3) = \frac{\Gamma 6\ \Gamma 3}{\Gamma 6+3}$ $\quad$ $\downarrow$ $\beta(2,7) = \frac{\Gamma 2\ \Gamma 7}{\Gamma 2+7}$

$\therefore \beta(6,3) = \frac{5!\,2!}{8!}$ $\quad \left(\because \Gamma n = (n-1)!\right)$

$\beta(2,7) = \frac{1!\,6!}{8!}$

$$\therefore \int_0^\infty \frac{x^2(1+x^4)}{(1+x)^9} dx = \frac{\Gamma 6\ \Gamma 3}{\Gamma 9} - \frac{\Gamma 2\ \Gamma 7}{\Gamma 9}$$
$$= \frac{5!\ 2!}{8!} - \frac{1!\ 6!}{8!}$$
$$= \frac{1}{42}$$