Q4. Solve: $(D^2+a^2)y = \cot ax$

Sol^n:- A.E:- $m^2+a^2=0$

$\Rightarrow m^2=-a^2$

$\Rightarrow m=\pm\sqrt{-a^2}=\pm ai$ $\therefore m=0+ai, 0-ai$

$\alpha=0, \beta=a$

C.F:- $CF = e^0(C_1\cos ax + C_2 \sin ax)$

$= C_1 \cos ax + C_2\sin ax$

Here, $y_1=\cos ax$ $\quad y_2=\sin ax$

$y_1' = -a\sin ax$ $\quad y_2' = a.\cos ax$

$X=\cot ax$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax+\sin^2 ax)$

$\therefore W = a$

- $P = -\int \frac{y_2 X\,dx}{W}$ $\qquad$ - $Q = -\int \frac{y_1 X\,dx}{W}$

$= -\int \frac{(\sin ax)(\cot ax)\,dx}{a}$

$= -\frac{1}{a}\int \cancel{\sin ax}\cdot\frac{\cos ax}{\cancel{\sin ax}}.dx$

$= -\frac{1}{a}\cdot\frac{\sin ax}{\frac{d}{dx}(ax)}$

$\therefore P = \frac{-\sin ax}{a^2}$

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- $\int \operatorname{cosec}\theta\,d\theta = \ln(\operatorname{cosec}\theta - \cot\theta$
- $\int \sec\theta\,d\theta = \ln(\sec\theta+\tan\theta)$
- $\int \sin\theta\,d\theta = -\cos\theta$

$$Q = \int \frac{y_1 X \, dx}{W}$$

$$= \int \frac{(\cos ax)}{a} \cdot \left(\frac{\cos ax}{\sin ax}\right) dx$$

$$= \frac{1}{a} \int \frac{\cos^2 ax}{\sin ax} dx = -\frac{1}{a} \int \frac{1 - \sin^2 ax}{\sin ax} dx$$

$$= \frac{1}{a} \int (\operatorname{cosec} ax - \sin ax) dx$$

$$= \frac{1}{a} \left[\frac{\ln(\operatorname{cosec} ax - \cot ax)}{a} - \left(\frac{-\cos ax}{a} \right) \right]$$

$$\therefore Q = \frac{1}{a^2} \left[\ln(\operatorname{cosec} ax - \cot ax) + \cos ax \right]$$

P.I :- $P.I = P y_1 + Q y_2$

$$= \left(\frac{-\sin ax}{a^2} \right) (\cos ax) + \frac{1}{a^2} \left[\ln(\operatorname{cosec} ax - \cot ax) + \cos ax \right] (\sin ax)$$

$$= \cancel{\frac{-\sin ax \cdot \cos ax}{a^2}} + \frac{1}{a^2} \sin ax \cdot \ln(\operatorname{cosec} ax - \cot ax) + \cancel{\frac{\sin ax \cos ax}{a^2}}$$

$$\Rightarrow P.I = \frac{\sin ax}{a^2} \ln(\operatorname{cosec} ax - \cot ax)$$

$$\therefore y = C.F + P.I$$

$$= C_1 \cos ax + C_2 \sin ax + \frac{\sin ax}{2} \ln(\operatorname{cosec} ax - \cot ax)$$

Q5. Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec}x \cot x$

Solⁿ:- $(D^2+1)y = \operatorname{cosec}x \cot x$

A.E :- $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm\sqrt{-1} = \pm i \quad \therefore m = 0+i, 0-i$

$\alpha = 0, \beta = 1$

C.F :- $CF = e^0(C_1\cos x + C_2 \sin x)$

$= C_1 \cos x + C_2 \sin x$

Here, $y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$X = \operatorname{cosec}x \cot x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \quad \therefore W = 1$

- $P = -\int \frac{y_2 X}{W} dx$

$= -\int \frac{\sin x \cdot (\operatorname{cosec}x \cot x)}{1} dx$

$= -\int \cancel{\sin x} \cdot \frac{1}{\cancel{\sin x}} \cdot \cot x \, dx$

$= -\int \cot x \, dx$

$\cot x = \frac{\cos x}{\sin x}$

Let $t = \sin x$

$\frac{dt}{dx} = \cos x$

$\Rightarrow dx = \frac{dt}{\cos x}$

$\therefore \int \frac{\cancel{\cos x}}{\sin x} \cdot \frac{dt}{\cancel{\cos x}}$

$= \int \frac{1}{t} \cdot dt$

$= \ln(t)$

$= \ln(\sin x)$

$\therefore P = -\ln(\sin x)$

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$$Q = \int \frac{y_1 X}{W}\, dx$$

$$= \int \frac{\cos x \cdot (\operatorname{cosec} x) \cdot \cot x}{1}\, dx$$

$$= \int \cot^2 x\, dx$$

$$= \int (\operatorname{cosec}^2 x - 1)\, dx$$

$$\therefore Q = -\cot x - x$$

(*) $\int \operatorname{cosec}^2 x \cdot dx = -\cot x$

We know $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\therefore \int \operatorname{cosec}^2 x \cdot dx = -\cot x$

P.I:- $PI = P y_1 + Q y_2$

$$= -\ln(\sin x) \cdot \cos x + (-\cot x - x)(\sin x)$$

$$= -\ln(\sin x) \cos x - (\cot x + x) \sin x$$

$$\therefore y = CF + P.I$$

$$= C_1 \cos x + C_2 \sin x - \ln(\sin x) \cos x - (\cot x + x) \sin x$$

(*) Q6. Solve $\frac{d^2y}{dx^2} + y = x \sin x$

Sol^n:- $(D^2 + 1) y = x \sin x$

A.E:- $m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm\sqrt{-1} = \pm i$ $\therefore m = 0 + i, 0 - i$

$\alpha = 0, \beta = 1$

C.F:- $CF = e^{0x}(C_1 \cos x + C_2 \sin x)$

$$= C_1 \cos x + C_2 \sin x$$

Here, $y_1 = \cos x$ $y_2 = \sin x$

$y_1' = -\sin x$ $y_2' = \cos x$

$X = x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x \quad \therefore W = 1$$

$$P = -\int \frac{y_2 X}{W} dx$$

$$= -\int \frac{(\sin x)(x \sin x)}{1} dx$$

$$= -\int x \sin^2 x \, dx$$

We know, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= 1 - 2\sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\therefore P = -\int x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= -\frac{1}{2}\left[\int x\,dx - \int x\cos 2x\,dx\right]$$

(*)

$$= -\frac{1}{2}\left[\frac{x^2}{2} - \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right]$$

$$\therefore P = -\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$$

(*) $\int uv = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$

$u = x$

$v = \cos 2x$

$\therefore u' = \frac{du}{dx} = \frac{dx}{dx} = 1 \quad \therefore u' = 1$

$\int v\,dx = \int \cos 2x\,.dx$

$= \frac{\sin 2x}{\frac{d}{dx}(2x)}$

$\therefore \int v\,dx = \frac{\sin 2x}{2}$

Now, $\int x\cos 2x\,dx$

$= x\int \cos 2x\,dx - \int\left[(1)\int \cos 2x\,dx\right]dx$

$= x\cdot\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}dx$

$= \frac{x}{2}\sin 2x - \frac{1}{2}\left(\frac{-\cos 2x}{\frac{d}{dx}(2x)}\right)$

$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$

$\therefore -\int x\cos 2x\,dx = -\frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x$

$$Q = \int \frac{y_1 X}{W}\,dx$$

$$= \int \frac{(\cos x)(x \sin x)}{1}\,dx$$

We know, $\sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin\theta = \frac{\sin 2\theta}{2\cos\theta}$

$$\therefore Q = \int \left(\frac{\sin 2x}{2\cancel{\cos x}}\right)\cdot\cancel{\cos x}\cdot x\,dx$$

$$= \int \frac{x \sin 2x\,dx}{2}$$ #

$$\therefore Q = \frac{1}{2}\int x \sin 2x\,dx$$

$$= \frac{1}{2}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$\therefore Q = \frac{-x\cos 2x}{4} + \frac{\sin 2x}{8}$$

$u = x$, $v = \sin 2x$

$u' = 1$

$\therefore \int v\,dx = \int \sin 2x\,dx = \frac{-\cos 2x}{2}$

& $\int \frac{-\cos 2x}{2}\,dx = -\frac{1}{2}\cdot\frac{\sin 2x}{2}$

$= -\frac{\sin 2x}{4}$

Now, $\int x \sin 2x\,dx$

$= x\cdot\left(\frac{-\cos 2x}{2}\right) - \int (1)\cdot\left(\frac{-\cos 2x}{2}\right)dx$

$= \frac{-x\cos 2x}{2} + \left(\frac{\sin 2x}{4}\right)$

$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}$

P.I:- $P.I = Py_1 + Qy_2$

$$= \left[-\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}\right](\cos x)$$
$$+ \left[\frac{\sin 2x}{8} - \frac{x\cos 2x}{4}\right](\sin x)$$

$$= -\frac{x^2}{4}\cos x + \frac{x}{4}\left[\sin 2x\cos x - \cos 2x \sin x\right]$$
$$+ \frac{1}{8}\left[\cos 2x\cos x + \sin 2x \sin x\right]$$

$$\therefore P.I = -\frac{x^2}{4}\cos x + \frac{x}{4}\sin(2x - x) + \frac{1}{8}\cos(2x - x)$$

$$= -\frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

$$\therefore y = CF + P.I$$

$$= C_1\cos x + C_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$