Implicit Function:- It is a function written in terms of both dependent and independant variables

Eg:- $x^2+y^2=4xy$, find $\frac{dy}{dx}$

Normally, for this type of problem, we differentiate both sides with x

$$\therefore 2x + 2y\cdot\frac{dy}{dx} = 4\cdot\frac{d}{dx}(x\cdot y)$$

$$= 4\left[x\frac{dy}{dx} + y\cdot(1)\right]$$

$$\Rightarrow \cancel{2}x + \cancel{2}y\frac{dy}{dx} = 2\{x\frac{dy}{dx} + 2y$$

$$\Rightarrow y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - x$$

$$\Rightarrow \frac{dy}{dx}\left[y-2x\right] = 2y-x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2y-x}{\cancel{2}y-2x}}$$

To apply partial derivative method, we do the following :-

$$x^2+y^2=4xy$$

$$\Rightarrow x^2+y^2-4xy=0$$

$\therefore$ Let $f(x,y) = x^2+y^2-4xy$

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

Now, we find partial derivative of $f(x,y)$ in terms of x & y

$$\therefore \cancel{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x}}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2-4xy) = 2x - 4y$$ (y → treated as constant)

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2-4xy) = 2y - 4x$$

Now, we can write :-

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$= \frac{-(2x-4y)}{(2y-4x)} = \frac{-(x-2y)}{(y-2x)}$$

$$\Rightarrow \frac{dy}{dx} = \cancel{\frac{2y-x}{2x}} \frac{2y-x}{y-2x}$$

Q. Find $\frac{dy}{dx}$ for the following :-

1. $\sin xy = \log(x+y)$

Solⁿ $\log(x+y) - \sin xy = 0$

Let $f(x,y) = \log(x+y) - \sin y$

$$\therefore \frac{\partial f}{\partial x} = \frac{1}{x+y} \cancel{- \sin y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x+y} - \cos y$$

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$= \frac{\frac{-1}{(x+y)}}{\frac{1 - \cos y(x+y)}{(x+y)}}$$

$$= \frac{(-1)}{\cancel{(x+y)}} \cdot \frac{\cancel{(x+y)}}{(1 - x\cos y - y\cos y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x\cos y + y\cos y - 1}$$

$$\cancel{\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial x} = -\left[\frac{\frac{1}{x+y} - \sin y}{\frac{1}{x+y} - \cos y}\right] = -\frac{1 + (x+y)\sin y}{1 - (x+y)\cos y}}$$

2. $x^y = e^{x-y}$

Let $f(x,y) = x^y - e^{x-y}$

$$\frac{\partial f}{\partial x} = y\cdot x^{y-1} - e^{x-y}(1) = y\cdot x^{y-1} - e^{x-y}$$

$$\frac{\partial f}{\partial y} = x^y \cdot \log x - e^{x-y}(-1) = x^y \log x + e^{x-y}$$

$x^y = e^{x-y} \Rightarrow y \log x = (x-y) \cancel{\log e}$

$\Rightarrow y = \dfrac{x-y}{\log x}$

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{y\cdot x^{y-1} - e^{x-y}}{x^y \log x + e^{x-y}}$$

$$= -\frac{\left(\frac{x-y}{\log x}\right)\cdot\frac{e^{x-y}}{x} - e^{x-y}}{e^{x-y}\cdot\log x + e^{x-y}} \qquad \left[\because y = \frac{x-y}{\log x} \ \& \ x^y = e^{x-y}\right]$$

$$= -\frac{\cancel{(e^{x-y})}\left(\frac{x-y}{x\log x} - 1\right)}{\cancel{(e^{x-y})}(\log x + 1)}$$

$$= -\left(\frac{x-y-x\log x}{x\log x}\right)\cdot\frac{1}{(\log x+1)}$$

Here, $y\log x = x - y$

$$\therefore \frac{dy}{dx} = \frac{-(y\log x - x\log x)}{x\log x\ (\log x+1)}$$

$$= \frac{x-y}{x(\log x+1)}$$

$$= \frac{\cancel{y}\log x}{\cancel{y}(\log x+1)(\log x+1)} \qquad \left[\begin{array}{l}\because x - y = y\log x \\ \Rightarrow x = y + y\log x \\ \quad = y(\log x + 1)\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(\log x+1)^2}$$

3. $x^p \cdot y^q = (x+y)^{p+q}$

Let $f(x,y) = x^p y^q - (x+y)^{p+q}$

~~df~~ Logarithimic approach :- $x^p y^q = (x+y)^{p+q}$

$\Rightarrow \log(x^p) + \log(y^q) = \log(x+y)^{p+q}$

$\Rightarrow p\log x + q\log y = (p+q)\log(x+y)$

$\therefore f(x,y) = p\log x + q\log y - (p+q)\log(x+y)$

$$\frac{\partial f}{\partial x} = \frac{p}{x} + \cancel{\frac{q}{y}}^{\,0} - \frac{p+q}{x+y} \cdot \cancel{(1+0)}$$

$$\frac{\partial f}{\partial y} = \cancel{\frac{p}{x}}^{\,0} + \frac{q}{y} - \frac{p+q}{x+y}$$

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\left[\frac{p}{x} - \frac{p+q}{x+y}\right] \Big/ \left[\frac{q}{y} - \frac{p+q}{x+y}\right]$$

$$= \frac{(p+q)x - p(x+y)}{x\cancel{(x+y)}} \times \frac{y\cancel{(x+y)}}{q(x+y) - (p+q)y}$$

$$= \frac{\cancel{px} + qx - \cancel{px} - py}{x} \cdot \frac{y}{qx + \cancel{qy} - py - \cancel{qy}}$$

$$= \frac{\cancel{(qx - py)}}{x} \cdot \frac{y}{\cancel{(qx - py)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$