Comparision Test :-

If $\sum u_n$ & $\sum v_n$ are two series of positive terms :-

① $\left.\begin{array}{l}\sum v_n \text{ is convergent} \\ \sum u_n \text{ is also convergent}\end{array}\right\}$ if $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (finite +ve)

② $\left.\begin{array}{l}\sum v_n \text{ is divergent} \\ \sum u_n \text{ is also divergent}\end{array}\right\}$ if $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (finite +ve)

p-series Test :-

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots \frac{1}{n^p}$

① If $p > 1 \Rightarrow$ Convergent

② If $p \leq 1 \Rightarrow$ Divergent

Q1. Test for convergence of $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots\frac{1}{2n-1}\dots\infty$

Solⁿ:- $u_n = \frac{1}{2n-1}$

$v_n = \frac{1}{n^{a-b}}$ $\left[\begin{array}{l} a \to \text{highest power in denominator} \\ b \to \text{" " " numerator} \end{array}\right]$

$\Rightarrow v_n = \frac{1}{n^{1-0}} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{2n-1} \cdot n$

$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(2-\frac{1}{n}\right)}$

$= \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}}$

$= \frac{1}{2-\frac{1}{\infty}}$

$\Rightarrow l = \frac{1}{2}$ (finite & +ve)

But, $\Sigma v_n = \Sigma \frac{1}{n^1}$ i.e. ~~p~~

$\therefore p = 1$ (By p-series Test)
$(p \leq 1)$

$\therefore \Sigma v_n \to$ divergent $\Rightarrow \Sigma u_n \to$ divergent

$\therefore$ Series is divergent

Q2 Test for convergence of series $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \ldots \infty$

Sol:- $u_n = \frac{2n-1}{n(n+1)(n+2)}$ [Even no. general form - $2n$; Odd no. general form - $2n-1$]

In u_n, Higest power in numerator $= 1$ $\therefore b = 1$

Higest power in denominator $= 3$ $a = 3$

$\therefore v_n = \frac{1}{n^{a-b}}$

$\Rightarrow v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \cdot n^2$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(2 - \frac{1}{n}\right)\cdot \cancel{n^2}}{n \cdot n\left(1+\frac{1}{n}\right) n\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{2-\frac{1}{\infty}}{\left(1+\frac{1}{\infty}\right)\left(1+\frac{2}{\infty}\right)}$$

$\Rightarrow l = 2$ (finite & +ve)

But, $\Sigma v_n = \Sigma \frac{1}{n^2}$

$\therefore p = 2$ (By p - series Test)

$(p > 1)$

$\therefore \Sigma v_n \to$ convergent $\Rightarrow \Sigma u_n \to$ convergent

$\therefore$ Series is convergent.

(*) Q3. Test for convergence of the series $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$

Sol:- $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$= \frac{\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}$$

$$= \frac{n^4+1-(n^4-1)}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$\Rightarrow u_n = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$$

$$= \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^4}}\right] + n^2\left[\sqrt{1+\frac{1}{n^4}}\right]}$$

$$\Rightarrow u_n = \frac{2}{n^2\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1+\frac{1}{n^4}}\right]}$$

Here,

Highest Power in numerator $(b) = 0$

Highest Power in denominator $(a) = 2$

$\therefore V_n = \frac{1}{n^{a-b}}$

$$\Rightarrow V_n = \frac{1}{n^{2-0}} = \frac{1}{n^2}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{2}{\cancel{n^2}\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1+\frac{1}{n^4}}\right]} \times \cancel{n^2}$$

$$= \lim_{n\to\infty} \frac{2}{\left[\sqrt{1+\frac{1}{n^4}} + \sqrt{1+\frac{1}{n^4}}\right]}$$

$$= \frac{2}{\sqrt{1+\left(\frac{1}{\infty}\right)^4} + \sqrt{1+\left(\frac{1}{\infty}\right)^4}}$$

$\therefore l = \frac{2}{1+1}$

$= \frac{\cancel{2}}{\cancel{2}}$

$\Rightarrow l = 1$ (finite & +ve)

But, $\Sigma v_n = \Sigma \frac{1}{n^2}$

$\therefore p = 2$ (By p-series Test)
$(p > 1)$

$\therefore \Sigma v_n \to$ convergent $\Rightarrow \Sigma u_n \to$ convergent

$\therefore$ Series is convergent.

(*) Q4. Test the convergence of the series whose n^{th} term is $\frac{1}{n} \cdot \sin\left(\frac{1}{n}\right)$

Sol^n:- $u_n = \frac{1}{n} \cdot \sin\left(\frac{1}{n}\right)$

We know, $\sin x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty$

$\Rightarrow \sin x = \frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \ldots \infty$

Here, Highest Power in numerator $(b) = 0$
Highest Power in denominator $(a) = 2$

$\therefore v_n = \frac{1}{n^{a-b}} = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \sin(1/n) \times n^2$

$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \ldots \infty\right] \cdot n^2$

$= \lim_{n\to\infty} \frac{1}{\cancel{n}} \cdot \frac{1}{\cancel{n}}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \ldots \infty\right] \times \cancel{n^2}$

$= \left[1 - \frac{1}{3!(\infty)^2} + \frac{1}{5!(\infty)^4} - \ldots \infty\right]$

$\Rightarrow l = [1 - 0]$

$= 1$

$\Sigma v_n = \frac{1}{n^2}$

$\therefore p = 2$ (By p-series Test)

$(p > 1)$

$\therefore \Sigma v_n \to$ convergent $\Rightarrow \Sigma u_n \to$ convergent

$\therefore$ Series is converegent [By Comparision Test]

By Maclaurin's series Expansion :-

If $y = f(x)$ is a function :-

$$\rightarrow \quad y = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \infty$$

For :-

(1) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

(2) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$