Cauchy's Root Test :-

A series $\sum u_n$ of positive terms is :-

(i) Convergence if $\lim_{n\to\infty} (u_n)^{1/n} < 1$

(ii) Divergence if $\lim_{n\to\infty} (u_n)^{1/n} > 1$

(iii) Test fails if $\lim_{n\to\infty} (u_n)^{1/n} = 1$

Q1. Test convergence of the series :-

$$\sum_{n=1}^{\infty} \left[\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}\right]$$

Sol:- Here,

$$u_n = \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}\right]^{1/n}$$

$$\therefore u_n = \frac{\left(\frac{n+1}{n}\right)^{n}}{3}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)^{n}}{3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{3\cancel{n}}$$

$$= \frac{\left(1+\frac{1}{\infty}\right)}{3}$$

$$\Rightarrow l = \frac{1}{3} < 1$$

By Cauchy's Root Test, $\sum u_n$ is convergence

Q2. Test the convergence of the series whose n^{th} term is :-

$$u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$$

Solⁿ :- Here, $u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\Rightarrow (u_n)^{1/n} = \left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n}$$

$$= \frac{n^n}{(n+1)^n}$$

$$\therefore (u_n)^{1/n} = \frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1 \quad [\text{where } e \approx 2.7\ldots]$$

By Cauchy's Root Test, Σu_n is convergent

Prove :- $\lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ (or) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

Solⁿ :-

P. T. O.

To prove this :-

1) L'Hospital's rule → applied

~~(diffe~~ → derivative of numerator & denominator

2) Logarithmic to Exponential Form → applied

→ $\log_e N = x$

$\Rightarrow e^x = N$

To prove :- $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

Proof :-

Let $k = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

Takes $\log_e$ (ln) on both sides :-

$$\Rightarrow \ln k = \ln \left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right]$$

$$= \lim_{n \to \infty} \ln \left[1 + \frac{1}{n}\right]^n$$

$$= \lim_{n \to \infty} n \cdot \ln\left(1 + \frac{1}{n}\right)$$

$$\Rightarrow \ln k = \lim_{n \to \infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \frac{\ln\left(1 + \frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{0}{0} \quad \Big\} \; \times$$

$\therefore$ We apply L'Hospital's Rule :- (differentiate numerator & denominator w.r.t n)

$$\therefore \ln k = \lim_{n \to \infty} \frac{\frac{1}{\left(1 + \frac{1}{n}\right)} \cdot \cancel{\left(0 - \frac{1}{n}\right)}}{\cancel{\left(-\frac{1}{n}\right)}}$$

$$= \lim_{n \to \infty} \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$= \frac{1}{1 + \frac{1}{\infty}} \quad \left(\because \frac{1}{\infty} = 0\right)$$

$$\Rightarrow \boxed{\ln k = 1}$$

$$\Rightarrow \log_e k = 1$$

$$\Rightarrow k = e^1 \quad \text{[Log to Exponential Form]}$$

$$\therefore \boxed{\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e}$$

Hence proved

Q3. Test convergence of series :-

$$\frac{1^3}{3} + \frac{2^3}{3^2} + \dots \frac{n^3}{3^n}$$

Sol^n: $u_n = \dfrac{n^3}{3^n}$

$\Rightarrow (u_n)^{1/n} = \left(\dfrac{n^3}{3^n}\right)^{1/n}$

$= \dfrac{n^{3/n}}{3}$

$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \dfrac{(n^{1/n})^3}{3}$

$= \dfrac{\lim_{n\to\infty} (n^{1/n})^3}{3} \rightarrow = 1 \quad \left[\because \lim_{n\to\infty} n^{1/n} = 1\right]$

$\Rightarrow l = \dfrac{1^3}{3} = \dfrac{1}{3}$

$\Rightarrow l = \dfrac{1}{3} < 1$

$\therefore$ By Cauchy Root Test, Σu_n is convergent

(*) Prove: $\lim_{n\to\infty} n^{1/n} = 1$

Proof:- Let $k = \lim_{n\to\infty} n^{1/n}$

Take log both sides :-

$\Rightarrow \ln k = \lim_{n\to\infty} \frac{1}{n} \ln n$

$= \lim_{n\to\infty} \dfrac{\ln n}{1/n}$

By L'Hospital's rule :- $= \lim_{n\to\infty} \dfrac{\frac{1}{n}}{-\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{1}{n}(-n^2)$

$\Rightarrow \log_e k = \lim_{x\to 0} (-x)$

$\Rightarrow k = e^0$

$\boxed{k = 1} \Rightarrow \therefore \boxed{\lim_{n\to\infty} n^{1/n} = 1}$

Hence proved