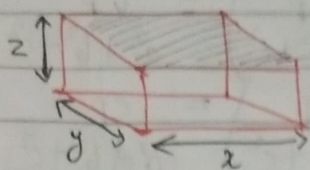
Q2. A rectangular box open at the top, is to have a volume of 32cc. Find the dimensions of the box that required the least material for construction

Sol:-



Here, volume $= xyz = 32$ cc (given)

$\therefore$ Surface area $= xy + 2yz + 2xz$ (top open)

$\therefore$ We have:- $V(x,y,z) = xyz - 32$

& $S(x,y,z) = xy + 2yz + 2xz$

$\therefore F(x,y,z) = V(x,y,z) + \lambda S(x,y,z)$

$\Rightarrow F(x,y,z) = xyz - 32$ ✗ A/Q, $V \to$ (given), $S \to$ unknown

$\therefore$ In $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ $f \to$ unknown (to be founded), $\phi \to$ given function

$f(x,y,z) \to S(x,y,z)$ & $\phi(x,y,z) \to V(x,y,z)$

Correct form $F(x,y,z) = S(x,y,z) + \lambda V(x,y,z)$

$= xy + 2yz + 2xz + \lambda(xyz - 32)$

where $\lambda \to$ lagrange multiplier

- $F_x = \frac{\partial F}{\partial x} = y + 2z + \lambda(yz)$
- $F_y = \frac{\partial F}{\partial y} = x + 2z + \lambda(xz)$
- $F_z = \frac{\partial F}{\partial z} = 2y + 2x + \lambda(xy)$

Now:- (Find λ)

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow y + 2z + \lambda(yz) = 0$	$\Rightarrow x + 2z + \lambda(xz) = 0$	$\Rightarrow 2y + 2x + \lambda(xy) = 0$
$\Rightarrow \lambda(yz) = -(y + 2z)$	$\Rightarrow \lambda(xz) = -(x + 2z)$	$\Rightarrow \lambda(xy) = -2(x + y)$
$\Rightarrow -\lambda = \frac{y + 2z}{yz}$ —— (1)	$\Rightarrow -\lambda = \frac{x + 2z}{xz}$ —— (2)	$\Rightarrow -\lambda = \frac{2x + 2y}{xy}$ —— (3)

(Instead of writing x, y, z in terms of λ and putting in ϕ and then find x, y, z)

Equating ① & ②:- $\dfrac{y+2z}{yz} = \dfrac{x+2z}{xz}$

$\Rightarrow \cancel{1} + \dfrac{2z}{y} = \cancel{1} + \dfrac{2z}{x}$

$\Rightarrow \left(\dfrac{2z}{y}\right) = \left(\dfrac{2z}{x}\right) \Rightarrow x = y$

Equating ① & ③:- $\dfrac{y+2z}{yz} = \dfrac{2x+2y}{xy}$

$\Rightarrow \dfrac{y}{z} + \cancel{2} = \cancel{2} + \dfrac{2y}{x}$

$\Rightarrow \dfrac{\cancel{y}}{z} = \dfrac{2\cancel{y}}{x} \Rightarrow x = 2z$

Hence,

$x = y$ & $x = 2z$

$\Rightarrow y = x$ $\quad \Rightarrow z = x/2$

Now:-

Volume of box $= xyz = 32$

$\Rightarrow x(x)\left(\dfrac{x}{2}\right) = 32$

$\Rightarrow x^3 = 64$

$\Rightarrow x = \sqrt[3]{64}$

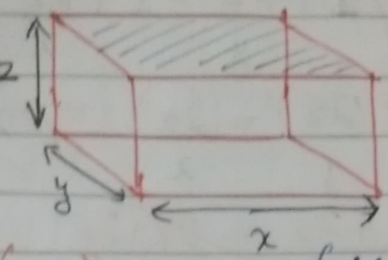
$= 4$

$\therefore x = 4, \; y = 4, \; z = 2$

∴ Dimension of box required:- Length = 4 cm
Breadth = 4 cm
Height = 2 cm

Q3. Find the dimensions of the rectangular box without top of maximum capacity with surface area $432\ m^2$

Sol^n:-



Here, Volume $= xyz$ (unknown)
Surface area $= xy + 2yz + 2zx = 432$ (known)

We have:- $V(x,y,z) = xyz$ $\quad (f(x,y,z)) \rightarrow$ unknown 1st
$S(x,y,z) = xy + 2yz + 2zx - 432$ $\quad (\phi(x,y,z))$ known 2nd

$\therefore F(x,y,z) = V(x,y,z) + \lambda S(x,y,z)$
$\Rightarrow F(x,y,z) = xyz + \lambda(xy + 2yz + 2zx - 432)$

$F_x = 0$	$F_y = 0$	$F_z = 0$
$\Rightarrow yz + \lambda y + 2\lambda z = 0$	$\Rightarrow xz + \lambda x + 2\lambda z = 0$	$\Rightarrow xy + 2\lambda y + 2\lambda x = 0$
$\Rightarrow \lambda(y+2z) = -yz$	$\Rightarrow \lambda(x+2z) = -xz$	$\Rightarrow \lambda(2y+2x) = -xy$
$\Rightarrow \lambda = \dfrac{-yz}{y+2z}$	$\Rightarrow \lambda = \dfrac{-xz}{x+2z}$	$\Rightarrow \lambda = \dfrac{-xy}{2x+2y}$
$\Rightarrow \dfrac{-1}{\lambda} = \dfrac{y+2z}{yz}$ — (1)	$\Rightarrow \dfrac{-1}{\lambda} = \dfrac{x+2z}{xz}$ — (2)	$\Rightarrow \dfrac{-1}{\lambda} = \dfrac{2x+2y}{xy}$ — (3)

For easier simplification

Equating (1) & (2) $\Rightarrow \dfrac{y+2z}{yz} = \dfrac{x+2z}{xz}$

$\Rightarrow \cancel{\dfrac{1}{z}} + \dfrac{2}{y} = \cancel{\dfrac{1}{z}} + \dfrac{2}{x} \Rightarrow \dfrac{\cancel{2}}{y} = \dfrac{\cancel{2}}{x}$ $\quad \therefore x = y$
$\Rightarrow y = x$

Equating (1) & (3) $\Rightarrow \dfrac{y+2z}{yz} = \dfrac{2x+2y}{xy}$

$\Rightarrow \dfrac{1}{z} + \cancel{\dfrac{2}{y}} = \cancel{\dfrac{2}{y}} + \dfrac{2}{x} \Rightarrow \dfrac{1}{z} = \dfrac{2}{x}$ $\quad \therefore x = 2z$
$\Rightarrow z = x/2$

$\therefore$ We have :- $y = x$ & $z = x/2$

Put em in $S(x,y,z) = 432$

$$S(x,y,z) = 432$$

$$\Rightarrow x \cdot x + 2x \cdot \frac{x}{2} + 2 \cdot \frac{x}{2} \cdot x = 432$$

$$\Rightarrow \frac{2x^2 + 2x^2 + 2x^2}{2} = 432$$

$$\Rightarrow \frac{6x^2}{2} = 432 \qquad \cancel{\Rightarrow x^2 = \overset{144}{432} \times \frac{2}{6\,3}}$$

$$\Rightarrow x^2 = \overset{144}{\cancel{432}} \times \frac{\cancel{2}}{\cancel{6}\,\cancel{3}}$$

$$\Rightarrow x = \sqrt{144} = \pm 12$$

Here, $x = -12$ is invalid as length can't be -ve

$\therefore x = 12$ & $y = 12$ & $z = 6$

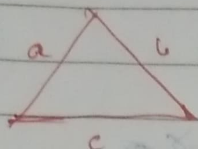
$\therefore$ Dimensions are :- Length = 12 m

Breadth = 12 m

Height = 6 m

Q4. Show that if perimeter of a triangle is a constant, then the triangle has maximum area when its equilateral

Sol^n:-



Here,

Perimeter $= a+b+c = k$ (constant)

Area $= \sqrt{s(s-a)(s-b)(s-c)}$ $\left[s = \frac{a+b+c}{2} = \frac{k}{2} \right]$

$$\Rightarrow A^2 = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)$$

We have:-

$$P(a,b,c) = a+b+c-k \quad \text{(known)}$$

$$A^2(a,b,c) = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right) \quad \text{(unknown)}$$

$$F(a,b,c) = A(a,b,c) + \lambda P(a,b,c) \quad \left[\begin{array}{l}\text{Unknown} \rightarrow 1^{st} \\ \text{Known} \rightarrow 2^{nd}\end{array}\right]$$

$$\cancel{= a+b+c-k+\lambda}$$

$$\Rightarrow F(a,b,c) = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right) + \lambda(a+b+c-k)$$

- $F_a = \frac{k}{2}(-1)\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)+\lambda = -\frac{k}{2}\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)+\lambda$
- $F_b = \frac{k}{2}\left(\frac{k}{2}-a\right)(-1)\left(\frac{k}{2}-c\right)+\lambda = -\frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-c\right)+\lambda$
- $F_c = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)(-1)+\lambda = -\frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)+\lambda$

$F_a = 0$

$\Rightarrow -\frac{k}{2}\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)+\lambda = 0$

$\Rightarrow \lambda = \frac{k}{2}\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)$ ——— (1)

$F_b = 0$

$\Rightarrow -\frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-c\right)+\lambda = 0$

$\Rightarrow \lambda = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-c\right)$ ——— (2)

$F_c = 0$

$\Rightarrow -\frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)+\lambda = 0$

$\Rightarrow \lambda = \frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)$ ——— (3)

Equating (1) & (2) $\Rightarrow$

$$\cancel{\frac{k}{2}}\left(\frac{k}{2}-b\right)\cancel{\left(\frac{k}{2}-c\right)} = \cancel{\frac{k}{2}}\left(\frac{k}{2}-a\right)\cancel{\left(\frac{k}{2}-c\right)}$$

$$\Rightarrow \frac{k}{2} - b = \frac{k}{2} - a$$

$$\Rightarrow a = b$$

Equating (2) & (3) $\Rightarrow$

$$\cancel{\frac{k}{2}}\cancel{\left(\frac{k}{2}-a\right)}\left(\frac{k}{2}-c\right)=\cancel{\frac{k}{2}}\cancel{\left(\frac{k}{2}-a\right)}\left(\frac{k}{2}-b\right)$$

$$\Rightarrow \cancel{\frac{k}{2}}-c=\cancel{\frac{k}{2}}-b$$

$$\Rightarrow c=b$$

~~= a = b = c~~

$\therefore a=b=c$ (Put in $P(a,b,c)=k$)

$$\therefore P(a,b,c)=a+b+c=k$$

$$\Rightarrow a+a+a=k$$

$$\Rightarrow 3a=k$$

$$\Rightarrow a=\frac{k}{3} \qquad \therefore a=b=c=\frac{k}{3}$$

$$\therefore \text{Area}=\sqrt{\frac{k}{2}\left(\frac{k}{2}-a\right)\left(\frac{k}{2}-b\right)\left(\frac{k}{2}-c\right)}$$

$$=\sqrt{\frac{k}{2}\left(\frac{k}{2}-\frac{k}{3}\right)\left(\frac{k}{2}-\frac{k}{3}\right)\left(\frac{k}{2}-\frac{k}{3}\right)}$$

$$=\sqrt{\frac{k}{2}\left(\frac{k}{2}-\frac{k}{3}\right)^2\left(\frac{k}{2}-\frac{k}{3}\right)}$$

$$\therefore \text{Area}=\left(\frac{k}{2}-\frac{k}{3}\right)^2\cdot\sqrt{\frac{k}{2}\left(\frac{k}{2}-\frac{k}{3}\right)}$$

$\therefore$ ~~As~~ $a=b=c$ $\therefore$ Its an equilateral triangle (when perimeter is constant)