D'Alembert's Ratio Test :-

The series Σu_n of +ve terms is :-

(i) Convergence :- If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$

(ii) Divergence :- If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$

(iii) Failed :- If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

Concepts :- • $\lim_{n\to\infty} u_n = 0$ (convergent)

• $\lim_{n\to\infty} u_n > 0$ (divergent)

Q1. Test the Convergence of $\underset{1^2\cdot x^0}{1^2} + \underset{2^2\cdot x^1}{2^2 x^2} + \underset{3^2\cdot x^2}{3^2x^2} + \ldots\ldots$ where x is positive

Solⁿ :- Let $u_n = n^2 x^{n-1}$

$\therefore u_{n+1} = (n+1)^2 \cdot x^{n+1-1}$

$= (n+1)^2 \cdot x^n$

$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^2 \cdot x^n}{n^2 \cdot x^{n-1}}$

Take n common on numerator :-

$$= \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)^2 \cdot \cancel{x^n}}{\cancel{n^2} \cdot \cancel{x^n} \cdot x^{-1}} \qquad \left[\because (n+1)^2 = \left[n\left(1+\frac{1}{n}\right)\right]^2 = n^2\left(1+\frac{1}{n}\right)^2\right]$$

$\Rightarrow \frac{u_{n+1}}{u_n} = x\left(1+\frac{1}{n}\right)^2$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x\left(1+\frac{1}{n}\right)^2$

$= x\left(1+\frac{1}{\infty}\right)^2 \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$

By D'Alembert's Ratio Test, • $x < 1$:- convergent

• $x > 1$:- divergent

• $x = 1$:- Test fails

Q2 Test the Convergence of $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \ldots\ldots$

Solⁿ:- Let $u_n = \frac{x^n}{(2n-1)2n}$

$\therefore u_{n+1} = \frac{x^{n+1}}{(2(n+1)-1)\,2(n+1)}$

$= \frac{x^{n+1}}{2(2n+1)(\cancel{2})(n+1)}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{2(2n+1)(n+1)} \times \frac{(2n-1)2n}{x^n}$

Take n common on both numerator and denominator

$= \frac{\cancel{x^n}\cdot x}{\cancel{2}\cdot\cancel{n}\left(2+\frac{1}{n}\right)\cdot\cancel{n}\left(1+\frac{1}{n}\right)} \times \frac{\cancel{n}\left(2-\frac{1}{n}\right)\cdot\cancel{2}\cancel{n}}{\cancel{x^n}}$

$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{x\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$

$= \frac{x\left(2-\frac{1}{\infty}\right)}{\left(2+\frac{1}{\infty}\right)\left(1+\frac{1}{\infty}\right)}$

$= \frac{\cancel{2}\cdot x}{\cancel{2}\times 1}$

$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$

$\therefore$ By D'Alembert's Ratio Test,
- $x < 1 \rightarrow$ convergent
- $x > 1 \rightarrow$ divergent
- $x = 1 \rightarrow$ Test fails

Q3. Test the convergence of $1+\frac{x}{2}+\frac{x^2}{5}+\ldots\ldots$

Sol:- Let $u_n = \frac{x^n}{n^2+1}$

$\therefore u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{x^n}$

$= \frac{\cancel{x^n}\cdot x}{n^2+2n+2} \cdot \frac{n^2+1}{\cancel{x^n}}$

$= \frac{x}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{2}{n^2}\right)} \cdot \cancel{n^2}\left(1+\frac{1}{n^2}\right)$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x\left(1+\frac{1}{n^2}\right)}{1+\frac{2}{n}+\frac{2}{n^2}}$

$= \frac{x\left(1+\frac{1}{\infty^2}\right)}{1+\frac{2}{\infty}+\frac{2}{\infty^2}}$

$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$

By D'Alembert's Ratio Test:-

- $x>1$:- divergent
- $x<1$:- convergent
- $x=1$:- test fails