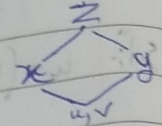
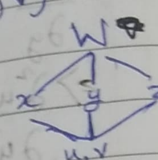
Partial differentiation of Composite Function (2 variables)



- $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \left[\text{where } \cancel{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x \partial y}} \right.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$



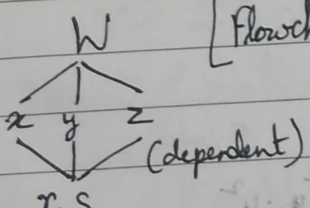
- $W = f(x,y,z)$ where $x = g_1(u,v)$, $y = g_2(u,v)$, $z = g_3(u,v)$.

$$\therefore \frac{\partial W}{\partial u} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial W}{\partial v} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial v}$$

Q1. $\frac{\partial W}{\partial r}$ and $\frac{\partial W}{\partial s}$ in terms of r and s if $W = x + 2y + z^2$; $x = r/s$; $y = r^2 + \log S$, $z = 2r$

Sol:-



[Flowchart]

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial r} \quad \text{— (1)}$$

$$\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial s} \quad \text{— (2)}$$

$$(1) \Rightarrow \cancel{\frac{\partial W}{\partial r} = (2y+z^2)\cdot\frac{1}{s} + (x+2+z^2)\cdot(2r)} + \quad \frac{\partial W}{\partial x} = \frac{\partial}{\partial x}\left(x + \underbrace{2y + z^2}_{\text{constant}}\right) = 1$$

$$(1) \Rightarrow \frac{\partial W}{\partial r} = 1\left(\frac{1}{s}\right) + 2(2r) + (2z)\cdot 2 \Rightarrow \frac{\partial W}{\partial r} = \frac{1}{s} + 4r + 4(2r)$$

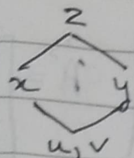
$$\therefore \frac{\partial W}{\partial r} = \frac{1}{s} + 12r$$

$$(2) \Rightarrow \frac{\partial W}{\partial s} = 1\left(-r\cdot\frac{1}{s^2}\right) + 2\left(\frac{1}{s}\right) + (2z)\cdot(0) = \frac{-r}{s^2} + \frac{2}{s} + 0$$

$$\therefore \frac{\partial W}{\partial s} = \frac{-r}{s^2} + \frac{2}{s}$$

Q2. $z = f(x,y)$ $\left[x = e^{u} + e^{-v}, \; y = e^{-u} - e^{v}\right]$ P.T:- $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Sol^n:-



(Flowchart) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ —— (1)

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$ —— (2)

Here, we do not have expression for z ∴ we keep it as $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

(1) $\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\left(e^{u} + e^{-v}\cdot(0)\right) + \frac{\partial z}{\partial y}\cdot\left(-e^{-u} - e^{-v}(0)\right)$

$= \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial y}\cdot e^{-u}$

(2) $\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^{-v} + \frac{\partial z}{\partial y}\cdot e^{v}$

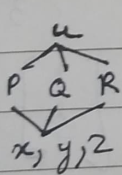
LHS $= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot e^{u} + \frac{\partial z}{\partial y}\cdot e^{-u} - \frac{\partial z}{\partial x}\cdot e^{-v} - \frac{\partial z}{\partial y}\cdot e^{v}$

$= \frac{\partial z}{\partial x}\cancelto{x}{\left[e^{u} + e^{-v}\right]} - \frac{\partial z}{\partial y}\cancelto{y}{\left[e^{-u} - e^{v}\right]} =$ RHS

Hence proved

Q3. If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$ P.T:- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol^n:- Let $P = e^{x-y}$, $Q = e^{y-z}$, $R = e^{z-x}$ ∴ $u = f(P, Q, R)$



(Flowchart) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial P}\cdot\frac{\partial P}{\partial x} + \frac{\partial u}{\partial Q}\cdot\frac{\partial Q}{\partial x} + \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial x}$ — (1)

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial P}\cdot\frac{\partial P}{\partial x} + \frac{\partial u}{\partial Q}\cdot\frac{\partial Q}{\partial x} + \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial y}$ — (2)

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial P}\cdot\frac{\partial P}{\partial z} + \frac{\partial u}{\partial Q}\cdot\frac{\partial Q}{\partial z} + \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial z}$ — (3)

(1) $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial P}\left(e^{x-y}\right) + \frac{\partial u}{\partial Q}(0) + \frac{\partial u}{\partial R}\left(-e^{z-x}\right)$

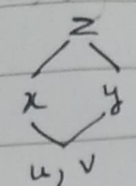
(2) $\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial P}\left(-e^{x-y}\right) + \frac{\partial u}{\partial Q}\left(e^{y-z}\right) + \frac{\partial}{\partial R}(0)$

(3) $\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial P}(0) + \frac{\partial u}{\partial Q}\left(-e^{y-z}\right) + \frac{\partial u}{\partial R}\cdot\left(e^{z-x}\right)$

Q.4. If $z = f(x,y)$, $x = e^u \cos v$ & $y = e^u \sin v$

P.T. :- i) $x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = e^{2u}\dfrac{\partial z}{\partial y}$ ii) $\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2\right]$

Sol:-



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(e^u \cdot \cos v) + \frac{\partial z}{\partial y}\cdot(e^u \sin v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

(i) $\cancel{LHS = x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = x\dfrac{\partial z}{\partial x}(-e^u}$ We know, $x = e^u \cos v$

$y = e^u \sin v$

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot x + \frac{\partial z}{\partial y}\cdot y \quad \& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-y) + \frac{\partial z}{\partial y}(x)$$

(i) $LHS = x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = \cancel{x^2\dfrac{\partial z}{\partial x} + xy\dfrac{\partial z}{\partial y} +}$

$$= -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} + xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y}$$

$$= (x^2 + y^2)\frac{\partial z}{\partial y}$$

$$= \left[(e^u)^2\left[\underbrace{\sin^2 v + \cos^2 v}_{1}\right]\right]\frac{\partial z}{\partial y}$$

$$= e^{2u}\cdot\frac{\partial z}{\partial y} = RHS$$

(ii) $LHS = \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$

$$= \left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]^2 + \left[x \cdot \frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right]^2$$

$$= x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2xy\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + x^2\left(\frac{\partial z}{\partial y}\right)^2 + y^2\left(\frac{\partial z}{\partial x}\right)^2 - 2xy\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(x^2+y^2) + \left(\frac{\partial z}{\partial y}\right)^2(x^2+y^2)$$

$$= (x^2+y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$= \cancel{(x e^{2u} \cdot \frac{\partial z}{\partial y}}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

$= RHS$