Radius of curvature :-

(2) $\rho = \dfrac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$

Q1. Find radius of curvature for curve $x = at^2$, $y = 2at$ [Parabola]

Sol :- Given Parametric Form :- $\cdot\ x = at^2$

$\cdot\ y = 2at$

Differentiate w.r.t t

For x_1 :- $x_1 = 2at$

For x_2 :- $x_2 = 2a$

For y_1 :- $y_1 = 2a$

For y_2 :- $y_2 = 0$

$\therefore \rho = \dfrac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$

$= \dfrac{[(2at)^2 + (2a)^2]^{3/2}}{2at(0) - 2a(2a)}$

$= \dfrac{[4a^2(t^2+1)]^{3/2}}{\ominus 4a^2}$ can't be -ve

$= \dfrac{8a^3(t^2+1)^{3/2}}{4a^2}$

$\Rightarrow \rho = 2a(t^2+1)^{3/2}$

(*) By Formula 1 :-

1) Find $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

Differentiate y w.r.t t

Differentiate x w.r.t t

2) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx}$

Put value in $\dfrac{dt}{dx}$

3) Use :- $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

Q2. Find the radius of curvature for the curve at any point 't'

$x = a\left(\cos t + \log \tan \frac{t}{2}\right)$, $y = a \sin t$

Sol:- By Method 1 :- $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ [Method (2) not recommended in this question] ↳ long calculation that's why

(1) To find $\frac{dy}{dx}$:- $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

- For $\frac{dy}{dt}$:- $\frac{dy}{dt} = \frac{d}{dt}(a \sin t) \Rightarrow \frac{dy}{dt} = a\cos t$
- For $\frac{dx}{dt}$:- $\frac{dx}{dt} = \frac{d}{dt}\left[a\left(\cos t + \log \tan \frac{t}{2}\right)\right]$

$$= a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \cdot \sec^2(t/2) \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\frac{\sin(t/2)}{\cos(t/2)} \times \cos^2(t/2)} \cdot 1/2\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$\sin 2\theta = 2\sin\theta\cos\theta \longrightarrow \sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$$\Rightarrow \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1-\sin^2 t}{\sin t}\right] \qquad 1-\sin^2\theta = \cos^2\theta$$

$$\Rightarrow \frac{dx}{dt} = a\frac{\cos^2 t}{\sin t} \Rightarrow \frac{dx}{dt} = a\frac{\cos^2 t}{\sin t}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \cancel{a}\cos t \times \frac{\sin t}{\cancel{a}\cos^{\cancel{2}} t}$$

$$\Rightarrow \frac{dy}{dx} = \tan t \qquad \therefore y_1 = \tan t$$

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(2) To find $\frac{d^2y}{dx^2}$:- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

- For $\frac{d}{dt}\left(\frac{dy}{dx}\right)$:- $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(\tan t)$

$= \sec^2 t$

$\therefore \frac{d}{dt}\left(\frac{dy}{dx}\right) = \sec^2 t \rightarrow \frac{1}{\cos^2 t}$

- For $\frac{dt}{dx}$:- $\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \sec^2 t \times \cancel{\frac{a\cos^2 t}{\sin t}} \quad \frac{\sin t}{a\cos^2 t}$$

$$= \frac{1}{\cos^2 t} \times \cancel{\frac{a\cos^2 t}{\sin t}}$$

$$= \frac{1}{\cos^2 t} \times \frac{\sin t}{a\cos^2 t}$$

$$\therefore y_2 = \frac{\sin t}{a\cos^4 t}$$

(3) Radius of Curvature :-

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+\tan^2 t]^{3/2}}{\left(\frac{\sin t}{a\cos^4 t}\right)}$$

$$= (\sec^2 t)^{3/2} \times \frac{a\cos^4 t}{\sin t}$$

$$= \sec^3 t \times \frac{1}{\sec^4 t} \cdot \frac{a}{\sin t}$$

$$= \frac{1}{\sec t} \cdot \frac{a}{\sin t}$$

$$= a\frac{\cos t}{\sin t}$$

$$\Rightarrow \rho = a\cot t$$

Q3. Find radius of curvature at θ on the curve :-

$$x = a(\theta - \sin\theta) \quad , \quad y = a(1-\cos\theta)$$

Sol :- $\rho = \dfrac{(x_1^2 + y_1^2)^{3/2}}{x_1 y_2 - y_1 x_2}$

- For x_1 :- $x_1 = \dfrac{d}{d\theta}\left(a(\theta - \sin\theta)\right)$ $\quad$ $\therefore x_1 = a(1-\cos\theta)$

$= a - a\cos\theta$

- For x_2 :- $x_2 = \dfrac{d}{d\theta}\left(a(1-\cos\theta)\right)$ $\quad$ $\therefore x_2 = a\sin\theta$

$= a(-(-\sin\theta))$

- For y_1 :- $y_1 = \dfrac{d}{d\theta}\left(a(1-\cos\theta)\right)$ $\quad$ $\therefore y_1 = a\sin\theta$

$= a\sin\theta$

- For y_2 :- $y_2 = \dfrac{d}{d\theta}\left(a\sin\theta\right)$ $\quad$ $\therefore y_2 = a\cos\theta$

$= a\cos\theta$

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Now,
$$\rho = \frac{(x_1^2+y_1^2)^{3/2}}{x_1y_2-y_1x_2} = \frac{\left[a^2(1-\cos\theta)^2+a^2\sin^2\theta\right]^{3/2}}{\left[a(1-\cos\theta)\cdot a\cos\theta - a\sin\theta\cdot a\sin\theta\right]}$$

$$= \frac{\left[a^2(1-2\cos\theta+\cos^2\theta)+a^2\sin^2\theta\right]^{3/2}}{a^2\cos\theta - a^2\cos^2\theta - a^2\sin^2\theta}$$

$$= \frac{\left[a^2-2a^2\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta\right]^{3/2}}{a^2\cos\theta - a^2(\sin^2\theta+\cos^2\theta)}$$

$$= \frac{\left[a^2-2a^2\cos\theta+a^2(\sin^2\theta+\cos^2\theta)\right]^{3/2}}{a^2\cos\theta - a^2}$$

$$= \frac{\left[a^2-2a^2\cos\theta+a^2\right]^{3/2}}{a^2(\cos\theta-1)}$$

$$= \frac{(2a^2-2a^2\cos\theta)^{3/2}}{a^2(\cos\theta-1)}$$

$$= \frac{\left[2a^2(1-\cos\theta)\right]^{3/2}}{-a^2(\cancel{1}-\cos\theta)}$$

$$= \frac{2^{3/2}\cdot a^3}{a^2}(1-\cos\theta)^{3/2-1}$$

$$\Rightarrow \rho = 2\sqrt{2}\,a\,(1-\cos\theta)^{1/2} \quad \text{———} \quad (*)$$

Let's evaluate $(1-\cos\theta)^{1/2}$:-

We know, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\therefore \cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos^2(\theta/2) - (1-\cos^2(\theta/2))$

$= 2\cos^2(\theta/2) - 1$

$\Rightarrow 1-\cos\theta = 2 - 2\cos^2(\theta/2) = 2\left[1-\cos^2(\theta/2)\right]$

(in $(*)$)

$$\therefore \rho = 2\sqrt{2}\,a\left[\sqrt{2(1-\cos^2(\theta/2))}\right]$$

$$= 2\sqrt{2}\,a\,(\sqrt{2})(\sqrt{\sin^2(\theta/2)})$$

$$= 4a\sin\frac{\theta}{2}$$

$$\therefore \rho = 4a\sin\frac{\theta}{2}$$