Gamma Function :-

For a positive value of n $(n>0)$,

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma n$$

where $\Gamma n = (n-1)!$ Eg:- $\Gamma 5 = 4!$

$\Gamma 4 = 3!$

Here, Γ is not root/square root $\rightarrow$ gamma symbol

Q1. Find value of $\int_0^\infty e^{-x} x^7 dx$

Solⁿ:- Here, $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$

$x^{n-1} \Rightarrow x^7$ $\therefore$ $n-1 = 7$

$\Rightarrow n = 8$

$\therefore \int_0^\infty e^{-x} x^{7} dx = \Gamma 8$

We know, $\Gamma n = (n-1)!$

$\Gamma 8 = (8-1)!$

$= 7!$

$\therefore \int_0^\infty e^{-x} x^7 dx = 7!$

(*) In $\int_0^\infty e^{-x} x^{n-1} dx$, if there is a constant 'a' over e i.e. e^{-ax}

$$\therefore \int_0^\infty e^{-ax} \cdot x^{n-1} dx = \frac{\Gamma n}{a^n}$$

Q2. Find value of $\int_0^\infty e^{-4x} x^{10} dx$

Sol:- $\int_0^\infty e^{-4x} x^{10} dx = \frac{\Gamma 11}{4^{11}}$ (where $n = 11$ & $a = 4$)

Important Tricks :-

1) $\Gamma(n+1) = n\,\Gamma n$ 2) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (root π)

Eg:

Q3. Find $\Gamma\frac{5}{2}$ value

Solⁿ:- $\Gamma n = \Gamma\frac{5}{2}$ $\therefore n = \frac{5}{2}$

Represent $\frac{5}{2}$ in the form of $n+1$

$\therefore n+1 = \frac{5}{2} \Rightarrow n = \frac{5}{2} - 1$

$\Rightarrow n = \frac{3}{2}$

Hence, $\frac{5}{2} = \frac{3}{2} + 1$ ($n \to n+1$ form)

$\therefore$ Now we apply, $\Gamma(n+1) = n\,\Gamma n$

$\therefore \Gamma\frac{5}{2} = \Gamma\left(\frac{3}{2}+1\right)$

We convert $\Gamma(n+1) \longrightarrow n\,\Gamma n$

where $n+1 = \frac{3}{2} \xrightarrow{-1} (n+1)-1 = n = 3/2$

$\Rightarrow \frac{3}{2} - 1 = \frac{1}{2}$ $\therefore \frac{3}{2} = \frac{1}{2} + 1$

$\therefore \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\,\Gamma\left(\frac{1}{2}+1\right)$

write in form of $n\,\Gamma n$

where $n+1 = \frac{1}{2} \xrightarrow{-1} (n+1)-1 = n = 1/2$

$\Rightarrow \frac{1}{2} - 1 = -ve$ $\therefore$ we stop here & write $\Gamma\frac{1}{2}$

$\therefore \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{3}{4}\sqrt{\pi}$

$\therefore \Gamma\frac{5}{2} = \frac{3}{4}\sqrt{\pi}$

Q4. Find value of $\int_0^\infty e^{-x} x^3 dx$

Sol:- Here, in $x^3 \rightarrow n-1 = 3$

$\Rightarrow n = 4$

$\therefore \int_0^\infty e^{-x} x^3 dx = \Gamma n$

$= \Gamma 4$

But we know, $\Gamma n = (n-1)!$

$\therefore \Gamma 4 = 3! = 6$

Q5. Find value of $\int_0^\infty e^{-x} x^{1/2} dx$

Sol:- Here, in $x^{1/2} \longrightarrow n-1 = \frac{1}{2}$

$\Rightarrow n = \frac{3}{2}$

$\therefore \int_0^\infty e^{-x} x^{1/2} dx = \Gamma n = \Gamma \frac{3}{2}$

But we know, $\Gamma n+1 = n\Gamma n$

To get n :- $n+1 = \frac{3}{2} \Rightarrow n = \frac{3}{2} - 1$

$\Rightarrow \boxed{n = \frac{1}{2}}$

$\therefore \Gamma \frac{3}{2} = \frac{1}{2} \Gamma \frac{1}{2}$

But we know, $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\therefore \frac{1}{2} \Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2}$

Hence, $\int_0^\infty e^{-x} x^{1/2} dx = \frac{\sqrt{\pi}}{2}$