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Maxima and Minima :-

Steps 1) Find f_x, f_y, f_{xx}, f_{xy}, f_{yy}

2) Find stationary points :- $f_x = 0$, $f_y = 0$

3) $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$ [Put possible points combinations into this eq^n]

4) If $rt - s^2 > 0$ && $r > 0 \longrightarrow$ minima

If $rt - s^2 > 0$ && $r < 0 \longrightarrow$ maxima

If $rt - s^2 < 0 \longrightarrow$ Neither maxima nor minima

If $rt - s^2 = 0 \longrightarrow$ Inconclusive point

$\Rightarrow$ 4 cases

Q1. Find all stationary values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Sol^n:- $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Step 1 :- $f_x = \frac{df}{dx} = 3x^2 + 3y^2 - 30x + 72$

$f_y = \frac{df}{dy} = 6xy - 30y$

$\rightarrow$ Use in Step 2

$f_{xx} = \frac{d}{dx}\left(\frac{df}{dx}\right) = 6x - 30$

$f_{xy} = \frac{d}{dx}\left(\frac{df}{dy}\right) = 6y$

$f_{yy} = \frac{d}{dy}\left(\frac{df}{dy}\right) = 6x - 30$

$\rightarrow$ Use in Step 3

Step 2 :- $f_x = 0$

$\Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$

At $y=0$

$3x^2 + 3(0)^2 - 30x + 72 = 0$

$\Rightarrow 3x^2 - 30x + 72 = 0$

$\Rightarrow x^2 - 10x + 24 = 0$ $\therefore \boxed{x = 6, 4}$

$\longrightarrow (6,0)$ & $(4,0)$ are stationary points

At $x=5$

$3(5)^2 + 3y^2 - 30(5) + 72 = 0$

$\Rightarrow 75 + 3y^2 - 150 + 72 = 0$

$\Rightarrow 3y^2 - 3 = 0$

$\Rightarrow y^2 - 1 = 0$ $\therefore \boxed{y = 1, -1}$

$\longrightarrow (5,1)$ & $(5,-1)$ are stationary points

$f_y = 0$

$\Rightarrow 6xy - 30y = 0$

$\Rightarrow 6y(x - 5) = 0$

$\Rightarrow y(x-5) = 0$

$\therefore x = 5$ & $y = 0$

Here, since we are using $x=5$ & $y=0$ in $f_x = 0$

We do not consider $(5,0)$ as a stationary point

$\left[\begin{array}{l} f_x(5,0) = 3(5)^2 + 3(0)^2 - 30(5) + 72 \\ = 75 - 150 + 72 \\ = -3 \\ \therefore f_x \neq 0 \text{ at } (5,0) \therefore \text{Its not stationary point} \end{array}\right]$

Step 3 :- $r = f_{xx} = 6x - 30$ & $s = f_{xy} = 6y$ & $t = f_{yy} = 6x - 30$

- For $(6,0) \rightarrow$ • $r = 6(6) - 30 \Rightarrow r = 6$ | • $s = 6(0) \Rightarrow s = 0$ | • $t = 6(6) - 30 \Rightarrow t = -6$

$\therefore rt - s^2 = (6)(-6) - 0^2 = -36$

$\because rt - s^2 < 0$ && $r > 0 \longrightarrow$ Minima/minimum point is $(6,0)$

- For $(4,0) \rightarrow$ • $r = 6(4) - 30 \Rightarrow r = -6$ | • $s = 6(0) \Rightarrow s = 0$ | • $t = 6(4) - 30 \Rightarrow t = -6$

$\therefore rt - s^2 = (-6)(-6) - 0^2 = 36$

$\because rt - s^2 > 0$ && $r < 0 \longrightarrow$ Maxima/maximum point is $(4,0)$

• For $(5,1) \rightarrow$ • $r = 6(5) - 30$ $\Rightarrow r = 0$ | • $s = 6(1)$ $\Rightarrow s = 6$ | • $t = 6(5) - 30$ $\Rightarrow t = 0$

$\therefore rt - s^2 = (0)(0) - (6)^2 = -36$

$\therefore rt - s^2 < 0 \longrightarrow$ Neither maxima / minima point

• For $f(5,-1) \rightarrow$ • $r = 6(5) - 30$ $\Rightarrow r = 0$ | • $s = 6(-1)$ $\Rightarrow s = -6$ | • $t = 6(5) - 30$ $\Rightarrow t = 0$

$\therefore rt - s^2 = (0)(0) - (-6)^2 = -36$

$\therefore rt - s^2 < 0 \longrightarrow$ Neither maxima / minima point

Hence :- $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ —— (4)

1. Maxima / Maximum Point $= (4,0)$

At $(4,0)$,

~~$f(4,0) = 3(4)^2 + 3(0)^2 - 30(4) + 72$~~
~~$= 48 - 120 + 72$~~

do not put it in f_x

(Put values in (4))

$f(4,0) = (4)^3 + 3(4)(0)^2 - 15(4)^2 - 15(0)^2 + 72(4)$
$= 64 - 240 + 288$
$\Rightarrow f(4,0) = 112$ $\therefore$ Max value $\rightarrow 112$

2. Minima / Minimum Point $= (6,0)$

At $(6,0)$,

$f(6,0) = (6)^3 + 3(6)(0)^2 - 15(6)^2 - 15(0)^2 + 72(6)$
$= 216 - 540 + 432$
$\Rightarrow f(6,0) = 108$ $\therefore$ Min value $\rightarrow 108$

3. Stationary points of $f(x)$ are :-

4 points :- $(6,0)$, $(4,0)$, $(5,1)$, $(5,-1)$

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Q2. Find the maximum and minimum values of $x^3+3xy^2-3x^2-3y^2+4$

Sol^n:- $f(x,y) = x^3+3xy^2-3x^2-3y^2+4$

Step 1 $f_x = 3x^2+3y^2-6x$ & $f_y = 6xy-6y$

Step 2

$\therefore f_x = 0$

$\Rightarrow 3x^2+3y^2-6x=0$

At $x=1$

$3(1)^2+3y^2-6(1)=0$

$\Rightarrow 3y^2=3$

$\Rightarrow y=\sqrt{1}=\pm 1$

$\therefore (1,1)$ & $(1,-1) \rightarrow$ stationary points

At $y=0$

$3x^2+3(0)^2-6x=0$

$\Rightarrow x^2-2x=0$

$\Rightarrow x(x-2)=0 \quad \therefore x=0,2$

$\therefore (0,0)$ & $(1,0) \rightarrow$ stationary points

$f_y = 0$

$\Rightarrow 6xy-6y=0$

$\Rightarrow 6y(x-1)=0$

$\Rightarrow y(x-1)=0 \quad \therefore \boxed{x=1,\ y=0}$

Check for $f_x(1,0)$

$f_x(1,0) = 3(1)^2+3(0)^2-6(1)$

$= -3$

$\because f_x(1,0) \neq 0$, $\therefore (1,0) \rightarrow$ not a stationary point

$\therefore$ We use only $f_x = 0$ to find stationary points.

Step 3

$r = f_{xx} = \frac{d}{dx}(3x^2+3y^2-6x)$

$\Rightarrow r = 6x-6$

$s = f_{xy} = \frac{d}{dx}(6xy-6y)$

$\Rightarrow s = 6y$

$t = f_{yy} = \frac{d}{dy}(6xy-6y)$

$\Rightarrow t = 6x-6$

- For $(1,1)$:- $r=6(1)-6$ & $s=6(1)$ & $t=6(1)-6$

  $\Rightarrow r=0$ $\quad \Rightarrow s=6$ $\quad \Rightarrow t=0$

  $\therefore rt-s^2 = (0)(0)-(6)^2 = -36$

  $\because rt-s^2<0 \longrightarrow$ Neither maxima/minima point

- For $(1,-1)$:- $r=6(1)-6$ & $s=6(-1)$ & $t=6(1)-6$

  $\Rightarrow r=0$ $\quad \Rightarrow s=-6$ $\quad \Rightarrow t=0$

  $\therefore rt-s^2 = (0)(0)-(-6)^2 = -36$

  $\because rt-s^2<0 \longrightarrow$ Neither maxima/minima point

- For $(0,0)$:- $r=6(0)-6$ & $s=6(0)$ & $t=6(0)-6$

  $\Rightarrow r=-6$ $\quad \Rightarrow s=0$ $\quad \Rightarrow t=-6$

  $\therefore rt-s^2 = (-6)(-6)-(0)^2 = 36$

  $\because rt-s^2>0$ & $r<0 \longrightarrow$ Maxima/maximum point is $(0,0)$

- For $(2,0)$:- $r = 6(2) - 6$ & $s = 6(0)$ & $t = 6(2) - 6$
$\Rightarrow r = 6$ $\Rightarrow s = 0$ $\Rightarrow t = 6$

$\therefore rt - s^2 = (6)(6) - 0^2 = 36$
$\because rt - s^2 > 0$ & $r > 0$ $\longrightarrow$ Minima/minimum point is $(2,0)$

Hence :- $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ —— (*)

1. Maxima/Maximum point $= (0,0)$
At $(0,0)$:-

$f(0,0) = (0)^3 + 3(0)(0)^2 - 3(0)^2 - 3(0)^2 + 4$
$\Rightarrow f(0,0) = 4$ $\therefore$ Max value $= 4$

2. Minima/Minimum point $= (2,0)$
At $(2,0)$:-

$f(2,0) = (2)^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4$
$= 8 - 12 + 4$
$\Rightarrow f(2,0) = 0$ $\therefore$ Min value $= 0$

3. Stationary points of $f(x)$ are :-

$(1,1)$, $(1,-1)$, $(0,0)$, $(2,0)$