Q1. Find the points on the parabola $y^2 = 8x$ which is the radius of curvature is $\frac{125}{16}$

Sol:- Given curve - $y^2 = 8x$

For y_1:- Differentiate both sides w.r.t x

$$\Rightarrow 2y\frac{dy}{dx} = 8$$

$$\Rightarrow y_1 = \frac{4}{y}$$

For y_2:- $y_2 = \frac{d}{dx}\left(\frac{4}{y}\right)$

$$= 4 \cdot \left(-\frac{1}{y^2}\right) \cdot \frac{dy}{dx} \quad \rightarrow y_1 = 4/y$$

$$= \frac{-4}{y^2} \times \frac{4}{y}$$

$$\Rightarrow y_2 = -\frac{16}{y^3}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \Rightarrow \left[1+\left(\frac{4}{y}\right)^2\right]^{3/2} \times \frac{y^3}{-16} = \frac{125}{16}$$

$$= \cancel{(1+y^2)}$$

$\rightarrow$ can't be -ve

$$\Rightarrow \frac{(y^2+16)^{3/2}}{(y^2)^{3/2}} \times \frac{y^3}{16} = \frac{125}{16}$$

$$\Rightarrow (y^2+16) = (125)^{2/3}$$

$$= 5^{3 \times 2/3}$$

$$= 5^2$$

$$\Rightarrow y^2 + 16 = 25$$

$$\Rightarrow y^2 = 9$$

$$\Rightarrow y = \pm 3$$

$\therefore$ For $y = +3$:- $x = \frac{y^2}{8} = \frac{9}{8}$

For $y = -3$:- $x = \frac{9}{8}$

$\therefore$ Point is $\left(\frac{9}{8}, 3\right)$ & $\left(\frac{9}{8}, -3\right)$

Q2. Show that the curve $y = \frac{ax}{a+x}$, the radius of curvature ρ at (x, y) related as $\left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2} + \frac{y^2}{x^2}$

Sol:- Given curve: $y = \frac{ax}{a+x}$

For y_1:- $y_1 = \frac{d}{dx}\left(\frac{ax}{a+x}\right)$

$$= \frac{(a+x)\frac{d}{dx}(ax) - ax \cdot \frac{d}{dx}(a+x)}{(a+x)^2}$$

$$= \frac{(a+x)a - ax}{(a+x)^2}$$

$$= \frac{a^2 + \cancel{ax} - \cancel{ax}}{(a+x)^2} \Rightarrow y_1 = \frac{a^2}{(a+x)^2}$$

For y_2:- $y_2 = \frac{d}{dx}\left(\frac{a^2}{(a+x)^2}\right)$

$$\cancel{= a^2 \cdot \frac{2}{(a+x)^3} \cdot \frac{d}{dx}(a+x)}$$

$$= a^2(-2)(a+x)^{-2-1} \Rightarrow y_2 = \frac{-2a^2}{(a+x)^3}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \left[1 + \left(\frac{a^2}{(a+x)^2}\right)^2\right]^{3/2} \times \frac{(a+x)^3 \times a}{-2a^2 \times a}$$

can't be -ve

A/Q,

$$y = \frac{ax}{a+x} \Rightarrow \frac{a}{a+x} = \frac{y}{x}$$

Square both sides:-

$$\Rightarrow \left(\frac{a}{a+x}\right)^2 = \frac{y^2}{x^2}$$

$$\rho = \left[1 + \left[\left(\frac{a}{a+x}\right)^2\right]^2\right]^{3/2} \times \cancel{(a+x)} \frac{1}{\frac{2}{a}\left(\frac{a}{a+x}\right)^3}$$

$$\rho = \left[1 + \left(\frac{y^2}{x^2}\right)^2\right]^{3/2} \times \frac{1}{\frac{2}{a}\left(\frac{y}{x}\right)^3}$$

Multiply both sides by $\left(\frac{2}{a}\right)$:-

$$\Rightarrow \frac{2}{a}\rho = \left[1 + \frac{y^4}{x^4}\right]^{3/2} \times \cancel{\left(\frac{2}{a}\right)} \cdot \frac{1}{\cancel{\left(\frac{2}{a}\right)} \cdot \frac{y^3}{x^3}}$$

$$= \frac{(x^4 + y^4)^{3/2}}{(x^4)^{3/2}} \times \frac{x^3}{y^3}$$

$$= \frac{(x^4 + y^4)^{3/2}}{x^3 y^3}.$$

Raise to the power '2/3' on both sides:-

$$\Rightarrow \left(\frac{2}{a}\rho\right)^{2/3} = \left[\frac{(x^4 + y^4)^{3/2}}{x^3 y^3}\right]^{2/3}$$

$$= \left[\frac{x^4 + y^4}{x^{3\times 2/3} \cdot y^{3\times 2/3}}\right]$$

$$= \frac{x^4 + y^4}{x^2 y^2}$$

Now seperate the terms:-

$$\left(\frac{2}{a}\rho\right)^{2/3} = \frac{x^4}{x^2 y^2} + \frac{y^4}{x^2 y^2}$$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2}$$

Hence proved