Radius of Curvature :-

The amount of bending of a curve at given point on it $\rightarrow$ Curvature

- For Cartesian curve :- $\rho = \dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$

- If parametric coordinate given :- $\rho = \dfrac{\left(x'^2+y'^2\right)^{3/2}}{x'y''-y'x''}$ $\left[\begin{array}{l} x' = \frac{dx}{dt} \rightarrow x = f(t) \\ y' = \frac{dy}{dt} \rightarrow y = g(t) \end{array}\right]$

Q1. Find radius of curvature of curve :- $x^2+y^2=a^2$

Sol:- $x^2+y^2=a^2$ } $\rightarrow$ Equation of circle so ans should be a

Find $\frac{dy}{dx}$:- Differentiate both sides w.r.t x

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} \Rightarrow \boxed{y' = \frac{-x}{y}}$$

Find $\frac{d^2y}{dx^2}$:- Differentiate y' again w.r.t x

$$\Rightarrow y'' = \frac{d}{dx}\left(\frac{-x}{y}\right)$$

$$= \frac{y\cdot\frac{d}{dx}(-x) - (-x)\cdot\frac{dy}{dx}}{y^2} \qquad \frac{dy}{dx} = y' = -\frac{x}{y}$$

$$= \frac{-y + x\cdot\left(\frac{-x}{y}\right)}{y^2} = \frac{\frac{-y^2-x^2}{y}}{y^2}$$

$x^2+y^2=a^2$

$$\Rightarrow y'' = -\frac{(y^2+x^2)}{y^3} \Rightarrow \boxed{y'' = -\frac{a^2}{y^3}}$$

Now, Radius of curvature $(\rho) = \dfrac{\left(1+(y')^2\right)^{3/2}}{y''}$

$$= \frac{\left[1+\left(\frac{-x}{y}\right)^2\right]^{3/2}}{\left(-\frac{a^2}{y^3}\right)}$$

$$y' = \frac{dy}{dx} = \frac{-x}{y} \qquad\qquad y'' = \frac{d^2y}{dx^2} = -\frac{a^2}{y^3}$$

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$$\therefore \rho = \frac{\left[1 + \frac{x^2}{y^2}\right]^{3/2} \cdot y^2}{a^2}$$

(Radius always +ve ∴ −a → +a taken)

$$= \left(\frac{y^2 + x^2}{y^2}\right)^{3/2} \cdot \frac{y^3}{a^2}$$

$$= \left(\frac{a^2}{y^2}\right)^{3/2} \cdot \frac{y^3}{a^2}$$

$$= \frac{a^3}{\cancel{y^3}} \cdot \frac{\cancel{y^3}}{a^2}$$

$$\Rightarrow \boxed{\rho = a}$$

∴ Radius of curvature of the curve = a

Q2. Find radius of curve where :- $x = a\cos t$ & $y = a\sin t$

Sol:- We have, $x = a\cos t$, $y = a\sin t$ } ∴ Find derivative for these two w.r.t t

$$x' = a\frac{d}{dt}(\cos t) \Rightarrow \boxed{x' = -a\sin t}$$

$$x'' = -a\frac{d}{dt}(\sin t) \Rightarrow \boxed{x'' = -a\cos t}$$

} For x

$$y' = a\frac{d}{dt}(\sin t) \Rightarrow \boxed{y' = a\cos t}$$

$$y'' = a\frac{d}{dt}(\cos t) \Rightarrow \boxed{y'' = -a\sin t}$$

} For y

Now, Radius of curvature $(\rho) = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

$$\rho = \frac{[(-a\sin t)^2 + (a\cos t)^2]^{3/2}}{(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)}$$

$$= \frac{[a^2(\sin^2 t + \cos^2 t)]^{3/2}}{a^2(\sin^2 t + \cos^2 t)} = \frac{a^3}{a^2}$$

$$\Rightarrow \boxed{\rho = a}$$

∴ Radius of curvature of curve = a

Q3. Find radius of curvature of curve :- $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$

Sol^n:- $\sqrt{x}+\sqrt{y}=1$

$\frac{dy}{dx}$:- Differentiate both sides w.r.t x :-

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\cancel{2}\sqrt{x}}\times\cancel{2}\sqrt{y} \Rightarrow \boxed{y'=-\frac{\sqrt{y}}{\sqrt{x}}}$$

$\frac{d^2y}{dx^2}$:- Differentiate y' w.r.t x :-

$$y''=-\frac{\left(\sqrt{x}\cdot\frac{d}{dx}(\sqrt{y})-\sqrt{y}\cdot\frac{d}{dx}(\sqrt{x})\right)}{(\sqrt{x})^2}$$

$$=\frac{-\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}+\sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}$$

$$=\frac{-\frac{\sqrt{x}}{2\sqrt{y}}\cdot\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)+\frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$=\frac{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{y}{x}}}{x} \Rightarrow \boxed{y''=\frac{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{y}{x}}}{x}}$$

At $\left(\frac{1}{4},\frac{1}{4}\right)$:-

- $y'=-\frac{\sqrt{1/4}}{\sqrt{1/4}} \Rightarrow \boxed{y'=-1}$
- $y''=\frac{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1/4}{1/4}}}{1/4}=\frac{1}{1/4} \Rightarrow \boxed{y''=4}$

Radius of curvature :- $\rho=\frac{\left(1+(y')^2\right)^{3/2}}{y''}$

$$=\frac{\left(1+(-1)^2\right)^{3/2}}{4}$$

$$=\frac{2^{3/2}}{2^2}=2^{3/2-2}=2^{-1/2}$$

$$\Rightarrow \rho=\frac{-1}{\sqrt{2}}$$

Q4. In curve $y = ae^{x/a}$, prove that $\rho = a\sec^2\theta\,\mathrm{cosec}\,\theta$ where $\theta = \tan^{-1}\left(\frac{y}{a}\right)$

Sol:- $y = ae^{x/a}$

$\frac{dy}{dx}$:- $\frac{dy}{dx} = a \cdot e^{x/a} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$

$= \cancel{a} \cdot e^{x/a} \cdot \frac{1}{\cancel{a}} \Rightarrow \boxed{y' = e^{x/a}}$

$\frac{d^2y}{dx^2}$:- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{x/a}\right)$

$= e^{x/a} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) \Rightarrow \boxed{y'' = \frac{e^{x/a}}{a}}$

Radius of curvature $(\rho) = \frac{(1+y'^2)^{3/2}}{y''}$

$\Rightarrow \rho = \frac{\left(1+\left(e^{x/a}\right)^2\right)^{3/2}}{\left(\frac{e^{x/a}}{a}\right)}$ — #

But, A.Q,

∴ we need to prove → $\rho = a\sec^2\theta\,\mathrm{cosec}^2\theta$

where $\theta = \tan^{-1}(y/a)$

∴ We need to find relation betⁿ :- • $y = ae^{x/a}$

• $\theta = \tan^{-1}(y/a)$

Put $(y = ae^{x/a}) \xrightarrow{in} \theta = \tan^{-1}(y/a)$

∴ $\theta = \tan^{-1}\left(\frac{\cancel{a}\,e^{x/a}}{\cancel{a}}\right)$

$\Rightarrow \boxed{\tan\theta = e^{x/a}}$

$\frac{y}{a} = e^{x/a}$

$\Rightarrow \tan\theta = \frac{y}{a}$ ⊛

$\Rightarrow \rho = \frac{\left(1+(y/a)^2\right)^{3/2}}{\left(\frac{y/a}{a}\right)} \xrightarrow{\text{in terms of } \theta} \rho = \frac{(1+\tan^2\theta)^{3/2}}{\frac{\tan\theta}{a}}$

$\therefore \rho = \frac{(\sec^2\theta)^{3/2}}{\tan\theta} \cdot a$ $(\because 1+\tan^2\theta = \sec^2\theta)$

$= \sec^3\theta \cdot \frac{a}{\tan\theta} = \sec^2\theta \cdot \frac{1}{\cancel{\cos\theta}}\left(\frac{\cancel{\cos\theta}}{\sin\theta}\right) \cdot a$

$\Rightarrow \rho = a\sec^2\theta\,\mathrm{cosec}\,\theta$

∴ Radius of curvature of curve $= a\sec^2\theta\,\mathrm{cosec}^2\theta$

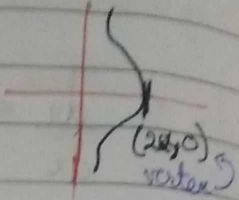
Hence proved

Q. Find radius of curvature of following curve $y^2 = \frac{4a^2(2a-x)}{x}$ at its vertex

Sol: $y^2 = \frac{4a^2(2a-x)}{x} \Rightarrow xy^2 = 4a^2(2a-x)$

$\Rightarrow x(y^2+4a^2) = 8a^3$

$\Rightarrow x = \frac{8a^3}{y^2+4a^2}$



(Curve tracing)

$\frac{dx}{dy}$:- $\frac{dx}{dy} = 8a^3 \cdot \frac{d}{dy}\left((y^2+4a^2)^{-1}\right)$

$= \frac{8a^3}{(y^2+4a^2)^2}(2y) \Rightarrow y' = \frac{16a^3y}{(y^2+4a^2)^2}$

$\frac{d^2x}{dy^2}$:- $\frac{d^2x}{dy^2} = 16a^3 \frac{d}{dy}\left(\frac{y}{(y^2+4a^2)^2}\right)$

$= 16a^3\left[\frac{(y^2+4a^2)^2 \cdot \frac{d}{dy}(y) - y \cdot \frac{d}{dy}\left((y^2+4a^2)^2\right)}{\left((y^2+4a^2)^2\right)^2}\right]$

$\Rightarrow y'' = 16a^3\left[\frac{(y^2+4a^2)^2 - y \cdot 2(y^2+4a^2)\cdot(2y)}{(y^2+4a^2)^4}\right]$

At vertex (2a, 0) :-

$y' = \frac{16a^3(0)}{(0^2+4a^2)^2} \Rightarrow \boxed{y' = 0}$

$y'' = 16a^3\left[\frac{0^2+16a^4-(0)2(0^2+4a)(2\cdot(0))}{(0^2+4a)^4}\right]$

$= 16a^3\left(\frac{16a^4}{(4a)^4}\right)$

$= (4)^2a^3\left(\frac{(4a^2)^2}{(4a)^4}\right) \Rightarrow \boxed{y'' = \frac{1}{a}}$

$\therefore$ Radius of curvature $(\rho) = \frac{(1+y'^2)^{3/2}}{y''} = \frac{(1+0)^{3/2}}{1/a}$

$\Rightarrow \boxed{\rho = a}$

$\therefore$ Radius of curvature of curve at vertex (2a, 0) $= a$