(*) Q9. Find the maximum & minimum value of $\sin x \sin y \sin(x+y)$ $\begin{bmatrix} 0 < x < \pi \\ 0 < y < \pi \end{bmatrix}$

Solⁿ:- $f(x,y) = \sin x \sin y \sin(x+y)$

$f_x = \sin y \left[\frac{d}{dx}\left(\sin x \sin(x+y)\right)\right]$ & $f_y = \sin x \left[\frac{d}{dy}\left(\sin y \sin(x+y)\right)\right]$

$= \sin y \left[\sin x \cos(x+y) + \cos x \sin(x+y)\right]$ $= \sin x \left[\sin y \cos(x+y) + \cos y \sin(x+y)\right]$

We know, $\sin(A+B) = \sin A \cos B + \cos A \sin B$ Here, $A = x$ and y & $B = x+y$

$f_x = \sin y \sin(x+x+y)$ & $f_y = \sin x \sin(y+x+y)$

$\Rightarrow f_x = \sin y \sin(2x+y)$ & $f_y = \sin x \sin(2y+x)$

$f_x = 0$	$f_y = 0$
$\Rightarrow \sin y \sin(2x+y) = \sin 0$	$\Rightarrow \sin x \sin(2y+x) = \sin 0$
$\therefore \sin y = \sin 0 \Rightarrow y = 0, \pi, 2\pi$	$\therefore \sin x = \sin 0 \Rightarrow x = 0, \pi, 2\pi$
& $\sin(2x+y) = 0 \Rightarrow 2x+y = 0, \pi, 2\pi$	& $\sin(2y+x) = 0 \Rightarrow 2x+y = 0, \pi, 2\pi$

1. Points such as $(0,0), (0,\pi), (0,2\pi), (\pi,0), (\pi,\pi), (\pi,2\pi) \ldots (2\pi,2\pi)$ are not possible because we are given the range:-

$0 < x < \pi$
$0 < y < \pi$

$\left(\because y = 0, \pi, 2\pi \quad x = 0, \pi, 2\pi\right) \rightarrow$ False

2. $2x+y = 0$ —— (1) $x+2y = 0$ —— (2)
$2x+y = \pi$ —— (3) $x+2y = \pi$ —— (4)
$2x+y = 2\pi$ —— (5) $x+2y = 2\pi$ —— (6)

$\because$ The range is from $0 \rightarrow \pi$

~~Take~~ ~~(1)~~ Compare (1) & (2)
(3) & (4)
(5) & (6)

Do not take (1) & (3) or (2) & (6)

∵ RHS should have same value to ensure range not exceeded classmate

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For ① & ②

$① - ② \times 2 \Rightarrow 2x + y - 2x - 4y = 0$

$\Rightarrow -3y = 0$

$\Rightarrow y = 0$

$① \Rightarrow 2x + 0 = 0$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

But $(0,0)$ is not valid (Mentioned in 1)

For ③ & ④

$③ - ④ \times 2 \Rightarrow 2x + y - 2x - 4y = \pi - 2\pi$

$\Rightarrow -3y = -\pi$

$\Rightarrow y = \frac{\pi}{3}$ $\longrightarrow$ Satisfies $0 < y < \pi$

$③ \Rightarrow 2x + \frac{\pi}{3} = \pi$

$\Rightarrow 2x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\Rightarrow x = \frac{\pi}{3}$ $\longrightarrow$ Satisfies $0 < x < \pi$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is valid

For ⑤ & ⑥

$⑤ - ⑥ \times 2 \Rightarrow 2x + y - 2x - 4y = 2\pi - 4\pi$

$\Rightarrow -3y = -2\pi$

$\Rightarrow y = \frac{2\pi}{3}$ $\longrightarrow$ Satisfies $0 < y < \pi$

$⑤ \Rightarrow 2x + \frac{2\pi}{3} = 2\pi$

$\Rightarrow 2x = 2\pi - \frac{2\pi}{3}$

$\Rightarrow x = \frac{4\pi}{3 \times 2} = \frac{2\pi}{3}$ $\longrightarrow$ Satisfies $0 < x < \pi$

$\therefore$ Stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ & $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

- $r = f_{xx} = \dfrac{d}{dx}\left(\sin y \sin(2x+y)\right) \Rightarrow r = 2\sin y \cos(2x+y)$
- $s = f_{xy} = \dfrac{d}{dx}\left(\sin x \sin(x+2y)\right) \Rightarrow s = \sin x \cos(x+2y) + \cos x \sin(x+2y)$
  $= \sin(2x+2y)$ [sin(A+B) formula]
- $t = f_{yy} = \dfrac{d}{dy}\left(\sin x \sin(x+2y)\right) \Rightarrow t = 2\sin x \cos(x+2y)$

here cos → -ve

- For $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$; $\bullet\ r = 2\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{2\pi}{3} + \dfrac{\pi}{3}\right)$ $\left[\begin{aligned}&\cos(\pi)\\ &= \cos(\pi - 0)\\ &= -\cos(0) = -1\end{aligned}\right]$

$= \cancel{2}\cdot\dfrac{\sqrt{3}}{\cancel{2}}(-1) \Rightarrow r = -\sqrt{3}$

sin → +ve $(\pi-\theta)$

$\bullet\ s = \sin\left(\dfrac{2\pi}{3} + \dfrac{2\pi}{3}\right)$ $\left[\begin{aligned}&\sin\frac{4\pi}{3}\\ &= \sin\left(\pi + \frac{\pi}{3}\right)\\ &= -\sin\left(\frac{\pi}{3}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}\right]$

sin → -ve $(\pi+\theta)$ $\quad \dfrac{4\pi}{3} - \pi = \dfrac{\pi}{3}$

$= \sin\dfrac{4\pi}{3} \Rightarrow s = -\dfrac{\sqrt{3}}{2}$

$\bullet\ t = 2\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{3} + \dfrac{2\pi}{3}\right) \Rightarrow t = -\sqrt{3}$

$\therefore rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\dfrac{\sqrt{3}}{2}\right)^2$

$= 3 - \dfrac{3}{4} = \dfrac{9}{4}$

$\because rt - s^2 > 0$ & $r < 0 \Rightarrow$ Maxima

$\therefore$ Maximum point is at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$

$\sin\dfrac{2\pi}{3} = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

Max value $= f\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right) = \sin\left(\dfrac{\pi}{3}\right)\sin\left(\dfrac{\pi}{3}\right)\sin\left(\dfrac{\pi}{3} + \dfrac{\pi}{3}\right)$

$= \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{2}$

$= \dfrac{3\sqrt{3}}{8}$

$\cancel{\therefore\, r = f_{xx}}$

$$3\pi, \pi \quad \begin{array}{c|c} S & A \\ \hline T & C \end{array} \quad 0, 2\pi$$

- For $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$:- • $r = 2\sin\left(\frac{2\pi}{3}\right)\cos\overbrace{\left(\frac{4\pi}{3}+\frac{2\pi}{3}\right)}^{2\pi}$

$$= 2\sin\left(\pi - \frac{\pi}{3}\right)\cos(2\pi + 0)$$

$$= \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\cdot (1) \Rightarrow r = \sqrt{3}$$

• $s = \sin\left(\frac{4\pi}{3} + \frac{4\pi}{3}\right) = \sin\left(\frac{8\pi}{3}\right)$

$$= \sin\left(3\pi - \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} \Rightarrow s = \frac{\sqrt{3}}{2}$$

• $t = 2\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{2\pi}{3} + \frac{4\pi}{3}\right) \Rightarrow t = \sqrt{3}$

$$\therefore rt - s^2 = (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4} = \frac{9}{4}$$

$\because rt - s^2 > 0$ & $r > 0 \longrightarrow$ Minima

$\therefore$ Minimum point is at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

Min value $= f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$

$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

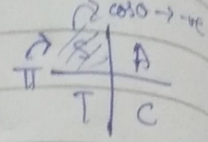
$$= \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\cdot\sin\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\left(\frac{-\sqrt{3}}{2}\right)$$

$$= -\frac{3\sqrt{3}}{8}$$

Q10. In a plane triangle ABC, find the maximum value of $\cos A \cos B \cos C$

Sol:- $\because$ ABC $\rightarrow$ triangle $\therefore$ Sum of $\angle$'s $= 180^\circ$



$$\therefore A + B + C = \pi$$
$$\Rightarrow C = \pi - (A+B)$$

$$\therefore f(A,B) = \cos A \cos B \cos C$$
$$= \cos A \cos B \cos(\pi - (A+B))$$
$$\Rightarrow f(A,B) = -\cos A \cos B \cos(A+B)$$

1. $f_A = -\cos B \left[\frac{d}{dA}\left(\cos A \cos(A+B)\right)\right]$

$$= -\cos B \left[\cos A[-\sin(A+B)] + \cos(A+B)(-\sin A)\right]$$
$$= \cos B \left[\sin(A+B)\cos A + \cos(A+B)\sin A\right]$$ [Use $\sin(A+B)$ formula]

$$\Rightarrow f_A = \cos B \sin(2A+B)$$

2. $f_B = -\cos A \left[\frac{d}{dB}\left(\cos B \cos(A+B)\right)\right]$

$$= -\cos A \left[\cos B \cdot (-\sin(A+B)) + \cos(A+B)(-\sin B)\right]$$
$$= \cos A \left[\sin(A+B)\cos B + \cos(A+B)\sin B\right]$$

$$\Rightarrow f_B = \cos A \sin(A+2B)$$

Now

$f_A = 0$ & $f_B = 0$

$\Rightarrow \cos B \cdot \sin(2A+B) = 0$

$\therefore \cos B = \cos(\pi/2)$

$\Rightarrow B = \pi/2$

(or)

$\sin(2A+B) = \sin 0$

$\Rightarrow 2A + B = 0, \pi, 2\pi$

$\underset{\text{invalid}}{0}$

$\Rightarrow \cos A \sin(A+2B) = 0$

$\therefore \cos A = \cos(\pi/2)$

$\Rightarrow A = \pi/2$

(or)

$\sin(A+2B) = \sin 0$

$\Rightarrow A + 2B = 0, \pi, 2\pi$

$\underset{\text{invalid}}{0}$

However, since ABC is a triangle

$\therefore \angle(s)$ can't be 0

For $f_A = 0$:- $B = \pi/2$
& $2A + B = \pi, 2\pi$ —— (1)

For $2A+B=\pi$:-
Put $A = \pi/2$:-
$\cancel{2}\left(\frac{\pi}{\cancel{2}}\right) + B = \pi$
$\Rightarrow B = 0$ (invalid)

For $f_B = 0$:- $A = \pi/2$
& $A + 2B = \pi, 2\pi$ —— (2)

For $A+2B=\pi$:-
Put $B = \pi/2$:-
$A + 2\left(\frac{\pi}{2}\right) = \pi$
$\Rightarrow A = 0$ (invalid)

1. Let's evaluate (1) & (2) :- (Both $= \pi$) $\boxed{\begin{matrix} 2A+B=\pi \\ A+2B=\pi \end{matrix}}$

(1) − (2) × 2 $\Rightarrow 2A + B - 2A - 4B = \pi - 2\pi$
$\Rightarrow \cancel{-}3B = \cancel{-}\pi$
$\Rightarrow B = \frac{\pi}{3} = \frac{180}{3} = 60^\circ$

(1) $\Rightarrow 2A + \frac{\pi}{3} = \pi$
$\Rightarrow \cancel{2}A = \frac{\cancel{2}\pi}{3} \Rightarrow A = \frac{\pi}{3} = 60^\circ$

$\therefore$ A + B
$60 + 60 < 180$
$\Rightarrow 120 < 180$

$\therefore$ Stationary point :-
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

2. Let's evaluate (1) & (2) :- (Both $= 2\pi$) $\boxed{\begin{matrix} 2A+B=2\pi \\ A+2B=2\pi \end{matrix}}$

(1) − (2) × 2 $\Rightarrow 2A + B - 2A - 4B = 2\pi - 4\pi$
$\Rightarrow \cancel{-}3B = \cancel{-}2\pi$
$\Rightarrow B = \frac{2\pi}{3} = \frac{2 \times 180}{3} = 120^\circ$

(1) $\Rightarrow 2A + \frac{2\pi}{3} = 2\pi$
$\Rightarrow 2A = \frac{4\pi}{3} \Rightarrow A = \frac{2\pi}{3}$
$= \frac{2 \times 180}{3}$
$= 120^\circ$

$\therefore$ No stationary point in this case

A + B
$120^\circ + 120^\circ > 180$
$\Rightarrow 240 > 180$

But $A + B = 120^\circ + 120^\circ = 240^\circ > 180$
(invalid)

- $r = f_{AA} = \frac{d}{dA}\left(\cos B \sin(2A+B)\right) \Rightarrow r = 2\cos B \cos(2A+B)$
- $s = f_{AB} = \frac{d}{dA}\left(\cos A \sin(A+2B)\right) \Rightarrow s = \cos A \cos(A+2B) - \sin A \sin(A+2B)$
  $= \cos(2A+2B)$ [$\cos(A+B)$ formula]
- $t = f_{BB} = \frac{d}{dB}\left(\cos A \sin(A+2B)\right) \Rightarrow t = 2\cos A \cos(A+2B)$

$\cos\pi = \cos(\pi - 0) = -\cos 0 = -1$

- For $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$:-
  - $r = 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$
    $= 2\left(\frac{1}{2}\right)\cdot(-1) \Rightarrow r = -1$
  - $s = \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right)$
    $= \cos\left(\pi + \frac{\pi}{3}\right)$ $\left[\cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}\right]$
    $= -\cos\frac{\pi}{3} \Rightarrow s = -\frac{1}{2}$
  - $t = 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) \Rightarrow t = -1$

$\therefore rt - s^2 = (-1)(-1) - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore rt - s^2 > 0$ & $r < 0$ $\longrightarrow$ Maxima at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$\therefore$ Maximum value $= f(\pi/3, \pi/3)$
$= -\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$ $\left[\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}\right]$
$= -\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)$
$= \frac{1}{8}$