Sample

Q1 $(x^3D^3+3x^2D^2+xD+8)y = 65\cos(\log x)$

Sol^n:- $xD = D'$

$x^2D^2 = D'(D'-1)$

$x^3D^3 = D'(D'-1)(D'-2)$

$D = \frac{d}{dy}$ & $D' = \frac{d}{dz}$

$\therefore [D'(D'-1)(D'-2)+3D'(D'-1)+D'+8]y = 65\cos(\log x) \rightarrow 65\cos z$

Let $e^z = x \Rightarrow \log x = z$

Q2. $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y = 3x^2+4x+1$

Sol^n:- Here, in $xD \longrightarrow$ take x from highest degree D

$\rightarrow (3x+2)$ (variable coefficient)

Let $e^z = 3x+2 \Rightarrow \log(3x+2) = z$

$(3x+2)D = 3D' \longleftrightarrow (3x+2)\frac{dy}{dx} = 3xD = 3D'$

$(3x+2)^2D = 9D'(D'-1) \longleftrightarrow (3x+2)^2\frac{d^2y}{dx^2} = 3^2D'(D'-1) = 9D'(D'-1)$

$(3x+2)^3 \times \rightarrow \because$ highest degree is 2nd

$\therefore [9D'(D'-1)+3(3D')-36]y = 3x^2+4x+1$

From:- $e^z = 3x+2 \Rightarrow x = \frac{e^z-2}{3}$

$\therefore [9D'(D'-1)+3(3D')-36]y = 3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$

Q1. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Sol^n:- $(x^2D^2 - xD + 1)y = 0$

Let $x = e^z \Rightarrow z = \log x$

$\therefore$ We have :- $\cdot\ xD = D'$
$\cdot\ x^2D^2 = D'(D'-1)$ } Till 2nd degree

$\therefore \left[D'(D'-1) - D' + 1\right]y = 0$

$\Rightarrow \left[(D')^2 - D' - D' + 1\right]y = 0$

$\Rightarrow \left[(D')^2 - 2D' + 1\right]y = 0$

A.E:- $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)(m-1) = 0 \quad \therefore m = 1, 1$ (repeated)

CF:- $CF = (C_1 + C_2\cancel{x})e^{\cancel{x}} \Rightarrow CF = (C_1 + C_2 z)e^z$

→ we write in terms of z $\left(D' = \frac{d}{dz}\right)$

* Before, we had eq^n(s) w.r.t x $\therefore CF = (C_1 + C_2 x)e^x$
Here, we have eq^n(s) w.r.t z $\therefore CF = (C_1 + C_2 z)e^z$

But $z = \log x$ $\therefore CF = (C_1 + C_2\log x)e^{\log x}$
But $e^z = x$ $\therefore CF = (C_1 + C_2\log x)x$

P.I:- $P.I = 0$ $[\because RHS = 0]$

$\therefore y = CF + \cancelto{0}{PI}$

$\Rightarrow y = (C_1 + C_2 \log x)e^z$

Q. Solve $x^2y'' + 2xy' + 2y = 0$

Sol^n:- $(x^2D^2 + 2xD + 2)y = 0$

Let $x = e^z \Rightarrow z = \log x$

$\therefore$ We have :- • $xD = D'$
• $x^2D^2 = D'(D'-1)$ } Till 2^{nd} degree

$\therefore [D'(D'-1) + 2D' + 2]y = 0$
$\Rightarrow [(D')^2 - D' + 2D' + 2]y = 0$
$\Rightarrow [(D')^2 + D' + 2]y = 0$

A.E :- $m^2 + m + 2 = 0$
$\Rightarrow$ here, $a = 1, b = 1, c = 2$
$\therefore m = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm \sqrt{-7}}{2}$ $\therefore m = \frac{-1 \pm \sqrt{7}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{7}}{2}i$

$\therefore \alpha = -1/2$ & $\beta = \sqrt{7}/2$

C.F :- $CF = e^{\alpha z}(C_1 \cos\beta z + C_2 \sin\beta z)$
$= e^{-\frac{1}{2}z}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}z\right) + C_2 \sin \frac{\sqrt{7}z}{2}\right)$

$\Rightarrow CF = e^{-\frac{1}{2}\log x}\left(C_1 \cos \frac{\sqrt{7}}{2}\log x + C_2 \sin \frac{\sqrt{7}}{2}\log x\right)$
$= e^{\log x^{-1/2}}\left(C_1 \cos \frac{\sqrt{7}}{2}\log x + C_2 \sin \frac{\sqrt{7}}{2}\log x\right)$

Here, $\because e^{\log_e x^{-1/2}} \rightarrow e^{\log a} = a$ $\therefore e^{\log_e x^{-1/2}} = x^{-1/2} = \frac{1}{\sqrt{x}}$

$\therefore CF = \frac{1}{\sqrt{x}}\left(C_1 \cos \frac{\sqrt{7}}{2}\log x + C_2 \sin \frac{\sqrt{7}}{2}\log x\right)$

P.I :- $PI = 0$ ($\because$ RHS $= 0$)

$\therefore y = CF + \cancel{PI}^{\,0}$ $\Rightarrow y = \frac{1}{\sqrt{x}}\left(C_1 \cos \frac{\sqrt{7}}{2}\log x + C_2 \sin \frac{\sqrt{7}}{2}\log x\right)$