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Linear differential eq^n with constant coefficients [Only CF]

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + P_n y = X \quad \text{[Some function } \to X]$$
(1)

General sol^n (G S)

$$\text{General sol}^n \text{ (G S)} \to \text{C F (Complementary Function)},\ \text{P.I (Particular Integral)}$$

If in eq^n (1), $X = 0$ then :- G.S $\to$ C.F

Operators :-
1) $\frac{d}{dx} = D \quad \therefore \frac{dy}{dx} = Dy$

2) $\frac{d^2}{dx^2} = D^2 \quad \therefore \frac{d^2y}{dx^2} = D^2 y$

$\frac{d^n}{dx^n} = D^n \quad \therefore \frac{d^n y}{dx^n} = D^n y$

Auxillary equation :- Write the differential eq^n in terms of D (using Operators #)

Eg:- $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

$\Rightarrow D^3y - 6D^2y + 11Dy - 6y = 0$ [In D-form]

$\Rightarrow y(D^3 - 6D^2 + 11D - 6) = 0$

$\therefore D^3 - 6D^2 + 11D - 6 = 0$ [Solve this cubic eq^n]

$D = 1, 2, 3$ [Real & Distinct]

Composite Function Cases [For Auxiliary Eqⁿ roots] :-

1) Real & Distinct → $D = 1, 2, 3$

2) Real & Repeated → $D = 1, 2, 2$

3) Imaginary & Distinct → $D = \alpha \pm i\beta$ eg:- $D = 3 \pm 4i$

4) Imaginary & Repeated → eg:- $D = \underbrace{3 \pm 4i, 3 \pm 4i}_{\text{(repeated)}}, 2i$

Type 1 (Real & Distinct)

Q. $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

Suppose, $D = 1, 2, 3$ [$m_1, m_2, m_3 = 1, 2, 3$ respectively]

$\therefore \boxed{y = c_1 e^{1x} + c_2 e^{2x} + c_3 e^{3x}}$ → CF [if $y = 0$]

Type 2 (Real & Repeated)

Case I:- $D = 2, 2$ then :- $y = (c_1 + c_2 x) e^{2x}$ ($m = 2$)

Write 'x' after the 1st c of repeated roots

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case II:- $D = 1, 2, \underline{3, 3}$ then :- $y = \underbrace{c_1 e^{1x} + c_2 e^{2x}}_{\text{non repeated}} + \underbrace{(c_3 + c_4 x)}_{\text{repeated}} e^{3x}$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 x e^{m_3 x} + c_4 e^{m_4 x}$

Case III:- $D = 1, \underline{3, 3, 3}$ then :- $y = c_1 e^{1x} + (c_2 + c_3 x + c_4 x^2) e^{3x}$

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Type 3 (Imaginary & Distinct)

→ 2 roots

Suppose, $D = 2 \pm 5i$ Here, $\alpha = 2$ & $\beta = 5$

$$\therefore \boxed{y = [c_1 \cos(\beta x) + c_2 \sin(\beta x)]\, e^{\alpha x}}$$

$$\Rightarrow y = e^{2x}[c_1 \cos 5x + c_2 \sin 5x]$$

Type 4 (Imaginary & Repeated)

→ 4 roots

Suppose, $D = 5 \pm 6i, 5 \pm 6i$ Here, $\alpha = 5$ & $\beta = 6$

$$\therefore y = [(c_1 + c_2 x)\cos(\beta x) + (c_3 + c_4 x)\sin(\beta x)] \cdot e^{\alpha x}$$

$$\Rightarrow y = e^{5x}[(c_1 + c_2 x)\cos(\beta x) + (c_3 + c_4 x)\sin(\beta x)]$$

Q1. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ $\cancel{\Rightarrow D^3}$

Solⁿ:- Auxillary eqⁿ:- $(D^3 - 6D^2 + 11D - 6)y = 0$

$\Rightarrow D^3 - 6D^2 + 11D - 6 = 0$ $\therefore D = 1, 2, 3$

$\therefore$ It is of Type 1

$$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$$
$$= c_1 e^{1x} + c_2 e^{2x} + c_3 e^{3x}$$

Q2. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$

Solⁿ:- Auxillary eqⁿ:- $(D^3 - 5D^2 + 8D - 4)y = 0$

$\Rightarrow D^3 - 5D^2 + 8D - 4 = 0$ $\therefore D = 1, 2, 2$

$\therefore$ It is of Type 2

$$\therefore y = c_1 e^{m_1 x} + (c_2 + c_3 x)e^{m_2 x}$$
$$= c_1 e^{x} + (c_2 + c_3 x)e^{2x}$$

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Q3. $\cancel{(D^2+2D+5)y=0}$ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Sol:- Auxillary Function :- $(D^2+2D+5)y = 0$

$\Rightarrow D^2+2D+5 = 0 \quad \therefore D = -1 \pm 2i$

Here, $\alpha = -1$, $\beta = 2$

It's of Type 3

$\therefore y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$= e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$

Q4. $\frac{d^3y}{dx^3} + y = 0$

Sol:- Auxillary Function :- $(D^3+1)y = 0$

$\Rightarrow D^3+1 = 0 \quad \therefore D = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

(Degree 3 → calculate in calculator)

It's of Type 3

$(\alpha = 1/2, \beta = \sqrt{3}/2)$

$\therefore y = c_1 e^{\alpha x} + e^{\alpha x}[c_2 \cos\beta x + c_3 \sin\beta x]$

$= c_1 e^{-x} + e^{1/2 x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

Q5. $D = 1 \pm \sqrt{3}i, 1 \pm \sqrt{3}i$

Sol:- Here, $\alpha = 1$, $\beta = \sqrt{3}$

It's of Type 4

$\therefore y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

$= e^{x}[(c_1 + c_2 x)\cos\sqrt{3}x + (c_3 + c_4 x)\sin\sqrt{3}x]$