Cauchy's Integral Test :-

If for $x \geq 1$, $f(x)$ be a positive monotonic integrable function such that :-

- $f(n) = u_n$

For +ve integer value of n, $\sum_{n=1}^{\infty} u_n$: convergent

- $\int_1^{\infty} f(x)\,dx$

If value is finite, it is convergent
If value is infinite, it is divergent

Q1. Test Convergence of series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$

Soln:- Let $u_n = \frac{1}{n^2} = f(n)$

$\therefore f(x) = \frac{1}{x^2}$

$\therefore \int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx$

$= \int_1^{\infty} x^{-2}\,dx \qquad \left[\int x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right] \right]$

$= \left[\frac{x^{-2+1}}{-2+1}\right]_1^{\infty} = -\left[\frac{1}{x}\right]_1^{\infty}$

$= -\left(\frac{1}{\infty} - \frac{1}{1}\right)$

$\Rightarrow \int_1^{\infty} f(x)\,dx = 1$ (finite)

$\therefore$ By Cauchy's Integral Test, $\sum u_n$ is convergent

Q2. Test Convergence of the series $\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\ldots$

Sol^n:- Let $u_n = \frac{1}{(n+1)(n+2)} = f(n)$

$$\therefore f(x) = \frac{1}{(x+1)(x+2)}$$

$$= \frac{1}{x+1} - \frac{1}{x+2}$$

$$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x+1}\,dx - \int_1^\infty \frac{1}{x+2}\,dx$$

$$= \left[\log(x+1)\right]_1^\infty - \left[\log(x+2)\right]_1^\infty$$

$$= \left[\log(x+1) - \log(x+2)\right]_1^\infty$$

$$= \left[\log\frac{(x+1)}{(x+2)}\right]_1^\infty$$

Take x common on both numerator & denominator

$$\therefore \int_1^\infty f(x)\,dx = \left[\log\frac{\cancel{x}(1+\frac{1}{x})}{\cancel{x}(1+\frac{2}{x})}\right]_1^\infty$$

$$= \log\frac{(1+\frac{1}{\infty})}{(1+\frac{2}{\infty})} - \log\frac{(1+\frac{1}{1})}{(1+\frac{2}{1})}$$

$$= \log\frac{(1+0)}{(1+0)} - \log\frac{(1+1)}{(1+2)}$$

$$= \log 1 - \log\frac{2}{3}$$

$$= \log\left(\frac{1}{2/3}\right)$$

$$\Rightarrow \int_1^\infty f(x)\,dx = \log\left(\frac{3}{2}\right) \text{ (finite)}$$

By Cauchy's Integral Test, Σu_n is convergent

Q3. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$

& divergent if $0<p<1$

Sol:- Let $u_n = \frac{1}{n^p} = f(x)$

$\therefore f(x) = \frac{1}{x^p}$

$\therefore \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx \implies$

$p=1$:- $[\log x]_1^\infty \longrightarrow \int_1^\infty \frac{1}{x}\,dx$

$p>1$:- $\left(\frac{x^{-p+1}}{-p+1}\right)_1^\infty \longrightarrow \int_1^\infty x^{-p}\,dx$

$p<1$:- $\left[\frac{x^{1-p}}{1-p}\right]_1^\infty \longrightarrow \int_1^\infty x^{p}\,dx$

$\therefore \int_1^\infty f(x)\,dx =$

$p=1$:- $\log(\infty) - \log(1) = \infty$ (infinite)

$p>1$ (let's say 2) :- $\frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{-1}{x} = \frac{-1}{\infty} = 0$ (finite)

$p<1$ (let's say 0) :- $\frac{x^{1-0}}{1-0} = x = \infty$ (infinite)

$\therefore$ By Cauchy's Integral test,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$:- • convergent at $p>1$

• divergent at $p\leq 1$ or $0<p<1$