(*) Q1. At any point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Prove that :- (i) $\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$

(ii) $\rho = \frac{a^2b^2}{p^3}$ where p is the length of perpendicular drawn from center on the tangent.

(iii) $\rho = \frac{(CD)^3}{ab}$ where CP & CD are pair of conjugate diameter

(iv) $(ab)^{2/3}(\rho_1^{2/3}+\rho_2^{2/3}) = a^2+b^2$ where ρ_1 & ρ_2 are radii of curvature at the end of extremities of conjugate diameters

Sol:- Parametric coordinate of ellipse :-

(i) $x = a\cos t$ & $y = b\sin t$ } → Satisfies the ellipse eqn

$\therefore x' = -a\sin t$ | $y' = b\cos t$

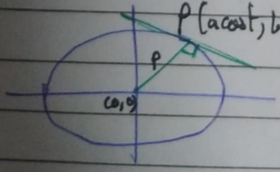
$x'' = -a\cos t$ | $y'' = -b\sin t$

$\therefore$ Radius of curvature $(\rho) = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$

$$\rho = \frac{[(-a\sin t)^2 + (b\cos t)^2]^{3/2}}{(-a\sin t)(-b\sin t)-(b\cos t)(-a\cos t)}$$

$$= \frac{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}{ab[\sin^2 t+\cos^2 t]} = \cancel{\frac{(a^2)^{3/2}}{ab}}\ \frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$$

$$\Rightarrow \cancel{\rho} = \frac{a^3}{ab} = \frac{a^2}{b} \quad \therefore \boxed{\rho = \frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}}$$

(ii) 

Eqn of Tangent :- $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$

$\Rightarrow \frac{x(a\cos t)}{a^2} + \frac{y(b\sin t)}{b^2} = 1$

where, x_1 → Coordinate x of parametric form of curve

y_1 → " y " " " "

∴ Eqⁿ of Tangent :- $\frac{x(a\cos t)}{a^2} + \frac{y(\cancel{b}\sin t)}{b^{\cancel{2}}} = 1$

$$\Rightarrow \frac{x\cos t}{a} + \frac{y\sin t}{b} = 1$$

Here, $T_{(0,0)}$ ∴ Distance formula b/w two line eqⁿ :-

-1

$$p = \frac{|ax+by-RHS|}{(\text{Eq}^n \text{ of Tangent})^{1/2}} \rightarrow (\text{coeff of } x)^2 + (\text{coeff of } y)^2$$

$(a\cos t, b\sin t)$

$$\therefore p = \frac{|0+0-1|}{\sqrt{\left(\frac{\cos t}{a}\right)^2 + \left(\frac{\sin t}{b}\right)^2}} = \frac{1}{\sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{b^2}}}$$

$$\Rightarrow p = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{1/2}} \qquad \therefore \frac{1}{p} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}{ab}$$

To prove :- $\rho = \frac{a^2b^2}{p^3} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$

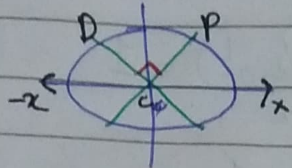
Proof :- $LHS = \frac{a^2b^2}{p^3}$

$$= a^2b^2 \cdot \left[\frac{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}{ab}\right]^3$$

$$= \frac{a^2b^2\,(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{a^3b^3}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} = RHS$$

Hence proved

(iii)  Conjugate diameters form $90°$ to each other

$\therefore P(a\cos t, b\sin t)$ $C(0,0)$
$D(-a\cos t, b\sin t)$

$\therefore CD = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{(-a\cos t-0)^2+(b\sin t-0)^2}$

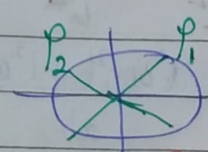
$\Rightarrow CD = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$

To prove:- $\rho = \dfrac{(CD)^3}{ab} = \dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$

Proof:- LHS $= \dfrac{(CD)^3}{ab}$

$= \dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} = RHS$

Hence proved

(iv)  $\rho_2 = \dfrac{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}{ab}$ & $\rho_1 = \dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}$

sign change

To prove:- $(ab)^{2/3}(\rho_1^{2/3}+\rho_2^{2/3}) = a^2+b^2$

Proof:- LHS $= (ab)^{2/3}(\rho_1^{2/3}+\rho_2^{2/3})$

$= (ab)^{2/3}\left[\left\{\dfrac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}\right\}^{2/3} + \left\{\dfrac{(a^2\cos^2 t + b^2\sin^2 t)^{3/2}}{ab}\right\}^{2/3}\right]$

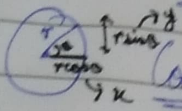
$= (ab)^{2/3} \cdot \dfrac{a^2(\sin^2 t + \cos^2 t) + b^2(\sin^2 t + \cos^2 t)}{(ab)^{2/3}}$

$= a^2 + b^2 = RHS$

Hence proved

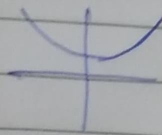
Parametric Coordinates :-

1) **Circle :-** **Eqn :-** $(x-h)^2 + (y-k)^2 = r^2$ where $(h,k) \rightarrow$ centre of circle



Coordinates :- $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} (r\cos\theta, r\sin\theta)$

2) **Parabola :-** 1) **Eqn 1 :-** $y^2 = 4ax$ 3) $y^2 = -4ax$

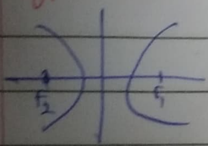


Coordinates :- $\left.\begin{array}{l} x = at^2 \\ y = 2at \end{array}\right\} (at^2, 2at)$ $\left.\begin{array}{l} x = -at^2 \\ y = 2at \end{array}\right\}$

2) **Eqn 2 :-** $x^2 = 4ay$ 4) $x^2 = -4ay$

Coordinate :- $\left.\begin{array}{l} x = 2at \\ y = at^2 \end{array}\right\} (2at, at^2)$ $\left.\begin{array}{l} x = 2at \\ y = -at^2 \end{array}\right\}$

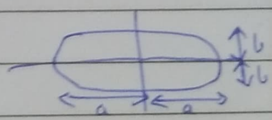
3) **Hyperbola :-** 1) **Eqn 1 :-** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ 2) **Eqn 2 :-** $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



Coordinates :- $\left.\begin{array}{l} x = a\sec\theta \\ y = b\tan\theta \end{array}\right\}$ **Coordinates :-** $\left.\begin{array}{l} x = a\tan\theta \\ y = b\sec\theta \end{array}\right\}$

4) **Ellipse :-** 1) **Eqn 1 :-** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Major axis $\rightarrow$ x-axis)

Coordinates :- $\left.\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\}$



2) **Eqn 2 :-** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Major axis $\rightarrow$ y-axis)

Coordinates :- $\left.\begin{array}{l} x = b\cos\theta \\ y = a\sin\theta \end{array}\right\}$