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Date
Page

Homogeneous Differential Equation With Variable Coefficients

~~Euler Type~~:-

Variable Coefficient $\left\{ x^n \dfrac{d^n y}{dx^n} + a_1 x^{n-1} \dfrac{d^{n-1}}{dx^{n-1}} + a_2 x^{n-2} \dfrac{d^{n-2} y}{dx^{n-2}} + \dots\dots + a_n y = X \right.$ $\quad x^{n-n} = x^0 = (1)$

$\Downarrow$ conversion to

Constant Coefficient $\left\{ \dfrac{d^n y}{dz^n} + k_1 \dfrac{d^{n-1} y}{dx^{n-1}} + k_2 \dfrac{d^{n-2} y}{dx^{n-2}} + \dots\dots + k_n y = X \right.$

Euler Type:-

1) Let $\boxed{x = e^z}$ $\xrightarrow[\text{both sides}]{\text{Take log}}$ $\log x = z \cancel{\log_e e}^{1} \Rightarrow \boxed{\log x = z}$

2) $\dfrac{dz}{dx} = \dfrac{1}{x}$ [Differentiate $\log x = z$ w.r.t x]

3) $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$ ——— ①

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{1}{x}$ ——— ①

$\Rightarrow x \dfrac{dy}{dx} = \dfrac{dy}{dz}$

4) Let $\dfrac{d}{dx} = D$ & $\dfrac{d}{dz} = D'$ $\quad \therefore \boxed{x D y = D' y}$

5) Differentiate ① w.r.t x

$$\frac{d^2 y}{dx^2} = \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x}\underbrace{\frac{d}{dx}\left(\frac{dy}{dz}\right)}_{\text{write w.r.t } z}$$

$$\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

Chain rule

$\because$ $\dfrac{dy}{dz}$ $\downarrow$ $z = \log(x)$ $\downarrow$ x $\quad \dfrac{d}{dz}\left(\dfrac{dy}{dz}\right)$ $\quad \dfrac{dz}{dx}$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\underbrace{\frac{d}{dz}\left(\frac{dy}{dz}\right)}\cdot\underbrace{\frac{dz}{dx}} \quad \text{--- (2)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\cdot\left(\frac{1}{x}\right) \quad \left[\because \frac{dz}{dx} = \frac{1}{x}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}\left[-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \cancel{\frac{dy}{dz}} - \frac{dy}{dz} + \frac{d^2y}{dz^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\therefore \boxed{x^2D^2y = (D'^2 - D')y} \quad \text{or} \quad \boxed{x^2D^2y = D'(D'-1)y}$$

6) Differentiate (2) w.r.t x

$$\cancel{\frac{d^3y}{dx^3}}$$

$$\therefore \boxed{x^3D^3y = D'(D'-1)(D'-2)y} \quad \ldots\ldots \text{ till } n$$

$$\boxed{x^nD^ny = D'(D'-1)(D'-2)\ldots(D'-n+1)}$$

Eg:- $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$

Here,

Each term $x^n\frac{d^ny}{dx^n}$ has x^n → variable coefficient

→ $x^3D^3y \quad \left[D = \frac{d}{dx}\right]$

By Euler-Cauchy:-

$$x^nD^ny = \underbrace{D'(D'-1)(D'-2)\ldots\ldots(D'-n+1)}_{\text{constant coefficient}}$$

By converting x-dependent terms to expressions in D' with constants, Hence, the differential equation now has constant coefficients in terms of (z or D')