Total derivative of Composite Implicit Functions

Q1. Find $\frac{dy}{dx}$ when $x^3+y^3=3axy$

Solⁿ:- Here, $f(x,y)=x^3+y^3-3axy$

$$\therefore \frac{dy}{dx}=\frac{-\partial f/\partial x}{\partial f/\partial y} \quad \left[\begin{array}{l} \frac{\partial f}{\partial x}=3x^2+0-3ay=3x^2-3ay \\ \frac{\partial f}{\partial y}=\cancel{0}+3y^2-3ax=3y^2-3ax \end{array}\right]$$

$$=\frac{-\cancel{3}(x^2-ay)}{\cancel{3}(y^2-ax)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x^2-ay)}{(y^2-ax)}$$

Q2. If $u=x\log(xy)$ where $x^3+y^3+3xy=1$ find $\frac{du}{dx}$ & $\frac{du}{dy}$

Solⁿ:- Here,

$u \rightarrow$ function of (x,y)

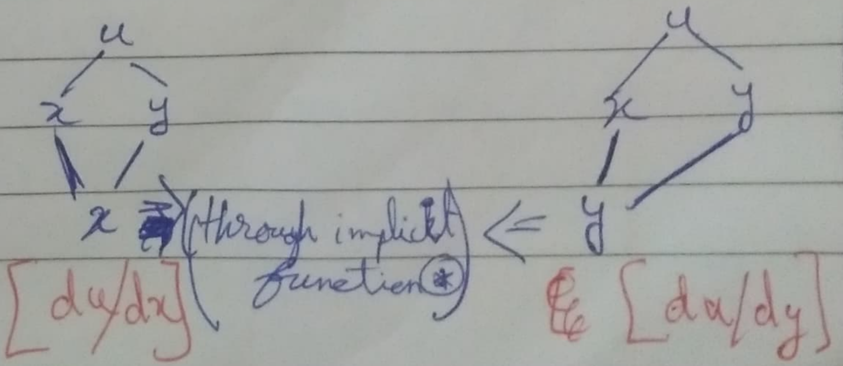
Also, we have $\underbrace{x^3+y^3+3xy=1}_{}$ $\rightarrow f(x,y)=x^3+y^3+3xy-1$ — (*)

here, we assume y is a function of x

x is a function of x

$\therefore$ x, y are depended on each other

Multi-chain rule tree :-



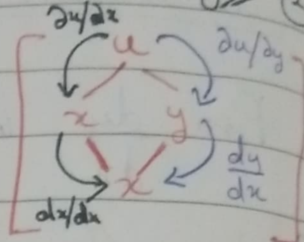
du/dx:- 1^{st} Let's find $\frac{dy}{dx}$ $\left[u = x\log(xy) = x[\log x + \log y] \right]$

- $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x \cdot \log(xy)\right) = x \frac{\partial}{\partial x}\left[\log x + \log y\right] + \left(\log x + \log y\right) \cdot \frac{\partial x}{\partial x}$

$\Rightarrow \frac{\partial u}{\partial x} = x \cdot \frac{1}{x} + (\log x + \log y) = 1 + \log x + \log y$ — ①

- $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x \log(xy)\right) = x \cdot \frac{\partial}{\partial y}\left[\log x + \log y\right] = x \cdot \frac{1}{y} = \frac{x}{y}$ — ②

$\therefore \cancel{\frac{dy}{dx} = \partial u} \quad \frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$



2^{nd} Let's find $\frac{dy}{dx}$ $\left[f(x,y) = x^3 + y^3 + 3xy - 1 \right]$

- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^3 + 3xy - 1\right) = 3x^2 + 0 + 3y = 3(x^2 + y)$
- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^3 + 3xy - 1\right) = 0 + 3y^2 + 3x = 3(x + y^2)$

$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{\cancel{3}(x^2+y)}{\cancel{3}(x+y^2)} = -\frac{(x^2+y)}{(x+y^2)}$ — ③

Using ①, ②, ③ in $\frac{du}{dx}$ we get:-

$\frac{du}{dx} = (1 + \log x + \log y)(1) + \frac{x}{y}\left[\frac{-(x^2+y)}{(x+y^2)}\right]$

$\Rightarrow \frac{du}{dx} = 1 + \log x + \log y - \frac{x}{y}\frac{(x^2+y)}{(x+y^2)}$

classmate

Date

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du/dy:- 1^{st} Let's find $\frac{du}{dx}$ $\left[u = x\left[\log x + \log y\right]\right]$

- $\frac{\partial u}{\partial x} = 1 + \log x + \log y$ — ①

- $\frac{\partial u}{\partial y} = \frac{x}{y}$ — ②

$\therefore \frac{du}{dy} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dy}$

2^{nd} Let's find $\frac{dx}{dy}$ $\left[f(x,y) = x^3 + y^3 + 3xy - 1\right]$

- $\frac{\partial f}{\partial x} = 3(x^2 + y)$

- $\frac{\partial f}{\partial y} = 3(x + y^2)$

$\frac{dx}{dy}$ = Reciprocal of $\frac{dy}{dx}$

$\therefore \frac{dx}{dy} = \frac{-\partial f/\partial y}{\partial f/\partial x} = \frac{-3(x+y^2)}{3(x^2+y)} = \frac{-(x+y^2)}{(x^2+y)}$ — ③

Using ①, ②, ③ in $\frac{du}{dy}$ we get:-

$$\frac{du}{dy} = \left(1 + \log x + \log y\right) \cdot \left[\frac{-(x+y^2)}{(x^2+y)}\right] + \frac{x}{y}(1)$$

$$\Rightarrow \frac{du}{dy} = \frac{-(x+y^2)}{(x^2+y)}\left[1 + \log x + \log y\right] + \frac{x}{y}$$

Q3. Find $\frac{dy}{dx}$ when $y \sin x = x \cos y$

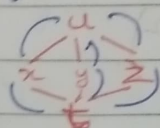
Sol:- Here, $f(x,y) = y\sin x - x\cos y$

- $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(y\sin x - x\cos y\right) = y\cos x - \cos y$
- $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y\sin x - x\cos y\right) = \sin x - x(-\sin y) = \sin x + x\sin y$ ~~$= \sin x$~~

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial x} = \frac{-(y\cos x - \cos y)}{(\sin x + x\sin y)} = \frac{\cos y - y\cos x}{\sin x + x\sin y}$$

Q4. Find $\frac{du}{dt}$ where $u = x^2+y^2+z^2$ $\left\{\begin{array}{l} x = e^t \\ y = e^t \sin t \\ z = e^t \cos t \end{array}\right\}$

Sol:- In terms of multi-chain rule:-



Since we already have x, y, z in terms of t we do not need to define $f(x,y,z)$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

For $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$:-

- $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) = 2x + 0 + 0 = 2x$ — (1)
- $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) = 0 + 2y + 0 = 2y$ — (2)
- $\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) = 0 + 0 + 2z = 2z$ — (3)

For dx/dt, dy/dt, dz/dt :-

- $x = e^t \Rightarrow \frac{dx}{dt} = \frac{d}{dt}(e^t) = e^t \cdot \frac{dt}{dt} = e^t$ — (4)

- $y = e^t \sin t \Rightarrow \frac{dy}{dt} = \frac{d}{dt}(e^t \sin t) = e^t \frac{d}{dt}(\sin t) + \sin t \frac{d}{dt}(e^t) = \cancel{e^t \cos t + e^t}$

$\Rightarrow \frac{dy}{dt} = e^t \cos t + e^t \sin t = e^t(\sin t + \cos t)$ — (5)

- $z = e^t \cos t \Rightarrow \frac{dz}{dt} = \frac{d}{dt}(e^t \cos t) = e^t \frac{d}{dt}(\cos t) + \cos t \frac{d}{dt}(e^t)$

$\Rightarrow \frac{dz}{dt} = e^t(-\sin t) + e^t \cos t = e^t(\cos t - \sin t)$ — (6)

Using (1), (2), (3), (4), (5), (6) in $\frac{du}{dt}$ we get :-

$\therefore \frac{du}{dt} = (2x)\cdot(e^t) + (2y)\left[e^t(\sin t + \cos t)\right] + (2z)\left[e^t(\cos t - \sin t)\right]$

But we know :- $\left\{ \begin{array}{l} x = e^t \\ y = e^t \sin t \\ z = e^t \cos t \end{array} \right\}$

$\therefore \frac{du}{dt} = 2(e^t)(e^t) + 2(e^t \sin t)\left[e^t(\sin t + \cos t)\right]$

$+ 2(e^t \cos t)\left[e^t(\cos t - \sin t)\right]$

$= 2e^{2t}\left[1 + \underbrace{\sin^2 t + \cancel{\sin t \cos t} + \cos^2 t}_{\sin^2\theta + \cos^2\theta = 1} - \cancel{\sin t \cos t}\right]$

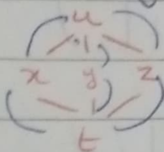
$= 2e^{2t}[1+1]$

$= 2e^{2t}(2)$

$\Rightarrow \frac{du}{dt} = 4e^{2t}$

Q5. Find $\frac{du}{dt}$ where $u = xy + yz + zx$ $\left\{\begin{array}{l} x = 1/t \\ y = e^{t} \\ z = e^{-t} \end{array}\right\}$

Sol^n:- In terms of multi-chain rule:-



$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

$[u = xy + yz + zx]$:-

- $\frac{\partial u}{\partial x} = y + z$
- $\frac{\partial u}{\partial y} = x + z$
- $\frac{\partial u}{\partial z} = y + x$

For $x = 1/t$:-

$$\frac{dx}{dt} = \frac{d}{dt}\left(t^{-1}\right) = (-1)\cdot\left(t^{-2}\right)$$

$$\Rightarrow \frac{dx}{dt} = -\frac{1}{t^2}$$

For $y = e^{t}$:-

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^{t}\right) = e^{t}\cdot\frac{dt}{dt}$$

$$\Rightarrow \frac{dy}{dt} = e^{t}$$

For $z = e^{-t}$:-

$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{-t}\right) = e^{-t}\cdot\frac{d(-t)}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -e^{-t}$$

$$\therefore \frac{du}{dt} = (y+z)\left(-\frac{1}{t^2}\right) + (x+z)\left(e^{t}\right) + (y+x)\left(-e^{-t}\right)$$

But we know:- $\left\{\begin{array}{l} x = 1/t \\ y = e^{t} \\ z = e^{-t} \end{array}\right\}$

$$\therefore \frac{du}{dt} = \left(e^{t} + e^{-t}\right)\left(-\frac{1}{t^2}\right) + \left(\frac{1}{t} + e^{-t}\right)\left(e^{t}\right) + \left(e^{t} + \frac{1}{t}\right)\left(-e^{-t}\right)$$

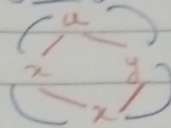
$$= -\frac{e^{t}}{t^2} - \frac{e^{-t}}{t^2} + \frac{e^{t}}{t} + \underbrace{\left(e^{-t}\right)\left(e^{t}\right)}_{+1} - \underbrace{\left(e^{t}\right)\left(e^{-t}\right)}_{-1} - \frac{e^{-t}}{t}$$

$$= -\frac{1}{t^2}\left(e^{t} + e^{-t}\right) + \frac{1}{t}\left(e^{t} - e^{-t}\right) + \cancel{1} - \cancel{1}$$

$$\Rightarrow \frac{du}{dt} = -\frac{1}{t^2}\left(e^{t} + e^{-t}\right) + \frac{1}{t}\left(e^{t} - e^{-t}\right)$$

Q6. Find $\frac{du}{dx}$ where $u = \sin(x^2+y^2)$ & $a^2x^2 + b^2y^2 = c^2$ $[a, b, c \to \text{constants}]$

Sol^n:- From $a^2x^2+b^2y^2=c^2$, x is dependent on y & y is dependent on x

For $\frac{du}{dx}$, multi chain dependency :- 

(tree)

$$\therefore \frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \to (\#)$$

- $u = \sin(x^2+y^2)$:-

- $$\frac{\partial u}{\partial x} = \cos(x^2+y^2)\,\frac{\partial}{\partial x}(x^2+y^2) = 2x\cos(x^2+y^2)$$

- $$\frac{\partial u}{\partial y} = \cos(x^2+y^2)\,\frac{\partial}{\partial y}(x^2+y^2) = 2y\cos(x^2+y^2)$$

(#) For $\frac{dy}{dx}$:- Let $f(x,y) = a^2x^2+b^2y^2-c^2$

- $$\frac{\partial f}{\partial x} = 2a^2x + 0 - 0 = 2a^2x$$
- $$\frac{\partial f}{\partial y} = 0 + 2b^2y - 0 = 2b^2y$$

$$\therefore \frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{\cancel{2}a^2x}{\cancel{2}b^2y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a^2x}{b^2y}$$

$$\therefore \frac{du}{dx} = \left[2x\cos(x^2+y^2)\right](1) + \left[2y\cos(x^2+y^2)\right]\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2\cos(x^2+y^2)\left[x + \cancel{y}\left(-\frac{a^2x}{b^2\cancel{y}}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = 2\cos(x^2+y^2)\cdot\frac{(b^2x - a^2x)}{b^2}$$

$$= 2\cos(x^2+y^2)\cdot\frac{x(b^2-a^2)}{b^2}$$

$$\Rightarrow \frac{du}{dx} = 2x\cos(x^2+y^2)\cdot\left(\frac{b^2-a^2}{b^2}\right)$$