Q1. Solve : $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

Solⁿ:- Let $D = \frac{d}{dx}$ & $D' = \frac{d}{dz}$

$\left[(x+2)^2 D^2 - (x+2)D + 1\right] y = 3x+4$

Let $x+2 = e^z$ —— (1) $\Rightarrow z = \log(x+2)$

(here, $a = 1$ & $b = 2$)

∴ We have :-
- $(x+2)D = D'$
- $(x+2)^2 D^2 = D'(D'-1)$

From (1) $\Rightarrow x = e^z - 2$

$\therefore \left[D'(D'-1) - D' + 1\right] y = 3(e^z - 2) + 4$

$\Rightarrow (D'^2 - D' - D' + 1) y = 3e^z - 6 + 4$

$\Rightarrow (D'^2 - 2D' + 1) y = 3e^z - 2$

$\underline{A.E}$:- $m^2 - 2m + 1 = 0$
$\Rightarrow (m-1)^2 = 0 \quad \therefore m = 1, 1$

$\underline{CF}$:- $CF = (C_1 + C_2 z) e^{1z}$
But $z = \log(x+2)$ & $e^z = x+2$

$\Rightarrow CF = (C_1 + C_2 \log(x+2)) \cdot (x+2)$

$\underline{P.I}$:- $PI = \dfrac{1}{D'^2 - 2D' + 1} \cdot (3e^z - 2)$ (Type 1)

$= \dfrac{3e^z}{D'^2 - 2D' + 1} + \dfrac{(-2e^{0z})}{D'^2 - 2D' + 1}$

- $PI_1 = \dfrac{3e^z}{D'^2 - 2D' + 1}$ (Type 1, $D' = 1$)

$D' = 1 \therefore D'^2 - 2D' + 1 = 1^2 - 2(1) + 1 = 0$
$\therefore$ We differentiate denominator w.r.t D' & multiply numerator by z

$PI_1 = \dfrac{3e^z}{2D' - 2} \cdot z$ At $D' = 1$, $2D' - 2 = 2(1) - 2 = 0$

$= \dfrac{3e^z \cdot z}{2(1) - 2}$ Again we diff w.r.t D' & multiply numerator by z

$\Rightarrow PI_1 = \dfrac{3e^z z}{2} \cdot z = \dfrac{3z^2 e^z}{2}$

- $PI_2 = \dfrac{-2e^{0z}}{D'^2 - 2D' + 1}$ (Type 1, $D' = 0$)

$= \dfrac{-2(1)}{0^2 - 2(0) + 1}$

$\Rightarrow PI_2 = -2$

$z = \log(x+2)$ & $x = e^z - 2$
$\Rightarrow e^z = x+2$

$\therefore PI = PI_1 + PI_2 = \dfrac{3(\log(x+2))^2 (x+2)}{2} - 2$

$\therefore y = CF + P.I$
$= (C_1 + C_2 \log(x+2))(x+2) + \dfrac{3(\log(x+2))^2 (x+2)}{2} - 2$

Q2. Solve:- $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\dfrac{dy}{dx}-36y=3x^2+4x+1$

Sol^n:- Let $D=\dfrac{d}{dx}$ & $D'=\dfrac{d}{dz}$

$\therefore\ [(3x+2)^2D^2+3(3x+2)D-36]\,y=3x^2+4x+1$

Let $3x+2=e^z$ —— (1) $\Rightarrow z=\log(3x+2)$

$(a=3,\ b=2)$

From (1) $\Rightarrow x=\dfrac{e^z-2}{3}$

We have:- $\cdot\ (3x+2)D=3D'$

$\cdot\ (3x+2)^2D^2=3^2D'(D'-1)=9D'(D'-1)$

$\therefore\ [9D'(D'-1)+3(3D')-36]\,y=3\left(\dfrac{e^z-2}{3}\right)^2+4\left(\dfrac{e^z-2}{3}\right)+1$

$\Rightarrow (9D'^2-\cancel{9D'}+\cancel{9D'}-36)\,y=\dfrac{\dfrac{(e^z-2)^2}{\cancel{3}}+4(e^z-2)+3}{3}$

$\Rightarrow 9(D'^2-4)\,y=\dfrac{e^{2z}+4-\cancel{4e^z}+\cancel{4e^z}-8+3}{3}$

$\Rightarrow (D'^2-4)\,y=\dfrac{e^{2z}-1}{3\times 9}$

$\Rightarrow (D'^2-4)\,y=\dfrac{e^{2z}-1}{27}$

A.E:- $m^2-4=0$

$\Rightarrow m^2=4$

$\Rightarrow m=\pm\sqrt{4}=\pm 2 \quad \therefore m=2,-2$

CF:- $CF=C_1e^{2z}+C_2e^{-2z}$

$=C_1(e^z)^2+C_2\times\dfrac{1}{(e^z)^2}$

$\Rightarrow CF=C_1(3x+2)^2+\dfrac{C_2}{(3x+2)^2}$

P.I:- $PI = \dfrac{\dfrac{e^{2z}-1}{27}}{D'^2-4}$

$$= \frac{e^{2z}}{27(D'^2-4)} + \frac{-e^{0z}}{27(D'^2-4)} \qquad (\text{Type I})$$

- $PI_1 = \dfrac{e^{2z}}{27(D'^2-4)} \qquad (D'=2)$

$D'=2$, $\therefore D'^2-4 = (2)^2-4 = 0$

We differentiate denominator by D' and multiply numerator by z

$$PI_1 = \frac{e^{2z}\cdot z}{27(2D')} \qquad (D'=2)$$

$$= \frac{e^{2z}\cdot z}{27(2\times 2)} = \frac{(e^{z})^2\cdot z}{108}$$

But $e^z = 3x+2$ & $z = \log(3x+2)$

$$PI_1 = \frac{(3x+2)^2(\log(3x+2))}{108}$$

- $PI_2 = \dfrac{-e^{0z}}{27(D'^2-4)} \qquad (D'=0)$

$$= \frac{-1}{27(0-4)} \qquad \cancel{\therefore PI =}$$

$$PI_2 = \frac{1}{108}$$

$\therefore PI = PI_1 + PI_2$

$$= \frac{(3x+2)^2(\log(3x+2))}{108} + \frac{1}{108}$$

$y = CF + PI$

$$= C_1(3x+2)^2 + \frac{C_2}{(3x+2)^2} + \frac{(3x+2)^2(\log(3x+2))}{108} + \frac{1}{108}$$

Q3. Solve: $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Soln:- Let $D = \frac{d}{dx}$ & $D' = \frac{d}{dz}$

$[(2x+3)^2 D^2 - (2x+3) D - 12]y = 6x$

$\rightarrow x = \frac{e^z - 3}{2}$

Let $2x+3 = e^z \Rightarrow z = \log(2x+3)$

$(a=2, b=3)$

We have:- • $(2x+3)D = 2D'$

• $(2x+3)^2 D^2 = 4D'(D'-1)$

$\therefore [4D'(D'-1) - 2D' - 12]y = 6x$

$\Rightarrow [4D'^2 - 4D' - 2D' - 12]y = \cancel{6}^{3}\left(\frac{e^z - 3}{\cancel{2}}\right)$

$\Rightarrow (4D'^2 - 6D' - 12)y = 3(e^z - 3)$

$\Rightarrow 2(2D'^2 - 3D' - 6)y = 3(e^z - 3)$

$\Rightarrow (2D'^2 - 3D' - 6)y = \frac{3}{2}(e^z - 3)$

A.E:- $2m^2 - 3m - 6 = 0$

Here, $a = 2, b = -3, c = -6$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-6)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+48}}{4}$$

$\Rightarrow m = \frac{3 \pm \sqrt{57}}{4}$ $\therefore m = \frac{3}{4} + \frac{\sqrt{57}}{4}, \frac{3}{4} - \frac{\sqrt{57}}{4}$

CF:- $CF = e^{\alpha z}(C_1 \cos\beta z + C_2 \sin\beta z)$ $\left(\alpha = \frac{3}{4}, \beta = \frac{\sqrt{57}}{4}\right)$

$= e^{3/4 z}\left(C_1 \cos\frac{\sqrt{57}}{4}z + C_2 \sin\frac{\sqrt{57}}{4}z\right)$ $\left[\begin{array}{l} z = \log(2x+3) \\ \& \ e^z = 2x+3 \end{array}\right]$

$\Rightarrow CF = (2x+3)^{3/4}\left(C_1 \cos\frac{\sqrt{57}}{4}\log(2x+3) + C_2 \sin\frac{\sqrt{57}}{4}\log(2x+3)\right)$

P-I :- ~~e^{2x}~~

$$PI = \frac{\frac{3}{2}(e^{z}-3)}{2D'^2-3D'-6}$$

$$= \frac{3}{2}\left[\frac{e^{z}}{2D'^2-3D'-6} - \frac{3e^{0z}}{2D'^2-3D'-6}\right]$$

$(D'=1)$ $(D'=0)$

$$= \frac{3}{2}\left[\frac{e^{z}}{2(1)^2-3(1)-6} - \frac{3(1)}{2(0)^2-3(0)-6}\right]$$

$$= \frac{3}{2}\left[\frac{e^{z}}{-7} - \frac{3}{-6}\right]$$

But $e^{z} = 2x+3$

$$\therefore PI = \frac{3}{\cancel{2}}\left[\frac{(2x+3)}{-14} - \frac{\cancel{3}}{\cancel{-12}}\right]$$

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$$\Rightarrow PI = \frac{3(2x+3)}{-14} + \frac{3}{4}$$

$\therefore y = CF + PI$

$$= (2x+3)^{3/4}\left(C_1\cos\frac{\sqrt{57}}{4}\left(\log(2x+3)\right) + C_2\sin\frac{\sqrt{57}}{4}\left(\log(2x+3)\right)\right)$$

$$-\frac{3(2x+3)}{14} + \frac{3}{4}$$