2^{nd} order derivative :- • $\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$ • $\frac{\partial^2 z}{\partial y \cdot \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

In terms of f/u :

* Eg :- $\frac{\partial^2 z}{\partial x^2} = f_{xx} = u_{xx}$ • $\frac{\partial^2 z}{\partial y \cdot \partial x} = f_{yx} = u_{yx}$

$\hookrightarrow \frac{\partial^2 f}{\partial x^2}$ $\hookrightarrow \frac{\partial u}{\partial y \partial x}$

1) If $u = \log(x^2+y^2)$, P.T $\frac{\partial^2 u}{\partial x \cdot \partial y} = \frac{\partial^2 u}{\partial y \cdot \partial x}$
(u_{xy}) (u_{yx})

Solⁿ :- $u = \log(x^2+y^2)$

In $u_{xy} \rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$\therefore \frac{\partial u}{\partial y} = \frac{(0+2y)}{x^2+y^2} = \frac{2y}{x^2+y^2} \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \cancel{(x^2+y^2)\frac{\partial}{\partial x}(2y)}\ 2y \cdot \frac{\partial}{\partial x}(x^2+y^2)^{-1}$

$\Rightarrow u_{xy} = \frac{2y \cdot (-1) \cdot \frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$

$\therefore LHS = u_{xy} = \frac{-4xy}{(x^2+y^2)^2}$

In $u_{yx} \rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$\therefore \frac{\partial u}{\partial x} = \cancel{\log(x^2+y^2)}\ \frac{2x}{x^2+y^2} \Rightarrow \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) = \frac{2x\ (-1)}{(x^2+y^2)^2} \cdot (2y)$

$\therefore RHS = u_{yx} = \frac{-4xy}{(x^2+y^2)^2}$ $\therefore LHS = RHS$

2) $z = \tan(y+ax) + (y-ax)^{3/2}$ S.T (show that) $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

Solⁿ :- $z = \tan(y+ax) + (y-ax)^{3/2}$ —— ①

$\frac{\partial z}{\partial x} = \sec^2(y+ax) \cdot a + \frac{3}{2}(y-ax)^{1/2} \cdot (-a)$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = a \cdot [2\sec(y+ax)] \cdot \sec(y+ax)\tan(y+ax) \cdot a$

$+ \frac{3}{2}(-a) \cdot \frac{1}{2} \cdot \frac{1}{(y-ax)^{1/2}} \cdot (-a)$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = a^2\left[2\sec^2(y+ax)\cdot\tan(y+ax) + \frac{3}{4(y-ax)^{1/2}}\right] \longrightarrow LHS$$
(z_{xx})

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}\cdot(y-ax)^{1/2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2\sec(y+ax)\cdot\sec(y+ax)\tan(y+ax) + \frac{3}{2}\cdot\frac{1}{2}\frac{1}{(y-ax)^{1/2}}$$

$$\Rightarrow a^2\cdot\frac{\partial^2 z}{\partial y^2} = a^2\left[2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4(y-ax)^{1/2}}\right] \longrightarrow RHS$$
(z_{yy})

$\therefore LHS = RHS$

3) $u = e^{x^2+y^2+z^2}$ P.T. $\frac{\partial^3 u}{\partial x\partial y\partial z} = 8xyzu$

Sol^n:- $u = e^{x^2+y^2+z^2}$ $[u_{xyz}]$ $z\to y\to x$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(e^{x^2+y^2+z^2}\right) = e^{x^2+y^2+z^2}\cdot(2z)$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = 2z\cdot e^{x^2+y^2+z^2}\cdot(2y)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\partial z}\right) = 4zy\cdot\underbrace{e^{x^2+y^2+z^2}}_{u}\cdot(2x)$$

$$\Rightarrow \frac{\partial^3 u}{\partial x\partial y\partial z} = 8xyzu$$

* 4) $u = \log(x^3+y^3+z^3-3xyz)$ P.T $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2\cdot u = \frac{-9}{(x+y+z)^2}$

Sol^n:- $u = \log(x^3+y^3+z^3-3xyz)$

• $\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$ • $\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$ • $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$LHS = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2\cdot u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot u\right]$$

$$\rightarrow u\left(\frac{\partial}{\partial x}/\frac{\partial}{\partial y}/\frac{\partial}{\partial z}\right)=\frac{\partial u}{\partial x}/\frac{\partial u}{\partial y}/\frac{\partial u}{\partial z}$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\frac{3x^2-3yz+3y^2-3zx+3z^2-3xy}{x^3+y^3+z^3-3xyz}\right]$$

$$=\left(\text{"} \quad \text{"} \quad \text{"}\right)\left\{3\left[\frac{x^2+y^2+z^2-xz-yz-xy}{x^3+y^3+z^3-3xyz}\right]\right\}$$

We know,

$$x^3+y^3+z^3-3xyz=\left(x^2+y^2+z^2-xz-yz-xy\right)\cdot(x+y+z)$$

$$=\left(\text{"} \quad \text{"} \quad \text{"}\right)\cdot 3\left[\frac{\cancel{(x^3+y^3+z^3-3xyz)}}{\cancel{(x^3+y^3+z^3-3xyz)}}\cdot\frac{1}{(x+y+z)}\right]$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot 3\left(\frac{1}{x+y+z}\right)$$

Let $3\left(\frac{1}{x+y+z}\right)=v$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot v$$

$$=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial}{\partial z}$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=\frac{3(-1)}{(x+y+z)^2}+\frac{3(-1)}{(x+y+z)^2}+\frac{3(-1)}{(x+y+z)^2}$$

$$=\frac{-9}{(x+y+z)^2}$$

$= RHS$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot u=\frac{-9}{(x+y+z)^2}$$