Q4. If $u = \log(x^3+y^3+z^3-3xyz)$ then

Show that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol^n:- Multichain :- $u \to x, y, z$ We have to find $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

For ~~$\left(\frac{\partial}{\partial x}\right)^2$~~ :-

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-\cancel{3}xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z} \quad \text{——— (1)}$$

$$\therefore LHS = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\cdot u\right]$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad [\text{Using (1)}]$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

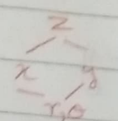
$$= \frac{-9}{(x+y+z)^2} = RHS$$

Hence proved

Q5. If $z = f(x,y)$ where $x = r\cos\theta$, $y = r\sin\theta$

Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Solⁿ:- Multi-chain:-



~~We have to find~~ The following will remain partials : $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$

$$\therefore \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(\cos\theta) + \frac{\partial z}{\partial y}(\sin\theta)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta + 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \quad \text{--- (I)}$$

$$\therefore \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{\partial^2 z}{\partial x^2}\cdot r^2\sin^2\theta + \frac{\partial^2 z}{\partial y^2} r^2\cos^2\theta - 2r^2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \frac{\partial^2 z}{\partial x^2}\sin^2\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta - 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \quad \text{--- (I)}$$

$\therefore$ (I) + (II) $\Rightarrow$

$$RHS = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta + 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$+ \frac{\partial^2 z}{\partial x^2}\sin^2\theta + \frac{\partial^2 z}{\partial y^2}\cos^2\theta - 2\sin\theta\cos\theta\,\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$= \frac{\partial^2 z}{\partial x^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial^2 z}{\partial y^2}(\sin^2\theta + \cos^2\theta)$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$$= LHS$$

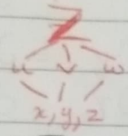
Hence proved

Q6. If $Z = f(y-z, z-x, x-y)$

Show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$

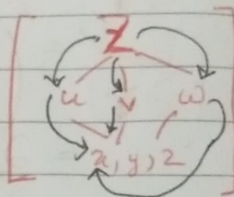
Solⁿ:- Let $u = y-z$, $v = z-x$, $w = x-y$

Multi chain rule



$\therefore z = f(u, v, w)$

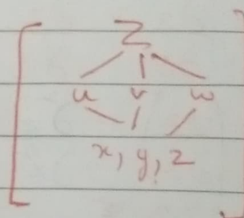
1. $\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial Z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial Z}{\partial w}\cdot\frac{\partial w}{\partial x}$



$= \frac{\partial Z}{\partial u}(0) + \frac{\partial Z}{\partial v}\cdot(-1) + \frac{\partial Z}{\partial w}(1)$

$\Rightarrow \frac{\partial Z}{\partial x} = -\frac{\partial Z}{\partial v} + \frac{\partial Z}{\partial w}$ —— (1)

2. $\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial Z}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial Z}{\partial w}\cdot\frac{\partial w}{\partial y}$



$= \frac{\partial Z}{\partial u}(1) + \frac{\partial Z}{\partial v}(0) + \frac{\partial Z}{\partial w}(-1)$

$\Rightarrow \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial u} - \frac{\partial Z}{\partial w}$ —— (2)

3. $\frac{\partial Z}{\partial z} = \frac{\partial Z}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial Z}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial Z}{\partial w}\cdot\frac{\partial w}{\partial z}$

$= \frac{\partial Z}{\partial u}(-1) + \frac{\partial Z}{\partial v}(1) + \frac{\partial Z}{\partial w}(0)$

$\Rightarrow \frac{\partial Z}{\partial z} = -\frac{\partial Z}{\partial u} + \frac{\partial Z}{\partial v}$ —— (3)

(1) + (2) + (3) $\Rightarrow$ LHS $= \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z}$

$= -\frac{\partial Z}{\partial v} + \frac{\partial Z}{\partial w} + \frac{\partial Z}{\partial u} - \frac{\partial Z}{\partial w} - \frac{\partial Z}{\partial u} + \frac{\partial Z}{\partial v}$

$= 0$

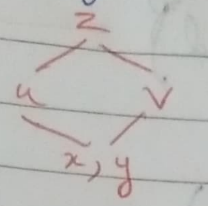
$=$ RHS

Hence proved

Q7. If $z = f(u,v)$ where $u = lx + my$, $v = ly - mx$

Show that :- $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}\right)$

Solⁿ :- Multi-chain :-



$$\begin{bmatrix} u = lx + my \\ v = ly - mx \end{bmatrix}$$

1. For $\dfrac{\partial^2 z}{\partial x^2}$:-

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(l) + \frac{\partial z}{\partial v}\cdot(-m) \quad \text{—— (1)}$$

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial u}(l) - \frac{\partial}{\partial v}(m) \quad \text{—— (2)}$$

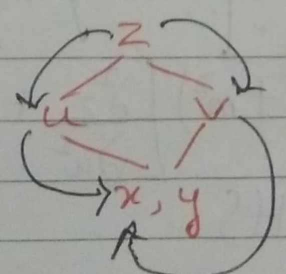
We know, $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$\therefore$ Using (1) & (2) we get :-

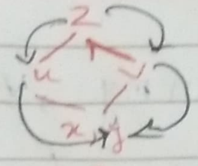
$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \left\{\left[\frac{\partial}{\partial u}(l) - \frac{\partial}{\partial v}(m)\right]\left[\frac{\partial z}{\partial u}(l) - \frac{\partial z}{\partial v}(m)\right]\right.$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - lm\frac{\partial^2 z}{\partial u\,\partial v} - lm\frac{\partial^2 z}{\partial v\,\partial u} + m^2\frac{\partial^2 z}{\partial v^2} \quad \text{—— (3)}$$

~~2. For~~ Here :-



2. For $\frac{\partial^2 z}{\partial y^2}$:-



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(m) + \frac{\partial z}{\partial v}(l) \quad \text{--- (4)}$$

$$\therefore \frac{\partial}{\partial y} = \frac{\partial}{\partial u}(m) + \frac{\partial}{\partial v}(l) \quad \text{--- (5)}$$

We know, $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

Using (4) & (5) we get :-

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = \left[\frac{\partial}{\partial u}(m) + \frac{\partial}{\partial v}(l)\right]\left[\frac{\partial z}{\partial u}(m) + \frac{\partial z}{\partial v}(l)\right]$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = m^2 \frac{\partial^2 m}{\partial u^2} + lm \frac{\partial^2 z}{\partial u \partial v} + lm \frac{\partial^2 z}{\partial v \partial u} + l^2 \frac{\partial^2 z}{\partial v^2} \quad \text{--- (6)}$$

$$(3) + (6) \Rightarrow LHS = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$$= l^2 \frac{\partial^2 z}{\partial u^2} \cancel{- lm \frac{\partial^2 z}{\partial u \partial v}} \cancel{- lm \frac{\partial^2 z}{\partial v \partial u}} + m^2 \frac{\partial^2 z}{\partial v^2}$$
$$+ m^2 \frac{\partial^2 z}{\partial u^2} \cancel{+ lm \frac{\partial^2 z}{\partial u \partial v}} \cancel{+ lm \frac{\partial^2 z}{\partial v \partial u}} + l^2 \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2}(l^2 + m^2) + \frac{\partial^2 z}{\partial v^2}(l^2 + m^2)$$

$$= (l^2 + m^2)\left[\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right]$$

$$= RHS$$

Hence proved