Q9. Solve: $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Sol^n:- Let $D = \frac{d}{dx}$

$\therefore (x^2D^2 + 4xD + 2)y = e^x$

Let $x = e^z \Rightarrow \log x = z$

$\therefore$ We have:- $\cdot\ xD = D'$

$\cancel{\therefore (x}$ $\qquad \cdot\ x^2D^2 = D'(D'-1)$

$\cancel{\therefore (x^2D^2 + 4xD}$

$\therefore [D'(D'-1) + 4D' + 2]y = e^{e^z}$

$\Rightarrow (D'^2 + 3D' + 2)y = e^{e^z}$

A.E:- $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \qquad \therefore m = -1, -2$

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$\underline{CF}$:- $CF = C_1 e^{-2} + C_2 e^{-2z}$

$= \dfrac{C_1}{e^2} + \dfrac{C_2}{(e^2)^2}$

$\Rightarrow CF = \dfrac{C_1}{x} + \dfrac{C_2}{x^2}$

P.I:- $PI = \dfrac{e^{e^z}}{D'^2+3D'+2}$ ~~(Type 1)~~ [a=

$= \dfrac{e^{e^z}}{(D'+1)(D'+2)}$ [Now, we seperate them]

$= \dfrac{e^{e^z}}{D'+1} - \dfrac{e^{e^z}}{D'+2}$

We know, $x^1 \times x^{-1} = 1$

∴ We can write $e^{e^z} = e^{-z} \cdot e^{z} \cdot e^{e^z}$

$\therefore PI = \dfrac{e^{-z} \cdot e^{z} \cdot e^{e^z}}{D'+1} - \dfrac{e^{-2z} e^{2z} e^{e^z}}{D'+2}$ [Take $e^a \cdot e^{-a}$ such that we get D' in denominator]

Now,

$$P.I = \frac{e^{-z} e^{z} e^{e^z}}{D'+1} - \frac{e^{-2z} e^{2z} e^{e^z}}{D'+2} \quad \left[\begin{array}{l} \text{Type } 4 \ , \ a=-1 \\ D' \to D'+a = D'-1 \end{array}\right]$$

$$= \frac{e^{-z} e^{z} e^{e^z}}{D'-1+1} - \frac{e^{-2z} e^{2z} e^{e^z}}{D'-2+2} \quad \text{(Ensure only } D' \text{ remains in denominator)}$$

$$= \frac{e^{-z} e^{z} e^{e^z}}{D'} - \frac{e^{-2z} e^{2z} e^{e^z}}{D'\cancel{+2}}$$

$$\Rightarrow PI = e^{-z}\int e^{z} e^{e^z} dz - e^{-2z}\int e^{2z} e^{e^z} dz$$

Let $\boxed{e^z = t}$

$$\Rightarrow \frac{dt}{dz} = e^z \Rightarrow \boxed{dt = e^z dz}$$

$$\therefore PI = e^{-z}\int \underbrace{e^{e^z}}_{e^t} \cdot \underbrace{e^z dz}_{dt} - e^{-2z}\int \underbrace{e^z}_{t} \cdot \underbrace{e^{e^z}}_{e^t} \cdot \underbrace{e^z dz}_{dt}$$

$$= e^{-z}\int e^t dt - e^{-2z}\int \cancel{te} \, t e^t dt$$

$$= e^{-z}(e^t) - e^{-2z}\left[t\int e^t dt - \int\left(\frac{dt}{dt}\cdot\int e^t dt\right)dt\right]$$

u.v rule :- (integration) $\int uv\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$

$$\Rightarrow P.I = e^{-z}\, e^{e^z} - e^{-2z}\left[t e^t - \int e^t dt\right]$$

$$= e^{-z} e^{x} - e^{-2z}\left[e^z \cdot e^{e^z} - e^{e^z}\right]$$

But $z = \log x$ & $e^z = x$

$$\therefore PI = \frac{1}{x}\cdot e^x - \frac{1}{x^2}\cdot x \cdot e^x + \frac{1}{x^2}\cdot e^x$$

$$\Rightarrow PI = \frac{e^x}{x^2}$$

$$\therefore y = CF + PI$$

$$= \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{e^x}{x^2}$$

Q10 Solve: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\log x\sin(\log x)$

Sol:- Let $D=\frac{d}{dx}$

$\therefore (x^2D^2+xD+1)y=\log x\sin(\log x)$

Let $x=e^z \Rightarrow \log x=z$

$\therefore$ We have:- $\cdot\ xD=D'$

$\cdot\ x^2D^2=D'(D'-1)$

$\therefore [D'(D'-1)+D'+1]y=z\sin z$

$\Rightarrow (D'^2+1)y=z\sin z$

A.E:- $m^2+1=0$

$\Rightarrow m^2=-1$

$\Rightarrow m=\pm\sqrt{-1}=\pm i$ $\quad\therefore m=0+i,\ 0-i$

$(\alpha=0,\ \beta=1)$

CF:- $CF=e^{\alpha z}(C_1\cos\beta z+C_2\sin\beta z)$

$=e^{0z}(C_1\cos(1)z+C_2\sin(1)z)$

But, $z=\log x$

$\therefore CF=C_1\cos(\log x)+C_2\sin(\log x)$

PI:- $PI=\frac{z\sin z}{D'^2+1}$ for this $\left[\text{Type 5}\quad e^{i\theta}=\underset{\text{Real}}{\cos\theta}+i\underset{\text{Imaginary}}{\sin\theta}\right]$

$=\frac{\text{I.P of } e^{iz}\cdot z}{D'^2+1}$ $\quad[D'\to D'+i]$

$=\frac{\text{I.P of } e^{iz}\cdot z}{(D'+i)^2+1}$

$=\frac{\text{IP of } e^{iz}\cdot z}{D'^2+i^2+2D'i+1}$

Here, $i=\sqrt{-1}$

$\therefore i^2=-1$

$$\therefore PI = \frac{I.P \text{ of } e^{iz}\cdot z}{D'^2 + 2iD' - \cancel{1} + \cancel{1}}$$

$$= \frac{I.P \text{ of } e^{iz}\cdot z}{D'^2 + 2iD'} \rightarrow \text{for this}$$

Now, take common such that we validate Type 3

$$\Rightarrow PI = \frac{I.P \text{ of } e^{iz}\cdot z}{2iD'\left[\frac{D'^2}{2iD'} + 1\right]}$$

$$= \frac{IP \text{ of } e^{iz}\cdot z}{2iD'}\left[\cancel{1} + \frac{D'}{2i}\right]^{-1}$$

$(1+x)^{-1} = 1 - x + x^2 + \dots$

$$\Rightarrow PI = \frac{IP \text{ of } e^{iz}}{2iD'}\left[1 - \frac{D'}{2i}\right]\cdot z \qquad \left[\because D'(z) = 1, \; D'^2(z) = 0\right]$$

$$= \frac{IP \text{ of } e^{iz}}{2iD'}\left[z - \frac{1}{2i}\right]$$

$$= \frac{IP \text{ of } e^{iz}}{2i}\left[\int z\,dz - \int \frac{1}{2i}\,dz\right]$$

$$= IP \text{ of } \frac{e^{iz}}{2i}\left[\frac{z^2}{2} - \frac{z}{2i}\right]$$

Here, $e^{iz} = \cos z + i\sin z$

$$\Rightarrow P.I = IP \text{ of } \frac{(\cos z + i\sin z)}{2i}\left[\frac{z^2}{2} - \frac{z}{2i}\right]$$

$$= IP \text{ of } (\cos z + i\sin z)\left[\frac{z^2}{4i} - \frac{z}{4(i)^2}\right] \quad (i^2 = -1)$$

$$= IP \text{ of } (\cos z + i\sin z)\left[\frac{-z^2 i}{4} + \frac{z}{4}\right]$$

$$\Rightarrow P.I = \frac{-z^2\cos z}{4} + \frac{z\sin z}{4}$$

$\therefore y = CF + PI$

$$= C_1\cos(\log x) + C_2\sin(\log x) - \frac{z^2\cos z}{4} + \frac{z\sin z}{4}$$

$$\Rightarrow y = C_1\cos(\log x) + C_2\sin(\log x) - \frac{(\log x)^2\cos(\log x)}{4} + \frac{(\log x)\sin(\log x)}{4}$$

Q11. Solve: $\left[x^2D^2-(2m-1)xD+(m^2+n^2)\right]y = n^2x^m\log x$

Sol^n:- $\left[x^2D^2-(2m-1)xD+(m^2+n^2)\right]y = n^2x^m\log x$

Let $x=e^z \Rightarrow \log x = z$

$\therefore$ We have :- $\bullet\ xD = D'$

$\bullet\ x^2D^2 = D'(D'-1)$

$\therefore \left[D'(D'-1)-(2m-1)D'+(m^2+n^2)\right]y = n^2e^{mz}z$

$\Rightarrow \left[D'^2-\cancel{D'}-2mD'+\cancel{D'}+m^2+n^2\right]y = n^2e^{mz}z$

$\therefore (D'^2-2mD'+m^2+n^2)y = n^2e^{mz}z$

AE :- $t^2-2mt+m^2+n^2=0$.

Here, $a=1$, $b=-2m$, $c=m^2+n^2$

$\therefore t = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-(-2m)\pm\sqrt{(-2m)^2-4(1)(m^2+n^2)}}{2(1)}$

$= \dfrac{2m\pm\sqrt{4m^2-4m^2-4n^2}}{2}$

$= \dfrac{2m\pm\sqrt{-4n^2}}{2}$

$= \dfrac{2m\pm 2ni}{2}$ $\therefore t = m+ni, m-ni$

$(\alpha = m,\ \beta = n)$

CF :- $CF = e^{\alpha z}(C_1\cos\beta z + C_2\sin\beta z)$

$= e^{mz}(C_1\cos nz + C_2\sin nz)$

But $z=\log x$ & $x=e^z$

$\Rightarrow CF = x^m(C_1\cos(n\log x) + C_2\sin(n\log x))$

P.I.: $P.I = \dfrac{x^2 e^{mz} z}{D'^2 - 2mD' + m^2 + x^2}$ → for this $\left[\begin{array}{l}\text{Type 4,} \\ D' \to D' + m\end{array}\right]$

$$= \frac{x^2 e^{mz} z}{(D'+m)^2 - 2m(D'+m) + m^2 + x^2}$$

$$= \frac{x^2 e^{mz} z}{D'^2 + \cancel{m^2} + \cancel{2mD'} - \cancel{2mD'} - \cancel{2m^2} + \cancel{m^2} + x^2}$$

$$\Rightarrow PI = \frac{x^2 e^{mz} z}{D'^2 + x^2}$$

$$= \frac{\cancel{x^2} e^{mz} z}{\cancel{x^2}\left(\frac{D'^2}{x^2} + 1\right)}$$

$$= e^{mz} z \left(1 + \frac{D'^2}{m^2}\right)^{-1}$$

$$= e^{mz} \left(1 + \frac{D'^2}{m^2}\right) z$$

$$= e^{mz}\left(z + \cancel{\frac{D'^2(z)}{m^2}}\right) \quad 0$$

$$\Rightarrow PI = (e^{z})^{m} \cdot z$$

Here, $e^z = x$ & $z = \log x$

$$\therefore PI = x^m (\log x)$$

$$\therefore y = CF + PI$$

$$= x^m\left(C_1 \cos(x \log x) + C_2 \sin(x \log x)\right) + x^m(\log x)$$