Q6. Solve: $(x^2D^2 - xD + 1)y = \left[\frac{\log x}{x}\right]^2$

Solⁿ:- $(x^2D^2 - xD + 1)y = \left[\frac{\log x}{x}\right]^2$

Let $x = e^z \Rightarrow \log x = z$

$\therefore$ We have :- $xD = D'$

$x^2D^2 = D'(D'-1)$

$\therefore \left[D'(D'-1) - D' + 1\right]y = \left(\frac{z}{e^z}\right)^2$

$\Rightarrow (D'^2 - 2D' + 1)y = \frac{z^2}{e^{2z}} = e^{-2z} \cdot z^2$

A.E:- $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0 \quad \therefore m = 1, 1$

CF:- $CF = (C_1 + C_2 z)e^{z}$

$\Rightarrow CF = [C_1 + C_2(\log x)]x$

P.I:- $P.I = \frac{e^{-2z} z^2}{D'^2 - 2D' + 1}$ $\left[D' \to D' + a \quad (a = -2) = D' - 2\right]$ (Type 4)

For this

$= \frac{e^{-2z} z^2}{(D'-2)^2 - 2(D'-2) + 1}$

$= \frac{e^{-2z} z^2}{D'^2 - 4D' + 4 - 2D' + 4 + 1}$

$= \frac{e^{-2z} z^2}{D'^2 - 6D' + 9}$ For this (Type 3)

$= \frac{e^{-2z} z^2}{9\left(\frac{D'^2 - 6D'}{9} + 1\right)}$

$= \frac{e^{-2z}}{9}\left(1 + \frac{D'^2 - 6D'}{9}\right)^{-1} z^2$

$$PI = \frac{e^{-2z}}{9}\left[1-\left(\frac{D'^2-6D'}{9}\right)+\left(\frac{D'^2-6D'}{9}\right)^2\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[1-\frac{D'^2}{9}+\frac{6D'}{9}+\frac{36D'^2}{81}\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2-\frac{D'^2(z^2)}{9}+\frac{6D'(z^2)}{9}+\frac{36D'^2(z^2)}{81}\right] \quad \left(D'=\frac{d}{dz}\right)$$

$$= \frac{e^{-2z}}{9}\left[z^2-\frac{2}{9}+\frac{6(2z)}{9}+\frac{36(2)}{81}\right]$$

$\rightarrow$ [Here, $D'(z^2) = \frac{d}{dz}(z^2) = 2z$ & $D'^2(z^2) = \frac{d}{dz}(2z) = 2$]

$$PI = \frac{1}{9(e^z)^2}\left[z^2+\frac{4}{3}z+\frac{2}{3}\right]$$

$\rightarrow$ [Here, $e^z = x$ & $z = \log x$]

$$\therefore PI = \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

$$\therefore y = CF + PI$$

$$= (C_1+C_2\log x)x + \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Q7. Solve: $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Solⁿ:- $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Let $x = e^z \Rightarrow \log x = z$

$\therefore$ We have:- $\cdot\ xD = D'$

$\cdot\ x^2D^2 = D'(D'-1)$

$\therefore [D'(D'-1) - 2D' - 4]y = e^{2z} + 2z$

$\Rightarrow (D'^2 - 3D' - 4)y = e^{2z} + 2z$

AE:- $m^2 - 3m - 4 = 0$

$\Rightarrow m^2 - 4m + m - 4 = 0$

$\Rightarrow m(m-4) + 1(m-4) = 0$

$\Rightarrow (m+1)(m-4) = 0$ $\quad \therefore m = -1, 4$

CF:- $CF = C_1e^{-z} + C_2e^{4z}$ (in terms of z)

$= C_1(e^z)^{-1} + C_2(e^z)^4$

$= \dfrac{C_1}{x} + C_2x^4$

P.I:- $PI = \dfrac{e^{2z} + 2z}{D'^2 - 3D' - 4}$

$\cdot\ PI_1 = \dfrac{e^{2z}}{D'^2 - 3D' - 4}$ (Type 1, $D = a = 2$)

$= \dfrac{e^{2z}}{(2)^2 - 3(2) - 4}$

$= \dfrac{(e^z)^2}{-6}$ $\cancel{= x^2}$

$\Rightarrow PI_1 = \dfrac{-x^2}{6}$

$$\cdot \quad P.I_2 = \frac{2z}{D'^2 - 3D' - 4} \quad (\text{Type } 3)$$

$$= \frac{2z}{-4\left[\frac{D'^2 - 3D'}{-4} + 1\right]}$$

$$= \frac{-\cancel{2}\cancel{4}}{2}\left[1 \cancel{+} \frac{-D'^2 - 3D'}{4}\right]^{-1} z$$

$$= -\frac{1}{2}\left[1 + \frac{D'^2 - 3D'}{4}\right] z \qquad \left[(1-x)^{-1} = 1 + x + x^2 + \dots\right]$$

$$= -\frac{1}{2}\left[z + \cancel{\frac{D'^2(z)}{4}}^{\to 0} - \frac{3D'(z)}{4}\right]$$

$$= -\frac{1}{2}\left[z - \frac{3(1)}{4}\right]$$

$$\Rightarrow PI_2 = -\frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

$$\therefore PI = PI_1 + PI_2$$

$$= -\frac{x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

$$y = CF + PI$$

$$= \frac{C_1}{x} + C_2 x^4 - \frac{x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

$$\Rightarrow y = \frac{C_1}{x} + C_2 x^4 - \frac{x^2}{6} - \frac{\log x}{2} - \frac{3}{8}$$

Q5. Solve:- $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Sol:- $D = \frac{d}{dx}$

$\therefore \left(D^2 + \frac{1}{x}D\right)\cdot y = \frac{12\log x}{x^2}$

$\Rightarrow \left(x^2D^2 + \frac{x^2}{x}D\right)y = 12\log x$ (Multiply both sides by x^2)

$\Rightarrow (x^2D^2 + xD)y = 12\log x$

Let $x = e^z \Rightarrow \log x = z$

$\therefore$ We have :-
- $xD = D'$
- $x^2D^2 = D'(D'-1)$

~~$\therefore D' +$~~

$\therefore [D'(D'-1) + D']y = 12\log x$

$\Rightarrow [D'^2 - \cancel{D'} + \cancel{D'}]y = 12\log x$ [$\log x = z$]

$\Rightarrow D'^2 y = 12z$

A.E :- $m^2 = 0$ $\quad \therefore m = 0, 0$

~~CF :- $CF = C_1 e + C_2 e^{0x}$~~

C.F :- $CF = (C_1 + C_2 z)e^{0x}$

$= (C_1 + C_2\log x)$

$\underline{PI}$:- $PI = \frac{12z}{D'^2}$ (Type-3)

$$= \frac{12}{D'} \cdot \left(\frac{1}{D'} z\right)$$

$$= \frac{12}{D'} \int z \, dz$$

$$= \frac{\cancel{12}^{6}}{D'} \frac{z^2}{\cancel{2}}$$

$$= 6 \cdot \left(\frac{1}{D'} z^2\right)$$

$$= 6 \int z^2 \, dz$$

$$= \cancel{6}^{2} \frac{z^3}{\cancel{3}}$$

$$\Rightarrow PI = 2z^3$$

$$\therefore y = CF + PI$$

$$= C_1 + C_2(\log x) + 2z^3$$

$$\Rightarrow y = C_1 + C_2 \log x + 2(\log x)^3$$