(*)

Q10. Solve $(D^2 - D)y = e^x \cos x$

Solⁿ:- A.E:- $m^2 - m = 0$

$\Rightarrow m(m-1) = 0$ $\quad m = 0, 1$

CF:- $CF = C_1 e^{0x} + C_2 e^x = C_1 + C_2 e^x$

Here, $y_1 = 1$ $\quad y_2 = e^x$

$y_1' = 0$ $\quad y_2' = e^x$

$X = e^x \cos x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix} = e^x + 0$ $\quad \therefore W = e^x$

- $P = -\int \frac{y_2 X}{W} dx = -\int \frac{(\cancel{e^x})(e^x \cos x)}{\cancel{e^x}} dx$ $\quad$ Apply uv rule $\int uv\,dx = u\int v\,dx - \int (u' \int v\,dx)\,dx$

$= -\left[e^x \int \cos x\,dx - \int \left(e^x \int \cos x\,dx \right) dx \right]$ $\quad \int e^x \sin x\,dx$

$= e^x \int \sin x\,dx - \int \left(e^x \int \sin x\,dx \right) dx$

$= -\left[e^x \sin x - \int e^x \sin x\,dx \right]$ $\quad = e^x \cdot (-\cos x) - \int e^x (-\cos x)\,dx$

$= e^x(-\cos x) + \int e^x \cos x\,dx$

Instead use formula:-

$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2} \left[a\cos bx + b\sin bx \right]$ $\quad$ continues till ∞

Here, $P = -\int e^x \cos x\,dx$ $\quad \left[a = 1, b = 1 \right]$

$\Rightarrow P = -\left[\frac{e^x}{1^2+1^2} (\cos x + \sin x) \right] = -\frac{e^x}{2} (\cos x + \sin x)$

- $Q = \int \frac{y_1 X}{W} dx = \int \frac{(1)(\cancel{e^x} \cos x)}{\cancel{e^x}} dx$ $\quad \cancel{Q = 0}$ $\quad \therefore Q = \sin x$

PI:- $PI = P y_1 + Q y_2$

$= -\frac{e^x}{2} (\cos x + \sin x) + e^x \sin x$

$\therefore y = CF + PI = C_1 + C_2 e^x - \frac{e^x}{2} (\cos x + \sin x) + e^x \sin x$

Q11. Solve $y'' + 4y = 4\sec^2 2x$

Sol^n:- $(D^2+4)y = 4\sec^2 2x$

A.E:- $m^2+4=0$

$\Rightarrow m = \pm\sqrt{-4} = \pm 2i$ $\quad$ $\therefore m = 0+2i, 0-2i$ $\quad$ $[\alpha = 0, \beta = 2]$

CF:- $CF = e^{0}(C_1\cos 2x + C_2\sin 2x)$

$\Rightarrow CF = C_1\cos 2x + C_2\sin 2x$

Here, $\quad y_1 = \cos 2x \qquad y_2 = \sin 2x$

$\qquad y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$

$X = 4\sec^2 2x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x)$

$\therefore W = 2$

$P = -\int \frac{y_2 X}{W}\,dx = -\int \frac{(\sin 2x)(\overset{2}{\cancel{4}}\sec^2 2x)}{\cancel{2}}\,dx = -2\int \tan 2x \sec 2x\,dx$

$\therefore P = -2\left(\frac{\sec 2x}{\frac{d}{dx}(2x)}\right) = -\sec 2x$ $\quad$ $\left[\because \frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta \quad \therefore \int \sec\theta\tan\theta\,d\theta = \sec\theta\right]$

$Q = \int \frac{y_1 X}{W}\,dx = \int \frac{(\cos 2x)(\overset{2}{\cancel{4}}\sec^2 2x)}{\cancel{2}}\,dx = 2\int \frac{1}{\cos 2x}\,dx = 2\int \sec 2x\,dx$

$= \frac{2\log(\sec 2x + \tan 2x)}{\frac{d}{dx}(2x)} = \frac{\cancel{2}\log(\sec 2x + \tan 2x)}{\cancel{2}}$

$\therefore Q = \log(\sec 2x + \tan 2x)$

PI:- $PI = Py_1 + Qy_2$

$= (-\sec 2x)(\cos 2x) + \log(\sec 2x + \tan 2x)(\sin 2x)$

$\Rightarrow PI = -1 + \sin 2x \log(\sec 2x + \tan 2x)$

$\therefore y = CF + PI$

$= C_1\cos 2x + C_2\sin 2x - 1 + \sin 2x\log(\sec 2x + \tan 2x)$

(*) $\int \sec 2x\,dx = \log(\sec 2x + \tan 2x)$

$\int \operatorname{cosec} 2x\,dx = \log(\operatorname{cosec} 2x - \cot 2x)$

Q12. Solve $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

Sol^n:- $(D^2-1)y = \dfrac{2}{1+e^x}$

A.E:- $m^2-1=0$

$\Rightarrow m = \pm\sqrt{1} = \pm 1$ $\quad \therefore m = +1, -1$

CF:- $CF = C_1 e^{1x} + C_2 e^{-x}$

Here, $y_1 = e^x \qquad y_2 = e^{-x}$

$y_1' = e^x \qquad y_2' = -e^{-x}$

$X = \dfrac{2}{1+e^x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -1-1$

$\therefore W = -2$

$\dfrac{1}{a(a+1)} = \dfrac{1}{a} - \dfrac{1}{a+1}$

$P = -\int \dfrac{y_2 X}{W} dx = -\int \dfrac{(e^{-x})}{-2}\left(\dfrac{2}{1+e^x}\right) dx = \int \dfrac{1}{e^x(1+e^x)} dx$

$= \int \left(\dfrac{1}{e^x} - \dfrac{1}{e^x+1}\right) dx$

$= \int e^{-x} dx - \dfrac{dx}{e^x+1}$

$= \int e^{-x} dx - \int \dfrac{e^{-x} dx}{e^{-x}+1}$ ##

$\int e^{-x} dx - \dfrac{dx}{\frac{1}{e^{-x}}+1}$

$= \int e^{-x} dx - \dfrac{e^{-x} dx}{e^{-x}+1}$

Let $t = 1+e^{-x}$

$\dfrac{dt}{dx} = 0 + (-1)e^{-x}$

$\Rightarrow dx = \dfrac{-dt}{e^{-x}}$

$\therefore \int \dfrac{\cancel{e^{-x}}}{e^{-x}+1} \cdot \dfrac{-dt}{\cancel{e^{-x}}}$

$= -\int \dfrac{1}{t} dt$

$= -e^{-x} - \left(-\int \dfrac{1}{t} dt\right) = -e^{-x} + \log(t) \quad [t = 1+e^{-x}]$

$\therefore P = -e^{-x} + \log(1+e^{-x})$

$$Q = \int \frac{y_1 X}{W} dx = \int \frac{e^x}{-\cancel{2}} \cdot \frac{\cancel{2}}{(1+e^x)} dx = -\int \frac{e^x}{1+e^x} dx$$

Let $e^x + 1 = t$

$$\therefore \frac{dt}{dx} = 0 + e^x \Rightarrow dx = \frac{dt}{e^x}$$

$$\therefore Q = -\int \frac{\cancel{e^x}}{1+e^x}\left(\frac{dt}{\cancel{e^x}}\right) = -\int \frac{1}{t} dt$$

$$= -\log(t) \quad [t = e^x + 1]$$

$$\therefore Q = -\log(e^x + 1)$$

PI :- $P.I = Py_1 + Qy_2$

$$= [-e^{-x} + \log(1+e^x)](e^x) + [-\log(e^x+1)](e^{-x})$$

$$\Rightarrow P.I = -1 + \log(1+e^x)\, e^x - e^{-x}\log(1+e^x)$$

$$\therefore y = CF + PI$$

$$= C_1 e^x + C_2 e^{-x} - 1 + e^x \log(1+e^x) - e^{-x}\log(1+e^x)$$