Particular Integral (Type 3 → $X = x^m$)

For this type we need to know the following:-

* **Binomial Expansion**:- $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ [~~Traditional~~ $n \geq 0$, $n \geq k \geq 0$]

$= \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$ [Generalized Binomial Coefficient. Here, $n \to$ any value, $k \to$ non-negative]

* Here, $k \to$ no. of terms in this expansion

Example:-

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!} = \frac{120}{2\times 6} = 10$ [$5 \geq 0$, $5 \geq 2 \geq 0$]

- $\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$

$\binom{-1}{2}$:- $n = -1$, $k = 2$ (no. of terms in numerator expansion $= 2$)

$\therefore \binom{-1}{2} = \frac{(-1)(\cancel{-1}-2+\cancel{1})}{2!}$

$= \frac{(-1)(-2)}{2}$

$= \frac{2}{2} \Rightarrow \binom{-1}{2} = 1$

Here, we will be using the formula $\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$

To compute the following:-

- $(1+x)^{-1}$
- $(1-x)^{-1}$

$\binom{-1}{0} \rightarrow k = 0 \rightarrow$ no. of terms $= 0$

$\therefore \binom{-1}{0} = \underline{1}$

* We can use :- ~~$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{}$~~

$(1+x)^n = \sum_{k=0}^{\infty} (-1)^k x^k \; [\forall\, |x| < 1]$ (Direct method)

$\therefore$ For the following :-

1) $(1+x)^{-1} = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

$= \binom{-1}{0} x^0 + \binom{-1}{1} x^1 + \binom{-1}{2} x^2 + \dots\dots$

[Using Generalized Binomial Coefficient]

$\rightarrow$ no. of terms $= 0$

$\therefore \binom{-1}{0} = 1$

But here, $k = 1 \;\therefore$ no. of terms is 1

Hence, we take last element $\rightarrow (-1-1+1)$ $[n-k+1]$

$\therefore \binom{-1}{1} = \frac{(-1-\cancel{1}+\cancel{1})}{1!} = -1$

- $\binom{-1}{2} \rightarrow k = 2$, $\therefore$ no. of terms is 2 $<$ $n = -1$; $(n-k+1) = (-1-2+1) = -2$

$\therefore \binom{-1}{2} = \frac{(-1)(-2)}{2!} = \frac{(-1)(-\cancel{2})}{\cancel{2}} \Rightarrow \binom{-1}{2} = 1$

Therefore, its going in a sequence of :-

$+1 \rightarrow -1 \rightarrow +1 \rightarrow -1 \rightarrow \dots\dots$

$$\boxed{\begin{aligned} \therefore (1+x)^{-1} &= 1 + (-1)x^1 + (1)x^2 + (-1)x^3 + \dots\dots \\ \Rightarrow (1+x)^{-1} &= 1 - x + x^2 - x^3 + x^4 + \dots\dots \end{aligned}}$$

$\Rightarrow (1-x)^{-1} = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

$= \binom{-1}{0}(x)^0 + \binom{-1}{1}(x)^1 + \binom{-1}{2}(x)^2 + \binom{-1}{3}\overset{(-x)^3}{\cancel{x}} + \ldots\ldots$

Here,

- $\binom{-1}{0}$: $k = 0$ No terms / zero terms

$\therefore \binom{-1}{0} = 1$

- $\binom{-1}{1}$: $k = 1$ No. of terms $= 1 \rightarrow n = -1$

$\therefore \binom{-1}{1} = \frac{n}{k!} = \frac{-1}{1!} \Rightarrow \binom{-1}{1} = -1$

- $\binom{-1}{2}$: $k = 2$ No. of terms $= 2$ $\begin{cases} n = -1 \\ (n-k+1) = (-1-2+1) = -2 \end{cases}$

$\therefore \binom{-1}{2} = \frac{n(n-k+1)}{k!} = \frac{(-1)(-2)}{2!}$

$\Rightarrow \binom{-1}{2} = \frac{2}{1\times2} = 1$

$+1 \rightarrow -1 \rightarrow +1 \rightarrow -1 \rightarrow \ldots\ldots\ldots$

$(1-x)^{-1} = [1+(-x)]^{-1}$

Now, we apply $(1+x)^{-1}$ formula here

$\boxed{\begin{aligned} &\therefore (1-x)^{-1} = +1(-x)^0 + (-1)(-x)^1 + (+1)(-x)^2 + (-1)(-x)^3 + \ldots\ldots \\ &\Rightarrow (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots \end{aligned}}$

Q. Solve $(D^3-3D+2)y=x$

Sol^n A.E :- $D^3-3D+2=0$

$\Rightarrow D(D^2-3)=-2$

$\therefore D=-2,1,1$ (Real & repeated)

C.F :- $C.F = c_1e^{-2x}+(c_2+c_3x)e^{x}$

P.I :- $P.I = \frac{1}{f(D)}\cdot x$ [$f(D) = D^3-3D+2$]

Here,

we need to write $f(D)$ in terms of $(1+x)^{-1}$ or $(1-x)^{-1}$

$$P.I = \frac{1}{D^3-3D+2}\cdot x$$

$$= \frac{1}{2\left(\frac{D^3-3D}{2}+1\right)}\cdot x$$

We know, $\frac{1}{a^m}=a^{-m}$

$$\therefore P.I = \frac{1}{2}\left(1+\frac{D^3-3D}{2}\right)^{-1}\cdot x \quad \left[\text{In the form of } (1+x)^{-1}\right]$$

$(1+x)^{-1} = (1-x+x^2-x^3+\dots)$

$$\therefore P.I = \frac{1}{2}\left[1-\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^3-3D}{2}\right)^2-\left(\frac{D^3-3D}{2}\right)^3+\dots\right]\cdot x$$

Here, $Dx = \frac{d(x)}{dx} = 1 \xrightarrow{\text{next}} D^2x = \frac{d}{dx}\left(\frac{dx}{dx}\right) = \frac{d}{dx}(1) = 0 \xrightarrow{\text{next}} D^3x = 0$. so on

$$\therefore P.I = \frac{1}{2}\left[x-\left(\frac{0^2-3(1)}{2}\right)+\left(\frac{D^6+9D^2-6D^4}{2}\right)-\dots\right]$$

$$= \frac{1}{2}\left[x-\left(-\frac{3}{2}\right)\right]$$

$$\Rightarrow P.I = \frac{1}{2}\left[x+\frac{3}{2}\right]$$

$\because$ $Dx = 1$ & $D^n x = 0$ $[n > 1]$

$\therefore$ It stops at $\left(\frac{D^3 - 3D}{2}\right)$

General sol^n: $y = c_1 e^{-2x} + (c_2 + c_3 x)e^{x} + \frac{1}{2}\left[x + \frac{3}{2}\right]$

Q2. Solve $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2$

Sol^n:- $(D^3 - 2D + 4)y = 3x^2 - 5x + 2$ [In D form]

A.E:- $D^3 - 2D + 4 = 0$

$\Rightarrow \cancel{D(D^2-2) = -4}$ $\therefore D = -2, 1 \pm i$

$[\alpha = 1, \beta = 1]$

C.F:- $C.F = c_1 e^{-2x} + e^{x}(c_2 \cos 1x + c_3 \sin 1x)$

$= c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

P.I:- $P.I = \frac{1}{f(D)} \cdot (3x^2 - 5x + 2)$ $[f(D) = D^3 - 2D + 4]$

$= \frac{3x^2 - 5x + 2}{D^3 - 2D + 4}$

Let's write it in the form $(1+x)^{-1}$

$\therefore P.I = \frac{3x^2 - 5x + 2}{4\left(\frac{D^3 - 2D}{4} + 1\right)}$

$= \frac{1}{4}\left(1 + \frac{D^3 - 2D}{4}\right)^{-1} \cdot (3x^2 - 5x + 2)$

Now, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$\therefore P.I = \frac{1}{4}\left(1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \dots\right)(3x^2 - 5x + 2)$

Here, $\cancel{Dx = \frac{d}{dx}}$ $D(3x^2 - 5x + 2) = 6x - 5$

$\downarrow$

$D^2(6x - 5) = 6$

$D^3(6) = 0$

$\therefore D^3(3x^2 - 5x + 2)$ onwards, its 0

Now lets compute :-

1. $\left(\frac{D^3-2D}{4}\right)(3x^2-5x+2) = \frac{0-2(6x-5)}{4}$ $\quad 2D(3x^2-5x+2) = 6x-5$

$= \cancel{\frac{-6x+5}{4}} \quad \frac{-12x+10}{4}$

2. $\left(\frac{D^3-2D}{4}\right)^2(3x^2-5x+2) = \left(\frac{D^6+4D^2-4D^4}{16}\right)\cdot(3x^2-5x+2)$

$= \frac{0+4D^2(3x^2-5x+2)-0}{16}$

$= \frac{\cancel{4}\times\cancel{6}^{3}}{\cancel{16}_{\cancel{4}\,2}}$

$\Rightarrow \left(\frac{D^3-2D}{4}\right)^2(3x^2-5x+2) = \frac{3}{2}$

3. $\left(\frac{D^3-2D}{4}\right)^3(3x^2-5x+2) = \left(\frac{D^9-8D^3-3D^4}{64}\right)\cdot(3x^2-5x+2)$

$= \frac{0-8D^3(3x^2-5x+2)-0}{64}$

$= \frac{8\times 0}{64} = 0$ $\quad$ [∴ We stop till $\left(\frac{D^3-2D}{4}\right)^2$]

$\therefore P.I = \frac{1}{4}\left[1+\left(\frac{D^3-2D}{4}\right)+\left(\frac{D^3-2D}{4}\right)^2\right](3x^2-5x+2)$

$= \frac{1}{4}\left[3x^2-5x+2-\frac{\cancel{6x+10}}{4}_{-12x}+\frac{3}{2}\right]$

$= \frac{1}{4}\left[3x^2-5x+2\,\cancel{-6x}\,\frac{+10}{4}\right]$

• General sol :-

$\cancel{y=c_1e^{2x}+e^{x}(c_2\cos x+c_3\sin x)+\frac{1}{4}\left[3x^2-5x+2-\frac{6x+8}{4}\right]}$

$y = c_1e^{-2x}+e^{x}(c_2\cos x+c_3\sin x)+\frac{1}{4}\left[3x^2-5x+2-\frac{12x+10}{4}+\frac{3}{2}\right]$

$+\frac{1}{4}\left[3x^2-2x+1\right]$ (After simplifying)