Q8. If $z = f(x, y)$ where $x = e^{u} + e^{-v}$, $y = e^{-u} + e^{v}$.
Show that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$

Soln:- Multichain :-

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & u, v & \end{tikzcd} \qquad \begin{bmatrix} x = e^{u} + e^{-v} \\ y = e^{-u} + e^{v} \end{bmatrix}$$

1. $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(e^{u}\right) + \frac{\partial z}{\partial y}\left(-e^{-u}\right) \quad \text{--- (1)}$$

2. $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^{-v}\right) + \frac{\partial z}{\partial y}\left(e^{v}\right) \quad \text{--- (2)}$$

$\therefore$ (1) $-$ (2) $\Rightarrow$ $LHS = \dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$

$$= \frac{\partial z}{\partial x}\left(e^{u}\right) + \frac{\partial z}{\partial y}\left(-e^{-u}\right) + \frac{\partial z}{\partial x}\left(e^{-v}\right) - \frac{\partial z}{\partial y}\left(e^{v}\right)$$

$$= \frac{\partial z}{\partial x}\left(e^{u} + e^{-v}\right) - \frac{\partial z}{\partial y}\left(e^{-u} + e^{v}\right)$$

But $x = e^{u} + e^{-v}$ & $y = e^{-u} + e^{v}$

$$= \frac{\partial z}{\partial x} \cdot x - \frac{\partial z}{\partial y} \cdot y$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

$= RHS$

Hence proved

Q9. If $z = f(x,y)$ where $\left\{ \begin{array}{l} x = u\cos\alpha - v\sin\alpha \\ y = u\sin\alpha + v\cos\alpha \end{array} \right\}$

Prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$

Sol^n:- Multi chain :-

$$\begin{array}{ccc} & z & \\ x & & y \\ & u, v & \end{array} \qquad \left[\begin{array}{l} x = u\cos\alpha - v\sin\alpha \\ y = u\sin\alpha + v\cos\alpha \end{array} \right]$$

1. For $\dfrac{\partial^2 z}{\partial u^2}$:- i) $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$= \frac{\partial z}{\partial x}(\cos\alpha) + \frac{\partial z}{\partial y}(\sin\alpha) \quad \text{—— (1)}$$

~~For~~ Here, $\dfrac{\partial}{\partial u} = \dfrac{\partial}{\partial x}(\cos\alpha) + \dfrac{\partial}{\partial y}(\sin\alpha)$ —— (2)

$\therefore$ ii) $\dfrac{\partial^2 z}{\partial u^2} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial u}\right)$

Using (1) & (2), we get :-

$$= \left[\frac{\partial}{\partial x}(\cos\alpha) + \frac{\partial}{\partial y}(\sin\alpha)\right]\left[\frac{\partial z}{\partial x}(\cos\alpha) + \frac{\partial z}{\partial y}(\sin\alpha)\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 z}{\partial x^2}(\cos^2\alpha) + \frac{\partial^2 z}{\partial x\partial y}(\cos\alpha\sin\alpha) + \frac{\partial^2 z}{\partial y\partial x}(\sin\alpha\cos\alpha) + \frac{\partial^2 z}{\partial y^2}(\sin^2\alpha) \quad \text{—— (I)}$$

2. For $\dfrac{\partial^2 z}{\partial v^2}$:- i) $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}$

$$= \frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}(\cos\alpha) \quad \text{—— (3)}$$

Here, $\dfrac{\partial}{\partial v} = \dfrac{\partial}{\partial x}(-\sin\alpha) + \dfrac{\partial}{\partial y}(\cos\alpha)$ —— (4)

$\therefore$ ii) $\dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial z}{\partial v}\right)$

Using (3) & (4), we get :-

$$= \left[\frac{\partial}{\partial x}(-\sin\alpha) + \frac{\partial}{\partial y}(\cos\alpha)\right]\left[\frac{\partial z}{\partial x}(-\sin\alpha) + \frac{\partial z}{\partial y}(\cos\alpha)\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial x^2}(\sin^2\alpha) + \frac{\partial^2 z}{\partial x\partial y}(-\sin\alpha\cos\alpha) + \frac{\partial^2 z}{\partial y\partial x}(-\cos\alpha\sin\alpha) + \frac{\partial^2 z}{\partial y^2}(\cos^2\alpha) \quad \text{—— (II)}$$

$\therefore$ (I) + (II) $\Rightarrow$ RHS $= \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$

$$= \frac{\partial^2 z}{\partial x^2}(\cos^2\alpha) + \frac{\partial^2 z}{\partial x \partial y}(\cos\alpha \sin\alpha) + \frac{\partial^2 z}{\partial y \partial x}(\sin\alpha\cos\alpha) + \frac{\partial^2 z}{\partial y^2}(\sin^2\alpha)$$

$$+ \frac{\partial^2 z}{\partial x^2}(\sin^2\alpha) - \frac{\partial^2 z}{\partial x \partial y}(\cos\alpha \sin\alpha) - \frac{\partial^2 z}{\partial y \partial x}(\sin\alpha\cos\alpha) + \frac{\partial^2 z}{\partial y^2}(\cos^2\alpha)$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\cos^2\alpha + \sin^2\alpha\right) + \frac{\partial^2 z}{\partial y^2}\left(\sin^2\alpha + \cos^2\alpha\right)$$

Here, $\sin^2\alpha + \cos^2\alpha = 1$

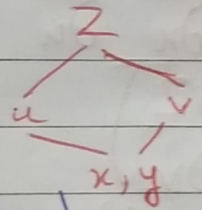
$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

$= \cancel{\text{RHS}}$ LHS

Hence proved

(*) Q10 Transform the eqⁿ $z_{xx}+2z_{xy}+z_{yy}=0$ by changing independent variables using $u=x-y$ & $v=x+y$.

Solⁿ:- Multichain :-



$$\begin{bmatrix} u = x-y \\ v = x+y \end{bmatrix}$$

1. For $z_{xx}\left(\frac{\partial^2 z}{\partial x^2}\right)$:-

(i) $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1) \quad \text{—— (1)}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Here, $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad \text{—— (2)}$$

(ii) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

Using (1) & (2) we get :-

$$= \left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]\left[\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$$

$$\Rightarrow \underbrace{\frac{\partial^2 z}{\partial x^2}}_{z_{xx}} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} \quad \text{——} \quad (I)$$

2. For z_{xy} $\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right]$:-

(i) $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$

$$= \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}(1)$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{——} \quad (3)$$

(ii) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

But from 1.(ii), we know;- $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$

$$= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$\Rightarrow \underbrace{\frac{\partial^2 z}{\partial x \partial y}}_{z_{xy}} = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} \quad \text{——} \quad (II)$$

3. For z_{yy} $\left(\frac{\partial^2 z}{\partial y^2}\right)$:-

(i) $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$

$$= \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}(1)$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{[From 2(i)]}$$

(ii) $\cancel{\frac{\partial^2 z}{\partial y^2}}$ Here, $\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ —— (4)

$\therefore \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$= \left[-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]\left[-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$ (Using (2) & (4))

$\Rightarrow \underbrace{\frac{\partial^2 z}{\partial y^2}}_{\to z_{yy}} = \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$ —— III

$\therefore$ I $+ 2$ II $+$ III $= 0$ [A/c, $z_{xx} + 2z_{xy} + z_{yy}$]

$\Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} + 2\left[-\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}\right]$

$+ \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2} = 0$

// $\to$ cancels each other

~~$\Rightarrow 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} = 0$~~

$\Rightarrow 4\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} = 0$

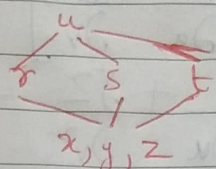
$\Rightarrow 2z_{vv} + z_{uv} - z_{vu} = 0$

$\therefore$ The transformed eqn $\Rightarrow 2z_{vv} + z_{uv} - z_{vu} = 0$

Q11. $u = f(r, s, t)$ where $\left\{ r = \frac{x}{y},\ s = \frac{y}{z},\ t = \frac{z}{x} \right\}$

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Sol^n:- Multichain :-



1. For $\frac{\partial u}{\partial x}$:-

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) - \frac{\partial u}{\partial t}\left(\frac{z}{x^2}\right) \quad \text{--- (1)}$$

2. For $\frac{\partial u}{\partial y}$:-

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r}\left(\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) \quad \text{--- (2)}$$

3. For $\frac{\partial u}{\partial z}$:-

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s}\left(\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right) \quad \text{--- (3)}$$

Now,

$$LHS = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

Using (1), (2), (3), we get :-

$$\cancel{= x\left[\frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) - \frac{\partial u}{\partial s}\left(\frac{z}{x^2}\right)\right] + y\left[-\frac{\partial u}{\partial s}\left(\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)\right] + z\Big[}$$

$$= x\left[\frac{\partial u}{\partial r}\left(\frac{1}{y}\right) - \frac{\partial u}{\partial t}\left(\frac{z}{x^2}\right)\right] + y\left[-\frac{\partial u}{\partial r}\left(\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right)\right] + z\left[-\frac{\partial u}{\partial s}\left(\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)\right]$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{y}\right) - \frac{\partial u}{\partial t}\left(\frac{z}{x}\right) - \frac{\partial u}{\partial r}\left(\frac{x}{y}\right) + \frac{\partial u}{\partial s}\left(\frac{y}{z}\right) - \frac{\partial u}{\partial s}\left(\frac{y}{z}\right) + \frac{\partial u}{\partial t}\left(\frac{z}{x}\right)$$

Cancel each other out

$$= 0$$

$$= RHS$$

Hence proved