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Q6. Discuss the maxima & minima of the function $f(x,y) = x^3+y^3-3axy$

Sol^n:- $f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay$ & $f_y = 3y^2-3ax$

$f_x = 0$

$\Rightarrow 3x^2-3ay = 0$

$\Rightarrow x^2 = ay$

$\Rightarrow y = \frac{x^2}{a}$ —— ①

For $x=0$

$y = \frac{0^2}{a}$

$\Rightarrow y = 0$ } $(0,0)$

For $x=a$

$y = \frac{a^2}{a}$

$\Rightarrow y = a$ } (a,a)

$f_y = 0$

$\Rightarrow 3y^2-3ax = 0$

$\Rightarrow y^2 = ax$ —— ②

② $\Rightarrow \left(\frac{x^2}{a}\right)^2 = ax$

$\Rightarrow \frac{x^4}{a^2} = ax$

$\Rightarrow x^4 = a^3x$

$\Rightarrow x^4 - a^3x = 0$

$\Rightarrow x(x^3-a^3) = 0$ $\therefore x = 0, a$ (Use it in ①)

$x^3-a^3=0$
$\Rightarrow x^3 = a^3$
$\Rightarrow x = a$

$\therefore$ Stationary points are:- ~~(0,0), (a,a); (0,a), (a,0)~~

$(0,0)$ & (a,a)

Here,

$(0,a)$ & $(a,0)$ are not stationary points because it doesn't satisfy $f_x = 0$ and take combo of x & y when writing points (x,y)

$\therefore x = 0$ is put on $f_x = 0 \rightarrow y = 0$ $\therefore (0,0)$

$\therefore x = a$ " " " " $\rightarrow y = a$ $\therefore (a,a)$

For r, s, t:-

- $r = f_{xx} = \frac{d}{dx}(3x^2-3ay) \Rightarrow r = 6x$
- $s = f_{xy} = \frac{d}{dx}(3y^2-3ax) \Rightarrow s = -3a$
- $t = f_{yy} = \frac{d}{dy}(3y^2-3ax) \Rightarrow t = 6y$

Now;-

- For $(0,0)$:- $\bullet\ r=0$
  $\bullet\ s=-3a$
  $\bullet\ t=0$

$\therefore rt-s^2=(0)(0)-(-3a)^2=-9a^2$
$\therefore rt-s^2<0 \longrightarrow$ saddle point

- For (a,a):- $\bullet\ r=6a$
  $\bullet\ s=-3a$
  $\bullet\ t=6a$

$\therefore rt-s^2=(6a)(6a)-(-3a)^2=36a^2-9a^2=27a^2$
If $a>0$ $\therefore rt-s^2>0$ & $r>0 \longrightarrow$ Minima
If $a<0$ $\therefore rt-s^2>0$ & $r<0 \longrightarrow$ Maxima

Maximum / Minimum value $=f(a,a)$

$=(a)^3+(a)^3-3a(a)(a)$
$=2a^3-3a^3$
$=-a^3$

$a\left(\frac{a}{3}\right)-2x\left(\frac{a}{3}\right)-\left(\frac{a}{3}\right)^2=0$

$\Rightarrow \frac{a^2}{3}-\frac{2ax}{3}-\frac{a^2}{9}$

$\Rightarrow \frac{3a^2-6ax-a^2}{9}=0$

$\Rightarrow 2a^2-6ax=0$

$\Rightarrow a^2-3ax=0$

$\Rightarrow a(a-3x)=0 \quad \therefore x=\frac{a}{3}$

$ax-x^2-2x\left(\frac{a}{3}\right)=0$

$\Rightarrow 3ax-3x^2-2ax=0$

$\Rightarrow ax-3x^2=0$

$\Rightarrow x(a-3x)=0$

$\rightarrow a-3x=0$
$\Rightarrow 3x=a$

$\therefore x=0,\ x=\frac{a}{3}$

Q7. Find the maximum and minimum value of $xy(a-x-y)$

Solⁿ:- $f(x,y) = xy(a-x-y) = axy - x^2y - xy^2$

$f_x = ay - 2xy - y^2$ & $f_y = ax - x^2 - 2xy$

$f_x = 0$	$f_y = 0$
$\Rightarrow ay - 2xy - y^2 = 0$	$\Rightarrow ax - x^2 - 2xy = 0$
$\Rightarrow y(a-2x-y) = 0$	$\Rightarrow x(a-x-2y) = 0$
$\therefore y = 0$	$\therefore x = 0$
$a - 2x - y = 0$ —— ①	$a - x - 2y = 0$ —— ②

① $\cancel{+}$ ② $\times 2 \Rightarrow a - \underline{2x} - y \cancel{+} 2a + \underline{2x} + 4y = 0$

$\Rightarrow -a + 3y = 0$

$\Rightarrow y = \frac{a}{3}$ $\quad \therefore y = \frac{a}{3}$

~~Put $y = 0$ in ① $\Rightarrow a - 2x - 0 = 0$ $\Rightarrow a - 2x = 0$ $\Rightarrow x = a$~~ } Cannot do this as we have $x=0$ & $y=0$ (in $f_y=0$) (in $f_x=0$)

① Put $y = \frac{a}{3}$ in ① $\Rightarrow a - 2x - \left(\frac{a}{3}\right) = 0$

$\left(\frac{a}{3}, \frac{a}{3}\right)$ $\quad \Rightarrow 2x = \frac{3a - a}{3} = \frac{2a}{3}$

$\Rightarrow x = \frac{a}{3}$ $\quad \therefore x = \frac{a}{3}$

$\therefore y = 0, \frac{a}{3}$ & $x = 0, \frac{a}{3}$

② Put $x = 0$ in ① $\Rightarrow a - 2(0) - y = 0$

$(0, a)$ $\quad \Rightarrow y = a$

③ Put $y = 0$ in ② $\Rightarrow a - x - 2(0) = 0$

$(a, 0)$ $\quad \Rightarrow x = a$

④ $\because y = 0$ & $x = 0$ $\quad \therefore (0, 0)$

Stationary points are :- $(0,0)$, $(a,0)$, $(0,a)$, $\left(\frac{a}{3},\frac{a}{3}\right)$

- $r = f_{xx} = \frac{d}{dx}\left(ay-2xy-y^2\right) \Rightarrow r = -2y$
- $s = f_{xy} = \frac{d}{dx}\left(ax-x^2-2xy\right) \Rightarrow s = a-2x-2y$
- $t = f_{yy} = \frac{d}{dy}\left(ax-x^2-2xy\right) \Rightarrow t = -2x$

- For $(0,0)$:- $r = -2(0) = 0$, $s = a-2(0)-2(0) = a$, $t = -2(0) = 0$

$\therefore rt-s^2 = (0)(0)-a^2 = -a^2$

$\because rt-s^2 < 0$ ~~& r~~ $\rightarrow$ Saddle point

- For $(a,0)$:- $r = -2(a) = -2a$, $s = a-2(a)-2(0) = -a$, $t = -2(0) = 0$

$\therefore rt-s^2 = (-2a)(0)-(-a)^2 = -a^2$

$\because rt-s^2 < 0 \rightarrow$ Saddle point

- For $(0,a)$:- $r = -2(0) = 0$, $s = a-2(0)-2(a) = -a$, $t = -2(a) = -2a$

$\therefore rt-s^2 = (0)(-2a)-(-a)^2 = -a^2$

$\because rt-s^2 < 0 \rightarrow$ Saddle point

- For $\left(\frac{a}{3},\frac{a}{3}\right)$:-
  - $r = -2\left(\frac{a}{3}\right) = \frac{-2a}{3}$
  - $s = a-2\left(\frac{a}{3}\right)-2\left(\frac{a}{3}\right) = \frac{-a}{3}$
  - $t = -2\left(\frac{a}{3}\right) = \frac{-2a}{3}$

$\therefore rt-s^2 = \left(\frac{-2a}{3}\right)\left(\frac{-2a}{3}\right)-\left(\frac{-a}{3}\right)^2$

$= \frac{4a}{9} - \frac{a^2}{9}$

$\Rightarrow rt-s^2 = \frac{3a^2}{9} = \frac{a^2}{3}$

If $a>0$ $\because rt-s^2>0$ & $r<0 \rightarrow$ Maxima point

If $a<0$ $\because rt-s^2>0$ & $r>0 \rightarrow$ Minima point

$\therefore$ Maximum/Minimum value $= f\left(a/3, a/3\right)$

$= a\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)-\left(\frac{a}{3}\right)^2\left(\frac{a}{3}\right)-\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)^2$

$= \frac{a^3}{27}$

Q8. Examine the maxima/minima of following function :-

$$f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$$

Sol:- $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}$

$f_x = 2x + y - \frac{1}{x^2}$ & $f_y = x + 2y - \frac{1}{y^2}$

$f_x = 0$

$\Rightarrow 2x + y - \frac{1}{x^2} = 0$

$\Rightarrow 2x^3 + x^2 y - 1 = 0$ —— (1)

$f_y = 0$

$\Rightarrow x + 2y - \frac{1}{y^2} = 0$

$\Rightarrow xy^2 + 2y^3 - 1 = 0$ —— (2)

(1) − (2) $\Rightarrow 2x^3 + x^2 y - 2y^3 - xy^2 = 0$

$\Rightarrow 2(x^3 - y^3) + xy(x - y) = 0$

$\Rightarrow 2(x-y)(x^2 + xy + y^2) + xy(x-y) = 0$

$\Rightarrow (x-y)\left[2x^2 + 2xy + 2y^2 + xy\right] = 0$

$\Rightarrow (x-y)(2x^2 + 3xy + 2y^2) = 0$

$\therefore x - y = 0$ & $2x^2 + 3xy + 2y^2 = 0$

$\Rightarrow x = y$

Here, $a = 2$, $b = 3$, $c = 2$

$\therefore D = b^2 - 4ac$

$= (3)^2 - 4(2)(2)$

$\Rightarrow D = -7 < 0$ $\therefore$ No real roots

$\therefore x = y$ —— (3)

(3) in (1) or (2) $\Rightarrow 2y^3 + y^2 y - 1 = 0$

$\Rightarrow 3y^3 = 1$

$\Rightarrow y^3 = \frac{1}{3}$ $\Rightarrow y = \left(\frac{1}{3}\right)^{1/3}$ $\therefore x = \left(\frac{1}{3}\right)^{1/3}$

$\therefore$ Stationary point is $\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

- $r = f_{xx} = \frac{d}{dx}\left(2x + y - \frac{1}{x^2}\right) \Rightarrow r = 2 + \frac{2}{x^3}$
- $s = f_{xy} = \frac{d}{dx}\left(x + 2y - \frac{1}{y^2}\right) \Rightarrow s = 1$
- $t = f_{yy} = \frac{d}{dy}\left(x + 2y - \frac{1}{y^2}\right) \Rightarrow t = 2 + \frac{2}{y^3}$

- For $\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$: $\cdot\ r = 2 + \frac{2}{(1/3)^{1/3 \times 3}} = 2 + 2(3) = 8$

$\cdot\ s = 1$

$\cdot\ t = 2 + \frac{2}{(1/3)^{\frac{1}{3} \times 3}} = 2 + 2(3) = 8$

$\therefore rt - s^2 = (8)(8) - 1^2 = 63$

$\because rt - s^2 > 0$ & $r > 0 \longrightarrow$ Minima

$\therefore$ Minimum value $= f\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

$= \left(\frac{1}{3}\right)^{2/3} + \left(\frac{1}{3}\right)^{1/3 + 1/3} + \left(\frac{1}{3}\right)^{2/3} + \frac{1}{(1/3)^{1/3}} + \frac{1}{(1/3)^{1/3}}$

$= 3 \cdot \left(\frac{1}{3}\right)^{2/3} + 2(3)^{1/3}$

$= 3\left(\frac{1}{3}\right)^{2/3} + 2\left(\frac{1}{3}\right)^{-1/3}$

$= 3\left(\frac{1}{3}\right)^{2/3} + 2\left(\frac{1}{3}\right)^{-1+\frac{2}{3}}$

$= 3\left(\frac{1}{3}\right)^{2/3} + 2\left(\frac{1}{3}\right)^{-1}\left(\frac{1}{3}\right)^{2/3}$

$= \left(\frac{1}{3}\right)^{2/3}\left[3 + \frac{2}{(3)^{-1}}\right]$

$= \left(\frac{1}{3}\right)^{2/3}(3 + 2 \times 3) = 9 \times 3^{-2/3} = 3^2 \times 3^{-2/3}$

$= 3^{2 - 2/3}$

$= 3^{4/3}$