Method of Variation of parameters

Step 1 :- $CF = C_1 y_1 + C_2 y_2$ $\quad$ $y_1 \rightarrow$ coefficient of C_1

$\boxed{y_1 =}$ & $\boxed{y_2 =}$ $\quad$ $y_2 \rightarrow$ coefficient of C_2

Step 2 :- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Step 3 :- $u = -\int \frac{y_2 X}{W} dx$ & $v = \int \frac{y_1 X}{W} dx$

where $X \rightarrow$ RHS of ~~ff~~ function

Step 4 :- $PI = u y_1 + v y_2$

Q1. Solve $\frac{d^2y}{dx^2} + y = \sec x \tan x$

Solⁿ:- $(D^2+1)y = \sec x \tan x$ —— ①

A.E:- $m^2+1=0 \Rightarrow m = \pm\sqrt{-1} = \pm i$ $\therefore m = 0+i, 0-i$

$\alpha = 0$ & $\beta = 1$

C.F:- $CF = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$= e^0(C_1 \cos 1x + C_2 \sin 1x)$

$\Rightarrow CF = C_1 \cos x + C_2 \sin x$

Here,

y_1 = coefficient of C_1 & y_2 = coefficient of C_2

$\Rightarrow y_1 = \cos x$ $\Rightarrow y_2 = \sin x$

$\therefore y_1' = -\sin x$ $y_2' = \cos x$

In eqⁿ ①, $X = \sec x \tan x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = \sin^2 x + \cos^2 x$$

$\Rightarrow W = 1$

$$u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{(\sin x)(\sec x \tan x)}{1}\,dx$$

$$= -\int \sin x \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot dx$$

$$= -\int \frac{\sin^2 x}{\cos^2 x}\,dx$$

$$= -\int \tan^2 x\,dx$$

[We know, $\tan^2 x + 1 = \sec^2 x$ $\Rightarrow \tan^2 x = \sec^2 x - 1$]

$$= -\int (\sec^2 x - 1)\,dx$$

[We know, $\frac{d}{dx}(\tan x) = \sec^2 x$ $\therefore \int \sec^2 x\,dx = \tan x$]

$$= -\int \sec^2 x\,dx + \int dx$$

$\Rightarrow u = -\tan x + x$

$$v = \int \frac{y_1 X}{W} dx = \int \frac{(\cos x)(\sec x \tan x)}{1} dx$$

$$= \int \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} \cdot \tan x \, dx$$

$$= \int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

$$\left[\text{Let } t = \cos x \quad \frac{dt}{dx} = -\sin x \Rightarrow dx = \frac{-dt}{\sin x} \right]$$

$$= \int \frac{\cancel{(\sin x)}}{t} \cdot \frac{-dt}{\cancel{(\sin x)}}$$

$$= -\int \frac{1}{t} dt$$

$$\left[\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \therefore \int \frac{1}{x} dx = \ln(x) \right]$$

$$\Rightarrow v = -\ln(t)$$

But $t = \cos x$

$$\therefore v = -\ln(\cos x)$$

We know,

$$a \log b = \log b^a \quad \therefore -\log b = \log b^{-1} = \log\left(\frac{1}{b}\right)$$

$$\therefore v = \log \cos^{-1} x$$

$$\Rightarrow v = \log\left(\frac{1}{\cos x}\right) = \ln(\sec x).$$

P.I :- $P.I = u y_1 + v y_2$

$$= (x - \tan x)(\cos x) + \ln\left(\frac{1}{\cos x}\right) \cdot (\sin x)$$

$$\Rightarrow P.I = x \cos x - \sin x + \ln\left(\sec x\right) \sin x$$

$$\therefore y = C.F + P.I$$

$$= C_1 \cos x + C_2 \sin x + x \cos x - \sin x (1 - \ln(\sec x))$$

(*) Q2. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

Sol^n:- $(D^2+3D+2)y = e^{e^x}$ —— (i)

A.E:- $m^2+3m+2=0$

$\Rightarrow (m+2)(m+1)=0$ $\therefore m=-2,-1$

CF:- $CF = C_1e^{m_1x}+C_2e^{m_2x}$

$\Rightarrow CF = C_1e^{-x}+C_2e^{-2x}$

Here, $y_1 = e^{-x}$ & $y_2 = e^{-2x}$

$y_1' = e^{-x}\cdot\frac{d}{dx}(-x)$ $y_2' = e^{-2x}\cdot\frac{d}{dx}(-2x)$

$\Rightarrow y_1' = -e^{-x}$ $\Rightarrow y_2' = -2e^{-2x}$

From eq^n (1), $X = e^{e^x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = (e^{-x})(-2e^{-2x}) - (-e^{-x})(e^{-2x}) = -2e^{-x-2x}+e^{-x-2x}$$

$\Rightarrow W = -e^{-3x}$

$$u = -\int \frac{y_2X}{W}dx = -\int \frac{(e^{-2x})(e^{e^x})}{(-e^{-3x})}dx = \int e^x e^{e^x}dx$$

Let $t = e^x$

$\frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x}$

$\Rightarrow u = \int \cancel{e^x}\cdot e^t\cdot\frac{dt}{\cancel{e^x}} = \int e^t dt$ [We know, $\frac{d}{dx}(e^x) = e^x$ $\therefore \int e^x dx = e^x$]

$= e^t$

But $t = e^x$

$\Rightarrow u = e^{e^x}$

$$v = \int \frac{y_1 X}{W}\, dx = \int \frac{(e^{-x})(e^{e^x})}{(-e^{-3x})}\, dx$$

$$= -\int e^{2x} \cdot e^{e^x}$$

Let $t = e^x$ $\therefore \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x}$

$$\Rightarrow v = -\int e^{2x} \cdot e^t \cdot \frac{dt}{e^x} = -\int e^x e^t\, dt = -\int \underset{u}{t}\, \underset{v}{e^t}\, dt$$

(Apply u·v rule)

$$= -\left[t \int e^t dt - e^t \int t\, dt \right]$$

In u v rule for $\int$:- 1^{st} term $\rightarrow$ +ve $= u \int v\, dt$
2^{nd} term $\rightarrow$ -ve $= v \int u\, dt$

$$\therefore v = -\left[t \cdot e^t - e^t (1) \right]$$

$$\Rightarrow v = e^t - te^t = e^t(1-t)$$

But $t = e^x$

$$\Rightarrow v = e^{e^x}(1 - e^x)$$

<u>P.I</u> :- $P.I = uy_1 + vy_2$

$$= (e^{e^x})(e^{-x}) + e^{e^x}(1-e^x)(e^{-2x})$$

Let $e^x = q$

$$\Rightarrow P.I = e^q e^{-q} + e^q(1-q)(e^{-2q})$$

$$\cancel{= e^q e^{-q} + e^q - e^q q - q e^q}$$

$$= e^q q^{-q} + e^q e^{-2q} - q e^q e^{-2q}$$

$$\Rightarrow PI = e^{-2x} \cdot e^{e^x}$$

$$y = CF + PI$$

$$= C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} e^{e^x}$$