Q1. Find the equation of the evolute of the curve $y^2=4ax$

Solⁿ:- Parametric form:-

$$x = at^2 \quad \& \quad y = 2at$$

$$\therefore \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

For $\frac{dy}{dx}$:- $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$$= \cancel{2a} \cdot \frac{1}{\cancel{2a}t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{t} \qquad \left(y_1 = \frac{1}{t}\right)$$

For $\frac{d^2y}{dx^2}$:- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

In terms / w.r.t t :-

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{1}{2at}$$

$$= -\frac{1}{t^2} \cdot \left(\frac{1}{2at}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{2at^3} \qquad \left(y_2 = \frac{-1}{2at^3}\right)$$

We know :-

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

For $\bar{x}$: $\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$

$= \cancel{2}at^2 - \left[\left(\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)\right] \times \frac{1}{\left(-\frac{1}{2at^3}\right)}$ $[x = at^2]$

$= \cancel{2}at^2 - \frac{t^2+1}{\cancel{t^3}} \times (-2a\cancel{t^3})$

$\Rightarrow \bar{x} = 3at^2 + 2a$

For $\bar{y}$: $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$= 2at + \left[1+\frac{1}{t^2}\right] \times \frac{1}{\left(-\frac{1}{2at^3}\right)}$ $[y = 2at]$

$= 2at + \frac{t^2+1}{\cancel{t^2}} \times (-2at^{\cancel{3}})$

$\Rightarrow \bar{y} = \cancel{2at} - 2at^3 - \cancel{2at}$

$\Rightarrow \bar{y} = -2at^3$

To eliminate 't':-

$\bar{x} = 3at^2 + 2a$ $\bar{y} = -2at^3$

$\Rightarrow t = \left(\frac{\bar{x}-2a}{3a}\right)^{1/2}$ $\Rightarrow t = \left(\frac{-\bar{y}}{2a}\right)^{1/3}$

$\therefore \left(\frac{\bar{x}-2a}{3a}\right)^{1/2} = \left(\frac{-\bar{y}}{2a}\right)^{1/3}$

$\Rightarrow \left(\frac{\bar{x}-2a}{3a}\right)^{3} = \left(\frac{\bar{y}}{2a}\right)^{2}$ [Exchange powers]

$\Rightarrow \frac{(\bar{x}-2a)^3}{27a^{\cancel{3}}} = \frac{(\bar{y})^2}{4a^2}$

$\Rightarrow 4(\bar{x}-2a)^3 = 27a\bar{y}^2$

$\therefore$ Evolute is $4(x-2a)^3 = 27ay^2$

$\hookrightarrow$ locus of centre of curvature

Q2. Find the equation of the evolute of the curve $x^2 = 4ay$

Sol^n:- Parametric form:-

$$x = 2at \quad \& \quad y = at^2$$

$$\therefore \frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$$

For $\frac{dy}{dx}$:- $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$$= \cancel{2a}t \times \frac{1}{\cancel{2a}}$$

$$\Rightarrow \frac{dy}{dx} = t$$

For $\frac{d^2y}{dx^2}$:- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$= \frac{d}{dt}(t) \times \left(\frac{1}{2a}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2a}$$

We know :-

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

For $\bar{x}$:

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= 2at - \frac{t(1+t^2)}{\left(\frac{1}{2a}\right)}$$

$$= 2at - (t+t^3)(2a)$$

$$= 2at - 2at - 2at^3$$

$$\Rightarrow \bar{x} = -2at^3$$

For $\bar{y}$:

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= at^2 + \frac{(1+t^2)}{(1/2a)}$$

$$= at^2 + 2a + 2at^2$$

$$\Rightarrow \bar{y} = 3at^2 + 2a$$

Eliminate 't':

$$\bar{x} = -2at^3$$

$$\Rightarrow t^3 = \frac{\bar{x}}{-2a}$$

$$\Rightarrow t = \left(\frac{-\bar{x}}{2a}\right)^{1/3}$$

$$\bar{y} = 3at^2 + 2a$$

$$\Rightarrow t^2 = \frac{\bar{y}-2a}{3a}$$

$$\Rightarrow t = \left(\frac{\bar{y}-2a}{3a}\right)^{1/2}$$

$$\therefore \left(\frac{-\bar{x}}{2a}\right)^{1/3} = \left(\frac{\bar{y}-2a}{3a}\right)^{1/2}$$

$$\Rightarrow \left(\frac{-\bar{x}}{2a}\right)^2 = \left(\frac{\bar{y}-2a}{3a}\right)^3$$

$$\Rightarrow \cancel{27a}\ \frac{\bar{x}^2}{4a^2} = \frac{(\bar{y}-2a)^3}{27a^3}$$

$$\Rightarrow 27a\bar{x}^2 = 4(\bar{y}-2a)^3$$

$\therefore$ Evolute is $4(y-2a)^3 = 27ax^2$

Q3. Find the equation of evolute of curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol:- Parametric form of ellipse :-

$$x = a\cos\theta \qquad\qquad y = b\sin\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta \qquad\qquad \Rightarrow \frac{dy}{d\theta} = b\cos\theta$$

For $\frac{dy}{dx}$:-
$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$
$$= b\cos\theta \cdot \frac{1}{-a\sin\theta}$$
$$\Rightarrow y_1 = -\frac{b}{a}\cot\theta$$

For $\frac{d^2y}{dx^2}$:-
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$
$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\left(\frac{1}{-a\sin\theta}\right)$$
$$= +\frac{b}{a}\left(+\operatorname{cosec}^2\theta\right)\cdot\frac{1}{-a\sin\theta}$$
$$\Rightarrow y_2 = \frac{-b}{a^2\sin^3\theta}$$

We know :-

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$
$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

r $\bar{x}$:- $\bar{x} = x \mp \dfrac{y_1(1+y_1^2)}{y_2}$

$$= a\cos\theta \mp \frac{\left(-\frac{b}{a}\cot\theta\right)\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\left(\frac{-b}{a^2\sin^3\theta}\right)}$$

$$= a\cos\theta \mp \frac{\frac{b}{a}\,\frac{\cos\theta}{\sin\theta}\left(1+\frac{b^2}{a^2}\,\frac{\cos^2\theta}{\sin^2\theta}\right)}{\frac{b}{a^2\sin^3\theta}}$$

$$= a\cos\theta \mp \frac{b}{a}\,\frac{\cos\theta}{\sin\theta}\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right) \times \frac{a^2\sin^3\theta}{b}$$

$$\Rightarrow \bar{x} = a\cos\theta \mp \frac{\cos\theta}{a}\left(a^2\sin^2\theta + b^2\cos^2\theta\right)$$

$$= \frac{a^2\cos\theta - a^2\sin^2\theta\cos\theta \mp b^2\cos^3\theta}{a}$$

$$\cancel{= a^2\cos\theta\left(\cos\theta - \sin^2\theta\right)}$$

$$= \frac{a^2\cos\theta\left(1-\sin^2\theta\right) \mp b^3\cos^3\theta}{a}$$

$$= \frac{a^2\cos\theta\left(\cos^2\theta\right) \mp b^3\cos^3\theta}{a}$$

$$\Rightarrow \bar{x} = \frac{a^3\cos^3\theta \mp b^3\cos^3\theta}{a}$$

In terms of $\cos^3\theta$:-

$$a\bar{x} = \cos^3\theta\left(a^2 \mp b^2\right)$$

$$\Rightarrow \cos^3\theta = \cancel{\frac{a\bar{x}}{a^2+b^2}} \quad \frac{a\bar{x}}{a^2-b^2}$$

$$\Rightarrow \cos\theta = \left(\frac{a\bar{x}}{a^2 \mp b^2}\right)^{1/3}$$

For $\bar{y}$:- $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$

$$= b\sin\theta + \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2\sin^3\theta}}$$

$$= b\sin\theta + \left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right) \times \frac{a^2\sin^3\theta}{-b}$$

$$= b\sin\theta + \left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right) \times \frac{a^2\sin^3\theta}{-b}$$

$$= \frac{-b^2\sin\theta + a^2\sin^3\theta + b^2\cos^2\theta\cdot\sin\theta}{-b}$$

$$= \frac{-b^2\sin\theta\left(1-\cos^2\theta\right) + a^2\sin^3\theta}{-b}$$

$$\Rightarrow \bar{y} = \frac{-b^2\sin\theta\left(\sin^2\theta\right) + a^2\sin^3\theta}{-b} = \frac{-b^2\sin^3\theta + a^2\sin^3\theta}{-b}$$

In terms of $\sin^3\theta$:-

$$-b\bar{y} = \sin^3\theta\left(a^2-b^2\right)$$

$$\Rightarrow \sin^3\theta = \frac{-b\bar{y}}{a^2-b^2}$$

$$\Rightarrow \sin\theta = \left(\frac{-b\bar{y}}{a^2-b^2}\right)^{1/3}$$

We have to eliminate θ from:-

$$\cos\theta = \left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}$$

$$\& \sin\theta = \left(\frac{-b\bar{y}}{a^2-b^2}\right)^{1/3}$$

We apply $\boxed{\sin^2\theta + \cos^2\theta = 1}$

$\because \sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \left[\left(\frac{-b\bar{y}}{a^2-b^2}\right)^{1/3}\right]^2 + \left[\left(\frac{a\bar{x}}{a^2-b^2}\right)^{1/3}\right]^2 = 1$$

$$\Rightarrow \left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(\frac{-b\bar{y}}{a^2-b^2}\right)^{2/3} = 1$$

$$\Rightarrow (a\bar{x})^{2/3} + (-b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$$

$\therefore$ Evolute is $(a\bar{x})^{2/3} + (-b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$

Q. Find radius of curvature on $y = e^x$ at the point where the curve cuts the y-axis. {MCQ (5)}

Sol:- $y = e^x$

The curve cuts the y-axis $\Rightarrow x = 0$

At point $(0, y)$ we find radius of curvature

$$y_1 = \frac{d}{dx}(e^x) = e^x = e^0 = 1$$

$$y_2 = \frac{d^2y}{dx^2} = e^x = e^0 = 1$$

Now,

Radius of curvature $(\rho) = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$\therefore \rho = \frac{(1+1^2)^{3/2}}{1}$$

$$= 2^{3/2}$$

$$\therefore \boxed{\rho = 2\sqrt{2}}$$