# RCF 2 Evaluation and Consistency\* $\varepsilon \& \mathcal{C} * \pi_O \mathbf{R} * \pi_O^{\bullet} \mathbf{R}$

Michael Pfender<sup>†</sup>

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Abstract: We construct here an iterative evaluation of all (coded) PR maps: progress of this iteration can be measured by descending complexity, within Ordinal  $O := \mathbb{N}[\omega]$ , of polynomials in one indeterminate, called " $\omega$ ". As (well) order on this Ordinal we choose the lexicographical one. Non-infinit descent of such iterations is added as a mild additional axiom schema ( $\pi_O$ ) to Theory  $\mathbf{PR_A} = \mathbf{PR} + (abstr)$  of Primitive Recursion with predicate abstraction, out of foregoing part RFC 1. This then gives (correct) on-termination of iterative evaluation of argumented deduction trees as well: for theories  $\mathbf{PR_A}$  and  $\pi_O\mathbf{R} = \mathbf{PR_A} + (\pi_O)$ . By means of this constructive evaluation the Main Theorem is proved, on Termination-conditioned (Inner) Soundness for Theories  $\pi_O\mathbf{R}$ , O extending  $\mathbb{N}[\omega]$ . As a consequence we get in fact Self-Consistency for theories  $\pi_O\mathbf{R}$ , namely  $\pi_O\mathbf{R}$ -derivability of  $\pi_O\mathbf{R}$ 's own free-variable Consistency formula

 $\operatorname{Con}_{\pi_O \mathbf{R}} = \operatorname{Con}_{\pi_O \mathbf{R}}(k) =_{\operatorname{def}} \neg \operatorname{Prov}_{\pi_O \mathbf{R}}(k, \lceil \operatorname{false} \rceil) : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free.}$ Here PR predicate  $\operatorname{Prov}_{\mathbf{T}}(k, u)$  says, for an arithmetical theory  $\mathbf{T}$ : number  $k \in \mathbb{N}$  is a  $\mathbf{T}$ -Proof code proving internally  $\mathbf{T}$ -formula code u, arithmetised  $\operatorname{Proof}$  in Gödel's sense.

As to expect from classical setting, Self-Consistency of  $\pi_O \mathbf{R}$  gives (unconditioned) Objective Soundness. Eventually we show Termination-Conditioned Soundness "already" for  $\mathbf{PR_A}$ . But it turns out that present derivation of Self-Consistency, and already that of Consistency formula of  $\mathbf{PR_A}$  from this conditioned Soundness "needs" schema  $(\tilde{\pi})$  of non-infinit descent in Ordinal  $\mathbb{N}[\omega]$ , which is presumably not derived by  $\mathbf{PR_A}$  itself.

<sup>&</sup>lt;sup>0</sup> Legend of LOGO:  $\varepsilon$  for Constructive evaluation,  $\mathcal{C}$  for Self-Consistency to be derived for suitable theories  $\pi_O \mathbf{R}$ ,  $\pi_O^{\bullet} \mathbf{R}$  strengthening in a "mild" way the (categorical) Free-Variables Theory  $\mathbf{PR_A}$  of Primitive Recursion with predicate abstraction

<sup>\*</sup>Consideration of implicational version  $(\pi_O^{\bullet})$  of Descent axiom added

<sup>&</sup>lt;sup>†</sup>TU Berlin, Mathematik, pfender@math.tu-berlin.de

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## 1 Summary

Gödel's first Incompleteness Theorem for *Principia Mathematica* and "verwandte Systeme", on which in particular is based the second one, on non-provability of **PM**'s own Consistency formula Con<sub>**PM**</sub>, exhibits a (closed) **PM** formula  $\varphi$  with property that

$$\mathbf{PM} \vdash [\varphi \iff \neg (\exists k \in \mathbb{N}) \mathit{Prov}_{\mathbf{PM}}(k, \lceil \varphi \rceil)], \text{ in words:}$$

Theory **PM** derives  $\varphi$  to be equivalent to its "own" coded, arithmetised non-Provability.

Since this equivalence needs already for its statement "full" formal, "not testable" quantification, the  $Consistency\ Provability$  issue is not settled for Free-Variables Primitive Recursive Arithmetic and its strengthenings – Theories  ${\bf T}$  which express (formalised, "internal") Consistency as free-variable formula

$$\operatorname{Con}_{\mathbf{T}} = \operatorname{Con}_{\mathbf{T}}(k) = \neg \operatorname{Prov}_{\mathbf{T}}(k, \lceil \operatorname{false} \rceil) : \mathbb{N} \to 2 :$$

"No  $k \in \mathbb{N}$  is a *Proof* code proving  $\lceil \text{false} \rceil$ ."

This is the point of depart for investigation of "suitable" strengthenings  $\pi_O \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O)$  of categorical Theory  $\mathbf{P} \mathbf{R}_{\mathbf{A}}$  of Primitive Recursion, enriched with predicate abstraction Objects  $\{A \mid \chi\} = \{a \in A \mid \chi(a)\}$ : Plausibel axiom schema  $(\pi_O)$ , more presisely: its contraposition  $\tilde{\pi}_O$ , states "weak" impossibility of infinite descending chains in any Ordinal O extending polynomial semiring  $\mathbb{N}[\omega]$ , with its canonical, lexicographical order.

Central Non-Infinite Descent Schema, Descent Schema for short:

We need an axiom-schema for expressing – in free variables – Finite descent (endo-driven) chains, descending in complexity value out of Ordinal  $O \succeq \mathbb{N}[\omega]$ , a schema called  $(\pi_O)$ , which gives the "name" to  $Descent^1$  Theory  $\pi_O \mathbf{R} = \mathbf{PR_A} + (\pi_O)$ : This theory is a pure strengthening of  $\mathbf{PR_A}$ , it has the same language.

Easier to interprete logically is  $(\pi_O)$ 's equivalent, Free-Variables contraposi-

 $<sup>^0\</sup>mathrm{extended}$  Poster Abstract "Arithmetical Consistency via Constructive evaluation", Conference celebrating Kurt Gödel's 100th birthday, Vienna april 28, 29, 2006

<sup>&</sup>lt;sup>1</sup>notion added 2 JAN 2009

tion, on "absurdity" of infinite descending chains, namely:

[The first four lines of the antecedent constitute (p, c) as (the data of) a  $CCI_O$ : of a Complexity Controlled Iteration, with (stepwise) descending order values in Ordinal O. Central **example:** General Recursive, ACCKERMANN type PR-code evaluation  $\varepsilon$  will be resolved into such a  $CCI_O$ ,  $O := \mathbb{N}[\omega] \subset \mathbb{N}$ .]

My **Thesis** then is that these theories  $\pi_O \mathbf{R}$ , weaker than **PM**, **set theories** and even Peano Arithmetic **PA** (when given its *quantified* form), <u>derive</u> their own internal (Free-Variable) Consistency formula  $\operatorname{Con}_{\pi_O \mathbf{R}}(k) : \mathbb{N} \to 2$ , see above.

Notions and Arguments for Self-Consistency of  $\pi_O \mathbf{R}$ : In order to obtain constructive Theories – candidates for self-Consistency – we introduce first, into fundamental Theory  $\mathbf{PR}$  of (categorical) Free-Variables Primitive Recursion, predicate abstraction of PR maps  $\chi = \chi(a) : A \to 2$  (A a finite power of NNO N), into defined Objects  $\{A \mid \chi\}$ , and then strengthen Theory  $\mathbf{PR_A}$  obtained this way, by a free-variables, (inferential) schema ( $\pi_O$ ) of "on"-terminating descent, into Theorie(s)  $\pi_O \mathbf{R}$ , on-terminating descent of Complexity Controlled Iterations (CCI<sub>O</sub>'s, see above), with (descending) complexity values in Ordinal  $O \succeq \mathbb{N}[\omega]$ .

Strengthened Theory  $\pi_O \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O)$ , with its language equal to that of  $\mathbf{P} \mathbf{R}_{\mathbf{A}}$ , is asserted to derive the (Free-Variable) formula  $\mathrm{Con}_{\pi_O \mathbf{R}}(k)$  which expresses internally: within  $\pi_O \mathbf{R}$  itself, Consistency of Theory  $\pi_O \mathbf{R}$ , see above.

**Proof** is by  $CCI_{\mathbb{N}[\omega]}$  (descent) property of a suitable, *atomic* PR evaluation step e applied to PR-map-code/argument pairs  $(u, x) \in PR_A \times \mathbb{X}$ .

[Here  $\mathbb{X} \subset \mathbb{N}$  denotes the *Universal Object* of all (codes of) *singletons* and (nested) *pairs* of natural numbers, enriched by a shymbol  $\underline{\perp}$  equally coded in  $\mathbb{N}$ , to designate *undefined values*, of *defined partially defined* PR maps. Objects A of  $\mathbf{PR_A}$ ,  $\pi_O \mathbf{R}$  admit a *natural embedding*  $A \subseteq \mathbb{X}$  into this this universal Object.]

Iteration  $\varepsilon$ , of step e, is in fact controlled by a syntactic complexity  $c_{PR}(u) \in \mathbb{N}[\omega]$ , descending with each application of e as long as minimum complexity  $0 = c_{PR}(\lceil \operatorname{id} \rceil)$  is not "yet" reached.

Strengthening of  $\mathbf{PR_A}$  by schema  $(\pi_O)$  – cf. its free-variables contraposition  $(\tilde{\pi}_O)$  above – into Theory  $\pi_O \mathbf{R} = \mathbf{PR} + (\pi_O)$ , is "just" to allow for a so to say sound, canonical evaluation "algorithm" for  $\pi_O \mathbf{R}$ :

On one hand it is proved straight forward that evaluation  $\varepsilon$  above has the expected recursive properties of an *evaluation*, this within (categorical, Free-Variables) Theory  $\mu \mathbf{R}$  of  $\mu$ -Recursion.

On the other hand,  $\pi_O \mathbf{R}$  has the same **Language** as  $\mathbf{PR_A}$ , so that this  $\varepsilon$  is a natural candidate for likewise – sound – evaluation of internal version of theory  $\pi_O \mathbf{R}$ , and for being totally defined in a suitable Free-Variables sense, technically: to on-terminate, this just by its property to be a Complexity Controlled Iteration, with order values in  $\mathbb{N}[\omega]$ .

In fact, by schema  $(\pi_O)$  itself  $(O \text{ extending } \mathbb{N}[\omega])$ ,  $\varepsilon$  preserves the **extra** equation instances inserted by internalisation of  $(\pi_O)$ .

**Dangerous bound:** is there a good reason that this evaluation is not a self-evaluation for Theory  $\pi_O \mathbf{R}$ ?

Answer:  $\varepsilon$  is – by definition – not PR: If you take the diagonal

$$diag(n) =_{def} \varepsilon(\operatorname{enum}_{PR}(n), \operatorname{cantor}_{\mathbb{X}}(n)) : \mathbb{N} \to \mathbb{N},$$

enum<sub>PR</sub> an internal PR count of all PR map codes, and cantor<sub>X</sub> :  $\mathbb{N} \stackrel{\cong}{\longrightarrow} \mathbb{X}$  "the" Cantor's count of  $\mathbb{X} \subset \mathbb{N}$ , then you get ACKERMANN's original diagonal function<sup>2</sup> which grows faster than any PR function: but  $\pi_O \mathbf{R}$  has only PR maps as its maps, it is a (pure) strengthening of  $\mathbf{PR}_{\mathbf{A}}$ .

On the other hand,  $\varepsilon$  is *intuitively* total, since, intuitively, complexity  $c e^m(u, x)$  "must" reach 0 in *finitely many e*-steps. The latter intuition can be, in free variables (!), expressed *formally* by  $\pi_O \mathbf{R}$ 's **schema**  $(\tilde{\pi}_O)$ : Free-Variables contraposition of  $(\pi_O)$ . Schema  $(\tilde{\pi}_O)$  says that a condition which implies *infinite descent* of such a chain (on all x), must be *false* (on all x), "absurd".

Complexity Controlled Iteration  $\varepsilon$  of e extends canonically into a Complexity Controlled evaluation  $\varepsilon_d$ , of argumented deduction trees,  $\varepsilon_d$  again defined by  $\mathrm{CCI}_{\mathbb{N}[\omega]}$ : this time by iteration of a *tree evaluation step*  $e_d$  suitably extending basic evaluation step e to argumented deduction trees.

Deduction-tree evaluation starts on trees of form  $dtree_k/x$ , obtained as follows from k and x: Call  $dtree_k$  the (first)  $deduction\ tree$  which (internally)  $proves\ k$  th internal equation u = k v of theory  $\pi_O \mathbf{R}$ , enumeration of proved equations being (lexicographically) by code of (first) Proof. This argument-free deduction tree  $dtree_k$  then is provided – node-wise top down from given  $x \in \mathbb{X}$  – with its  $spread\ down$  arguments in  $\mathbb{X}_{\square} = \text{def}\ \mathbb{X} \cup \{\square\} = \mathbb{X} \cup \{\square\} \subset \mathbb{N}$ ; (empty list  $\square = \square \subset \mathbb{N}$ ) refers to a not yet known argument, not "yet" at a given time of stepwise  $evaluation\ e_d$ .)

<sup>&</sup>lt;sup>2</sup> for a two-parameter, simple genuine ACKERMANN function cf. Eilenberg/Elgot 1970

Spreading down arguments this way eventually converts argument-free k th deduction tree  $dtree_k$  into (partially non-dummy) argumented deduction tree  $dtree_k/x$ .

Iteration  $\varepsilon_d$ , of tree evaluation step  $e_d$ , again is Complexity Controlled descending in Ordinal  $\mathbb{N}[\omega]$ , when controlled by deduction tree complexity  $c_d$ . This complexity is defined essentially as the (polynomial) sum of all (syntactical) complexities  $c_{PR}(u)$  of map codes appearing in the deduction tree.

So, as it does to *basic* evaluation  $\varepsilon$ , schema  $\tilde{\pi}_{\mathbb{N}[\omega]}$  applies to complexity controlled evaluation  $\varepsilon_d$  of argumented deduction-trees as well, and gives

**Deduction-Tree Evaluation non-infinit Descent:** Infinit strict descent of endo map  $e_d$  – with respect to complexity  $c_d$  – is *absurd*.

This deduction-tree evaluation  $\varepsilon_d$  externalises, as far as terminating, k th internal equation u = k v of theory  $\pi_O \mathbf{R}$  into complete evaluation  $\varepsilon(u, x) = \varepsilon(v, x)$ :

Termination-Conditioned Inner Soundness, our Main Theorem.

For a given PR predicate  $\chi = \chi(x) : \mathbb{X} \to 2$ , the **Main Theorem** reads:

Theory  $\pi_O \mathbf{R}$  derives: If for  $k \in \mathbb{N}$  and for  $x \in \mathbb{X} \setminus \{\underline{\perp}\}$  given,  $Prov_{\pi_O \mathbf{R}}(k, \lceil \chi \rceil)$  "holds", and if argumented  $\pi_O \mathbf{R}$  deduction tree  $dtree_k/x$  admits complete evaluation by m ("say") deduction-tree evaluation-steps  $e_d$ ,

Then the pair (k, x) is a Soundness-Instance, i. e. then k th given (internal)  $\pi_O \mathbf{R}$ -Provability  $Prov_{\pi_O \mathbf{R}}(k, \lceil \chi \rceil)$  implies  $\chi(x)$ , for the given argument  $x \in \mathbb{X} \setminus \{\underline{\perp}\}$ . All this within Theory  $\pi_O \mathbf{R}$  itself.

# Corollary: Self-Consistency Derivability for Theory $\pi_O \mathbf{R}$ :

$$\pi_O \mathbf{R} \vdash \operatorname{Con}_{\pi_O \mathbf{R}}$$
, i.e. "necessarily" in *Free-Variables* form:  $\pi_O \mathbf{R} \vdash \neg \operatorname{Prov}_{\pi_O \mathbf{R}}(k, \lceil \operatorname{false} \rceil) : \mathbb{N} \to 2$ , i.e. equationally:  $\pi_O \mathbf{R} \vdash \neg \lceil \lceil \operatorname{false} \rceil \stackrel{\sim}{=}_k \lceil \operatorname{true} \rceil \rceil : \mathbb{N} \to 2$ ,  $k \in \mathbb{N}$  free:

Theory  $\pi_O \mathbf{R}$  derives that no  $k \in \mathbb{N}$  is the internal  $\pi_O \mathbf{R}$ -Proof for  $\lceil \text{false} \rceil$ .

**Proof** of this Corollary to Termination-Conditioned Soundness:

By the last assertion of the **Theorem**, with  $\chi = \chi(x) := \text{false}_{\mathbb{X}}(x) : \mathbb{X} \to 2$ , and  $x := \langle 0 \rangle \in \mathbb{X}$ , we get:

Evaluation-effective internal inconsistency of  $\pi_O \mathbf{R}$ , i.e. availability of an evaluation-terminating internal deduction tree of  $\lceil \text{false} \rceil$ , implies false:

$$\pi_O \mathbf{R} \vdash \lceil \text{false} \rceil \stackrel{.}{=}_k \lceil \text{true} \rceil \land c_d \ e_d^m(dtree_k/\langle 0 \rangle) \stackrel{.}{=} 0 \implies \text{false}_{\mathbb{X}}(\langle 0 \rangle).$$

Contraposition to this, still with  $k, m \in \mathbb{N}$  free:

$$\pi_O \mathbf{R} \vdash \text{true}_{\mathbb{X}}(\langle 0 \rangle) \implies \neg \lceil \lceil \text{false} \rceil \overset{\text{``}}{=}_k \lceil \text{true} \rceil \rceil \lor c_d e_d^m(dtree_k/\langle 0 \rangle) > 0,$$

i.e. by Free-Variables (Boolean) tautology:

$$\pi_O \mathbf{R} \vdash \lceil \text{false} \rceil = k \lceil \text{true} \rceil \implies c_d \ e_d^m(dtree_k/\langle 0 \rangle) > 0 : \mathbb{N}^2 \to 2.$$

This  $\pi_O \mathbf{R}$  derivative invites to apply schema  $(\tilde{\pi}_{\mathbb{N}[\omega]})$  of  $\pi_O \mathbf{R}$ :

"infinite endo-driven descent with order values in  $\mathbb{N}[\omega]$  is absurd."

We apply this schema to deduction tree evaluation  $\varepsilon_d$  given by  $step\ e_d$  and complexity  $c_d$  which descends – this is Argumented-Tree  $Evaluation\ Descent$  – with each application of  $e_d$ , as long as complexity 0 is not ("yet") reached. We combine this with choice of "overall"  $absurdity\ condition$ 

$$\psi = \psi(k) := [ \lceil \text{false} \rceil = k \lceil \text{true} \rceil ] : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free (!)}$$

and get, by schema  $(\tilde{\pi}_{\mathbb{N}[\omega]})$ , overall negation of this (overall) "absurd" predicate  $\psi$ , namely

$$\pi_O \mathbf{R} \vdash \neg [ \lceil \text{false} \rceil =_k \lceil \text{true} \rceil ] : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free.}$$

This is  $\pi_O \mathbf{R}$ -derivation of the free-variable Consistency Formula of  $\pi_O \mathbf{R}$  itself.

From this Self-Consistency of Theorie(s)  $\pi_O \mathbf{R}$ , which is equivalent to injectivity of (special) internal numeralisation  $\nu_2: 2 \mapsto [\mathbb{1}, 2]_{\pi_O \mathbf{R}}$ , we get immediately injectivity of all these numeralisations  $\nu_A = \nu_A(a): A \mapsto [\mathbb{1}, A] = [\mathbb{1}, A]/\check{=}$ , and from this, with naturality of this family, "full" objective **Soundness** of Theory  $\pi_O \mathbf{R}$  which reads:

Formalised  $\pi_O \mathbf{R}$ -Provability of (code of) PR predicate  $\chi : \mathbb{X} \to 2$  implies – within Theory  $\pi_O \mathbf{R}$  – "validity"  $\chi(x)$  of  $\chi$  at "each" of  $\chi$ 's arguments  $x \in \mathbb{X}$ .

But for derivation of Self-Consistency from Termination-conditioned Soundness, a suitable **strengthening** of  $\mathbf{PR_A}$ , here by schema  $(\tilde{\pi}) = (\tilde{\pi}_{\mathbb{N}[\omega]})$ , stating absurdity of infinite descent in Ordinal  $\mathbb{N}[\omega]$ , seems to be necessary: my guess is that Theories  $\mathbf{PRA}$  as well as  $\mathbf{PR}$  and hence  $\mathbf{PR_A}$ , are not strong enough to derive their own (internal) Consistency. On the other hand, we know from Gödel's work that Principia Mathematica "und verwandte Systeme" are too strong for being self-consistent. This is true for any (formally) quantified Arithmetical Theory  $\mathbf{Q}$ , in particular for the (classical, quantified) version  $\mathbf{PA}$  of Peano Arithmetic: Such theory  $\mathbf{Q}$  has all ingredients for Gödel's Proof of his two Incompleteness Theorems.

In section 7 We discuss<sup>3</sup> a formally stronger, *implicational*, "local" variant  $(\pi_O^{\bullet})$  of <u>inferential</u> Descent axiom  $(\pi_O)$ , with respect to Self-Consistency and (Objective) Soundness: In particular, Self-Consistency **Proof** becomes technically easier for corresponding theory  $\pi_O^{\bullet} \mathbf{R}$ .

The final section  $8^4$  gives a **proof** of (Objective) <u>Consistency</u> for Theorie(s)  $\pi_O^{\bullet}\mathbf{R}$  (hence  $\pi_O\mathbf{R}$ ) relative to basic Theory  $\mathbf{PR_A}$  of Primitive Recursion and hence relative to fundamental Theory  $\mathbf{PR}$  of Primitive Recursion "itself".

For **proof** of this (relative) Consistency, we use a schema,  $(\rho_O)$ , of recursive *reduction* for predicate validity, reduction along a Complexity Controlled Iteration (CCI<sub>O</sub>), admitted by Theory **PR<sub>A</sub>** (and its strengthenings.)

<sup>&</sup>lt;sup>3</sup>insertion? JAN 2009

<sup>&</sup>lt;sup>4</sup>inserted 2 JAN 2009

# 2 Iterative Evaluation of PR Map Codes

Object- and map terms of all our theories are coded straight ahead, in particular since formally we have no (individual) variables on the Object Language level: We code all our terms just as prime power products "over" the LaTeX source codes describing these terms, this externally in <u>naive numbers</u>, out of  $\underline{\mathbb{N}}$  as well as into the NNO  $\mathbb{N}$  of the (categorical) arithmetical theory itself.

Equality Enumeration: As "any" theories, fundamental Theory **PR** of Primitive Recursion as well as basic Theory **PR**<sub>A</sub> = **PR** + (abstr), definitional enrichement of **PR** by the schema of predicate abstraction:  $\langle \chi : A \to 2 \rangle \mapsto \{A \mid \chi\}$ , a "virtual", abstracted Object in **PR**<sub>A</sub>, admit an (external) primitive recursive enumeration of their respective **theorems**, ordered by length (more precisely: by lexicographical order) of the first **proofs** of these (equational) Theorems, here:

$$=^{\mathbf{PR}} (\underline{k}) : \underline{\mathbb{N}} \to \mathbf{PR} \times \mathbf{PR} \subset \underline{\mathbb{N}} \times \underline{\mathbb{N}} \text{ and}$$
$$=^{\mathbf{PR}_{\mathbf{A}}} (k) : \mathbb{N} \to \mathbf{PR}_{\mathbf{A}} \times \mathbf{PR}_{\mathbf{A}} \subset \mathbb{N} \times \mathbb{N}$$

respectively.

By the PR Representation Theorem 5.3 of Romàn 1989, these <u>enumerations</u> give rise to their internal versions

$$\stackrel{\sim}{=}_k^{\mathbf{PR}} : \mathbb{N} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N}^2$$
 and  $\stackrel{\sim}{=}_k^{\mathbf{PR}_{\mathbf{A}}} : \mathbb{N} \to \mathrm{PR}_{\mathbf{A}} \times \mathrm{PR}_{\mathbf{A}} \subset \mathbb{N}^2$ ,

with internalisation (representation) property

$$\mathbf{PR} \vdash \stackrel{.}{=}_{\operatorname{num}(\underline{k})} = \operatorname{num}(=_{\underline{k}}^{\mathbf{PR}}) : \mathbb{1} \to \operatorname{PR} \times \operatorname{PR} \subset \mathbb{N}^2 \text{ and}$$
  
 $\mathbf{PR} \vdash \stackrel{.}{=}_{\operatorname{num}(\underline{k})} = \operatorname{num}(=_{\underline{k}}^{\mathbf{PR}_{\mathbf{A}}}) : \mathbb{1} \to \operatorname{PR}_{\mathbf{A}} \times \operatorname{PR}_{\mathbf{A}} \subset \mathbb{N}^2.$ 

Here (external) numeralisation is given externally PR as

$$\begin{aligned} &\operatorname{num}(\underline{n}) = s^{\underline{n}} : \mathbb{1} \stackrel{0}{\longrightarrow} \mathbb{N} \stackrel{s}{\longrightarrow} \dots \stackrel{s}{\longrightarrow} \mathbb{N}, \\ &\operatorname{num}(\underline{m},\underline{n}) = (\operatorname{num}(\underline{m}),\operatorname{num}(\underline{n})) : \mathbb{1} \to \mathbb{N} \times \mathbb{N}, \ \underline{m},\underline{n} \ (\text{``meta''}) \ \underline{\text{free}} \ \text{in } \underline{\mathbb{N}}, \end{aligned}$$

 $PR = \{ \mathbb{N} \mid PR \}$  is the predicative, PR decidable subset of  $\mathbb{N}$  "of all PR codes" (a  $PR_A$ -Object), internalisation of  $PR \subset \underline{\mathbb{N}}$  of all PR-terms on Object Language level. Analogeous meaning for internalisation  $PR_A \subset \mathbb{N}$  of  $PR_A \subset \underline{\mathbb{N}}$ .

For discussion of "constructive" evaluation, we need representation of all  $\mathbf{PR_A}$  maps within one  $\mathbf{PR}$  endo map <u>monoid</u>, namely within  $\mathbf{PR}(\mathbb{X}_{\pm}, \mathbb{X}_{\pm})$ , where  $\mathbb{X} \subset \mathbb{N}$ ,  $\mathbb{X} = {\mathbb{N} \mid \mathbb{X} : \mathbb{N} \to 2}$  is the (predicative) *Universal Object* of  $\mathbb{N}$ -singletons  ${\langle n \rangle \mid n \in \mathbb{N}}$ , possibly nested  $\mathbb{N}$ -pairs  ${\langle a; b \rangle \mid a, b \in \mathbb{X}}$ , and

$$\mathbb{X}_{\perp} =_{\operatorname{def}} \mathbb{X} \dot{\cup} \{\underline{\perp}\} = \mathbb{X}(a) \dot{\vee} a \doteq \underline{\perp} : \mathbb{N} \to 2$$

is X augmented by symbol (code)  $\underline{\perp} : \mathbb{1} \to \mathbb{N}, \underline{\perp}$  taking care of defined undefined arguments of defined partial maps.<sup>5</sup>

Here we view (formally)  $\mathbb{X} = \mathbb{X}(a)$ ,  $\mathbb{X}_{\underline{\perp}} = \mathbb{X}_{\underline{\perp}}(a) : \mathbb{N} \to \mathbb{N}$  as **PR**-predicates, not "yet" as abstracted Objects  $\mathbb{X} = \{\mathbb{N} \mid \mathbb{X}\}, \mathbb{X}_{\underline{\perp}} = \{\mathbb{N} \mid \mathbb{X}_{\underline{\perp}}\},$  of Theory **PR**<sub>A</sub> = **PR** + (abstr).

We allow us to write " $a \in \mathbb{X}$ " instead of  $\mathbb{X}(a) \doteq \text{true} : \mathbb{N} \to \mathbb{N}$ , and " $a \in \mathbb{X}_{\perp}$ " for  $\mathbb{X}_{\perp}(a) \doteq \text{true}$ , and similarly for other predicates.

This way we introduce – à la Reiter – "Object" 2 just as target for predicates  $\chi:A\to 2$ , meaning  $\chi:A\to \mathbb{N}$  to be a *predicate* in the exact sense that  $\chi:A\to \mathbb{N}$  satisfies

$$\chi \circ \operatorname{sign} \ =_{\operatorname{by\, def}} \ \chi \circ \neg \circ \neg = \chi : \mathbb{N} \stackrel{\chi}{\longrightarrow} \mathbb{N} \stackrel{\operatorname{sign}}{\longrightarrow} \mathbb{N}, \text{ "still" } A \ fundamental.$$

We **define**, within endo map <u>set</u>  $\mathbf{PR}(\mathbb{N}, \mathbb{N})$  a subTheory  $\mathbf{PR}\mathbb{X}$  externally PR as follows, by mimikry of schema (abstr) for the special case of predicate  $\mathbb{X} = \mathbb{X}(a) : \mathbb{N} \to \mathbb{N}$ , but *without* introduction of a coarser notion of equality, as in case of schema of abstraction constituting Theory  $\mathbf{PR_A} = \mathbf{PR} + (abstr)$ .

So Theory  $\mathbf{PR}\mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$  comes in, by external PR <u>enumeration</u> of its Object and map <u>terms</u> as follows:

Objects of **PR**X are predicates  $\chi: \mathbb{X} \to 2$ , i.e. **PR**-predicates  $\chi: \mathbb{N} \to 2$  such that

$$\mathbf{PR} \vdash \chi(a) \implies \mathbb{X}(a) : \mathbb{N} \to 2$$
, i.e. such that  $\mathbf{PR} \vdash \chi(a) \implies \mathbb{X}_{\perp}(a) \land a \neq \underline{\perp} : \mathbb{N} \to 2$ .

 $\mathbf{PR}\mathbb{X}$ -maps in  $\mathbf{PR}\mathbb{X}(\chi,\psi)$  are  $\mathbf{PR}$ -maps  $f:\mathbb{N}\to\mathbb{N}$  such that

$$\neg \mathbb{X}(a) \implies f(a) \doteq \bot$$
, and  $\chi(a) \implies \psi \circ f(a) : \mathbb{N} \to 2$ ,

observe the "truncated" parallelism to **definition** of  $\mathbf{PR_A}$ -maps  $f: \{A \mid \chi\} \to \{B \mid \psi\}$ .

Then "assignment"  $\mathbf{I}: \mathbf{PR} \xrightarrow{\square} \mathbf{PR} \mathbb{X}$  is **defined** as follows externally PR:

$$\begin{split} \mathbf{I} & \mathbb{1} = \dot{\mathbb{1}} &=_{\mathrm{def}} \{\langle 0 \rangle\} : \mathbb{N} \supset \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \to 2, \\ \mathbf{I} & \mathbb{N} = \dot{\mathbb{N}} &=_{\mathrm{def}} \langle \mathbb{N} \rangle &=_{\mathrm{def}} \{\langle n \rangle \mid n \in \mathbb{N}\} : \mathbb{N} \supset \mathbb{X}_{\underline{\perp}} \supset \mathbb{X} \to 2, \\ & \text{and further } \underline{\text{recursively}} : \end{split}$$

$$\mathbf{I}\left(A\times B\right) \ =_{\mathrm{def}} \ \left\langle A\times B\right\rangle \ =_{\mathrm{def}} \ \left\{\left\langle a;b\right\rangle \mid (a,b)\in (A\times B)\right\} : \mathbb{N}\supset \mathbb{X}\to 2,$$

Functorial **definition** of **I** on **PR** maps:

$$\mathbf{PR}(A,B)\ni f\overset{\mathbf{I}}{\mapsto}\mathbf{I}\,f=\dot{f}\in\mathbf{PR}\mathbb{X}$$

 $<sup>^5</sup>$  cf. Ch. 1, final section X

then is "canonical", by external PR on the structure of **PR**-map  $f: A \to B$ , in particular by mapping all "arguments" in  $\mathbb{N} \setminus \dot{A} = \mathbb{N} \setminus \mathbf{I} A$  into  $\underline{\perp} \in \mathbb{X}_{\underline{\perp}}$ : one waste basket outside all Objects of **PR** $\mathbb{X}$ .

Interesting now is that we can extend embedding I above into an <u>embedding</u>  $I: \mathbf{PR_A} \longrightarrow \mathbf{PRX}$ , by the following

**Definition:** For a (general) **PR<sub>A</sub>** Object, of form  $\{A \mid \chi\}$ , define

$$\mathbf{I} \{ A \mid \chi \} =_{\text{def}} \{ \dot{A} \mid \dot{\chi} \} =_{\text{by def}} \{ \mathbf{I} A \mid \mathbf{I} \chi \}$$
$$=_{\text{bv def}} \{ a \in \mathbf{I} A \mid \mathbf{I} \chi(a) \doteq \langle \text{true} \rangle \} : \mathbb{N} \supset \mathbb{X}_{\perp} \to 2.$$

We replace here "don't-worry arguments" in the complement  $\neg \chi$  of  $\mathbf{PR_A}$ -Object  $\{A \mid \chi\}$  by cutting them out in the definition of replacing  $\mathbf{PR}\mathbb{X}$ -Object  $\mathbf{I}\{A \mid \chi\} = \{\dot{A} \mid \dot{\chi}\}$ . "Coarser" notion  $=^{\mathbf{PR_A}}$  (coarser then  $=^{\mathbf{PR}}$ ) is then replaced by original notion of equality,  $=^{\mathbf{PR}}$  itself, notion of map-equality of roof  $\mathbf{PR}\mathbb{X}$  " $\subset$ "  $\mathbf{PR}(\mathbb{N},\mathbb{N})$ : This formal "sameness" of PR equality was the goal of the considerations above: The new version  $\mathbf{PR_A}$  replacing  $\mathbf{PR_A}$  isomorphically, is a  $\mathbf{sub}$ Theory of  $\mathbf{PR}$  with notion of equality – objectively as well as (then) internally – inherited from fundamental Theory  $\mathbf{PR}$ .

# Universal Embedding Theorem:<sup>7</sup>

- (i)  $I : \mathbf{PR} \longrightarrow \mathbf{PR} \mathbb{X} \subset \mathbf{PR}(\mathbb{N}, \mathbb{N})$  above is an <u>embedding</u> which preserves composition.
- (ii) (Enumerative) Restriction  $\mathbf{I}: \mathbf{PR} \xrightarrow{\cong} \mathbf{PR}^{\mathbb{X}} =_{\mathrm{def}} \mathbf{I}[\mathbf{PR}]$  of this embedding to its (<u>enumerated</u>) Image defines an <u>isomorphism</u> of categories. It is **defined** above as

$$\langle f : A \to B \rangle \xrightarrow{\mathbf{I}} \langle \dot{f} : \dot{A} \to \dot{B} \rangle,$$

by the "natural" (primitive) <u>recursion</u> on the structure of f as a <u>map</u> in fundamental Theory **PR** of (Cartesian) Primitive Recursion.

(iii)  ${\bf PR}$  embedding I "canonically" extends into an  $\underline{\rm embedding}$  (!)

$$I: \mathbf{PR}_A \longrightarrow \mathbf{PR}(\mathbb{N}, \mathbb{N})$$

of Theory  $\mathbf{PR_A} = \mathbf{PR} + (abstr)$  – Theory  $\mathbf{PR}$  with abstraction of predicates into ("new", "virtual") Objects  $\{A \mid \chi : A \to 2\}$  – to the Set of  $\mathbf{PR}$  endomaps of  $\mathbb{N}$ , of which – by the way –  $\mathbf{PR_A}(\mathbb{X}_{\underline{\perp}}, \mathbb{X}_{\underline{\perp}})$  is (formally) a SubQuotient.

[Equality = $^{\mathbf{PR_A}}$  of (distinguished)  $\mathbf{PR}$  endo maps when viewed as  $\mathbf{PR_A}$  endo maps on  $\mathbb{X}_{\underline{\perp}} = \{ \mathbb{N} \mid \mathbb{X}_{\underline{\perp}} : \mathbb{N} \to 2 \}$ , is <u>embedded</u> to  $\mathbf{PRX}$ - ( $\mathbf{PR}$ -)equality by  $\mathbf{I} : \mathbf{PR_A} \longrightarrow \mathbf{PRX}$  " $\subset$ "  $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ .]

<sup>&</sup>lt;sup>6</sup>for the details see Ch. 1, final section X.

<sup>&</sup>lt;sup>7</sup>from Ch. 1, final section X

(iv) Main assertion: Embedding I above defines an isomorphism of categories

$$I: PR_A \xrightarrow{\cong} PR_A^X$$

onto a "naturally choosen" (<u>emumerated</u>) category  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$  of  $\mathbf{PR}$  predicates on *Universal Object* ( $\mathbf{PR}$ -predicate)  $\mathbb{X}_{\underline{\perp}}: \mathbb{N} \to \mathbb{N}$ , with canonical maps in between (see above), and whith composition inherited from that of  $\mathbf{PR}(\mathbb{N}, \mathbb{N})$ . This isomorphism is defined (naturally) by

$$\mathbf{I}(f: \{A \mid \chi\} \to \{B \mid \psi\}) = \langle \dot{f}: \dot{\chi} \to \dot{\psi} \rangle,$$

$$\dot{\chi}: \mathbb{N} \supset \mathbb{X}_{\perp} \supset \mathbb{X} \supset \dot{A} \to 2,$$

$$\dot{\psi}: \mathbb{N} \supset \mathbb{X}_{\perp} \supset \mathbb{X} \supset \dot{B} \to 2,$$

$$\dot{f} = \underset{\text{by def}}{\mathbf{I}_{\mathbf{PR}}(f)}: \mathbb{N} \supset \dot{A} \to \dot{B} \subset \mathbb{N} \text{ above.}$$

By this isomorphism of categories,  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$  inherits from category  $\mathbf{PR}_{\mathbf{A}}$  all of its (categorically described) structure: the isomorphism transports Cartesian PR structure, equality predicates on all Objects, schema of predicate abstraction, equalisers, and – trivially – the whole algebraic, logic and order structure on NNO  $\mathbb{N}$  and truth Object 2.

We have furthermore:

(v) For each fundamental Object A, embedded Object  $\dot{A} = \mathbf{I} A \subset \mathbb{X}_{\underline{\perp}}$  comes with a retraction  $\operatorname{retr}_{A}^{\mathbb{X}} : \mathbb{X}_{\underline{\perp}} \to \dot{A} \dot{\cup} \{\underline{\perp}\}$ , **defined** by  $\operatorname{retr}_{A}^{\mathbb{X}}(a) =_{\operatorname{def}} a$  for  $a \in \dot{A}$ ,  $\operatorname{retr}_{A}^{\mathbb{X}}(a) =_{\operatorname{def}} \underline{\perp}$  otherwise.

This family of retractions clearly extends to a retraction family

$$\operatorname{retr}_{\{A \mid \chi\}}^{\mathbb{X}} : \mathbb{X}_{\underline{\perp}} \to \{\dot{A} \mid \dot{\chi}\} \dot{\cup} \{\underline{\perp}\} = \mathbf{I} \{A \mid \chi\} \dot{\cup} \{\underline{\perp}\}$$

for all  $\mathbf{PR}_{\mathbf{A}}$ -Objects  $\{A \mid \chi\}$ : This is what  $\underline{\perp} \in \mathbb{X}_{\underline{\perp}}$  is good for.

(vi) For each Object  $\{A \mid \chi\}$  of  $\mathbf{PR_A}$ , in particular for each fundamental Object  $A \equiv \{A \mid \mathrm{true}_A\}$ ,  $\mathbf{PR_A}$  comes with the characteristic (predicative) subset  $\dot{\chi} : \mathbf{I} \{A \mid \chi\} : \mathbb{X}_{\underline{\perp}} \to 2$  of  $\mathbb{X}_{\underline{\perp}}$  defined PR above, isomorphic to  $\{A \mid \chi\}$  within  $\mathbf{PR_A}$  (!) via "canonical"  $\mathbf{PR_A}$ -isomorphism

$$\operatorname{iso}_{\{A\mid\chi\}}^{\mathbb{X}}:\{A\mid\chi\}\stackrel{\cong}{\longrightarrow} \mathbf{I}\,\{A\mid\chi\}=\{\dot{A}\mid\dot{\chi}\},$$

the  $\mathbf{PR_A}$ -isomorphism **defined** PR on the "structure" of  $\{A \mid \chi\}$ , as restriction of  $\mathrm{iso}_A^{\mathbb{X}} : A \to \mathbf{I}A$  for fundamental Object A, in turn (externally/internally) PR defined by

$$\begin{split} & \mathrm{iso}_{\mathbb{I}}^{\mathbb{X}}(0) \ =_{\mathrm{def}} \ \langle 0 \rangle : \mathbb{1} \to \mathbf{I} \, \mathbb{1} \subset \mathbb{X}_{\underline{\perp}}, \\ & \mathrm{iso}_{\mathbb{N}}^{\mathbb{X}}(0) \ =_{\mathrm{def}} \ \langle 0 \rangle : \mathbb{1} \to \ [ \ \mathbf{I} \, \mathbb{1} \subset \ ] \, \mathbf{I} \, \mathbb{N} \subset \mathbb{X}_{\underline{\perp}}, \end{split}$$

further externally PR:

$$\operatorname{iso}_{(A\times B)}^{\mathbb{X}}(a,b) =_{\operatorname{def}} \langle \operatorname{iso}_{A}^{\mathbb{X}}(a); \operatorname{iso}_{B}^{\mathbb{X}}(b) \rangle : A \times B \stackrel{\cong}{\longrightarrow} \mathbf{I}(A\times B) \subset \mathbb{X}_{\underline{\perp}}.$$

We name the inverse isomorphism  $jso_{\{A \mid \chi\}}^{\mathbb{X}} : \mathbf{I} \{A \mid \chi\} \xrightarrow{\cong} \{A \mid \chi\}.$ 

(vii)  $\underline{\text{family}} \text{ iso}_{\{A \mid \chi\}}^{\mathbb{X}} : \{A \mid \chi\} \xrightarrow{\cong} \mathbf{I} \{A \mid \chi\} \subset \mathbb{X}_{\underline{\perp}} \subset \mathbb{N} \text{ above, } \{A \mid \chi\} \text{ Object of } \mathbf{PR_{A}}, \text{ is } natural, \text{ in the sense of the following commuting } \mathbf{PR_{A}}\text{-DIAGRAM} \text{ for a } \mathbf{PR_{A}}\text{-map } f : \{A \mid \chi\} \to \{B \mid \psi\} :$ 

$$\begin{cases}
A \mid \chi \rangle & \xrightarrow{f} \{B \mid \psi \} \\
& \text{iso}_{\{A \mid \chi \}}^{\mathbb{X}} \downarrow \cong = \cong \bigvee_{\text{iso}_{\{B \mid \psi \}}}^{\mathbb{X}} \\
\{\dot{A} \mid \dot{\chi} \} & \xrightarrow{\text{If}} \mathbf{I} \{B \mid \psi \} \xrightarrow{\subset} \mathbf{I} \{B \mid \psi \} \dot{\cup} \{\underline{\bot} \} \\
& \downarrow \subset & \downarrow \subset \\
& \mathbb{X}_{\underline{\bot}} & \xrightarrow{f = \text{by def } \mathbf{I_{PR}} f} \\
& \downarrow \subset & \downarrow \subset \\
& \mathbb{X}_{\underline{\bot}} & \downarrow \subset \\
& \mathbb{N} & \xrightarrow{f} & \mathbb{N}
\end{cases}$$

 $\mathbf{PR_{A}} \text{ Embedding diagram for } \mathbf{I} f = \mathbf{I_{PR_{A}}} f$   $\in \mathbf{PR_{A}^{\mathbb{X}}}(I \{A \mid \chi\}, \{B \mid \psi\}) = \mathbf{PR}\mathbb{X}(I \{A \mid \chi\}, \{B \mid \psi\}).$ 

In particular

$$\mathbf{I} f(a) =_{\text{by def}} \begin{cases} \operatorname{iso}_{B}^{\mathbb{X}} \circ f \circ \operatorname{jso}_{A}^{\mathbb{X}}(a) : \dot{A} \xrightarrow{\cong} A \xrightarrow{f} B \xrightarrow{\cong} \dot{B} \\ \operatorname{if} \dot{\chi}(a) \doteq \langle \operatorname{true} \rangle_{A}, \ i.e. \ \operatorname{if} \ \chi(\operatorname{jso}_{A}^{\mathbb{X}}(a)), \\ \underline{\bot} \in \dot{B} \cup \{\underline{\bot}\} \subset \mathbb{X}_{\underline{\bot}} \ otherwise, \\ i.e. \ \operatorname{if} \ \neg \chi(\operatorname{jso}_{A}^{\mathbb{X}}(a)). \end{cases}$$

By PR internalisation we get from the above the following

Internal Embedding Theorem: With Internalisitions  $PR: \mathbb{N} \to 2$  of  $\mathbf{PR} \subset \underline{\mathbb{N}}$ ,  $PR_A: \mathbb{N} \to 2$  of  $\mathbf{PR}_A \subset \underline{\mathbb{N}}$ ,  $PR_A^{\mathbb{X}} \subset PR\mathbb{X} \subset [\mathbb{N}, \mathbb{N}]_{\mathbf{PR}}: \mathbb{N} \to 2$ , and the corresponding internalised notions of equality

$$\check{=}_{k}^{\mathbf{PR}},\ \check{=}_{k}^{\mathbf{PR_{A}}},\ \check{=}^{\mathbf{PR_{A}}}^{\mathbb{X}}\subset\check{=}^{\mathbf{PR}\mathbb{X}}:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$$

we get  $PR_A$  injections

$$I = I(u) : \operatorname{PR} \xrightarrow{\cong} I[\operatorname{PR}] \subset \operatorname{PRX}/\check{=}^{\operatorname{PRX}} =$$

$$= \operatorname{PRX}/\check{=}^{\operatorname{PR}} \subset [\mathbb{N}, \mathbb{N}] =_{\operatorname{def}} [\mathbb{N}, \mathbb{N}]_{\operatorname{PR}}/\check{=}^{\operatorname{PR}},$$

as well as an extension of this I into

$$I = I(u) : \operatorname{PR}_{\mathcal{A}} \xrightarrow{\cong} \operatorname{PR}_{\mathcal{A}}^{\mathbb{X}} = I[\operatorname{PR}_{\mathcal{A}}] \subset \operatorname{PR}\mathbb{X}/\check{=}^{\mathbf{PR}}\mathbb{X} \subset [\mathbb{N}, \mathbb{N}] = [\mathbb{N}, \mathbb{N}]_{\mathbf{PR}}/\check{=}^{\mathbf{PR}}.$$

Both injections are internal (Cartesian PR) functors, isomorphic onto their (enumerated) images  $\operatorname{PR}^{\mathbb{X}} = I[\operatorname{PR}]$  and  $\operatorname{PR}^{\mathbb{X}}_{A} = I[\operatorname{PR}^{\mathbb{X}}_{A}] \subset \mathbb{N}$  respectively.

(*Enumerated*) injectivity of I is meant injectivity as a  $\mathbf{PR_A}$  map, more precisely: as a map in Theory  $\mathbf{PR_A}Q = \mathbf{PR_A} + (\mathrm{Quot})$ : Theory  $\mathbf{PR_A}$  definitionally

(and conservatively) enriched with *Quotients* by (enumerated) equivalence relations (cf. Reiter 1980), such as in particular the different internal notions  $\check{=}_k : \mathbb{N} \to \mathbb{N}^2$  above. The "mother" of all these is here  $\check{=} = \check{=}_k^{\mathbf{PR}} : \mathbb{N} \to \mathrm{PR} \times \mathrm{PR} \subset \mathbb{N}^2$ .

The second *injectivity* – corresponding to theories  $\mathbf{PR}_{\mathbf{A}}$ ,  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$ , and  $\mathbf{PR}_{\mathbf{A}}$  reads, in terms of  $\mathbf{PR}$  and  $\mathbf{PR}_{\mathbf{A}}$  alone:

$$I(u) \stackrel{\mathbf{PR}}{=}_{k} I(v) \implies u \stackrel{\mathbf{PR_A}}{=}_{j(k)} v : \mathbb{N} \times \lceil A, B \rceil^2 \to 2,$$
  
 $k \in \mathbb{N} \text{ free}, \ u, v \in \lceil A, B \rceil^2 \text{ free}, \ j = j(k) : \mathbb{N} \to \mathbb{N} \text{ available in } \mathbf{PR},$   
 $A, B \text{ in } \mathbf{PR_A} \text{ (meta) } \underline{\text{free}};$ 

analogeous meaning for the former internal (parallel: objective) injectivity properties  $\mathbf{q.e.d.}$ 

[As mentioned above,  $Coding \ PR = PR/\check{=}^{\mathbf{PR}}$  of Theory  $\mathbf{PR} = \mathbf{PR}/=\mathbf{PR}$  restricts to coding  $PR\mathbb{X} = PR\mathbb{X}/\check{=} = PR\mathbb{X}/\check{=}^{\mathbf{PR}} \subset [\mathbb{N}, \mathbb{N}]_{\mathbf{PR}}/\check{=}^{\mathbf{PR}}$ : coding of Object and map terms of  $\mathbf{PR}\mathbb{X}$  as well as internalising its inherited (enumerated) notion of equality.]

We now have all formal ingredients for **stating** Recursive Characterisation of (wanted) – double recursive – evaluation algorithms

$$\begin{split} \varepsilon^{\mathbf{PR}} &= \varepsilon^{\mathbf{PR}}(u,a) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \cong \mathrm{PR}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}, \\ \text{and its extension} \\ \varepsilon &= \varepsilon^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u,a) : \mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}. \end{split}$$

These evaluations are to become formally partial  $\mathbf{PR_A}$ -maps, i.e. maps of Theory  $\mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}}$ , see Ch. 1.

(Formal) partiality will be here not of PR decidable nature, in contrast to that of defined partial –  $\mathbf{PR_A}$  – maps, of form  $f: \{A \mid \chi\} \to \{B \mid \psi\}$  discussed above.

### Double Recursive Characterisation of Evaluation Algorithms

$$\varepsilon^{\mathbf{PR}}: \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}} \quad \text{and} \quad \varepsilon = \varepsilon(u,a): \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}$$

to evaluate all map codes in  $PR \cong PR^{\mathbb{X}}$  on all arguments of – free variable on – Universal Object  $\mathbb{X}_{\perp}$ .

The (wanted) **characterisation** is the following:

- Exceptional case of  $x = \underline{\perp} \in \mathbb{X}_{\underline{\perp}}$  undefined argument case:  $\varepsilon(u,\underline{\perp}) \doteq \underline{\perp} : \operatorname{PR}_{A} \to \operatorname{PR}_{A} \times \mathbb{X}_{\underline{\perp}} \to \mathbb{X}_{\underline{\perp}} : \operatorname{Once} \text{ a value is } defined undefined, it remains so under evaluation of any map code.$
- case of basic map constants bas :  $A \to B$ , namely bas one of  $0: \mathbb{1} \to \mathbb{N}$ ,  $s: \mathbb{N} \to \mathbb{N}$ ,  $\mathrm{id}_A: A \to A$ ,  $\Delta_A: A \to A \times A$ ,  $\Theta_{A,B}: A \times B \to B \times A$ ,

 $\ell_{A,B}: A \times B \to A$ , and  $r_{A,B}: A \times B \to B$ , first A,B fundamental Objects, in **PR**:

$$\varepsilon^{\mathbf{PR}}(\lceil \mathsf{bas} \rceil, a) = \mathsf{bas}(a) : \mathbb{X}_{\underline{\perp}} \supset A \to B \sqsubset \mathbb{X}_{\underline{\perp}},$$
i. e. (formally) in terms of theory  $\mathbf{PR}^{\mathbb{X}} \cong \mathbf{PR} :$ 

$$\varepsilon^{\mathbf{PR}^{\mathbb{X}}}(\lceil \mathbf{I} \mathsf{bas} \rceil, a) = \mathbf{I} \mathsf{bas}(a) = \mathbf{I}_{\mathbf{PR}} \mathsf{bas}(a) :$$

$$\mathbb{X}_{\underline{\perp}} \supset \dot{A} \to \dot{B} \subset \mathbb{X}_{\underline{\perp}}.$$

Extension  $\varepsilon = \varepsilon^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}$  to the case of all – basic – Objects of  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}} \supset \mathbf{PR}^{\mathbb{X}} \cong \mathbf{PR}$ :

$$\varepsilon(\lceil \mathbf{I} \operatorname{bas} \rceil, a) = \mathbf{I} \operatorname{bas}(a) : \mathbb{X}_{\underline{\perp}} \supset \mathbf{I} A \to \mathbf{I} B \subset \mathbb{X}_{\underline{\perp}} \text{ ("again")},$$

$$= \operatorname{by} \operatorname{def} \begin{cases} \operatorname{iso}_{B}^{\mathbb{X}} \circ \operatorname{bas} \circ \operatorname{jso}_{A}^{\mathbb{X}}(a) : \\ \mathbf{I} A \xrightarrow{\operatorname{jso}} A \xrightarrow{\operatorname{bas}} B \xrightarrow{\cong} \mathbf{I} B \text{ if } a \in \mathbf{I} A, \\ \underline{\perp} \text{ otherwise, i. e. if } a \in \mathbb{X}_{\underline{\perp}} \setminus \mathbf{I} A \end{cases}$$

$$: \mathbb{X}_{\perp} \supset \mathbf{I} A \to \mathbf{I} B \subset \mathbb{X}_{\perp},$$

this time A and B (suitable, basic) Objects, of  $PR_A$ .

## Example:

$$\begin{split} &\varepsilon(\,\,\lceil \mathbf{I}\,\ell_{\{\mathbb{N}\,|\, \mathrm{even}\},\mathbb{N}\times\mathbb{N}}^{\,\,\rceil}\,,x) \\ &= \begin{cases} \langle x_1 \rangle \in \langle \mathbb{N} \rangle = \mathbf{I}\,\mathbb{N} \text{ if } x = \langle x_1; \langle x_{2\,1}; x_{2\,2} \rangle \rangle \in \langle \mathbb{N}\times\mathbb{N}^2 \rangle \ \wedge \ 2|x_1, \\ \underline{\perp} \text{ otherwise} \\ &: \mathbb{X}_{\underline{\perp}} \supset \langle \{\mathbb{N}\,|\, \mathrm{even}\} \times \mathbb{N}^2 \rangle \to \langle \{\mathbb{N}\,|\, \mathrm{even}\} \rangle \subset \langle \mathbb{N} \rangle \subset \mathbb{X}_{\underline{\perp}}\,. \end{cases} \end{split}$$

The *compound* cases are the following ones:

- case of evaluation of internally composed

$$\langle v \odot u \rangle =_{\text{by def}} \langle v \cap u \rangle, \text{ for } u \in [A, B]_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}, v \in [B, C]_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}} \text{ "$\subset$" } [\mathbb{N}, \mathbb{N}]_{\mathbf{PR}} :$$

Characterisation in this composition case is (is wanted):

$$\begin{split} \varepsilon(\langle v\odot u\rangle,a) &= \varepsilon(v,\varepsilon(u,a)) = \varepsilon\ \widehat{\circ}\ (v,\varepsilon\ \widehat{\circ}\ (u,a)): \\ \lceil B,C\rceil\times\lceil A,B\rceil\times\mathbb{X}_{\underline{\bot}} &\to \mathbb{X}_{\underline{\bot}}, \text{ in particular} \\ \varepsilon(\langle v\odot u\rangle,a) &\doteq \underline{\bot} \iff a\in\mathbb{X}_{\underline{\bot}}\smallsetminus A, \ defined\ undefined. \end{split}$$

[Formally we cannot "yet" guarantee that  $\varepsilon$  be enumeratively terminating at "all" regular arguments, "termination" in a sense still to be **defined.**]

**Remark:** "Definition" in this – central – composition case is recursively legitimate, by structural recursion on depth $\langle v \odot u \rangle$  down to depth(u) and depth(v),  $u, v \in PR_A^X$ , PR **definition** of depth(u) for (general)

$$u = \langle \dot{\chi}, \mathring{u}, \dot{\psi} \rangle \in [\mathbf{I} \{A \mid \chi\}, \mathbf{I} \{B \mid \psi\}]_{\mathbf{PR}^{\mathbb{X}}} \subset [\mathbb{X}_{\underline{\perp}} \setminus \{\underline{\perp}\}, \{B \mid \psi\}]_{\mathbf{PR}^{\mathbb{X}}}$$

see below.

- cylindrified  $\langle A \times v \rangle$ ,  $v \in [B, B']_{\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}}}$ :

$$\varepsilon(\langle A \times v \rangle, x) = \begin{cases} \langle x_1; \varepsilon(v; x_2) \rangle \in \langle A \times B' \rangle \subset \mathbb{X}_{\underline{\perp}} & (\ulcorner \times \urcorner) \\ \text{if } x = \langle a; b \rangle \in \langle A \times B \rangle \subset \mathbb{X}_{\underline{\perp}}, \\ \underline{\perp} & \text{otherwise} \end{cases}$$

$$: \mathbb{X}_{\underline{\perp}} \supset \langle A \times B \rangle \rightarrow \langle A \times B' \rangle \subset \mathbb{X}_{\underline{\perp}} :$$

evaluation in the cylindrified component.

- internally iterated  $u^{\lceil \S \rceil}$ , for  $u \in [A, A]$ :

$$\begin{split} \varepsilon(u^{\lceil \S \rceil}, \langle a; 0 \rangle) &= a, \\ \varepsilon(u^{\lceil \S \rceil}, \langle a; s \, n \rangle) &= \varepsilon(u, \varepsilon(u^{\lceil \S \rceil}, \langle a; n \rangle)) \\ &= \varepsilon \mathbin{\widehat{\circ}} (u, \varepsilon \mathbin{\widehat{\circ}} (u^{\lceil \S \rceil}, \langle a; n \rangle)) : \\ (\operatorname{PR}_{\mathcal{A}}^{\mathbb{X}} \times \mathbb{N}) \times \mathbb{X}_{\underline{\bot}} \supset (\lceil A, A \rceil \times \mathbb{N}) \times A \rightharpoonup A \subset \mathbb{X}_{\underline{\bot}}, \end{split}$$
 (iteration step)

"⊃" meaning "again":  $\varepsilon(u^{\lceil \S \rceil}, x) \doteq \underline{\perp}$  in all other cases. This case distinction is always here PR.

- abstracted map code u, of form

$$u = \langle \dot{\chi}, \mathring{u}, \dot{\psi} \rangle \in \lceil \mathbf{I} \{A \mid \chi\}, \mathbf{I} \{B \mid \psi\} \rceil_{\mathbf{PR}_{\mathbf{A}}^{\times}} :$$

$$\varepsilon(u, a) = \begin{cases} \varepsilon^{\mathbf{PR}}(\mathring{u}, a) \in \{\dot{B} \mid \dot{\psi}\} = \mathbf{I} \{B \mid \psi\} \\ \text{if } \chi(a) \doteq \text{true} \\ \underline{\perp} \text{ otherwise } i.e. \text{ if } a \in \mathbb{X}_{\underline{\perp}} \setminus \mathbf{I} \{A \mid \chi\} \end{cases} : \operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \supset \lceil \{\dot{A} \mid \dot{\chi}\}, \{\dot{B} \mid \dot{\psi}\} \rceil \rightharpoonup \{\dot{B} \mid \dot{\psi}\} \subset \mathbb{X} \subset \mathbb{X}_{\underline{\perp}}.$$

**Remark:** If we restrict (wanted) evaluation  $\varepsilon$  to fundamental map codes, out of

$$\mathrm{PR}\;[\;\;\sqsubset\mathrm{PR}^{\mathbb{X}}_{\mathrm{A}}\;]\;\; \sqsubset\mathrm{PR}\mathbb{X}\;\; \subset \lceil\mathbb{N},\mathbb{N}\rceil_{\mathbf{PR}},$$

– omit last case above and the "I" in description of  $\varepsilon$  above throughout – we get, by  $\mathbf{PR_A}$  implications in cases above for basic map constants, composition, cylindrification, as well as of iteration characterisation of (wanted)

fundamental evaluation

$$\varepsilon^{\mathbf{PR}} = \varepsilon^{\mathbf{PR}}(u, a) : \operatorname{PR} \times \mathbb{X}_{\underline{\perp}} \supset [A, B]_{\mathbf{PR}} \times A \rightharpoonup B \subset \mathbb{X}_{\underline{\perp}},$$

$$A, B \subset \mathbb{X}_{\underline{\perp}} \text{ fundamental, restriction of}$$

$$\varepsilon = \varepsilon(u, a) = \varepsilon^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u, a) : \operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}} \text{ above,}$$

both to be characterised (within Theorie(s)  $\pi_O \mathbf{R}$  to come), as formally partial  $\mathbf{PR_A}$  maps – out of Theory  $\mathbf{PR_A}$  –, but on-terminating in  $\pi_O \mathbf{R}$ , and to be defined below as Complexity Controlled Iterations "CCI<sub>O</sub>'s" with complexity values in Ordinal  $\mathbb{N}[\omega]$ .

Considering this restricted, fundamental evaluation  $\varepsilon^{\mathbf{PR}}: \mathrm{PR} \times \mathbb{X}_{\perp} \to \mathbb{X}_{\perp}$  will be helpfull, in particular since the Objects of  $\mathbf{PR_A}$  are nothing else then fundamental predicates  $\chi: A \to 2$ , still more formal: fundamental maps  $\chi: A \to \mathbb{N}$  such that  $\neg \circ \neg \circ \chi = {}^{\mathbf{PR}} \chi: A \to \mathbb{N} \to \mathbb{N}$ .

Recursive Legitimacy for "definition" above of evaluation  $\varepsilon$  is obvious for all cases above, except for second subcase of case of *iterated*, since in the other cases recursive reference is made (only) to map terms of lesser depth.

Here depth $(u): \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \to \mathbb{N}$  is **defined** PR as follows:

$$\begin{aligned} &\operatorname{depth}(\,\ulcorner\operatorname{id}_A^{\,\urcorner}\,) \ =_{\operatorname{def}} \ 0 \ \operatorname{for} \ A \ \operatorname{fundamental}, \\ &\operatorname{as} \ \operatorname{well} \ \operatorname{as} \ \operatorname{for} \ A = \{A' \,|\, \chi\} \ \operatorname{basic}, \ \operatorname{in} \ \mathbf{PR_A}. \\ &\operatorname{depth}(\,\ulcorner\operatorname{bas}'^{\,\urcorner}\,) \ =_{\operatorname{def}} \ 1 \ \operatorname{for} \ \operatorname{bas}' : A \to B \\ &\operatorname{one} \ \operatorname{of} \ \operatorname{the} \ \operatorname{other} \ \operatorname{basic} \ \operatorname{map} \ \operatorname{constants}, \ \operatorname{in} \ \mathbf{PR_A}; \ \operatorname{further} \ \operatorname{PR}: \\ &\operatorname{depth}(\langle v \odot u \rangle) \ =_{\operatorname{def}} \ \operatorname{depth}(u) + \operatorname{depth}(v) + 1 : \\ & [B,C]_{\mathbf{PR_A}^{\,\boxtimes}} \times [A,B]_{\mathbf{PR_A}^{\,\boxtimes}} \to \mathbb{N}^2 \to \mathbb{N}. \end{aligned}$$

We then get automatically

$$\begin{split} \operatorname{depth}_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}} \langle \, \, \lceil \{ \dot{A} \, | \, \dot{\chi} \} \, \rceil \, , u, \, \, \lceil \{ \dot{B} \, | \, \dot{\psi} \} \, \rceil \, \rangle \\ &= \operatorname{depth}_{\mathbf{PR}^{\mathbb{X}}} \langle \, \, \lceil \dot{A} \, \rceil \, , \, \, \lceil \dot{B} \, \rceil \, \rangle = \operatorname{depth}_{\mathbf{PR}}(u) : \, [A,B]_{\mathbf{PR}} \subset \operatorname{PR} \to \mathbb{N} : \\ & \text{forget about (depth of) Domain and Codomain.} \end{split}$$

Using this depth = depth(u) :  $\operatorname{PR}_{A}^{\mathbb{X}} \to \mathbb{N}$ , (wanted) characterisation above of  $\varepsilon^{\operatorname{PR}}$  and  $\varepsilon = \varepsilon^{\operatorname{PR}_{A}^{\mathbb{X}}}$  is recursively *legitimate* for all cases except – a priori – the iteration case, since in those cases it recurs to its "definition" for map terms with (strictly) lesser depth.

In case of an iterated, reference is made to a term with equal depth, but with decreased iteration counter: from

$$\operatorname{iter}(u^{\lceil \S \rceil}, \langle a; s \, n \rangle) \ =_{\operatorname{def}} \ s \, n \quad \operatorname{down to} \quad \operatorname{iter}(u^{\lceil \S \rceil}, \langle a; n \rangle) \ =_{\operatorname{def}} \ n.$$

This shows double recursive, (intuitive) legitimacy of our "definition", more precisely: (double recursive) description of formally partial evaluation

 $\varepsilon: \operatorname{PR}_A^{\mathbb{X}} \times \mathbb{X}_{\perp} \longrightarrow \mathbb{X}_{\perp}$ . A possible such (formally partial) map is *characterised* by the above *general recursive* equation system. This system constitutes a *definition* by a (nested) *double recursion* à la ACKERMANN, and hence in particular it constitutes a **definition** in classical recursion theory.

We now attempt to **resolve** basic evaluation  $\varepsilon$ , to be **characterised** by the above *double recursion*, into a **definition** as an *iteration* of a suitable evaluation step

$$e = e(u, x) : \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\perp} \to \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\perp},$$

first of a step  $e = e^{\mathbf{PR}}(u, x) : \mathrm{PR} \times \mathbb{X}_{\underline{\perp}} \to \mathrm{PR} \times \mathbb{X}_{\underline{\perp}}.$ 

In fact resolution into a Complexity Controlled Iteration, CCI, which is to give, upon reaching complexity 0, evaluation result  $\varepsilon(u,x) \in \mathbb{X}_{\perp}$  in its right component.

For discussion of termination of this (content driven) iteration, we consider

Complexity Controlled Iterations in general: Such a  $CCI_O$  is given – in Theory  $\mathbf{PR_A}$  by data a ("predecessor")  $step\ p:A\to A$  coming with a  $complexity\ c:A\to O$ , such that  $\mathbf{PR_A}\vdash DeSta\lceil p \mid c\rceil\ (a):A\to 2$ , where

$$DeSta[p | c](a) =_{def} [c(a) > 0 \implies p \ c(a) < c(a)]$$

$$(strict \ \underline{Descent} \ above \ complexity \ zero)$$

$$\wedge [c(a) \doteq 0 \implies p(a) \doteq_{A} a]$$

$$(\underline{Stationarity} \ at \ complexity \ zero).$$

O is an *Ordinal*, here a suitable extension  $O \succeq \mathbb{N}[\omega]$  of the semiring of polynomials in one indeterminate, with lexicographical order. *Suitable* in the sense that we are convinced that it does not allow for infinitely descending chains.

# Examples of such "Ordinals", besides $\mathbb{N}[\omega]$ :

- [ $\mathbb{N}$  itself as well as  $\mathbb{N} \times \mathbb{N}$ ,  $\mathbb{N}^{\underline{m}}$  with hierarchical order are Ordinals below  $\mathbb{N}[\omega]$ , but we will need for our complexity values Ordinals  $O \succeq \mathbb{N}[\omega] \cong \mathbb{N}^+$ ]:
- $O = \mathbb{N}^+ \equiv \mathbb{N}[\xi] \equiv \mathbb{N}[\omega]$ :  $\mathbb{N}^+$  is the set of non-empty strings, ordered lexicographically, and to be interpreted here as *coefficient strings* of (the semiring of) polynomials over  $\mathbb{N}$  in one indeterminate. The order choosen on  $\mathbb{N}[\omega]$  is in fact the lexicographical one on its coefficient strings in  $\mathbb{N}^+$ .
- O the semiring  $O = \mathbb{N}[\xi_1, \dots, \xi_{\underline{m}}]$  in  $\underline{m}$  indeterminates, the *later* indeterminates having *higher priority* with respect to O's order.
- O the semiring  $\mathbb{N}[\vec{\xi}] = \bigcup_m \mathbb{N}[\xi_1] \dots [\xi_m]$  in several variables (in arbitrary finitely many ones). Order "extrapolated" from foregoing example.
- O the *ultimate* (?) (countable) Ordinal  $\mathbb{E}$  given by arbitrarily balanced bracketing of strings of natural numbers:

All of the above examples can be given the form of such sets of balanced-bracketed strings, but not containing *singletons of singletons*, of form  $\langle \langle \ldots \rangle \rangle$ .

Admitting these pairs of double, triple,... brackets leads to interpretation of  $\mathbb{E}$  as the semi-algebra of strings of polynomials in (finitely many) indeterminates out of (countable) families of families of ... families of (candidates for) indeterminates: indeterminates out of later families then get higher priority with respect to the order of  $\mathbb{E}$ .

Abbreviating predicate  $DeSta[p|c](a): A \to 2$  given, "positive" **axiom** schema  $(\pi_O)$ , of all  $CCI_O$ 's to on-terminate – whose equivalent contraposition is schema  $(\tilde{\pi}_O)$  of non-infinit descent of the  $CCI_O$ 's –, reads:

$$c: A \to O, \ p: A \to A \ \mathbf{PR_A} \ \text{maps} \\ \mathbf{PR_A} \vdash \ DeSta\left[p \mid c\right](a): A \to 2 \ \text{(see above)}; \\ \text{furthermore: for } \chi: A \to 2 \ \text{"test" predicate, in } \mathbf{PR_A}: \\ \text{"test on reaching } 0_O \text{" by chain } p^n(a): \\ \mathbf{PR_A} \vdash \ TerC[p, c, \chi] = TerC[p, c, \chi] (a, n): A \times \mathbb{N} \to 2, \\ =_{\text{def}} \left[c \ p^n(a) \doteq 0 \implies \chi(a)\right]: A \times \mathbb{N} \to 2, \\ \left(\underline{Termination} \ \underline{Comparison} \ \text{condition)}, \\ \text{with } quantifier \ decoration:} \\ (\pi_O) \ \ \underline{\mathbf{PR_A}} \vdash \ (\forall a) \left[\left(\exists n\right) c \ p^n(a) \doteq 0_O \implies \chi(a)\right] \\ \overline{\pi_O} \mathbf{R} \vdash \chi: A \to 2, \quad \text{i. e. } \chi =^{\pi_O} \mathbf{R} \ \text{true}_A: A \to 2.$$

It is important to note in context of evaluation – that "emerging" Theory  $\pi_O \mathbf{R}$  has same language as basic PR Theory  $\mathbf{PR_A}$ . It just adds equations forced by the additional schema. Axis case is  $O := \mathbb{N}[\omega]$ ,  $(\pi) =_{\mathrm{def}} (\pi_{\mathbb{N}[\omega]})$ ,  $\pi \mathbf{R} =_{\mathrm{def}} \mathbf{PR_A} + (\pi)$ . Theory  $\pi_{\mathbb{N}} \mathbf{R}$  would be just Theory  $\mathbf{PR_A}$ .

Characterisation Theorem for  $CCI_O$ 's: Let complexity  $c = c(a) : A \to O$  and predecessor  $p = p(a) : A \to A$  be given, as in the antecedent of  $(\pi_O)$  above. Then (formally partial)  $\mathbf{P}\hat{\mathbf{R}}_{\mathbf{A}}$  map

$$f(a) = p^{\S} \, \widehat{\circ} \, (a, \mu \, [\, c \, | \, p \, ] \, \widehat{\circ} \, a) : A \rightharpoonup A \times \mathbb{N} \to A$$

is nothing else then the  $\widehat{\mathbf{PR}}_{\mathbf{A}}$  map (while loop)  $f = \text{wh}[c > 0_O \mid p] : A \rightharpoonup A$ , and we "name" it  $\text{wh}_O[c \mid p] : A \rightharpoonup A$ .

Written with free variable, and dynamically:

$$\operatorname{wh}_{O}[c \mid p](a) \stackrel{\triangle}{=} \operatorname{wh}[c(a) > 0_{O} \mid a := p(a)] : A \rightharpoonup A.$$

By while loop Characterisation in RFC1, this complexity controlled iteration  $(CCI_O)$  is characterised by

$$\operatorname{wh}_{O} = \operatorname{wh}_{O}[c \mid p] \, \widehat{\circ} \, a = \begin{cases} a & \text{if } c(a) \stackrel{.}{=} 0_{O} \\ \operatorname{wh} \, \widehat{\circ} \, p(a) & \text{if } c(a) > 0_{O} \end{cases} : A \stackrel{.}{\rightharpoonup} A.$$

The standard  $\mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}}$  form of this  $\mathbf{CCI}_O$  reads:

wh<sub>O</sub> = wh<sub>O</sub>[
$$c \mid p$$
] =  $\langle (d_{wh_O}, \widehat{wh}_O) : D_{wh_O} \to A \times A \rangle : A \to A$ , with  $D_{wh_O} = \{(a, n) \mid p^n(a) \doteq 0_O\}$   
 $d_{wh_O} = d_{wh_O}(a, n) = \ell(a, n) = a : D_{wh_O} \to A$ , and  $\widehat{wh}_O(a, n) = p^{\S}(a, \min\{m \leq n \mid p^m(a) \doteq 0_O\}) = p^n(a) : D_{wh_O} \to A$ ,

the latter because of stationarity of  $p: A \to A$  at zero-complexity.

Comment: In terms of these while loops, equivalently: formally partial PR maps, schema ( $\pi_O \mathbf{R}$ ) says map theoretically: Defined-arguments enumeration of the  $\mathrm{CCI}_O$ 's have image predicates, and these predicative images equal true, on the common Domain, A, of the given step and complexity. By definition, this means that these enumerations are onto, become so by axiom; and by this, all  $\mathrm{CCI}_O$ 's on-terminate. In our context – use equality definability – this is equivalent with epi property of the defined-arguments enumerations of the  $\mathrm{CCI}_O$ 's – but not with these enumerations to be retractions.

**Dangerous bound:**<sup>8</sup> For complexity  $c: A \to O$  above, descending with "each" step  $p: A \to A$ , we have

$$\widehat{\operatorname{wh}}_O[c \mid p] \widehat{\circ} (\operatorname{id}_A, \mu_O) \widehat{=} \operatorname{wh}_O : A \rightharpoonup D_{\operatorname{wh}_O} \to A, \text{ where}$$

$$\mu_O = \mu_O[c \mid p] (a) =_{\operatorname{def}} \mu\{n \mid c p^n \doteq_O 0\} : A \rightharpoonup \mathbb{N}.$$

But this  $\mu_O = \mu_O[c \mid p] : A \to \mathbb{N}$  cannot in general be a  $(\mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}})$  section to  $d_{\mathrm{wh}_O[c \mid p]} : D_{\mathrm{wh}_O[c \mid p]} \to A$ , since otherwise – by **Section Lemma** in Ch. 1 –  $\mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}}$  map  $\mu_O : A \to D_{\mathrm{wh}_O[c \mid p]}$  would become a PR (!) section to defined-arguments (PR) enumeration  $d_{\mathrm{wh}_O[c \mid p]}$ , and hence  $\mathrm{wh}_O[c \mid p] : A \to A$  would become PR itself. But at least for evaluation  $\varepsilon$ , which is of CCI<sub>O</sub> form, this is excluded by ACKERMANN's result that diagonalisation of  $\varepsilon$  – "evaluate n-th (unary) map at argument n" – grows faster than any PR map.

[Here we use the Church type result of Ch. 1, that any  $\mu$ -recursive map has a representation as a partial PR<sub>A</sub> map, i.e. that it can be viewed as a map within Theory  $\mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}}$ , as well as Objectivity of evaluation  $\varepsilon$  which will be **proved** below.]

With motivation above, we now **define**  $PR_A$  maps

$$e = e^{\mathbf{PR}}(u, a) : \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \to \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp}$$

evaluation step, and  $c = c_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}} : \mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \to \mathbb{N}[\omega]$  (evaluation) complexity, to give **evaluation** in fact as a formally partial map

$$\varepsilon = \varepsilon^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u, a) : \mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \longrightarrow \mathbb{X}_{\underline{\perp}}, \text{ within theory } \mathbf{P}\widehat{\mathbf{R}}_{\mathbf{A}},$$

 $<sup>^8</sup>$ added 2 Nov 2008

e and c maps within Theory  $\mathbf{PR}_{\mathbf{A}}$ .

Partial evaluation map  $\varepsilon$  then will be **defined** by iteration of PR evaluation step  $e: PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}$ , descending in complexity

$$c = c(u, x) = c_{\varepsilon}(u, x) =_{\operatorname{def}} c_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u) : \operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \twoheadrightarrow \operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \to \mathbb{N}[\omega].$$

The (endo) evaluation step

$$e = e(u, x) = (e_{\text{map}}(u, x), e_{\text{arg}}(u, x)) : PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}$$

is **defined** below as a  $PR_A$  map. Here left component

 $e_{\text{map}}(u, x) : \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to \operatorname{PR}_{A}^{\mathbb{X}}$  designates the by-one-step evaluated, reduced map code, and right component

 $e_{\text{arg}}(u, x) : \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to \mathbb{X}_{\underline{\perp}} \text{ is to designate}$ the by-one-step ("in part") evaluated argument.

So here is the **definition** of evaluation step  $e = (e_{\text{map}}, e_{\text{arg}})$ , endo map of  $PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}$ , by **PR**<sub>A</sub> case distinction, cf. (wanted) characterisation of  $\varepsilon$  above:

- case of **basic** maps, of form bas :  $A \to B$  in  $\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}}(A, B)$  :

$$e(\lceil \operatorname{d\dot{a}s} \rceil, a) =_{\operatorname{def}} (\lceil \operatorname{id}_{\dot{B}} \rceil, \operatorname{d\dot{a}s}(a)) : \mathbb{X}_{\underline{\perp}} \supset \dot{A} \xrightarrow{\operatorname{d\dot{a}s}} \dot{B} \xrightarrow{\subset} \mathbb{X}_{\underline{\perp}},$$
  
 $\dot{A} =_{\operatorname{bv}\operatorname{def}} \mathbf{I} A, A = \{A' \mid \chi\} \text{ in } \mathbf{PR}_{\mathbf{A}}, \text{ analogeously for } \dot{B}.$ 

"finished".

**Recall:** bas:  $A \to B$  is one out of the basic map constants

$$id_A$$
,  $0: \mathbb{1} \to \mathbb{N}$ ,  $s: \mathbb{N} \to \mathbb{N}$ ,  $!_A$ ,  $\Theta_{A,B}$ ,  $\Delta_A$ ,  $\ell_{A,B}$ ,  $r_{A,B}$ ,

A, B Objects of  $\mathbf{PR}_{\mathbf{A}}$ , in particular: A, B  $\mathbf{PR}$ -Objects.

- composition cases: "for" (free variable)  $v \in [A, B], [A, B] = [A, B]_{\mathbf{PR}_{A}}^{\mathbb{X}}$ :

$$e(\langle v \odot \lceil \mathrm{id}_A \rceil \rangle, a) =_{\mathrm{def}} (v, a) \qquad (\odot \text{ anchoring})$$
  
$$\in [A, B] \times A \subset \mathrm{PR}_{A}^{\mathbb{X}} \times \mathbb{X} \subset \mathrm{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\perp}.$$

For 
$$((u, v), a) \in \lceil B, C \rceil \times (\lceil A, B \rceil \setminus \{\lceil \operatorname{id}_A \rceil\}) \times A \subset (\operatorname{PR}_A^{\mathbb{X}})^2 \times \mathbb{X}_{\underline{\perp}} :$$

$$e(\langle v \odot u \rangle, a) =_{\operatorname{def}} (\langle v \odot e_{\operatorname{map}}(u, x) \rangle, e_{\operatorname{arg}}(u, x))$$

$$\in \lceil \operatorname{Dom}(e_{\operatorname{map}}(u, x)), C \rceil \times \mathbb{X}_{\perp} \subset \operatorname{PR}_{\Delta}^{\mathbb{X}} \times \mathbb{X}_{\perp},$$

where  $\text{Dom}(e_{\text{map}}(u,x))$ , Object of  $\text{PR}_{\mathbf{A}}^{\mathbb{X}}$ , is "known" – **defined** PR on depth, in particular – "anchoring" – for  $e_{\text{map}}(u,x) = \text{bås}$  above, Dom of form  $\dot{A}$  in  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$  (A in  $\mathbf{PR}_{\mathbf{A}}$ ) is known, "etc." PR.

So **definition** of e in this composition case in toto, is PR on depth( $\langle v \odot u \rangle$ ), "down to" depth( $\langle v \odot e_{\text{map}}(u, x) \rangle$ .

#### - cylindrified cases:

- "trivial", termination (sub)case:

$$e(\langle \lceil \operatorname{id}_A \rceil \lceil \times \rceil \lceil \operatorname{id}_B \rceil \rangle, \langle a; b \rangle) =_{\operatorname{def}} (\lceil \operatorname{id}_{(A \times B)} \rceil, \langle a; b \rangle)$$

"finished", and

– genuine cylindrified case: for  $v \in [B, B'] \setminus \{ \lceil id_B \rceil \}$ :

$$e(\langle \lceil \mathrm{id}_A \rceil \lceil \times \rceil v \rangle, \langle a; b \rangle)$$

$$=_{\mathrm{def}} (\langle \lceil \mathrm{id}_A \rceil \lceil \times \rceil e_{\mathrm{map}}(v, b) \rangle, \langle a; e_{\mathrm{arg}}(v, b) \rangle) :$$

apply evaluation (step) to right component v and its argument b.

#### - iteration case

$$u^{\lceil \S \rceil} \in \lceil \langle A \times \mathbb{N} \rangle, A \rceil, \ \langle a; n \rangle \in \langle A \times \mathbb{N} \rangle \text{ (free)}:}$$
 $e(u^{\lceil \S \rceil}, \langle a; n \rangle) =_{\text{def}} (u^{[n]}, a), \text{ where, by PR definition}$ 
 $u^{[0]} =_{\text{def}} \lceil \text{id}_A \rceil \in \operatorname{PR}_A^{\mathbb{X}}, \text{ and } u^{[s\,n]} =_{\text{def}} \langle u^{[n]} \odot u \rangle \in \operatorname{PR}_A^{\mathbb{X}}$ 
is  $code \ expansion \ \text{"at run time"}.$ 

[This latter case of **definition** by *code expansion*, is not very "effective", but logically simple.]

**Definition** of evaluation complexity, to descend with each application of evaluation (endo) step, first of **PR** map codes  $u \in PR$ :

$$c(u) = c_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}(u)} : \mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \to \mathbb{N}[\omega]$$
, is **defined** as a  $\mathbf{PR}_{\mathbf{A}}$ -map as follows:

$$c \operatorname{rid}_{A} =_{\operatorname{def}} 0 \cdot \omega^{0} = \min_{\mathbb{N}[\omega]}, \ A \mathbf{PR}_{\mathbf{A}}^{\mathbb{X}} - Object,$$
  
 $c \operatorname{rbas}' =_{\operatorname{def}} 1 \cdot \omega^{0} : \mathbb{1} \to \mathbb{N}[\omega],$ 

for bas' one of the other basic map constants of  $\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$ ;

for 
$$(u, v) \in [B, C] \times [A, B] = [B, C]_{\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}}} \times [A, B]_{\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}}}$$
:

$$c \langle v \odot u \rangle =_{\text{def}} c(u) + c(v) + 1 \cdot \omega^0 \in \mathbb{N}[\omega]$$

(internal composition  $\odot$ );

$$c \langle A \times v \rangle = c \langle \dot{A} \upharpoonright \times \urcorner v \rangle =_{\text{def}} c(v) + 1 \cdot \omega^0 : \operatorname{PR}_{A}^{\mathbb{X}} \to \mathbb{N}[\omega]$$
 (internal cylindrification;)

for 
$$u \in [A, A]_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}$$
:

$$c(u^{\lceil \S \rceil}) =_{\operatorname{def}} \omega^{1} \cdot (c(u) + 1) = (c(u) + 1) \cdot \omega^{1}$$
:

$$\operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \supset \lceil A, A \rceil \to \mathbb{N}[\omega]$$
 (internal iteration),

where 
$$\omega=\omega^1\equiv 0;1\,,\ \omega^2\equiv 0;0;1\,,\ \omega^3\equiv 0;0;0;1$$
 etc. in  $\mathbb{N}[\omega],$ 

$$\mathbb{N}[\omega] \equiv \mathbb{N}^+ = \mathbb{N}^* \setminus \{\bot\} \equiv \mathbb{N}_{>0}, \text{ Ch. 1.}$$

Motivation for above definition – in particular for this latter iteration case – will become clear with the corresponding case in **proof** of **Descent Lemma** below for *basic evaluation* 

$$\varepsilon = \varepsilon(u, v) =_{\text{def}} \text{wh}[c_{\varepsilon} | e] : \mathbf{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \rightharpoonup \mathbf{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp}.$$

**Remark:** As pointed out already above, restriction of a  $\mathbf{PR}^{\mathbb{X}}$  map code  $u \in [\dot{A}, \dot{B}]$  to  $u' \in [\{\dot{A} \mid \dot{\chi}\}, \{\dot{B} \mid \dot{\psi}\}]$  has no effect to complexity: If u restricts this way, then

$$c(u') = c^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u') = c^{\mathbf{PR}^{\mathbb{X}}}(u) = c^{\mathbf{PR}}(u) = c^{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}}(u).$$

**Example:** Complexity of *addition*, with  $+ =_{\text{by def}} s^{\S} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ , identified with  $\dot{+} : \langle \mathbf{I} \mathbb{N} \times \mathbf{I} \mathbb{N} \rangle \to \mathbf{I} \mathbb{N}$  within  $\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}} :$ 

$$c \vdash \vdash \neg = c \vdash s^{\$ \neg} = c (\vdash s^{\neg \neg \$ \neg})$$
  
=  $\omega^1 \cdot (c \vdash s^{\neg} + 1) = 2 \cdot \omega \in \mathbb{N}[\omega] [\equiv 0; 2 \in \mathbb{N}^+].$ 

Evaluation step and complexity above are the right ones to give

**Descent Lemma** for formally *partially defined* and "nevertheless" *on-terminating* evaluation map

$$\varepsilon = \varepsilon(u, a) = \text{by def} \text{ wh} [c_{\varepsilon} \mid e] : PR_{\Lambda}^{\mathbb{X}} \times \mathbb{X}_{\perp} \to PR_{\Lambda} \times \mathbb{X}_{\perp} \xrightarrow{r} \mathbb{X}_{\perp},$$

i.e. for step

$$e = e(u, a) = (e_{\text{map}}, e_{\text{arg}}) : PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}, \text{ and complexity}$$

$$c_{\varepsilon} = c_{\varepsilon}(u, a) =_{\operatorname{def}} c(u) : \operatorname{PR}_{A}^{\mathbb{X}} \to \mathbb{N}[\omega]$$

we have Descent above  $0 \in \mathbb{N}[\omega]$ , and Stationarity at complexity 0:

$$\mathbf{PR}_{\mathbf{A}} \vdash c_{\varepsilon}(u, a) > 0 \implies c_{\varepsilon} e(u, a) < c_{\varepsilon} c(u, a) :$$

$$\mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to \mathbb{N}[\omega] \times \mathbb{N}[\omega] \xrightarrow{< \times <} 2^{2} \stackrel{\Longrightarrow}{\longrightarrow} 2, \ i. e.$$

$$\mathbf{PR}_{\mathbf{A}} \vdash c(u) > 0 \implies c \, e_{\mathrm{map}}(u, a) < c(u) : \mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \to 2, \qquad (Desc)$$
as well as

$$\mathbf{PR}_{\mathbf{A}} \vdash c(u) \doteq 0 \ [ \iff u \text{ of form } u = \mathrm{id}_{A} \ ]$$
$$\implies c_{\varepsilon} e(u, a) \doteq 0 \ [ \land e(u, a) \doteq (u, a) \ ], \tag{Sta}$$

this with respect to the canonical, "lexicographic", and – intuitively – finite-descent order of the polynomial semiring  $\mathbb{N}[\omega]$ .

**Proof:** The only non-trivial case  $(v, b) \in PR_A^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}$  for the descent condition  $c \ e(v, b) < c(v, b)$  is the iteration case

$$(u^{\lceil \S^{\rceil}}, \langle a; n \rangle) \in \lceil \langle A \times \mathbb{N} \rangle, A \rceil \times A \subset \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}}.$$

In this "acute" iteration case we have in fact by induction on n,

$$c(u^{[n]}) = n \cdot c(u) + (n - 1), \text{ since - recursion:}$$

$$c(u^{n+1}) = c \langle u \odot u^{[n]} \rangle = c(u^{[n]}) + c(u) + 1 = (n+1) \cdot c(u) + n,$$
whence
$$c_{\varepsilon} e(u^{\lceil \S \rceil}, \langle a; n \rangle) = c(u^{[n]}) \text{ (definition of } e)$$

$$= n \cdot c(u) + (n - 1) < \omega \cdot (c(u) + 1),$$
since  $\omega > m, m \in \mathbb{N}$ .

["+1" in  $c(u^{\lceil \S \rceil}) =_{\operatorname{def}} \omega \cdot c(u) + 1$  is to account for the (trivial) case  $\lceil \operatorname{id} \rceil^{\lceil \S \rceil}$ .] Stationarity at complexity  $0 \in \mathbb{N}[\omega]$  is obvious **q.e.d.** 

This Basic Descent Lemma makes plausible **global termination** of the ( $\mu$ -recursive) version of evaluation  $\varepsilon = \varepsilon(u, x) : \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\perp} \to \mathbb{X}_{\perp}$ , in a suitable framework, here: it **proves** that this basic (formally) partial evaluation map out of  $\widehat{\mathbf{PR}}_{A}$ :

$$\varepsilon = \varepsilon(u,x) : \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \rightharpoonup \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \twoheadrightarrow \mathbb{X}_{\perp}$$

on-terminates within Theory  $\pi_O \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O \mathbf{R})$ , for Ordinal  $O \succeq \mathbb{N}[\omega]$ . This means that evaluation  $\varepsilon$  has an onto, epi defined arguments enumeration

$$d_{\varepsilon} = d_{\varepsilon}(n, (u, x)) =_{\operatorname{def}} (u, x) :$$

$$D_{\varepsilon} = \{ (m, (u, x)) \mid c \ell \ e^{n}(u, a) \doteq 0 \} \to \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}_{\perp}$$

within  $\pi \mathbf{R} =_{\text{def}} \pi_{\mathbb{N}[\omega]} \mathbf{R}$ , and a fortiori in  $\pi_O \mathbf{R}$ , Ordinal  $O \succeq \mathbb{N}[\omega]$ , such choice of O taken always here.

**Remark:** Even if intuitively terminating, and derivably on-terminating, partial map  $\varepsilon$  does not give (by isomorphic translation), a self-evaluation of Theory

$$\pi \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi) = \pi \mathbf{R} + (\pi_{\mathbb{N}[\omega]}),$$

"Dangerous bound" in Summary above. Nothing is said (above) on evaluation of Theory  $\pi_O \hat{\mathbf{R}} = \widehat{\pi_O \mathbf{R}}$ .

In present context, we need an "explicit"

Free-Variable Termination Condition, in particular for our *basic* evaluation  $\varepsilon$ , and later for its extension,  $\varepsilon_d$ , into an evaluation for *argumented deduction* trees.

For a while loop in general, of form

wh[
$$\chi \mid f$$
]( $a$ ):  $A \rightarrow A$  (read: while  $\chi(a)$  do  $a := f(a)$ ),  
define  $[m \ def \ wh[ $\chi \mid f$ ]( $a$ )] = def  $[\neg \chi \ f^m(a)] : \mathbb{N} \times A \rightarrow 2$ :$ 

m "defines" argument a for while loop wh[ $\chi | f$ ], to terminate on this defined argument after at most m steps.

This gives in addition:

$$[m \ def \ \operatorname{wh}[\chi \mid f](a)] \implies \operatorname{wh}(a) \doteq_A \widehat{\operatorname{wh}}(a,m) : \mathbb{N} \times A \to 2;$$

$$[\operatorname{wh}(a) \doteq_A \widehat{\operatorname{wh}}(a,m)] =_{\operatorname{by} \operatorname{def}} f^\S(a, \min\{n \leq m \mid \neg \chi \ f^n(a)\}) : \mathbb{N} \times A \to 2.$$

Things become more elegant for  $CCI_O$ 's, because of stationarity of CCI's at complexity  $0 = 0_O \in O$ :

$$\mathbf{PR}_{\mathbf{A}} \vdash [m \ def \ \mathrm{wh}_{O}[c \mid p](a)] = [c \ p^{m}(a) \doteq 0_{O} \ \land \ \mathrm{wh}_{O}(a) \doteq_{A} p^{m}(a)] :$$

$$A \times \mathbb{N} \to 2, \quad \text{in particular:}$$

$$\mathbf{PR}_{\mathbf{A}} \vdash [m \ def \ \varepsilon(u, x)] = [c \ell e^{m}(u, x) \doteq 0 \ \land \ \varepsilon(u, x) \doteq r e^{m}(u, x)] :$$

$$\mathbb{N} \times (\mathrm{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp}) \to 2.$$

We will use this given termination counter "m def ..." only as a (termination) condition (!), in implications of form m def wh<sub>O</sub>(a)  $\Longrightarrow \chi(a)$ ,  $\chi = \chi(a)$  a termination conditioned predicate. And we will make assertions on formally partial maps such as evaluation  $\varepsilon$  and argumented deduction-tree evaluation  $\varepsilon_d$  below, mainly in this termination-conditioned, "total" form.

So the main stream of our story takes place in theory  $\mathbf{PR_A}$ : we go back usually to the  $\mathbf{PR_A}$ -building blocks of formally partial maps occurring, in particular to those of basic evaluation  $\varepsilon$  as well as those of tree evaluation  $\varepsilon_d$  to come.

**Iteration Domination** above, applied to the *Double Recursive* equations for  $\varepsilon$ , makes out of these the following

#### Dominated Characterisation Theorem for evaluation

$$\begin{split} \varepsilon &= \varepsilon(u,a) : \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}_{\underline{\perp}} \rightharpoonup \mathbb{X}_{\underline{\perp}}, \\ \text{and hence equally for its } isomorphic translation \\ \varepsilon &= \varepsilon(u,a) : \mathrm{PR}_{\mathrm{A}} \times \mathbb{X} \rightharpoonup \mathbb{X} : \end{split}$$

$$\begin{aligned} \mathbf{PR_A} &\vdash \left[ \varepsilon (\lceil \mathsf{bas} \rceil, a) \doteq \mathsf{bas}(a) \text{ resp } \varepsilon (\lceil \mathsf{bas} \rceil, a) \doteq \mathsf{bas}(a) \right] \wedge : \\ \left[ m \ def \ \varepsilon (v \odot u, a) \right] \implies \varepsilon (\langle v \odot u \rangle, a) \doteq \varepsilon (v, \varepsilon (u, a)) \\ &\land \left[ m \ def \ \varepsilon (v, b) \right] \implies \varepsilon (\langle \lceil \mathsf{id} \rceil \rceil \lceil \times \rceil v \rangle, \langle a; b \rangle) \doteq \langle a; \varepsilon (v, b) \rangle \\ &\land \varepsilon (u^{\lceil \S \rceil}, \langle a; 0 \rangle) \doteq e^1 (u^{\lceil \S \rceil}, \langle a; 0 \rangle) \doteq a \\ &\land \left[ m \ def \ \varepsilon (u^{\lceil \S \rceil}, \langle a; s \ n \rangle) \right] \implies : \\ &m \ defines \ all \ \varepsilon \ instances \ below, \ and : \\ &\varepsilon (u^{\lceil \S \rceil}, \langle a; s \ n \rangle) \doteq \varepsilon (u^{\lceil \S \rceil}, \langle \varepsilon (u, a); n \rangle) \doteq \varepsilon (u, \varepsilon (u^{\lceil \S \rceil}, \langle a; n \rangle)) : \\ & \mathbb{N} \times (\mathrm{PR_A}^{\mathbb{X}})^2 \times \mathbb{X}^2 \times \mathbb{N} \to 2, \\ &m \in \mathbb{N} \ \mathrm{free}, \ u, v \in \mathrm{PR_A}^{\mathbb{X}} \subset \ \mathbb{N} \ \mathrm{free} \ \mathrm{resp.} \ u, v \in \mathrm{PR_A} \subset \mathbb{N} \ \mathrm{free}, \\ &a, b \in \mathbb{X} \subset \mathbb{N}, \ n \in \mathbb{N} \ \mathrm{free}. \end{aligned}$$

**Proof** of this **Theorem** by Primitive Recursion (Peano Induction) on  $m \in \mathbb{N}$  free, via case distinction on codes w,

$$w \in \mathrm{PR}^{\mathbb{X}}_{\mathrm{A}} \subset [\mathbb{X}, \mathbb{X}]_{\mathbf{PR}\mathbb{X}} \subset [\mathbb{N}, \mathbb{N}]_{\mathbf{PR}} \subset \mathbb{N},$$

and arguments  $z \in \mathbb{X}$  appearing in the different cases of the asserted conjunction, as follows, case w one of the basic map constants being trivial:

All of the following – **induction step** – is situated in  $\mathbf{PR}_{\mathbf{A}}$ , read:  $\mathbf{PR}_{\mathbf{A}} \vdash etc.$ :

- case  $(w, z) = (\langle v \odot u \rangle, a)$  of an (internally) composed, subcase  $u = \lceil id \rceil$ : obvious.

Non-trivial subcase  $(w, z) = (\langle v \odot u \rangle, a), u \neq \lceil id \rceil$ :

$$m+1 \ def \ \varepsilon(w,a) := \varepsilon(\langle v \odot u \rangle,a) \implies :$$

$$\varepsilon(w, a) =_{\text{by def}} e^{\S}((\langle v \odot e_{\text{map}}(u, x) \rangle, e_{\text{arg}}(u, a)), m)$$

by iterative definition of  $\varepsilon$  in this case

$$\doteq \varepsilon(v, \varepsilon(e_{\text{map}}(u, a), e_{\text{arg}}(u, a)))$$

by induction hypothesis, namely:

$$m \ def \ \mu[c \mid e] (\langle v \odot e_{\text{map}}(u, a) \rangle, e_{\text{arg}}(u, a)), \ [i.e. \ \mu \leq m]$$

 $\Longrightarrow$ :

$$m+1$$
 def  $\varepsilon(v,\varepsilon(e_{\mathrm{map}}(u,a),e_{\mathrm{arg}}(u,a))) \doteq \varepsilon(v,\varepsilon(u,a))$ :

Same way back, by the same induction hypothesis, on m, map code v unchanged, "passive", in both directions of reasoning.

- case  $(w, z) = (\langle \lceil \operatorname{id} \rceil \lceil \times \rceil v \rangle, \langle a; b \rangle)$  of an (internally) cylindrified: Obvious by definition of  $\varepsilon$  on a cylindrified map code.
- case  $(w, z) = (u^{\lceil \S \rceil}, \langle a; 0 \rangle)$   $\in [\langle A \times \mathbb{N} \rangle, A] \times \langle A \times \mathbb{N} \rangle \subset \mathrm{PR}_{\mathrm{A}}^{\mathbb{X}} \times \mathbb{X}$ of a null-fold (internally) iterated: again obvious.
- case  $(w, z) = (u^{\lceil \S \rceil}, \langle a; n+1 \rangle)$   $\in [\langle A \times \mathbb{N} \rangle, A] \times \langle A \times \mathbb{N} \rangle \subset \operatorname{PR}_{A}^{\mathbb{X}} \times \mathbb{X}$ of a genuine (internally) iterated: for  $a \in \dot{A}$ ,  $n \in \mathbb{N}$  free:

$$(w,z) \doteq (u^{\lceil \S \rceil}, \langle a; n+1 \rangle) \implies :$$

$$m+1 \ def \ \varepsilon(w,z) \implies$$

$$\varepsilon(w,z) \doteq \varepsilon(e_{\mathrm{map}}(u^{\,\lceil \S^{\,\rceil}}\,,\langle a;n+1\rangle),e_{\mathrm{arg}}(u^{\,\lceil \S^{\,\rceil}}\,,\langle a;n+1\rangle))$$

$$\doteq \varepsilon(u^{[n+1]}, a) \doteq \varepsilon(\langle u^{[n]} \odot u \rangle, a) \doteq \varepsilon(u^{[n]}, \varepsilon(u, a))$$

the latter by induction hypothesis on m,

case of internal composed

$$\doteq \varepsilon(u^{\lceil \S \rceil}, \langle \varepsilon(u, a); n \rangle) :$$

same way back – using bottom up characterisation of the iterated – with  $\varepsilon(u, a)$  in place of a, and n in place of n + 1.

This shows the (remaining) predicative—truncated—iteration equations "anchor" and "step", for an (internally) iterated  $u^{\lceil \S \rceil}$ , and so **proves** fullfillment of the above **Double Recursive** system of **truncated equations** for  $\varepsilon: \operatorname{PR}_A^{\mathbb{X}} \times \mathbb{X} \to \mathbb{X}$ , as well "then" for isomorphic translation  $\varepsilon: \operatorname{PR}_A \times \mathbb{X} \to \mathbb{X}$ , in terms of its defining components, within basic theory  $\operatorname{PR}_A \sqsubset \operatorname{P}\widehat{\mathbf{R}}_A$  "itself" **q.e.d.** 

Characterisation Corollary: Evaluations –  $\widehat{PR}_A$ -maps –

$$\varepsilon = \varepsilon(u, a) : \operatorname{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X} \supset [\mathbf{I} A, \mathbf{I} B]_{\mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}} \times \mathbf{I} A \rightharpoonup \mathbf{I} B \subset \mathbb{X}$$
 as well as  $-$  back-translation  $-$  
$$\varepsilon = \varepsilon(u, a) : \operatorname{PR}_{\mathbf{A}} \times \mathbb{X} \supset [A, B]_{\mathbf{PR}_{\mathbf{A}}} \times A \rightharpoonup B \subset \mathbb{X},$$

now (both) **defined** as Complexity Controlled iterations – CCI's – with complexity values in Ordinal  $O := \mathbb{N}[\omega]$ , on-terminate in Theorie(s)  $\pi_O \mathbf{R}$  ( $O \succeq \mathbb{N}[\omega]$ ), by **definition** of these theory strengthenings of  $\mathbf{PR_A}$ ,  $\mathbf{PR_A}$ , and satisfy there the **characteristic** Double-Recursive equations stated for  $\varepsilon$  at begin of section.

**Evaluation Objectivity:** We "rediscover" here the logic *join* between the Object Language level and the external PR Metamathematical level, join by externalisation via evaluation  $\varepsilon$  above. The corresponding, very plausible Theorem says that evaluation  $\varepsilon$  mirrors "concrete" codes,  $\lceil f \rceil$  of maps  $f: A \to B$  of Theories  $\mathbf{PR}$  (via  $\mathbf{PR}^{\mathbb{X}} = \mathbf{I}[\mathbf{PR}]$ ),  $\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}}$  as well as  $\mathbf{PR}_{\mathbf{A}}$ , the latter via  $\mathbf{PR}^{\mathbb{X}}_{\mathbf{A}} \cong \mathbf{PR}_{\mathbf{A}}$ , back into these maps themselves.

**Objectivity Theorem:** Evaluation  $\varepsilon$  is *objective*, i. e.: for each *single*, (meta <u>free</u>)  $f: \mathbb{X}_{\perp} \supset A \rightarrow B \sqsubset \mathbb{X}_{\perp}$  in Theory  $\mathbf{PR_A}$  itself, we have, with "isomorphic translation" of evaluation from  $\mathbf{PR_A^{\mathbb{X}}}$ :

$$\mathbf{PR_A} \vdash \varepsilon(\lceil f \rceil, a) = f(a) : \mathbb{X} \supset A \to B \sqsubset \mathbb{X}, \text{ symbolically:}$$
  
 $\mathbf{PR_A} \vdash \varepsilon(\lceil f \rceil, \bot) = f : A \to B,$ 

a fortiori:  $\pi_O \mathbf{R} \vdash \varepsilon(\lceil f \rceil, a) = f(a) : \mathbb{X} \supset A \to B \sqsubset \mathbb{X}.$ 

**Remark:** For such f fixed,

$$\varepsilon(\,\lceil f \rceil\,,a) = \varepsilon \mathbin{\widehat{\circ}} (\,\lceil f \rceil\,,a) : A \to \lceil A,B \rceil \times A \rightharpoonup B$$

is in fact a  $\mathbf{PR_A}$  map  $\varepsilon(\lceil f \rceil, \bot) = \varepsilon(\lceil f \rceil, a) : A \to B$ , although in the **Proof** of the **Theorem** intermediate steps are formally  $\mathbf{PR_A}$  equations " $\widehat{=}$ ": But  $\mathbf{PR_A} \sqsubseteq \mathbf{PR_A}$  is a diagonal monoidal PR *Embedding*.

**Proof** of Evaluation Objectivity by first: External structural recursion on the nesting depth  $\underline{depth}[f]$  ("bracket depth") of  $\mathbf{PR_A}$ -map  $f:A \to B$  in

question, seen as external <u>code</u>:  $f \in \underline{\mathbb{N}}$ , and second: in case of an <u>iterated</u>,  $g^{\S} = g^{\S}(a, n) : A \times \mathbb{N} \to A$ , by  $\mathbf{PR_A}$ -recursion on iteration count  $n \in \mathbb{N}$ . This uses (dominated) Double Recursive Characterisation of evaluation  $\varepsilon$  **q.e.d.** 

**Finally** here: as forshadowed above, *evaluations* "split" into (externally) indexed Objective evaluation families

$$[\varepsilon_{A,B} = \varepsilon_{A,B}(u,a) : [A,B] \times A \rightarrow B]_{A,B \text{ Objects}},$$

with all of the above characteristic properties "split".

Central for all what follows is (Inner) Soundness Problem for evaluation

$$\varepsilon = \varepsilon(u, a) : \mathbf{PR}_{\mathbf{A}}^{\mathbb{X}} \times \mathbb{X}_{\perp} \rightharpoonup \mathbb{X}_{\perp}, \text{ namely:}$$

Is there a "suitable" Condition  $\Gamma = \Gamma(k, (u, v)) : \mathbb{N} \times (PR_A^{\mathbb{X}})^2 \to 2$ , under which Theory  $\mathbf{PR_A}$  exports internal equality u = k v into Objective, predicative equality  $\varepsilon(u, a) = \varepsilon(v, a)$ ? Formally: such that

$$\mathbf{PR}_{\mathbf{A}} \vdash \Gamma(k, (u, v)) \implies \left[ u \stackrel{\cdot}{=}_{k} v \implies \varepsilon(u, a) \stackrel{\cdot}{=} \varepsilon(v, a) \right] :$$

$$\mathbb{N} \times (\mathrm{PR}_{\mathbf{A}}^{\mathbb{X}})^{2} \times \mathbb{X} \rightharpoonup \mathbb{X} \times \mathbb{X} \stackrel{\stackrel{\cdot}{=}}{\longrightarrow} 2?$$

Such ("suitably conditioned") evaluation Soundness is strongly expected, and <u>derivable</u> without condition in classical Recursion Theory (and **set** theory) – the latter two in the rôle of frame theory  $\mathbf{PR_A}$  above:

The formal **problem** here lies in *termination*.

# 3 Deduction Trees and Their Top Down Argumentation

As a first step for "solution" of the (Conditioned) Soundness Problem for evaluation  $\varepsilon : \operatorname{PR}_A^{\mathbb{X}} \times \mathbb{X} \to \mathbb{X}$ , we fix in present section internal, "formalised"  $\operatorname{Proofs} \operatorname{Proof}_{\mathbf{T}}$  of map Theorie(s)  $\mathbf{T} := \pi_O \mathbf{R}$  as (internal) deduction trees dtree<sub>k</sub> with nodes labeled by map-code internal equations. These deduction trees are ordered by tree nesting-depth, and – second priority – code length:  $\operatorname{dtree}_k$  is the k th deduction tree in this order, it (internally) proves, deduces  $\pi_O \mathbf{R}$ -equation u = k.

For reaching our goal of **Termination-Conditioned Soundness** for evaluation

$$\varepsilon = \varepsilon(u, x) : \pi_{\mathcal{O}} \mathbf{R} \times \mathbb{X} = \mathbf{P} \mathbf{R}_{\mathcal{A}} \times \mathbb{X} \cong \mathbf{P} \mathbf{R}_{\mathcal{A}}^{\mathbb{X}} \times \mathbb{X} \to \mathbb{X}, \text{ with}$$

$$\pi_{\mathcal{O}} \mathbf{R} \vdash \Gamma(k, (u, v)) \implies [u \stackrel{*}{=}_{k}^{\pi_{\mathcal{O}} \mathbf{R}} v \implies \varepsilon(u, a) \stackrel{!}{=} \varepsilon(v, a)],$$

below,  $\Gamma$  "the" suitable Termination condition, we consider evaluation of argumented deduction trees  $dtree_k/a$ , top down "argumented" starting with given argument, to wanted equation  $\varepsilon(u, a) \doteq \varepsilon(v, a)$ .

For fixing ideas, we redefine – with the above counting  $dtree_k$  of deduction trees – internal proving as

$$Prov_{\pi_{O}\mathbf{R}}(k, \ u \stackrel{\cdot}{=} v) =_{\operatorname{def}} Prov_{\pi_{O}\mathbf{R}}(dtree_{k}, \ u \stackrel{\cdot}{=} v)$$
$$=_{\operatorname{by \ def}} [u \stackrel{\cdot}{=}_{k}^{\pi_{O}\mathbf{R}} v] : \mathbb{N} \times \operatorname{PR}_{A}^{2} \cong \mathbb{N} \times (\operatorname{PR}_{A}^{\mathbb{X}})^{2} \to 2.$$

Each such deduction tree, deducing - root - internal equation u = v can canonically be argumented top down with suitable arguments for each of its (node) equations, when given - just one - argument to its root equation u = v.

**Example:** Internal version of equational "simplification" Theorem  $s \ a \ \dot{} \ s \ b = a \ \dot{} \ b$ , namely  $\langle \lceil s \rceil \odot \lceil \ell \rceil \lceil \dot{} - \rceil \rceil \lceil s \rceil \odot \lceil r \rceil \rangle \ \check{=}_k \langle \lceil \ell \rceil \rceil \lceil \dot{} - \rceil \rceil \rceil \rangle$ , "still" more formal – we omit from now on Object subscripts (for  $\pi_O^{\mathbb{X}} \mathbf{R} = \mathbf{PR}_{\mathbf{A}}^{\mathbb{X}}$ -Objects):

$$\ulcorner \dot{-} \urcorner \odot \langle \ulcorner s \urcorner \odot \ulcorner \ell \urcorner ; \ulcorner s \urcorner \odot \ulcorner r \urcorner \rangle \check{=}_k \ulcorner \dot{-} \urcorner \odot \langle \ulcorner \ell \urcorner ; \ulcorner r \urcorner \rangle,$$

 $k \in \mathbb{N}$  suitable.

Internal deduction tree  $dtree_k$  in this case:

 $dtree_k =$ 

$$\begin{array}{c|c} & \langle \ulcorner s \, \ell \urcorner \, \ulcorner \neg \urcorner \, s \, r \urcorner \rangle \stackrel{\check{=}}{=}_k \langle \ulcorner \ell \urcorner \, \ulcorner \neg \urcorner \, r \rangle \\ & \stackrel{\check{=}}{=}_i \langle \ulcorner \operatorname{pre} s \, \ell \urcorner \, \ulcorner \neg \urcorner \, r \urcorner \rangle & \stackrel{\check{=}}{=}_j \langle \ulcorner \ell \urcorner \, \ulcorner \neg \urcorner \, r \urcorner \rangle \\ & \stackrel{\check{=}}{=}_i \langle \ulcorner r \, \neg \urcorner \, r \, r \urcorner \rangle & \stackrel{\check{=}}{=}_i \langle \ulcorner \ell \urcorner \, \neg \neg \urcorner \, r \urcorner \rangle \\ & \stackrel{\check{=}}{=}_{ii} \langle \ulcorner r \, \neg \urcorner \, r \, r \, r \rangle & (\operatorname{definition of pre}) \, . \\ & \stackrel{\check{=}}{=}_{iii} \langle \ulcorner \operatorname{pre} s \, \ell \, \neg \, \neg \neg \, r \, r \, r \rangle \\ & \stackrel{\check{=}}{=}_{iiii} \langle \ulcorner \operatorname{pre} \ell \, \neg \, \neg \, \neg \, r \, r \rangle \\ & \stackrel{\check{=}}{=}_{iiii} \langle \ulcorner \operatorname{pre} \ell \, \neg \, \neg \, \neg \, r \, r \rangle \\ & (\operatorname{definition of} \stackrel{\check{=}}{=}) \, . \end{array}$$

When evaluated – by deduction tree evaluation  $\varepsilon_d$  – on argument  $\langle a; 7 \rangle \in \langle \mathbb{N}^2 \rangle$  above – this deduction tree, say  $dtree_k$ , should (and will) give the following inference tree  $\varepsilon_d(dtree_k/\langle a; 7 \rangle)$  in Object Level Language:

$$\varepsilon_d(dtree_k/\langle a;7\rangle) =$$

 $\sim \neg \neg \odot \langle \neg pre \neg /a; 7 \rangle$ 

(definition of  $\dot{-}$ ).

$$s \ a \dot{-} s \ 7 = a \dot{-} 7$$

$$s \ a \dot{-} s \ 7 = \operatorname{pre}(s \ a) \dot{-} 7 \qquad \operatorname{pre}(s \ a) \dot{-} 7 = a \dot{-} 7$$

$$(U_3) \frac{s \ a \dot{-} s \ 7 = (s \ a \dot{-} s \ 0) \dot{-} 7}{s \ a \dot{-} s \ 7 = \operatorname{pre}(s \ a \dot{-} s \ 7)}$$

$$a \dot{-} s \ 7 = \operatorname{pre}(a \dot{-} 7)$$

$$pre \ s \ a = a$$

Deduction- and <u>Inference</u> trees above contain some "macros", for example GOODSTEIN's uniqueness rule (U<sub>3</sub>), which is a **Theorem** of **PR**, **PR**<sub>A</sub>, and hence of  $\pi_O$ **R**. Without such macros, concrete <u>inferences</u>/deductions would become very deep and long. But theoretically, we can describe these trees and their evaluation rather effectively by (primitive) Recursion on **axioms** and axiom **schemata** of our Theorie(s),  $\pi_O$ **R**.

**Deduction Trees for Theory**  $\pi_O \mathbf{R}$ : We introduce now the family  $dtree_k$ ,  $k \in \mathbb{N}$  of  $\pi_O \mathbf{R}$ 's (internal) – "fine grain" –  $deduction \ trees$ : "fine grain" is to mean, that each (HORN type) implication in such a tree falls in one of the following cases:

- Node entry is an equation directly given by (internalised) axiom.
- A bar stands for an implication of at most two "down stairs" (internal) premise-equations implying "upwards" a conclusion-equation, directly by a suitable (internal) instance of an **axiom** schema of the Theory considered, here Theorie(s)  $\pi_O \mathbf{R}$ .

So we are lead to **define** the natural-numbers-indexed family  $dtree_k$  as follows:

$$dtree_k = dtree_k^{\pi_O \mathbf{R}} : \mathbb{N} \to Bintree_{PR_A} \subset \mathbb{X}$$

is PR given by

$$dtree_0 = t_0 = \langle \lceil id \rceil \stackrel{.}{=}_0 \lceil id \rceil \rangle =_{\text{by def}} \langle \lceil id \rceil; \lceil id \rceil \rangle \in Bintree_{\text{PR}_A},$$
  
$$dtree_k = \langle \langle u_k; v_k \rangle; \langle dtree_{i(k)}; dtree_{j(k)} \rangle \rangle : \mathbb{N} \to Bintree_{\text{PR}_A}^2,$$

the latter written symbolically

$$dtree_k = \underbrace{\begin{array}{ccc} u_k \stackrel{.}{=}_k v_k \\ \hline u_i \stackrel{.}{=}_i v_i & \underbrace{u_j \stackrel{.}{=}_j v_k} \\ t_{ii} & t_{ji} & t_{ij} & t_{jj} \end{array}}_{}$$

with – as always below – left resp. right *predecessors* abbreviated  $i := i(k), j := j(k) : \mathbb{N} \to PR^2$ , and recursively: ii := i(i) = i(i(k)) etc.

 $Bintree_{PR_A} \subset \mathbb{X}$  above denotes the (predicative) subset of those (nested) lists of natural numbers which code binary trees with nodes labeled by  $\mathbf{PR_A}$  code pairs, meant to code internal  $\mathbf{PR_A} \cong \mathbf{PR_A}^{\mathbb{X}}$  equations.

Argumented Deduction Trees as Similarity Trees: Things become easier, in particular so evaluation of argumented, instantiated deduction trees, if treated in the wider frame of Similarity trees

$$Stree =_{def} Bintree_{(PR \times \mathbb{X}_{\square})^2} \subset \mathbb{N}.$$

By **definition**, Stree is the predicative set of (coded) binary trees with nodes labeled by Similarity pairs  $u/x \sim v/y$ , of pairs of map-code/argument pairs, called "Similarity pairs", since in the interesting, legitimate cases, they are expected to be converted into equal pairs, by (deduction-) tree evaluation  $\varepsilon_d$ .

General form of  $t \in Stree$ :

$$t = \frac{u/x \sim v/y}{\frac{u'/x' \sim v'/y'}{t' \quad \tilde{t}'} \quad \frac{u''/x'' \sim v''/y''}{t'' \quad \tilde{t}''}}$$

 $t', \ldots, \tilde{t}'' \in Stree$  have (strictly) lesser depth than t.

In the legitimate cases these pairs are "expected" to become equal under Stree-evaluation  $\varepsilon_d$  below – argumented deduction tree evaluation: legitimate are just argumented deduction trees, of form  $dtree_k/x$ .

We will **define** Stree-evaluation  $\varepsilon_d$ : Stree  $\rightarrow$  Stree iteratively as  $CCI_O$  via a PR evaluation step  $e_d = e_d(t)$ : Stree  $\rightarrow$  Stree and a complexity  $c_d = c_d(t)$ : Stree  $\rightarrow \mathbb{N}[\omega]$ .

[Ordinal O is here always choosen to extend  $\mathbb{N}[\omega]$ . Notation  $\varepsilon_d$ ,  $e_d$ ,  $e_d$ , is choosen because restriction to argumented <u>deduction</u> trees "is meant".]

This construction of  $\varepsilon_d$  will extend basic evaluation  $\varepsilon : \operatorname{PR} \times \mathbb{X} \to \operatorname{PR} \times \mathbb{X} \to \mathbb{X}$ , by suitable extension of basic  $step\ e : \operatorname{PR} \times \mathbb{X} \to \operatorname{PR} \times \mathbb{X}$ , and basic descending  $complexity\ c_{\varepsilon}(u,a) = c_{\operatorname{PR}}(u) : \operatorname{PR} \times \mathbb{X} \to \operatorname{PR} \to \mathbb{N}[\omega]$ .

We will see in next section that **definition** of tree evaluation step  $e_d = e_d(t)$  needs formal definition of argumentation of arbitrary (legitimate) deduction trees,  $(dtree_k, x) \mapsto TreeArg(dtree_k, x) = dtree_k/x \in Stree$ .

This will be the first, formally long, task to accomplish. For making things homogeneous, we identify pure, argument-free trees, node-labeled with map pairs  $u \sim v$ , with dummy argumented trees, in dumTree  $\subset$  Stree, dummy arguments given to (left and right sides of) all of its similarity pairs:

 $\langle u \sim v \rangle \mapsto \langle u/\square \sim v/\square \rangle$ , in particular  $dtree_k$  is identified with  $dtree/\square \in dumTree \subset Stree$  obtained this way.

We now give **Tree-Argumentation** – by **case distinction** PR on *nesting* depth of (arbitrary)  $t \in dumTree$ , for suitable arguments to be spread down, from root of t, arguments out of  $\mathbb{X}$ , in particular out  $f(\mathbb{X} \times \mathbb{N}) \subset \mathbb{X}$  etc.

Cases of Tree-Argumentation, by equation resp. HORN clause meant to deduce root (or branch) equation  $u \sim v$  from left and right antecedents, see figure above of t with this (general) root,

This type of display of up-to-two explicit (binary) levels, plus recursive mention of lower branches, will suffice all our needs: two levels are enough for dislay of HORN type implications, from (up to two) equations to one equation.

- (unconditioned) equational case  $EquCase \subset Stree$  for TreeArg:

$$\langle u/\square \sim v/\square \rangle/x =_{\text{def}} \langle u/x \sim v/x \rangle$$
  
=  $_{\text{by def}} \langle \langle u; x \rangle; \langle v; x \rangle \rangle : (\text{PR}_{\text{A}}^{\mathbb{X}})^2 \times \mathbb{X} \to \textit{Stree} :$ 

replace the "waiting" dummy arguments by two equal (!) "real" ones.

This case covers in particular reflexivity of equality, associativity of composition, bi-neutrality of identities, terminality of !, Godements and Fourman's equations for the induced, as well as the *equations* for iteration.

- symmetry of equality case SymCase: straight forward.
- transitivity-of-equality case (basic forking case): for  $t \in dumTree$  of form

$$t = \frac{u/\square \sim w/\square}{\frac{u/\square \sim v/\square}{t' \quad \tilde{t'}} \quad \frac{v/\square \sim w/\square}{t'' \quad \tilde{t''}}}$$

(hence t',  $\tilde{t}'$ , t'',  $\tilde{t}''$  all in dumTree), we **define** recursively:

$$t/x =_{\text{def}} \frac{u/x \sim w/x}{\frac{u/x \sim v/x}{t'/x \quad \tilde{t}'/x} \quad \frac{v/x \sim w/x}{t''/x \quad \tilde{t}''/x}}$$

- composition compatibility case:  $t \in dumTree$  of form

$$t = \frac{v \odot u/\square \sim v' \odot u'/\square}{\underbrace{v/\square \sim v'/\square}_{t'} \underbrace{u/\square \sim u'/\square}_{\tilde{t}''}}$$

with all branches in dumTree (or empty). Here we define

$$t/x =_{\text{def}} \frac{v \odot u/x \sim v' \odot u'/x}{\underbrace{v/\square \sim v'/\square}_{t'} \underbrace{u/x \sim u'/x}_{t''/x}}$$

[Actual argument is given to pair  $u \sim u'$  of first factors, and – recursively – to its deduction tree.]

- compatibility-of-cylindrification case: straight forward

Remain the following two cases:

- FR! Case, of Uniqueness of initialised iterated:

for 
$$t =$$

$$\frac{w/\square \sim \langle v^{\S} \odot \langle \lceil id \rceil \lceil \times \rceil u \rangle \rangle / \square}{\langle w \odot \langle u; \lceil 0 \rceil \rangle \rangle / \square \sim u/\square} \frac{\langle w \odot \langle v \lceil \times \rceil \lceil s \rceil \rangle \rangle / \square \sim \langle v \odot w \rangle / \square}{t'}$$

we define

$$t/\langle x; n \rangle =_{\text{def}}$$

$$\frac{w/\langle x;n\rangle \sim v^{\S} \odot \langle \lceil id \rceil \lceil \times \rceil u\rangle / \langle x;n\rangle}{\frac{w \odot \langle u; \lceil 0 \rceil \rangle / x \sim u/x}{t'/x} \frac{w \odot \langle v \lceil \times \rceil \lceil s \rceil \rangle / \langle x;n\rangle \sim \langle v \odot w\rangle / \langle x;n\rangle}{t''/\langle x;n\rangle}}$$

$$t/\langle a;7\rangle =_{\text{def}}$$

$$\frac{w/\langle a;7\rangle \sim v^{\S} \odot \langle \lceil id \rceil \lceil \times \rceil u \rangle / \langle a;7\rangle}{\frac{w \odot \langle u; \lceil 0 \rceil \rangle / a \sim u/a}{t'/a} \frac{w \odot \langle v \lceil \times \rceil \lceil s \rceil \rangle / \langle a;7\rangle \sim \langle v \odot w \rangle / \langle a;7\rangle}{t''/\langle a;7\rangle}$$

- final, extra case  $\pi_O Case$ , of on-terminating ("finite") descent, extra for axis Theory  $\pi_O \mathbf{R}$  - corresponding to schema ( $\pi_O$ ) of on-termination of descending chains in Ordinal  $O \succeq \mathbb{N}[\omega]$ . This case is hard – and logically not self-evident, because it is self-referential in a sense:

The first thing to do is internalisation of (HORN) clause  $(\pi_O \mathbf{R})$ . We begin with internalisation of **definitions**  $DeSta[c|p](a): A \to 2$ , – of  $\underline{Descent} + \underline{Stationarity}$  – of complexity c, with each application of (predecessor) step p, as well as  $\underline{Termination}$   $\underline{C}$ omparison formula (predicate) into – obvious –

**Definitions** – "abbreviations" – defining  $\mathbf{PR_A} \cong \mathbf{PR_A}^{\mathbb{X}}$  maps desta = desta(u, v):  $\mathrm{PR_A} \times \lceil \mathbb{X}, O \rceil \to \lceil \mathbb{X}, 2 \rceil$  (internal descent + stationarity), and terc = terc(u, v, w):  $\mathrm{PR_A} \times \lceil \mathbb{X}, O \rceil \times \lceil \mathbb{X}, 2 \rceil \to \lceil \mathbb{X}, 2 \rceil$  (internal  $termination\ comparison$ ), are immediate, "term by term."

Free variable  $w \in [X, 2]$  stands for an internal *comparison* predicate, and terc(u, v, w) says – internally – that reaching <u>c</u>omplexity zero: <u>ter</u>minating, when iterating u "sufficiently" often, makes *comparison* w (internally) true:

All this when "completely" evaluated on suitable argument out of X. The internal conclusion (root) equation for w then is  $w = \lceil \text{true} \rceil$ .

Putting all this together we arrive at the following type of dummy argumented tree t in the actual  $\pi_O Case$ :

$$t = \frac{w/\square \sim \lceil \text{true} \rceil / \square}{\frac{desta(u, v)/\square \sim \lceil \text{true} \rceil / \square}{t'} \frac{terc(u, v, w)/\square \sim \lceil \text{true} \rceil / \square}{\tilde{t}''}}$$

with, as always above, branches t',  $\tilde{t}'$ , t'',  $\tilde{t}'' \in dumTree \subset Stree$  all dummy argumented Similarity trees.

In analogy to the cases above, we are led to **define** for t of the actual form:

$$t/x =_{\mathrm{def}}$$

$$\frac{w/x \sim \lceil \text{true} \rceil / x}{\frac{desta(u,v)/x \sim \lceil \text{true} \rceil / x}{t'/x} \frac{terc(u,v,w)/\langle x; n_{+} \rangle \sim \lceil \text{true} \rceil / \langle x; n_{+} \rangle}{t''/\langle x; n_{+} \rangle}}$$

These are the regular cases. Cases not covered up to here are considered irregular, and aborted by deduction-tree evaluation step  $e_d = e_d(t)$ : Stree  $\rightarrow$  Stree to be **defined** below, into  $\langle id/\square \sim id/\square \rangle \in dumTree \subset Stree$ .

**Dangerous Bound** in case  $(\pi_O)$  above: If one wants to *spread down* a given argument, down from the *root* of a dummy argumented tree to (the nodes of) its *branches*, one may think that it be necessary to give all arguments needed on the way top down already to the *root equation*.

In our actual "argumentation case" above, we did **not** give right component of a pair  $\langle x; n \rangle \in \langle \mathbb{X} \rangle^2$  to the *root* equation, only its left component x. Only right subtree gets "full" argument – of form  $\langle x; n_+ \rangle$  – substituted at actual argumentation step.

Logically, argument (part)  $n_+ \in \mathbb{N}$  has the character of a bound variable, hidden to the equation on top, here

" $w/x \sim \lceil \text{true} \rceil$ ", and to all equations way up to the "global" root of the deduction tree provided with arguments so far.

"Free" variable  $n_+$  is to mean here classically a variable which is universally bound within an implication, more specifically: a variable which is existentially bound in the premise of (present) implication, since this variable does not appear within the conclusion of the implication.

In classical Free-Variables Calculus, we would have to make sure that the fresh Free Variable – here "over"  $\mathbb{N}$  – given to the right hand branch above, i. e. to terc(u,v,w) and its deductive descendants, gets not the name of any (free) variable already occurring as a component of "x" in the present context. This possible conflict would be resolved classically by counting names of Free Variables – here of  $type \mathbb{N}$  – given during argumentation, and by giving to such a variable to be introduced in fresh – as in present case – an indexed name with index not used so far: this motivates notation " $n_+$ " for this "fresh" variable.

In our *categorical* Free-Variables Calculus – with Free Variables <u>interpreted</u> as (nested) *projections*, we interprete this *fresh* variable  $n_+$  introduced in "critical" argumentation case above, as – additional – right projection

$$\langle n_+ \rangle := \langle r_{\mathbb{X}, \mathbb{N}} \rangle : \mathbb{X} \supset \langle \mathbb{X} \times \mathbb{N} \rangle \to \langle \mathbb{N} \rangle,$$

of extended Cartesian product  $\langle \mathbb{X} \times \mathbb{N} \rangle$ , extending argument domain  $\mathbb{X}$  for root  $\langle w/\square \sim \lceil \text{true} \rceil / \square \rangle$ . This way, categorically, variable  $\langle n_+ \rangle$  behaves in fact – intuitively – as a fresh Free Variable in the actual context.

# 4 Evaluation Step on Map-Code/Argument Trees

We attempt now to extend basic evaluation  $\varepsilon$  of map-code argument pairs which has been given above as iteration of step

$$e = e(u, x) = (e_{map}(u, x), e_{arg}(u, x)) : PR \times X \rightarrow PR \times X,$$

into a – terminating (?) – evaluation  $\varepsilon_d$  of Similarity trees t, of general form displayed earlier.

This evaluation comes – in the present framework – as a  $(CCI_O)$  iteration of a suitable (descent) step

$$e_d = e_d(t) : Stree \to Stree,$$

on the set  $Streesubset\mathbb{N}$  of  $Similarity\ trees.$ 

[ Stree will host – see below – in particular all the intermediate results of (iteratively) applying **deduction-tree evaluation step**  $e_d$  to trees of form  $t = dtree_k/x$ : pure decuction trees, argumented by (suitable) constants or variables, argumentation see foregoing section.]

**Definition** of argumented-deduction-tree evaluation step

$$e_d = e_d(t) : Stree \rightarrow Stree$$

recursively (PR) on depth(t), i.e. on the *nesting depth* of t, as a (binary) tree. More precisely: by recursive case distinction on the form of the two upper layers of t.

\* For t near flat, i.e. of form

$$t = \frac{u/x \sim v/y}{\langle \lceil id \rceil / x' \sim \lceil id \rceil / y' \rangle \quad \langle \lceil id \rceil / x' \sim \lceil id \rceil / y' \rangle}$$

we define  $e_d(t) =_{\text{def}} root(t) = \langle u/x \sim v/y \rangle \in Stree$ . [In real deduction-life we expect here  $x' \doteq y'$ .]

"The" **exception** is the following **argument shift simplification** case – arising in *deduction* context below from the (internalised) schema of composition **compatibility** with equality (*between* maps):

• Exceptional tree  $t \in Stree$  is one of form

$$t \ = \ \frac{v \odot \lceil id \rceil / x \sim v' \odot \lceil id \rceil / x}{\underbrace{v / \square \sim v' / \square}_{t'} \qquad \lceil id \rceil / x \sim \lceil id \rceil / x}$$

 $t', t'' \in dumTree$ , pure map code trees, dummy argumented at each argument place. t' and/or t'' may be empty.

**Note** that in this – at least at surface – *legitimate* case, left and right argument, x, of root "equation" of t is the same. If not, t would be considered illegitimate, and aborted by  $e_d$  into  $t_0/\square =_{\text{def}} \langle \operatorname{id}/\square \sim \operatorname{id}/\square \rangle$ .

For t of exceptional (but regular) form above, we now **define** recursively:

$$e_d(t) =_{\text{def}} \frac{\langle v/x \sim v'/x \rangle}{t'/x \quad t''/x}$$

This is **shift** and **simplification:** right branch with its pair of identities is obsolete, its (common) argument x is shifted, formally substituted, into v and v' as well as into the trees "responsable for the proof" of hitherto not (yet) argumented equation, formally: "Similarity"  $v/\square \sim v'/\square$ .

**Comment:** Present **case** is the first and only "surface" case, where **definition** for evaluation step  $e_d$  on "deduction trees" coming nodewise with variables, needs *substitution*, *instantiation* of a (general) variable – here  $x \in \mathbb{X}$  – into a general (!) "deduction tree".

By that reason, we had to consider the whole bunch of (quasi) legitimate cases of "deduction" trees and their "natural" spread down argumentation into Similarity trees:  $dtree_k/x \in Stree$ .

\* Standard Case which applies "en cours de route" of stepwise tree-evaluation  $\varepsilon_d$ , step  $e_d$ , where step  $e_d: Stree \to Stree$  is to apply basic evaluation step  $e: \operatorname{PR} \times \mathbb{X} \to \operatorname{PR} \times \mathbb{X}$  to all map-code/argument pairs labeling the nodes of tree  $t \in Stree$  in question:

This is the case when  $t \in Etree$  is of form  $t = \frac{u/x \sim v/y}{t'}$ 

and not exceptional. Here we define – PR on depth(t):

$$e_d(t) =_{\text{def}} \frac{e(u/x) \sim e(v/y)}{e_d(t') e_d(t'')}$$

**SubException:** For  $t' \in dumTree$  we **define** in this standard superCase:

$$e_d(t) =_{\text{def}} \frac{e(u/x) \sim e(v/y)}{t' e_d(t'')}$$

Dummy tree t' waits for *later argumentation*, to come from evaluated right branch; an empty tree t' in this case remains empty under  $e_d$ .

What we still need, to become (intuitively) sure on **termination** of iteration

$$e_d^m(t): Stree \times \mathbb{N} \to Stree,$$

i. e. to become sure that this iteration (stationarily) results in a tree t of form  $t = \langle \lceil id \rceil / \bar{x} \sim \lceil id \rceil / \bar{y} \rangle$ , this for m "big enough", is a suitable tree **complexity** 

$$c_d = c_d(t) : Stree \to O\mathbb{N}[\omega],$$

which **strictly descends** – above complexity zero – with each application of  $step \ e_d$ .

This just in order to give within  $\pi_O \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O)$ , by its schema  $(\pi) = (\pi_{\mathbb{N}[\omega]})$   $(O \succeq \mathbb{N}_{\omega})$ , on-terminating descent of argumented (deduction) tree evaluation  $\varepsilon_d$ , which is **defined** – analogeously to basic evaluation  $\varepsilon$  – as the formally partial map

$$\varepsilon_d = \varepsilon_d(t/x) = \text{by def} \ e_d^{\S}(t/x, \mu\{m \mid c_d \ e_d^m(t/x) \doteq 0\}) : Stree \rightharpoonup Stree.$$

**Definition** of (argumented-)deduction tree complexity

$$c_d = c_d(t) : Stree \to \mathbb{N}[\omega] \preceq O$$

as natural extension of basic map complexity

$$c = c_{\varepsilon}(u, x) = c_{PR}(u) : PR \times \mathbb{X} \rightarrow PR \rightarrow \mathbb{N}[\omega]$$

to argumented "deduction" trees, definition in words:

 $c_d(t)$  is t's number of inference bars plus the sum of all map code complexities  $c_{PR}(u)$  for  $u \in PR$  appearing in t's node labels (including the dummy argumented ones). The sum is the sum of polynomials in  $\mathbb{N}[\omega]$  – just here we need the polynomial structure of Ordinal  $O := \mathbb{N}[\omega]$ .

[Formally this **definition** is PR on depth of tree t. As in case  $c_{\varepsilon}$  for basic evaluation  $\varepsilon = \varepsilon(u, x) : \text{PR} \times \mathbb{X} \to \text{PR} \times \mathbb{X}$ , the arguments of the trees do not enter in this complexity.]

An easy (recursive) calculation of the – different structural cases for – trees  $t \in Stree$  proves

**Deduction-Tree Evaluation Descent Lemma:** Extended PR evaluation step  $e_d = e_d(t)$ :  $Stree \to Stree$  strictly descends with respect to (PR) extended map code complexity  $c_d = c_d(t)$ :  $Stree \to \mathbb{N}[\omega]$  above complexity zero, i. e.

$$c_d(t) > 0 \implies c_d \ e_d(t) < c_d(t) : Stree \to \mathbb{N}[\omega]^2 \to 2,$$

and is stationary at complexity zero:

$$c_d(t) \doteq 0 \implies e_d(t) \doteq t : Stree \rightarrow 2.$$

[We have choosen complexity  $c_d$  just in a manner to make sure this stepwise descent.]

So intuitively we expect – and can <u>derive</u> in **set theory** – that argumented-deduction-tree evaluation  $\varepsilon_d: Stree \to Stree$  for  $\pi_O \mathbf{R}$ , **defined** as Complexity Controlled Iteration (CCI<sub>O</sub>) of step  $e_d$  – descending complexity  $c_d: Stree \to \mathbb{N}[\omega] \leq O$  – always terminates, with a correct result of form  $\langle \mathrm{id}/\bar{x} \sim \mathrm{id}/\bar{y} \rangle$ , with  $\bar{x} \doteq \bar{y}$ , the latter when applied to a given argumented deduction tree of form  $t = dtree_k/x$ .

We will not **prove** this termination: Termination will be only a **Condition** in *Main Theorem* next section.

#### 5 Termination-Conditioned Soundness

Termination Condition – a  $\mathbf{PR_{A}}$ -predicate – for  $\mathrm{CCI}_O$ 's was introduced above, and reads for (basic, iterative) evaluation

$$\varepsilon = \varepsilon(u, x) = e^{\mu\{n \mid c_{PR}e^n \doteq 0\}} : PR \times \mathbb{X} \rightharpoonup \mathbb{X} :$$

$$[m \ def \ \varepsilon(u, x)] =_{def} [c_{\varepsilon} \ e^m(u, x) \doteq 0] : \mathbb{N} \times PR \times \mathbb{X} \to 2,$$

$$m \in \mathbb{N}, \ u \in PR, \ x \in \mathbb{X} \text{ all free.}$$

Analogously for Argumented Deduction Tree evaluation defined as CCI "over" step  $e_d = e_d(t) : Stree \to Stree$ , t an "argumented deduction tree", frame Stree, complexity  $c_d : Stree \to \mathbb{N}[\omega]$  measuring descent.

Here domination, truncation, quantitative "definedness" of termination reads

$$[m \ def \ \varepsilon_d(t)] =_{\text{bv def}} [c_d \ e_d^m(t) \doteq 0] : \mathbb{N} \times Stree \rightarrow 2, \ m, \ t \ free.$$

By definition of  $\varepsilon$  and  $\varepsilon_d$  – in particular by stationarity at complexity zero, we obtain with this "free" truncation  $(m \in \mathbb{N} \text{ free})$ :

$$[m \ def \ \varepsilon(u,x)] \implies [c_{PR} e^m(u,x) \doteq 0] \land [\varepsilon(u,x) \doteq r \ e^m(u,x)], \ \underline{and}$$
$$[m \ def \ \varepsilon_d(t)] \implies [c_d e_d^m(t) \doteq 0] \land [\varepsilon_d(t) \doteq e_d^m(t)].$$

Using the above abbreviations, we state the

#### Main Theorem, on Termination-Conditioned Soundness:

For theories  $\pi_O \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O)$ , of Primitive Recursion with (predicate abstraction and) on-terminating descent in Ordinal  $O \succeq \mathbb{N}[\omega]$  extending  $\mathbb{N}[\omega]$ , we have

(i) Termination-Conditioned Inner Soundness:

$$\pi_{O}\mathbf{R} \vdash [u \stackrel{.}{=}_{k} v] \land [m \ def \ \varepsilon_{d}(dtree_{k}/a)]$$

$$\implies m \ def \ \varepsilon(u, a), \ \varepsilon(v, a) \land :$$

$$\varepsilon(u, a) \stackrel{.}{=} r \ e^{m}(u, a) \stackrel{.}{=} r \ e^{m}(v, a) \stackrel{.}{=} \varepsilon(v, a),$$

$$u, v \in \mathrm{PR}, \ a \in \mathbb{X}, \ m \in \mathbb{N} \ \mathrm{free}.$$

In words, this Truncated Inner Soundness says: Theory  $\pi_{\mathcal{O}}\mathbf{R}$  derives:

If for an internal  $\pi_O \mathbf{R}$  equation u = k v the (minimal) argumented deduction tree  $\mathrm{dtree}_k/a$  for u = k v, top down argumented with  $a \in \mathbb{X}$  admits complete argumented-tree evaluation – i. e. If tree-evaluation becomes stationary after a finite number m of evaluation steps  $e_d$  –,

**Then** both sides of this internal (!) equation are completely evaluated on a, by (at most) m steps e of original, basic evaluation  $\varepsilon$ , into equal values.

Substituting in the above "concrete" codes into u resp. v, we get, by Objectivity of evaluation  $\varepsilon$ :

(ii) Termination-Conditioned Objective Soundness for Map Equality:

For  $\pi_O \mathbf{R}$  maps (i. e.  $\mathbf{PR}_{\mathbf{A}}$  maps)  $f, g : \mathbb{X} \supseteq A \to B \subseteq \mathbb{X}$ :

$$\pi_{O}\mathbf{R} \vdash [ \lceil f \rceil \stackrel{.}{=}_{k} \lceil g \rceil \land m \ def \ \varepsilon_{d}(dtree_{k}/a) ]$$

$$\implies f(a) \stackrel{.}{=}_{B} r \ e^{m}(\lceil f \rceil, a) \stackrel{.}{=}_{B} r \ e^{m}(\lceil g \rceil, a) \stackrel{.}{=}_{B} g(a) :$$

If an internal deduction-tree for (internal) equality of  $\lceil f \rceil$  and  $\lceil g \rceil$  is available, and If on this tree – top down argumented with a given  $a \in A$  – tree-evaluation **terminates**, will say: iteration of evaluation step  $e_d$  becomes **stationary** after a finite number m of steps, **Then** equality  $f(a) \doteq_B g(a)$  of f and g at this argument is the consequence.

Specialising this to case  $f := \chi : A \to 2$ ,  $g := \text{true}_A : A \to 2$ , we eventually get

(iii) Termination-Conditioned Objective Logical Soundness:

$$\pi_O \mathbf{R} \vdash Prov_{\pi_O \mathbf{R}}(k, \lceil \chi \rceil) \land m \ def \ \varepsilon_d(dtree_k/a) \implies \chi(a) : \mathbb{N}^2 \to 2 :$$

If tree-evaluation of a deduction tree of a predicate  $\chi: \mathbb{X} \to 2$  – the tree top down argumented with "an"  $a \in \mathbb{X}$  – **terminates** after a finite number m of tree-evaluation steps, **Then**  $\chi(a) \doteq \text{true}$  is the consequence.

[The latter statement reminds at the Second Uniform Reflection Principle RFN'(T) in SMORYNSKI 1977.]

**Proof** of "axis" Termination-Conditioned Inner Soundness:

Without reference to formally partial maps  $\varepsilon : \operatorname{PR} \times \mathbb{X} \to \mathbb{X}$  and  $\varepsilon_d : Stree \to Stree$  – alone in  $\pi_O \mathbf{R}$  terms  $e : \operatorname{PR} \times \mathbb{X} \to \operatorname{PR} \times \mathbb{X}$ ,  $c_{\operatorname{PR}} : \operatorname{PR} \to \mathbb{N}[\omega]$ , as well as  $e_d : Stree \to Stree$  and  $c_d : Stree \to \mathbb{N}[\omega]$  – this **Theorem** reads:

$$\pi_{O}\mathbf{R} \vdash u \stackrel{.}{=}_{k} v \wedge c_{d} e_{d}^{m}(dtree_{k}/a) \stackrel{.}{=} 0$$

$$\implies c_{PR} r e^{m}(u, a) \stackrel{.}{=} 0 \stackrel{.}{=} c_{PR} r e^{m}(v, a)$$

$$\wedge r e^{m}(u, a) \stackrel{.}{=} r e^{m}(v, a) : \mathbb{N}^{2} \times PR^{2} \rightarrow 2 \qquad (\check{\bullet})$$

**Proof** of  $(\check{\bullet})$  is by (primitive) recursion on depth( $dtree_k$ ) of k th (internal) deduction tree  $\pi_O \mathbf{R}$ -proving its root  $u \check{=}_k v$ . Argumented tree  $dtree_k/a$  then has same depth, and strictly speaking, we argue PR on depth( $dtree_k/a$ ), by recursive case distinction on the form of  $dtree_k/a$ .

Flat SuperCase depth( $dtree_k$ ) = 0, i.e. SuperCase of unconditioned, axiomatic (internal) equations u = v:

We demonstrate our Proof strategy on the first involved of these cases, namely associativity of (internal) composition:

$$AssCase =_{\operatorname{def}} [dtree_k \doteq \langle \langle w \odot v \rangle \odot u \rangle \check{=}_k \langle w \odot \langle v \odot u \rangle \rangle] : \mathbb{N} \times \mathrm{PR}^3 \to 2.$$

Here we first evaluate left hand side of equation substituted, "instantiated" with (Free-Variable) argument  $a \in A$ :

```
\pi_{O}\mathbf{R} \vdash AssCase \implies :
m \ def \ \varepsilon_{d}(dtree_{k}/a)
\implies [m \ def \ \varepsilon(\langle w \odot v \rangle \odot u, a)]
\implies [m \ def \ \varepsilon(u, a)] \land [m \ def \ \varepsilon(w \odot v, \varepsilon(u, a))]
\land \varepsilon(\langle w \odot v \rangle \odot u, a) \stackrel{.}{=} \varepsilon(w \odot v, \varepsilon(u, a))
[\implies \text{the above }]
\land [m \ def \ \varepsilon(v, \varepsilon(u, a))] \land \varepsilon(v \odot u, a) \stackrel{.}{=} \varepsilon(v, \varepsilon(u, a))
\land [m \ def \ \varepsilon(w, \varepsilon(v \odot u, a))]
\land \varepsilon(w \odot v, \varepsilon(u, a)) \stackrel{.}{=} \varepsilon(w, \varepsilon(v \odot u, a))
```

Same way – evaluation on a composed works step e by step e successively, it does not care here on brackets  $\langle \ldots \rangle$  – we get for the right hand side of the equation:

$$\pi_O \mathbf{R} \vdash AssCase \implies [m \ def \ \varepsilon_d(dtree_k/a) \implies : \\ m \ def \ \varepsilon(w \odot \langle v \odot u \rangle, a) \wedge \varepsilon(w \odot \langle v \odot u \rangle, a) \stackrel{\cdot}{=} \varepsilon(w, \varepsilon(v, \varepsilon(u, a)))].$$

Put together:

$$\pi_{O}\mathbf{R} \vdash \langle\langle w \odot v \rangle \odot u \rangle \stackrel{.}{=}_{k} \langle w \odot \langle v \odot u \rangle\rangle \implies [m \ def \ \varepsilon_{d}(dtree_{k}/a) \implies :$$

$$[m \ def \ \varepsilon(\langle w \odot v \rangle \odot u, a)] \land [m \ def \ \varepsilon(w \odot \langle v \odot u \rangle, a)]$$

$$\land \varepsilon(\langle w \odot v \rangle \odot u, a) \stackrel{.}{=} \varepsilon(w, \varepsilon(v, \varepsilon(u, a))) \stackrel{.}{=} \varepsilon(w \odot \langle v \odot u \rangle, a).]$$

This proves assertion  $(\check{\bullet})$  in this associativity-of-composition case.

Analogeous **Proof** for the other **flat**, equational cases, namely Reflexivity of Equality, Left and Right Neutrality of Identities, Functor property of Cylindrification, Godement equations for induced into Cartesian (!) product, Fourmann's equation for uniqueness of the induced, and finally, the two equations (!) for the (internally) iterated.

We give the **Proof** for the latter case explicitely, since it is logically the most involved one for Theory **PR**, and "characteristic" for treatment of (internal) potential infinity.

For commodity, we choose – equivalent – "bottom up" presentation of this iteration case, namely *iteration step* equation  $f^{\S}(a, s, n) = f^{\S}(f(a), n)$  instead of earlier axiom  $f^{\S}(a, s, n) = f^{\S}(f(a), n)$ , formally:

$$f^{\S}\circ (A\times s)=f^{\S}\circ (f\times \mathbb{N}): A\times \mathbb{N}\to A\times \mathbb{N}\to A.$$

The **anchor** case statement for the internal iterated  $u^{\lceil \S \rceil}$  is trivial: apply evaluation step e once.

Bottom up iteration step, Case of genuine iteration equation:

$$\pi_{O}\mathbf{R} \vdash iteqCase(k, u)$$

$$[ =_{def} [dtree_{k} \doteq \langle u^{\lceil \S \rceil} \odot \langle \lceil id \rceil \lceil \times \rceil \rceil \rangle \check{=}_{k} u^{\lceil \S \rceil} \odot \langle u \lceil \times \rceil \rceil id \rceil \rangle \rangle]]$$

$$\Longrightarrow : m \ defines \ all \ instances \ of \ \varepsilon \ below, \ and:$$

$$\varepsilon(u^{\lceil \S \rceil} \odot \langle \lceil id \rceil \lceil \times \rceil \rceil \lceil s \rceil \rangle, \langle a; n \rangle) \qquad (1)$$

$$\dot{=} \varepsilon(u^{\lceil \S \rceil}, \varepsilon(\lceil id \rceil \lceil \times \rceil \rceil \lceil s \rceil, \langle a; n \rangle))$$

$$\dot{=} \varepsilon(u^{\lceil \S \rceil}, \langle a; s \ n \rangle) \dot{=} \varepsilon(u^{\lceil \S \rceil} \odot \langle u \lceil \times \rceil \rceil \lceil id \rceil \rangle, \langle a; n \rangle). \qquad (2)$$

This common (termination conditioned) evaluation result for both sides – (1) and (2) – of  $\check{=}_k \in PR^2$ , is what we wanted to show in this general iteration equality case.

[Freyd's uniqueness case, to be treated below, is not an equational case, it is a genuine HORN case.]

Let us turn to the – remaining – genuine Horn cases for assertion ( $\check{\bullet}$ ).

**Comment:** All of our arguments below are to be formally just Free Variables – "undefined elements" – or map constants such as  $0, s0: \mathbb{I} \to \mathbb{N}$ . But since the variables usually occur in premise and conclusion of the HORN clauses – to be <u>derived</u> – of assertion ( $\check{\bullet}$ ), they mean the same throughout such a clause: In this sense their "multiple" occurences are bounded together, with meaning: for all. "But" if such a variable occurs – within an implication – only in the premise, it means intuitively an existence, to imply the conclusio, cf. discussion of tree-argumentation in the  $(\pi_O)$ -case.

**Proof** of Termination-Conditioned Soundness for the "deep", genuine HORN cases of  $dtree_k$ , HORN type (at least) at deduction of root:

Symmetry- and Transitivity-of-equality cases are immdediate.

- Compatibility Case of composition with equality:

$$\frac{dtree_k/a}{v/\square \sim v'/\square} = \frac{\langle v \odot u \rangle/a \sim \langle v' \odot u' \rangle/a}{v/\square \sim v'/\square \qquad dtree_j/a}$$

$$\frac{dtree_{ii(k)}/\square \qquad dtree_{ji(k)}/\square}{dtree_{ji(k)}/\square}$$

with two subcases:

- exceptional, shift case  $u = u' = \lceil \operatorname{id} \rceil$ ,  $dtree_j = t_0 = \langle \lceil \operatorname{id} \rceil \sim \lceil \operatorname{id} \rceil \rangle$ : In this subcase, to be treated separately because of exceptional definition of step  $e_d$  in this case, namely – recursively –

$$e_d(dtree_k/a) = _{\mathrm{by\,def}} dtree_i/a \ (shift\ to\ left\ branch), \ \mathrm{and\ hence}$$
 "then"  $\pi_O\mathbf{R} \vdash m\ def\ \varepsilon_d(dtree_k/a) \Longrightarrow :$   $\varepsilon_d(dtree_k/a) \doteq \varepsilon_d(e_d(dtree_k/a)) \doteq \varepsilon_d(dtree_i/a)$  whence, by induction hypothesis  $(\check{\bullet}_i)$  also:  $\wedge\ \varepsilon(v,a) \doteq \varepsilon(v',a), \ \mathrm{and\ hence}, \ \mathrm{trivially}:$   $\wedge\ \varepsilon(v\odot \lceil id \rceil, a) \doteq \varepsilon(v'\odot \lceil id \rceil, a): \ Soundness \ (\check{\bullet}_k).$ 

Genuine Composition Compatibility Case: not both u, u' code of identity: This case is similar to – and combinatorially simpler than the above. It is easily **proved** by recursion on depth $(dtree_k)$ : we have just to evaluate – truncated soundly – argumented tree  $dtree_j/a$ . This branch evaluation is given by hypothesis because of depth $(dtree_j/a) < depth(dtree_k/a)$ .

- Case of Freyd's (internal) uniqueness of the iterated, is case of tree  $t = dtree_k/\langle a; n \rangle$  of form

**Comment:** w is here an internal comparison candidate fullfilling the same internal PR equations as  $\langle v^{\lceil \S \rceil} \odot \langle u^{\lceil \times \rceil} \cap \operatorname{id} \rangle / \langle a; n \rangle \rangle$ . It should is - Soundness - evaluated identically to the latter, under condition that evaluation of the corresponding argumented deduction tree terminates after finitely many steps, say after m steps  $e_d$ .

Soundness **assertion**  $(\check{\bullet}_k)$  for the present Freyd's uniqueness **case** is **proved** PR on depth( $dtree_i$ ), depth( $dtree_j$ ) < depth( $dtree_k$ ), by established "double recursive" equations – this time for evaluation of the iterated – established above for our dominated, truncated case. These equations give in fact:

$$\pi_{O}\mathbf{R} \vdash fr! Case \implies : m \text{ defines all the following } \varepsilon\text{-terms, and}$$

$$\varepsilon(w, \langle a; 0 \rangle) \doteq \varepsilon(u, a) \doteq \varepsilon(v^{\lceil \S \rceil} \odot \langle u \lceil \times \rceil \rceil \operatorname{rid} \rangle, \langle a; 0 \rangle), \text{ as well as} \quad (\bar{0})$$

$$\pi_{O}\mathbf{R} \vdash fr! Case \implies : m \text{ defines all the following } \varepsilon\text{-terms, and}$$

$$\varepsilon(w, \langle a; s \, n \rangle) = \varepsilon(w \odot \langle \lceil \operatorname{id} \rceil \rceil \times \rceil \rceil \rangle, \langle a; n \rangle) \doteq \varepsilon(v \odot w, \langle a; n \rangle)$$

$$\dot{=} \varepsilon(v, \varepsilon(w, \langle a; n \rangle)) \qquad (\bar{s}).$$

But the same is true for  $v^{\lceil \S \rceil} \odot \langle u^{\lceil \times \rceil} \cap \operatorname{id} \rceil \rangle$  in place of w, once more by (truncated) double recursive equations for  $\varepsilon$ , this time with respect to the *initialised internal iterated*.

( $\bar{0}$ ) and ( $\bar{s}$ ) put together show, by **induction** on iteration count  $n \in \mathbb{N}$  – all other free variables k, m, u, v, w, a together form the passive parameter for this induction – truncated Soundness assertion ( $\check{\bullet}$ ) of the Theorem for this Freyd's uniqueness case, namely:

$$\pi_O \mathbf{R} \vdash fr!Case \implies : m \text{ defines all the following } \varepsilon\text{-terms, and}$$

$$\varepsilon(w, \langle a; n \rangle) \doteq \varepsilon(v^{\lceil \S \rceil} \odot \langle u^{\lceil \times \rceil} \cap id^{\rceil} \rangle, \langle a; n \rangle). \tag{\bullet}_k)$$

Final Case, not so "direct", is internal version of case  $(\pi_O)$  of "finite" descent – in Ordinal  $O \succeq \mathbb{N}[\omega]$  – of ("endo driven")  $\mathrm{CCI}_O$ 's: Complexity Controlled Iterations with complexity values in O. In a sense, treatment of this **axiom** has something of reflexive, since it constitutes theory  $\pi_O \mathbf{R} = \mathbf{PR}_{\mathbf{A}} + (\pi_O)$ , and since on-termination of evaluations  $\varepsilon$  and – "derived" –  $\varepsilon_d$  is forced by "just" this axiom, for  $O := \mathbb{N}[\omega]$ .

**Proof** strategy for this case is "construction" of "super" predecessor  $p_{\pi} = p_{\pi_O}$ , "super" complexity  $c_{\pi}$ , and test predicate  $\chi_{\pi}$ , such that  $p_{\pi}$  descends as long as  $c_{\pi} > 0$ , is stationary at 0 and **proves** Termination Conditioned Soundness in present case by application of schema  $(\pi_O)$  itself (!) to data  $p_{\pi}$ ,  $c_{\pi}$ ,  $\chi_{\pi}$ .

For treatment of this final case, we rely on *internalisation* of **Abbreviations**  $DeSta[p,c]:A\to 2:\underline{De}scent+\underline{Sta}tionarity$  of  $CCI_O$  (given for step  $p:A\to A$  and Complexity  $c:A\to O$ ), as well as  $TerC[p,c,\chi]:A\to 2:\underline{Ter}mination$   $\underline{C}omparison$ .

The internal version of "the above" is – with

 $u \in PR = [X, X]_{\mathbf{PR_A}}$  internalising iteration step  $p : A \to A$ ,

 $v \in [X, O]$  internalising complexity  $c : A \to O$ , and

 $w \in [\mathbb{X},2]$  internalising test  $\chi:A \to 2$  – present argumented deduction tree

 $dtree_k/a =$ 

$$\frac{w/a \sim \lceil \text{true} \rceil}{dtree_{ii}/a \ dtree_{ji}} \frac{terc(u, v, w)/\langle a; n_{+} \rangle \sim \lceil \text{true} \rceil}{dtree_{ij}/\langle a; n_{+} \rangle} \frac{terc(u, v, w)/\langle a; n_{+} \rangle \sim \lceil \text{true} \rceil}{dtree_{ij}/\langle a; n_{+} \rangle}$$

**Comment:** In the present  $\pi_O Case$ , (Free-Variable) argument argument  $n_+ \in \mathbb{N}$  for logical (right) predecessor-branch  $dtree_j$  within present instance  $dtree_k/a$  above, is not part of argument argument "given" to (root of)  $dtree_k$ .

It is thought to be universally quantified within "its" (argumented) right branch  $dtree_j/\langle a; n_+ \rangle$ , so in fact it is thought to be existentially quantified since it appears there just in the premise, cf. discussion – Dangerous Bound –

in foregoing section, on deduction-tree argumentation:  $n_+$  is here a fresh NNO variable, categorically seen as "fresh" name of a right projection.

In what follows, we name this *fresh* NNO-variable  $n_+$  "back" into n. As you will see, there will result from this no confusion, since we work just on two *actual* levels of our argumented deduction tree  $dtree_k/a$ , only the right (argumented) branch comes with a "visible" "extra" NNO variable, now called n, giving substitution, *instantiation*  $dtree_i/\langle a; n \rangle$ .

We now attempt to show the assertion proper,  $(\check{\bullet})$ , for present  $\pi_O Case$ , via the original, *objective*, schema  $(\pi_O)$  *itself*. We use for this the following "super" **instance** of this schema:

- First we choose the (common) complexity/step **Domain**  $A_{\pi} \subset \mathbb{N} \times PR^3 \times A$  - short for " $A_{\pi_O}$ " - predicatively **defined** as

$$A_{\pi} = A_{\pi}(a_{\pi}) = A_{\pi}(m, (u, v, w), a)$$

$$=_{\text{def}} [m \text{ def } \varepsilon(u, a), \ \varepsilon(v, a), \ \varepsilon(v \odot u, a), \varepsilon(w, a)]$$

$$\mathbb{N} \times \text{PR}^{3} \times A \supseteq \mathbb{N} \times (\lceil A, O \rceil \times \lceil A, A \rceil \times \lceil A, 2 \rceil) \times A) \to 2,$$
and composit Free Variable
$$a_{\pi} =_{\text{def}} (m, (u, v, w), a) [= \text{id}_{A_{\pi}}] : A_{\pi} \to A_{\pi} :$$

All of  $a_{\pi}$ 's components free – (nested) projections – in particular so "dominating", formally: truncating,  $m \in \mathbb{N}$ , as well as  $u \in [A, A]$ ,  $v \in [A, O]$ ,  $w \in [A, 2]$ , and  $a \in A$ .

[ $A \subseteq \mathbb{X}$  (as well as O) are considered as <u>meta-variables</u>, ranging over the subobjects of  $\mathbb{X}$ , "i. e." over the Objects of  $\mathbf{PR_A}$  – and the Ordinals (of  $\mathbf{PR_A}$ ) extending  $\mathbb{N}[\omega]$  respectively.]

In present internal proof,  $deduction\ tree$ , we have, with respect to  $left\ predecessor$  branch

$$dtree_i = dtree_{i(k)} \in Stree,$$

of actual deduction tree  $dtree_k$ , in particular with regard to its root:

$$\pi_O Case(k,(u,v,w))/a \implies root dtree_i/a \doteq \langle desta(u,v)/a \sim \lceil true \rceil /a \rangle.$$

- Next ingredient for present application of **descent** schema is **complexity** 

$$c_{\pi} = c_{\pi}(a_{\pi}) : A_{\pi} \to O :$$

Here we choose Objectivisation of *internal* complexity v by **dominated**, **truncated evaluation**, namely

$$c_{\pi} = c_{\pi}(a_{\pi}) = c_{\pi}(m, (u, v, w), a) =_{\text{def}} r e^{m}(v, a) = \varepsilon(v, a) : A_{\pi} \to O.$$

The latter equation – termination with m – follows by **definition** of Domain  $A_{\pi}$  of  $c_{\pi}$ .

[(Just) here we need Ordinal  $O \succeq \mathbb{N}[\omega]$  to extend  $\mathbb{N}[\omega]$ : In the present approach, syntactical complexity of PR map codes takes values in  $\mathbb{N}[\omega]$ . But it is not excluded a priori that in another attempt e.g. Ordinal  $\mathbb{N}^2$  would do.]

- As **predecessor step**  $p_{\pi}$  for present application of **descent** schema  $(\pi_O)$ , again within Theory **PR<sub>A</sub>**, we choose  $p_{\pi} = p_{\pi}(a_{\pi}) : A_{\pi} \to A_{\pi}$ , dominated, truncated by Free Variable  $m \in \mathbb{N}$ , as

$$p_{\pi}(a_{\pi}) = p_{\pi}(m, (u, v, w), a)$$
  
=\def (m, (u, v, w), r e^m(v, a)) = (m, (u, v, w), \varepsilon(v, a)) : A\_{\pi} \to A\_{\pi}.

Here again, as for complexity  $c_{\pi}$  above, **definition** of Domain  $A_{\pi}$  provides termination m def  $\varepsilon(v, a) \doteq_A r e^m(v, a)$  of (iterative) evaluation  $\varepsilon$ .

– In choice of comparison predicate  $\chi_{\pi} = \chi_{\pi}(a) : A_{\pi} \to 2$  we are free: a suitable choice – suitable for the needs of **proof** in the actual case – leads, analogeously to the other " $(\pi_O)$ -data", to externalisation via **evaluation** of an arbitrary internal predicate (free variable)  $w \in [A, 2] \subset PR$ , as follows – same receipt:

$$\chi_{\pi}(a_{\pi}) = \chi_{\pi}(m, (u, v, w), a) =_{\text{def}} r e^{m}(w, a) = \varepsilon(w, a) : A_{\pi} \to 2.$$

Termination m def  $\varepsilon(w, a) \doteq r \ e^m(w, a)$  of  $\varepsilon(w, a) : A_{\pi} \to 2$  is as for complexity  $c_{\pi}$  and predecessor  $p_{\pi}$  above.

For due application of this – now completely defined – **instance** of schema  $(\pi_O)$  – which constitutes Theory  $\pi_O \mathbf{R}$  – we check the two **antecedents**, as follows:

$$\pi_O \mathbf{R} \vdash DeSta_{\pi}(a_{\pi}) : A_{\pi} \to 2 : left \ antecedent, \ \mathbf{and}$$
  
 $\pi_O \mathbf{R} \vdash TerC_{\pi}(a_{\pi}, n) : A_{\pi} \times \mathbb{N} \to 2 \ right \ antecedent:$ 

By definition – with *composit* Free Variable  $a_{\pi} = (m, (u, v, w), a) \in A_{\pi}$  above, actual **Left antecedent** reads:

$$DeSta_{\pi}(a_{\pi}) = [c_{\pi}(a_{\pi}) > 0 \implies c_{\pi} p_{\pi}(a_{\pi}) < c_{\pi}(a_{\pi})]$$

$$\wedge [c_{\pi}(a_{\pi}) \doteq 0_{O} \implies p_{\pi}(a_{\pi}) \doteq_{A_{\pi}} a_{\pi}] : A_{\pi} \to 2,$$

explicitely:

 $DeSta_{\pi}(m,(u,v,w),a) = [m \ defines \ all \ of \ the \ following \ instances \ of \ \varepsilon] \ and \ [\varepsilon(v,a) > 0 \implies \varepsilon(v,\varepsilon(u,a)) < \varepsilon(v,a)] \ \land \ [\varepsilon(v,a) \doteq 0 \implies \varepsilon(u,a) \doteq_A a] : A_{\pi} \to 2,$ 

the latter m-terminations again by choice of Domain  $A_{\pi}$ .

#### - Right Antecedent

$$TerC_{\pi}(a_{\pi}, n) = TerC((m, (u, v, w), a), n) : A_{\pi} \times \mathbb{N} \to 2$$

then is – for present  $(\pi_O)$ -proof instance "necessarily" – **defined** as

$$TerC_{\pi}(a_{\pi}, n) =_{\operatorname{def}} \left[ c_{\pi} \ p_{\pi}^{\S}(a_{\pi}, n) \doteq 0 \Longrightarrow \chi_{\pi}(a_{\pi}) \right]$$
$$= \left[ c_{\pi} \ p_{\pi}^{n}(a_{\pi}) \doteq 0 \Longrightarrow \chi(a_{\pi}) \right] : A_{\pi} \to 2.$$

[(Free) iteration count  $n \in \mathbb{N}$  – formally:  $n_+ \in \mathbb{N}$ , see above – comes in (only) here. n is to count the number of iterated "applications" of e – formally: evaluation steps – applied to internal endo u, on a given argument  $a \in A$ , for Comparison with (evaluation of) internal test predicate w, again evaluated on a.]

We spell out **premise** equation  $c_{\pi} p_{\pi}^{n}(a_{\pi}) \doteq 0$ :

$$[c_{\pi} p_{\pi}^{n}(a_{\pi}) \doteq 0] \quad [= [c_{\pi} p_{\pi}^{n}(m, (u, v, w), a) \doteq 0]]$$

$$= [m \ def \ \varepsilon(v, \bar{a}) \doteq 0] \quad \text{with } \bar{a} = r \ e^{n}(u, a) : A_{\pi} \to A_{\pi} \to A;$$
with auxiliary, dependent variable  $\bar{a}$  eliminated:
$$= [m \ def \ \varepsilon(v \odot u^{[n]}, a) \doteq \varepsilon(v, \varepsilon(u^{[n]}, a)) \doteq 0].$$

[  $u^{[n]} = u \odot ... \odot u$  is – PR defined – n-fold code expansion, see intermediate map-argument in iterative (basic) evaluation  $\varepsilon$  above.]

The above defines – formally PR – premise equation  $c_{\pi} p_{\pi}^{n}(a_{\pi}) \doteq 0$ .

Test predicate  $\chi_{\pi}: A_{\pi} \to 2$  in right antecedent  $TerC(a_{\pi}): A_{\pi} \to 2$  is – by *choice* above –

$$\chi_{\pi}(a_{\pi}) = \chi_{\pi}(m, (u, v, w), a) = \underset{\text{by def}}{\text{lef}} [m \text{ def } \varepsilon(w, a) \stackrel{.}{=} r e^{m}(w, a)] : A_{\pi} \rightarrow 2.$$

Putting things together into the actual right antecedent gives

$$TerC(a_{\pi}, n) = [c_{\pi} p_{\pi}^{n}(a_{\pi}) \doteq 0 \implies \chi_{\pi}(a_{\pi})]$$

$$= [c_{\pi} p_{\pi}^{n}(m, (u, v, w), a) \doteq 0 \implies \chi_{\pi}(m, (u, v, w), a)]$$

$$= [m \ def \ \varepsilon(v, \varepsilon(u^{[n]}, a)) \land m \ def \ \varepsilon(w, a)$$

$$\land [\varepsilon(v, \varepsilon(u^{[n]}, a)) \doteq 0 \implies \varepsilon(w, a)]] : A_{\pi} \times \mathbb{N} \to 2.$$

"Regular" Termination of all instances of  $\varepsilon : \operatorname{PR} \times \mathbb{X} \to \mathbb{X}$  is here given again by choice of  $A_{\pi} : \mathbb{N} \times (\operatorname{PR}^3 \times A) \to 2$ .

**Comment:** Free Variable  $m \in \mathbb{N}$  – occurring in our *premises* only – means here intuitively assumption of "existence" of a sufficiently large number – m – such that m iterations of evaluation step  $e: \operatorname{PR} \times A \to \operatorname{PR} \times A$  suffice for regular – not genuinely truncated – m fold iteration of step e to give the wanted result  $\varepsilon(u,a):=e^m(a)$ .

Intuitively such m "disappears" – better: is hidden into the potentially infinite – in all of our (complexity controlled) iterations considered; and axiom schema  $(\pi_O)$  which constitutes Theory  $\pi_O \mathbf{R}$  – has just the sense to approximate – without enriching the language (of Theory  $\mathbf{PR_A}$ ) – this intuition of finite termination of  $\mathbf{PR_A}$  based, formally partial evaluation.

So far the data.

We now verify the needed **properties** of the two Antecedents of schema  $(\pi_O)$  for the actual instance

$$A_{\pi}$$
,  $DeSta_{\pi}(a_{\pi}): A_{\pi} \to 2$ , and  $TerC_{\pi}(a_{\pi}, n): A_{\pi} \times \mathbb{N} \to 2$ :

- Strict Descent above complexity 0, and Stationarity at 0:

$$\pi_O \mathbf{R} \vdash \pi_O \mathit{Case}(k, (u, v, w)) / a \Longrightarrow :$$
 $m \ \mathit{def} \ \varepsilon_d(\mathit{dtree}_i, a) \land (\text{``and gives further''})$ 
 $m \ \mathit{def} \ \varepsilon(\mathit{desta}(u, v), a) \land \dot{=} \varepsilon(\lceil \mathsf{true} \rceil, a) \dot{=} \mathsf{true}.$ 

This gives in particular  $\pi_O \mathbf{R} \vdash DeSta_{\pi}(m, (u, v, w), a) : A_{\pi} \to 2$ , the latter in particular by  $\varepsilon$ -Objectivity applied to **definition** (\*) of desta(u, v)above, and by m-dominated (formally: m-truncated) **Double Recursive** equations for (iterative) evaluation  $\varepsilon : \operatorname{PR} \times \mathbb{X} \to \mathbb{X}$ .

- Termination Comparison for comparison predicate  $\chi_{\pi}: A_{\pi} \to 2:$ 

$$\pi_{O}\mathbf{R} \vdash \pi_{O} Case(k, (u, v, w)) / \langle a; n \rangle \implies :$$

$$m \ def \ \varepsilon_{d}(dtree_{j}, \langle a; n \rangle) \ \land \ (\text{"gives further"})$$

$$m \ def \ \varepsilon(terc(u, v, w), \langle a; n \rangle) \doteq \text{true, whence}$$

$$\pi_{O}\mathbf{R} \vdash TerC_{\pi}((m, (u, v, w), a), n) : A_{\pi} \to 2.$$

The latter again by – dominated, formally: truncated – "characteristic" (Double Recursive) equations for  $\varepsilon : \operatorname{PR} \times \mathbb{X} \to \mathbb{X}$ .

So we have verified **both Antecedents** for (objective) schema  $(\pi_O)$ , in its here needed **instance**  $A_{\pi_O}$ ,  $DeSta_{\pi_O}$ ,  $TerC_{\pi_O}$ .

**Postcedent** of this *on-terminating descent* schema for theory  $\pi_O \mathbf{R}$  then gives

$$\pi_{O}\mathbf{R} \vdash \chi_{\pi}(m, (u, v, w), a) : A_{\pi} \to 2$$
, namely  
 $\pi_{O}\mathbf{R} \vdash \pi_{O} Case(k, (u, v, w))/a \implies \chi_{\pi}$ , and hence in particular  
 $\pi_{O}\mathbf{R} \vdash \pi_{O} Case(k, (u, v, w))/a \implies :$   
 $m \ def \ \varepsilon_{d}(dtree_{k}/a) \implies \varepsilon(w, a) \doteq \text{true} \doteq \varepsilon(\lceil \text{true}_{A} \rceil, a) : (\check{\bullet}_{k}).$ 

So in this **final case** too, (internal) root equation

$$root \ dtree_k = \underset{\text{by def}}{=} \langle w \, \check{=}_k \, \lceil true_A \rceil \rangle$$

is evaluated – formally: *termination-conditioned* evaluated – into expected **objective** predicative equation:

$$\pi_O \mathbf{R} \vdash [m \ def \ \varepsilon_d(dtree_k/a)] \implies \varepsilon(w, a) \doteq_A \varepsilon(\lceil \operatorname{true}_A \rceil, a).$$

This means that dominated, formally: truncated evaluation  $\varepsilon_d$  of argumented deduction trees evaluates – in case of Termination – not only the map code/argument pairs in  $dtree_i/a = dtree_{i(k)}/a$  as well as in  $dtree_j(k)/\langle a;n\rangle$  into equal values, but – recursion – by this also those of  $dtree_k/a$ ,  $a \in A \subseteq \mathbb{X}$ , all this in the present, last regular case of  $(k,a) \in \mathbb{N} \times A \subseteq \mathbb{N} \times \mathbb{X}$ , and its associated deduction tree  $dtree_k/a$ , a (recursively) substituted, instantiated into pure, variable-free internal (equational) deduction tree  $dtree_k$  for any internal equation, general form u = k.

This – exhaustive – recursive case distinction shows Dominated, formally: truncated, and more intuitive: **Termination-Conditioned**, **Soundness** for Theory  $\pi_O \mathbf{R}$ , relative to itself, and hence also the other assertions of **Main Theorem**, on Termination-Conditioned Soundness q.e.d.

**Remark:** Universal set  $\mathbb{X} \subset \mathbb{N}$  seems to give a good service: without it, we would have be forced (?) to define evaluation  $\varepsilon$  as a <u>family</u>

$$\varepsilon = [\varepsilon_{A,B} : \lceil A, B \rceil \times A \rightharpoonup B]_{A,B \in \mathbf{Obj}_{\mathbf{PR}_A}}$$

<u>meta-indexed</u> over pairs of Objects of Theory  $\mathbf{PR_A}$ , as is usual in Category Theory for *axiomatically* given evaluation

$$\epsilon = [\epsilon_{A,B} : B^A \times A \to B]_{A,B \in \mathbf{Obj_C}},$$

C a (Cartesian) Closed Category in the sense of EILENBERG & KELLY 1966 and LAMBEK & SCOTT 1986. (Observe our typographic distinction between the two "evaluations").

At least formally, a constructive **definition** of evaluation as one single – formally partial –  $\mathbf{PR_A}$  map  $\varepsilon = \varepsilon(u, x) : \lceil \mathbb{X}, \mathbb{X} \rceil \times \mathbb{X} \to \mathbb{X}$  is "necessary" or at least makes things simpler.

So both, the typified approach – traditional in Categorical main stream, as well as the Ehresmann type one starting with just one *class* of maps – and partially defined composition – are usefull in our context:  $Universal\ set\ \mathbb{X}$  – of  $(codes\ of)\ strings$  of natural numbers here makes the join.

From this *Main Theorem*, we get, as shown in detail in **Summary** above – use of schema  $(\tilde{\pi}_O)$ , on absurdity of infinitely descending  $CCI_O$ 's "in" Ordinal O, contraposition of and therefore equivalent to schema  $(\pi_O)$  – the following

Self-Consistency Corollary for Theories  $\pi_O \mathbf{R}$ :

$$\pi_O \mathbf{R} \vdash \neg Prov_{\pi_O \mathbf{R}}(k, \lceil \text{false} \rceil) : \mathbb{N} \to 2 :$$

Theory  $\pi_O \mathbf{R}$ ,  $O \succeq \mathbb{N}[\omega]$ , <u>derives</u> its own – Free-Variable – (internal) non-Provability of 「false¬, i. e. it <u>derives</u> its own (Free-Variable) Consistency Formula.

## 6 An Implicational, Local Variant of Axiom of Descent

We consider an <u>alternative</u> Descent axiom over  $\mathbf{PR_A}$ , namely the following *implicational*, by that equational schema, to replace Descent axiom  $(\pi_O)$ , namely

$$c = c(a) : A \to O \ (complexity),$$

$$p = p(a) : A \to A \ ("predecessor" step)$$

$$\chi = \chi(a) : A \to 2$$

$$(arbitrary) "test" predicate for circumscription of "\exists n",$$

$$(\pi_O^{\bullet}) \frac{\text{logically: } \chi \text{ a free meta-Variable over } \mathbf{PR_A}\text{-predicates on } A}{[[[DeSta^{\bullet}[c \mid p](a, n) \implies c p^n(a) \stackrel{.}{=} 0_O]} \\ \implies \chi(a)] \implies \chi(a)] = \text{true : } A \times \mathbb{N} \to 2 :$$

For "each" a "exists"  $n \in \mathbb{N}$  terminating  $p^n$  into  $c p^n(a) \doteq 0$ , existence expressed "locally" via 2 implications, local at "given"  $a \in A$ , and concerning "test" predicate (free predicate Variable)  $\chi = \chi(a) : A \to 2$ .

**Definition** of *individualised Descent condition*, above, descent condition concerning "only" a "given", (finite) *sequence* of length n, starting at given a:

**Strengthening Remark:** This (equational) **axiom** infers "original" schema  $(\pi_O)$  by inferential <u>modus ponens</u>: <u>Antecedent</u> of  $(\pi_O)$  makes true (first) premise  $DeSta^{\bullet}[c \mid p](a, n)$  of  $(\pi_O^{\bullet})$ 's <u>Postcedent</u>, for  $a \in A$  free (!), and then gives – by boolean Free Variables tautology – <u>Postcedent</u>

$$\pi_O^{\bullet} \mathbf{R} \vdash \chi(a) = \operatorname{true}_A : A \to 2, \ a \in A \text{ free, of schema } (\pi_O) \text{ for theory } \pi_O^{\bullet} \mathbf{R}.$$

We turn to (equivalent) Free-Variables Contraposition to local, implicational schema  $(\pi_O^{\bullet})$ . It reads:

$$\begin{array}{l} c = c(a): A \to O, \, p = p(a): A \to A \text{ in } \mathbf{PR_A} \text{ "given"}, \\ \frac{\psi = \psi(a): A \to 2 \text{ (meta \underline{free}) "absurdity test" predicate}}{\pi_O^{\bullet} \mathbf{R} \; \vdash \; \left[ \left[ \psi(a) \Rightarrow DeSta^{\bullet} \left[ \; c \, \middle| \; p \, \right] (a,n) \, \land \, c \, p^n(a) > 0 \, \right] \Rightarrow \neg \, \psi(a) \, \right]:} \\ A \times \mathbb{N} \to 2. \end{array}$$

Interpretation of  $(\pi_O^{\bullet})$  and  $(\tilde{\pi}_O^{\bullet})$ :

(i) Implicational schema  $(\pi_O^{\bullet})$  says intuitively: for any  $a \in A$  "given", there "exists"  $n \in \mathbb{N}$  such that descent  $c p^0(a) > \ldots > c p^n(a)$  during n steps, implies (stationary) termination  $c p^n(a) \doteq 0_O$  after n steps.

- (ii) In particular: If chain  $[c \mid p]$  satisfies earlier descent condition  $DeSta[c \mid p](a)$ , mainly:  $c(a') > 0 \implies cp(a') < c(a')$  for all (consecutive) arguments of form  $a' = p^{n'}(a)$ ,  $n' \leq n$ , "any" n given, then this chain must become stationary after finitely many steps  $n' \mapsto n' + 1$ . All this individually, "locally" for  $a \in A$  given.
- (iii) If  $[c \mid p]$  satisfies DeSta globally: for  $a \in A$  free, then chain above must be stationary after finitely many steps for all a (with termination index still individual for each a.) This case is just (Interpretation of) **Strengthening Remark** above:  $(\pi_O^*)$  infers  $(\pi_O)$ .
- (iv) (Equivalent) Free-Variables Contraposition  $(\tilde{\pi}_{O}^{\bullet})$  of  $(\pi_{O}^{\bullet})$ :

$$[\psi(a) \Rightarrow [DeSta^{\bullet}(a,n) \land cp^{n}(a) > 0]] \Rightarrow \neg \psi(a) \text{ interprets:}$$

DeSta  $[c|p](p^n(a))$  for (individual)  $a \in A$  and for all  $n \in \mathbb{N}$ , but nevertheless infinite descent at "this" a, is absurd: any condition  $\psi = \psi(a)$  on A which implies that absurdity for the given a, must be false on that a.

Theorie(s)  $\pi_O^{\bullet} \mathbf{R} = \mathbf{P} \mathbf{R}_{\mathbf{A}} + (\pi_O^{\bullet})$  now **inherit** directly all of the assertions on formally partial,  $\mathbf{P} \hat{\mathbf{R}}_{\mathbf{A}}$  evaluation  $\varepsilon = \varepsilon(u, a) : \mathbf{P} \mathbf{R}_{\mathbf{A}} \times \mathbb{X} \to \mathbb{X}$  as well as of argumented-deduction-tree evaluation  $\varepsilon_d : Stree \to Stree$ , with the following exceptions, where schema  $(\pi_O \mathbf{R})$  enters explicitly:

Tree Argumentation, extra Case: For this we need "abbreviation"

$$DeSta^{\bullet}[c \mid p](a, n) : A \times \mathbb{N} \to 2,$$

this predicate reads more formally:

$$=_{\text{by def}} \text{ pr} \left[ \text{true} : A \to 2, \ b \land \textit{DeSta} \left[ \left. c \, \right| p \right] \! \left( p^{n'}(a) \right) \right] : A \times \mathbb{N} \to 2.$$

Here  $b := r_{A \times \mathbb{N}, 2} : (A \times \mathbb{N}) \times 2 \to 2$  is right projection, and

$$\operatorname{pr}\left[g:A\to B,\ h:(A\times\mathbb{N})\times B\to B\right]:A\times\mathbb{N}\to B$$

is (unique) **definition** of a PR<sub>A</sub> map, out of anchor g and  $step\ h$ , by the full schema (pr) of Primitive Recursion.

Still more formally, without use of Free Variables, we have

$$\begin{split} \operatorname{DeSta}^{\bullet}[\,p\,|\,c\,] &= \operatorname{pr}[\,\operatorname{true}_{A}\,,\,\,r_{A\times\mathbb{N},2}\,\wedge\,\left[\,\operatorname{DeSta}\left[\,c\,|\,p\,\right]\circ p^{\S}\circ\ell_{A\times\mathbb{N},2}\,\right]\,\right] : \\ A\times\mathbb{N} &\to 2. \end{split}$$

We internalise this sequential descent, DeSta, into

$$\begin{aligned} \operatorname{desta}^{\bullet}(u,v) \ =_{\operatorname{def}} \ \lceil \operatorname{pr} \rceil \ [ \ \lceil \operatorname{true}_{A} \rceil \ ; \ \lceil r \rceil \ \ \lceil \wedge \rceil \ [ \ \operatorname{desta}(u,v) \odot v \ \rceil \S \rceil \odot \ \lceil \ell \rceil \ ] \ ] \ : \\ \lceil A,O \rceil \times \lceil A,A \rceil \to \lceil A \times \mathbb{N},2 \rceil, \end{aligned}$$

where desta = desta(u, v) is internal version of DeSta[c|p] defined and used frequently above: no change here.

This gives the following **type** of dummy argumented tree t in the actual  $\pi_O^{\bullet}$  Case, with just one explicit level:

with branches t',  $\tilde{t}' \in dum Tree \subset Stree dummy argumented$  Similarity trees. In analogy to the other equational cases (for theorie(s)  $\pi_O \mathbf{R}$ , we are led to **define** for t the actual, argumented form:

$$t/\langle a;n\rangle \ =_{\operatorname{def}} \ \frac{\langle \langle \langle \operatorname{desta}^{\bullet}(u,v) \ \lceil \Rightarrow \rceil \ \langle u \odot v^{\lceil \S \rceil} / \langle a;n \rangle \ \lceil \dot{=} \ 0 \rceil \, \rangle \rangle}{t'/\langle a;n \rangle} \\ \frac{\lceil \Rightarrow \rceil \ w/a \rangle \ \lceil \Rightarrow \rceil \ w/a \rangle \ \sim \ \lceil \operatorname{true} \rceil}{t'/\langle a;n \rangle}$$

This completes tree argumentation, by consideration of the **final**, extra case, final case here treating schema  $(\pi_O^{\bullet})$  for theorie(s)  $\pi_O^{\bullet} \mathbf{R}$ , replacing original one(s)  $(\pi_O)$ , for theorie(s)  $\pi_O \mathbf{R}$ .

**Definition** of map-code/argument trees, Stree, of (PR) tree-complexity  $c_d$ : Stree  $\to O$  as well as (PR) tree-evaluation step  $e_d$ : Stree  $\to$  Stree carry over – suitably modified – from theorie(s)  $\pi_O \mathbf{R}$  to present theorie(s)  $\pi_O^{\bullet}$ . The same then is true for the "finite" **Descent** of map-code/argument tree evaluation  $\varepsilon_d$ : Stree  $\to$  Stree. This  $\varepsilon_d$  is the CCI<sub>O</sub> **defined** by these (modified) complexity  $c_d$  and iteration of step  $e_d$ : iteration as long as complexity  $0_O$  is not "yet" reached.

From this we get, in analogy to that for theorie(s)  $\pi_O \mathbf{R}$ , the (modified)

*Main Theorem* for theorie(s)  $\pi_O^{\bullet} \mathbf{R}$ , again on Termination-Conditioned Soundness:

It is conceptually unchanged: replace Descent Theory  $\pi_O \mathbf{R}$  by "even" local Descent Theory  $\pi_O^{\bullet} \mathbf{R}$ , and read internal equality (enumeration)  $\stackrel{.}{=}_k : \mathbb{N} \to \mathrm{PR_A}^2$  as internal equality of  $\pi_O^{\bullet} \mathbf{R}$  (just this makes the difference.)

Termination-Conditioned Inner Soundness reads, for theories  $\pi_O^{\bullet} \mathbf{R} = \mathbf{PR_A} + (\pi_O^{\bullet})$ :

$$\pi_O^{\bullet} \mathbf{R} \vdash [u \stackrel{.}{=}_k v] \land [m \ def \ \varepsilon(u, a), \ \varepsilon(v, a)] \Longrightarrow :$$

$$\varepsilon(u, a) \stackrel{.}{=} r \ e^m(u, a) \stackrel{.}{=} r \ e^m(v, a) \stackrel{.}{=} \varepsilon(v, a),$$

$$u, v \in \mathrm{PR}_A, \ a \in \mathbb{X}, \ m \in \mathbb{N} \ \mathrm{free}.$$

Interpretation: Unchanged, see Main Theorem for theorie(s)  $\pi_O \mathbf{R}$  above. Same for the **consequences**:

- Termination-Conditioned Objective Soundness for Map-Equality, which gives in particular

- Termination-Conditioned Objective Logical Soundness:

$$\pi_O^{\bullet} \mathbf{R} \vdash Prov_{\pi_O^{\bullet} \mathbf{R}}(k, \lceil \chi \rceil) \land [m \ def \ \varepsilon_d(dtree_k/a)] \implies \chi(a) : \mathbb{N}^2 \times A \to 2.$$

(Modified) Proof of Termination-Conditioned Inner Soundness:

There is no change necessary in all **Cases** except the **extra**, final case characterising theory  $\pi_O \mathbf{R}$  resp.  $\pi_O^{\bullet} \mathbf{R}$ : The standard, non-**extra** cases can be **proved** already within  $\mathbf{PR_A}$ , with u = k v designating  $\mathbf{PR_A}$ 's internal-equality enumeration, as well when designating the *stronger* ones of  $\pi_O \mathbf{R}$  resp. the still stronger ones of present theorie(s)  $\pi_O^{\bullet} \mathbf{R}$ .

Remains to prove Termination-Conditioned Inner Soundness for

Extra Case for theory  $(\pi_O^{\bullet})$ , corresponding to its characteristic, extra axiom  $(\pi_O^{\bullet})$ .

For this, recall:

$$desta = desta(u, v) =_{\text{by def}} \langle u \, \lceil > 0 \, \rceil \, \, \lceil \Rightarrow \rceil \, u \, \odot v \, \lceil < \rceil \, u \rangle \wedge \langle v \, \lceil \dot{=} \, 0 \, \rceil \, \, \lceil \Rightarrow \rceil \, u \, \lceil \dot{=} \, \rceil \, \lceil \text{id} \, \rceil \rangle :$$
$$[\mathbb{X}, O] \times [\mathbb{X}, \mathbb{X}] \to [X, 2] = [\mathbb{X}, 2]_{\mathbf{PR}_{\mathbf{A}}}.$$

Free variable  $w \in [X, 2]$  is to internalise test predicate  $\chi : A \to 2$ .

Finally **recall** from above completely formal internalisation

$$desta^{\bullet}(u,v): \lceil \mathbb{X},O \rceil \times \lceil \mathbb{X},\mathbb{X} \rceil \to \lceil \mathbb{X} \times \mathbb{N},2 \rceil$$
 given by  $desta^{\bullet}(u,v) =_{\operatorname{def}} \lceil \operatorname{pr} \rceil \lceil \operatorname{true} \rceil; \lceil r \rceil \lceil \wedge \rceil \lceil \operatorname{desta}(u,v) \odot v \rceil \rceil$   $\mathbb{X}, \mathbb{X} \cap \mathbb{X} \cap$ 

What we have to **prove** in this case – taking into account just the only explicit equation in the corresponding deduction tree – is

$$\pi_{O}^{\bullet}\mathbf{R} \vdash m \text{ def all } \varepsilon \text{ terms below } \Longrightarrow :$$

$$[[\varepsilon(\text{desta}^{\bullet}(u,v),\langle a;n\rangle) \Longrightarrow [\varepsilon(u \odot v^{\lceil \S \rceil},\langle a;n\rangle) \doteq 0]$$

$$\Longrightarrow \varepsilon(w,a)] \Longrightarrow \varepsilon(w,a)] \doteq \text{true } :$$

$$\mathbb{N} \times ([\mathbb{X},O] \times [\mathbb{X},X] \times [\mathbb{X},2]) \times \langle \mathbb{X} \times \mathbb{N} \rangle \to 2.$$

$$(\bullet^{\bullet})$$

For reduction of this case "to itself", we **define** here – in (simpler) parallel to the  $\pi_O \mathbf{R}$  setting – a special **instance** for schema  $(\pi_O^{\bullet})$ , "consisting" out of a "super Domain"  $A_{\pi}$ , a "super complexity"  $c_{\pi}: A_{\pi} \to O$ , a "super step"  $p_{\pi}: A_{\pi} \to A_{\pi}$ , as well as a "super test predicate"  $\chi_{\pi}: A_{\pi} \to 2$ , such that in fact "finite descent" is given – and such that this instance of  $(\pi_O^{\bullet})$  is able to <u>derive</u> our assertion  $(\bullet^{\bullet})$  in present case. Here are the data for this instance:

$$A_{\pi} =_{\text{def}} \{ (m, (u, v, w), a) \in \mathbb{N} \times (\lceil \mathbb{X}, O \rceil \times \lceil \mathbb{X}, X \rceil \times \lceil \mathbb{X}, 2 \rceil) \times \mathbb{X} \mid m \text{ def } \varepsilon(u, a), \varepsilon(v, a), \varepsilon(\text{desta}^{\bullet}(u, v), a), \varepsilon(w, a) \} \subset \mathbb{N} \times \operatorname{PR}_{A}^{3} \times \mathbb{X}.$$

Introduce Free Variable  $a_{\pi} =_{\text{def}} (m, (u, v, w), a) \in A_{\pi} \subset \mathbb{N} \times \text{PR}_{A}^{3} \times \mathbb{X},$  and **define** 

$$c_{\pi} = c_{\pi}(a_{\pi}) =_{\text{def}} r e^{m}(u, a) : A_{\pi} \to O, \ c_{\pi}(a_{\pi}) = \varepsilon(u, a) : A_{\pi} \to O \text{ for short,}$$
  
(termination property of  $m$  "fixed" within  $a_{\pi} \in A_{\pi}$ .)  
 $p_{\pi}(a_{\pi}) = p_{\pi}(m, (u, v, w), a) =_{\text{def}} (m, (u, v, w), \varepsilon(v, a)) : A_{\pi} \to A_{\pi}$ .

Finally, externalised "super test predicate" is taken, suitable for actual **proof**,

$$\chi_{\pi} = \chi_{\pi}(a_{\pi}) = \chi(m, (u, v, w), a) = \varepsilon(w, a) = \operatorname{by def} r e^{m}(w, a) : A_{\pi} \to 2.$$

These fixed, next step is calculation of *DeSta* for above "super" data:

$$DeSta[c_{\pi} | p_{\pi}] (a_{\pi})$$

$$= [c_{\pi}(a_{\pi}) > 0_{O} \implies c_{\pi} p_{\pi}(a_{\pi}) < c_{\pi}(a_{\pi})] \qquad (Descent)$$

$$\wedge [c_{\pi}(a_{\pi}) \doteq 0 \implies c_{\pi}(a_{\pi}) \doteq a_{\pi}]. \qquad (Stationarity)$$

By **definition** of these data, this calculation gives:

$$DeSta[c_{\pi} | p_{\pi}] (a_{\pi})$$

$$= [m \ def \ all \ instances \ of \ \varepsilon \ below] \land :$$

$$[\varepsilon(u, a) > 0_{O} \implies \varepsilon(u, \varepsilon(v, a)) < \varepsilon(u, a)]$$

$$\land [\varepsilon(u, a) \doteq 0 \implies \varepsilon(v, a) \doteq_{A} a] : \mathbb{N} \times PR_{A}^{3} \times \mathbb{N} \supset A_{\pi} \to 2.$$

But this is equality between (m-dominated) iteration predicates

$$DeSta[c_{\pi} | p_{\pi}](m, (u, v, w), a) \Longrightarrow :$$

$$[m \ def \ \varepsilon(desta^{\bullet}(u, v), a)]$$

$$\wedge \ DeSta[c_{\pi} | p_{\pi}](m, (u, v, w), a) \doteq \varepsilon(desta^{\bullet}(u, v), a) :$$

$$\mathbb{N} \times ([\mathbb{X}, O] \times [\mathbb{X}, \mathbb{X}] \times [\mathbb{X}, 2]) \times \mathbb{X} \to 2,$$

We Objectivise internal continous descent desta(u, v), via evaluation  $\varepsilon$  on  $\langle a; n \rangle \in \langle \mathbb{X}; \mathbb{N} \rangle$ : we expect to get just instance  $DeSta^{\bullet}[c_{\pi} | p_{\pi}] \langle a; n \rangle$  of Objective sequen-

tial Descent:

```
 \begin{array}{l} m \ def \ \text{all } \varepsilon \ \text{terms in } (\bullet^{\bullet}) \ implies: \\ m \ def \ \text{all } \varepsilon \ \text{terms below } \wedge : \\ \varepsilon (desta^{\bullet}(u,v),\langle a;n\rangle) \\ & \doteq \varepsilon (\lceil \operatorname{pr} \rceil [\lceil \operatorname{true}_{\mathbb{X}} \rceil; \lceil r \rceil \rceil \lceil \wedge \rceil [desta(u,v) \odot v^{\lceil \S \rceil} \odot \lceil \ell \rceil]], \langle a;n\rangle) \\ & \doteq \varepsilon (\lceil \operatorname{pr} \rceil [\lceil \operatorname{true}_{\mathbb{X}} \rceil; \lceil r \rceil \rceil \lceil \wedge \rceil [desta(u,v) \odot v^{\lceil \S \rceil} \odot \lceil \ell \rceil]], \langle a;n\rangle) \\ & \doteq \varepsilon (\lceil \operatorname{pr} \rceil [\lceil \operatorname{true}_{\mathbb{X}} \rceil; \lceil r \rceil \rceil \lceil \wedge \rceil [desta(u,v) \odot v^{\lceil \S \rceil}, \langle a;n'\rangle)) \\ & \doteq \varepsilon (\lceil \operatorname{pr} \rceil [\lceil \operatorname{desta}(u,v) \odot v^{\lceil \S \rceil}, \langle a;n'\rangle)) \\ & \doteq \underset{n' \leq n}{\wedge} \varepsilon (desta(u,v), \varepsilon (v^{\lceil \S \rceil}, \langle a;n'\rangle)) \\ & \doteq \underset{n' \leq n}{\wedge} \varepsilon (desta(u,v), p_{\pi}^{n'}(m,(u,v,w),a)) \\ & \text{with } a_{\pi} := (m,(u,v,w),a), p_{\pi}^{n'}(a_{\pi}) \in A_{\pi} \subset \mathbb{N} \times \operatorname{PR}_{\mathbb{A}}^{3} \times \mathbb{X}, \text{ for } n' \leq n \\ & = \underset{by \operatorname{def}}{\operatorname{by def}} \underset{n' \leq n}{\wedge} \operatorname{DeSta}[c_{\pi} | p] (p_{\pi}^{n'}(a_{\pi})) \\ & = \underset{by \operatorname{def}}{\operatorname{DeSta}^{\bullet}} [c_{\pi} | p_{\pi}] (a_{\pi},n) \\ & = \operatorname{DeSta^{\bullet}}[c_{\pi} | p_{\pi}] ((m,(u,v,w),a),n) : \\ \mathbb{N} \times (\lceil \mathbb{X}, O \rceil \times \lceil \mathbb{X}, \mathbb{X} \rceil \times \lceil \mathbb{X}, 2 \rceil) \times \langle A \times \mathbb{N} \rangle \to 2. \end{array}
```

This is wanted externalisation

```
m \ def \ all \ \varepsilon \ terms \ in \ (\bullet^{\bullet}) \ implies:
\varepsilon(desta^{\bullet}(u,v),\langle a;n\rangle) \doteq DeSta^{\bullet}[c_{\pi} | p_{\pi}] ((m,(u,v,w),a),n) : \quad (\varepsilon \ desta)
\mathbb{N} \times ([\mathbb{X},O] \times [\mathbb{X},\mathbb{X}] \times [\mathbb{X},2]) \to 2.
```

This given, we attempt, again by Objectivisation via  $\varepsilon$  of  $(\bullet^{\bullet})$ , to show the "finite" descent property for our **instance**  $A_{\pi}$  etc., i. e. essentially for  $DeSta^{\bullet}$ , as follows:

```
m \ def \ all \ \varepsilon \ terms \ in \ (ullet^{ullet}) \ implies:
[[DeSta^{ullet}[c_{\pi} | p_{\pi}] (a_{\pi}, n) \implies \chi_{\pi}(a_{\pi})] \implies \chi_{\pi}(a_{\pi})]
= [[DeSta^{ullet}[c_{\pi} | p_{\pi}] ((m, (u, v, w), a), n) \implies \varepsilon(w, a)] \implies \varepsilon(w, a)]
\stackrel{\cdot}{=} [[\varepsilon (desta^{ullet}(u, v), \langle a; n \rangle) \implies \varepsilon(w, a)] \implies \varepsilon(w, a)] : \quad (just \ (ullet^{ullet}))
\mathbb{N} \times ([\mathbb{X}, O] \times [\mathbb{X}, \mathbb{X}] \times [\mathbb{X}, 2]) \times \langle A \times \mathbb{N} \rangle \to 2.
```

This shows that our hypothesis ( $\bullet \bullet$ ) is equivalent to "finite" sequential descent of instance  $\langle \langle A_{\pi}, c_{\pi}, p_{\pi} \rangle, \chi_{\pi} \rangle$ .

But this is an instance "for" **axiom**  $(\pi_O^{\bullet}\mathbf{R})$  of our Theory  $\pi_O^{\bullet}\mathbf{R} = \mathbf{P}\mathbf{R}_{\mathbf{A}} + (\pi_O^{\bullet})$ . So that axiom shows remaining assertion  $(\bullet^{\bullet})$ , *Inner Soundness* for the final, "self-referential" case. This **proves** the **Main Theorem** for theorie(s)  $\pi_O^{\bullet}\mathbf{R}$ .

By use of (contrapositive) characteristic schema  $(\tilde{\pi}_O^{\bullet})$  of theory  $\pi_O^{\bullet} \mathbf{R} = \mathbf{P} \mathbf{R}_A + (\pi_O^{\bullet})$  (absurdity of infinitely descending iterative *O*-chains), we get – in complete analogy to the **proof** for theorie(s)  $\pi_O \mathbf{R}$  in **Summary** above:

Self-Consistency Corollary for Theories  $\pi_O^{\bullet} \mathbf{R}$ :

$$\pi_O^{\bullet} \mathbf{R} \vdash \neg Prov_{\pi_O^{\bullet} \mathbf{R}}(k, \lceil \text{false} \rceil) : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free :}$$

Theory  $\pi_O^{\bullet}\mathbf{R}$ ,  $O \succeq \mathbb{N}[\omega]$ , <u>derives</u> its own – Free Variable – (internal) non-Provability of 「false¬, i.e. it <u>derives</u> its own (Free Variable) Consistency Formula.

# 7 Unconditioned Objective Soundness

As is well known, Consistency Provability and Soundness are strongly tied together. Above we have shown that already Termination-Conditioned Soundness entails Consistency Provability. Here we "easily"  $\underline{\text{derive}}$  Full, Unconditioned Objective (!) Soundness from Consistency Provability, for all of our Descent  $Theories \Pi$ , strengthenings of  $\mathbf{PR_A}$ ,  $\Pi$  standing from now on for one arbitrary such theory, namely  $\pi_O \mathbf{R}$  of on-terminating Complexity Controlled Iterations, or  $\pi_O^{\bullet} \mathbf{R}$  of "on-terminating"  $CCI_O$ 's, with complexity values in Ordinal O, O one of the (Order) extensions of Ordinal  $\mathbb{N}[\omega]$  introduced above, i. e. one of  $\mathbb{N}[\omega]$ ,  $\mathbb{N}[\xi_1, \ldots \xi_m]$ ,  $\mathbb{X}$ , and  $\mathbb{E}$ .

We start with the observation that Consistency(-formula) Derivability  $\Pi \vdash \neg [0 = 1] : \mathbb{N} \to 2$  is equivalent to derivability

$$\Pi \vdash [\nu_2(a) \stackrel{\cdot}{=}_k \nu_2(b)] \implies a \stackrel{\cdot}{=} b : \mathbb{N} \times (2 \times 2) \to 2 : (*)$$

Test with  $(a,b) \in \{(0,0),(0,1),(1,0),(1,1)\}$ . Cases (0,1) and (1,0) are (each) just Consistency derivability, the remaining two are trivial.

Formally this test is based on the fact, that

$$(0,0), (0,s0), (s0,0), (s0,s0): \mathbb{1} \to 2 \times 2$$

are the 4 coproduct injections of coproduct (sum)  $2 \oplus 2 =_{\text{def}} 2 \times 2$ .

Now (\*) is – by **definition** – just injectivity of internal numeralisation

$$\nu_2 = \nu_2(a) : 2 \to [1, 2]_{\mathbf{\Pi}} = [1, 2]_{\mathbf{PR}_{\mathbf{A}}}/\check{=}^{\mathbf{\Pi}}.$$

This numeralisation is defined within general Arithmetical theories by

$$\begin{split} \nu_{\mathbb{N}} &= \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}] = \lceil \mathbb{1}, \mathbb{N} \rceil / \check{=} \text{ PR as follows:} \\ \nu(0) &=_{\operatorname{def}} \quad \lceil 0 \rceil : \mathbb{1} \to [\mathbb{1}, \mathbb{N}], \\ \nu(s\,n) &=_{\operatorname{def}} \quad \lceil s \rceil \odot \nu(n) : \mathbb{N} \to [\mathbb{1}, \mathbb{N}], \text{ whence in particular:} \\ \nu(\operatorname{num}(\underline{n})) &= \lceil \operatorname{num}(\underline{n}) \rceil = \lceil s \dots s \circ 0 \rceil \\ \text{for external numeralisation num} : \underline{\mathbb{N}} \longrightarrow \mathbf{S}(\mathbb{1}, \mathbb{N}). \end{split}$$

Further – externally PR:

$$\nu_{A\times B} = \nu_{A\times B}(a,b) =_{\text{def}} \langle \nu_A(a); \nu_B(b) \rangle : A \times B \to [\mathbb{1}, A] \times [\mathbb{1}, B] \xrightarrow{\cong} [\mathbb{1}, A \times B].$$

For an abstraction Object  $\{A \mid \chi\}$ , as in particular  $2 = \{\mathbb{N} \mid \langle s \, 0\}$ ,  $\nu_{\{A \mid \chi\}}$  is defined by (double) restriction, of  $\nu_A : A \to [\mathbb{1}, A]$ .

Naturality Lemma for Internal Numeralisation: For each  $\Pi$  map  $(\mathbf{PR_A} \text{ map})$   $f: A \to B$  the following DIAGRAM commutes – in category  $\Pi Q = \Pi + \text{Quot} \supset \Pi$ : Theory  $\Pi$  enriched by (virtual) Quotients by equivalence Relations, such as in particular  $\check{=} = \check{=}_k : \mathbb{N} \to [\mathbb{X}, \mathbb{X}]^2$ :

**Proof:** We have to show equality in the following Free-Variable setting which displays the assertion, by **definition** of functor  $[1, f] : [1, A] \to [1, B]$ :

$$A \ni a \longmapsto f \qquad f(a) \in B$$

$$\downarrow^{\nu_A} \qquad \qquad \downarrow^{\nu_B}$$

$$[\mathbb{1}, A] \ni \nu_A(a) \longmapsto^{[\mathbb{1}, f]} f \odot \nu_A(a) \qquad \check{=} \nu_B(f(a)) \in [\mathbb{1}, B]$$

This internal equality  $\lceil f \rceil \odot \nu_A(a) = \nu_B(f(a))$  is **proved** straightforward by external structural <u>recursion</u> on the structure of  $f: A \to B$  in  $\mathbf{PR_A}$ , beginning with the maps constants 0, s,  $\ell$ , using internal associativity of " $\odot$ ", and (objective) PR on the iteration count for the case of an iterated.

Injectivity Lemma for Internal Numeralisation: Injectivity of  $\nu_2$ :  $2 \to [\mathbbm{1},2]_{\Pi}$ , given by Consistency <u>derivability</u>, extends to injectivity of all  $\nu_A = \nu_A(a): A \to [\mathbbm{1},2]$ , first to  $\nu_{\mathbb{N}} = \nu(n): \mathbbm{N} \to [\mathbbm{1},\mathbbm{N}]$  essentially by considering truncated subtracction, and then immediately to the other Objects of  $\mathbf{PR}$  and  $\mathbf{PR}_{\mathbf{A}}$ .

This leads to our final result here, namely

(Unconditioned) Objective Soundness Theorem for  $\Pi$ :

- For each pair  $f, g: A \to B$  of  $\mathbf{PR_A}$ -maps:

$$\Pi \vdash \lceil \lceil f \rceil \stackrel{.}{=}_k \lceil g \rceil \rceil \implies \lceil f(a) \stackrel{.}{=}_B g(a) \rceil : \mathbb{N} \times A \to 2,$$

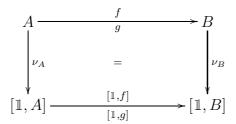
whence by specialision:

- For each  $\mathbf{PR}_{\mathbf{A}}$  predicate  $\chi = \chi(a) : A \to 2$ :

$$\Pi \vdash Prov_{\Pi}(k, \lceil \chi \rceil) \implies \chi(a) : A \to 2 :$$

Availability of an (Internal) *Proof* of (code of) a predicate implies *truth* of this predicate at each argument.

**Proof** of first **assertion:** Consider the following commutative DIAGRAM – in Theory  $\Pi Q \supset \Pi$ :



This gives

by injectivity of  $\nu_B$ .

This taken together gives first – and then second – assertion of the Theorem q.e.d.

Analysis of **Proof** above shows that we can take (internal) Consistency as an additional condition for a an arithmetical theory **S** instead using it as <u>derived</u> property of our (self-consistent) theories  $\Pi$ . This then gives, for such general theory **S**, with  $\mathbf{S}^+ =_{\text{def}} \mathbf{S} + \operatorname{Con}_{\mathbf{S}}$ :

Consistency Conditioned Injectivity of Internal Numeralisation:

$$\mathbf{S}^+ \vdash \nu_A(a) \stackrel{\cdot}{=} \stackrel{\mathbf{S}}{=} \nu_A(a') \implies a \stackrel{\cdot}{=}_A a' : \mathbb{N} \times A^2 \to 2.$$

[ Note the difference between frame  $S^+$  and internal equality taken within weaker theory S itself.]

Consistency Conditioned Soundness:

- for  $\mathbf{PR_A}$ -maps  $f, g: A \to B$ :

- in particular for a predicate  $\chi = \chi(a) : A \to 2$ :

$$\mathbf{S}^+ \vdash Prov_{\mathbf{S}}(k, \lceil \chi \rceil) \implies \chi(a) : N \times A \to 2.$$

Again: Here (internal) **S**-Provability is the premise. It coincides with Provability of frame  $S^+$  only for self-consistent S, as for example for theorie(s)  $\Pi = \Pi^+$  considered above.

(Conditioned) injectivity of internal numeralisation, and naturality invite to consider an <u>inferential</u> form of (conditioned)  $\omega$ -Completeness:

#### $\omega$ -Completeness Theorem, Inference Form:

- Strengthenings **S** of **PR**<sub>A</sub> are Consistency-conditioned  $\omega$ -inference-complete, i. e.

$$(\underline{\operatorname{Comp}}_{\omega}^{\mathbf{S}/\mathbf{S}^{+}}) \quad \frac{\chi = \chi(a) : A \to 2 \text{ in } \mathbf{PR_{A}}, \\ k = k(a) : \mathbb{N} \to \operatorname{Proof_{\mathbf{S}}} \text{ in } \mathbf{PR_{A}}, \\ \mathbf{S}^{+} \vdash \operatorname{Prov_{\mathbf{S}}}(k(a), \lceil \chi \rceil \odot \nu_{A}(a)) : A \to 2}{\mathbf{S}^{+} \vdash \chi : A \to 2.}$$

- Axis case: Self-consistent theories  $\Pi$  are ("unconditioned") inferential  $\omega$ -self-complete, they admit the special schema derived from the above:

$$(\underline{\operatorname{Comp}}_{\omega}^{\mathbf{\Pi}}) \begin{tabular}{l} $\chi = \chi(a) : A \to 2$ in $\mathbf{PR_A}$, \\ $k = k(a) : \mathbb{N} \to Proof_{\mathbf{\Pi}}$ in $\mathbf{PR_A}$, \\ $\frac{\mathbf{\Pi} \vdash Prov_{\mathbf{\Pi}}(k(a), \lceil \chi \rceil \odot \nu_A(a)) : A \to 2$}{\mathbf{\Pi} \vdash \chi : A \to 2$, and hence, by internalisation: } \\ $\mathbf{\Pi} \vdash Prov_{\mathbf{\Pi}}(k[\chi], \lceil \chi \rceil) : \mathbb{1} \to 2$, \\ $k[\chi] : \mathbb{1} \to Proof_{\mathbf{\Pi}}$ the code of $\mathbf{\Pi}$ $\underbrace{\operatorname{Proof}}_{\mathbf{\Pi}}$ of $\chi$. \\ \end{tabular}$$

[The latter internalisation of  $\Pi$  – <u>derivation</u> of  $\chi$  into an (internal) Proof of  $\Pi$  itself for  $\lceil \chi \rceil$  is decisive: it works because of self-consistency  $\Pi = \Pi^+$ . Schema ( $\underline{\operatorname{Comp}}_{\omega}^{\Pi}$ ), with last poscedent, almost says that  $\mathbbm{1}$  is a separator Object for internalised theory  $\Pi$ : test with all internal points, even: with all internal numerals, establishes internal equality, at least for "concrete" code pairs  $\lceil f \rceil$ ,  $\lceil g \rceil \in \lceil A, B \rceil$ , coming coded from objective map pairs  $f, g: A \to B$  of  $\Pi$ .]

**Proof:** Look at  $\nu$ -naturality DIAGRAM in foregoing section, and take special case  $\chi: A \to 2$  for  $f: A \to B$ . Then consider Free-Variable DIAGRAM chase for this f, subsequent DIAGRAM. By commutativity of that rectangle we have

$$\lceil \chi \rceil \odot \nu_A(a) \stackrel{\cdot}{=}_{j(a)}^{\mathbf{S}} \nu_2(\chi(a)),$$

suitable  $j = j(a) : A \to Proof_{\mathbf{S}} \subset \mathbb{N}$ . But by antecedent, we have also

$$\lceil \chi \rceil \odot \nu_A(a) \stackrel{\mathbf{S}}{=}_{k(a)}^{\mathbf{S}} \lceil \text{true} \rceil$$
, whence  $\nu_2(\chi(a)) \stackrel{\mathbf{S}}{=}_{k'(a)}^{\mathbf{S}} \lceil \text{true} \rceil = \nu_2(\text{true})$ .

(Consistency conditioned) injectivity of internal numeralisation  $\nu$  then gives  $\chi(a) \doteq true$ ,  $a \in A$  free. Taken together: Given the antecedent  $\mathbf{S}^+$  derivation, we get  $\mathbf{S}^+ \vdash \chi(a) : A \to 2$ ,  $a \in A$  free. This is what we wanted to show.

The "axis" case of a self-consistent theory, such as  $\Pi$ , then is trivial, and gives (Unconditioned) inferential  $\omega$ -Completeness.

## Coda: Termination Conditioned Soundness for Theory PR<sub>A</sub>

Termination-conditioned (!) (Objective) Soundness holds "already" for basic PR Theory  $\mathbf{PR_A}$ , and hence also for its embedded Free-Variables fundamental (categorical) Theory  $\mathbf{PR} \sqsubset \mathbf{PR_A}$ . The argument is use of following **Reduction** schema ( $\rho_O$ ) of predicate-truth, Reduction "along" a given  $\mathrm{CCI}_O$ .

Eventually we will **prove** by this schema of  $\mathbf{PR_A}$  (!) Consistency of Descent Theories  $\Pi$  relative to  $\mathbf{PR_A}$ .

**Theorem:** Theory  $\mathbf{PR_A}$  admits the following **Schema** of *Reduction* along  $\mathbf{CCI_O}$ 's for *Ordinal O*:

$$[c: A \to O \mid p: A \to A] \text{ is a CCI}_O \text{ in } \mathbf{PR_A},$$

$$\chi = \chi(a): A \to 2 \ \mathbf{PR_A}\text{-predicate to be investigated},$$

$$\mathbf{PR_A} \vdash c(a) \doteq 0_O \implies \chi(a): A \to 2 \ \textit{predicate anchor},$$

$$(\rho_O) \ \frac{\mathbf{PR_A} \vdash \chi(p(a)) \implies \chi(a): A \to 2 \ \textit{reduction step}}{\mathbf{PR_A} \vdash [m \ \textit{def } \text{wh}_O[c \mid p](a) \implies \chi(a)]: A \times \mathbb{N} \to 2.}$$

<u>Postcedent</u> meaning: Termination-of-while-loop conditioned truth of  $\chi(a)$ , "individual" a.

**Proof** by (Free-Variables) Peano induction on free variable  $m \in \mathbb{N}$ : Anchor  $m \doteq 0$ : obvious by Antecedent (anchor). Induction "hypothesis" on m: m def  $\mu_O[c|p](a) \Longrightarrow \chi(a)$ .

Peano Induction Step:

$$\mathbf{PR_A} \vdash m + 1 \ def \ \mu_O[c \mid p] \ (a')$$

$$\implies m \ def \ \mu_O[c \mid p] \ (p(a')) \doteq m$$
by iterative definition of  $\mu_O[c \mid p]$ 

$$\implies \chi(p(a')) \text{ by induction hypothesis}$$

$$\implies \chi(a') : A \times \mathbb{N} \to 2,$$

the latter by **Antecedent** Reduction step q.e.d.

For **Proof** of *Termination-Conditioned Objective Soundness* of  $\mathbf{PR_A}$  by itself, we now consider the following instance of this Reduction schema  $(\rho_{\mathring{O}})$  of  $\mathbf{PR_A}$ :

- Domain  $\mathring{A} =_{\text{def}} \mathbb{N} \times Stree = \mathbb{N} \times Stree_{\mathbf{PR_A}}$ , Stree above without the additional data coming in by schema  $(\pi_O)$  with its "added" (internal) deduction structure.
- Ordinal  $\mathring{O}=_{\mathrm{def}}\mathbb{N}\times\mathbb{N}[\omega]$  with hierarchical order: first priority to left component.
- "Predecessor" step  $p := \mathring{e} = \mathring{e}(m,t) =_{\text{def}} (m 1, e_d(t)) : \mathring{A} \to \mathring{A}$ , (deduction) tree evaluation  $e_d$  above, again "truncated" to the (internal) deduction data of  $\mathbf{PR}_{\mathbf{A}}$ .
- Tree complexity  $\mathring{c} = \mathring{c}(m,t) =_{\text{def}} (m, c_d(t)) : \mathring{A} \to \mathring{O}, \mathbf{PR_A}$  truncation as for  $\mathring{e}$  above.
- Finally the predicate to be reduced with respect to its truth:

$$\mathring{\varphi} = \mathring{\varphi}(m,t) =_{\text{def}} \left[ m \ def \ \varepsilon(root_{\ell}(t)) \doteq \varepsilon(root_{r}(t)) \right] :$$

$$\mathbb{N} \times Stree \to 2 \times \mathbb{X}^{2} \xrightarrow{2 \times \dot{=}} 2 \times 2 \xrightarrow{\wedge} 2.$$

Here  $root_{\ell}(t)$  and  $root_{r}(t)$  are the left and right entries, of form u/x resp. v/y, of  $root(t) = \langle u/x \sim v/x \rangle$  say.

**Verification** of this instance of reduction schema  $(\rho_{\mathring{O}})$  is now as follows: *Anchoring*:

$$\mathbf{PR}_{\mathbf{A}} \vdash \mathring{c}(m, c_d(t)) \doteq (0, 0) \implies :$$

$$\mathring{\varphi}(m, t) \doteq \begin{bmatrix} 0 \text{ } def \text{ } \varepsilon(\lceil \operatorname{id} \rceil/x \doteq \varepsilon(\lceil \operatorname{id} \rceil/y) \doteq [x \doteq y] \doteq \operatorname{true},$$
the latter necessarily for (flat) legimate  $t$  of this form.

Reduction Step for  $\mathring{\varphi}$ :

$$\mathbf{PR}_{\mathbf{A}} \vdash \mathring{\varphi} \stackrel{\circ}{e}(m,t) =_{\text{by def}} \left[ m \doteq 1 \ def \ \varepsilon(root_{\ell} \ e_{d}(t)) \doteq \varepsilon(root_{r} \ e_{d}(t)) \right]$$

$$\implies \left[ m \ def \ \varepsilon(root_{\ell}(t)) \doteq \varepsilon(root_{r}(t)) \right].$$

This implication is **proved** – logically – by recursive case distinction on the two surface levels of t, cases given in the main text above, the  $(\pi_O)$  case truncated. Formally, this recursion is PR on (minimal) number m of steps  $e_d$  for complete tree evaluation of t.

Out of this **Antecedent**, schema  $(\rho_{\mathring{O}})$  gives as its **Postcedent** 

$$\mathbf{PR}_{\mathbf{A}} \vdash [m \ def \ \text{wh}_{\mathring{\mathcal{O}}}[\mathring{c} \mid \mathring{e}] \ (m', dtree_k^{\mathrm{PR}_{\mathrm{A}}}/x)] \implies :$$

$$[m' \ def \ \varepsilon(root_{\ell}(dtree_k^{\mathrm{PR}_{\mathrm{A}}}/x)) \doteq \varepsilon(root_r(dtree_k^{\mathrm{PR}_{\mathrm{A}}}/x))] :$$

$$\mathbb{N}^2 \times (\mathbb{N} \times \mathbb{X}) \to 2, \ m, m', k \in \mathbb{N}, \ x \in \mathbb{X} \ \text{free},$$

in particular, with m := m':

$$\mathbf{PR}_{\mathbf{A}} \vdash [m \ def \ \operatorname{wh}_{\mathbb{N}[\omega]}[c_d \mid e_d] (dtree_k/x)] \Longrightarrow :$$

$$[m \ def \ \varepsilon(root_{\ell}(dtree_k/x)) \doteq \varepsilon(root_r(dtree_k/x))] :$$

$$\mathbb{N} \times (\mathbb{N} \times \mathbb{X}) \to 2, \ m, k \in \mathbb{N}, \ x \in \mathbb{X} \text{ free.}$$

This is in fact

Termination-Conditioned Soundness Theorem for basic PR Theory  $\mathbf{PR_A}$ , which holds by consequence also for fundamental PR Theory  $\mathbf{PR}$   $\square$   $\mathbf{PR_A}$ .

Can we reach from this Self-Consistency for  $\mathbf{PR_A}$  as well, in the manner we have got it for theorie(s)  $\pi_O \mathbf{R} = \mathbf{PR_A} + (\pi_O) = \mathbf{PR_A} + (\tilde{\pi}_O)$ ?

If you look at this derivation in the **Summary** above, you find as the final, decisive step, <u>inference</u> from

$$\pi_O \mathbf{R} \vdash \lceil \text{false} \rceil \stackrel{\text{`=}}{=}_k \lceil \text{true} \rceil \implies c_d \ e_d^m(dtree_k/0) > 0 : \mathbb{N}^2 \to 2, \ \underline{\text{to}}$$
  
 $\pi_O \mathbf{R} \vdash \neg \lceil \lceil \text{false} \rceil \stackrel{\text{`=}}{=}_k \lceil \text{true} \rceil \rceil : \mathbb{N} \to 2, \ k \in \mathbb{N} \text{ free (!)}.$ 

This comclusion gets its legitimacy by application of schema  $(\tilde{\pi})$  to its suitable Antecedent with in particular absurdity condition  $\psi$  – for infinite descent – choosen as

$$\psi = \psi(k) := [ \lceil \text{false} \rceil \overset{\circ}{=}_k \lceil \text{true} \rceil ] : \mathbb{N} \to 2.$$

Same for a general one out of theories  $\Pi$ , namely  $\Pi$  one of  $\pi_O \mathbf{R}$ ,  $\pi_O^{\bullet} \mathbf{R}$ .

If such – formal, axiomatic – absurdity of infinite descent is *not* available in the theory, infinite descent of in particular  $c_d \ e_d^m(dtree_k/0) > 0$  ("for all" m) could not be excluded: internal provability  $\lceil \text{false} \rceil \ \check{=} \ \lceil \text{true} \rceil$  could "happen" formally by just "the fact" that (internal) deduction tree for (internal) Theorem  $\lceil \text{false} \rceil \ \check{=} \ \lceil \text{true} \rceil$  cannot be externalised, by (iterative) deduction tree evaluation  $\varepsilon_d$ , in a finite number of its steps  $e_d$ .

So, in this sense, addition of highly plausible schema  $(\tilde{\pi})$  resp.  $(\tilde{\pi}^{\bullet})$  is "necessary" – at least it is sufficient – for <u>derivation</u> of (internal) *Consistency*, this already for derivation of internal Consistency of Theory  $\mathbf{PR_A}$ .

This latter result is not that astonishing, since Theory  $\pi \mathbf{R} = \pi_{\mathbb{N}[\omega]} \mathbf{R}$  is stronger than  $\mathbf{PR_A}$ , at least <u>formally</u>. Not to expect – the Gödel Theorems – was finding of any *Self-Consistent* (necessarily *arithmetical*) theory, here theorie(s)  $\mathbf{\Pi}$ ,  $\mathbf{\Pi}$  one of  $\pi_O \mathbf{R}$ ,  $\pi_O^* \mathbf{R}$ ,  $O \succeq \mathbb{N}[\omega]$ :

The most involved cases in the **proofs** leading to this Self-Consistency for theorie(s)  $\Pi$  – in particular in (the two) Main Theorem(s) on *Termination-Conditioned Inner Soundness*, and in the constructions leading to the notions used – all come from "this" additional schema ( $\Pi$ ), schema ( $\Pi$ ) one of the schemata ( $\pi_O$ ) and ( $\pi_O^{\bullet}$ ) which constitute theorie(s)  $\Pi$  as ("pure") strengthenings of  $\mathbf{PR}_{\mathbf{A}} \supset \mathbf{PR}$ .

"Same" discussion for (Unconditioned) Objective Soundness for  $\Pi$ , derived in the above from Self-Consistency. Conversely, this Objective Soundness contains Self-Consistency as a particular case.

**Problem:** Is Theory  $\pi \mathbf{R}$ , more general: are theories  $\Pi$  (Objectively) Consistent <u>relative</u> to *basic* Theory  $\mathbf{PR_A}$ , and – by that – relative to *fundamental* Theory  $\mathbf{PR} \sqsubseteq \mathbf{PR_A}$  of Primitive Recursion "itself"?

In other words (case  $\pi \mathbf{R}$ ): do Descent data  $c: A \to O := \mathbb{N}[\omega], p: A \to A$ , and availability of a  $\mathbf{PR}_{\mathbf{A}}$  point  $a_0: \mathbb{1} \to A$  such that

$$\mathbf{PR}_{\mathbf{A}} \vdash c \, p^{\S}(a_0, n) > 0_O :$$

$$\mathbb{1} \times \mathbb{N} \xrightarrow{a_0 \times \mathbb{N}} A \times \mathbb{N} \xrightarrow{p^{\S}} A \xrightarrow{c} O \xrightarrow{>0_O} 2,$$

 $(n \in \mathbb{N} \text{ free, intuitively: } for all \ n \in \mathbb{N} : \underline{\text{derived}} \ non-termination \text{ at } a_0), \text{ lead to a } \underline{\text{contradiction}} \text{ within Theory } \mathbf{PR_A}?$ 

We will take up this (relative) **Consistency Problem** again in terms of (recursive) *Decision*, RCF 5.

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Address of the author:

M. PFENDER Institut für Mathematik Technische Universität Berlin D-10623 Berlin

pfender@math.TU-Berlin.DE