BESSEL SEQUENCES IN HILBERT C*-MODULES

ABDELILAH KARARA* AND KHADIJA MABROUK

ABSTRACT. In this peaper we stady certain Bessel sequences $\{f_k\}_{k=1}^{\infty}$ in Hilbert C*- modules \mathcal{H} for which operator S defined by 1.2 is of the form $\mathcal{T} + \xi I$, for some real number ξ and a adjointable linear operator \mathcal{T} . Additionally, we investigate frames known as compact-tight frames, which have frame operators that are compact perturbations of constant multiples of the identity. As a conclusion, we provide a theory regarding the weaving of specific compact-tight frames.

1. Introduction

The notion of frame is a recent active mathematical research topic, signal processing, computer science, etc. Frames for Hilbert spaces were first introduced in **1952** by Duffin and Schaefer [2] for study of nonharmonic Fourier series. Daubechies, Grossmann, and Meyer [1] revived and developed them in **1986**, and popularized from then on.

Many mathematicians have recently generalized frame theory from Hilbert spaces to Hilbert C^* -modules. For find details of frames in Hilbert C^* -modules we refer to [3, 5, 6, 7, 8, 9, 10, 11]. The purpose of this paper is to investigate.

Throughout this paper, Let \mathcal{H} be a countably generated Hilbert \mathcal{A} -module and $\{\mathcal{K}_k\}_{k\in\mathbb{Z}}$ be the collection of Hilbert \mathcal{A} -module, we also reserve the notation $\operatorname{End}_{\mathcal{A}}^*(\mathcal{H},\mathcal{K}_k)$ for the collection of all adjointable \mathcal{A} -linear maps from \mathcal{H} to \mathcal{K}_k and $\operatorname{End}_{\mathcal{A}}^*(\mathcal{H},\mathcal{H})$ is denoted by $\operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$. We will use $\mathcal{N}(\mathcal{T})$ for the kernel of \mathcal{T} .

$$\ell^2 = \left\{ \{f_k\}_{k \in \mathbb{N}} : \|\sum_{k \in \mathbb{N}} \langle f_k, f_k \rangle \| < \infty \right\}$$

that is, $\mathcal{T}(\{f_k\}_{k\in\mathbb{N}}) = \{f_{k+1}\}_{k\in\mathbb{N}}$. Let $f = \{f_k\}_{k\in\mathbb{N}}$ and $g = \{g_k\}_{k\in\mathbb{N}}$, the inner product is defined by $\langle f, g \rangle = \sum_{k\in\mathbb{N}} \langle f_k, g_k \rangle$, clearly ℓ^2 is a Hilbert \mathcal{A} -module.

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^{*}Corresponding author.

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In the following we briefly recall the definitions and basic properties of Hilbert C^* -modules.

Definition 1.1. [4]. Let \mathcal{A} be a unital C^* -algebra and \mathcal{H} be a left \mathcal{A} -module, such that the linear structures of \mathcal{A} and \mathcal{H} are compatible. \mathcal{H} is a pre-Hilbert \mathcal{A} -module if \mathcal{H} is equipped with an \mathcal{A} -valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

- (i) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle = 0$ if and only if x = 0.
- (ii) $\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
- (iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$. If \mathcal{H} is complete with ||.|| is a norm on \mathcal{H} , it is called a Hilbert \mathcal{A} -module or a Hilbert C^* -module over \mathcal{A} . For every a in C^* -algebra \mathcal{A} , we have $||a|| = (a^*a)^{\frac{1}{2}}$.

Lemma 1.2. [7]. Let \mathcal{H} be Hilbert \mathcal{A} -module. If $\mathcal{T} \in End^*_{\mathcal{A}}(\mathcal{H})$, then

$$\langle \mathcal{T}x, \mathcal{T}x \rangle_{\mathcal{A}} \leq \|\mathcal{T}\|^2 \langle x, x \rangle_{\mathcal{A}}, \forall x \in \mathcal{H}.$$

Definition 1.3. [4] A family $\{f_k\}_{k\in\Theta}$ of elements of \mathcal{H} , if there exist two positive constants A and B such that for all $f \in \mathcal{H}$,

$$A\langle f, f \rangle_{\mathcal{A}} \le \sum_{k \in \Theta} \langle f, f_k \rangle_{\mathcal{A}} \langle f_k, f \rangle_{\mathcal{A}} \le B\langle f, f \rangle_{\mathcal{A}}. \tag{1.1}$$

The numbers A and B are called lower and upper bounds of the frame, respectively. If $A = B = \alpha$, the frame is α -tight. If A = B = 1, it is called a normalized tight frame or a Parseval frame.

Let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ be a frame for \mathcal{H} . The operator

$$\mathcal{T}_{\mathfrak{F}}:\ell^2 o\mathcal{H},\quad \mathcal{T}_{\mathfrak{F}}\left(\left\{\xi_k
ight\}_{k\in I}
ight)=\sum_{k=1}^\infty\xi_kf_k,$$

is called the synthesis operator of \mathfrak{F} , and

$$\mathcal{T}_{\mathfrak{F}}^*: \mathcal{H} \to \ell^2, \quad \mathcal{T}_{\mathfrak{F}}^* f = \{\langle f, f_k \rangle\}_{k \in I}$$

is called the analysis operator of \mathfrak{F} . The frame operator $S:\mathcal{H}\to\mathcal{H}$, defined by

$$Sf = \mathcal{T}_{\mathfrak{F}} \mathcal{T}_{\mathfrak{F}}^* f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \tag{1.2}$$

is a positive, self-adjoint, and invertible operator.

Definition 1.4. Let \mathcal{H} be a finitely or countably generated Hilbert \mathcal{A} -module \mathcal{H} over a unital C^* -algebra \mathcal{A} , and let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathfrak{G} = \{g_k\}_{k=1}^{\infty}$ be Bessel sequences in \mathcal{H} . If for $\mu > 0$,

$$\|\mathcal{T}_{\mathfrak{F}} - \mathcal{T}_{\mathfrak{G}}\| \leq \mu.$$

Then \mathfrak{G} is called a μ -perturbation of \mathfrak{F}

Definition 1.5. [12] For a given natural number m. A finite family of frames $\left\{\left\{x_{ij}\right\}_{j\in\mathbb{J}}, i\in\{1,\ldots,m\}\right\}$ in a Hilbert \mathcal{A} -module \mathcal{H} is said to be woven if there are universal constants C and D such that, for every partition $\left\{\sigma_{1},\ldots,\sigma_{m}\right\}$ of \mathbb{J} , the family $\left\{x_{ij}\right\}_{i=1,j\in\sigma_{i}}^{m}$ is a frame for \mathcal{H} with lower and upper frame bounds C and D, respectively. In this case, we usually call $\left\{\left\{x_{ij}\right\}_{j\in\mathbb{J}}, i\in\{1,\ldots,m\}\right\}$ woven with universal bounds (C,D). Each family $\left\{x_{ij}\right\}_{i=1,j\in\sigma_{i}}^{m}$ is called a weaving.

2. From Bessel sequences to frames

Proposition 2.1. Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence in \mathcal{H} with bound η and S be the frame operator. if we can write S in the following form $S = \mathcal{T} + \xi I$ for some real number ξ and a linear operator \mathcal{T} , then we have:

- (1) $\{f_k\}_{k=1}^{\infty}$ is a frame. If \mathcal{T} is a positive operator and $\xi > 0$,
- (2) \mathcal{T} is bounded and self-adjoint.
- (3) Let A > 0, the operator \mathcal{T} is a positive whenever $\xi \leq A$. If $\{f_k\}_{k=1}^{\infty}$ is a frame with lower bound A

Proof. (1) For every $f \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle = \langle Sf, f \rangle = \langle \xi f, f \rangle + \langle Tf, f \rangle$$

Since \mathcal{T} is a positive operator then,

$$\xi \langle f, f \rangle \le \sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle \le \eta \langle f, f \rangle$$

which implies the sequence $\{f_k\}_{k=1}^{\infty}$ is a frame for \mathcal{H} .

(2) for $f \in \mathcal{H}$ we have,

$$\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle \le \langle f, f \rangle.$$

Thus S is a bounded operator. Hence, $S - \xi I$ is also bounded. Since ξ is a real number the operator \mathcal{T} satisfies the following equalities

$$\mathcal{T} = \mathcal{T}^* = S - \xi I$$

is obvious.

(3) For every $f \in \mathcal{H}$,

$$\langle \mathcal{T}f, f \rangle = \langle Sf, f \rangle - \langle \xi f, f \rangle$$

$$\geq A \langle f, f \rangle - \xi \langle f, f \rangle$$

$$\geq (A - \xi) \langle f, f_k \rangle$$

Morever, the hypothesis $\xi \leq A$ shows that for every $f \in \mathcal{H}$,

$$\langle Tf, f \rangle \ge 0.$$

In the following theorem we admit the following hypothesis:

$$||f|||g|| \le \sqrt{1+\eta^2} ||\langle f, g \rangle|| \iff ||\alpha f - g|| \le \sqrt{\frac{\eta^2}{1+\eta^2}} ||g||, \quad f, g \in \mathcal{H}.$$
 (2.1)

Where $\alpha \in \mathbb{R}$ and $\eta \geq 0$.

Theorem 2.2. Let $\mathcal{T} \in \operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ and $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H} \text{ such that }$,

$$\sum_{k=1}^{\infty} \langle f, f_k \rangle f_k = \mathcal{T}f + \xi f, \quad \forall f \in \mathcal{H}$$

where ξ is a real scalar.

- (1) The famille $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence.
- (2) Suppose that α is a real number and $\eta \geq 0$ are such that

$$\|\alpha f - \mathcal{T}f\| \le \sqrt{\frac{\eta^2}{1+\eta^2}} \|\mathcal{T}f\|, \quad \forall f \in \mathcal{H}.$$
 (2.2)

If \mathcal{T} is self-adjoint and bounded from below by ρ with

$$\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi| > 0$$

then $\{f_k\}_{k=1}^{\infty}$ is a frame.

(3) With (2.2), assume that $\{g_k\}_{k=1}^{\infty}$ is a μ -perturbation of $\{f_k\}_{k=1}^{\infty}$. If $\mu < \sqrt{\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi|}$, then $\{g_k\}_{k=1}^{\infty}$ is a frame for \mathcal{H} with bounds

$$\left(\left(\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi|\right)^{\frac{1}{2}} - \mu\right)^2 \ and \ (\mu + (\|\mathcal{T}\| + |\xi|)^{\frac{1}{2}})^2.$$

Proof. (1) Using Cauchy-Schwarz inequality we get

$$\|\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle \| = \|\langle \mathcal{T}f, f \rangle + \langle \xi f, f \rangle \|$$

$$\leq \|\langle \mathcal{T}f, f \rangle \| + |\xi| \|f\|^2$$

$$\leq \|\langle \mathcal{T}f, \mathcal{T}f \rangle\|^{\frac{1}{2}} \|\langle f, f \rangle\|^{\frac{1}{2}} + |\xi| \|f\|^2$$

$$\leq \|\mathcal{T}\| \|\langle f, f \rangle\| + |\xi| \|f\|^2$$

$$= (\|\mathcal{T}\| + |\xi|) \|\langle f, f \rangle\|.$$

(2) In view of 2.1 and the assumption that $\mathcal{T} \in \operatorname{End}_{\mathcal{A}}^*(\mathcal{H})$ is self-adjoint we can write

$$\|\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle \| = \|\langle \xi f, f \rangle + \langle \mathcal{T} f, f \rangle \|$$

$$\geq \|\langle \mathcal{T} f, f \rangle \| - |\xi| \|\langle f, f \rangle \|$$

$$= \|\langle f, \mathcal{T} f \rangle \| - |\xi| \|\langle f, f \rangle \|$$

$$\geq \frac{1}{\sqrt{1+n^2}} \|\mathcal{T} f \| \|f\| - |\xi| \|\langle f, f \rangle \|,$$

Since the operator \mathcal{T} is bounded from below by ρ , we get

$$\|\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle \| \ge \sqrt{\frac{\rho^2}{1+\eta^2}} \|\langle f, f \rangle \| - |\xi| \|\langle f, f \rangle \|$$
$$= \left(\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi| \right) \|\langle f, f \rangle \|.$$

(3) Since $\{f_k\}_{k=1}^{\infty}$ is a frame with bounds $\|\mathcal{T}\| + |\xi|$ and $\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi|$, for every $f \in \mathcal{H}$,

$$\left(\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi|\right)^{\frac{1}{2}} \|f\| \le \|U_{|}^* f\| \le (|\xi| + \|\mathcal{T}\|)^{\frac{1}{2}} \|f\|.$$

Therefore,

$$\|\mathcal{T}_{\mathfrak{G}}^{*}(f)\| \ge \|\mathcal{T}_{\mathfrak{F}}^{*}(f)\| - \|(\mathcal{T}_{\mathfrak{F}}(f) - \mathcal{T}_{\mathfrak{G}}(f))^{*}\| \ge \left(\left(\sqrt{\frac{\rho^{2}}{1+\eta^{2}}} - |\xi|\right)^{\frac{1}{2}} - \mu\right) \|f\|.$$

Furthermore,

$$||U_{\dot{G}}^{*}(f)|| \leq ||(\mathcal{T}_{\mathfrak{F}}(f) - \mathcal{T}_{\mathfrak{G}}(f))^{*}|| + ||\mathcal{T}_{\mathfrak{F}}^{*}(f)|| \leq (\mu + (||\mathcal{T}|| + |\xi|)^{\frac{1}{2}})||f||.$$

Hence,

$$\left(\sqrt{\frac{\rho^2}{1+\eta^2}} - |\xi| - \mu\right)^2 \langle f, f \rangle \leq \sum_{i=1}^{\infty} \langle f, g_k \rangle \langle g_k, f \rangle \leq (\mu + (\|\mathcal{T}\| + |\xi|)^{\frac{1}{2}})^2 \langle f, f \rangle.$$

3. Compact and finite-rank-tight frames

We assume in this section that the compact self-adjoint operator \mathcal{T} is written in the following form

$$\mathcal{T} = \sum_{k=1}^{\infty} \alpha_k \langle \cdot, e_k \rangle e_k. \tag{3.1}$$

Where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for \mathcal{H} formed of the eigenvectors of \mathcal{T} . This fact is used to give conditions in the following theorem that enable us to assume a Bessel sequence is a frame.

Theorem 3.1. Let \mathcal{T} be a compact self-adjoint operator and $\{f_k\}_k^{\infty}$ be a Bessel sequence such that for $\xi > 0$, $S = \mathcal{T} + \xi I$. Assume that \mathcal{T} has the forme of (3.1) such that

$$\alpha := \inf_{k} \alpha_k + \xi > 0.$$

Then, $\{f_k\}_{k=1}^{\infty}$ is a frame.

Proof. For every $f \in \mathcal{H}$, we have

$$Sf = \mathcal{T}f + \xi f$$

$$= \sum_{k=1}^{\infty} \alpha_k \langle f, e_k \rangle e_k + \xi f$$

$$= \sum_{k=1}^{\infty} \alpha_k \langle f, e_k \rangle e_k + \xi \sum_{k=1}^{\infty} \langle f, e_k \rangle e_k$$

$$= \sum_{k=1}^{\infty} (\alpha_k + \xi) \langle f, e_k \rangle e_k.$$

Consequently,

$$\sum_{k=1}^{\infty} \langle f, f_k \rangle \langle f_k, f \rangle = \langle Sf, f \rangle$$

$$= \sum_{k=1}^{\infty} (\alpha_k + \xi) \langle f, e_k \rangle \langle e_k, f \rangle$$

$$\geq \sum_{k=1}^{\infty} \left(\inf_k \alpha_k + \xi \right) \langle f, e_k \rangle \langle e_k, f \rangle$$

$$\geq \alpha \langle f, f \rangle.$$

Where $\alpha = \xi + \inf_k \alpha_k > 0$.

Definition 3.2. We say that a frame is compact-tight. If its frame operator S satisfies the following condition

$$S = \mathcal{K} + \xi I \tag{3.2}$$

where \mathcal{K} being a compact operator.

The previous definition remains true for finite-rank.

Let $\{f_k\}_{k=1}^{\infty}$ be a compact-tight frame such that

$$S = \mathcal{K}_1 + \xi_1 I = \mathcal{K}_2 + \xi_2 I,$$

where \mathcal{K}_1 and \mathcal{K}_2 being compact operators. Then,

$$\mathcal{K}_2 - \mathcal{K}_1 = (\xi_1 - \xi_2) I. \tag{3.3}$$

Thus

$$\xi_1 = \xi_2$$
.

The operator to the right of is indeed a compact operator, but the operator to the left cannot be compact unless

$$\xi_1 - \xi_2 = 0.$$

Consequently

$$\mathcal{K}_1 = \mathcal{K}_2$$
.

Proposition 3.3. The representation of the frame operator S as a compact perturbation of the form (3.2) is unique. If $\{f_k\}_{k=1}^{\infty}$ is compact-tight frame.

We can formulate the proposition 3.3 for tight finite rank frames like this we say that $\{f_k\}_{k=1}^{\infty}$ is a (ξ, \mathcal{K}) -compact-tight frame. If $\{f_k\}_{k=1}^{\infty}$ is a compact-tight frame such that the frame operator is written in the following form

$$S = \mathcal{K} + \xi I,$$

The method for creating compact-tight frames based on a given orthonormal basis is described in our following theorem.

Theorem 3.4. Let $\{l_k\}_{k=1}^{\infty}$ be a sequence of real numbers greater than ξ such that $\lim_{k\to\infty} l_k = \xi$ and ξ be a positive real number. If $\{e_k\}_{k=1}^{\infty}$ is any orthonormal basis of \mathcal{H} , then $\{l_k^{\frac{1}{2}}e_k\}_{k=1}^{\infty}$ is a compact-tight frame for \mathcal{H} with frame bounds ξ and $\sup_k l_k$.

Proof. Let $f \in \mathcal{H}$, we have

$$\sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle \left\langle l_k^{\frac{1}{2}} e_k, f \right\rangle = \sum_{k=1}^{\infty} l_k \left\langle f, e_k \right\rangle \left\langle e_k, f \right\rangle$$

implies that

$$\xi\langle f, f \rangle \le \sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle \left\langle l_k^{\frac{1}{2}} e_k, f \right\rangle \le \left(\sup_{k} l_k \right) \langle f, f \rangle.$$

Thus, $\left\{l_k^{\frac{1}{2}}e_k\right\}_{k=1}^{\infty}$ is a frame for \mathcal{H} with frame bounds ξ and $\sup_k l_k$. To show that this frame is compact-tight, note that

$$Sf = \sum_{k=1}^{\infty} \left\langle f, l_k^{\frac{1}{2}} e_k \right\rangle l_k^{\frac{1}{2}} e_k$$

$$= \sum_{k=1}^{\infty} l_k \left\langle f, e_k \right\rangle e_k$$

$$= \sum_{k=1}^{\infty} (l_k + \xi - \xi) \left\langle f, e_k \right\rangle e_k$$

$$= \sum_{k=1}^{\infty} (l_k - \xi) \left\langle f, e_k \right\rangle e_k + \xi \sum_{k=1}^{\infty} \left\langle f, e_k \right\rangle e_k$$

$$= \mathcal{K}f + \xi f.$$

Since $\lim_{k\to\infty} l_k = \xi$, Thus

$$\mathcal{K} := \sum_{k=1}^{\infty} (l_k - \xi) \langle ., e_k \rangle e_k$$

is a compact operator.

Example 3.5. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for \mathcal{H} . Since

$$l_k := 1 + \exp\left(-\frac{k^2}{2}\right) > 1$$

for every k and

$$\lim_{k \to \infty} l_k = 1,$$

Theorem 3.4 proves the sequence $\left\{l_k^{\frac{1}{2}}e_k\right\}_{k=1}^{\infty}$ is a compact-tight frame with bounds 1 and 2 .

Proposition 3.6. Let $\{g_k\}_{k=1}^{\infty}$ be a frame in \mathcal{H} It is created from an orthonormal basis by repeatedly iterating a set number of times each basis member. Then, $\{g_k\}_{k=1}^{\infty}$ is a finite-rank-tight frame.

Proof. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for \mathcal{H} . Assume that $\{g_k\}_{k=1}^{\infty}$ is obtained by repeating the basis elements to create the basis. e_{j_1}, \ldots, e_{j_n} , such

that e_{j_m} is repeated $\theta_m > 1$ times for each $m \in \{1, \ldots, n\}$. Then,

$$Sf = \sum_{k=1}^{\infty} \langle f, g_k \rangle g_k$$

$$= \sum_{m=1}^{n} (\theta_m - 1) \langle f, e_{j_m} \rangle e_{j_m} + \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i$$

$$= \mathcal{K}f + f.$$

Then, $\mathcal{K} = \sum_{m=1}^{n} (\theta_m - 1) \langle \cdot, e_{j_m} \rangle e_{j_m}$ is a finite-rank operator.

Theorem 3.7. The canonical dual of a compact-tight frame, is compact-tight.

Proof. Let $\{f_k\}_{k=1}^{\infty}$ be a (ξ, \mathcal{K}) -compact-tight frame such that the frame operator $S = \mathcal{K} + \xi I$.

In which case $\xi \neq 0$ and \mathcal{K} is a compact operator. Then,

$$S^{-1} = \mathcal{T} + \xi^{-1}I$$

where \mathcal{T} is a compact operator. So all we have to do is choose \mathcal{T} in such a way that

$$\mathcal{T}S = -\xi^{-1}\mathcal{K}.$$

So it is evident that \mathcal{T} is a compact operator. Furthermore,

$$(\mathcal{T} + \xi^{-1}I) S = \mathcal{T}S + \xi^{-1}S$$
$$= \xi^{-1}(\mathcal{K} + \xi I) - \xi^{-1}\mathcal{K}$$
$$= I.$$

Theorem 3.8. Let $\mathfrak{F} = \{f_k\}_{k=1}^{\infty}$ and $\mathfrak{G} = \{\tilde{f}_k\}_{k=1}^{\infty}$ be $(1, \mathcal{K}_1)$ - and $(1, \mathcal{K}_2)$ compact-tight frames, respectively. If there exists an infinite subset σ of \mathbb{N} such that $\{f_k\}_{k\in\sigma}$ and $\{\tilde{f}_k\}_{k\in\sigma^c}$ are orthonormal bases. Then, $\{f_k\}_{k=1}^{\infty}$ and $\{\tilde{f}_k\}_{k=1}^{\infty}$ cannot be woven.

Proof. Assume on the contrary that $\{f_k\}_{k=1}^{\infty}$ and $\{\tilde{f}_k\}_{k=1}^{\infty}$ are woven. Then, for the partition $\{\sigma, \sigma^c\}$, we obtain the frame $\{f_k\}_{k \in \sigma^c} \cup \{\tilde{f}_k\}_{k \in \sigma}$. According to the

definition 3.2, the frame operators for $\{f_k\}_{k=1}^{\infty}$ and $\{\tilde{f}_k\}_{k=1}^{\infty}$ can be written as

$$S_F f = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$$

$$= \sum_{k \in \sigma} \langle f, f_k \rangle f_k + \sum_{k \in \sigma^c} \langle f, f_k \rangle f_k$$

$$= \mathcal{K}_1 f + f$$

and

$$S_G f = \sum_{k=1}^{\infty} \left\langle f, \tilde{f}_k \right\rangle \tilde{f}_k$$
$$= \sum_{k \in \sigma^c} \left\langle f, \tilde{f}_k \right\rangle \tilde{f}_k + \sum_{k \in \sigma} \left\langle f, \tilde{f}_k \right\rangle \tilde{f}_k$$
$$= \mathcal{K}_2 f + f.$$

Make clear the frame operator of $\{f_k\}_{k\in\sigma^c}\cup\{\tilde{f}_k\}_{k\in\sigma}$ by S_{F+G} . Then for $f\in\mathcal{H}$,

$$S_{F+G}f = \sum_{k \in \sigma^c} \langle f, f_k \rangle f_k + \sum_{k \in \sigma} \langle f, \tilde{f}_k \rangle \tilde{f}_k$$
$$= \mathcal{K}_1 f + \mathcal{K}_2 f + f$$
$$= (\mathcal{K}_1 + \mathcal{K}_2) f + f.$$

Thus,

$$S_{F+G} = \mathcal{K}_1 + \mathcal{K}_2.$$

Since \mathcal{K}_1 and \mathcal{K}_2 are compact operators, S_{F+G} is a compact operator, which is absurd.

DECLARATIONS

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Not applicable.

Ethics Approval and Consent to Participate

It is important to note that this article does not involve any studies with animals or human participants.

Competing Interests

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Authors' Contributions

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References

- Daubechies I., Grossmann A., Meyer Y., 1986. Painless nonorthogonal expansions, J. Math. Phys. 27, 1271–1283.
- [2] Duffin R. J, Schaeffer A. C, Trans. Amer. Math. Soc. 72, 341-366, 1952. Doi: doi.org/10.1090/S0002-9947-1952-0047179-6
- [3] Ghiati, M., Rossafi, M., Mouniane, M. et al. Controlled continuous *-g-frames in Hilbert C*-modules. J. Pseudo-Differ. Oper. Appl. 15, 2 (2024). https://doi.org/10.1007/s11868-023-00571-1
- [4] Kaplansky I, Modules over operator algebras, Amer. J. Math. 75, 839-858, 1953. Doi: doi.org/10.2307/2372552
- Khorsavi A, Khorsavi B, Fusion frames and g-frames in Hilbert C*-modules, Int.
 J. Wavelet, Multiresolution and Information Processing 6, 433-446, 2008. Doi: doi.org/10.1142/S0219691308002458
- [6] H. Massit, M. Rossafi, C. Park, Some relations between continuous generalized frames. Afr. Mat. 35, 12 (2024). https://doi.org/10.1007/s13370-023-01157-2
- [7] Paschke, W.: Inner product modules over B* -algebras. Trans. Am. Math. Soc. 182, 443–468 (1973)
- [8] Rossafi M, Kabbaj S, *-K-operator Frame for $End^*_{\mathcal{A}}(\mathcal{H})$, Asian-Eur. J. Math. 13 (2020). Doi: doi.org/10.1142/S1793557120500606.
- [9] Rossafi, M., Ghiati, M., Mouniane, M., Chouchene, F., Touri, A., Kabbaj, S.: Continuous frame in Hilbert C*-modules. J. Anal. (2023). https://doi.org/10.1007/s41478-023-00581-8
- [10] Rossafi, M., Nhari, F. D., Park, C., Kabbaj, S.: Continuous g-frames with C*-valued bounds and their properties. Complex Anal. Operator Theory 16, 44 (2022). https://doi.org/10.1007/s11785-022-01229-4
- [11] N. K. Sahu, Controlled g-frames in Hilbert C*-modules, Mathematical Analysis and its Contemporary Applications, 3 (2021), 65–82.
- [12] X. Zhao, P. Li, Weaving Frames in Hilbert C*- Modules, Hindawi Journal of Mathematics Volume 2021, Article ID 2228397, 13 pages, https://doi.org/10.1155/2021/2228397.

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCES, UNIVERSITY OF IBN TOFAIL, KENITRA, MOROCCO

Email address: abdelilah.karara@uit.ac.ma; khadija.mabrouk@uit.ac.ma