[Digital Signal Processing]

Homework 2

1. Consider a continuous-time signal $x_c(t)$ with Fourier transform $X_c(j\Omega)$ shown in Figure 1-1.

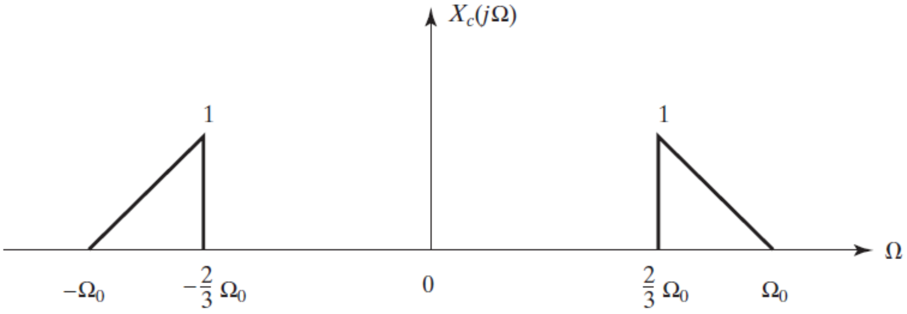


Figure 1-1

(a) A continuous-time signal $x_r(t)$ is obtained through the process shown in Figure 1-2. First, $x_c(t)$ is multiplied by an impulse train of period T_1 to produce the waveform $x_s(t)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_1).$$

Next, $x_s(t)$ is passed through a low-pass filter with frequency response $H_r(j\Omega)$. $H_r(j\Omega)$ is shown inf Figure 1-3. Determine the range of values for T_1 for which $x_r(t) = x_c(t)$.

Nyquist 조건.

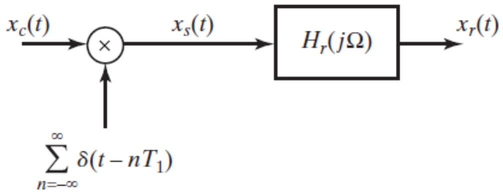


Figure 1-2

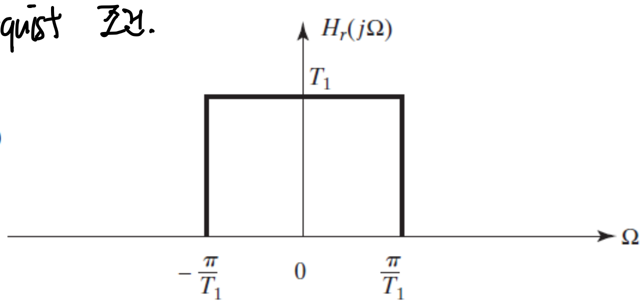
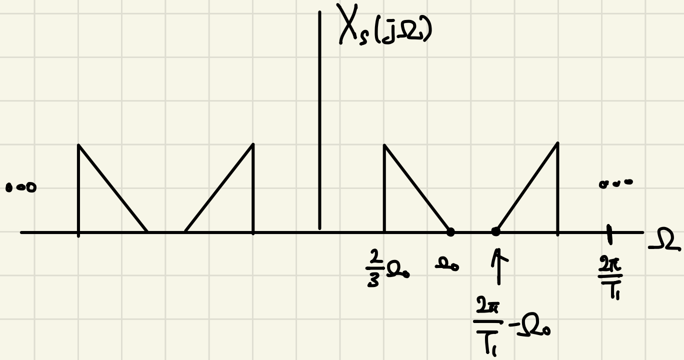


Figure 1-3

(b) Consider the system in Figure 1-4. The system in this case is the same as the one in part (a), except that the sampling period is now T_2. The system $H_s(j\Omega)$ is some continuous-time ideal LTI filter. We want $x_o(t)$ to be equal to $x_c(t)$ for all t, i.e., $x_o(t) = x_c(t)$ for some choice of $H_s(j\Omega)$. Find all values of T_2 for which $x_o(t) = x_c(t)$ is possible.

1

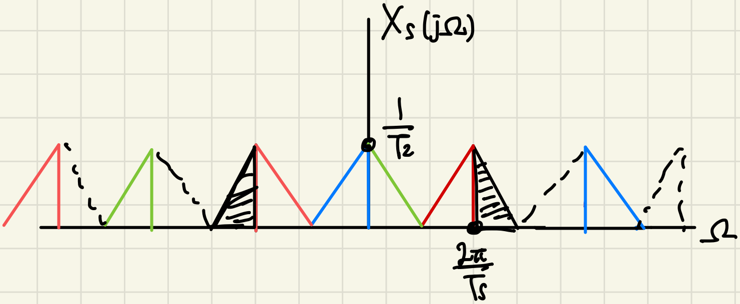
(a)



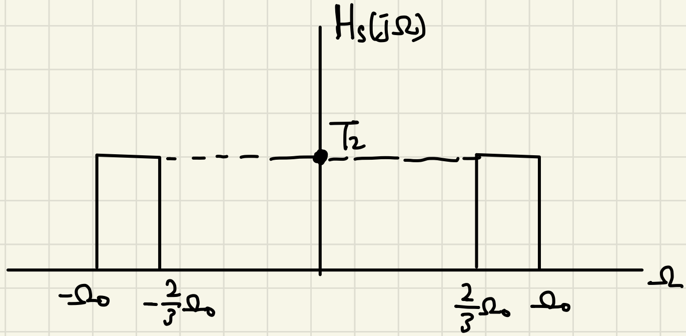
$\therefore \frac{2\pi}{T_1} - \Omega_0 \geq \Omega_0$

; $\boxed{T_1 \leq \frac{\pi}{\Omega_0}}$

(b) $H_s(j\Omega)$는 굳이 LPF가 아니어도 되므로:



$\therefore \frac{2\pi}{T_s} = \frac{2}{3}\Omega_0$; $\boxed{T_s = \frac{3\pi}{\Omega_0}}$



For the largest T_2 you determined that would still allow recovery of $x_c(t)$, choose $H_s(j\Omega)$ so that $x_o(t) = x_c(t)$. Sketch $H_s(j\Omega)$.

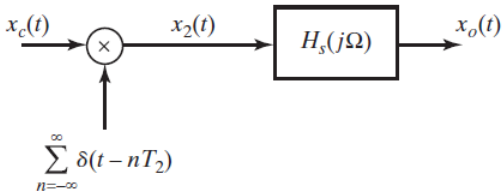


Figure 1-4

2. Figure 2-1 shows a continuous-time filter that is implemented using an LTI discrete-time ideal lowpass filter with frequency response over $-\pi \leq \omega \leq \pi$ as

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases}$$

(a) If the continuous-time Fourier transform of $x_c(t)$, namely $X_c(j\Omega)$, is as shown in Figure 2-2 and $\omega_c = \pi/5$, sketch and label $X(e^{j\omega})$, $Y(e^{j\omega})$ and $Y_c(j\Omega)$ when $1/T_1 = 1/T_2 = 2 \times 10^4$.

(b) For $1/T_1 = 1/T_2 = 6 \times 10^3$, and for input signals $x_c(t)$ whose spectra are bandlimited to $|\Omega| < 2\pi \times 5 \times 10^3$ (but otherwise unconstrained), what is the maximum choice of the cutoff frequency ω_c of the filter $H(e^{j\omega})$ for which the overall system is LTI? For this maximum choice of ω_c, specify $H_c(j\Omega)$.

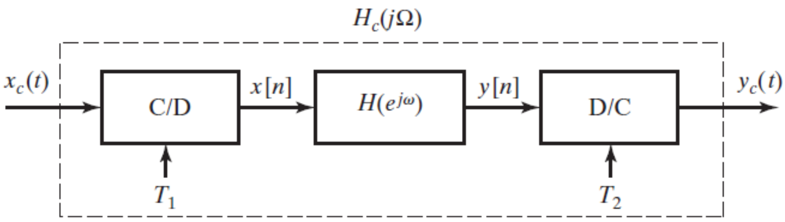


Figure 2-1

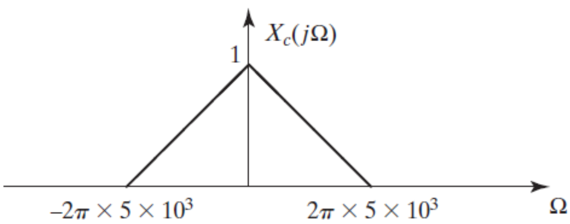
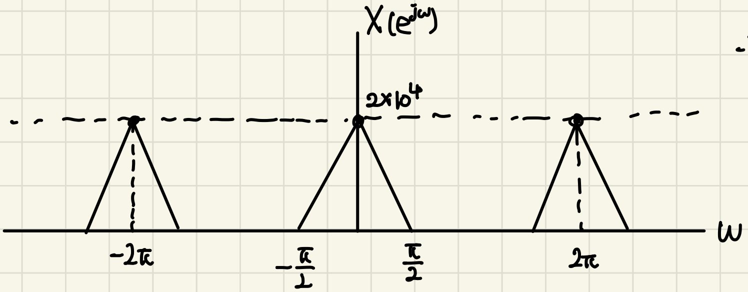
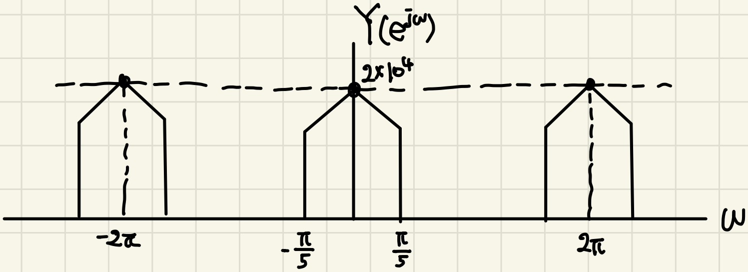


Figure 2-2

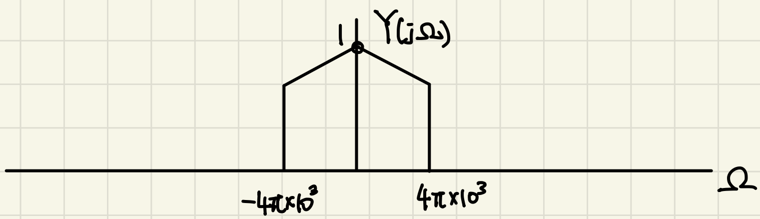
$\boxed{2}$ (a) $1/T_1 = 1/T_2 = 2\times10^4$ 일 때:



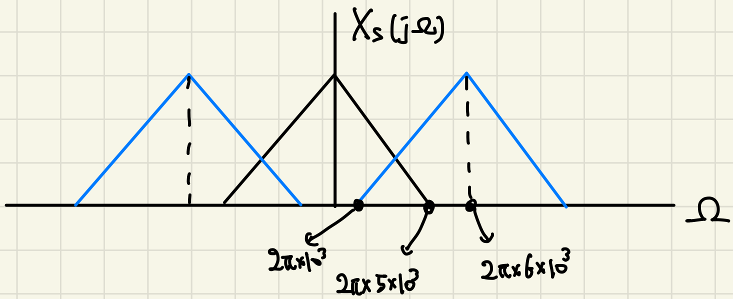
$$\therefore \omega = \Omega T_s = (2\pi\times5\times10^3)\cdot\frac{1}{2\times10^4} = \frac{\pi}{2}$$

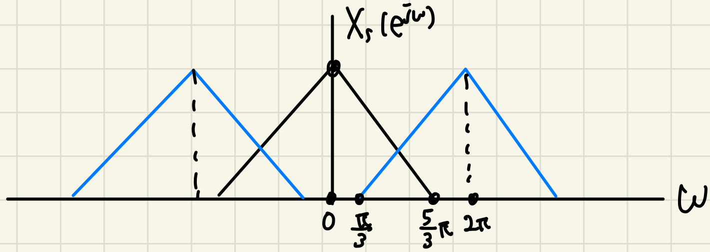


$$\therefore \Omega = \frac{\omega}{T_2} = \left(\frac{\pi}{5}\right)\cdot(2\times10^4) = 4\pi\times10^3$$



(b)





$$\boxed{\therefore \omega_{c,max} = \frac{\pi}{3}}$$

3. The system shown in Figure 3 is intended to approximate a differentiator for bandlimited continuous-time input waveforms.

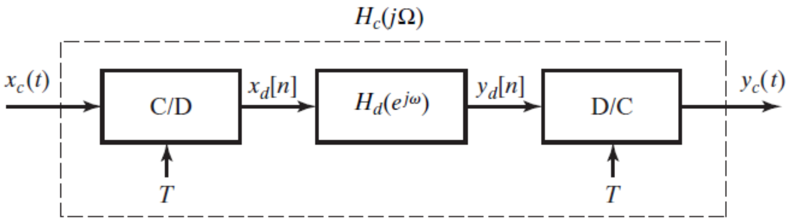


Figure 3

- The continuous-time input signal $x_c(t)$ is bandlimited to $|\Omega| < \Omega_M$.
- The C/D converter has sampling rate $T = \frac{\pi}{\Omega_M}$, and produces the signal $x_d[n] = x_c(nT)$.
- The discrete-time filter has frequency response

$$H_d(e^{j\omega}) = \frac{e^{j\omega/2} - e^{-j\omega/2}}{T}, \quad |\omega| \leq \pi$$

- The ideal D/C converter is such that $y_d[n] = y_c(nT)$.

(a) Find the continuous-time frequency response $H_c(j\Omega)$ of the end-to-end system.

(b) Find $x_d[n]$, $y_c(t)$, and $y_d[n]$, when the input signal is

$$x_c(t) = \frac{\sin(\Omega_M t)}{\Omega_M t}.$$

4. Consider the representation of the process of sampling followed by reconstruction shown in Figure 4.

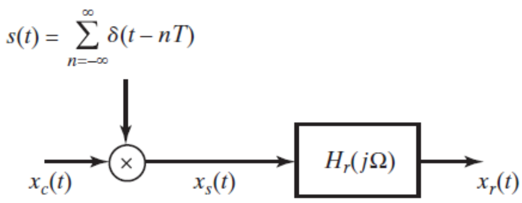


Figure 4

Assume that the input signal is

$$x_c(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3), \quad -\infty < t < \infty$$

The frequency response of the reconstruction filter is

$\boxed{3}$ (a) • $X_d(e^{j\omega}) = \frac{1}{T} X_c(j\frac{\omega}{T}) \quad (|\omega| < \pi)$

• $Y_d(e^{j\omega}) = \frac{1}{T} H_d(e^{j\omega}) X_c(j\frac{\omega}{T}) \quad (|\omega| < \pi)$

• $Y_c(j\Omega) = H_d(e^{j\Omega T}) X_c(j\Omega)$

• $H_c(j\Omega) = \begin{cases} \frac{Y_c(j\Omega)}{X_c(j\Omega)} & (|\Omega| < \frac{\pi}{T}) \\ 0 & (\text{o.w}) \end{cases}$

$= \begin{cases} H_d(e^{j\Omega T}) & (|\Omega| < \frac{\pi}{T}) \\ 0 & (\text{o.w}) \end{cases}$

$= \begin{cases} \frac{e^{j\Omega T/2} - e^{-j\Omega T/2}}{T} & (|\Omega| < \frac{\pi}{T}) \\ 0 & (\text{o.w}) \end{cases} = \boxed{\begin{cases} j\frac{2}{T}\sin(\frac{\Omega T}{2}) & (|\Omega| < \frac{\pi}{T}) \\ 0 & (\text{o.w}) \end{cases}}$

(b) $x_d[n] = x_c(nT)$

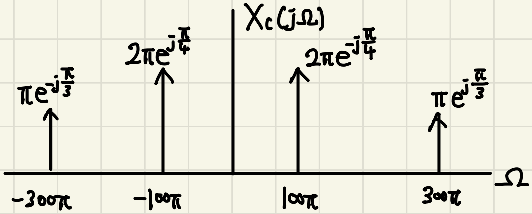
$= \frac{\sin(\Omega_M \cdot nT)}{\Omega_M \cdot nT}$

$= \frac{\sin(\pi n)}{\pi n} = 0 \quad (\because T = \frac{\pi}{\Omega_M} ; \Omega_M T = \pi)$

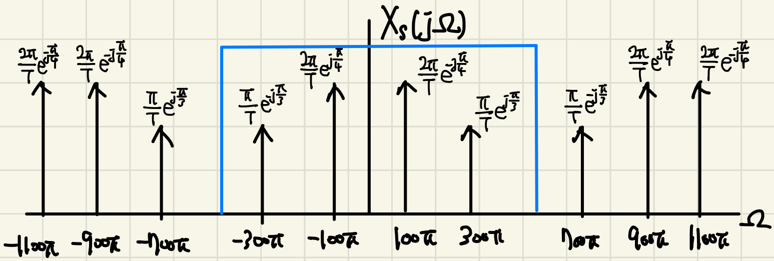
따라서 $y_d[n] = 0, \quad y_c(t) = 0$

4 (a) $X_c(j\Omega) = 2\pi e^{-j\frac{\pi}{4}}\delta(\Omega-100\pi) + 2\pi e^{j\frac{\pi}{4}}\delta(\Omega+100\pi)$

$+\ \pi e^{j\frac{\pi}{3}}\delta(\Omega-300\pi) + \pi e^{-j\frac{\pi}{3}}\delta(\Omega+300\pi)$



(b)

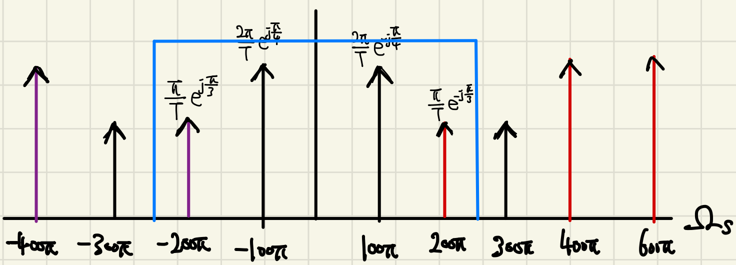


$f_s = \frac{1}{T} = 500$ (sample/sec) $\longrightarrow \frac{2\pi}{T} = 1000\pi$, $\frac{\pi}{T} = 500\pi$

∴ Aliasing 이 발생하지 않으므로, $x_r(t) = x_c(t)$

$\rightarrow \boxed{x_r(t) = 2\cos\left(100\pi t - \frac{\pi}{4}\right) + \cos\left(300\pi t + \frac{\pi}{3}\right) \quad (-\infty < t < \infty)}$

(c) $f_s = \frac{1}{T} = 250$ (sample/sec) $\longrightarrow \frac{2\pi}{T} = 500$, $\frac{\pi}{T} = 250$

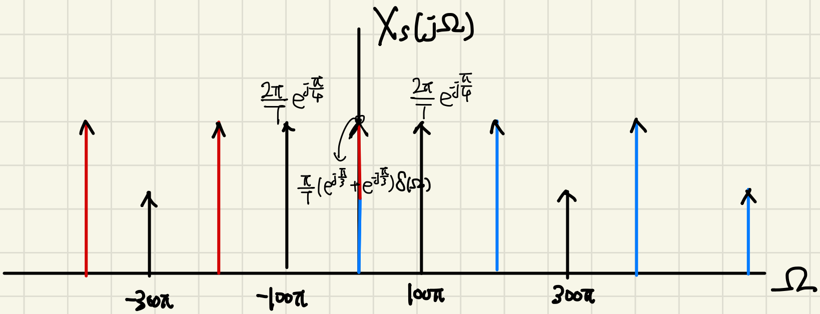


∴ Aliasing 발생 : $X_r(j\Omega) = 2\pi\left[e^{-j\frac{\pi}{4}}\delta(\Omega-100\pi) + e^{j\frac{\pi}{4}}\delta(\Omega+100\pi)\right]$

$+\ \pi\left[e^{-j\frac{\pi}{3}}\delta(\Omega-200\pi) + e^{j\frac{\pi}{3}}\delta(\Omega+200\pi)\right]$

$\Rightarrow x_r(t) = 2\cos\left(100\pi t - \frac{\pi}{4}\right) + \cos\left(200\pi t - \frac{\pi}{3}\right)$

(d)



$\Omega = \pm 100\pi, 0$ 인 성분만 추출하고 동시에 DC 성분이 존재해야 하므로

이를 만족하는 경우는 $\frac{2\pi}{T} = 300\pi$; $\frac{1}{T} = 150$ 인 경우뿐이다.

가능한가? (O)

$f_s = \frac{1}{T} = 150 \rightarrow \Omega_s = 300\pi \ (rad/s)$

$x_r(t) = \frac{1}{2}\left\{\cos\left(\frac{\pi}{3}\right) + \cos\left(-\frac{\pi}{3}\right)\right\} + 2\cos\left(100\pi t - \frac{\pi}{4}\right)$

$= \frac{1}{2} + 2\cos\left(100\pi t - \frac{\pi}{4}\right)$ $\therefore A = \frac{1}{2}$

$$H_r(j\Omega) = \begin{cases} T & |\Omega| \le \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

(a) Determine the continuous-time Fourier transform $X_c(j\Omega)$ and plot it as a function of Ω.

(b) Assume that $f_s = 1/T = 500$ samples/sec and plot the Fourier transform $X_s(j\Omega)$ as a function of Ω for $-2\pi/T \le \Omega \le 2\pi/T$. What is the output $x_r(t)$ in this case? (You should be able to give an exact equation for $x_r(t)$.)

(c) Now, assume that $f_s = 1/T = 250$ samples/sec. Repeat part (b) for this condition.

(d) Is it possible to choose the sampling rate so that

$$x_r(t) = A + 2\cos(100\pi t - \pi/4)$$

where A is a constant? If so, what is the sampling rate $f_s = 1/T$, and what is the numerical value of A?

5. Consider a discrete-time LTI system for which the frequency response $H(e^{j\omega})$, is described by:

$$H(e^{j\omega}) = -j, \qquad 0 < \omega < \pi$$

$$H(e^{j\omega}) = j, \qquad -\pi < \omega < 0$$

(a) Is the impulse response of the system $h[n]$ real-valued? (i.e., is $h[n] = h^*[n]$ for all n)

(b) Calculate the following

$$\sum_{n=-\infty}^{\infty} |h[n]|^2$$

(c) Determine the response of the system to the input $x[n] = s[n]\cos(\omega_c n)$, where $0 < \omega_c < \pi/2$ and $S(e^{j\omega}) = 0$ for $\omega_c/3 \le |\omega| \le \pi$.

6. A causal LTI system has the system function

$$H(z) = \frac{(1 - e^{j\pi/3}z^{-1})(1 - e^{-j\pi/3}z^{-1})(1 + 1.1765z^{-1})}{(1 - 0.9e^{j\pi/3}z^{-1})(1 - 0.9e^{-j\pi/3}z^{-1})(1 + 0.85z^{-1})}.$$

(a) Write the difference equation that is satisfied by the input $x[n]$ and output $y[n]$ of this system.

5 (a) $H^*(e^{j\omega}) = H(e^{-j\omega})$ 여부 확인하기.

$\boxed{\text{답: Yes}}$

- $0<\omega<\pi$ 에서 : $H(e^{j\omega}) = -j \rightarrow H(e^{-j\omega}) = j$
- 그런데 $e^{j(-\omega)}$ 에서 $-\pi < -\omega < 0$.
- 따라서 $H(e^{j(-\omega)}) = j \longrightarrow H(e^{-j\omega}) = H^*(e^{j\omega})$ (성립.)

(b) $\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|H(e^{j\omega})|^2}_{\substack{|j|^2=1 \\ |-j|^2=1}} d\omega = \boxed{1}$

(c) $x[n] = s[n]\cos(\omega_c n)$

$$\therefore X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} s[n]\cos(\omega_c n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} s[n]e^{-j\omega n}\left(\frac{e^{j\omega_c n} + e^{-j\omega_c n}}{2}\right)$$

$-j \rightarrow e^{-j\frac{\pi}{2}}$

$j \rightarrow e^{j\frac{\pi}{2}}$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} s[n]e^{-j(\omega-\omega_c)n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} s[n]e^{-j(\omega+\omega_c)n} = \frac{1}{2}\left[S(e^{j(\omega-\omega_c)}) + S(e^{j(\omega+\omega_c)})\right]$$

- $S(e^{j\omega})$ 는 $0 \le |\omega| \le \frac{\omega_c}{3}$ 에서만 존재하므로

  - $\omega - \omega_c = \pm\frac{\omega_c}{3}$; $\omega = \frac{4}{3}\omega_c, \frac{2}{3}\omega_c$
  - $\omega + \omega_c = \pm\frac{\omega_c}{3}$; $\omega = -\frac{4}{3}\omega_c, -\frac{2}{3}\omega_c$

$\rightarrow$ $X(e^{j\omega})$ 의 non zero 성분은 $\underbrace{[-\frac{4}{3}\omega_c, -\frac{2}{3}\omega_c]}_{[-\pi,0]}$, $\underbrace{[\frac{2}{3}\omega_c, \frac{4}{3}\omega_c]}_{[0,\pi]}$ 에 존재.

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{2}\left[-jS(e^{j(\omega-\omega_c)}) + jS(e^{j(\omega+\omega_c)})\right]$$

$$= \sum_{n=-\infty}^{\infty} s[n]e^{-j\omega n} \cdot \frac{e^{j(\omega_c n - \frac{\pi}{2})} + e^{-j(\omega_c n - \frac{\pi}{2})}}{2}$$

$$= \sum_{n=-\infty}^{\infty} s[n]\cos\left(\omega_c n - \frac{\pi}{2}\right)e^{-j\omega n}$$

$\boxed{\text{답: } y[n] = s[n]\cos\left(\omega_c n - \frac{\pi}{2}\right)}$

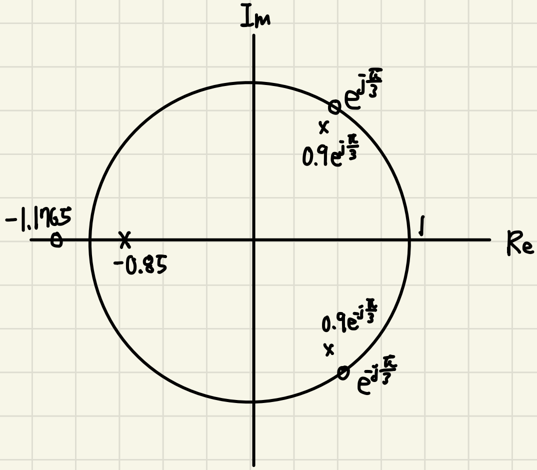
$\boxed{6}$ (a) $H(z) = \dfrac{(1-e^{j\pi/3}z^{-1})(1-e^{-j\pi/3}z^{-1})(1+1.1765z^{-1})}{(1-0.9e^{j\pi/3}z^{-1})(1-0.9e^{-j\pi/3}z^{-1})(1+0.85z^{-1})}$

$$= \frac{1+0.1765z^{-1}-0.1765z^{-2}+1.1765z^{-3}}{1-0.05z^{-1}+0.045z^{-2}+0.6855z^{-3}} = \frac{Y(z)}{X(z)}$$

$\therefore x[n]+0.1765x[n-1]-0.1765x[n-2]+1.1765x[n-3]$

$= y[n]-0.05y[n-1]+0.045y[n-2]+0.6855y[n-3]$

(b)



$\therefore$ system 이 causal.

$\leftrightarrow$ ROC 가 unit circle 포함

$\leftrightarrow$ $\boxed{\text{ROC} : |z| > 0.9}$

(b) Plot the pole–zero diagram and indicate the ROC for the system function.

7. Consider the system function

$$H(z) = \frac{rz^{-1}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}, \qquad |z| > r.$$

Assume first that $\omega_0 \neq 0$.

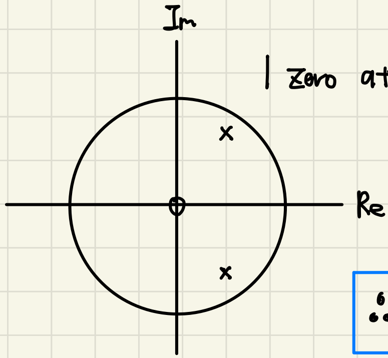
(a) Draw a labeled pole–zero diagram and determine $h[n]$.

(b) Repeat part (a) when $\omega_0 = 0$.

17 (a) $H(z) = \dfrac{rz}{z^2 - r(e^{j\omega} + e^{-j\omega})z + r^2}$

$\therefore$ pole : $z = re^{j\omega}, re^{-j\omega}$

zero : $z = 0, \infty$



1 zero at $z = \infty$.

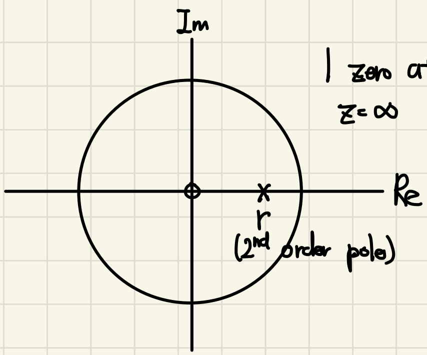
$$\boxed{\therefore h[n] = \left[\frac{1}{\sin\omega_0}\right]\left[r^n \sin(\omega_0 n)\, u[n]\right]}$$

(b) $\omega_0 = 0$ 일 때:

$$H(z) = \frac{rz^{-1}}{1 - 2rz^{-1} + r^2z^{-2}} = \frac{rz}{z^2 - 2rz + r^2} = \frac{rz}{(z-r)^2}$$

$\therefore$ pole : $z = r$ (중복),

zero : $z = 0, \infty$



1 zero at

$z = \infty$

$\therefore H(z) = \dfrac{rz^{-1}}{(1 - rz^{-1})^2}$ $(|z| > r)$

$\Updownarrow$

$$\boxed{h[n] = n r^n u[n]}$$