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Discrete Mathematics

CS 231

Assignment # 4

NAME : Shayan Rizwan Yazdanie

REG # : 2024585

Q27.

There are 50 states

↳ From EACH state, only one (1) representative is to be chosen.

⇒ CHOICES : Goverener or Senator 1 or Senator 2

↳ 3 choices

} multiplicative counting principle

Applying the product rule ; since the choices are independent (choosing from one state doesn't affect choices from another).

⇒ 3^{50}.

Q33.

(a) Vowels : $\{a, e, i, o, u\} = 5$ vowels

Consonants : $26 - 5 = 21$ consonants

∴ No vowels = consonants ;

With repetition = 21^8.

(b) Choosing 8 different consonants from 21 :

$P(21, 8) = \frac{21!}{(21-8)!} = \frac{21!}{13!} = 8.2 \times 10^9$

(c) $\underline{5}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26} = \cancel{5 \times 26^6}\ 5 \times 26^7 = 4.0159 \times 10^{10}$

(d) $\underline{5}\ \underline{25}\ \underline{24}\ \underline{23}\ \underline{22}\ \underline{21}\ \underline{20}\ \underline{19} = 5 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19$

↳ without repetition $= 1.21 \times 10^{10}$.

(e) At least one vowel (≥ 1 vowel)

COMPLEMENT (at least one vowel) = < 1 vowel = No vowels

∴ TOTAL − COMPLEMENT = at least one vowel

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Total strings with repetition $= 26^8$

Strings with no vowels $= 21^8$

By complement : $26^8 - 21^8 = 1.71 \times 10^{11}$

(f) $\boxed{5}$ $\underline{21}$ $\underline{21}$ $\underline{21}$ $\underline{21}$ $\underline{21}$ $\underline{21}$ $\underline{21}$ } EXACTLY one vowel

⇒ Choosing which one of the 8 positions will hold the vowel ;

$^8C_1 = 8$

⇒ For the chosen vowel position, we need to pick which vowel it is : 5

$\therefore$ $(^8C_1) \times 5 \times 21^7 = 7.204 \times 10^{10}$

(g) $\boxed{\times}$

at least one vowel } can be across whole string.

$\therefore$ TOTAL - COMPLEMENT = at least one string

⇒ Remaining 7 letters : total possibilities with repetition $= 26^7$

⇒ Remaining 7 letters : NO Vowels $= 21^7$

$\therefore$ $26^7 - 21^7$ = at least one string } remaining 7 letters with at least one vowel.

$\therefore$ $1 \times (26^7 - 21^7) = 26^7 - 21^7 = 6.23 \times 10^9$

Q34. A function from set A (domain, size m) to set B (codomain, size n) assigns each of the m elements in A to exactly one of the n elements in B.

$\therefore$ Number of such functions $= n^m$ (since for each of m inputs, there are n choices for the output, INDEPENDENTLY).

(a) $2^{10} = 1024$

(b) $3^{10} = 59049$

(c) $4^{10} = 1048576$

(d) $5^{10} = 9765625$

Each of the domain elements independently chooses one of the n codomain elements.

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Q36. Domain : $\{1, 2, \ldots, n\}$ → has n elements

Codomain : $\{0, 1\}$ → has 2 elements

Each of the n elements in the domain can be mapped to either 0 or 1.

$\therefore$ $\underline{2^n}$.

Q41. A palindrome reads the same forwards and backwards.
That means once you fix the first half of the string, the second half is forced.

We need to determine bit (1 or 0) strings of length n :-

<u>CASE 1 - n is even</u>

$\Rightarrow$ $n = 2k$; $k = n/2$.

A length-2k string looks like: $(a_1, a_2, a_3, \ldots, \cancel{a_k}, a_{k+1}, a_{2k})$.

$\therefore$ The palindrome condition : $a_1 = a_{2k}$, $a_2 = a_{2k-1}$, $a_k = a_{k+1}$.

So, the second half $(a_{k+1}, \ldots, a_{2k})$ is completely determined by the first half $(a_1, \ldots, a_k)$, and therefore, each of the options of the first half needs to be determined ;

$\Rightarrow$ 2^k ; $\underline{\underline{2^{n/2}}}$.

<u>CASE 2 - n is odd</u>

$\Rightarrow$ $n = 2k+1$; $k = n-1/2$

A length-(2k+1) string looks like : $(a_1, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_{2k+1})$

$\therefore$ The palindrome condition : $a_1 = a_{2k+1}$, $a_2 = a_{2k}$, $\ldots$, $a_k = a_{k+2}$.

The middle part, a_{k+1}, pairs with itself, so it can be chosen freely.

$\therefore \Rightarrow$ 2^{k+1} ; $2^{\frac{n-1}{2}+1} = 2^{\frac{n-1+2}{2}} = \underline{\underline{2^{\frac{n+1}{2}}}}$.

Q45. $434 + 883 + 43 = 1360$

1 section = 34 students

x = 1360 students

$1360 = 34x$; $\Rightarrow$ 40 sections.

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Q48. Arranging 6 people in a row from a group of 10 people;
Bride + groom } part of 10 people

a) $\boxed{B}$ _ _ _ _ _

⇒ Choose the other 5 people from the remaining 9 people (groom + 8 others): $({}^{9}C_{5}) \times 6! = 126 \times 720$
$= \underline{90720}$

b) BOTH BRIDE and GROOM must be in the picture;
⇒ Choose the other 4 people from the remaining 8 others:
$({}^{8}C_{4}) \times 6! = 70 \times 720 = \underline{50400}$.

c) CASE 1: Bride in picture, groom NOT in picture
⇒ Choose other 5 people from 8 others (not bride, not groom):
$({}^{8}C_{5}) \times 6! = 56 \times 720 = \underline{40320}$.

CASE 2: Groom in picture, bride not in picture
⇒ Choose other 5 people from 8 others (not groom, not bride):
$({}^{8}C_{5}) \times 6! = 56 \times 720 = 40320$.

⇒ $40320 + 40320 = \underline{80640}$.

Q58.
- Uppercase letters (i.e. A, B, C, ...) → 26 options
- Lowercase letters (i.e. a, b, c, ...) → 26 options
- digits (i.e. 0 - 9) → 10 options
- Underscores (_) → 1 option

$\boxed{\_}$ _ _ _ _ _ _ _

↓

$\underbrace{(26+26+1)}_{\text{reserved}}$

MAXIMUM LENGTH = 8; can contain fewer than 8 characters (i.e. ≤ 8)

∴ CASE 1: (8 characters): 53×63^7
CASE 2: (7 characters): 53×63^6
CASE 3: (6 characters): 53×63^5
CASE 4: (5 characters): 53×63^4
CASE 5: (4 characters): 53×63^3
CASE 6: (3 characters): 53×63^2

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CASE 7 : (2 characters) : 53×61

CASE 8 : (1 character) : 53

$\therefore (53) + (53\times61^1) + (53\times61^2) + (53\times61^2) + (53\times61^3) + (53\times61^4)$
$+ (53\times61^5) + (53\times61^6) + (53\times61^7)$

$\Rightarrow 53(1 + 63^1 + 63^2 + 63^3 + 63^4 + 63^5 + 63^6 + 63^7)$

$\Rightarrow \underline{\underline{2.12 \times 10^{14}}}$

Q60. Country code with between 1 and 3 digits ;

$\Rightarrow$ either 1 digit, 2 digits, 3 digits ;

Number part of the telephone with at most 15 digits

$\Rightarrow$ length of number part $\in \{0, 1, 2, \ldots, 15\}$.

$\Rightarrow$ Since both of these parts are independent of each other, we can solve for them seperately ;

COUNTRY CODES :-

CASE I :- 1 digits ;

$\Rightarrow \boxed{} \rightarrow$ CANNOT be a zero (0) ; $\therefore$ 1 – 9 } available options

$\therefore$ 9

CASE II :- 2 digits ;

$\Rightarrow$ _ _ } $9 \times 10 = \underline{90}$

CASE III :- 3 digits ;

$\Rightarrow$ _ _ _ } $9 \times 10 \times 10 = 900$

$\therefore$ Total country codes $= 9 + 90 + 900 = \underline{\underline{999}}$

SUBSCRIBER NUMBERS :-

The subscriber number has at most 15 digits, meaning it can have from 0 to 15 digits.

$\Rightarrow$ Each digit can be 0 – 9, so 10 choices per digit.

$\Rightarrow$ for length K (where K ranges from 0 to 15) : Number of sequences $= 10^k$.

TOTAL NUMBER of POSSIBLE SUBSCRIBERS = sum of 10^k for $k = 0$ to 15.

$\because$ 10 = options

0	$\underline{10}$	$\underline{10}\ \underline{10}$	$\underline{10}\ \underline{10}\ \underline{10}$	
10^0	10^1	10^2	10^3	...
1	10	10 (10)	(10×10)(10)	

A geometric series means that in order to get to the next term, you take the previous term, and multiply by a common ratio.

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Applying the geometric series formula :

$$S_n = \frac{t_n(r^{n+1} - 1)}{r-1} \quad \text{for } r > 1 \; ; \; t_n = a = \text{first term}$$

$$S_{15} = \frac{1\,(10^{16}-1)}{10-1} = \frac{10^{16}-1}{9}$$

$$\therefore\ 999 \times \left(\frac{(10^{16}-1)}{9}\right) = 111\,(10^{16}-1)$$

Q61. Telephone number = country code + 10-digit number

10-digit number :- of the form NXX - NXX - XXXX

$\Rightarrow$ N = digit 2 - 9 (8 choices)

X = digit 0 - 9 (10 choices)

$\therefore\ 8 \times 10 \times 10 \times 8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 8^2 \times 10^8 = 6.4 \times 10^9$

Country codes :-

CASE I :- 1 digit = 10

CASE II :- 2 digits = 10 × 10 = 100

CASE III :- 3 digits = 10 × 10 × 10 = 10^3 = 1000

$\therefore$ 10 + 100 + 1000 = 1110

Since the country code and the 10-digit numbers are independent of each other : $1110 \times (6.4 \times 10^9) = 7.104 \times 10^{12}$

Q63. Hexadecimal digits = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}

$\therefore$ 16 possible characters.

CASE I :- length 10 = 16^{10}

CASE II :- length 26 = 16^{26}

CASE III :- length 58 = 16^{58}

$\therefore\ 16^{10} + 16^{26} + 16^{58} = 16^{10} + 16^{26} + 16^{58}$

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SECTION 6.2

Q4. 10 red balls
10 blue balls

(a) $\left\lceil \frac{N}{2} \right\rceil = 3$; $\frac{N-1}{2} = 3-1$; $N-1 = 4$

$\Rightarrow N = 5$

(b) WORST-CASE argument :- 'To be sure' means that even in the worst luck/scenario, the condition still holds.

⇒ That is why we maximize the delay (worst case) before the condition is forced to be true.

∴ Worst-case = She keeps picking red balls as long as possible, before picking up those three blue balls (after exhausting all red balls);

$\Rightarrow 10 + 3 = 13.$

Q5. Years = 4
Majors = 21
TOTAL COMBINATIONS = $4 \times 21 = 84$

⇒ 2 students expected to graduate in the same year with the same major;

$\Rightarrow \cancel{\left\lceil \frac{n}{2} \right\rceil = 84 ; \; n-1 = 2(84-1) ; \; n-1 = 2(83) = 166 ; \; n =}$

If we have n students and 84 possible (year, major) combinations, then to guarantee at least 2 in one combination;

$\therefore n > 84$

$\Rightarrow n = 84 + 1 = 85.$

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Q6. Two (2) students with the SAME professor who earned the SAME final examination score ;

Applying the Pigeonhole principle ;

⇒ Each (professor, score) pair is a pigeonhole.

⇒ $6 \times 101 = 606$ = TOTAL pigeonholes

To guarantee at least 2 students in the same (professor, score) pair ; $n > 606$

$\therefore \quad n = 606 + 1 = 607$ students.

Q20. TOTAL STUDENTS = 9 ;

↳ There are two (2) groups : Male (M), Female (F).

(a) PROOF BY CONTRADICTION ;

⇒ Suppose the opposite case : fewer than 5 male AND fewer than 5 females.

$\therefore$ fewer than 5 males = 4 males

fewer than 5 females = 4 females

$\therefore$ Maximum total possible = $4 + 4 = 8$ students.

Since we have a total of 9 students, therefore the above statement is a contradiction, and hence, the original statement is true. Q.E.D.

(b) PROOF BY CONTRADICTION ;

⇒ Suppose the opposite case : fewer than 3 males AND fewer than 7 females.

$\therefore$ fewer than 3 males = 2 males

Fewer than 7 females = 6 females

$\therefore$ Maximum total possible = $2 + 6 = 8$ students.

Since we have a total of 9 students, therefore, the above statement is a contradiction, and hence, the original statement is true. Q.E.D

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Q21. TOTAL students = 25

↳ F (freshmen), S (sophomore), J (junior).

(a) PROOF BY CONTRADICTION ;

Suppose the opposite case : fewer than 9 F AND fewer than 9 S AND fewer than 9 J.

fewer than 9 F = 8 freshmen

Fewer than 9 S = 8 sophomores

fewer than 9 J = 8 juniors

∴ Maximum total possible = 8 + 8 + 8 = 24 students.

Since we have a total of 25 students, therefore, the above statement is a contradiction, and hence, the original statement is true. Q.E.D.

(b) PROOF BY CONTRADICTION ;

⇒ Suppose the opposite case : fewer than 3 F AND fewer than 19 S AND fewer than 5 J.

∴ Fewer than 3 F = 2 freshmen

fewer than 19 S = 18 sophomores

fewer than 5 J = 4 juniors

∴ Maximum total possible = 2 + 18 + 4 = 24 students.

Since we have a total of 25 students, therefore, the above statement is a contradiction, and hence the original statement is true. Q.E.D.

Q34. Applying the Pigeonhole principle ;

Pigeons = wage earners = 100,000,000

Holes = possible distinct earnings amounts (in pennies)

[CONTINUED --→]

~~1 / penny~~ ?

1 dollar = 100 pennies

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at least 1 penny to less than 1,000,000 dollars;

$\therefore$ $(1{,}000{,}000 \times 100) - 1 = 99{,}999{,}999$ pennies

Since there are 100,000,000 pigeons (earners) and 99,999,999 holes (possible earnings in pennies);

$\Rightarrow$ $100{,}000{,}000 > 99{,}999{,}999$;

by the pigeonhole principle, at least two earners must have exactly the same earnings to the penny.

Q37. Here, $n = 677$ classes (= pigeons) and $K = 38$ different time slots (= pigeonholes). By the generalized pigeonhole principle, there is a pigeonhole (time slot) with:

$$\left\lceil \frac{n}{K} \right\rceil = \left\lceil \frac{677}{38} \right\rceil = \left\lceil 17\frac{31}{38} \right\rceil = 18$$

$\therefore$ At least 18 classrooms are needed.