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DATE 10/18/2025 Discrete Mathematics

Assignment # 03

SECTION 3.2

Q19. 2^{n+1} is $O(2^n)$
2^{2n} is $O(2^n)$

$\Rightarrow$ $f(n) = O(g(n))$ means there exist constants $C>0$ and n_0 such that :

$$0 \leq f(n) \leq C\, g(n) \text{ for all } n \geq n_0$$

Here $g(n) = 2^n$

$2^{n+1} = 2(2^n)$

$\therefore$ $2^{n+1} \leq C(2^n)$

$\Rightarrow$ $2^n(2) \leq C(2^n)$

$$\frac{2^n(2)}{2^n} \leq \frac{C(2^n)}{2^n}$$

$\Rightarrow$ $2 \leq C$

$\Rightarrow$ $C \geq 2$

If we take $C=2$, then the inequality holds true.

$\therefore$ $2^{n+1} = O(2^n)$ with $C=2$, $n_0 = 0$.

$2^{2n} = (2^n)^2$

$\Rightarrow$ $(2^n)^{\cancel{2}} \leq C(\cancel{2^n})$

$\Rightarrow$ $2^n \leq C(1)$

$\Rightarrow$ $2^n \leq C$

$\Rightarrow$ 2^n grows without bound as $n \to \infty$, so no constant C can satisfy this for all large n.

$\therefore$ 2^{2n} IS NOT $O(2^n)$.

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Q23.

$f_1(n) = n\log n$

$f_2(n) = n^{3/2}$

Applying L'Hopitals' rule ;

$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{n\log n}{n^{3/2}} = \frac{n^{1-3/2}\log n}{1}$

$\Rightarrow \lim_{n\to\infty} \frac{n^{-1/2}\log n}{1} = \lim_{n\to\infty} \frac{\log n}{n^{1/2}}$

$\Rightarrow \lim_{n\to\infty} \frac{(\log n)'}{(n^{1/2})'} = \frac{1/n}{1/2 n^{-1/2}} = \left(\frac{1}{n}\right) \times \frac{2}{n^{1/2}}$

$\Rightarrow \lim_{n\to\infty} \frac{2}{n^{3/2}} = 2 \lim_{n\to\infty} \frac{1}{n^{3/2}}$

$\Rightarrow 2(0) = 0$

Since the ratio goes to 0, $n\log n$ ($f(x)$) grows slower than $n^{3/2}$, so Algorithm 1 uses FEWER operations for large n.

$\therefore$ Algorithm 1.

Q25

(a) $(n^2 + 8)(n+1)$

$\Rightarrow n^3$

The dominant term is n^3, so :

$O(n^3)$.

(b) $(n\log n + n^2)(n^3+2)$

$\Rightarrow n^2(n^3) = n^5$

$O(n^5)$.

(c) $(n! + 2^n)(n^3 + \log(n^2+1))$

$\Rightarrow n!(n^3)$.

$O(n!(n^3))$.

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Q30. One common method to determine whether two functions are of the same order :

Evaluate the limit of the ratio of the two functions as x approaches infinity :

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

If L is a positive constant (i.e. $0 < L < \infty$), then $f(x)$ and $g(x)$ are of the same order.

(a) $$\lim_{x\to\infty} \frac{3x+7}{x} = \frac{3\cancel{x}}{\cancel{x}} = 3$$

Since the limit is a positive constant (3), these pairs of functions are of the same order.

(b) $$\lim_{x\to\infty} \frac{2x^2+x-7}{x^2} = \frac{2\cancel{x^2}}{\cancel{x^2}} = 2$$

Since the limit is a positive constant (2), these pairs of functions are of the same order.

(c) $$\lim_{x\to\infty} \frac{\lfloor x+1/2 \rfloor}{x}$$

$\Rightarrow$ For any real number y,

$$y-1 < \lfloor y \rfloor \leq y$$

$$\therefore (x+1/2)-1 < \lfloor x+1/2 \rfloor \leq (x+1/2)$$

$$\Rightarrow x-1/2 < \lfloor x+1/2 \rfloor \leq x+1/2$$

$$\Rightarrow \frac{x-1/2}{x} < \frac{\lfloor x+1/2 \rfloor}{x} \leq \frac{x+1/2}{x}$$

$$\Rightarrow 1-1/2x < \frac{\lfloor x+1/2 \rfloor}{x} \leq 1+1/2x$$

Applying squeeze theorem ;

As $x\to\infty$: $1-\frac{1}{2x} \to 1$; $1+\frac{1}{2x} \to 1$

$$\therefore \lim_{x\to\infty} \frac{\lfloor x+1/2 \rfloor}{x} = 1$$

Since the limit is a positive constant (1), these pairs of functions are of the same order.

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(d) $\lim_{x\to\infty} \frac{\log(x^2+1)}{\log_2 x}$; $\log_2(x) = \frac{\ln x}{\ln 2}$

$\therefore \frac{\log(x^2+1)}{\log_2(x)} = \frac{\ln(x^2+1)}{\frac{\ln x}{\ln 2}} = \ln 2 \cdot \frac{\ln(x^2+1)}{\ln x}$

$\Rightarrow \ln(x^2+1) = \ln\left[x^2\left(1+\frac{1}{x^2}\right)\right] = \ln(x^2) + \ln\left(1+\frac{1}{x^2}\right)$

$\Rightarrow 2\ln x + \ln\left(1+\frac{1}{x^2}\right)$

$\therefore \frac{2\ln x + \ln(1+1/x^2)}{\ln x} = 2 + \frac{\ln(1+1/x^2)}{\ln x}$

$\Rightarrow \lim_{x\to\infty} 2 + \frac{\ln(1+1/x^2)}{\ln x} = \frac{1}{x^3} \div x = \frac{1}{x^3} \times \frac{1}{x} = \frac{1}{x^4}$

$= 0$

$\therefore 2+0 = 2$

$\Rightarrow \ln 2\,(2) = 2\ln 2$

Since the limit is a finite positive constant $(2\ln 2)$, these pairs of functions are of the same order.

(e) $\lim_{x\to\infty} \frac{\log_{10} x}{\log_2 x}$; $\log_{10}(x) = \frac{\ln x}{\ln 10}$; $\log_2(x) = \frac{\ln x}{\ln 2}$

$\Rightarrow \lim_{x\to\infty} \frac{\ln 2}{\ln 10} = \frac{\ln 2}{\ln 10}$

Since the limit is a positive constant $(\ln 2/\ln 10)$, these pairs of functions are of the same order.

Q33. Suppose $f(x) = \Theta(g(x))$

By definition, this means :

$\Rightarrow f(x) = O(g(x))$

$\Rightarrow f(x) = \Omega(g(x))$.

From $f(x) = O(g(x))$:

$\Rightarrow$ There exist $C_2 > 0$ and k_2 such that $|f(x)| \leq C_2|g(x)|$ for $x > k_2$.

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From $f(x) = \Omega(g(x))$:

$\Rightarrow$ There exist $C_1' > 0$ and k_1 such that $|f(x)| \geq C_1'|g(x)|$ for $x > k_1$.

Let $k = \max(k_1, k_2)$ and $C_1 = C_1'$, C_2 as above.

Then for $x > k$, both inequalities hold:

$$C_1 |g(x)| \leq |f(x)| \leq C_2 |g(x)|$$

- Suppose there exist $k, C_1, C_2 > 0$ such that for $x > k$:

$$C_1 |g(x)| \leq |f(x)| \leq C_2 |g(x)|$$

The right inequality $|f(x)| \leq C_2 |g(x)|$ means $f(x) = O(g(x))$.

The left inequality $|f(x)| \geq C_1 |g(x)|$ means $f(x) = \Omega(g(x))$.

Together, $f(x) = \Theta(g(x))$.

Q34.

(a) $3x^2 + x + 1 = \Theta(3x^2)$

When $x \geq 1$, $3x^2 + x + 1 \leq 3x^2 + x^2 + x^2 \leq 5x^2$

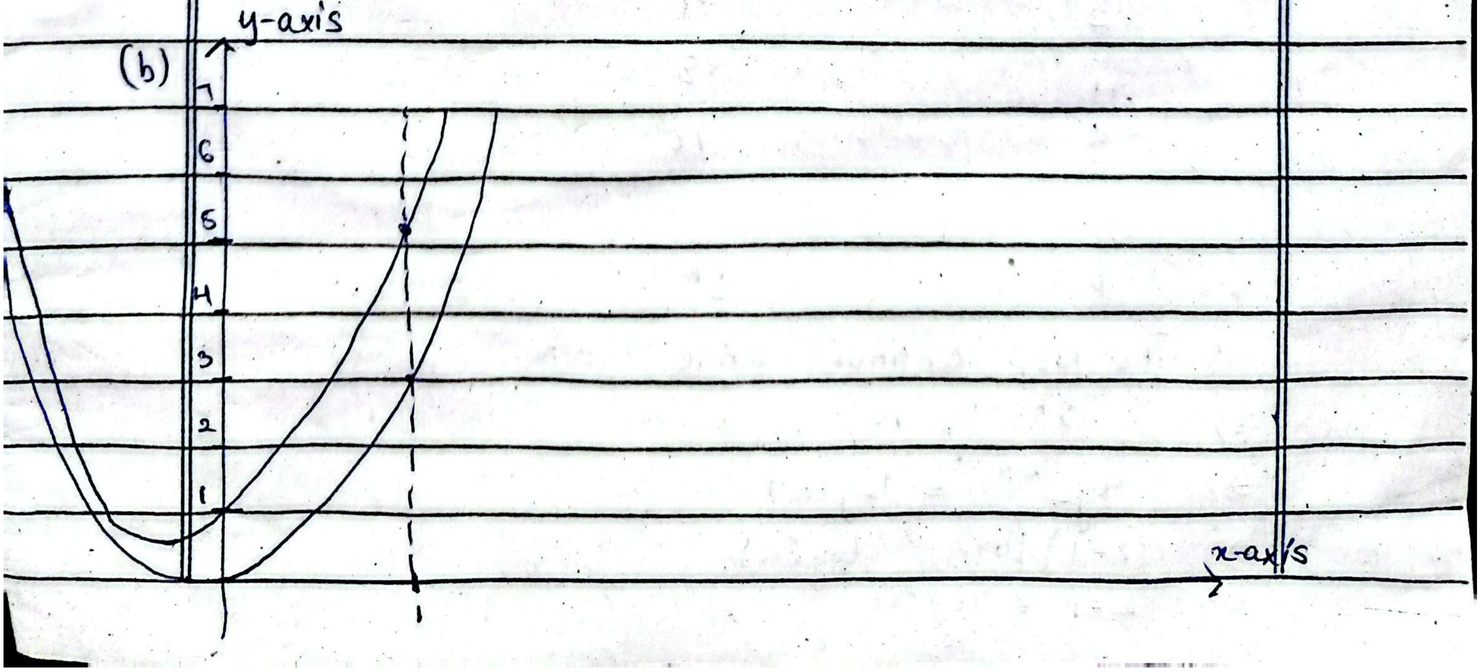
When $x \geq 0$, $3x^2 \leq 3x^2 + x + 1$

Combining these, we can see that when $x \geq 1$,

$\Rightarrow$ $3x^2 \leq 3x^2 + x + 1 \leq 5x$

$\therefore$ $k = 1$ $C_1 = 1$ $C_2 = 5$

(b)



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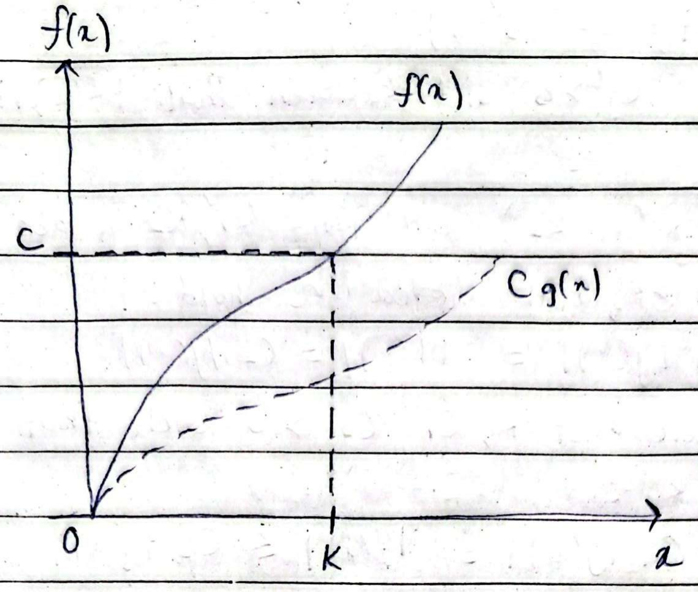
Q48.



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Q4.

```
i := 1
t := 0
while i ≤ n
    t := t+i
    i := 2i
```

⇒ 2 operations per iteration

Iteration	i value
1	1
2	2
3	4
4	8
5	16
...	
K	2^{K-1}

∴ The loop continues while $i \le n$.

⇒ $2^{K-1} \le n$

⇒ $\log 2^{K-1} \le \log(n)$

⇒ $(K-1)\log 2 \le \log(n)$

⇒ $K-1 \le \frac{\log(n)}{\log 2} \approx K-1 \le \log(n)$

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$\Rightarrow \quad k \leq \log_2(n) + 1$

$\therefore$ Number of iterations $= O(\log(n))$.

$\Rightarrow$ Each iteration $= 2$ operations

Number of iterations $= O(\log(n))$

$\Rightarrow \quad O(2 \times \log(n)) = \underline{O(\log n)}$

Q5. Algorithm Analysis :

```
int SmallestNum (int arr[], int size) {
    int tempSmallest = arr[0] ;
    for (int i=1 ; i< size ; i++) {
        if (tempSmallest > arr[i]) {
            tempSmallest = arr[i] ;
        }
    }
    return tempSmallest ;
}.
```

$\Rightarrow$ Loop : $(n-1)$ iterations

In each iteration, 1 comparison is made.

$\therefore$ Element comparisons : $(n-1)$ comparisons

$\Rightarrow$ Best case : Array in descending order ($(n-1)$ comparisons)

Worst case : Array in ascending order ($(n-1)$ comparisons)

Average case : $(n-1)$ comparisons

$\therefore \quad \underline{n-1}$ comparisons

Q7. List size $= n = 32$ elements

$\Rightarrow$ Target element is among the first 4 elements.

LINEAR SEARCH :

$\Rightarrow$ 4 comparisons.

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For a list of $n = 32$ elements, binary search takes exactly $\lfloor \log_2 n \rfloor + 1$ comparisons in the worst case.

$\therefore \quad \log_2 32 = 5$

$\Rightarrow \quad \lfloor 5 \rfloor + 1 = \underline{6}$ comparisons in the worst case.

$\therefore$ Linear search will locate the element more rapidly because it requires at most 4 comparisons, while binary search requires 6 comparisons in the worst case due to its $O(\log n)$ divide-and-conquer approach on the entire array.

Q19. $N = 2^{50}$; $2^{50} \approx 1.13 \times 10^{15}$ operations

$T = N \times t_{op}$

a) $T = N \times t_{op}$

$T = (1.13 \times 10^{15}) \times (10^{-6})\ s$

$T = 1.13 \times 10^{9}\ s$

$\therefore \quad T = \dfrac{1.13 \times 10^{9}}{3.156 \times 10^{7}} = \underline{35.8}$ years

b) $T = (1.13 \times 10^{15}) \times (10^{-9})\ s$

$T = 1.13 \times 10^{6}\ s$

$T = \dfrac{1.13 \times 10^{6}}{3.156 \times 10^{7}} = \underline{13.1}$ days

c) $T = (1.13 \times 10^{15}) \times (10^{-12})\ s$

$T = 1.13 \times 10^{3}\ s$

$T = \underline{18.8}$ mins.

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Q22. Algorithm Analysis :

```
max = a[1]
for i=2 to n :
    if a[i] > max :
        max = a[i]
```

⇒ Number of comparisons = $n-1$ (always same in best, worst, average case).

b) For linear search, the algorithm that is being applied is that we need to check each element from ~~last~~ first to last until target is found.

BEST CASE : Target is the first element.

∴ 1 comparison

c) BEST CASE : The middle element of the entire list is the target on the first check.

∴ 1 comparisons.

Q23. If there are 'n' number of elements in the list ;

⇒ If exactly half the time the element x is not in the lists;

BEST CASE : $n/2 + 1 = n+2/2$

WORST CASE : n.

AVERAGE-CASE : $\frac{n+2}{2} + n = \frac{n+2+2n}{2} = \frac{3n+2}{2} \times \frac{1}{2}$

⇒ $\frac{3n+2}{4}$ = average-case performance.

⇒ If x is in the list it is equally likely to be in any position ;

BEST CASE : 1 (at the first position)

WORST CASE : n (is at the last, requiring n comparisons)

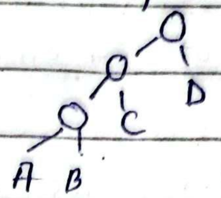
AVERAGE-CASE : $\frac{n+1}{2}$ = average-case performance.

Q49. Matrix Chain Multiplication

$$A \cdot B \cdot C \cdot D$$
$$30\times10 \quad 10\times40 \quad 40\times50 \quad 50\times30$$

⇒ There are multiple possibilities to apply parenthesis to obtain different multiplication outputs.

e.g. $((A\cdot B)\cdot C)\cdot D$ ⇒



} 3 nodes

In every case, we observe that there are three (3) nodes.

↳ Generalization of a formula : $T(n) = \dfrac{{}^{2n}C_n}{n+1} = \dfrac{{}^{2(3)}C_3}{3+1} = \dfrac{{}^{6}C_3}{4}$

$$= \frac{20}{4} = 5$$

∴ There are 5 possible combinations of matrix multiplications.

$m[1,1] = A$ = not being multiplied with anything $= 0$

$m[1,2] = A\cdot B = 30\times10\times40 = 12000$
$30\times\boxed{10 \; 10}\times40$

$m[2,3] = B\cdot C = 10\times40\times50 = 20000$
$10\times\boxed{40 \; 40}\times50$

$m[3,4] = C\cdot D = 40\times50\times30 = 60000$
$40\times\boxed{50 \; 50}\times30$

m	1	2	3	4
1	0	12000	35000	44000
2		0	20000	35000
3			0	60000
4				0

s	1	2	3	4
1		1	1	1
2			2	3
3				3
4				

$m[1,3] = A\cdot B\cdot C$

⇒ (1) $(A\cdot B)\cdot C = m[1,2] + m[3,3] + (30\times40\times50)$
$30\times10 \;\; 10\times40 \;\; 40\times50$

⇒ (2) $A\cdot(B\cdot C)$

$= m[1,1] + m[2,3] + (30\times10\times50)$

⇒ (1) $12000 + 0 + 60000 = 72000$

⇒ (2) $0 + 20000 + 15000 = 35000$

$m[2,4] = B\cdot C\cdot D$

⇒ (1) $(B\cdot C)\cdot D = m[2,3] + m[4,4] + (10\times50\times30) = 35000$
$10\times40 \;\; 40\times\boxed{50 \;\; 50}\times30$

$\Rightarrow$ (12) $B \cdot (C \cdot D)$ $= m[2,2] + m[3,4] + (10 \times 40 \times 30)$

$10 \times \boxed{40 \quad 40} \times 50 \quad 50 \times 30$

$= 0 + 60000 + 12000$

$= 72000$

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$\Rightarrow m[1,4] = \min \left\{ \begin{array}{l} m[1,1] + m[2,4] + (30 \times 10 \times 30), \\ m[1,2] + m[3,4] + (30 \times 40 \times 30), \\ m[1,3] + m[4,4] + (30 \times 50 \times 30) \end{array} \right\}$

$\Rightarrow m[1,4] = \min \left\{ \begin{array}{l} 0 + 35000 + 9000, \ 12000 + 60000 + 36000, \\ 35000 + 0 + 45000 \end{array} \right\}$

$\Rightarrow m[1,4] = \min \{ 44000, 108000, 80000 \}$

$\Rightarrow m[1,4] = 44000$

$m[i,j] = \min \{ m[i,k] + m[k+1,j] + (d_{i-1} \times d_k \times d_j) \}$

$(A) \cdot ((B \cdot C) \cdot D)$

$30 \times 10 \quad 10 \times 40 \quad 40 \times 50 \quad 50 \times 30$

$\Rightarrow \cancel{m[1,1] + m[2,3] + m[4,4] + (30 \times 10 \times 50 \times 30)}$

$\Rightarrow \cancel{0 + 20000 + 0 + (}$

$\Rightarrow (10 \times 40 \times 50) = BC = 20000$

$(10 \times 50 \times 30) = (BC)D = 15000$

$(30 \times 10 \times 30) = A((BC)D) = 9000$

$\Rightarrow 9000 + 15000 + 20000 = \underline{44000}$