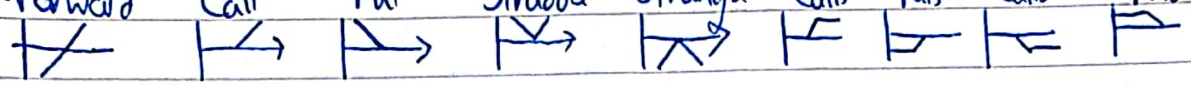
- Time Value of Money:
  - $FV_n = PV e^{rn}$ $\quad$ $FV_n = PV\left(1+\frac{r}{m}\right)^{nm}$ (m payments a year)
- Derivatives:
  - Forward, Call, Put, Straddle, Strangle, Spread: Bull Calls, Bull Puts, Bear Calls, Bear Puts



- Forward Pricing:
  - $F_t = S_t e^{r(T-t)}$ (no-arbitrage argument)
  - $f_t = (F_t - X)e^{-r(T-t)} = (F_t - F_0)e^{-r(T-t)}$
  - (under ~~valued~~: Short share, long forward / over priced: reverse)
- Binomial Model:
  - Risk free & Replicating portfolios (derivative & stock; bond & stock)
  - Risk neutral approach, $\hat{p} = \frac{e^{rT}-d}{u-d}$, $f_0 = \hat{E}(f_T)e^{-rT}$
- Brownian Motion:
  - 3 properties
  - Generalised & Geometric
- Stochastic Calculus:
  - In formula sheet (Ito's Lemma)
- Black Scholes:
  - 7 assumptions (3 no's + 4 other)
  - Self-financing portfolio, BSM PDE
  - Pricing options (in formula sheet), Put-Call parity ($P+S=C+Xe^{-rT}$)
- Hedging & The Greeks
  - $\Delta, \Gamma, \Theta, \nu, \rho$ $\quad$ $(S, S^2, t, \sigma, r)$
  - $\Delta$ & $\Gamma$ neutral
- Volatility:
  - historic data
  - implied vol, volatility smiles
- Risk Neutral Pricing:
  - Basic method
  - $\hat{E}(f_T)$,
  - $f_0 = \hat{E}(f_T)e^{-rT}$

# STATG017 - Stochastic Methods in Finance

## 2: Time Value of Money: Money now is worth more than money later

Future value $\rightarrow FV_n = PV(1+r)^n$ discrete compounding

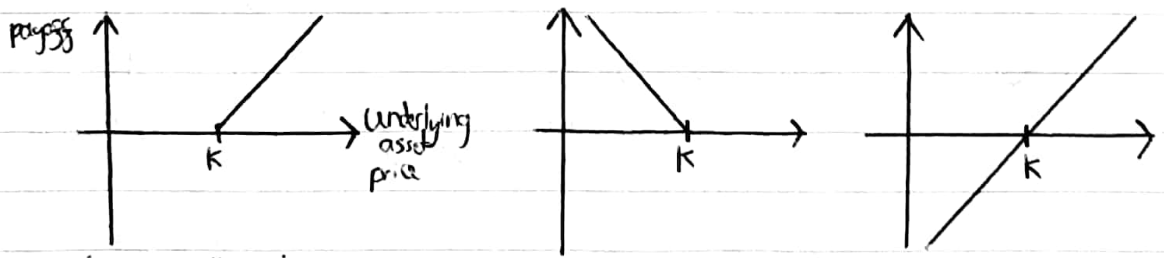
$FV_n = PV\, e^{rn}$ continuous compounding

$r$: risk free interest rate

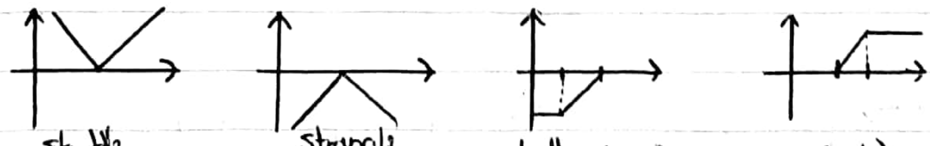
## 3: Introduction to Derivatives

- Forward: agree to buy (long) or sell (short) an asset at a set time in the future for a set price
- Future: similar to forward, but traded on an exchange & well specified, with value calculated daily
- Option: option to buy (call) or sell (put) an asset at a set time. ~~European~~ American can be exercised at any time, European only at expiry.

- Payoff Charts:



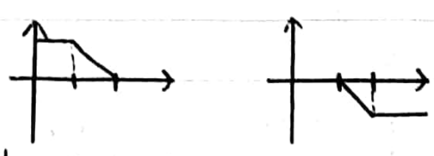

long call option | long put option | long forward



straddle (long put + call) | strangle (short put + call) | bull spread (puts) | (calls)

Spreads:
- short & long options of same type w/ diff strike prices

(short = reflect across $x$ axis)



bear spread (puts) | (calls)

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Forward Arbitrage:

If $F_t < S_t e^{-r(T-t)}$: $t=t$: short share, invest at $r$, enter forward long at $F_t$

$t=T$: buy share, get cash, profit

## 4: Arbitrage & Forward Pricing

- Arbitrage: opportunity to execute a series of trades guaranteed to make a profit with 0 risk
  - in pricing, we assume no arbitrage opportunities exist

forward price $\rightarrow F_t = S_t e^{r(T-t)}$ or $F_0 = (S_0 - I)e^{rT}$, $I = \sum_i c_i e^{-rt_i}$ ] discrete dividends

$F_t = S e^{(r-q)t}$ ] continuous dividends

forward value $\rightarrow f_t = (F_t - F_0)e^{-r(T-t)}$ $(f_0 = 0)$

$f_t = (F_t - X)e^{-r(T-t)}$

## 5,6: Pricing Options under the Binomial Model

$S, f_0 \rightarrow uS, f_u$ / $dS, f_d$

- 3 approaches:
  - Risk-free portfolio: 1 derivative short, $\Delta$ stocks long

$$\Pi_T = \begin{bmatrix} \Delta uS - f_u \\ \Delta dS - f_d \end{bmatrix} \Rightarrow \Delta = \frac{f_u - f_d}{uS - dS}$$ (as 2 payoffs are equal as risk free)

$$\Pi_0 = f_0 = \Pi_T e^{-rT}$$

Put-Call Parity:

$C - P = S - Xe^{-rT}$

for call & put w/ same underlying, strike & expiry

  - Replicating portfolio: $x$ cash in bond, $y$ stocks long

$$\Pi_T = \begin{bmatrix} yuS + xe^{rT} = f_u \\ ydS + xe^{rT} = f_d \end{bmatrix} \Rightarrow x = \left(\frac{uf_d - df_u}{u-d}\right)e^{-rT},\ y = \frac{f_u - f_d}{uS - dS}$$

$$\Pi_0 = f_0 = x + yS$$

  - Risk-neutral approach:

risk neutral probability $\longrightarrow \hat{p} = \frac{e^{rT} - d}{u - d}$, $f_0 = (\hat{p} f_u + (1-\hat{p}) f_d)e^{-rT}$

In 2 step model:

$f_0 = (\hat{p}^2 f_{uu} + \underline{2}\hat{p}(1-\hat{p})f_{ud} + (1-\hat{p})^2 f_{dd})e^{-rT}$

- Multi-step tree: work backwards from leaves

$S \rightarrow uS \rightarrow u^2S,\ udS$; $S \rightarrow dS \rightarrow udS,\ d^2S$

  - for American options, also take into account possibility of executing option at each node (ie take max of value if executed & value if not)

[never optimal to exercise American call early on non-dividend paying stock]

# 8: Brownian Motion (aka Wiener Process)

- continuous time, Markov stochastic process
- $Z_t$ is Brownian motion if:
  - ~~$Z_{t_1} - Z_{t_2}$ is independent of~~ non-overlapping increments of process are independent (ie $Z_{t_2} - Z_{t_1}$ independent of $Z_{S_2} - Z_{S_1}$, where $t_2 \geq t_1 \geq S_2 \geq S_1$)
  - for $0 \leq s \leq t$, $Z_t - Z_s \sim N(0, t-s)$
  - $Z_t \in \mathbb{R}$, $Z_0 = 0$
- Generalized Brownian Motion: $dS = \underset{\text{drift rate}}{\mu} dt + \underset{\text{variance}}{\sigma} \underset{\text{standard Brownian motion}}{dz}$

~~$S_t \sim N($~~

$$S_t - S_s \sim N((t-s)\mu, (t-s)\sigma^2), \quad 0 \leq s \leq t$$

- Geometric Brownian Motion: $dS = \underset{\text{mean rate of return}}{\mu} S dt + \underset{\text{volatility}}{\sigma} S dz$

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z\right]$$

$$S_t = S_0 e^{\mu t} \quad \text{when } \sigma = 0$$

- Can use approximation $\frac{\Delta S}{S} = \mu \Delta t + \sigma \underset{\varepsilon \sim N(0,1)}{\varepsilon} \Delta t$

ie $S_t \sim N(S_0 + \mu t, (S_0 \sigma \sqrt{t})^2)$

$$\ln S_t - \ln S_0 \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

9: Stochastic Calculus

$$dx_t = a(x_t, t)dt + b(x_t, t)dz_t \leftarrow$$ z is Brownian motion, not smooth, unbounded & non-differentiable. Need to use stochastic calculus

need some boundary condition → if process starts at $x_0$,

$$x_t = x_0 + \int_0^t a(x_s, s)ds + \int_0^t b(x_s, s)dz_s$$

• Ito's lemma: given $dx = a(x,t)dt + b(x,t)dz$ ← Ito process, z is Wiener process

and $G = G(x,t)$,

then:

$$dG = \left[\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right]dt + \frac{\partial G}{\partial x}b(x,t)dz$$

from Taylor expansion, dropping terms of order greater than dt (note $(dx)^2$ same order as dt)

ie, G follows Ito process ~~with~~ w/ $a(x_t, s)$, & $b(x_t, s)$ as above

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# 10: Black-Scholes Model

- Continuous time, continuous variable version of binomial tree
- Assume:
  1: Stock price process, $S_t$, follows geometric Brownian motion
  2: Can long or short stock (any amount)
  3: No transaction costs
  4: No dividends
  5: No arbitrage
  6: Trading continuous in time
  7: Risk free interest rate constant & same for all maturities

- By (1), $dS = \mu S dt + \sigma S dz$
  By (7), $dB = rBdt \leftarrow$ riskless 0 coupon bond $B_0 = 1$
- Goal: create replicating, self financing portfolio
  - $\phi_t$ of stock, $\psi_t$ of riskless asset

$$\Pi_t = \phi_t S_t + \psi_t B_t, \quad \Pi_T = f(S_T, T) = \text{payoff}(S_T)$$

$$dV_t = \phi_t dS(t) + \psi_t dB(t)$$ ← as self financing; changes in portfolio value driven by changes to stock & bond prices only

$$\Rightarrow \Pi_t = \int_0^t \phi_s dS_s + \int_0^t \psi_s dB_s$$

① $$\Rightarrow d\Pi_t = \phi_t dS + \psi_t dB = (\phi_t \mu S + \psi_t r B_t)dt + \phi_t \sigma S dz$$

② by Ito's lemma, $$d\Pi = \left[\frac{\partial \Pi}{\partial S}\mu S + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2\right]dt + \frac{\partial \Pi}{\partial S}\sigma dz$$

combine ① & ② $$\Rightarrow \phi_t \sigma S = \frac{\partial \Pi}{\partial S}\sigma S \Rightarrow \phi_t = \frac{\partial \Pi}{\partial S}$$

and

$$(\phi_t \mu S + r\psi_t B_t) = \frac{\partial \Pi}{\partial S}\mu S + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2$$

Eliminate $\psi_t$, substitute $\phi_t$,

Black-Scholes-Merton PDE → $$\boxed{r\Pi_t = r\frac{\partial \Pi}{\partial S}S_t + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2}$$

- To solve this PDE, need boundary conditions
  - $f_T = \max(S_T - X, 0)$ for European call
  - $f_T = \max(X - S_T, 0)$ " " put

- Pricing options:
  - European ~~call~~ options:

$$C = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$P = -S_0 N(-d_1) + Xe^{-rT} N(-d_2)$$

$$d_1 = \frac{\log \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{\log \frac{S_0}{X} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

(can verify $C - P = S_0 - Xe^{-rT}$, Put-Call parity)

## 11: Hedging & the Greeks

• Delta: $\Delta = \frac{\partial f}{\partial S}$

• Sensitivity of portfolio value ~~to~~ w/ respect to underlying
• given portfolio P, to make $\Delta$-neutral ($\Delta = 0$), add $x \times f$ (derivative), s.t.

$$\frac{\partial P}{\partial S} + x \cdot \frac{\partial f}{\partial S} = 0$$ (products

(need to constantly rebalance)

[0→1] → $\Delta_{call}$ +ve, $\Delta_{put}$ -ve [0→-1]

$\left( \begin{aligned} \Delta_c &= N(d_1) \\ \Delta_p &= N(d_1) - 1 \end{aligned} \right)$

• Gamma: $\Gamma = \frac{\partial^2 f}{\partial S^2}$

• balancing cannot be done continuously, but must be done at set time intervals
• small time intervals better but suffer from transaction costs
• $\Gamma$ can be used to see how quickly portfolio will become unbalanced; ie rate of change of $\Delta$ wrt S

⇒ aim to stay $\Gamma$ neutral too, s.t. $\Delta$ ~~change~~ less sensitive to S

$\Gamma_c = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T}} = \frac{Ke^{-rT}N'(d_2)}{S^2\sigma\sqrt{T}}$

$\Gamma_p = \Gamma_c$

• protect against large movements in S between $\Delta$ rebalances

• So now need to add to portfolio

$$V(S,t) = P(S,t) + x_F F + x_G G$$

$$\Rightarrow \Delta_V = 0 = \Delta_P + x_F \Delta_F + x_G \Delta_G$$
$$\Gamma_V = 0 = \Gamma_P + x_F \Gamma_F + x_G \Gamma_G$$

(can be solved if know all $\Delta$s & $\Gamma$s)
(one of F or G could be S; $\Delta_S = 1$, $\Gamma_S = 0$)

- Theta $\Theta = \frac{\partial f}{\partial t}$
- Vega $\mathcal{V} = \frac{\partial f}{\partial \sigma}$
- Rho $\rho = \frac{\partial f}{\partial r}$

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# 12: Volatility

- main challenge in pricing w/ Black-Scholes is finding correct volatility to use

- Method 1: Estimate $\sigma$ from historic data
  - if $S$ follows Geometric Brownian motion, $\log S$ follows Brownian motion

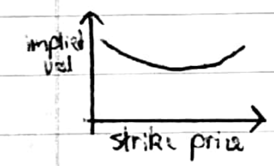
    $$\Rightarrow u_i = (\log S_i - \log S_{i-1}) = \log \frac{S_i}{S_{i-1}} \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right)$$

    $s^2 = \frac{1}{N-1}\sum(x_i - \bar{x})^2$

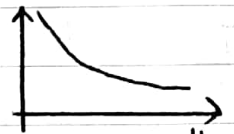
    $\tau$ = time in years between $S_i$ & $S_{i+1}$

  - So can find sample variance of $u_i = s^2$, then $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

- Method 2: Implied Volatility
  - work backwards from Market prices
  - cannot analytically solve; numerically solve easy enough
  - Volatility smiles:




    - should get straight line if Black-Scholes assumptions hold



'smirk' usually seen for call options

    - likely as changes in asset price don't follow log normal distribution, but have fat tails
    - fat tails lead to higher strike price calls / lower strike puts being more valuable, as large movements occur more often

      $\Rightarrow$ Black-Scholes gives lower implied vol

# 13: Risk Neutral Pricing

- ~~OA~~ continuous-time version of risk-neutral probabilities in binomial model
- Risk neutral process:

  in 'real' world, $dS = \mu S dt + \sigma S dz$, $E(S_t) = S_0 e^{\mu t}$

  in risk free world, $dS = r S dt + \sigma S dz$ $\hat{E}(S_t) = S_0 e^{rt}$
- Risk neutral pricing:

$$f_t = f(S,t) = e^{-r(T-t)} \cdot \hat{E}(f_T | S_t = S_t)$$

ie value is expected value at maturity discounted back

$$f_0 = e^{-rT} \hat{E}(f_T | S_0)$$

- eg, European Call

$$f_T = \max(S_T - X, 0)$$

$$dS = r S dt + \sigma S dz \Rightarrow \log S_T \sim N(\log S_0 + (r - \tfrac{\sigma^2}{2})T, \sigma^2 T)$$

$$\Rightarrow \log S_T = \log S_0 + (r - \tfrac{\sigma^2}{2})T + \sigma\sqrt{T}\, u, \quad u \sim N(0,1)$$

$$\Rightarrow f_T = \begin{cases} S_0 e^{(r-\frac{\sigma^2}{2})T} e^{\sigma\sqrt{T}u} - X & \text{if } \log S_T \geq \log X \\ 0 & \text{o/w} \end{cases}$$

$$\log S_T \geq \log X \Rightarrow u \geq \frac{\log \frac{X}{S_0} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_2$$

$$\Rightarrow \cancel{\hat{E}(f_T) = \int_{-d_2}^{\infty} f_T \times \phi(u)\, du}, \quad \phi(u) \text{ is pdf of standard normal}$$

$$\hat{E}(f_T) = \int_{-d_2}^{\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T} e^{\sigma\sqrt{T}u} - X) \times \phi(u)\, du$$

$$= S_0 e^{rT} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u - \sigma\sqrt{T})^2} du - XN(d_2)$$

let $v = u + \sigma\sqrt{T}$

$$\Rightarrow \hat{E}(f_T) = S_0 e^{rT} N(d_1) - XN(d_2), \quad f_0 = e^{-rT} \hat{E}(f_T)$$