	General BM	Geo BM
$Var(X_T \mid X_t)$	$\sigma^2(T-t)$	$X_t^2 e^{2\mu(T-t)}(e^{\sigma^2(T-t)}-1)$
$E(X_T \mid X_t)$	$X_t+\mu(T-t)$	$X_t e^{\mu(T-t)}$

$E[X^2] = Var[X] + (E[X])^2$

$\cancel{E[X^4] = Var[X]^3}$ (maybe)

VaR pros/cons:

- $P(\Pi_{t+\Delta t} - \Pi_t \leq -V_{1-\alpha}) = \alpha$

+ Simple, interpretable,
+ compare portfolios, set/monitor risk
+ determine capital bank must hold
- what when exceeded (ES)
- worse possible loss
- exponential cost w/ portfolio size (correlations)
- not additive
- dif results from dif methods
- no sensitivity to individual underlyings (Greeks)

Merton Model Pros/Cons:

+ Simple; treat equity/debt as functions of options; can use BS pricing
- only default at T, single bond for debt too simple, debt often in untraded bank loans, total asset value usually not directly observable (stock not full equity, bank loans) non-stochastic interest rates

- default at T
- non-stochastic r
- unobservable asset value

VaR data analysis:

- Count exceeded
- $Y \sim Bin(n, p=\alpha)$, test $P(Y \geq y \mid \text{model})$

VCV:

- $\sigma^2 = \sigma_A^2 + \sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B$, $\sigma_A^{(n)} = \text{Value}_A \times \frac{\text{Volatility}_A^{(m)}}{\sqrt{m}} \times \sqrt{n}$
- $VaR_{1-\alpha} = \sigma N^{-1}(1-\alpha)$
- assume linear combination, risk drivers geo brownian, $\underline{\mu}=0$

HisSim:

- $VaR_{1-\alpha} = 100\alpha$ percentile of losses, ES = avg. of times percentile exceeded

MC:

- determine risk drivers, assume stochastic dynamics, determine portfolio val as function, run many sim

- Probability of default, yield & spread:
  $B_G = Xe^{-rT}$, $B_C = Xe^{-yT} = (1-\hat{p})B_G$, $1-\hat{p} = e^{-sT}$, $s = r-y$
  - Risk neutral $\hat{p}$ for pricing, objective p for risk management
- Merton Model

+ Simple; can use BS valuation

$V_t = E_t + D_t$, $dV = \mu V dt + \sigma V dz \xrightarrow{(\mu=r)} \hat{p}(V_T < X) = N(-d_2)$,

$E_T = \begin{cases} 0 & \text{if } V_T \leq X \\ V_T - X & \text{o/w} \end{cases} = \max(0, V_T - X) = C_X^T(V_t)$ $\quad [\ln V_T - \ln V_0 \sim N((r-\frac{\sigma^2}{2})T, \sigma^2 T)]$

(as it's a bond) $\left(\begin{array}{l} X = D_t e^{(r+s)(T-t)} \\ D_t = Xe^{-y(T-t)} \end{array}\right)$

$D_T = \begin{cases} V_T & \text{if } V_T \leq X \\ X & \text{o/w} \end{cases} = \min(V_T, X) = X - \max(X - V_T, 0) = X - P_X^T(V_t)$

- Hazard Rates:
  - $S(t) = e^{-\int_0^t h(u)du}$, $f(t) = h(t)\,S(t)$
  - $P[\text{die in } [t_2, t_3] \mid S(t_1)'] = \frac{S(t_2) - S(t_3)}{S(t_1)}$ $\left(P(A|B) = \frac{P(A,B)}{P(B)}\right)$
- CDS:

  $F.L = \sum sM\Delta\, B(0,t_i)\, S(t_i)$

  $DL = \int_0^T (1-R) M B(0,u) f(u)\, du$

  $AP = \sum sM\Delta \int_{t_{i-1}}^{t_i} \frac{u - t_{i-1}}{t_i - t_{i-1}} B(0,u) h(u)\, du$

  $F.L + AP = DL$

- ~~Cour~~ Martingale: $E[X_{t+y} \mid X_t] = X_t$
- Definitions:
  - Complete economy (attainable derivative)
  - Self financing portfolio: $dV_t = \phi_t\, dS_t + \psi_t\, dB_t$
  - Martingale Prob World: R all martingales $\Rightarrow$
  - Fundamental theory:

    arbitrage free $\Leftrightarrow \exists$ empw

    complete $\Leftrightarrow$ empw is unique for every numeraire

Pricing Euro options:
$B_t = B_0 e^{rt}$, $R_t = \dots$
$K(t) = \dots$
$\Rightarrow dS = \dots$
$f_t = e^{-r(T-t)} \tilde{E}_t[X_T]$

- Martingale pricing

  $V_t = B_t\, \tilde{E}_t[B_T^{-1} X_T]$
- Changing prob world:

  $dz = dz^* - K(t)\, dt$
- Stochastic interest rates:
  - $P(t,T) = e^{-r(T-t)}$ if constant r
  - FRA:

    $r = \frac{-1}{T_2 - T_1} \ln k$, $k = \frac{P(t,T_2)}{P(t,T_1)}$
  - FRNP:
    - $f_t = B_t\, \tilde{E}_t[B_T^{-1} X_T]$, $B_T = P(t,T)$

      $f_t = P(t,T)\, \tilde{E}_t[X_T]$, and $\tilde{E}_t[X_T] = F_t$ in this world
  - bond options:

Blacks Model

$f_t = P(t,T)\, \tilde{E}_t[\max\{P(T,T_1) - K, 0\}] =$

$= P(t,T)\, [\tilde{E}_t[P(T,T_1)] N(d_1) - K N(d_2)]$

$f_t = P(t,T)\, [F_t N(d_1) - K N(d_2)]$

Feynman-Kac:

$$\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{1}{2} b(x,t)^2 \frac{\partial^2 F}{\partial x^2} = r(x,t) F$$

$F(x,T) = \phi(x,T)$

$\Leftrightarrow F(x,t) = E[e^{-\int_t^T r(x,u)du} \phi(X,T) \mid X_t = x]$

$dX = a(x,t)\,dt + b(x,t)\,dz$

if r too high, long 1 T_1 bonds, short $1/k$ T_2 bonds, long 1 FRA] time t] time T_1

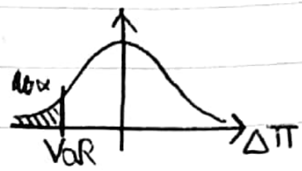
(reverse if too low)

1: Market Risk Measurement

Introduction to Market Risk

- risk associated w/ uncertainty about future value/earnings due to changes in asset prices / market rates
- Value at Risk:

  $(1-\alpha)$ VaR: $P(\Pi_{t+\Delta t} - \Pi_t \leq -V_\alpha) = \alpha$



Variance-Covariance Method:

1. Determine key risk drivers:
   - ie what causes changes in portfolio value, eg stock prices
2. Distribution assumptions:
   - (ie follows geo brownian motion) → risk factors follow multivariate normal distribution ($\Rightarrow$ log-normally distributed changes in risk factors)
3. Sensitivity of asset value to risk factors:
   - assume asset value linear combination of underlying risk drivers
4. Distribution parameters:
   - assume $\mu = 0$; ie no drift (ok as fairly on short time periods)
   - estimate $\Sigma = \frac{1}{n-1}\Sigma(X_i - \bar{X})^2$ [X_i = daily returns of i^{th} risk driver]

     (or $\sigma^2 = \sigma_A^2 + \sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B$)
5. Distribution of changes in portfolio value

   $VaR_\alpha = \langle Value \rangle \times \langle n\text{-day stdev} \rangle \times N^{-1}(\frac{\alpha}{100})$

Historical Simulation & Monte Carlo

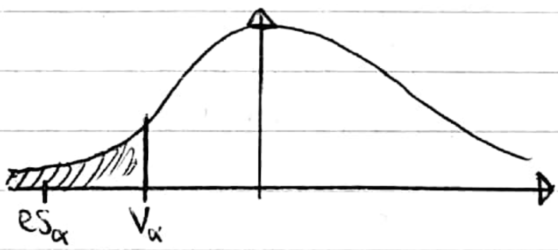
- VCV only works for linear products
- Historical simulation:
  - determine risk drivers; get series of historic values
  - convert to percentage changes ($v_i = \ln h_i/h_{i-1}$)
  - form empirical distribution of portfolio P/L
  - VaR is percentile of this distribution

- need lots of data to ensure we have outliers; 250+ days needed
- old data may not be representative of today
+ simple & intuitive, non-parametric (no distribution assumptions)

· Monte Carlo:
  · make assumptions on stochastic dynamics of risk drivers (including dependencies)
  · run simulations, determining distribution of portfolio value
  - time consuming to run; many simulations
  + works well for non-linear ~~& linear~~ products

Expected Shortfall

- $es_\alpha(X) = E[X \mid X \leq -V_\alpha(X)]$



· expected loss if VaR is exceeded
· Using VCV: conditional expectations on normal distribution
· Using HisSim: mean of all times VaR exceeded

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2: Credit Risk Measurement

Introduction to Credit Risk

- risk from a party defaulting on financial obligations
- Expected loss:

$$EL = PD \times EAD \times LGD = \text{probability of default} \times \underbrace{\text{exposure at default}}_{\text{eg amount of loan still not paid back}} \times \underbrace{\text{loss given default}}_{\text{\% of EAD}}$$

(1 − LGD = Recovery Rate)

- Corporate bonds:
  - have risk; bc company may go bankrupt
- Probability of default:
  - can be inferred from bond prices

$$B_G = Xe^{-rT}, \quad B_C = (1-\hat{p})B_G \Rightarrow \hat{p} = \frac{B_G - B_C}{B_G}$$

B_C: (zero-coupon) risky corporate bond; B_G: (zero-coupon) risk-free gov bond

- Yields & Spread: (effective rate of return on B_C)

$$B_C = Xe^{-yT}, \quad s = y - r$$

$$\Rightarrow \hat{p} = 1 - e^{-sT}$$

($\hat{p}$ is risk-neutral probability; use for pricing but not for risk management)

Counterparty Credit Risk

- Risk that counterparty defaults prior to expiration of contract
  - primarily applies to OTC contracts; exchange would guarantee payments
- Incorporating into derivative pricing:

assume independent
- f^* = value of derivative w/ no chance of default
- $\hat{p}$ = probability of default

$$\Rightarrow f = (1-\hat{p})f^* = f^* e^{-sT}$$

ie f is f^* discounted at the spread, $y - r$

(if $f^* = Xe^{-rT}$, $f = Xe^{-(s+r)T} = Xe^{-yT}$)

Structural ~~Risk~~ Credit Risk Modelling

- model firm value as stochastic process; default if falls below threshold value
- Debt & Equity in a firm:
  - on bankruptcy, debt holders take priority
  - equity holders benefit from increases in firm assets value
- Merton model for corporate liability valuation:

  Total value of firm assets = Equity + Debt

  $$V(t) = E(t) + D(t)$$

  default if $V_T < X$ → • assume all debt in form of single zero-coupon bond w/ face value X & maturity time T

  $$E_T = \begin{cases} 0 & \text{if } V_T < X \\ V_T - X & \text{if } V_T \geq X \end{cases}$$

  , ie equity can be a call option, strike price X (underlying V_t

  • assume V_t follows geometric brownian motion

  $$dV_t = \mu_V V\,dt + \sigma_V V\,dz$$

  $\Rightarrow$ can use Black-Scholes pricing for E

  - Probability of default:
    - use risk neutral world; $dV = rV\,dt + \sigma_V V\,dz$

  $\left[\ln V_T - \ln V_0 \sim N\left(\frac{r-\sigma^2}{2}T, \sigma^2 T\right)\right]$ $\hat{p} = \hat{P}(V_T < X) = 1 - N(d_2) = N(-d_2)$

    - in real world,

  $$p = P(V_T < X) = 1 - N\left(d_2 + \frac{\mu_V - r}{\sigma_V}\sqrt{T-t}\right)$$

  - Value of corporate debt:

  $$D_T = \begin{cases} X & \text{if } V_T > X \\ V_T & \text{if } V_T \leq X \end{cases}$$

  . ie debt can be ~~put option, strike X, underlying V_T~~ value $\min(V_T, X)$

  ~~$\Rightarrow$ can value by Black-Scholes, or~~

  - ~~$V_t = E_t + D_t$~~ · $\min(V_T, X) = X + \max(X - V_T, 0) = X - P_X$

  $\Rightarrow$ can value by Black-Scholes

  - Or,

  $\boxed{D(t) = e^{-r(T-t)} X N(d_2) + V_t(1 - N(d_1))}$

  ~~$V(t) = E(t) + D(t) = E(t) + X - P_X(t)$~~ $V(t) = E(t) + D(t)$

  ~~$V(T) =$~~ $D(t) = V(t) - E(t)$

(also note $E_T = V_T - D_T$

$$E_T = V_T - X + P_{XT} \Rightarrow E_T - P_{XT} = V_T - X \Rightarrow C - P = V_T - Xe^{-rT}$$

- Application of Merton model:
  - firm's total asset value not directly observable
    - stock processes only give part of information on equity
    - debt often in bank loans & not traded
  - need to estimate asset values & volatility
    - estimate σ_V from observable equity value process, E_t, using Ito's lemma

$$\sigma_E = \frac{\partial E}{\partial V} \frac{V_t}{E_t} \sigma_V \quad (\text{as } V_t = E_t + D_t)$$

Hazard Rate Credit Models

- Treat probability (or rate) of default as exogenous variable, which may depend on firm credit rating &/or other variables
- Intensity models:
  - model default process as point process; first point is default
  - (inhomogeneous) $\longrightarrow$ Poisson process, intensity (hazard rate) $h(t)$
    - prob default in $(t, t+\delta t) = h(t)\,\delta t$

$$h(t) = \frac{f(t) \leftarrow \text{pdf}}{1 - F(t) \leftarrow \text{cdf}}$$

$$1 - F(t) = \underset{\substack{\uparrow \\ \text{probability of 'survival' until time } t}}{S(t)} = \exp\left[-\int_0^t h(u)\,du\right] \qquad \left[\underbrace{f(t)}_{\text{density}} = \underbrace{h(t)}_{\text{instantaneous hazard}} \underbrace{S(t)}_{\text{chance of survival to } t}\right]$$

- Pricing credit default swaps:
  - transfer of risk between parties; ~~buyer~~ seller offers to pay buyer an amount lost in case of default of some entity
    - buyer pays a regular fee until default or maturity
  - Fixed Leg:
    - $FiL = E[\text{Fixed Leg}] = \sum_{i=1}^{n} s\Delta M B(0,t_i) S(t_i)$

expected value of all buyer payments

$$= sM\Delta \sum_{i=1}^{n} B(0,t_i) e^{-\int_0^{t_i} h(u)du}$$

(n payments of sM, $\frac{1}{\Delta}$ times a year
$B(t_i, t_j)$ is value at t_i of risk free bond, maturity t_j)

· Default Leg: considering possibility of default, expected value of payment gives this

$$DL = E[\text{Default Leg}] = \int_0^T (1-R)MB(0,u)f(u)du$$
$$= (1-R)M\int_0^T B(0,u)h(u)e^{-\int_0^u h(w)dw}du$$

(R = recovery rate, T = maturity)

· Accrual payment:
· seller deducts from default leg payment to account for fees from last scheduled payment until default time

$$AP = E[\text{Accrual payment}] = \sum_{i=1}^{n} s\Delta M\int_{t_{i-1}}^{t_i} \frac{u-t_{i-1}}{t_i-t_{i-1}} B(0,u)f(u)du$$
$$= \sum_{i=1}^{n} s\Delta M\int_{t_{i-1}}^{t_i} \frac{u-t_{i-1}}{t_i-t_{i-1}} B(0,u)h(u)e^{-\int_0^u h(w)dw}du$$

F.L. + AP = DL $\Rightarrow$ solve for s

3: Asset Pricing Techniques

Fundamental Theorem of Asset Pricing

- Asset price process S_t
- Derivative price process X_t, payoff $X_T(S_T)$ at maturity
- Additional tradeable asset with process B_t (numeraire asset)
- ϕ_t of underlying, ψ_t of numeraire
  $\Rightarrow V_t = \phi_t S_t + \psi_t B_t$, want V_t to replicate X_t
- Martingales: Fair Game processes:
  - Process X_n is a fair game if $E[X_{n+j} | X_n] = X_n$ (replace n w/ t for continuous time)
  - $dS = \mu S dt + \sigma S dz$
    $\Rightarrow E[S_{t+\tau} | S_t] = S_t e^{\mu\tau}$
    $\Rightarrow S$ is fair game if $\mu = 0$; $dX = \sigma X dz$
    (also must be bounded to be fair game)
- Numeriae, relative pricing process & self-financing:
  - Numeriae:
    - tradeable asset w/ strictly +ve price process
    - can express all other asset price processes in terms of numeriae
      $R_t = \frac{X_t}{B_t}$, relative price process
      (B_t is like measurement unit; one unit of some currency if not specified)
  - Self financing:
    - if funds enter/exit our replicating portfolio, arbitrage opportunities exist
    - any change in ϕ_t must be funded by changes in ψ_t (& vice versa)
    $\Rightarrow dV_t = \phi_t dS_t + \psi_t dB_t$
- Complete economies:
  - if self-financing strategy that ~~mimics~~ replicates derivative payoff X_T exists, derivative is attainable
  - if all derivatives in an economy are attainable, economy is complete

in Black-Scholes framework, N=R=1

$N+1$ ~~[illegible]~~ tradeable assets (1 risk free)
R sources of randomness (eg independent Brownian motions)

$N \leq R \Rightarrow$ Arbitrage free
$N \geq R \Rightarrow$ Complete
$\Rightarrow N = R \Rightarrow$ Arbitrage free & Complete

- Martingale Probability Worlds:
  - Equivalent Martingale Prob World if ~~all~~ relative price process of all tradeable assets are martingale processes
- Fundamental theorem of asset pricing:
  - Market arbitrage free iff exists equivalent martingale probability world measure
  - Arbitrage free market complete iff this empw is unique for every choice of numeriare
  - Tells us, given choice of numeriare, can find a probability world where ~~pr~~ relative price process are martingales, & self-financing trading strategies cannot outperform the market

by theory, exist unique Martingale probability world

Martingale Pricing Technique

- Consider complete & arbitrage free market, and numeraire with price process B_t
- Self financing trading strategy has process V_t, and $V_T = X_T$ for derivative w/ payoff X_T at maturity

$$\tilde{E}\left[\frac{V_T}{B_T} \middle| V_t, B_t\right] = \tilde{E}_t\left[\frac{V_T}{B_T}\right] = \frac{V_t}{B_t}$$

expectation in Martingale world

$$\tilde{E}_t\left[\frac{X_T}{B_T}\right] = \frac{V_t}{B_t} \Rightarrow V_t = B_t \tilde{E}_t\left[B_T^{-1} X_T\right]$$

(by no arbitrage, V_t is value of derivative)

- Constructing martingale measure & process:
  - need to find distribution of underlying asset price under a probability measure leading to it's relative price process being a martingale
  - Changing probability worlds:
    - Z_t is Brownian Motion in probability world P
    - $k(t)$ is a (possibly stochastic) process where $\int_0^t k(u)^2 du < \infty$
    - k defines change to new probability world P^*,

$$dZ = dZ^* - k(t)dt$$

ie $Z^*(t) = Z(t) + \int_0^t k(u)du$

$\Rightarrow$ changing probability world changes drift of process, but not volatility

$\Rightarrow$ can think of drift as same as probability measure

Example - Black-Scholes Framework

- Underlying price process:

$$dS = \mu S dt + \sigma S dz$$

- Numeriae:

$$dB = rBdt \quad (\Rightarrow B_t = B_0 e^{rt})$$

- Relative price process:

$$R_t = S_t / B_t = S(t) e^{-rt}$$

- Find martingale probability world:
  - By Ito's lemma, $dR = (\mu - r) R dt + \sigma R dZ$

    $\Rightarrow k(t) = \frac{\mu - r}{\sigma}$

    So, in world P^*, $dR = \sigma R dZ^*$ $(dZ = dZ^* - k(t)dt)$

    (0 drift, as required)

    and

    $$dS = rSdt + \sigma S dZ^* \quad \text{(drift rate now } r\text{)}$$

    providing us with probability distribution for S_T

- Pricing result:

$$\mathfrak{J}_t = B_t \tilde{E}_t [B_T^{-1} X_T]$$

$$= B_t \tilde{E}_t [(B_t e^{r(T-t)})^{-1} X_T]$$

$$\mathfrak{J}_t = e^{-r(T-t)} \tilde{E}_t [X_T] \quad \text{[risk neutral pricing result]}$$

Stochastic Interest Rate Environment

- constant r reasonable assumption for pricing equity derivatives, as uncertainty in r dominated by, eg, volatility in underlying asset / risk drivers
- not good assumption for products with greater dependence on cost of money, eg interest rate products
- Discount bonds:
  - pays 1 unit of currency at maturity at time T
  - follows price process $P(t,T)$ $\quad [P(t,T) = e^{-r(T-t)}$ if r constant$]$
- Forward pricing:
  - underlying S_t, payoff $X_T = S_T - F$
  - find F through no arbitrage:

replicating portfolio → $\cdot\ \Pi = 1$ long S, F short discount bonds maturity T

$$\Pi_T = S_T - F$$

$$\Pi_t = S_t - F\,P(t,T) = 0 \quad (\text{so } f_t = 0\text{, agreed forward at time } t)$$

$$\Rightarrow F = S_t / P(t,T)$$

- Forward rate agreements:
  - Party A pays Party B principle at time $T_1 > t$
  - B pays A principle + pre-agreed interest amount at $T_2 > T_1$
  - Assume A receive 1 unit at T_2; what should A pay at T_1?
    - $\Pi =$ short k T_1 bonds, long 1 T_2 bond

$$\Pi_t = P(t,T_2) - k\,P(t,T_1) = 0$$

$$\cancel{\Pi_{T_1} = P(T_1,T_2) - k} \Rightarrow k = \frac{P(t,T_2)}{P(t,T_1)}$$

(A pay k at T_1, receive 1 at T_2

~~$\Rightarrow$ agreed interest rate =~~

$$k e^{r(T_2-T_1)} = 1 \Rightarrow r = \frac{-1}{T_2-T_1}\ln k$$

Example - Forward Risk Neutral Pricing

- Derivative product payoff depends on future interest rate products
  $\Rightarrow$ cannot ignore uncertainty of future interest rates
- Many underlying assets ($P(t,T)$ for many T's) we can use as numeraire (don't need to use one based on spot interest rate)
- Forward risk-neutral process:
  - $f_t = B_t \tilde{E}_t[B_T^{-1} X_T]$; hard to evaluate expectation if B_T stochastic
    $\Rightarrow$ use $B_t = P(t,T)$, $B_T = 1$

  $$f_t = P(t,T)\tilde{E}_t[X_T]$$

  also, as seen $F_t = f_t/P(t,T) = f_t/P(t,T) = \tilde{E}_t(X_T)$
  $\Rightarrow$ in this world can assume expected payoff = forward value
- Bond options:
  - option to buy bond at future time for agreed price
    - value due to uncertain future interest rates
  - Euro call option on T_2 bond, strike price K; maturity $T<T_2$

    $X_T = \max\{P(T,T_2)-K, 0\}$

$$f_t = B_t \tilde{E}_t(f_T/B_T]$$

invalid if t close to T_2; close to T_2 value approaches 1

$$f_t = P(t,T)\tilde{E}_t[\max\{P(T,T_2)-K,0\}]$$

- Assume $P(t,T)$ log normally distributed

in formula sheet (?)

$$\Rightarrow f_t = P(t,T)(\tilde{E}[P(T,T_2)]N(d_1) - KN(d_2))$$

using $E[\max(V-X,0)] = E[V]N(d_1) - XN(d_2)$

$$f_t = P(t,T)[F_t N(d_1) - KN(d_2)]$$

Black's Model

$$\left(d_1 = \frac{\ln(F_t/K) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T}\right)$$

Relationship between PDEs & SDEs

· Feynman-Kac result: (relate ~~B~~PDE to SDE)

BSM PDE $\rightarrow$ · PDE $\frac{\partial G}{\partial t} + a(s,t)\frac{\partial G}{\partial s} + \frac{1}{2} b(s,t)^2 \frac{\partial^2 G}{\partial s^2} - r(x,t)G = 0$

payoff at maturity $\rightarrow$ · boundary conditions
$G(s,T) = \Psi(s,T)$

· has solution

risk neutral pricing formula $\longrightarrow$ $E[e^{-\int_t^T r(x,u)du}\Psi(S,T) | S_t = x]$, where $dS = a(t,S)dt + b(t,S)dZ$

$\Rightarrow$ can use certain SDEs to solve certain PDEs & vice versa
(eg MC simulation to find SDE expectation, then use to solve PDEs)

· Formulation of result:

$dS = a(S,t)dt + b(S,t)dz = a\,dt + b\,dz$, $S_t = S_0$, known S_0

$\Rightarrow dF = \left[\frac{\partial F}{\partial S}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2\right]dt + \frac{\partial F}{\partial z} b\,dz$, for $F(S,t)$

If $\frac{\partial F}{\partial S}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}b^2 = 0 \Rightarrow dF = \frac{\partial F}{\partial S} b\,dz$

$\Rightarrow F(S_T,T) - F(S_t,t) = \int_t^T \frac{\partial}{\partial S}F(S_u,u)\,b\,dz_u$

$\Rightarrow E[F(S_T,T)|S_t = S_0] - F(S_0,t) = E\left[\int_t^T \frac{\partial}{\partial S}F(S_u,u)\,b\,dz_u \,|\, S_t = S_0\right]$

$\Rightarrow E[F(S_T,T)|S_t = S_0] = F(S_0,t)$ [rhs 0 by properties of Brownian motion]

· Extension for BSM PDE:

$rF = \frac{\partial F}{\partial S}rS + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2$, $F(S_T,T) = \Psi(S_T)$

(eg $\Psi(S_T) = \max\{S_T - K, 0\}$ for a Euro call

· Let $G(S_t,t) = e^{r(T-t)}F(S_t,t)$

· Sub into BSM PDE

$\Rightarrow \frac{\partial G}{\partial S}rS + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 = 0$

⇒ can use Feynman-Kac result,

$$G(S_0,t) = E[G(S_T,T)|S_t = S_0]$$

$$G(S_0,t) = E[\Psi(S,T)|S_t = S_0] \quad (\text{where } dS = rSdt + \sigma Sdz)$$

· Substitute F back in,

$$F(S_0,t) = e^{-r(T-t)} E[F(S_T,T)|S_t = S_0]$$

$$F_t = e^{-r(T-t)} \hat{E}[F_T|S_t] \leftarrow \text{risk neutral pricing formula}$$